THE KK-THEORY OF FUNDAMENTAL C*-ALGEBRAS

PIERRE FIMA AND EMMANUEL GERMAIN

Abstract. Given a graph of C*-algebras as defined in [FF13], we prove a long exact sequence in KK-theory similar to the one obtained by Pimsner in [Pi86] for both the maximal and the vertex-reduced fundamental C*-algebras of the graph in the presence of possibly non GNS-faithful conditional expectations. We deduce from it the KK-equivalence between the full fundamental C*-algebra and the vertex-reduced fundamental C*-algebra even for non GNS-faithful conditional expectations. Our results unify, simplify and generalize all the previous results obtained before by Cuntz, Pimsner, Germain and Thomsen. It also generalizes the previous results of the authors on amalgamated free products.

1. Introduction

In 1986 the description of the KK-theory for some groups C*-algebras culminated in the computation by M. Pimsner of full and reduced crossed products by groups acting on trees [Pi86] (or by the fundamental group of a graph of groups in Serre's terminology). To go over the group situation has been difficult and it relied heavily on various generalizations of Voiculescu absorption theorem (see [Th03] for the most general results in that direction). Note also that G. Kasparov and G. Skandalis had another proof of Pimsner long exact sequence when studying KK-theory for buildings [KS91].

However the results we obtain here are based on a completely different point of view. Introduced in [FF13], the full or reduced fundamental C*-algebras of a graph of C*-algebras allows to treat on equal footings amalgamated free products and HNN extensions (and in particular cross-product by the integers). Let's describe its context. A graph of C*-algebras is a finite oriented graph with unital C*-algebras attached to its edges (B_e) and vertices (A_v) such that for any edge e there are embeddings r_e and s_e of B_e in $A_{r(e)}$ and $A_{s(e)}$ with $r(e)$ the range of e and $s(e)$ its source. As for groups, the full fundamental C*-algebra of the graph is a quotient of the universal C*-algebra generated by the A_v and unitaries u_e such that $u_e^* s_e(b) u_e = r_e(b)$ for all $b \in B_e$. In the presence of conditional expectations from $A_{s(e)}$ and $A_{r(e)}$ onto B_e, one can also construct various representations of the full fundamental C*-algebra on Hilbert modules over A_v or B_e. It is the interplay with the representations that yields the tools we need to prove our results.

In our previous paper [FG15], we first looked at one of the simplest graphs : one edge, two different endpoints. The full fundamental C*-algebra is then the full amalgamated free product. When the conditional expectations are *not* GNS-faithful, there are two possible reduced versions: the reduced free product of D. Voiculecscu, that we call the edge-reduced amalgamated free product and the vertex-reduced amalgamated free product we did construct in [FG15]. We did

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show that the full amalgamated free product and the vertex-reduced amalgamated free product are always K-equivalent and we did exhibit a long exact sequence in KK-theory for both of them.

In this paper, we extend the results of [FG15] to any fundamental C*-algebra of a finite graph of C*-algebras in the presence of conditional expectations, even non GNS-faithful ones.

Our first task is to introduce the good version of the reduced fundamental C*-algebra since there are several possible constructions of the reduced fundamental C*-algebra when the conditional expectations are not GNS-faithful and this fact was not clearly known to the authors in [FF13] in which it was always assumed that the conditional expectations are GNS-faithful. The construction of the vertex-reduced fundamental C*-algebra is made in section 2. We also describe in details its fundamental properties.

Our second task is to define the boundary maps in the long exact sequence. This will be done in a natural way: by multiplication, in the Kasparov product sense, by some elements in KK^1 that we construct in a geometric way in section 3. We also study the fundamental properties of these KK^1 elements which will be useful to prove the exactness of the sequence.

In section 4 we prove our main result: the exactness of the sequence. This is done by induction, using the analogue of Serre's devissage process, the properties of our KK^1 elements and the initial cases: the amalgamated free product case which was done in [FG15] and the HNN-extension case which can be deduced from the amalgamated free product case by a remark of Ueda [Ue08]. Explicitly, if C is any separable C*-algebra, P the full or reduced fundamental C*-algebra of the finite graph of C*-algebras $(\mathcal{G}, A_p, B_e)$ then we have the two 6-terms exact sequence, where E^+ is the set of positive edges and V is the vertex set of the graph $\mathcal{G}$,

$$\begin{array}{ccccc} \bigoplus_{e\in E^+} KK^0(C,B_e) & \xrightarrow{\sum s_e^*-r_e^*} & \bigoplus_{p\in V} KK^0(C,A_p) \longrightarrow & KK^0(C,P) \\ \uparrow & & & \downarrow \\ KK^1(C,P) & \longleftarrow \bigoplus_{p\in V} KK^1(C,A_p) & \xleftarrow{\sum s_e^*-r_e^*} & \bigoplus_{e\in E^+} KK^1(C,B_e) \end{array}$$

and

$$\begin{array}{ccccc} \bigoplus_{e\in E^+} KK^0(B_e,C) & \xleftarrow{\sum s_{e*}-r_{e*}} & \bigoplus_{p\in V} KK^0(A_p,C) \longleftarrow & KK^0(P,C) \\ \downarrow & & & \uparrow \\ KK^1(P,C) & \longrightarrow \bigoplus_{p\in V} KK^1(A_p,C) & \xrightarrow{\sum s_{e*}-r_{e*}} & \bigoplus_{e\in E^+} KK^1(B_e,C) \end{array}$$

In section 5 we give some applications of our results. A direct Corollary of our results is that the full and vertex-reduced fundamental C*-algebra of a graph of C*-algebras are K-equivalent. This generalizes and simplifies the results of Pimsner about the KK-theory of crossed-products by groups acting on trees [Pi86]. Also, our results imply that the fundamental quantum group of a graph of discrete quantum groups is K-amenable if and only if all the vertex quantum groups are K-amenable. This generalizes and simplifies the results of [FF13].

2. Preliminaries

2.1. **Notations and conventions.** All C*-algebras and Hilbert modules are supposed to be separable. For a C*-algebra A and a Hilbert A-module H we denote by $\mathcal{L}_A(H)$ the C*-algebra a A-linear adjointable operators from H to H and by $\mathcal{K}_A(H)$ the sub-C*-algebra of $\mathcal{L}_A(H)$ consisting of A-compact operators. We write $L_A(a) \in \mathcal{L}_A(A)$ the left multiplication operator by

$a \in A$. We use the term *ucp* for unital completely positive. When $\varphi : A \to B$ is a ucp map the *GNS construction of* is the unique, up to a canonical isomorphism, triple (H, π, ξ) such that H is a Hilbert B-module, $\pi : A \to \mathcal{L}_B(H)$ is a unital $*$-homomorphism and $\xi \in H$ is a vector such that $\pi(A)\xi \cdot B$ is dense in H and $\langle \xi, \pi(a)\xi \cdot b\rangle = \varphi(a)b$. We refer the reader to [Bl86] for basics notions about Hilbert C*-modules and KK-theory.

2.2. **Some homotopies.**

Lemma 2.1. *Let A, B be unital C*-algebras, H, K Hilbert B-modules, $\pi : A \to \mathcal{L}_B(H)$, $\rho : A \to \mathcal{L}_B(K)$ unital $*$-homomorphisms and $F \in \mathcal{L}_B(H,K)$ a partial isometry such that $F\pi(a) - \rho(a)F \in \mathcal{K}_B(H,K)$ for all $a \in A$ and $F^*F - 1 \in \mathcal{K}_B(H)$. Then, $[(K, \rho, V)] = 0 \in KK^1(A,B)$, where $V = 2FF^* - 1$.*

Proof. Let $\alpha := [(K, \rho, V)] \in KK^1(A,B)$. For $t \in [0,1]$, define

$$U_t = \begin{pmatrix} 1 - FF^* & 0 \\ 0 & 0 \end{pmatrix} + \cos(\pi t)\begin{pmatrix} FF^* & 0 \\ 0 & -1 \end{pmatrix} - \sin(\pi t)\begin{pmatrix} 0 & F \\ F^* & 0 \end{pmatrix} \in \mathcal{L}_B(K \oplus H).$$

We have $U_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $U_1 = -\begin{pmatrix} V & 0 \\ 0 & 1 \end{pmatrix}$. Note that, for all $t \in [0,1]$, $U_t^* = U_t$ and,

$$\begin{aligned} U_t^2 &= \begin{pmatrix} 1 - FF^* & 0 \\ 0 & 0 \end{pmatrix} + \cos(\pi t)^2\begin{pmatrix} FF^* & 0 \\ 0 & 1 \end{pmatrix} + \sin(\pi t)^2\begin{pmatrix} FF^* & 0 \\ 0 & F^*F \end{pmatrix} \\ &= \begin{pmatrix} 1 - FF^* & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} FF^* & 0 \\ 0 & 1 \end{pmatrix} + K_t = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + K_t, \end{aligned}$$

where $K_t = \sin(\pi t)^2 \begin{pmatrix} 0 & 0 \\ 0 & F^*F - 1 \end{pmatrix} \in \mathcal{K}_B(K \oplus H)$ for all $t \in [0,1]$, since $F^*F - 1 \in \mathcal{K}_B(H)$. Moreover, $U_t(\rho \oplus \pi)(a) - (\rho \oplus \pi)(a)U_t \in \mathcal{K}_B(K \oplus H)$ for all $a \in A$ since $F\pi(a) - \rho(a)F \in \mathcal{K}_B(H,K)$ for all $a \in A$. Consider the unique operators $U \in \mathcal{L}_{B\otimes C([0,1])}(K \oplus H) \otimes C([0,1]))$ and $K \in \mathcal{K}_{B\otimes C([0,1])}(K \oplus H) \otimes C([0,1]))$ such that the evaluation of U at t is U_t and the evaluation of K at t is K_t for all $t \in [0,1]$. In particular we have $U = U^*$ and $U^2 = 1 + K$ and, since $U_t(\rho \oplus \pi)(a) - (\rho \oplus \pi)(a)U_t \in \mathcal{K}_B(K \oplus H)$ for all $a \in A$ and all $t \in [0,1]$, we have

$$U(\rho \oplus \pi)(a) \otimes 1_{C([0,1])} - (\rho \oplus \pi)(a) \otimes 1_{C([0,1])}U \in \mathcal{K}_{B\otimes C([0,1])}((K \oplus H) \otimes C([0,1])) \quad \text{for all } a \in A.$$

Hence we get an homotopy

$$\gamma = [((K \oplus H) \otimes C([0,1]), (\rho \oplus \pi) \otimes 1_{C([0,1])}, U)] \in KK^1(A \otimes C([0,1]), B \otimes C([0,1]))$$

between $\gamma_0 = [(K \oplus H, \rho \oplus \pi, U_0)] = [(K \oplus H, \rho \oplus \pi, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix})] = 0$ since the triple is degenerated and $\gamma_1 = [(K \oplus H, \rho \oplus \pi, U_1)] = [(K \oplus H, \rho \oplus \pi, -\begin{pmatrix} V & 0 \\ 0 & 1 \end{pmatrix})]$. Hence, $\gamma_1 = x \oplus y$, where $x = [(K, \rho, -V)] = -\alpha$ and $y = [(H, \pi, -\mathrm{id}_H)] = 0$, since the triple is degenerated. □

2.3. **Fundamental C*-algebras.**

In this section we recall the results and notations of [FF13] and generalize the constructions to the case of non GNS-faithful conditional expectations.

If $\mathcal{G}$ is a graph in the sense of [Se77, Def 2.1], its vertex set will be denoted $\mathrm{V}(\mathcal{G})$ and its edge set will be denoted $\mathrm{E}(\mathcal{G})$. We will always assume that $\mathcal{G}$ is at most countable. For $e \in \mathrm{E}(\mathcal{G})$ we denote by $s(e)$ and $r(e)$ respectively the source and range of e and by $\overline{e}$ the inverse edge of

e. An *orientation* of $\mathcal{G}$ is a partition $\mathrm{E}(\mathcal{G}) = \mathrm{E}^+(\mathcal{G}) \sqcup \mathrm{E}^-(\mathcal{G})$ such that $e \in \mathrm{E}^+(\mathcal{G})$ if and only if $\overline{e} \in \mathrm{E}^-(\mathcal{G})$. We call $\mathcal{G}' \subset \mathcal{G}$ a *connected subgraph* if $V(\mathcal{G}') \subset V(\mathcal{G})$, $E(\mathcal{G}') \subset E(\mathcal{G})$ such that $e \in E(\mathcal{G}')$ if and only if $\overline{e} \in E(\mathcal{G}')$ and the graph $\mathcal{G}'$ with the source map and inverse map given map the ones of $\mathcal{G}$ restricted to $E(\mathcal{G}')$ is a connected graph.

Let $(\mathcal{G}, (A_q)_q, (B_e)_e)$ be a *graph of unital C*-algebras*. This means that

- $\mathcal{G}$ is a connected graph.
- For every $q \in \mathrm{V}(\mathcal{G})$ and every $e \in \mathrm{E}(\mathcal{G})$, A_q and B_e are unital C*-algebras.
- For every $e \in \mathrm{E}(\mathcal{G})$, $B_{\overline{e}} = B_e$.
- For every $e \in \mathrm{E}(\mathcal{G})$, $s_e : B_e \to A_{s(e)}$ is a unital faithful $*$-homomorphism.

For every $e \in \mathrm{E}(\mathcal{G})$, we set $r_e = s_{\overline{e}} : B_e \to A_{r(e)}$, $B_e^s = s_e(B_e)$ and $B_e^r = r_e(B_e)$. Given a maximal subtree $\mathcal{T} \subset \mathcal{G}$ the *maximal fundamental C*-algebra with respect to* $\mathcal{T}$ is the universal C*-algebra generated by the C*-algebras A_q, $q \in \mathrm{V}(\mathcal{G})$, and by unitaries u_e, $e \in \mathrm{E}(\mathcal{G})$, such that

- For every $e \in \mathrm{E}(\mathcal{G})$, $u_{\overline{e}} = u_e^*$.
- For every $e \in \mathrm{E}(\mathcal{G})$ and every $b \in B_e$, $u_{\overline{e}} s_e(b) u_e = r_e(b)$.
- For every $e \in \mathrm{E}(\mathcal{T})$, $u_e = 1$.

This C*-algebra will be denoted by P or $P_{\mathcal{G}}$. We will always view $A_p \subset P$ for all $p \in V(\mathcal{G})$ since, as explain in the following remark, the canonical unital $*$-homomorphisms from A_p to P are all faithful.

Remark 2.2. The C*-algebra P is not zero and the canonical maps $\nu_p : A_p \to P$ are injective for all $p \in V(\mathcal{G})$. This follows easily from the Voiculescu's absorption Theorem since we did assume all our C*-algebras separable and the graph $\mathcal{G}$ countable. Indeed, since A_p is separable for all $p \in V(\mathcal{G})$ and since $\mathcal{G}$ is at most countable we can representation faithfully all the A_p on the same separable Hilbert space H. Write $\pi'_p : A_p \to \mathcal{L}(H)$ the faithful representation. Replacing H by $H \otimes H$ and π'_p by $\pi'_p \otimes \mathrm{id}$ if necessary, we may and will assume that $\pi'_p(A_p) \cap \mathcal{K}(H) = \{0\}$ for all $p \in V(\mathcal{G})$. Write $C = \mathcal{L}(H)/\mathcal{K}(H)$ the Calkin algebra and $Q : \mathcal{L}(H) \to C$ the canonical surjection. Fix an orientation of $\mathcal{G}$. For $e \in E(\mathcal{G})$ we have two faithful representations $\pi'_{s(e)} \circ s_e$ and $\pi'_{r(e)} \circ r_e$ of B_e on H, both having trivial intersection with $K(H)$. By Voiculescu's absorption Theorem there exists, for all $e \in E^+(\mathcal{G})$, a unitary $V_e \in C$ such that $Q \circ \pi'_{r(e)}(r_e(b)) = V_e^* Q \circ \pi'_{s(e)}(s_e(b)) V_e$ for all $b \in B_e$ and all $e \in E^+(\mathcal{G})$. For $e \in E^-(\mathcal{G})$ define $V_e := (V_{\overline{e}})^*$ so that the relations $(V_e)^* = V_{\overline{e}}$ and $Q \circ \pi'_{r(e)}(r_e(b)) = V_e^* Q \circ \pi'_{s(e)}(s_e(b)) V_e$ holds for all $b \in B_e$ and all $e \in E(\mathcal{G})$. When $\omega = (e_1, \dots e_n)$ is a path in $\mathcal{G}$, we denote by V_ω the unitary $V_\omega := V_{e_1} \dots V_{e_n} \in C$ (if ω is the empty path we put $V_\omega = 1$). Fix a maximal subtree $\mathcal{T} \subset \mathcal{G}$. For $p, q \in V(\mathcal{G})$ let g_{pq} be the unique geodesic path in $\mathcal{T}$ from p to q (if $p = q$ then g_{pq} is the empty path by convention). Fix $p_0 \in V(\mathcal{G})$ and, for $e \in E(\mathcal{G})$, define $U_e := (V_{g_{s(e)p_0}})^* V_{(e, g_{r(e)p_0})}$ so that the relations $U_{\overline{e}} = U_e^*$ holds for any $e \in E(\mathcal{G})$ and $U_e = 1$ for any $e \in E(\mathcal{T})$. Finally, for $p \in V(\mathcal{G})$, define the faithful (since $\pi'_p(A_p) \cap \mathcal{K}(H) = \{0\}$) unital $*$-homomorphism $\pi_p : A_p \to C$ by $\pi_p := (V_{g_{p_0 p}})^* Q \circ \pi'_p(\cdot) V_{g_{p_0 p}}$. Then, it is easy to check that the relation $\pi_{r(e)}(r_e(b)) = U_e^* \pi_{s(e)}(s_e(b)) U_e$ holds for all $b \in B_e$ and all $e \in E(\mathcal{G})$. Hence, P is not zero and we have a unique unital $*$-homomorphism $\pi : P \to C$ such that $\pi(u_e) = U_e$ and $\pi \circ \nu_p = \pi_p$ for all $p \in V(\mathcal{G})$. In particular, the canonical map ν_p from A_p to P is faithful since π_p is faithful. Note that, when the C*-algebras A_p are not supposed to be separable and/or the graph $\mathcal{G}$ is not countable anymore the result is still true by considering the universal representation, as in the proof of [Pe99, Theorem 4.2] (which was inspired by [Bl78]).

Remark 2.3. Let $\mathcal{A} \subset P$ be the $*$-algebra generated by the A_q, for $q \in V(\mathcal{G})$, and the unitaries u_e, for $e \in E(\mathcal{G})$. Then $\mathcal{A}$ is a dense unital $*$-algebra of P. Moreover, since the graph $\mathcal{G}$ is supposed to be connected, for any fixed $p \in \mathrm{V}(\mathcal{G})$, $\mathcal{A}$ is the linear span of A_p and elements of the form $a_0 u_{e_1} \dots u_{e_n} a_n$ where $(e_1, \dots, e_n)$ is a path in $\mathcal{G}$ from p to p, $a_0 \in A_p$ and $a_i \in A_{r(e_i)}$ for $1 \leqslant i \leqslant n$.

We now recall the construction of the reduced fundamental C*-algebra, when there is a family of conditional expectations $E_e^s : A_{s(e)} \to B_e^s$, for $e \in E(\mathcal{G})$. Set $E_e^r = E_{\overline{e}}^s : A_{r(e)} \to B_e^r$ and note that, in contrast with [FF13], we do not assume the conditional expectations E_e^s to be GNS-faithful. However, as it was already mentioned in [FF13], all the constructions can be easily carried out without the non-degeneracy assumption. Let us recall these constructions now. We shall omit the proofs which are exactly the same as the GNS-faithful case and concentrate only on the differences with the GNS-faithful case.

For every $e \in \mathrm{E}(\mathcal{G})$ let $(K_e^s, \rho_e^s, \eta_e^s)$ be the GNS construction of the ucp map $s_e^{-1} \circ E_e^s : A_{s(e)} \to B_e$. This means that K_e^s is a right Hilbert B_e-module, $\rho_e^s : A_{s(e)} \to \mathcal{L}_{B_e}(K_e^s)$ and $\eta_e^s \in K_e^s$ are such that $K_e^s = \overline{\rho_e^s(A_{s(e)})\eta_e^s \cdot B_e}$ and $\langle \eta_e^s, \rho_e^s(a)\eta_e^s \cdot b\rangle = s_e^{-1} \circ E_e^s(a)b$. In particular, we have the formula $\rho_e^s(a)\eta_e^s \cdot b = \rho_e^s(a s_e(b))\eta_e^s$. Let us notice that the submodule $\eta_e^s.B_e$ of K_e^s is orthogonally complemented. In fact, its orthogonal complement $(K_e^s)^\circ$ is the closure of the set $\{\rho_e^s(a)\eta_e^s : a \in A_{s(e)}, E_e^s(a) = 0\}$ which is easily seen to be a Hilbert B_e-submodule of K_e^s. Similarly, the orthogonal complement of $\eta_e^r.B_e$ in K_e^r will be denoted $(K_e^r)^\circ$. Note that $\rho_e^s(B_e^s)(K_e^s)^\circ \subset (K_e^s)^\circ$.

Let $n \geqslant 1$ and $w = (e_1, \dots, e_n)$ a path in $\mathcal{G}$. We define Hilbert C*-modules K_i for $0 \leqslant i \leqslant n$ by

- $K_0 = K_{e_1}^s$
- If $e_{i+1} \neq \overline{e}_i$, then $K_i = K_{e_{i+1}}^s$
- If $e_{i+1} = \overline{e}_i$, then $K_i = (K_{e_{i+1}}^s)^\circ$
- $K_n = A_{r(e_n)}$

For $0 \leqslant i \leqslant n-1$, K_i is a right Hilbert $B_{e_{i+1}}$-module and K_n will be seen as a right Hilbert $A_{r(e_n)}$-module. We define, for $1 \leqslant i \leqslant n-1$, the unital $*$-homomorphism

$$\rho_i = \rho_{e_{i+1}}^s \circ r_{e_i} : B_{e_i} \to \mathcal{L}_{B_{e_{i+1}}}(K_i),$$

and, $\rho_n = L_{A_{r(e_n)}} \circ r_{e_n} : B_{e_n} \to \mathcal{L}_{A_{r(e_n)}}(K_n)$. We can now define the right Hilbert $A_{r(e_n)}$-module

$$H_w = K_0 \underset{\rho_1}{\otimes} \dots \underset{\rho_n}{\otimes} K_n$$

endowed with the left action of $A_{s(e_1)}$ given by the unital $*$-homomorphism defined by

$$\lambda_w = \rho_{e_1}^s \otimes \mathrm{id} : A_{s(e_1)} \to \mathcal{L}_{A_{r(e_n)}}(H_w).$$

For any two vertices $p, q \in \mathrm{V}(\mathcal{G})$, we define the Hilbert A_p-module $H_{q,p} = \bigoplus_w H_w$, where the sum runs over all paths w in $\mathcal{G}$ from q to p. By convention, when $q = p$, the sum also runs over the empty path, where $H_\emptyset = A_p$ with its canonical Hilbert A_p-module structure. We equip this Hilbert C*-module with the left action of A_q which is given by $\lambda_{q,p} : A_q \to \mathcal{L}_{A_p}(H_{q,p})$ defined by $\lambda_{q,p} = \bigoplus_w \lambda_w$, where, when $q = p$ and $w = \emptyset$ is the empty path, $\lambda_\emptyset := L_{A_p}$.

For every $e \in \mathrm{E}(\mathcal{G})$ and $p \in \mathrm{V}(\mathcal{G})$, we define an operator $u_e^p : H_{r(e),p} \to H_{s(e),p}$ in the following way. Let w be a path in $\mathcal{G}$ from $r(e)$ to p and let $\xi \in \mathcal{H}_w$.

- If $p = r(e)$ and w is the empty path, then $u_e^p(\xi) = \eta_e^s \otimes \xi \in H_{(e)}$.
- If $n = 1$, $w = (e_1)$, $\xi = \rho_{e_1}^s(a)\eta_{e_1}^s \otimes \xi'$ with $a \in A_{s(e_1)}$ and $\xi' \in A_p$, then

– If $e_1 \neq \overline{e}$, $u_e^p(\xi) = \eta_e^s \otimes \xi \in H_{(e,e_1)}$.
– If $e_1 = \overline{e}$, $u_e^p(\xi) = \begin{cases} \eta_e^s \otimes \xi & \in H_{(e,e_1)} & \text{if} \quad E_{e_1}^s(a) = 0, \\ r_{e_1} \circ s_{e_1}^{-1}(a)\xi' & \in A_p & \text{if} \quad a \in B_{e_1}^s. \end{cases}$

- If $n \geqslant 2$, $w = (e_1, \dots, e_n)$, $\xi = \rho_{e_1}^s(a)\eta_{e_1}^s \otimes \xi'$ with $a \in A_{s(e_1)}$ and $\xi' \in K_1 \underset{\rho_2}{\otimes} \dots \underset{\rho_n}{\otimes} K_n$, then
  – If $e_1 \neq \overline{e}$, $u_e^p(\xi) = \eta_e^s \otimes \xi \in H_{(e,e_1,\dots,e_n)}$.
  – If $e_1 = \overline{e}$, $u_e^p(\xi) = \begin{cases} \eta_e^s \otimes \xi & \in H_{(e,e_1,\dots,e_n)} & \text{if} \quad E_{e_1}^s(a) = 0, \\ (\rho_1(s_{e_1}^{-1}(a)) \otimes \mathrm{id})\xi' & \in H_{(e_2,\dots,e_n)} & \text{if} \quad a \in B_{e_1}^s. \end{cases}$

One easily checks that the operators u_e^p commute with the right actions of A_p on $H_{r(e),p}$ and $H_{s(e),p}$ and extend to unitary operators (still denoted u_e^p) in $\mathcal{L}_{A_p}(H_{r(e),p}, H_{s(e),p})$ satisfying $(u_e^p)^* = u_{\overline{e}}^p$. Moreover, for every $e \in \mathrm{E}(\mathcal{G})$ and every $b \in B_e$, the definition implies that

$$u_{\overline{e}}^p \lambda_{s(e),p}(s_e(b)) u_e^p = \lambda_{r(e),p}(r_e(b)) \in \mathcal{L}_{A_p}(H_{r(e),p}).$$

Let $w = (e_1, \dots, e_n)$ be a path in $\mathcal{G}$ and let $p \in \mathrm{V}(\mathcal{G})$, we set $u_w^p = u_{e_1}^p \dots u_{e_n}^p \in \mathcal{L}_{A_p}(H_{r(e_n),p}, H_{s(e_1),p})$.

The *p-reduced fundamental C*-algebra* is the C*-algebra

$$P_p = \langle (u_z^p)^* \lambda_{q,p}(A_q) u_w^p | q \in \mathrm{V}(\mathcal{G}), w, z \text{ paths from } q \text{ to } p \, \rangle \subset \mathcal{L}_{A_p}(H_{p,p}).$$

We sometimes write $P_p^{\mathcal{G}} = P_p$. Let us now explain how one can canonically view P_p as a quotient of P. Let $\mathcal{T}$ be a maximal subtree in $\mathcal{G}$. Given a vertex $q \in \mathrm{V}(\mathcal{G})$, we denote by g_{qp} the unique geodesic path in $\mathcal{T}$ from q to p. For every $e \in \mathrm{E}(\mathcal{G})$, we define a unitary operator $w_e^p = (u_{g_{s(e)p}}^p)^* u_{(e,g_{r(e)p})}^p \in P_p$.

For every $q \in \mathrm{V}(\mathcal{G})$, we define a unital faithful $*$-homomorphism $\pi_{q,p} : A_q \to P_p$ by

$$\pi_{q,p}(a) = (u_{g_{qp}}^p)^* \lambda_{q,p}(a) u_{g_{qp}}^p \quad \text{for all } a \in A_q.$$

It is not difficult to check that the following relations hold:

- $w_{\overline{e}}^p = (w_e^p)^*$ for every $e \in \mathrm{E}(\mathcal{G})$,
- $w_e^p = 1$ for every $e \in \mathrm{E}(\mathcal{T})$,
- $w_{\overline{e}}^p \pi_{s(e),p}(s_e(b)) w_e^p = \pi_{r(e),p}(r_e(b))$ for every $e \in \mathrm{E}(\mathcal{G})$, $b \in B_e$.

Thus, we can apply the universal property of the maximal fundamental C*-algebra P to get a unique surjective $*$-homomorphism $\lambda_p : P \to P_p$ such that $\lambda_p(u_e) = w_e^p$ for all $e \in \mathrm{E}(\mathcal{G})$ and $\lambda_p(a) = \pi_{q,p}(a)$ for all $a \in A_q$ and all $q \in V(\mathcal{G})$. We sometimes write $\lambda_p^{\mathcal{G}} = \lambda_p$.

Let $p_0, p, q \in V(\mathcal{G})$ and $a = \lambda_{p_0,p}(a_0) u_{e_1}^p \lambda_{s(e_2),p}(a_1) u_{e_2}^p \dots u_{e_n}^p \lambda_{q,p}(a_n) \in \mathcal{L}_{A_p}(H_{q,p}, H_{p_0,p})$, where $w = (e_1, \dots, e_n)$ is a (non-empty) path in $\mathcal{G}$ from p_0 to q, $a_0 \in A_{p_0}$ and, for $1 \leqslant i \leqslant n$, $a_i \in A_{r(e_i)}$. The operator a is said to be *reduced* (from p_0 to q) if for all $1 \leqslant i \leqslant n-1$ such that $e_{i+1} = \overline{e}_i$, we have $E_{e_{i+1}}^s(a_i) = 0$.

Let $w = (e_1, \dots, e_n)$ be a path from p to p. It is easy to check that any reduced operator of the form $a = \lambda_{p_0,p}(a_0) u_{e_1}^p \dots u_{e_n}^p \lambda_{q,p}(a_n)$ is in P_p and that the linear span $\mathcal{A}_p$ of A_p and the reduced operators from p to p is a dense $*$-subalgebra of P_p.

Remark 2.4. The notion of reduced operator also makes sense in the maximal fundamental C*-algebra (if we assume the existence of conditional expectations) and, for any fixed $p \in V(\mathcal{G})$, the linear span of A_p and the reduced operators from p to p is the $*$-algebra $\mathcal{A}$ introduced in Remark 2.3, which is dense in the maximal fundamental C*-algebra. Observe that, by definition, the morphism $\lambda_p : P \to P_p$ is the unique unital $*$-homomorphism which is formally equal to the identity on the reduced operators from p to p. More precisely, one has, for any reduced operator $a = a_0 u_{e_1} \dots u_{e_n} a_n \in P$ from p to p, $\lambda_p(a) = \lambda_{p,p}(a_0) u^p_{e_1} \dots u^p_{e_n} \lambda_{p,p}(a_n)$.

We will need the following purely combinatorial lemma which gives an explicit decomposition of the product of two reduced operators in P from p to p as a sum of reduced operators from p to p plus an element in A_p. For $e \in E(\mathcal{G})$ and $x \in A_{r(e)}$ we write $\mathcal{P}^r_e(x) := x - E^r_e(x)$.

Lemma 2.5. [FF13, Lemma 3.17] *Let $w = (e_n, \dots, e_1)$ and $\mu = (f_1, \dots, f_m)$ be paths from p to p. Set $n_0 = \max\{1 \leqslant k \leqslant \min(n,m) | e_i = \overline{f}_i, \forall i \leqslant k\}$. If the above set is empty, set $n_0 = 0$. Let $a = a_n u_{e_n} \dots u_{e_1} a_0 \in P$ and $b = b_0 u_{f_1} \dots u_{f_m} b_m \in P$ be reduced operators. Set $x_0 = a_0 b_0$ and, for $1 \leqslant k \leqslant n_0$, $x_k = a_k(s_{e_k} \circ r^{-1}_{e_k} \circ E^r_{e_k}(x_{k-1}))b_k$ and $y_k = \mathcal{P}^r_{e_k}(x_{k-1})$. The following holds :*

1. *If $n_0 = 0$, then $ab = a_n u_{e_n} \dots u_{e_1} x_0 u_{f_1} \dots u_{f_m} b_m$.*
2. *If $n_0 = n = m$, then $ab = \sum_{k=1}^n a_n u_{e_n} \dots u_{e_k} y_k u_{f_k} \dots u_{f_n} b_n + x_n$.*
3. *If $n_0 = n < m$, then $ab = \sum_{k=1}^n a_n u_{e_n} \dots u_{e_k} y_k u_{f_k} \dots u_{f_m} b_m + x_n u_{f_{n+1}} \dots u_{f_m} b_m$.*
4. *If $n_0 = m < n$, then $ab = \sum_{k=1}^m a_n u_{e_n} \dots u_{e_k} y_k u_{f_k} \dots u_{f_m} b_m + a_n u_{e_n} \dots u_{e_{m+1}} x_m$.*
5. *If $1 \leqslant n_0 < \min\{n, m\}$, then*

$$ab = \sum_{k=1}^n a_n u_{e_n} \dots u_{e_k} y_k u_{f_k} \dots u_{f_m} b_m + a_n u_{e_n} \dots u_{e_{n_0+1}} x_{n_0} u_{f_{n_0+1}} \dots u_{f_m} b_m.$$

Note that the preceding Lemma also holds in P_p, for all $p \in V(\mathcal{G})$, simply by applying the unital $*$-homomorphism λ_p which is formally the identity on the reduced operators from p to p, as explained in Remark 2.4.

In the following Proposition we completely characterize the p-reduced fundamental C*-algebra: it is the unique quotient of P for which there exists a GNS-faithful ucp map $P_p \to A_p$ which is zero on the reduced operators and "the identity on A_p". The proof of this result is contained in [FF13] in the GNS-faithful case but it is not explicitly stated. Since the proof is the same as the one of [FG15, Proposition 2.4] and all the necessary arguments are contained in [FF13], we will only sketch the proof of the next Proposition.

Proposition 2.6. *For all $p \in V(\mathcal{G})$ the following holds.*

1. *The morphism λ_p is faithful on A_p.*
2. *There exists a unique ucp map $\mathbb{E}_p : P_p \to A_p$ such that $\mathbb{E}_p \circ \lambda_p(a) = a$ for all $a \in A_p$ and*
$$\mathbb{E}_p(\lambda_p(a_0 u_{e_1} \dots u_{e_n} a_n)) = 0 \text{ for all } a = a_0 u_{e_1} \dots u_{e_n} a_n \in P \text{ a reduced operator from } p \text{ to } p.$$
Moreover, $\mathbb{E}_p$ is GNS-faithful.
3. *For any unital C*-algebra with a surjective unital $*$-homomorphism $\pi : P \to C$ and a GNS-faithful ucp map $E : C \to A_p$ such that $E \circ \lambda(a) = a$ for all $a \in A_p$ and*
$$E(\pi(a_0 u_{e_1} \dots u_{e_n} a_n)) = 0 \text{ for all } a = a_0 u_{e_1} \dots u_{e_n} a_n \in P \text{ a reduced operator from } p \text{ to } p$$
there exists a unique unital $*$-isomorphism $\nu : P_p \to C$ such that $\nu \circ \lambda_p = \pi$. Moreover, ν satisfies $E \circ \nu = \mathbb{E}_p$.

Proof. Assertion (1) follows from assertion (2), since $\mathbb{E}_p \circ \lambda_p(a) = a$ for all $a \in A_p$. Let us sketch the proof of assertion (2). Define $\xi_p = 1_{A_p} \in A_p \subset H_{p,p}$ and $\mathbb{E}_p(x) = \langle \xi_p, x\xi_p \rangle$ for all $x \in P_p$. Then $\mathbb{E}_p : P_p \to A_p$ is a ucp map and, for all $a \in A_p$, $\mathbb{E}_p(\lambda_p(a)) = \langle 1_{A_p}, L_{A_p}(a) 1_{A_p} \rangle = a$. Repeating the proof of [FF13, Proposition 3.18], we see that $\overline{P_p \xi_p \cdot A_p} = H_{p,p}$ and, for any reduced operator

$a \in A_p$, one has $\langle \xi_p, a\xi_p \rangle = 0$. It follows that the triple $(H_{p,p}, \mathrm{id}, \xi_p)$ is a GNS-construction of $\mathbb{E}_p$ (in particular $\mathbb{E}_p$ is GNS-faithful) and $\mathbb{E}_p(\lambda_p(x)) = 0$ for any reduced operator $x \in P$ from p to p, since the map λ_p sends reduced operators in P from p to p to reduced operators in P_p.

The proof of (3) is a routine. Since E is GNS-faithful on C we may and will assume that $C \subset \mathcal{L}_{A_p}(K)$, where (K, id, η) is a GNS-construction of E. By the properties of E and $\mathbb{E}_p$, the operator $U : H_{p,p} \to K$ defined by $U(\lambda_p(x)\xi_p) = \pi(x)\eta$ for all $x \in P$ reduced operator from p to p or $x \in A_p \subset P$ extends to a unitary operator $U \in \mathcal{L}_{A_p}(H_{p,p}, K)$. By the definition of U, the map $\nu(x) = UxU^*$, for $x \in P_p$, does the job. The uniqueness is obvious. □

Notation. We sometimes write $\mathbb{E}_p^{\mathcal{G}} = \mathbb{E}_p$.

For a connected subgraph $\mathcal{G}' \subset \mathcal{G}$ with a maximal subtree $\mathcal{T}' \subset \mathcal{G}'$ such that $\mathcal{T}' \subset \mathcal{T}$ we denote by $P_{\mathcal{G}'}$ the maximal fundamental C*-algebra of our graph of C*-algebras restricted to $\mathcal{G}'$ with respect to the maximal subtree $\mathcal{T}'$. By the universal property there exists a unique unital $*$-homomorphism $\pi_{\mathcal{G}'} : P_{\mathcal{G}'} \to P$ such that $\lambda_{\mathcal{G}'}(a) = a$ for all $a \in A_p$, $p \in V(\mathcal{G}')$ and $\pi_{\mathcal{G}'}(u_e) = u_e$ for all $e \in E(\mathcal{G}')$. The following Corollary says that we have a canonical identification of $P_p^{\mathcal{G}'}$ with the sub-C*-algebra of P_p generated by A_p and the reduced operators from p to p with associated path in $\mathcal{G}'$.

Proposition 2.7. *For all $p \in V(\mathcal{G}')$, there exists unique faithful $*$-homomorphism $\pi_p^{\mathcal{G}'} : P_p^{\mathcal{G}'} \to P_p$ such that $\pi_p^{\mathcal{G}'} \circ \lambda_p^{\mathcal{G}'} = \lambda_p \circ \pi_{\mathcal{G}'}$. The morphism $\pi_p^{\mathcal{G}'}$ satisfies $\mathbb{E}_p \circ \pi_p^{\mathcal{G}'} = \mathbb{E}_p^{\mathcal{G}'}$. Moreover, there exists a unique ucp map $\mathbb{E}_p^{\mathcal{G}'} : P_p \to P_p^{\mathcal{G}'}$ such that $\mathbb{E}_p^{\mathcal{G}'} \circ \pi_p^{\mathcal{G}'} = id$ and $\mathbb{E}_p^{\mathcal{G}'}(\lambda_p(a)) = 0$ for all $a \in P$ a reduced operator from p to p with associated path containing at least one vertex which is not in $\mathcal{G}'$.*

Proof. The uniqueness of $\pi_p^{\mathcal{G}'}$ being obvious, let us show the existence. Define $P' = \pi_p^{\mathcal{G}'} \circ \lambda_p^{\mathcal{G}'}(P_{\mathcal{G}'}) \subset P_p$ and let $E : P' \to A_p$ be the ucp map defined by $E = \mathbb{E}_p|_{P'}$. By the universal property of Proposition 2.6, assertion 3, it suffices to check that E is GNS-faithul. Let $x \in P'$ such that $E(y^*x^*xy) = \mathbb{E}_p(y^*x^*xy) = 0$ for all $y \in P'$. In particular $\mathbb{E}_p(x^*x) = 0$ and we may and will assume that x^*x is the image under λ_p of a sum of reduced operators from p to p with associated vertices in $\mathcal{G}'$. Let us show that $x = 0$. Since $\mathbb{E}_p$ is GNS-faitful and since P' contains the image under λ_p of A_p and of the reduced operators from p to p in P whose associated path from is in $\mathcal{G}'$, it suffices to show that $\mathbb{E}_p(y^*x^*xy) = 0$ for all $y = \lambda_p(a)$, where $a \in P$ is a reduced operator from p to p whose associated path contains at least one vertex which is not in $\mathcal{G}'$. It follows easily from Lemma 2.5 since this Lemma implies that, for all $z \in P$ a reduced operator from p to p with all edges in $\mathcal{G}'$ or $z \in A_p$ and for all $a \in P$ a reduced operator from p to p with at least one vertex which is not in $\mathcal{G}'$, the product a^*za is equal to a sum of reduced operators from p to p with at least one vertex which is not in $\mathcal{G}'$. In particular, $\mathbb{E}_p(\lambda_p(a^*za)) = 0$ for all such a and z. Hence, $\mathbb{E}_p(yx^*xy) = 0$ for all $y \in P_p$. By construction, $\pi_p^{\mathcal{G}'}$ satisfies $\mathbb{E}_p \circ \pi_p^{\mathcal{G}'} = \mathbb{E}_p^{\mathcal{G}'}$. Let us now construct the ucp map $\mathbb{E}_p^{\mathcal{G}'}$ (the uniqueness is obvious).

Let $H'_{p,p} = \bigoplus_{\omega \text{ a path in } \mathcal{G}' \text{ from } p \text{ to } p} H_\omega \subset H_{p,p}$. By convention the sum also contains the empty path for which $H_\emptyset = A_p$. Observe that $H'_{p,p}$ is a complemented Hilbert sub-A_p-module of $H_{p,p}$. Let $Q \in \mathcal{L}_{A_p}(H_{p,p})$ be the orthogonal projection onto $H'_{p,p}$ and define the ucp map $\mathbb{E}_p^{\mathcal{G}'} : P_p \to \mathcal{L}_{A_p}(H'_{p,p})$ by $\mathbb{E}_p^{\mathcal{G}'}(x) = QxQ$.

Since $xH'_{p,p} \subset H'_{p,p}$ for all $x \in P_p^{\mathcal{G}'}$, the projection Q commutes with every $x \in P_p^{\mathcal{G}'}$. Hence, after the identification $P_p^{\mathcal{G}'} \subset P_p$, we have $\mathbb{E}_p^{\mathcal{G}'}(x) = x$ for all $x \in P_p^{\mathcal{G}'}$.

Let $a = a_0 u_{e_1} \dots u_{e_n} a_n \in P$ be a reduced operator with $\omega = (e_1, \dots e_n)$ a path in $\mathcal{G}$ from p to p such that $e_k \notin E(\mathcal{G}')$ for some $1 \leq k \leq n$. Observe that, by Lemma 2.5, for all $b \in P$ a reduced operator from p to p with associated path in $\mathcal{G}'$ or for $b \in A_p$ the product ab is a sum of reduced operators from p to p whose associated path has at least one edge from $\mathcal{G}'$. Hence, $\lambda_p(ab)\xi_p \in H_{p,p} \ominus H'_{p,p}$ (where $\xi_p = 1_{A_p} \in H_{p,p}$). It follows now easily from this observation that $Q\lambda_p(a)Q\lambda_p(b)\xi_p = 0$ for all $b \in P$ a reduced operator from p to p or $b \in A_p$. Hence, $Q\lambda_p(a)Q = 0$ and this concludes the proof. □

The following definition is not contained in [FF13]. It is the correct version of the reduced fundamental C*-algebra in the case of non GNS-faithful conditional expectations in order to obtain the K-equivalence with the full fundamental C*-algebra. It is the main contribution of this preliminary section to the general theory of fundamental C*-algebras developed in [FF13].

Definition 2.8. The *vertex-reduced fundamental C*-algebra* P_{vert} is the C*-algebra obtained by separation completion of P for the C*-semi-norm $\|x\|_v = \text{Sup}\{\|\lambda_p(x)\| : p \in V(\mathcal{G})\}$ on P.

We sometimes write $P_{\text{vert}}^{\mathcal{G}} = P_{\text{vert}}$. We will denote by $\lambda : P \to P_{\text{vert}}$ (or $\lambda_{\mathcal{G}}$) the canonical surjection. Note that, by construction of P_{vert}, for all $p \in V(\mathcal{G})$, there exists a unique unital (surjective) $*$-homomorphism $\lambda_{v,p} : P_{\text{vert}} \to P_p$ such that $\lambda_{v,p} \circ \lambda = \lambda_p$. We sometimes write $\lambda_{v,p}^{\mathcal{G}} = \lambda_{v,p}$. We describe the fundamental properties of P_{vert} in the following Proposition. We call a family of ucp maps $\{\varphi_i\}_{i\in I}$, $\varphi_i : A \to B_i$ GNS-faithful if $\cap_{i\in I}\text{Ker}(\pi_i) = \{0\}$, where (H_i, π_i, ξ_i) is a GNS-construction for φ_i.

Proposition 2.9. *The following holds.*

1. *The morphism λ is faithful on A_p for all $p \in V(\mathcal{G})$.*
2. *For all $p \in V(\mathcal{G})$, there exists a unique ucp map $\mathbb{E}_{A_p} : P_{\text{vert}} \to A_p$ such that $\mathbb{E}_{A_p} \circ \lambda(a) = a$ for all $a \in A_p$ and all $p \in V(\mathcal{G})$ and,*

$$\mathbb{E}_{A_p}(\lambda_v(a_0 u_{e_1} \dots u_{e_n} a_n)) = 0 \text{ for all } a = a_0 u_{e_1} \dots u_{e_n} a_n \in P \text{ a reduced operator from } p \text{ to } p.$$

   Moreover, the family $\{\mathbb{E}_{A_p} : p \in V(\mathcal{G})\}$ is GNS-faithful.

3. *Suppose that C is a unital C*-algebra with a surjective unital $*$-homomorphism $\pi : P \to C$ and with ucp maps $E_{A_p} : C \to A_p$, for $p \in V(\mathcal{G})$, such that $E_{A_p} \circ \pi(a) = a$ for all $a \in A_p$, all $p \in V(\mathcal{G})$ and,*

$$E_{A_p}(\pi(a_0 u_{e_1} \dots u_{e_n} a_n)) = 0 \text{ for all } a = a_0 u_{e_1} \dots u_{e_n} a_n \in P \text{ a reduced operator from } p \text{ to } p$$

   and the family $\{E_{A_p} : p \in V(\mathcal{G})\}$ is GNS-faithful. Then, there exists a unique unital $*$-isomorphism $\nu : P_{\text{vert}} \to C$ such that $\nu \circ \lambda = \pi$. Moreover, ν satisfies $E \circ \nu = \mathbb{E}_p$ for all $p \in V(\mathcal{G})$.

Proof. (1). It follows from (2) since $\mathbb{E}_{A_p} \circ \lambda(a) = a$ for all $a \in A_p$ and all $p \in V(\mathcal{G})$.

(2). By Proposition 2.6, the maps $\mathbb{E}_{A_p} = \mathbb{E}_p \circ \lambda_{v,p}$ satisfy the desired properties and it suffices to check that the family $\{\mathbb{E}_{A_p} : p \in V(\mathcal{G})\}$ is GNS-faithful. This is done exactly as in the proof of assertion (2) of [FG15, Proposition 2.8].

(3). The proof is the same as the proof of assertion (3) of [FG15, Proposition 2.8], by using the universal property stated in Proposition 2.6 and the definition of P_{vert}. □

Notation. We sometimes write $\mathbb{E}^{\mathcal{G}}_{A_p} = \mathbb{E}_{A_p}$.

Proposition 2.10. *Let $\mathcal{G}' \subset \mathcal{G}$ be a connected subgraph with maximal subtree $\mathcal{T}' \subset \mathcal{T}$. There exists a unique faithful $*$-homomorphism $\pi^{\mathcal{G}'}_{\text{vert}} : P^{\mathcal{G}'}_{\text{vert}} \to P_{\text{vert}}$ such that $\pi^{\mathcal{G}'}_{\text{vert}} \circ \lambda_{\mathcal{G}'} = \lambda \circ \pi_{\mathcal{G}'}$. The morphism $\pi^{\mathcal{G}'}_{\text{vert}}$ satisfies $\mathbb{E}_{A_p} \circ \pi^{\mathcal{G}'}_{\text{vert}} = \mathbb{E}^{\mathcal{G}'}_{A_p}$ for all $p \in V(\mathcal{G})$. Moreover, there exists a unique ucp map $\mathbb{E}_{\mathcal{G}'} : P_{\text{vert}} \to P^{\mathcal{G}'}_{\text{vert}}$ such that $\lambda^{\mathcal{G}'}_{v,p} \circ \mathbb{E}_{\mathcal{G}'} = \mathbb{E}^{\mathcal{G}'}_p \circ \lambda_{v,p}$ for all $p \in V(\mathcal{G}')$.*

Proof. Define $P' = \lambda \circ \pi_{\mathcal{G}'}(P_{\mathcal{G}'}) \subset P_{\text{vert}}$ and consider, for $p \in V(\mathcal{G})$, the ucp map $E_{A_p} = \mathbb{E}_{A_p}|_{P'}$. Using the universal property of Proposition 2.9, assertion 3, it suffices to check that the family $\{E_{A_p} : p \in V(\mathcal{G})\}$ is GNS-faithful. Let $x \in P'$ such that $E_{A_p}(y^*x^*xy) = 0$ for all $y \in P'$ and all $p \in V(\mathcal{G})$. Arguing as in the proof of Proposition 2.7 we find that $\mathbb{E}_{A_p}(y^*x^*xy) = 0$ for all $y \in P_{\text{vert}}$ and all $p \in V(\mathcal{G})$. Since the family $\{\mathbb{E}_{A_p} : p \in V(\mathcal{G})\}$ is GNS faithful, the family $\{E_{A_p} : p \in V(\mathcal{G})\}$ is also GNS-faithful. The construction of the canonical ucp map $\mathbb{E}_{\mathcal{G}'} : P_{\text{vert}} \to P^{\mathcal{G}'}_{\text{vert}}$ is similar to the construction made in the proof of Proposition 2.7. Indeed, let $A = \bigoplus_{p \in V(\mathcal{G})} A_p$ and consider the Hilbert A-module $\bigoplus_{p \in V(\mathcal{G})} H_{p,p}$ with the (faithful) left action of P_{vert} given by $\nu = \bigoplus_{p \in V(\mathcal{G})} \lambda_{v,p}$. As in the proof of Proposition 2.7, given any $p \in V(\mathcal{G}')$, we identify the Hilbert module of path in $\mathcal{G}'$ from p to p, with the canonical Hilbert A_p-submodule $H'_{p,p} \subset H_{p,p}$ and we also view $\bigoplus_{p \in V(\mathcal{G}')} H'_{p,p} \subset \bigoplus_{p \in V(\mathcal{G})} H_{p,p}$ as a Hilbert A-submodule. Note that the left action $\bigoplus_{p \in V(\mathcal{G}')} \lambda^{\mathcal{G}'}_{v,p}$ of $P^{\mathcal{G}'}_{\text{vert}}$ on $\bigoplus_{p \in V(\mathcal{G}')} H'_{p,p}$ is faithful so that we may and will view $P^{\mathcal{G}'}_{\text{vert}} \subset \mathcal{L}_A(\bigoplus_{p \in V(\mathcal{G}')} H'_{p,p})$. Let $Q \in \mathcal{L}_A(\bigoplus_{p \in V(\mathcal{G})} H_{p,p})$ be the orthogonal projection onto $\bigoplus_{p \in V(\mathcal{G}')} H'_{p,p}$. Then it is not difficult to check that the ucp map $x \mapsto Q\nu(x)Q$ has the desired properties. □

Example 2.11. When the graph contains two edges: e and its opposite $\overline{e}$ then either $s(e) \neq r(e)$ and the construction considered above is the vertex-reduced amalgamated free product studied in [FG15, Section 2] or $s(e) = r(e)$ and the construction above is the vertex-reduced HNN-extension. Let us reformulate in details below our construction in that specific case. Note that the *edge-reduced HNN-extension* has been described in details in [Fi13].
Let A, B be unital C*-algebras and, for $\epsilon \in \{-1, 1\}$, let $\pi_\epsilon : B \to A$ be a unital faithful $*$-homomorphism and $E_\epsilon : A \to B$ a ucp map such that $E_\epsilon \circ \pi_\epsilon = \text{id}_B$. The full HNN-extension is the universal unital C*-algebra generated by A and a unitary u such that $u\pi_{-1}(b)u^* = \pi_1(b)$ for all $b \in B$. We denote this C*-algebra by $\text{HNN}(A, B, \pi_1, \pi_{-1})$. The (vertex) reduced HNN-extension C is the unique, up to isomorphism, unital C*-algebra satisfying the following properties:

1. There exists a unital $*$-homomorphism $\rho : A \to C$ and a unitary $u \in C$ such that $u\rho(\pi_{-1}(b))u^* = \rho(\pi_1(b))$ for all $b \in B$ and C is generated by $\rho(A)$ and u.
2. There exists a GNS-faithful ucp map $E : C \to A$ such that $E \circ \rho = \text{id}_A$ and $E(x) = 0$ for all $x \in C$ of the form $x = \rho(a_0)u^{\epsilon_1} \ldots u_{\epsilon_n}\rho(a_n)$ where $n \geq 1$, $a_k \in A$ and $\epsilon_k \in \{-1, 1\}$ are such that, for all $1 \leq k \leq n-1$, $\epsilon_{k+1} = -\epsilon_k \implies E_{-\epsilon_k}(a_k) = 0$.
3. If D is a unital C*-algebra with a unital $*$-homomorphism $\nu : A \to D$, a unitary $v \in D$ and a GNS-faithful ucp map $E' : D \to A$ such that
    - $v\nu(\pi_{-1}(b))v^* = \nu(\pi_1(b))$ for all $b \in B$ and D is generated by $\nu(A)$ and v.
    - $E' \circ \nu = \text{id}_A$ and $E'(x) = 0$ for all $x \in D$ of the form $x = \nu(a_0)v^{\epsilon_1} \ldots v^{\epsilon_n}\nu(a_n)$ with $n \geq 1$, $\epsilon_k \in \{-1, 1\}$, $a_k \in A$ such that, for all $1 \leq k \leq n-1$ one has $\epsilon_{k+1} = -\epsilon_k \implies E_{-\epsilon_k}(a_k) = 0$.

Then there exists a unique unital $*$-homomorphism $\widetilde{\nu} : C \to D$ such that $\widetilde{\nu} \circ \rho = \nu$ and $\widetilde{\nu}(u) = v$. Moreover, $E' \circ \widetilde{\nu} = E$. We denote this C*-algebra by $\mathrm{HNN}_{\mathrm{vert}}(A, B, \pi_1, \pi_{-1})$

We now describe the *Serre's devissage* process for our vertex-reduced fundamental C*-algebras.

For $e \in E(\mathcal{G})$, let $\mathcal{G}_e$ be the graph obtained from $\mathcal{G}$ by removing the edges e and $\overline{e}$ i.e., $V(\mathcal{G}_e) = V(\mathcal{G})$ and $E(\mathcal{G}_e) = E(\mathcal{G}) \setminus \{e, \overline{e}\}$. The source range and inverse maps are the restrictions of the one for $\mathcal{G}$. The *Serre's devissage* shows that, when $\mathcal{G}_e$ is not connected, the vertex-reduced fundamental C*-algebra is a vertex-reduced amalgamated free product and, when $\mathcal{G}_e$ is connected, the vertex-reduced fundamental C*-algebra is a vertex-reduced HNN-extension. We shall use freely the notations and results of [FG15, Section 2] about vertex-reduced amalgamated free products.

Case 1: $\mathcal{G}_e$ is not connected. Let $\mathcal{G}_{s(e)}$ (respectively $\mathcal{G}_{r(e)}$) the connected component of $s(e)$ (resp. $r(e)$) in $\mathcal{G}_e$. Since $\mathcal{G}_e$ is not connected $e \in E(\mathcal{T})$ and the graphs $\mathcal{T}_{s(e)} := \mathcal{T} \cap \mathcal{G}_{s(e)}$ and $\mathcal{T}_{r(e)} := \mathcal{T} \cap \mathcal{G}_{r(e)}$ are maximal subtrees of $\mathcal{G}_{s(e)}$ and $\mathcal{G}_{r(e)}$ respectively. Let $P_{\mathcal{G}_{s(e)}}$ and $P_{\mathcal{G}_{r(e)}}$ be the maximal fundamental C*-algebras of our graph of C*-algebras restricted to $\mathcal{G}_{s(e)}$ and $\mathcal{G}_{r(e)}$ respectively and with respect to the maximal subtrees $\mathcal{T}_{s(e)}$ and $\mathcal{T}_{r(e)}$ respectively. Recall that we have canonical maps $\pi_{\mathcal{G}_{s(e)}} : P_{\mathcal{G}_{s(e)}} \to P$ and $\pi_{\mathcal{G}_{r(e)}} : P_{\mathcal{G}_{r(e)}} \to P$.

Let $P_{\mathcal{G}_{s(e)}} \underset{B_e}{*} P_{\mathcal{G}_{r(e)}}$ be the full free product of $P_{\mathcal{G}_{s(e)}}$ and $P_{\mathcal{G}_{r(e)}}$ amalgamated over B_e relative to the maps $s_e : B_e \to P_{\mathcal{G}_{s(e)}}$ and $r_e : B_e \to P_{\mathcal{G}_{r(e)}}$. Observe that, since $e \in E(\mathcal{T})$, we have $u_e = 1 \in P$. Hence, we have $s_e(b) = r_e(b)$ in P, for all $b \in B_e$. By the universal property of the full amalgamated free product there exists a unique unital $*$-homomorphism $\nu : P_{\mathcal{G}_{s(e)}} \underset{B_e}{*} P_{\mathcal{G}_{r(e)}} \to P$ such that $\nu|_{P_{\mathcal{G}_{s(e)}}} = \pi_{\mathcal{G}_{s(e)}}$ and $\nu|_{P_{\mathcal{G}_{r(e)}}} = \pi_{\mathcal{G}_{r(e)}}$. Moreover, by the universal property of P, there exists also a unital $*$-homomorphism $P \to P_{\mathcal{G}_{s(e)}} \underset{B_e}{*} P_{\mathcal{G}_{r(e)}}$ which is the inverse of ν. Hence, ν is a $*$-isomorphism. Actually, this is also true at the vertex-reduced level.
Note that we have injective unital $*$-homomorphisms $\iota_{s(e)} = \lambda_{\mathcal{G}_{s(e)}} \circ s_e : B_e \to P_{\mathrm{vert}}^{\mathcal{G}_{s(e)}}$ and $\iota_{s(e)} = \lambda_{\mathcal{G}_{r(e)}} \circ r_e : B_e \to P_{\mathrm{vert}}^{\mathcal{G}_{r(e)}}$ and conditional expectations $E_{s(e)} = \lambda_{\mathcal{G}_{s(e)}} \circ E_e^s \circ \mathbb{E}_{A_{s(e)}}^{\mathcal{G}_{s(e)}}$ from $P_{\mathrm{vert}}^{\mathcal{G}_{s(e)}}$ to $\iota_{s(e)}(B_e)$ and $E_{r(e)} = \lambda_{\mathcal{G}_{r(e)}} \circ E_e^r \circ \mathbb{E}_{A_{r(e)}}^{\mathcal{G}_{r(e)}}$ from $P_{\mathrm{vert}}^{\mathcal{G}_{r(e)}}$ to $\iota_{r(e)}(B_e)$ so that we can perform the vertex-reduced amalgamated free product. Following [FG15, Section 2], we write

$$\pi : P_{\mathrm{vert}}^{\mathcal{G}_{s(e)}} \underset{B_e}{*} P_{\mathrm{vert}}^{\mathcal{G}_{r(e)}} \to P_{\mathrm{vert}}^{\mathcal{G}_{s(e)}} \overset{v}{\underset{B_e}{*}} P_{\mathrm{vert}}^{\mathcal{G}_{r(e)}}$$

the canonical surjection for the full amalgamated free product to the vertex-reduced amalgamated free product and $\mathbb{E}_1$ (resp. $\mathbb{E}_2$) the canonical ucp map from $P_{\mathrm{vert}}^{\mathcal{G}_{s(e)}} \overset{v}{\underset{B_e}{*}} P_{\mathrm{vert}}^{\mathcal{G}_{r(e)}}$ to $P_{\mathrm{vert}}^{\mathcal{G}_{s(e)}}$ (resp. to $P_{\mathrm{vert}}^{\mathcal{G}_{r(e)}}$).

Lemma 2.12. *There exists a unique $*$-isomorphism $\nu_e : P_{\mathrm{vert}}^{\mathcal{G}_{s(e)}} \overset{v}{\underset{B_e}{*}} P_{\mathrm{vert}}^{\mathcal{G}_{r(e)}} \to P_{\mathrm{vert}}$ such that:*

$$\nu_e \circ \pi \circ \lambda_{\mathcal{G}_{s(e)}} = \lambda \circ \pi_{\mathcal{G}_{s(e)}} \quad \text{and} \quad \nu_e \circ \pi \circ \lambda_{\mathcal{G}_{r(e)}} = \lambda \circ \pi_{\mathcal{G}_{r(e)}}.$$

Moreover we have $\mathbb{E}_{\mathcal{G}_{s(e)}} \circ \nu_e = \mathbb{E}_1$ and $\mathbb{E}_{\mathcal{G}_{r(e)}} \circ \nu_e = \mathbb{E}_2$.

Proof. The proof is the same as the proof of [FF13, Lemma 3.26], it suffices to prove that P_{vert} satisfies the universal property of $P_{\mathrm{vert}}^{\mathcal{G}_{s(e)}} \overset{v}{\underset{B_e}{*}} P_{\mathrm{vert}}^{\mathcal{G}_{r(e)}}$: the canonical ucp maps from P_{vert} to $P_{\mathrm{vert}}^{\mathcal{G}_{s(e)}}$ and

$P_{\rm vert}^{\mathcal{G}_{r(e)}}$ are the ones constructed in Proposition 2.10 i.e. $\mathbb{E}_{\mathcal{G}_{s(e)}}$ and $\mathbb{E}_{\mathcal{G}_{r(e)}}$. By [FG15, Proposition 2.8, assertion 3], the resulting isomorphism ν_e intertwines the canonical ucp maps. □

Case 2: $\mathcal{G}_e$ is connected.

Let $e \in E(\mathcal{G})$ and suppose that $\mathcal{G}_e$ is connected. Up to a canonical isomorphism of P we may and will assume that $\mathcal{T} \subset \mathcal{G}_e$. So that we have the canonical unital $*$-homomorphism $\pi_{\mathcal{G}_e} : P_{\mathcal{G}_e} \to P$. We consider the two unital faithful $*$-homomorphisms $s_e, r_e : B_e \to P_{\mathcal{G}_e}$. By definition, we have $u_e r_e(b) u_e^* = s_e(b)$ for all $b \in B_e$ and P is generated, as a C*-algebra, by $\pi_{\mathcal{G}_e}(P_{\mathcal{G}_e})$ and u_e. By the universal property of the maximal HNN-extension, there exists a unique unital (surjective) $*$-homomorphism $\nu : \text{HNN}(P_{\mathcal{G}_e}, B_e, s_e, r_e) \to P$ such that $\nu|_{P_{\mathcal{G}_e}} = \pi_{\mathcal{G}_e}$ and $\nu(u) = u_e$. Observe that, by the universal property of P, there exists a unital $*$-homomorphism $P \to \text{HNN}(P_{\mathcal{G}_e}, B_e, s_e, r_e)$ which is the inverse of ν. Hence ν is a $*$-isomorphism. Actually this is also true at the vertex-reduced level.

Define the faithful unital $*$-homomorphism $\pi_1, \pi_{-1} : B_e \to P_{\rm vert}^{\mathcal{G}_e}$ by $\pi_{-1} = \lambda_{\mathcal{G}_e} \circ s_e$ and $\pi_1 = \lambda_{\mathcal{G}_e} \circ r_e$. Note that the ucp maps $E_\epsilon : P_{\rm vert}^{\mathcal{G}_e} \to B_e$ defined by $E_1 = s_e^{-1} \circ E_e^s \circ \mathbb{E}_{s(e)}^{\mathcal{G}_e}$ and $E_{-1} = r_e^{-1} \circ E_e^r \circ \mathbb{E}_{r(e)}^{\mathcal{G}_e}$ satisfy $E_\epsilon \circ \pi_\epsilon = \text{id}_{B_e}$ for $\epsilon \in \{-1, 1\}$. Hence we may consider the vertex-reduced HNN-extension and the canonical surjection $\lambda_e : \text{HNN}(P_{\rm vert}^{\mathcal{G}_e}, B_e, s_e, r_e) \to \text{HNN}_{\rm vert}(P_{\rm vert}^{\mathcal{G}_e}, B_e, \pi_1, \pi_{-1})$. Write $v = \lambda_e(u)$, where $u \in \text{HNN}(P_{\rm vert}^{\mathcal{G}_e}, B_e, s_e, r_e)$ is the "stable letter". Recall that, by Proposition 2.10, we have the canonical faithful unital $*$-homomorphism $\pi_{\rm vert}^{\mathcal{G}_e} : P_{\rm vert}^{\mathcal{G}_e} \to P_{\rm vert}$. Let $\mathbb{E} : \text{HNN}_{\rm vert}(P_{\rm vert}^{\mathcal{G}_e}, B_e, \pi_1, \pi_{-1}) \to P_{\rm vert}^{\mathcal{G}_e}$ the canonical GNS-faithful ucp map.

Lemma 2.13. *There is a unique $*$-isomorphism $\nu_e : \text{HNN}_{\rm vert}(P_{\rm vert}^{\mathcal{G}_e}, B_e, \pi_1, \pi_{-1}) \to P_{\rm vert}$ such that $\nu_e \circ \lambda_e|_{P_{\rm vert}^{\mathcal{G}_e}} = \pi_{\rm vert}^{\mathcal{G}_e}$ and $\nu_e(u) = u_e$. Moreover $\mathbb{E}_{\mathcal{G}_e} \circ \nu_e = \mathbb{E}$.*

Proof. Since we have $u_e \pi_{\rm vert}^{\mathcal{G}_e}(\pi_{-1}(b)) u_e^* = \pi_{\rm vert}^{\mathcal{G}_e}(\pi_1(b))$ for all $b \in B_e$, it suffices, by the universal property of the vertex-reduced HNN-extension explained in Example 2.11, to check that we have a GNS-faithful ucp map $P_{\rm vert} \to P_{\rm vert}^{\mathcal{G}_e}$ satisfying the conditions described in Example 2.11. This ucp map is the one constructed in Proposition 2.10: it is the map $\mathbb{E}_{\mathcal{G}_e}$ and the conditions can be checked as in the proof of [FF13, Lemma 3.27]. The fact that the resulting isomorphism ν_e intertwines the ucp maps follows from the universal property. □

We end this preliminary section with an easy Lemma.

Lemma 2.14. *If $x = a_0 u_{e_1} \dots u_{e_n} a_n \in P$ is a reduced operator from p to p and $a_n \in B_{e_n}^r$ then*

$$\mathbb{E}_p(\lambda_p(x^* x)) = E_{e_n}^r \circ \mathbb{E}_p(\lambda_p(x^* x)).$$

Proof. Define $x_0 = a_0^* a_0$ and for $1 \leq k \leq n$, $x_k = a_k^*(r_{e_k} \circ s_{e_k}^{-1} \circ E_{e_k}^s(x_{k-1})) a_k$. We apply Lemma 2.5 to the pair $a = x^*$ and $b = x$ in case (2). It follows that $x^* x = y + x_n$, where y is a sum of reduced operators from p to p. Hence $\mathbb{E}_p(\lambda_p(y)) = 0$ and, since $a_n \in B_{e_n}^r$, we have $x_n = a_n^*(r_{e_n} \circ s_{e_n}^{-1} \circ E_{e_n}^s(x_{n-1})) a_n \in B_{e_n}^r$. □

3. Boundary maps

Define the ucp map $\mathbb{E}_e = E_e^r \circ \mathbb{E}_{A_{r(e)}} : P_{\rm vert} \to B_e^r$. Note that the GNS construction of $\mathbb{E}_e$ is given by $(H_{r(e), r(e)} \underset{E_e^r}{\otimes} B_e^r, \lambda_{v, r(e)} \otimes 1, \xi_{r(e)} \otimes 1)$. To simplify the notations, we will denote by

(K_e, ρ_e, η_e) the GNS construction of $\mathbb{E}_e$. We define $\mathcal{R}_e \subset K_e$ as the Hilbert B_e^r-submodule of K_e of the "words ending with e". More precisely,

$$\mathcal{R}_e := \overline{\text{Span}}\{\rho_e(\lambda(x))\eta_e \,|\, x = a_0 u_{e_1} \dots u_{e_n} a_n \in P \text{ reduced from } r(e) \text{ to } r(e)$$
$$\text{with } e_n = e \text{ and } a_n \in B_e^r\} \subset K_e.$$

It is easy to see from the definition that $\mathcal{R}_e$ is a Hilbert B_e^r-submodule of K_e. Moreover, it is complemented in K_e with orthogonal complement given by:

$$\mathcal{L}_e := \overline{\text{Span}}\{\rho_e(\lambda(x))\eta_e \,|\, x \in A_{r(e)} \text{ or } x = a_0 u_{e_1} \dots u_{e_n} a_n \in P \text{ reduced from } r(e) \text{ to } r(e) \text{ with}$$
$$e_n \neq e \text{ or } e_n = e \text{ and } a_n \in A_{r(e)} \ominus B_e^r\}.$$

Let $Q_e \in \mathcal{L}_{B_e^r}(K_e)$ be the orthogonal projection onto $\mathcal{R}_e$ and define

$$X_e = \{x = a_0 u_{e_1} \dots u_{e_n} a_n \in P \text{ reduced from } r(e) \text{ to } r(e) \text{ with } e_k \notin \{\overline{e}, e\} \text{ for all } 1 \leq k \leq n\},$$

Lemma 3.1. *The following holds.*

1. *For all reduced operator $a = a_n u_{e_n} \dots u_{e_1} a_0 \in P$ from $r(e)$ to $r(e)$ we have*

$$\text{Im}(Q_e \rho_e(\lambda(a)) - \rho_e(\lambda(a)) Q_e) \subset \overline{X_a}, \text{ where:}$$

$$X_a = \begin{cases} Y_a := \left(\bigoplus_{k \in \{1,\dots,n\}, e_k = e} \rho_e(\lambda(a_n u_{e_n} \dots u_{e_k}))\eta_e \cdot B_e^r \right) & \text{if } e \text{ is not a loop,} \\ Y_a \oplus \left(\bigoplus_{k \in \{1,\dots,n\}, e_k = \overline{e}} \rho_e(\lambda(a_n u_{e_n} \dots u_{e_{k+1}} a_k))\eta_e \cdot B_e^r \right) & \text{if } e \text{ is a loop.} \end{cases}$$

   (by convention, the term in the last direct sum is $\rho_e(\lambda(a_n))\eta_e \cdot B_e^r$, when $e_n = \overline{e}$ is a loop)
2. *Q_e commutes with $\rho_e(\lambda(a))$ for all $a \in \overline{\text{Span}}\left(A_{r(e)} \cup X_e\right)$.*
3. *$Q_e \rho_e(\lambda(a)) - \rho_e(\lambda(a)) Q_e \in \mathcal{K}_{B_e^r}(K_e)$ for all $a \in P$.*

Proof. During the proof we will use the notation $\mathcal{P}_e^r(x) = x - E_e^r(x)$ for $x \in A_{r(e)}$.

(1). Let $n \geq 1$ and $a = a_n u_{e_n} \dots u_{e_1} a_0 \in P$ a reduced operator from $r(e)$ to $r(e)$.
Suppose that $b \in A_{r(e)}$. We have $Q_e \rho_e(\lambda(b))\eta_e = 0$ and $ab = a_n u_{e_n} \dots u_{e_1} a_0 b \in P$ is reduced. Hence, if $e_1 \neq e$, we have $Q_e \rho_e(\lambda(ab))\eta_e = 0$ and, if $e_1 = e$, we have

$$ab = a_n u_{e_n} \dots u_e E_e^r(x_0) + a_n \dots u_e \mathcal{P}_e^r(x_0) \quad \text{where} \quad x_0 = a_0 b.$$

It follows that $Q_e \rho_e(\lambda(ab))\eta_e = \rho_e(\lambda(a_n u_{e_n} \dots u_e E_e^r(x_0)))\eta_e$. To conclude we have, $\forall b \in A_{r(e)}$,

$$(Q_e \rho_e(\lambda(a)) - \rho_e(\lambda(a)) Q_e) \rho_e(\lambda(b))\eta_e = \begin{cases} 0 \in X_a & \text{if } \quad e_1 \neq e, \\ \rho_e(\lambda(a_n u_{e_n} \dots u_{e_1}))\eta_e \cdot E_e^r(a_0 b) \in X_a & \text{if } \quad e_1 = e. \end{cases}$$

Suppose that $b = b_0 u_{f_1} \dots u_{f_m} b_m \in P$ is a reduced operator from $r(e)$ to $r(e)$. Let $0 \leq n_0 \leq \min\{n, m\}$ be the integer associated to the couple (a, b) in Lemma 2.5. This Lemma implies that, when $n_0 = 0$ or $n_0 = n < m$ or $1 \leq n_0 < \min\{n, m\}$, ab is a reduced word or a sum of reduced words that end by $u_{f_m} b_m$. Hence, in this cases, we have $\rho_e(\lambda(b))\eta_e \in \mathcal{R}_e \implies \rho_e(\lambda(ab))\eta_e \in \mathcal{R}_e$ and $\rho_e(\lambda(b))\eta_e \in \mathcal{L}_e \ominus A_{r(e)} \implies \rho_e(\lambda(ab))\eta_e \in \mathcal{L}_e \ominus A_{r(e)}$. It follows that $(Q_e \rho_e(\lambda(a)) - \rho_e(\lambda(a)) Q_e) \rho_e(\lambda(b))\eta_e = 0 \in X_a$.

Suppose now that $n_0 = m < n$. Lemma 2.5 implies that $ab = y + z$ where y is a sum of reduced words that end by $u_{f_m} b_m$ and $z = a_n u_{e_n} \dots u_{e_{m+1}} x_m$. Hence we have $\rho_e(\lambda(b))\eta_e \in \mathcal{R}_e \implies \rho_e(\lambda(y))\eta_e \in \mathcal{R}_e$ and $\rho_e(\lambda(b))\eta_e \in \mathcal{L}_e \ominus A_{r(e)} \implies \rho_e(\lambda(y))\eta_e \in \mathcal{L}_e \ominus A_{r(e)}$. It follows that

$$(Q_e\rho_e(\lambda(a)) - \rho_e(\lambda(a))Q_e)\rho_e(\lambda(b))\eta_e = \begin{cases} Q_e\rho_e(\lambda(z))\eta_e & \text{if } \rho_e(\lambda(b))\eta_e \in \mathcal{L}_e, \\ Q_e\rho_e(\lambda(z))\eta_e - \rho_e(\lambda(z))\eta_e & \text{if } \rho_e(\lambda(b))\eta_e \in \mathcal{R}_e. \end{cases}$$

We have $\rho_e(\lambda(z))\eta_e = \rho_e(\lambda(a_n u_{e_n} \dots u_{e_{m+1}} x_m))\eta_e$ hence,

$$Q_e\rho_e(\lambda(z))\eta_e = \begin{cases} 0 \in X_a & \text{if } e_{m+1} \neq e \text{ or } e_{m+1} = e \text{ and } x_m \in A_{r(e)} \ominus B_e^r, \\ \rho_e(\lambda(a_n u_{e_n} \dots u_{e_{m+1}}))\eta_e . x_m \in X_a & \text{if } e_{m+1} = e \text{ and } x_m \in B_e^r. \end{cases}$$

Hence $(Q_e\rho_e(\lambda(a)) - \rho_e(\lambda(a))Q_e)\rho_e(\lambda(b))\eta_e \in X_a$ if $\rho_e(\lambda(b))\eta_e \in \mathcal{L}_e$ and, if $\rho_e(\lambda(b))\eta_e \in \mathcal{R}_e$, we have $f_m = e$ and $b_m \in B_e^r$. Since $n_0 = m$ we conclude that $e_m = \overline{f_m} = \overline{e}$ and x_m is equal to $a_m(r_e \circ s_e^{-1} \circ E_e^s(x_{m-1}))b_m$. Note that, since $r(f_m) = r(e)$ and $f_m = \overline{e}$ we find that $s(e) = r(f_m) = r(e)$. Hence e must be a loop. Moreover, $\rho_e(\lambda(z))\eta_e = \rho_e(\lambda(a_n u_{e_n} \dots u_{e_{m+1}} a_m))\eta_e \cdot x'_m \in X_a$, where $x'_m = (r_e \circ s_e^{-1} \circ E_e^s(x_{m-1}))b_m \in B_e^r$. It follows that $(Q_e\rho_e(\lambda(a)) - \rho_e(\lambda(a))Q_e)\rho_e(\lambda(b))\eta_e \in X_a$ also when $\rho_e(\lambda(b))\eta_e \in \mathcal{R}_e$.

Suppose that $n_0 = n = m$. Lemma 2.5 implies that $ab = y + x_m$ where y is a sum of reduced words that end by $u_{f_m} b_m$. As before, we deduce that:

$$(Q_e\rho_e(\lambda(a)) - \rho_e(\lambda(a))Q_e)\rho_e(\lambda(b))\eta_e = \begin{cases} Q_e\rho_e(\lambda(x_m))\eta_e = 0 & \text{if } \rho_e(\lambda(b))\eta_e \in \mathcal{L}_e, \\ Q_e\rho_e(\lambda(x_m))\eta_e - \rho_e(\lambda(x_m))\eta_e & \text{if } \rho_e(\lambda(b))\eta_e \in \mathcal{R}_e. \end{cases}$$

And, if $\rho_e(\lambda(b))\eta_e \in \mathcal{R}_e$ then $f_m = e$ and $b_m \in B_e^r$. Since $n_0 = m = n$, we deduce that $e_n = \overline{f_m} = \overline{e}$ (hence e is a loop) and $x_m = a_n(r_e \circ s_e^{-1} \circ E_e^s(x_{n-1}))b_n \in a_n B_e^r$. Hence,

$$Q_e\rho_e(\lambda(x_m))\eta_e - \rho_e(\lambda(x_m))\eta_e = -\rho_e(\lambda(x_m))\eta_e = -\rho_e(\lambda(a_n))\eta_e \cdot x'_n \in X_a,$$

where $x'_n = (r_e \circ s_e^{-1} \circ E_e^s(x_{n-1}))b_n \in B_e^r$. This concludes the proof of the Lemma.

(2). It is obvious that $\rho_e(\lambda(a))$ commutes with Q_e for all $a \in A_{r(e)}$. Hence, (2) follows from (1).

(3). Again, it directly follows from the computations made in (1) but we write the details for the convenience of the reader. Since any reduced operator in P from $r(e)$ to $r(e)$ may be written as a product of reduced operators $a \in P$ from $r(e)$ to $r(e)$ of the form (I): the edges in a are all different from e or $\overline{e}$; (II): $a = u_{\overline{e}}x$, where x is a reduced operator from $s(e)$ to $r(e)$ whose edges are all different from e or $\overline{e}$; (III): $a = xu_e$, where x is a reduced operator from $r(e)$ to $s(e)$ whose edges are all different from e or $\overline{e}$. By (2) $\rho_e(\lambda(a))$ commutes with Q_e for a of type (I) and, since any element of type (II) is the adjoint of an element of type(III), it suffices to show that the commutator of Q_e and $\rho_e(\lambda(a))$ is compact for all a of type (III). First assume that e is a loop. In that case, it suffices to show that $Q_e\rho_e(\lambda(u_e)) - \rho_e(\lambda(u_e))Q_e$ is compact. Let $b \in P$. From the computations made in (1), we see that $(Q_e\rho_e(\lambda(u_e)) - \rho_e(\lambda(u_e))Q_e)\rho_e(\lambda(b))\eta_e = 0$ for any $b \in P$ reduced operator from $r(e)$ to $r(e)$ and, for $b \in A_{r(e)}$ one has

$$(Q_e\rho_e(\lambda(u_e)) - \rho_e(\lambda(u_e))Q_e)\rho_e(\lambda(b))\eta_e = \rho_e(\lambda(u_e)\eta_e \cdot E_e^r(b) = \rho_e(\lambda(u_e))\eta_e \cdot \langle \eta_e, \rho_e(\lambda(b))\eta_e \rangle.$$

Hence, the equality $(Q_e\rho_e(\lambda(u_e)) - \rho_e(\lambda(u_e))Q_e)\xi = \rho_e(\lambda(u_e))\eta_e \cdot \langle \eta_e, \xi \rangle$ holds for any $\xi = \rho_e(\lambda(b))\eta_e$ with b in the span of $A_{r(e)}$ and the reduced operators in P from $r(e)$ to $r(e)$. Hence, it holds for any $\xi \in K_e$. It follows that the commutator of Q_e and $\rho_e(\lambda(u_e))$ is a rank one operator, hence compact. Let us now assume that e is not a loop. Write $a = a_n u_{e_n} \dots u_{e_1} a_0 u_e$, where $n \geq 1$, $e_k \notin \{e, \overline{e}\}$ for all k. For $b \in P$ we write $X(b) = (Q_e\rho_e(\lambda(a)) - \rho_e(\lambda(a))Q_e)\rho_e(\lambda(b))\eta_e$.

As before, following the computations made in (1) we see that, since $e_k \notin \{e,\overline{e}\}$, we have $X(b)=0$ whenever b is a reduced operator from $r(e)$ to $r(e)$. Moreover, when $b\in A_{r(e)}$ we have $X(b)=\rho_e(\lambda(a))\eta_e\cdot\langle\eta_e,\rho_e(\lambda(b))\eta_e\rangle$. As before, it follows that the commutator of Q_e and $\rho_e(\lambda(a))$ is a rank one operator.

□

Define $V_e = 2Q_e-1\in\mathcal{L}_{B_e^r}(K_e)$. We have $V_e^2=1$, $V_e=V_e^*$ and, for all $x\in P_{\text{vert}}$, Lemma 3.1 implies that $V_e\rho_e(x)-\rho_e(x)V_e\in\mathcal{K}_{B_e^r}(K_e)$. Hence we get an element $y_e^{\mathcal{G}}\in KK^1(P_{\text{vert}},B_e^r)$. Define $x_e^{\mathcal{G}}=y_e^{\mathcal{G}}\underset{B_e^r}{\otimes}[r_e^{-1}]\in KK^1(P_{\text{vert}},B_e)$.

Remark 3.2. Note that we also have an element $z_e^{\mathcal{G}}=[\lambda]\underset{P_{\text{vert}}}{\otimes}x_e^{\mathcal{G}}\in KK^1(P,B_e)$.

Recall that for a subgraph $\mathcal{G}'\subset\mathcal{G}$ with a maximal subtree $\mathcal{T}'\subset\mathcal{G}'$ such that $\mathcal{T}'\subset\mathcal{T}$ we have the canonical unital faithful $*$-homomorphism $\pi_{\text{vert}}^{\mathcal{G}'}:P_{\text{vert}}^{\mathcal{G}'}\to P_{\text{vert}}$ defined in Proposition 2.10.

Proposition 3.3. *For all connected subgraphs $\mathcal{G}'\subset\mathcal{G}$ with maximal subtree $\mathcal{T}'\subset\mathcal{T}$, we have*

(1) *if $e\in E(\mathcal{G}')$ then $[\pi_{\text{vert}}^{\mathcal{G}'}]\underset{P_{\text{vert}}}{\otimes}x_e^{\mathcal{G}}=x_e^{\mathcal{G}'}\in KK^1(P_{\text{vert}}^{\mathcal{G}'},B_e)$,*

(2) *if $e\notin E(\mathcal{G}')$ then $[\pi_{\text{vert}}^{\mathcal{G}'}]\underset{P_{\text{vert}}}{\otimes}x_e^{\mathcal{G}}=0\in KK^1(P_{\text{vert}}^{\mathcal{G}'},B_e)$,*

(3) *$\sum_{r(e)=p}x_e^{\mathcal{G}}\underset{B_e}{\otimes}[r_e]=0\in KK^1(P_{\text{vert}},A_p)$ for all $p\in V(\mathcal{G})$,*

(4) *For all $e\in E(\mathcal{G})$ we have $x_{\overline{e}}^{\mathcal{G}}=-x_e^{\mathcal{G}}$.*

Proof. Let $\mathcal{G}'\subset\mathcal{G}$ be a connected subgraph with maximal subtree $\mathcal{T}'\subset\mathcal{T}$ and $e\in E(\mathcal{G})$.

(1). Suppose that $e\in E(\mathcal{G}')$ (hence $\overline{e}\in E(\mathcal{G}')$). Recall that we have the canonical ucp map $\mathbb{E}_{\mathcal{G}'}:P_{\text{vert}}\to P_{\text{vert}}^{\mathcal{G}'}$ from Proposition 2.10. Moreover, by definition of $\pi_{\text{vert}}^{\mathcal{G}'}$ of we have $\mathbb{E}_e^{\mathcal{G}'}=\mathbb{E}_e\circ\pi_{\text{vert}}^{\mathcal{G}'}$, where $\mathbb{E}_e^{\mathcal{G}'}=E_e^r\circ\mathbb{E}_{A_{r(e)}}^{\mathcal{G}'}$.

Let (K_e,ρ_e,η_e) be the GNS construction of $\mathbb{E}_e$ and define $K_e'=\overline{\rho_e\circ\pi_{\text{vert}}^{\mathcal{G}'}(P_{\text{vert}}^{\mathcal{G}'})\eta_e\cdot B_e^r}$. Observe that K_e' is complemented. Indeed, we have $K_e'\oplus L_e=K_e$, where

$$L_e=\overline{\text{Span}}\{\rho_e(x)\eta_e\cdot b\,:\,b\in B_e^r\text{ and }x\in P_{\text{vert}}\text{ such that }\mathbb{E}_{\mathcal{G}'}(x)=0\}.$$

Let $R_e\in\mathcal{L}_{B_e^r}(K_e)$ be the orthogonal projection onto K_e'. Since $\rho_e\circ\pi_{\text{vert}}^{\mathcal{G}'}(x)K_e'\subset K_e'$ for all $x\in P_{\text{vert}}^{\mathcal{G}'}$, R_e commutes with $\rho_e\circ\pi_{\text{vert}}^{\mathcal{G}'}(x)$ for all $x\in P_{\text{vert}}^{\mathcal{G}'}$. It is also easy to check that R_e commutes with Q_e hence with V_e.

Since $\mathbb{E}_e^{\mathcal{G}'}=\mathbb{E}_e\circ\pi_{\text{vert}}^{\mathcal{G}'}$ the triple (K_e',ρ_e',η_e'), where $\rho_e'(x)=\rho_e\circ\pi_{\text{vert}}^{\mathcal{G}'}(x)R_e$ for $x\in P_{\text{vert}}^{\mathcal{G}'}$ and $\eta_e'=\eta_e$, is a GNS construction of $\mathbb{E}_e^{\mathcal{G}'}$. Let $Q_e'\in\mathcal{L}_{B_e^r}(K_e')$ be the associated operator such that $x_e^{\mathcal{G}'}=[(K_e',\rho_e',V_e')]$, with $V_e'=2Q_e'-1$. By definition we have $Q_e'=Q_eR_e$ hence, $V_e'=V_eR_e$. It follows that $[\pi_{\text{vert}}^{\mathcal{G}'}]\underset{P_{\text{vert}}}{\otimes}x_e^{\mathcal{G}}=x_e^{\mathcal{G}'}\oplus y$, where $y\in KK^1(P_{\text{vert}}^{\mathcal{G}'},B_e)$ is represented by the triple $(L_e,\pi_e,V_e(1-R_e))$, where $\pi_e=\rho_e\circ\pi_{\text{vert}}^{\mathcal{G}'}(\cdot)(1-R_e)$. To conclude the proof of (1) it suffices to check that this triple is degenerated. Since V_e and $(1-R_e)$ commute, $V_e(1-R_e)$ is self-adjoint and $(V_e(1-R_e))^2=1-R_e=\text{id}_{L_e}$. Hence, it suffices to check that, for all $a\in P_{\text{vert}}^{\mathcal{G}'}$,

$$(Q_e\rho_e\circ\pi_{\text{vert}}^{\mathcal{G}'}(a)-\rho_e\circ\pi_{\text{vert}}^{\mathcal{G}'}(a)Q_e)(1-R_e)=0.$$

We already know from assertion (2) of Lemma 3.1 that $Q_e\rho_e(\lambda(a)) = \rho_e(\lambda(a))Q_e$ for all $a \in A_{r(e)}$ (and all $a \in X_e$). Let $a = a_n u_{e_n} \dots u_{e_1} a_0 \in P_{\mathcal{G}'}$ and $b = b_0 u_{f_1} \dots u_{f_m} b_m \in P$ be reduced operators from $r(e)$ to $r(e)$ and suppose that $\mathbb{E}_{\mathcal{G}'}(\lambda(b)) = 0$. Hence, there exists $k \in \{1, \dots m\}$ such that $f_k \notin E(\mathcal{G}')$ and it follows that the integer n_0 associated to the pair $(\pi_{\mathcal{G}'}(a), b)$ in Lemma 2.5 satisfies $n_0 < k$ since $e_l \in E(\mathcal{G}')$ for all $l \in \{1, \dots n\}$. Applying Lemma 2.5 in case (5), we see that $\pi_{\mathcal{G}'}(a)b$ is a sum of reduced operators that end with $u_{f_m}b_m$. Hence, $\rho_e(\lambda(b))\eta_e \in \mathcal{R}_e \implies \rho_e(\lambda(\pi_{\mathcal{G}'}(a)b))\eta_e \in \mathcal{R}_e$ and $\rho_e(\lambda(b))\eta_e \in \mathcal{L}_e \implies \rho_e(\lambda(\pi_{\mathcal{G}'}(a)b))\eta_e \in \mathcal{L}_e$. It follows that

$$[Q_e\rho_e(\pi_{\text{vert}}^{\mathcal{G}'}(\lambda_{\mathcal{G}'}(a))) - \rho_e(\pi_{\text{vert}}^{\mathcal{G}'}(\lambda_{\mathcal{G}'}(a)))Q_e]\rho_e(\lambda(b))\eta_e$$

$$= [Q_e\rho_e(\lambda(\pi_{\mathcal{G}'}(a))) - \rho_e(\lambda(\pi_{\mathcal{G}'}(a)))Q_e]\rho_e(\lambda(b))\eta_e = 0.$$

This concludes the proof of (1).

(2). Suppose that $e \notin E(\mathcal{G}')$ (hence $\overline{e} \notin E(\mathcal{G}')$). The element $[\pi_{\text{vert}}^{\mathcal{G}'}] \underset{P_{\text{vert}}}{\otimes} x_e^{\mathcal{G}}$ is represented by the triple (K_e, π_e, V_e), where $\pi_e = \rho_e \circ \pi_{\text{vert}}^{\mathcal{G}'}$. Since $V_e^2 = 1$ and $V_e^* = V_e$, it suffices to show that Q_e commutes with $\rho_e(\pi_{\text{vert}}^{\mathcal{G}'}(x))$ for all $x \in P_{\text{vert}}^{\mathcal{G}'}$. It follows from assertion (2) of Lemma 3.1 since $e, \overline{e} \notin E(\mathcal{G}')$ implies $\pi_{\text{vert}}^{\mathcal{G}'}(P_{\text{vert}}^{\mathcal{G}'}) \subset \overline{\text{Span}}\left(\lambda(A_{r(e)}) \cup \lambda(X_e)\right)$.

(3). For $p \in V(\mathcal{G})$ we use the notation $(H_p, \pi_p, \xi_p) := (H_{p,p}, \lambda_{v,p}, \xi_p)$ for the GNS construction of the canonical ucp map $\mathbb{E}_{A_p} : P_{\text{vert}} \to A_p$. Observe that $\xi_p \cdot A_p$ is orthogonally complemented in H_p and set $H_p^\circ = H_p \ominus \xi_p \cdot A_p$. Define $K_p = \bigoplus_{e \in E(\mathcal{G}), r(e) = p} K_e \underset{B_e^r}{\otimes} A_p$ and observe that, by Lemma 2.14, we have an isometry $F_p \in \mathcal{L}_{A_p}(H_p^\circ, K_p)$ defined by

$$F_p(\pi_p(\lambda(a_0 u_{e_1} \dots u_{e_n} a_n))\xi_p) = \rho_{e_n}(\lambda(a_0 u_{e_1} \dots u_{e_n}))\eta_{e_n} \otimes a_n,$$

for all $a_0 u_{e_1} \dots u_{e_n} a_n \in P$ reduced operator from p to p. We extend F_p to partial isometry, still denoted $F_p \in \mathcal{L}_{A_p}(H_p, K_p)$ by $F_p|_{\xi_p \cdot A_p} = 0$. Then $F_p^* F_p = 1 - Q_{\xi_p}$, where $Q_{\xi_p} \in \mathcal{L}_{A_p}(H_p)$ is the orthogonal projection onto $\xi_p \cdot A_p$. Moreover, $F_p F_p^* = \bigoplus_{e \in E(\mathcal{G}), r(e) = p} Q_e \otimes 1$.

Define $\rho_p = \bigoplus_{e \in E(\mathcal{G}), r(e) = p} \rho_e \otimes 1 : P_{\text{vert}} \to \mathcal{L}_{A_p}(K_p)$.

Lemma 3.4. *For any $a \in P$ we have $(F_p\pi_p(\lambda(a)) - \rho_p(\lambda(a))F_p) \in \mathcal{K}_{A_p}(H_p, K_p)$.*

Proof. It suffices to prove the lemma for any $a = a_n u_{e_n} \dots u_{e_1} a_0 \in P$ reduced operator from p to p since, for $a \in A_p$ one has $F_p\pi_p(\lambda(a)) = \rho_p(\lambda(a))F_p$. We may and will assume that $r(e_k) \neq p$ for all $k \neq 1$ since reduced operator from p to p may be written as the product of such operators. Fix such an operator a and, for $b \in P$ write $X(b) = (F_p\pi_p(\lambda(a)) - \rho_p(\lambda(a))F_p)(\pi_p(\lambda(b))\xi_p)$. If $b \in A_p$ then $F_p\pi_p(\lambda(b))\xi_p = 0$ and $ab = a_n u_{e_n} \dots u_{e_1} a_0 b \in P$ is reduced from p to p. Hence, $F_p\pi_p(\lambda(ab))\xi_p = \rho_{e_1}(\lambda(a_n u_{e_n} \dots u_{e_1}))\eta_{e_1} \otimes a_0 b$ and we have

$$\begin{aligned} X(b) &= (\rho_{e_1}(\lambda(a_n u_{e_n} \dots u_{e_1}))\eta_{e_1} \otimes 1) \cdot a_0 b \\ &= (\rho_{e_1}(\lambda(a_n u_{e_n} \dots u_{e_1}))\eta_{e_1} \otimes 1) \cdot \langle \pi_p(\lambda(a_0^*))\xi_p, \pi_p(\lambda(b))\xi_p \rangle. \end{aligned}$$

Suppose that $b = b_0 u_{f_1} \dots u_{f_m} b_m \in P$ is a reduced operator from p to p and write $b = b' b_m$, where $b' = b_0 u_{f_1} \dots u_{f_m}$. Let $0 \leq n_0 \leq \min\{n, m\}$ and, for $1 \leq k \leq n_0$, $x_k \in A_{s(e_k)}$ be the data associated to the couple (a, b') in Lemma 2.5. By Lemma 2.5 we can write $ab' = y + z$, where y

is either reduced and ends with u_{f_m} or is a sum of reduced operators that end with u_{f_m} and:

$$z = \begin{cases} a_n u_{e_n} \dots u_{e_{m+1}} x_m & \text{if } n_0 = m < n, \\ x_n & \text{if } n_0 = n = m, \\ 0 & \text{if } n_0 = 0 \text{ or } n_0 = n < m \text{ or } 1 \leq n_0 < \min\{n, m\}. \end{cases}$$

Since y is a sum of reduced operators ending with u_{f_m} we have $F_p\pi_p(\lambda(y))\xi_p = \rho_{f_m}(\lambda(y))\eta_{f_m} \otimes 1$ and,

$$\begin{aligned} X(b) &= F_p\pi_p(\lambda(ab'))\xi_p \cdot b_m - \rho_{f_m}(\lambda(ab'))\eta_{f_m} \otimes b_m \\ &= F_p\pi_p(\lambda(y))\xi_p \cdot b_m - \rho_{f_m}(\lambda(y))\eta_{f_m} \otimes b_m + F_p\pi_p(\lambda(z))\xi_p \cdot b_m - \rho_{f_m}(\lambda(z))\eta_{f_m} \otimes b_m \\ &= F_p\pi_p(\lambda(z))\xi_p \cdot b_m - \rho_{f_m}(\lambda(z))\eta_{f_m} \otimes b_m. \end{aligned}$$

Hence, if $n_0 = 0$, $n_0 = n < m$ or $1 \leq n_0 < \min\{n, m\}$ then $X(b) = 0$.
Note that if $n_0 = m < n$ then $\overline{f}_m = e_m$ which implies that $r(e_{m+1}) = s(e_m) = r(f_m) = p$ which does not happen with our hypothesis on a.
Finally, if $n_0 = n = m$ then $z = x_n = a_n s_{e_n} \circ r_{e_n}^{-1} \circ E_{e_n}^r(x_{n-1}) \in a_n B_{\overline{e}_n}^r$ and, since $f_m = f_n = \overline{e}_n$, we have $\rho_{f_m}(\lambda(z))\eta_{f_m} \otimes b_m = \rho_{\overline{e}_n}(\lambda(x_n))\eta_{\overline{e}_n} \otimes b_n \in (\rho_{\overline{e}_n}(\lambda(a_n))\eta_{\overline{e}_n} \otimes 1) \cdot A_p$ and $F_p\pi_p(\lambda(z))\xi_p \cdot b_m = F_p\pi_p(\lambda(x_n))\xi_p \cdot b_m = 0$. Hence,

$$\begin{aligned} X(b) &= -\rho_{\overline{e}_n}(\lambda(x_n))\eta_{\overline{e}_n} \otimes b_n = -\rho_{\overline{e}_n}(\lambda(a_n))\eta_{\overline{e}_n} \otimes s_{e_n} \circ r_{e_n}^{-1} \circ E_{e_n}^r(x_{n-1})b_n \\ &= -(\rho_{\overline{e}_n}(\lambda(a_n))\eta_{\overline{e}_n} \otimes 1) \cdot \langle \pi_p(\lambda(a')^*)\xi_p, \pi_p(\lambda(b))\xi_p \rangle, \end{aligned}$$

where $a' = u_{e_n} a_{n-1} \dots u_{e_1} a_0$. It follows that, for any reduced operator $b \in P$ from p to p and for any $b \in A_p$ the element $X(b)$ is equal to

$$(\rho_{e_1}(\lambda(a_n u_{e_n} \dots u_{e_1}))\eta_{e_1} \otimes 1) \cdot \langle \pi_p(\lambda(a_0^*))\xi_p, \pi_p(\lambda(b))\xi_p \rangle - (\rho_{\overline{e}_n}(\lambda(a_n))\eta_{\overline{e}_n} \otimes 1) \cdot \langle \pi_p(\lambda(a')^*)\xi_p, \pi_p(\lambda(b))\xi_p \rangle.$$

Hence, $F_p\pi_p(\lambda(a)) - \rho_p(\lambda(a))F_p$ is a finite rank operator. □

Since F_p is a partial isometry satisfying $F_pF_p^* - 1 = -Q_{\xi_p} \in \mathcal{K}_{A_p}(H_p)$, it follows from Lemma 3.4 that we can apply Lemma 2.1 to conclude that $[(K_p, \rho_p, V_p)] = 0 \in KK^1(P_{\text{vert}}, A_p)$, where $V_p = 2F_pF_p^* - 1 = \bigoplus_{e \in E(\mathcal{G}),\, r(e) = p} V_e \otimes 1$ and V_e has been defined previously by $V_e = 2Q_e - 1$. It follows from the definitions that (K_p, ρ_p, V_p) is a triple representing the element $\sum_{r(e)=p} x_e^{\mathcal{G}} \underset{B_e}{\otimes} [r_e]$. This concludes the proof of (3).

(4). Note that, for all $e \in E(\mathcal{G})$ and all $x \in P$, we have $\mathbb{E}_{\overline{e}}(\lambda(x)) = \lambda(u_e)\mathbb{E}_e(\lambda(u_e^* x u_e))\lambda(u_e^*)$. It follows from this formula that the operator $W_e : K_{\overline{e}} \underset{s_e^{-1}}{\otimes} B_e \to K_e \underset{r_e^{-1}}{\otimes} B_e$ defined by

$$W_e(\rho_{\overline{e}}(\lambda(x))\eta_{\overline{e}} \otimes b) = \rho_e(\lambda(x u_e))\eta_e \otimes b \quad \text{for } x \in P \text{ and } b \in B_e,$$

is a unitary operator in $\mathcal{L}_{B_e}(K_{\overline{e}} \underset{s_e^{-1}}{\otimes} B_e, K_e \underset{r_e^{-1}}{\otimes} B_e)$. Moreover, it is clear that W_e intertwines the representations $\rho_e(\cdot) \otimes 1$ and $\rho_{\overline{e}}(\cdot) \otimes 1$ and we have $W_e^*(Q_e \otimes 1)W_e = 1 \otimes 1 - Q_{\overline{e}} \otimes 1$.

□

Remark 3.5. Assertions (2) and (3) of the preceding Proposition obviously hold for the elements $z_e^{\mathcal{G}} = [\lambda] \underset{P_{\text{vert}}}{\otimes} x_e^{\mathcal{G}} \in KK^1(P, B_e)$ and also assertions (1) and (2) with $\pi_{\mathcal{G}'}$ instead of $\pi_{\text{vert}}^{\mathcal{G}'}$ since we have $\pi_{\text{vert}}^{\mathcal{G}'} \circ \lambda_{\mathcal{G}'} = \lambda \circ \pi_{\mathcal{G}'}$ for any connected subgraph $\mathcal{G}' \subset \mathcal{G}$, with maximal subtree $\mathcal{T}' \subset \mathcal{T}$.

We study now in details the behavior of our elements $x_e^{\mathcal{G}}$ under the *Serre's devissage* process.

The case of an amalgamated free product. Let A_1, A_2 and B be C*-algebras with unital faithful $*$-homomorphisms $\iota_k : B \to A_k$ and conditional expectations $E_k : A_k \to \iota_k(B)$ for $k = 1, 2$. Let $A_v = A_1 \overset{v}{\underset{B}{*}} A_2$ be the associated vertex-reduced amalgamated free product, $A_f = A_1 \underset{B}{*} A_2$ the full amalgamated free product and $\pi : A_f \to A_v$ the canonical surjection. Let (K, ρ, η) be the GNS construction of the canonical ucp map $E : A_v \to B$ (which is the composition of the canonical surjection from A to the edge-reduced amalgamated free product with the canonical ucp map from the edge-reduced amalgamated free product to B) and K_i, for $i = 1, 2$, be the closed subspace of K generated by $\{\rho(\pi(x))\eta : x = a_1 \dots a_n \in A_f \text{ reduced and ends with } A_i \ominus B\}$. Observe that K_i is a complemented Hilbert submodule of K. Actually we have $K = K_1 \oplus K_2 \oplus \eta \cdot B$. Let $Q_i \in \mathcal{L}_B(K)$ be the orthogonal projection onto K_i. The following Proposition is actually a special case of Lemma 3.1. In this special case the proof is very easy and left to the reader.

Proposition 3.6. *(K, ρ, V), where $V = 2Q_1 - 1$ defines an element $x_A = [(K, \rho, V)] \in KK^1(A_v, B)$.*

Let $e \in E(\mathcal{G})$ and suppose that $\mathcal{G}_e$ is not connected. We keep the same notations as the one used in the Serre's devissage process explained in the previous Section. In particular we have the $*$-isomorphism $\nu_e : A_{\mathcal{G}_e} := P_{\text{vert}}^{\mathcal{G}_{s(e)}} \overset{v}{\underset{B_e}{*}} P_{\text{vert}}^{\mathcal{G}_{r(e)}} \to P_{\text{vert}}$ from Lemma 2.12. We now have two canonical elements in $KK^1(P_{\text{vert}}, B_e)$: $x_e^{\mathcal{G}}$ and $x_{\mathcal{G}_e} := [\nu_e^{-1}] \underset{A_{\mathcal{G}_e}}{\otimes} y_{\mathcal{G}_e}$, where $y_{\mathcal{G}_e}$ is the element associated to the vertex-reduced amalgamated free product $A_{\mathcal{G}_e}$ constructed in Proposition 3.6. These two elements are actually equal.

Lemma 3.7. *We have $x_{\mathcal{G}_e} = x_e^{\mathcal{G}} \in KK^1(P_{\text{vert}}, B_e)$.*

Proof. The proof is a simple identification: there is not a single homotopy to write, only an isomorphism of Kasparov's triples. The key of the proof is to realize that the two ucp maps $P_{\text{vert}} \to B_e$ defined by $\varphi = r_e^{-1} \circ \mathbb{E}_e$ and $\psi = E \circ \nu_e^{-1}$ are equal, where $E : A_{\mathcal{G}_e} \to B_e$ is the canonical ucp map and it directly follows from the fact that ν_e intertwines the canonical ucp maps. Having this observation in mind, one construct an isomorphism of Kasparov's triples.
Recall that (K_e, ρ_e, η_e) denotes the GNS construction of the ucp map $\mathbb{E}_e : P_{\text{vert}} \to B_e^r$ and (K, ρ, η) denotes the GNS of the ucp map $E : A_{\mathcal{G}_e} \to B_e$.

Since $K = \overline{\rho \circ \nu_e^{-1}(P_{\text{vert}})\eta \cdot B_e}$, $K_e \underset{r_e^{-1}}{\otimes} B_e = \overline{\rho_e(P_{\text{vert}})\eta_e \otimes 1 \cdot B_e}$ and

$$\langle \eta, \rho \circ \nu_e^{-1}(x)\eta \rangle_K = \psi(x) = \varphi(x) = \langle \eta_e \otimes 1, \rho_e(x)\eta_e \otimes 1 \rangle_{K_e \underset{r_e^{-1}}{\otimes} B_e} \quad \text{for all } x \in P_{\text{vert}},$$

it follows that the map $U : K \to K_e \underset{r_e^{-1}}{\otimes} B_e$, $U(\rho \circ \nu_e^{-1}(x))\eta \cdot b) = \rho_e(x)\eta_e \otimes 1 \cdot b$ for $x \in P_{\text{vert}}$ and $b \in B_e$, defines a unitary $U \in \mathcal{L}_{B_e}(K, K_e \underset{r_e^{-1}}{\otimes} B_e)$. Moreover, U intertwines the representations $\rho \circ \nu_e^{-1}$ and $\rho_e(\cdot) \otimes 1$. Observe that $x_{\mathcal{G}_e}$ is represented by the triple $(K, \rho \circ \nu_e^{-1}, V)$, where $V = 2Q - 1$ and Q is the orthogonal projection on the closed linear span of the $\rho(\pi(x_1 \dots x_n))$, where $x_1 \dots x_n \in P_{\text{vert}}^{\mathcal{G}_{s(e)}} \underset{B_e}{*} P_{\text{vert}}^{\mathcal{G}_{r(e)}}$ is a reduced operator in the free product sense and $x_n \in P_{\text{vert}}^{\mathcal{G}_{s(e)}}$. Moreover, $x_e^{\mathcal{G}}$ is represented by the triple $(K_e \underset{r_e^{-1}}{\otimes} B_e, \rho_e(\cdot) \otimes 1, V_e)$, where $V_e = Q_e \otimes 1$ and Q_e

is the orthogonal projection onto the closed linear span of the $\rho_e(\lambda(a_0u_{e_1}\dots u_{e_n}a_n))\eta_e$, where $a_0u_{e_1}\dots u_{e_n}a_n \in P$ is reduced from $r(e)$ to $r(e)$ with $e_n = e$ and $a_n \in B_e^r$.

To conclude the proof, it suffices to observe that $UVU^* = V_e$. □

We study now the case of an HNN-extension.

The case of an HNN extension. For $\epsilon \in \{-1, 1\}$, let $\pi_\epsilon : B \to A$ be a unital faithful $*$-homomorphism $E_\epsilon : A \to B$ be a ucp map such that $E_\epsilon \circ \pi_\epsilon = \mathrm{id}_B$. Let C_f be the full HNN-extension with stable letter $u \in \mathcal{U}(C)$, C_v the vertex-reduced HNN-extension and $\pi : C_f \to C_v$ the canonical surjection. Let (K, ρ, η) be the GNS construction of the ucp map $E = E_1 \circ E_A : C_v \to B$, where $E_A : C_v \to A$ is the canonical GNS-faithful ucp map. Define the sub B-module

$$K_+ = \overline{\mathrm{Span}}\{\rho(\pi(x))\eta : x = a_0u^{\epsilon_1}\dots u^{\epsilon_n}a_n \in C_f \text{ is a reduced operator with } \epsilon_n = 1 \text{ and } a_n \in \pi_1(B)\}.$$

Observe that K_+ is complemented and let $Q_+ \in \mathcal{L}_B(K)$ be the orthogonal projection onto K_+. The following proposition, which is a special case of Lemma 3.1, is very easy to check.

Proposition 3.8. *(K, ρ, V), where $V = 2Q_+ - 1$, defines an element $x_C \in KK^1(C_v, B)$.*

Let $e \in E(\mathcal{G})$ and suppose that $\mathcal{G}_e$ is connected. Up to a canonical isomorphism of P we may and will assume that $\mathcal{T} \subset \mathcal{G}_e$. Recall that we have a canonical $*$-isomorphism $\nu_e : C_{\mathcal{G}_e} := \mathrm{HNN}_{\mathrm{vert}}(P_{\mathrm{vert}}^{\mathcal{G}_e}, B_e, \pi_1, \pi_{-1}) \to P_{\mathrm{vert}}$ defined in Lemma 2.13. As before, we get two canonical elements in $KK^1(P_{\mathrm{vert}}, B_e)$: $x_e^{\mathcal{G}}$ and $x_{\mathcal{G}_e} := [\nu_e^{-1}] \underset{C_{\mathcal{G}_e}}{\otimes} y_{\mathcal{G}_e}$, where $y_{\mathcal{G}_e} \in KK^1(C_{\mathcal{G}_e}, B_e)$ is the element associated to the vertex-reduced HNN-extension $C_{\mathcal{G}_e}$ constructed in Proposition 3.8. As before, these two elements are actually equal.

Lemma 3.9. *We have $x_{\mathcal{G}_e} = x_e^{\mathcal{G}} \in KK^1(P_{\mathrm{vert}}, B_e)$.*

Proof. Recall that (K, ρ, η) denotes the GNS construction of the canonical ucp map $E : C_{\mathcal{G}_e} \to B_e$. The proof is similar to the proof of Lemma 3.7 and is just a simple identification. Since ν_e intertwines the canonical ucp maps, the two ucp maps $\varphi, \psi : P_{\mathrm{vert}} \to B_e$ defined by $\varphi = \mathbb{E}_e$ and $\psi = E \circ \nu_e^{-1}$ are equal. As before, one can deduce easily from this equality an isomorphism of Kasparov's triples. Since the arguments are the same, we leave the details to the reader. □

Remark 3.10. The analogue of Lemmas 3.7, 3.9 are obviously still valid for the elements $z_e^{\mathcal{G}} \in KK^1(P, B_e)$ defined in Remark 3.2.

4. The exact sequence

For any separable C*-algebra C, let $F^*(-)$ be $KK^*(C, -)$. It is a $\mathbb{Z}_2$-graded covariant functor. If f is a morphism of C*-algebras, we will denote by f^* the induced morphism.

In the sequel $P_{\mathcal{G}}$ or simply P denotes either the full or the vertex reduced fundamental C*-algebra. We define the boundary maps $\gamma_e^{\mathcal{G}}$ from $F^*(P_{\mathcal{G}}) = KK^*(D, P_{\mathcal{G}})$ to $KK^{*+1}(D, B_e) = F^{*+1}(B_e)$ by $\gamma_e^{\mathcal{G}}(y) = y \otimes_P z_e^{\mathcal{G}}$ when P is the full fundamental C*-algebra or $\gamma_e^{\mathcal{G}}(y) = y \otimes_P x_e^{\mathcal{G}}$ when P is the vertex reduced one. In the sequel we simply write $x_e = x_e^{\mathcal{G}}$ and $z_e = z_e^{\mathcal{G}}$.

If $\mathcal{G}$ is a graph, then E^+ is the set of positive edges, V the set of vertices and for any $v \in V$, the map from A_v to $P_{\mathcal{G}}$ is π_v or sometimes $\pi_v^{\mathcal{G}}$ if it is necessary to indicate which graph we consider. If one removes an edge e_0 (and its opposite) to $\mathcal{G}$, the new graph is called $\mathcal{G}_0$, P_0 is the algebra associated to it and π_v^0 is the embedding of A_v in P_0. We also have for $\mathcal{G}_1 \subset \mathcal{G}$ a morphism $\pi_{\mathcal{G}_1}$ from $P_{\mathcal{G}_1}$ to $P_{\mathcal{G}}$.

Theorem 4.1. *In the presence of conditional expectations (not necessarily GNS -faithful), we have, for P the full or vertex reduced fundamental C*-algebra, a long exact sequence*

$$\longrightarrow \bigoplus_{e\in E^+} F^*(B_e) \overset{\sum_e s_e^* - r_e^*}{\longrightarrow} \bigoplus_{v\in V} F^*(A_v) \overset{\sum_v \pi_v^*}{\longrightarrow} F^*(P) \overset{\oplus_e \gamma_e^{\mathcal{G}}}{\longrightarrow} \bigoplus_{e\in E^+} F^{*+1}(B_e) \longrightarrow$$

Proof. First note that it is indeed a chain complex. Because s_e and r_e are conjugated in the full or reduced fundamental C*-algebra, we only have to check that $\gamma_e \circ \pi_v^* = 0$ (which is point 2 of prop 3.3) and (for P_{vert}), $\sum_{e\in E^+} x_e \otimes [r_e] - x_e \otimes [s_e] = 0$. As $x_e = -x_{\bar{e}}$ (point 4 of 3.3) and $s_{\bar{e}} = r_e$, this is the same as point 3 of 3.3. Because of remark 3.5, this is also true for the full fundamental C*-algebra.

Also if the graph contains only one geometric edge (i.e. two opposite oriented edges), we are in the case of the amalgamated free product or the HNN extension and the complex is known to be exact because of the results of [FG15]. For convenience we will briefly recall why and also we will identify the boundary map. Let's do the full amalgamated free product A_f first. Recall that in theorem 4.1 of [FG15], we proved that the suspension of $A_1 \underset{B}{*} A_2$ is KK-equivalent to D the cone of the inclusion of B in A_1 and A_2. Obviously D fits into a short exact sequence :

$$0 \to A_1 \otimes S \oplus A_2 \otimes S \longrightarrow D \overset{ev_0}{\longrightarrow} B \to 0.$$

Therefore there is a long exact sequence for our functor F^* :

$$F^*(A_1 \otimes S \oplus A_2 \otimes S) \to F^*(D) \to F^*(B) \to F^{*+1}(A_1 \otimes S \oplus A_2 \otimes S).$$

But $F^*(A_k \otimes S)$ identifies with $F^{*+1}(A_k)$ and $F^*(D)$ with $F^{*+1}(A_f)$. Via these identifications, the map from $F^*(B)$ to $F^*(A_k)$ becomes i_k^* or its opposite (this is seen using the mapping cone exact sequence) and the map from $F^*(A_k)$ to $F^*(A_f)$ is j_k^*. The only thing left is the identification of the boundary map from $F^*(A_f)$ to $F^{*+1}(B)$. It is obviously the Kasparov product by $x \otimes [ev_0]$ where x is the element in $KK^1(A_f, D)$ that implements the K-equivalence. The element $x \otimes [ev_0] \in KK^1(A_f, B)$ has been described in lemma 4.9 of [FG15] and it is equal to $[\pi] \otimes x_A$, where $x_A \in KK^1(A_v, B)$ is exactly the element of 3.6 and π is the canonical surjection from the full amalgamated free product A_f to the vertex-reduced amalgamated free product A_v. Therefore the boundary map is exactly given by the corresponding $\gamma_e^{\mathcal{G}}$ for the graph of the free product. Moreover, since x actually factorizes as $[\pi] \otimes_{A_v} z$ where $z \in KK^1(A_v, D)$, the same identifications and the same exact sequence hold for the vertex reduced free product A_v and theorem 4.1 is true for free products.

Now let's tackle the HNN extension case. Let's call C_m the full HNN extension of (A, B, θ) and E and E_θ the conditional expectations from A to B and $\theta(B)$. We also denote by C_v the vertex-reduced HNN-extension and $\pi : C_m \to C_v$ the canonical surjection. An explicit isomorphism is known to exist between C_m and the full amalgamated free product $e_{11}M_2(A) \underset{B\oplus B}{*} M_2(B)e_{11}$ where $B \oplus B$ imbeds diagonally in $M_2(A)$ via the canonical inclusion and θ, e_{11} is the matrix unit $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and the conditional expectations are $E_1 \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} = E(a_1) \oplus E_\theta(a_4)$ from $M_2(A)$ to $B \oplus B$ and $E_2 \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} = b_1 \oplus b_4$ from $M_2(B)$ to $B \oplus B$. The exact sequence for the HNN extension is then deduced from this isomorphism of C*-algebras (cf. [Ue08] for example).

If we call j_A and j_B the inclusions of $M_2(A)$ respectively $M_2(B)$ in the free product then the unitary u in C_m that implements θ is mapped to $j_A\begin{pmatrix}0&1\\0&0\end{pmatrix}j_B\begin{pmatrix}0&0\\1&0\end{pmatrix}$.
It is then clear that a reduced word in C_m that ends with u times b with b in B is mapped into a reduced word in the free product that ends with $j_B\begin{pmatrix}0&0\\1&0\end{pmatrix}\begin{pmatrix}b&0\\0&0\end{pmatrix}=j_B\begin{pmatrix}0&b'\\b&0\end{pmatrix}e_{11}$ i.e. that ends in $j_B(M_2(B))\ominus(B\oplus B)$. Therefore, in this situation and after a Kasparov product by $[\pi]$ on the left, the element described in 3.8 is the same as the element described in 3.6 and we have identified the correct boundary map.

Let's have a look now at the vertex reduced situation. Observe that the conditional expectation E_2 from $M_2(B)$ to $B\oplus B$ is GNS faithful. It follows from the constructions of [FG15, Section 2] that $M_2(A)\underset{B\oplus B}{\overset{2}{*}}M_2(B)$ is isomorphic to $M_2(A)\underset{B\oplus B}{\overset{e}{*}}M_2(B)$ and as a consequence $M_2(A)\underset{B\oplus B}{\overset{1}{*}}M_2(B)$ is isomorphic to $M_2(A)\underset{B\oplus B}{\overset{v}{*}}M_2(B)$. Using the universal properties it is now obvious that the vertex reduced HNN extension of (A,B,θ) is $e_{11}M_2(A)\underset{B\oplus B}{\overset{1}{*}}M_2(B)e_{11}$. Therefore the identification described earlier for the full free product and HNN extension is again true for the vertex reduced free product and corresponding vertex reduced HNN extension. Hence theorem 4.1 is again valid for HNN extensions.

We now prove exactness at each place by induction on the cardinal of edges and *devissage*. Note that 3.7 and 3.9 allow us to decompose our fundamental algebra in HNN or free product while using the same boundary maps γ_e.

Lemma 4.2. *We have the exactness of* $\bigoplus_{e\in E^+}F^*(B_e)\overset{\sum_e s_e^*-r_e^*}{\longrightarrow}\bigoplus_{v\in V}F^*(A_v)\overset{\sum_v\pi_v^*}{\longrightarrow}F^*(P)$.

Proof. Choose a positive edge e_0. Then without this edge (and its opposite), the graph $\mathcal{G}_0$ is either connected (Case I) or has two connected components $\mathcal{G}_1$ and $\mathcal{G}_2$ (Case II).

Case I. P is the HNN extension of $P_{\mathcal{G}_0}$ and B_{e_0}. The set of vertices of $\mathcal{G}$ is the same as the set of vertices of $\mathcal{G}_0$ and we may and will assume that $v_0=s(e_0)=r(e_0)$. Let $x=\oplus x_v$ be in $\oplus_{v\in V}F^*(A_v)$ such that $\sum_v\pi_v^*(x_v)=0$. If $y=\sum_v\pi_v^{0*}(x_v)$, then clearly $\pi_{\mathcal{G}_0}(y)=0$. Then, the long exact sequence for P seen as an HNN extension implies then that there exists $y_0\in F^*(B_{e_0})$ such that $(\pi_{v_0}\circ s_{e_0})^*(y_0)-(\pi_{v_0}\circ r_{e_0})^*(y_0)=y=\sum_v\pi_v^{0*}(x_v)$. Hence,

$$\sum_v\pi_v^{0*}(\oplus_{v\neq v_0}x_v\oplus(x_{v_0}-s_{e_0}^*(y_0)+r_{e_0}^*(y_0))=0.$$

Using the exactness for P_0 as $\mathcal{G}_0$ has one less edge, we get that there exists for any $e\neq e_0$ a y_e such that $\sum_{e\neq e_0}s_e^*(y_e)-r_e^*(y_e)=\oplus_{v\neq v_0}x_v\oplus(x_{v_0}-s_{e_0}^*(y_0)+r_{e_0}^*(y_0))$. Thus,

$$\sum_{e\neq e_0}s_e^*(y_e)-r_e^*(y_e)+s_{e_0}^*(y_0)-r_{e_0}^*(y_0)=x.$$

Case II. P is the amalgamated free product of $P_1=P_{\mathcal{G}_1}$ and $P_2=P_{\mathcal{G}_2}$ over B_{e_0}. For $i=1,2$, denote by V_i the vertices of $\mathcal{G}_i$. We know that V is the disjoint union of V_1 and V_2. The map π_v^i will be the embedding of A_v in P_i. We also write $v_1=s(e_0)$ and $v_2=r(e_0)$. Let $x=\oplus x_v$ be in $\oplus_{v\in V}F^*(A_v)$ such that $\sum_v\pi_v^*(x_v)=0$. Let $x_i=\oplus_{v\in V_i}{\pi_v^i}^*(x_v)$. Clearly $\pi_{\mathcal{G}_1}^*(x_1)+\pi_{\mathcal{G}_2}^*(x_2)=0$. Then, the long exact sequence for P seen as an amalgamated free product gives a $y_0\in F^*(B_{e_0})$

such that $(\pi^1_{v_1}\circ s_{e_0})^*(y_0)-(\pi^2_{v_2}\circ r_{e_0})^*(y_0)=x_1\oplus x_2$. Define $\bar{x}_1=\oplus_{v\in V_1}x_v-s^*_{e_0}(y_0)$ and $\bar{x}_2=\oplus_{v\in V_2}x_v+r^*_{e_0}(y_0)$. We have, for $i=1,2$, $\sum_{v\in V_i}{\pi^i_v}^*(\bar{x}_i)=0$. Therefore by induction as $\mathcal{G}_i$ has strictly less edges than $\mathcal{G}$, there exists for any $e\neq e_0$ a $y_e\in F^*(B_e)$ such that $\bar{x}_1\oplus\bar{x}_2=\sum_{e\neq e_0}s^*_e(y_e)-r^*_e(y_e)$. Hence, $x=\sum_{e\neq e_0}s^*_e(y_e)-r^*_e(y_e)+s^*_{v_0}(y_0)-r^*_{v_0}(y_0)$. □

Lemma 4.3. *The following chain complex is exact in the middle*

$$\bigoplus_{v\in V}F^*(A_v)\overset{\sum_v\pi^*_v}{\longrightarrow}F^*(P)\overset{\oplus_e\gamma^{\mathcal{G}}_e}{\longrightarrow}\bigoplus_{e\in E^+}F^{*+1}(B_e)$$

Proof. As in the previous Lemma, we separate in the proof in Case I and Case II.

Case I. Let x be in $F^*(P)$ such that for any e, $\gamma^{\mathcal{G}}_e(x)=0$. In particular for the edge e_0. Using the long exact sequence for P seen as an HHN extension, and since $\gamma^{\mathcal{G}}_{e_0}(x)=0$ we get that there exists x_0 in $F^*(P_0)$ such that $\pi^*_{\mathcal{G}_0}(x_0)=x$. For any edges $e\neq e_0$, one has $\gamma^{\mathcal{G}_0}_e(x_0)=\gamma^{\mathcal{G}}_e(\pi^*_{\mathcal{G}_0}(x_0))=0$. Hence by induction there exists for any $v\in V(\mathcal{G}_0)=V(\mathcal{G})$ a $y_v\in F^*(A_v)$ such that $\sum_v\pi^{0*}_v(y_v)=x_0$. Hence $x=\sum_v(\pi_{\mathcal{G}_0}\circ\pi^0_v)^*(y_v)=\sum_v\pi^*_v(y_v)$.

Case II. Using that P is the free product of P_1 and P_2, we get an $x_i\in F^*(P_i)$ for $i=1,2$ such that $x=\pi^*_{\mathcal{G}_1}(x_1)+\pi^*_{\mathcal{G}_2}(x_2)$. Now for $i=1,2$, and for any edge e of $\mathcal{G}_i$, we have

$$\gamma^{\mathcal{G}_i}_e(x_i)=\gamma^{\mathcal{G}}_e(\pi^*_{\mathcal{G}_i}(x_i))=\gamma^{\mathcal{G}}_e(x)-\gamma^{\mathcal{G}}_e(\pi^*_{\mathcal{G}_j}(x_j))\text{ for }j\neq i.$$

But e is not an edge of $\mathcal{G}_j$, so $\gamma^{\mathcal{G}}_e\circ\pi^*_{\mathcal{G}_j}=0$. Hence $\gamma^{\mathcal{G}_i}_e(x_i)=0$. By induction we get for any vertex of $V_1\cup V_2=V(\mathcal{G})$ a $y_v\in F^*(A_v)$ such that $x_i=\sum_{v\in V_i}{\pi^i_v}^*(y_v)$ for $i=1,2$. Therefore $x=\sum_v\pi^*_v(y_v)$.

□

Lemma 4.4. *The following chain complex is exact in the middle*

$$F^{*-1}(P)\overset{\oplus_e\gamma^{\mathcal{G}}_e}{\longrightarrow}\bigoplus_{e\in E^+}F^*(B_e)\overset{\sum_e s^*_e-r^*_e}{\longrightarrow}\bigoplus_{v\in V}F^*(A_v)$$

Proof. **Case I.** Let $x=\oplus_{e\in E^+}x_e$ such that $\sum_e s^*_e(x_e)-r^*_e(x_e)=0$. Then for the distinguished vertex v_0, one has ${\pi^0_{v_0}}^*(s^*_{e_0}(x_{e_0}))-{\pi^0_{v_0}}^*(r^*_{e_0}(x_{e_0}))=-\sum_{e\neq e_0}{\pi^0_{v_0}}^*(s^*_e(x_{e_0}))-{\pi^0_{v_0}}^*(r^*_e(x_{e_0}))$. But as e is an edge of $\mathcal{G}_0$, s_e and r_e are conjugated by a unitary of P_0. Therefore there difference are 0 in any KK-groups. Thus ${\pi^0_{v_0}}^*(s^*_{e_0}(x_{e_0}))-{\pi^0_{v_0}}^*(r^*_{e_0}(x_{e_0}))=0$. Using the long exact sequence for P as an HHN extension, we get a y_0 in $F^{*-1}(P)$ such that $\gamma^{\mathcal{G}}_{e_0}(y_0)=x_{e_0}$. Set now $\bar{x}_e=x_e-\gamma^{\mathcal{G}}_e(y_0)$ for any $e\neq e_0$ and compute

$$\begin{aligned}\sum_{e\neq e_0}s^*_e(\bar{x}_e)-r^*_e(\bar{x}_e) &= \sum_{e\neq e_0}s^*_e(x_e)-r^*_e(x_e)-\sum_e s^*_e(\gamma^{\mathcal{G}}_e(y_0))-r^*_e(\gamma^{\mathcal{G}}_e(y_0))\\ &\quad+s^*_{e_0}(\gamma^{\mathcal{G}}_{e_0}(y_0))-r^*_{e_0}(\gamma^{\mathcal{G}}_{e_0}(y_0))\\ &= \sum_e s^*_e(x_e)-r^*_e(x_e),\end{aligned}$$

by the third property of γ_e. Hence, $\sum_{e\neq e_0}s^*_e(\bar{x}_e)-r^*_e(\bar{x}_e)=0$. By induction there exists $\bar{y}_1$ in $F^{*-1}(P_0)$ such that for all $e\neq e_0$, $\gamma^{\mathcal{G}_0}_e(y_1)=\bar{x}_e$. Set at last $y_1=\pi^*_{\mathcal{G}_0}(\bar{y}_1)$ which is an element of $F^{*-1}(P)$. Now $\gamma^{\mathcal{G}}_{e_0}(y_0+y_1)=x_0+\gamma^{\mathcal{G}}_{e_0}\circ\pi^*_{\mathcal{G}_0}(\bar{y}_1)$. But e_0 is not an edge of $\mathcal{G}_0$ so $\gamma^{\mathcal{G}}_{e_0}\circ\pi^*_{\mathcal{G}_0}=0$.

Hence $\gamma^{\mathcal{G}}_{e_0}(y_0+y_1) = x_0$. On the other end, for $e \neq e_0$, $\gamma^{\mathcal{G}}_e(y_0+y_1) = \gamma^{\mathcal{G}}_e(y_0)+\bar{x}_e$ as $\gamma^{\mathcal{G}_0}_e = \gamma^{\mathcal{G}}_e \circ \pi^*_{\mathcal{G}_0}$. It follows that $\gamma^{\mathcal{G}}_e(y_0 + y_1) = x_e$.

Case II. Call E_i the edges of $\mathcal{G}_i$ for $i = 1, 2$. Note that for any positive edge e, if $s(e) \in V_1$ then either $e \in E_1$ or $e = e_0$ and if $r(e) \in V_2$ then $e \in E_2$. Let $x = \oplus_{e\in E^+}x_e$ such that $\sum_e s^*_e(x_e) - r^*_e(x_e) = 0$. The equality can be rewritten as $\sum_{e\in E_1^+} s^*_e(x_e) - r^*_e(x_e) + s^*_{e_0}(x_{e_0}) = 0$ in $\oplus_{v\in V_1}F^*(A_v)$ and $\sum_{e\in E_2^+} s^*_e(x_e) - r^*_e(x_e) - r^*_{e_0}(x_{e_0}) = 0$ in $\oplus_{v\in V_2}F^*(A_v)$. Let's compute now $\pi^1_{v_1}(x_{e_0})$. It is $-\sum_{e\in E_1^+}(\pi^1_{s(e)} \circ s_e)^*(x_e) - (\pi_r(e)^1 \circ r_e)^*(x_e)$ by the preceding remark. But as s_e and r_e are conjugated in P_1 because e is an edge of $\mathcal{G}_1$, this is 0. In the same way $\pi^2_{v_2}(x_{e_0}) = 0$. Therefore using the long exact sequence for P as a free product of P_1 and P_2, there is a y_0 in $F^{*-1}(P)$ such that $\gamma^{\mathcal{G}}_{e_0}(y_0) = x_{e_0}$. For all $e \neq e_0$ set $\bar{x}_e = x_e - \gamma^{\mathcal{G}}_e(y_0)$. Then,

$$\sum_{e\in E_1^+} s^*_e(\bar{x}_e) - r^*_e(\bar{x}_e) = \sum_{e\in E_1^+} s^*_e(x_e) - r^*_e(x_e) - \left(\sum_{e\in E_1^+} s^*_e \circ \gamma^{\mathcal{G}}_e(y_0) - r^*_e \circ \gamma^{\mathcal{G}}_e(y_0)\right).$$

But the third property of the $\gamma^{\mathcal{G}}_e$ implies that $0 = \sum_{e\in E_1^+} s^*_e \circ \gamma^{\mathcal{G}}_e + s^*_{e_0} \circ \gamma^{\mathcal{G}}_{e_0} - \sum_{e\in E_1^+} r^*_e \circ \gamma^{\mathcal{G}}_e$ using the remark made at the begining of this proof. Hence,

$$\sum_{e\in E_1^+} s^*_e(\bar{x}_e) - r^*_e(\bar{x}_e) = \sum_{e\in E_1^+} s^*_e(x_e) - r^*_e(x_e) + s^*_{e_0}(x_{e_0}) = 0.$$

Similarly, $\sum_{e\in E_2^+} s^*_e(\bar{x}_e)-r^*_e(\bar{x}_e) = 0$. Therefore by induction, there exists for $i = 1, 2$, an element y_i in $F^{*-1}(P_i)$ such that for all e in E_i^+, $\gamma^{\mathcal{G}_i}_e(y_i) = \bar{x}_e$. Set now $y = y_0 + \pi_{\mathcal{G}_1}(y_1) + \pi_{\mathcal{G}_2}(y_2)$ in $F^{*-1}(P)$. Then $\gamma^{\mathcal{G}}_{e_0}(y) = x_{e_0} + \gamma^{\mathcal{G}}_{e_0} \circ \pi^*_{\mathcal{G}_1}(y_1) + \gamma^{\mathcal{G}}_{e_0} \circ \pi^*_{\mathcal{G}_2}(y_2) = x_{e_0}$ as $\gamma^{\mathcal{G}}_{e_0} \circ \pi_{\mathcal{G}_i} = 0$ since e_0 is not an edge of $\mathcal{G}_1$ nor $\mathcal{G}_2$. On the other end, for $e \in E_1$, $\gamma^{\mathcal{G}}_e(y) = \gamma^{\mathcal{G}}_e(y_0) + \gamma^{\mathcal{G}_1}_e(y_1) + 0$ as e is not an edge of $\mathcal{G}_2$. Hence $\gamma^{\mathcal{G}}_e(y) = \gamma^{\mathcal{G}}_e(y_0) + \bar{x}_e = x_e$. The same is of course true for an edge in E_2. So we are done. □

The proof of Theorem 4.1 is now complete. □

Let's treat now the case $F^*(-) = KK(-, C)$. Again if f is a morphism of C*-algebras we will adopt the same notation f^* for the induced morphism. Now the map $\gamma^{\mathcal{G}}_e$ from $F(B_e)$ to $F(P)$ is defined as $\gamma^{\mathcal{G}}_e(a) = x^{\mathcal{G}}_e \otimes_{B_e} a$ if P is the vertex reduced fundamental C*-algebra or $\gamma^{\mathcal{G}}_e(a) = z^{\mathcal{G}}_e \otimes_{B_e} a$ if P is the full fundamental C*-algebra.

Theorem 4.5. *In the presence of conditional expectations, we have, for P the full or reduced fundamental C*-algebra, a long exact sequence*

$$\longleftarrow \bigoplus_{e\in E^+} F^*(B_e) \xleftarrow{\sum_e s^*_e - r^*_e} \bigoplus_{v\in V} F^*(A_v) \xleftarrow{\sum_v \pi^*_v} F^*(P) \xleftarrow{\oplus_e \gamma^{\mathcal{G}}_e} \bigoplus_{e\in E^+} F^{*+1}(B_e) \longleftarrow$$

Proof. As before this is a chain complex and the same identifications proves it for free products and HNN extension. We will now show exactness with the three following lemmas.

Lemma 4.6. *We have the exactness of* $\bigoplus_{e\in E^+} F^*(B_e) \xleftarrow{\sum_e s^*_e - r^*_e} \bigoplus_{v\in V} F^*(A_v) \xleftarrow{\sum_v \pi^*_v} F^*(P)$.

Proof. Let $x = \oplus x_v \in \oplus_v F(A_v)$ such that $\sum_e s^*_e(\oplus x_v) - r^*_e(\oplus x_v) = 0$.

Case I. We have $\sum_{e\neq e_0} s_e^*(\oplus x_v)-r_e^*(\oplus x_v)=0$ hence, there is a y_0 in $F(P_0)$ such that for all v, ${\pi_v^0}^*(y_0)=x_v$. But $s_{e_0}^*\circ{\pi_{v_0}^0}^*(y_0)=s_{e_0}^*(x_{v_0})=r_{e_0}^*(x_{v_0})=r_{e_0}^*\circ{\pi_{v_0}^0}^*(y_0)$. Using the exact sequence for P as an HNN extension of P_0 and the two copies of B_{e_0}, we get that there is $y\in F(P)$ such that ${\pi_{\mathcal{G}_0}}^*(y)=y_0$. Now for all v, $\pi_v^*(y)={\pi_v^0}^*(y_0)=x_v$.

Case II. We have, for $k=1,2$, $\sum_{e\in E_k^+} s_e^*(\oplus x_v)-r_e^*(\oplus x_v)=0$ hence there is $y_k\in F(P_k)$ such that ${\pi_v^k}^*(y_k)=x_v$ for any $v\in V_k$. As $s_{e_0}^*\circ{\pi_{v_1}^1}^*(y_1)=s_{e_0}^*(x_{v_1})=r_{e_0}^*(x_{v_2})=r_{e_0}^*\circ{\pi_{v_2}^2}^*(y_2)$. Using the exact sequence for P as a free product, we have a $y\in F(P)$ such that ${\pi_{\mathcal{G}_k}}^*(y)=y_k$ for $k=1,2$. Then for $k=1,2$ and all $v\in V_k$, $\pi_v^*(y)={\pi_v^k}^*(y_k)=x_v$. □

Lemma 4.7. *The following chain complex is exact in the middle*

$$\bigoplus_{v\in V}F^*(A_v)\overset{\sum_v\pi_v^*}{\longleftarrow}F^*(P)\overset{\oplus_e\gamma_e^{\mathcal{G}}}{\longleftarrow}\bigoplus_{e\in E^+}F^{*+1}(B_e)$$

Proof. Let y be in $F(P)$ such that $\pi_v^*(y)=0$ for all v.

Case I. Let $y_0=\pi_{\mathcal{G}_0}^*(y)$. Then for all v, ${\pi_v^0}^*(y_0)=\pi_v^*(y)=0$. Therefor there exists $x=\sum_{e\neq e_0}x_e$ such that $\sum_{e\neq e_0}\gamma_e^{\mathcal{G}_0}(x_e)=y_0$. Put $z=y-\sum_{e\neq e_0}{\gamma_e^{\mathcal{G}}}^*(x_e)$. Then,

$$\pi_{\mathcal{G}_0}^*(z)=y_0-\sum_{e\neq e_0}\gamma_e^{\mathcal{G}_0}(x_e)=0.$$

Hence there is a $x_{e_0}\in F(B_{e_0})$ such that $\gamma_{e_0}(x_{e_0})=z$ and $y=\sum_{e\neq e_0}\gamma_e^{\mathcal{G}}(x_e)+\gamma_{e_0}(x_{e_0})$.

Case II. Let $y_k=\pi_{\mathcal{G}_k}^*(y)$ for $k=1,2$. For all $v\in V_k$, ${\pi_v^k}^*(y_k)=\pi_v^*(y)=0$, hence there exists $x_k=\oplus_{e\in E_k^+}x_e$ such that $\sum_{e\in E_k^+}\gamma_e^{\mathcal{G}_k}(x_e)=y_k$. Let $z=y-\sum_{e\neq e_0}{\gamma_e^{\mathcal{G}}}^*(x_e)$. Then for $k=1,2$, $\pi_{\mathcal{G}_k}^*(z)=y_k-\sum_{e\in E_k^+}\gamma_e^{\mathcal{G}_k}(x_e)=0$ as $\pi_{\mathcal{G}_2}^*\circ\gamma_e^{\mathcal{G}_1}=0$ because of 3.3. Hence $z=\gamma_{e_0}(x_{e_0})$ for some x_{e_0} in $F(B_{e_0})$ and we are done.

□

Lemma 4.8. *The following chain complex is exact in the middle*

$$F^{*-1}(P)\overset{\oplus_e\gamma_e^{\mathcal{G}}}{\longleftarrow}\bigoplus_{e\in E^+}F^*(B_e)\overset{\sum_e s_e^*-r_e^*}{\longleftarrow}\bigoplus_{v\in V}F^*(A_v)$$

Proof. Let $x=\oplus x_e$ in $F(\oplus_e B_e)$ such that $\sum_{e\in E^+}\gamma_e^{\mathcal{G}}(x_e)=0$.

Case I. We have $0={\pi_{\mathcal{G}_0}}^*(\sum_{e\in E^+}\gamma_e^{\mathcal{G}}(x_e))=\sum_{e\neq e_0}\gamma_e^{\mathcal{G}_0}(x_e)$ as ${\pi_{\mathcal{G}_0}}^*\circ\gamma_{e_0}=0$. Hence by induction, there is a $z=\oplus z_v$ in $\oplus_v F(A_v)$ such that for all $e\neq e_0$, $x_e=s_e^*(z_{s(e)})-r_e^*(z_{r(e)})$. Put $x_0=x_{e_0}-s_{e_0}^*(z_{v_0})-r_{e_0}^*(z_{v_0})$. By 5.5 we have $\sum_{e\in E^+}\gamma_e\circ s_e^*-\gamma_e\circ r_e^*=0$ hence,

$$\gamma_{e_0}\circ(-s_{e_0}^*(z_{v_0})+r_{e_0}^*(z_{v_0}))=\sum_{e\neq e_0}\gamma_e(s_e^*(\oplus z_v))-\gamma_e(r_e^*(\oplus z_v))=\sum_{e\neq e_0}\gamma_e(x_e).$$

It follows that $\gamma_{e_0}(x_0)=\gamma_{e_0}(x_{e_0})+\sum_{e\neq e_0}\gamma_e(x_e)=0$. Using the long exact sequence for P as an HNN extension of P_0 and B_{e_0}, we get a $z_0\in F(P_0)$ such that $x_0=s_{e_0}^*({\pi_{v_0}^0}^*(z_0))-r_{e_0}^*({\pi_{v_0}^0}^*(z_0))$. So $x_{e_0}=s_{e_0}^*(z_{v_0}+{\pi_{v_0}^0}^*(z_0))-r_{e_0}^*(z_{v_0}+{\pi_{v_0}^0}^*(z_0))$ and we are done.

Case I. $0 = \pi_{\mathcal{G}_k}{}^*(\sum_{e\in E^+}\gamma_e^{\mathcal{G}}(x_e)) = \sum_{e\neq E_k^+}\gamma_e^{\mathcal{G}_k}(x_e)$ for $k = 1, 2$. Hence there is a $z = \oplus z_v$ such that for all $e \in E_k^+$, $x_e = s_e^*(z_{s(e)}) - r_e^*(z_{r(e)})$. Write $x_0 = x_{e_0} - s_{e_0}^*(z_{v_1}) - r_{e_0}^*(z_{v_2})$. As before we have that $\gamma_{e_0}(x_0) = 0$ and by exactness of the exact sequence for the free product of P_1 and P_2 there is $z_1 \in F(P_1)$ and $z_2 \in F(P_2)$ such that $x_0 = s_{e_0}^*(\pi_{v_1}^1{}^*(z_1)) - r_{e_0}^*(\pi_{v_2}^2{}^*(z_2))$. Finally $x_{e_0} = s_{e_0}^*(z_{v_1} + \pi_{v_1}^1{}^*(z_1)) - r_{e_0}^*(z_{v_2} + \pi_{v_2}^2{}^*(z_2))$. □

The proof of Theorem 4.5 is now complete. □

5. Applications

In this section we collect some applications of our results to K-equivalence and K-amenability of quantum groups.

Let $(\mathcal{G}, A_p, B_e)$ and $(\mathcal{G}, A'_p, B'_e)$ be two graphs of unital C*-algebras with maps s_e and s'_e and conditional expectations E_e^s and $(E_e^s)'$. Suppose that we have unital $*$-homomorphisms $\nu_p : A_p \to A'_p$ and $\nu_e : B_e \to B'_e$ such that $\nu_e = \nu_{\overline{e}}$ and $\nu_{s(e)} \circ s_e = s'_e \circ \nu_e$ for all $e \in E(\mathcal{G})$. Let P and P' be the associated full fundamental C*-algebras with canonical unitaries u_e and u'_e respectively. By the relations $\nu_e = \nu_{\overline{e}}$ and $\nu_{s(e)} \circ s_e = s'_e \circ \nu_e$ and the universal property of the full fundamental C*-algebra, there exists a unique unital $*$-homomorphism $\nu : P \to P'$ such that

$$\nu|_{A_p} = \nu_p \quad \text{and} \quad \nu(u_e) = u'_e \quad \text{for all } p \in V(\mathcal{G}),\ e \in E(\mathcal{G}).$$

Theorem 5.1. *If $(E_e^s)' \circ \nu_{s(e)} = \nu_{s(e)} \circ E_e^s$ and ν_p, ν_e are K-equivalences for all $p \in V(\mathcal{G}), e \in E(\mathcal{G})$ then ν is a K-equivalence.*

Proof. Consider the following diagrams with exact rows

$$\begin{array}{ccccccccc}
\bigoplus_{e\in E^+} KK(D,B_e) & \to & \bigoplus_{p\in V} KK(D,A_p) & \to & KK(D,P) & \to & \bigoplus_{e\in E^+} KK^1(D,B_e) & \to & \bigoplus_{p\in V} KK^1(D,A_p) \\
\downarrow \bigoplus \cdot \underset{B_e}{\otimes} [\nu_e] & & \downarrow \bigoplus \cdot \underset{A_p}{\otimes} [\nu_p] & & \downarrow \cdot \underset{P}{\otimes} [\nu] & & \bigoplus \cdot \underset{B_e}{\otimes} [\nu_e] & & \downarrow \bigoplus \cdot \underset{A_p}{\otimes} [\nu_p] \\
\bigoplus_{e\in E^+} KK(D,B'_e) & \to & \bigoplus_{p\in V} KK(D,A'_p) & \to & KK(D,P') & \to & \bigoplus_{e\in E^+} KK^1(D,B'_e) & \to & \bigoplus_{p\in V} KK^1(D,A'_p)
\end{array}$$

$$\begin{array}{ccccccccc}
\bigoplus_{e\in E^+} KK(B'_e,D) & \to & \bigoplus_{p\in V} KK(A'_p,D) & \to & KK(P',D) & \to & \bigoplus_{e\in E^+} KK^1(B'_e,D) & \to & \bigoplus_{p\in V} KK^1(A'_p,D) \\
\downarrow \bigoplus [\nu_e] \underset{B_e}{\otimes} \cdot & & \downarrow \bigoplus [\nu_p] \underset{A_p}{\otimes} \cdot & & \downarrow [\nu] \underset{P}{\otimes} \cdot & & \bigoplus [\nu_e] \underset{B_e}{\otimes} \cdot & & \downarrow \bigoplus [\nu_p] \underset{A_p}{\otimes} \cdot \\
\bigoplus_{e\in E^+} KK(B_e,D) & \leftarrow & \bigoplus_{p\in V} KK(A_p,D) & \leftarrow & KK(P,D) & \leftarrow & \bigoplus_{e\in E^+} KK^1(B_e,D) & \leftarrow & \bigoplus_{p\in V} KK^1(A_p,D)
\end{array}$$

By the Five Lemma and the hypothesis, it suffices to check that, for each D, every square of the two diagrams is commutative. We check that for the first diagram. The verification for the second diagram is similar. For a unital inclusion $X \subset Y$ of unital C*-algebras, we write $\iota_{X\subset Y}$ the inclusion map. The first square on the left and the last square on the right of the first diagram are obviously commutative since, by hypothesis, $\nu_{s(e)} \circ s_e = s'_e \circ \nu_e$ and $\nu_{r(e)} \circ r_e = r'_e \circ \nu_e$ for all $e \in E^+$. The second square on the left is commutative since, by definition of ν, we have $\nu \circ \iota_{A_p \subset P} = \iota_{A'_p \subset P'} \circ \nu_p$ for all $p \in V$. Hence, it suffices to check that the third square, starting from the left, is commutative. Note that the commutativity of this square is equivalent to the equality $z_e \underset{B_e}{\otimes} [\nu_e] = [\nu] \underset{P'}{\otimes} z'_e \in KK^1(P, B'_e)$, where $z_e \in KK^1(P, B_e)$ and $z'_e \in KK^1(P', B'_e)$ are the KK^1 elements constructed in Remark 3.7 associated with the graphs of C*-algebras $(\mathcal{G}, A_p, B_e)$ and $(\mathcal{G}, A'_p, B'_e)$ respectively. This equality follows easily from the assumption $(E_e^s)' \circ \nu_{s(e)} = \nu_{s(e)} \circ E_e^s$

since it gives a canonical isomorphism of Hilbert modules $K_e \otimes_{\nu_e} B'_e \simeq K'_e$ which is easily seen to implement an isomorphisms between the Kasparov triples representing $z_e \underset{B_e}{\otimes} [\nu_e]$ and $[\nu] \underset{P'}{\otimes} z'_e$. □

We write P_{vert} the vertex reduced fundamental C*-algebra of $(\mathcal{G}, A_p, B_e)$ and $\lambda : P \to P_{\text{vert}}$ the canonical surjective unital $*$-homomorphism. The following Theorem is an immediate consequence of the two 6 terms exact sequences we proved in this paper: one for the full fundamental C*-algebra P and one for the vertex-reduced fundamental C*-algebra P_{vert} and the Five Lemma.

Theorem 5.2. *Suppose that $\mathcal{G}$ is a finite graph then the class of the canonical surjection $[\lambda] \in KK(P, P_{\text{vert}})$ is invertible.*

Remark 5.3. The previous result is actually true without assuming the graph $\mathcal{G}$ to be finite. Indeed the inverse of $[\lambda]$ and the homotopy showing that it is an inverse can be constructed directly, without using induction. Since such a proof requires more work and does not bring any new ideas, we chose to not include it.

Corollary 5.4. *The following holds.*

1. *If G be the fundamental compact quantum group of a finite graph of compact quantum groups $(G_p, G_e, \mathcal{G})$ then $\widehat{G}$ is K-amenable if and only if $\widehat{G}_p$ is K-amenable for all p.*
2. *If G is the compact quantum group obtained from the (finite) graph product of the family of compact quantum groups G_p, $p \in V(\mathcal{G})$ (see* [CF14]*) then $\widehat{G}$ is K-amenable if and only if $\widehat{G}_p$ is K-amenable for all $p \in V(\mathcal{G})$.*

Proof. Using induction, (2) is a consequence of (1) since, as observed in [CF14], a graph product maybe written as an amalgamated free product using a kind of *devissage* strategy.

Let us prove (1). Consider the two graphs of C*-algebras $(\mathcal{G}, C_{max}(G_p), C_{max}(G_e))$ and $(\mathcal{G}, C_{red}(G_p), C_{red}(G_e))$ with full fundamental C*-algebra P_{max} and P respectively. Note that both graphs have natural families of conditional expectations by only the conditional expectations on $(\mathcal{G}, C_{red}(G_p), C_{red}(G_e))$ are GNS faithful (expect in the presence of co-amenability) L.et P_{red} be the vertex reduced fundamental C*-algebra of $(\mathcal{G}, C_{red}(G_p), C_{red}(G_e))$. We recall that $C_{max}(G) = P_{max}$ and $C_{red}(G) = P_{red}$ (see [FF13]). Let $\lambda : P \to P_{red}$ the canonical surjection, which is a K-equivalence by Theorem 5.2 and let $\nu : P_{max} \to P$ be the canonical surjection obtained from the canonical surjections $\nu_p := \lambda_{G_p} : C_{max}(G_p) \to C_{red}(G_p)$ and $\nu_e := \lambda_{G_e} : C_{max}(G_e) \to C_{red}(G_e)$ as explained in the discussion before Thereom 5.1. Since the hypothesis on the conditional expectations of this Theorem are obviously satisfied, it follows that, whenever $\widehat{G}_p$ is K-amenable for all p (hence $\widehat{G}_e$ is also K-amenable for all e as a quantum subgroup of $\widehat{G}_{s(e)}$), $\widehat{G}$ is K-amenable. The proof of the converse is obvious. □

Remark 5.5. The first assertion of the previous Corollary strengthens the results of [Pi86, Corollary 19] and also [FF13, Fi13, Ve04] and unify all the proofs.

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Emmanuel Germain
LMNO, CNRS UMR 6139, Université de Caen, France
E-mail address: emmanuel.germain@unicaen.fr

Pierre FIMA
Univ Paris Diderot, Sorbonne Paris Cité, IMJ-PRG, UMR 7586, F-75013, Paris, France
Sorbonne Universités, UPMC Paris 06, UMR 7586, IMJ-PRG, F-75005, Paris, France
CNRS, UMR 7586, IMJ-PRG, F-75005, Paris, France
E-mail address: pierre.fima@imj-prg.fr