Rapid decay and polynomial growth for bicrossed products

Pierre Fima and Hua Wang

Abstract

We study the rapid decay property and polynomial growth for duals of bicrossed products coming from a matched pair of a discrete group and a compact group.

1 Introduction

In the breakthrough paper paper [Ha78], Haagerup showed that the norm of the reduced C*-algebra \( C^*_\text{r}(F_N) \) of the free group on \( N \)-generators \( F_N \), can be controlled by the Sobolev \( l^2 \)-norms associated to the word length function on \( F_N \). This is a striking phenomenon which actually occurs in many more cases. Jolissaint recognized this phenomenon, called Rapid Decay (or property \((RD)\)), and studied it in a systematic way in [Jo89]. Property \((RD)\) has now many applications. Let us mention the remarkable one concerning K-theory. Property \((RD)\) allowed Jolissaint [Jo89] to show that the K-theory and \( C^*_\text{r}(\Gamma) \) equals the K-theory of subalgebras of rapidly decreasing functions on \( \Gamma \). This result was then used by V. Lafforgue in his approach to the Baum-Connes conjecture via Banach KK-theory [La00, La02].

In this paper, we view discrete quantum groups as duals of compact quantum groups. The theory of compact quantum groups has been developed by Woronowicz [Wo87, Wo88, Wo98]. Property \((RD)\) for discrete quantum groups has been introduced and studied by Vergnioux [Ve07]. Property \((RD)\) has been refined later [BVZ14] in order to fit in the context of non-unimodular discrete quantum groups.

In this paper, we study the permanence of property \((RD)\) under the bicrossed product construction. This construction was initiated by Kac [Ka68] in the context of finite quantum groups and was extensively studied later by many authors in different settings. The general construction, for locally compact quantum groups, was developed by Vaes-Vainerman [VV03]. In the context of compact quantum groups given by matched pairs of classical groups, an easier approach, that we will follow, was given by Fima-Mukherjee-Patri [FMP17].

Following [FMP17], the bicrossed product construction associates to a matched pair \((\Gamma, G)\) of a discrete group \( \Gamma \) and a compact group \( G \) (see Section 2.2) a compact quantum group \( \hat{G} \), called the bicrossed product. Given a length function \( l \) on the set of equivalence classes \( \text{Irr}(G) \) of irreducible unitary representations of \( G \) one can associate in a canonical way, as explained in Proposition 4.2, a pair of length functions \( (l_\Gamma, l_G) \) on \( \Gamma \) and \( \text{Irr}(G) \) respectively. Such a pair satisfies some compatibility relations and every pair of length functions \( (l_\Gamma, l_G) \) on \( (\Gamma, \text{Irr}(G)) \) satisfying those compatibility relations will be called matched (see Definition 4.1). Any matched pair \( (l_\Gamma, l_G) \) on \( (\Gamma, \text{Irr}(G)) \) allows one to reconstruct a canonical length function on \( \text{Irr}(G) \). The main result of the present paper is the following.

**Theorem A.** Let \((\Gamma, G)\) be a matched pair of a discrete group \( \Gamma \) and a compact group \( G \). Denote by \( \hat{G} \) the bicrossed product. The following are equivalent.

1. \( \hat{G} \) has property \((RD)\).

2. There exists a matched pair of length function \((l_\Gamma, l_G)\) on \((\Gamma, \text{Irr}(G))\) such that both \((\Gamma, l_\Gamma)\) and \((\hat{G}, l_G)\) have \((RD)\).

For amenable discrete groups, property \((RD)\) is equivalent to polynomial growth [Jo89] and the same occurs for discrete quantum groups [Ve07]. Hence, for the compact classical group \( G \) one has that \((\hat{G}, l_G)\) has \((RD)\) if and only if it has polynomial growth. Note that a bicrossed product of a matched pair \((\Gamma, G)\) is co-amenable if and only if \( \Gamma \) is amenable [FMP17]. The following theorem shows the permanence of polynomial growth under the bicrossed product construction.

**Theorem B.** Let \((\Gamma, G)\) be a matched pair of a discrete group \( \Gamma \) and a compact group \( G \). Denote by \( \hat{G} \) the bicrossed product. The following are equivalent.
1. $\hat{G}$ has polynomial growth.
2. There exists a matched pair of length function $(l_r, l_G)$ on $(\Gamma, \text{Irr}(G))$ such that both $(\Gamma, l_r)$ and $(\hat{G}, l_G)$ have polynomial growth.

The main ingredient to prove Theorem A and B is the classification of the irreducible unitary representation of a bicrossed product and the fusion rules.

The paper is organized as follows. Section 2 is a preliminary section in which we introduce our notations. In section 3 we classify the irreducible unitary representation of a bicrossed product and describe their fusion rules. Finally, in section 4, we prove Theorem A and Theorem B.

## 2 Preliminaries

### 2.1 Notations

For a Hilbert space $H$, we denote by $\mathcal{U}(H)$ its unitary group and by $\mathcal{B}(H)$ the $C^*$-algebra of bounded linear operators on $H$. When $H$ is finite dimensional, we denote by $\text{Tr}$ the unique trace on $\mathcal{B}(H)$ such that $\text{Tr}(1) = \dim(H)$. We use the same symbol $\otimes$ for the tensor product of Hilbert spaces, unitary representations of compact quantum groups, minimal tensor product of $C^*$-algebras. For a compact quantum group $G$, we denote by $\text{Irr}(G)$ the set of equivalence classes of irreducible unitary representations and $\text{Rep}(G)$ the collection of finite dimensional unitary representations. We will often denote by $[u]$ the equivalence class of an irreducible unitary representation $u$. For $u \in \text{Rep}(G)$, we denote by $\chi(u)$ its character, i.e., viewing $u \in \mathcal{B}(H) \otimes C(G)$ for some finite dimensional Hilbert space $H$, one has $\chi(u) = (\text{Tr} \otimes \text{id})(u) \in C(G)$. We denote by $\text{Pol}(G)$ the unital $C^*$-algebra obtained by taking the Span of the coefficients of irreducible unitary representation, by $C_m(G)$ the enveloping $C^*$-algebra of $\text{Pol}(G)$ and by $C(G)$ the $C^*$-algebra generated by the GNS construction of the Haar state on $C_m(G)$. We also denote by $\varepsilon : C_m(G) \to \mathbb{C}$ the counit and we use the same symbol $\varepsilon \in \text{Irr}(G)$ to denote the trivial representation and its class in $\text{Irr}(G)$. In the entire paper, the word representation means a unitary and finite dimensional representation.

### 2.2 Compact bicrossed products

In this section, we follow the approach and the notations of [FMP17].

Let $(\Gamma, G)$ be a pair of a countable discrete group $\Gamma$ and a second countable compact group $G$ with a left action $\alpha : \Gamma \to \text{Homeo}(G)$ of the compact space $G$ by homeomorphisms and a right action $\beta : G \to \text{S}(G)$ of $G$ on the discrete space $\Gamma$, where $\text{S}(G)$ is the Polish group of bijections of $\Gamma$, the topology being the one of pointwise convergence i.e., the smallest one for which the evaluation maps $\text{S}(G) \to \Gamma, \sigma \mapsto \sigma(\gamma)$ are continuous, for all $\gamma \in \Gamma$, where $\Gamma$ has the discrete topology. Here, $\alpha$ is a group homomorphism and $\beta$ is an antihomomorphism. The pair $(\Gamma, G)$ is called a matched pair if $\Gamma \cap G = \{e\}$ with $e$ being the common unit for both $G$ and $\Gamma$, and if the actions $\alpha$ and $\beta$ satisfy the following matched pair relations:

$$\forall g, h \in G, \gamma, \mu \in \Gamma, \quad \alpha_\gamma(gh) = \alpha_\gamma(g)\alpha_{\beta_\gamma(\mu)}(h), \quad \beta_\gamma(\mu) = \beta_{\alpha_{\beta_\gamma(\mu)}}(\gamma)\beta_\gamma(\mu) \quad \text{and} \quad \alpha_\gamma(e) = \beta_\gamma(e) = e.$$  

We also write $\gamma \cdot g := \beta_\gamma(g)$. From now on, we assume $(\Gamma, G)$ is matched. It is shown in [FMP17] that $\beta$ is automatically continuous. By continuity of $\beta$ and compactness of $G$, every $\beta$ orbit is finite. Moreover, the sets $G_{r,s} := \{g \in G : r \cdot g = s\}$ are clopen. Let $v_{rs} = 1_{G_{r,s}} \in C(G)$ be the characteristic function of $G_{r,s}$. It is shown in [FMP17] that, for all $\beta$-orbits $\gamma \in \Gamma/G$, the unitary $v_{\gamma,G} := \sum_{r,s \in \gamma \cdot G} v_{rs} \otimes v_{rs} \in B(l^2(\gamma \cdot G)) \otimes C(G)$ is a unitary representation of $G$ as well as a magic unitary, where $e_{rs} \in B(l^2(\gamma \cdot G))$ are the canonical matrix units and the Haar probability measure $\nu$ on $G$ is $\alpha$-invariant.

It is shown in [FMP17] that there exists a unique compact quantum group $\hat{G}$, called the bicrossed product of the matched pair $(\Gamma, G)$, such that $C(\hat{G}) = \Gamma_\alpha \ltimes C(G)$ is the reduced $C^*$-algebraic crossed product, generated by a copy of $C(G)$ and the unitaries $u_\gamma, \gamma \in \Gamma$ and $\Delta : C(\hat{G}) \to C(\hat{G}) \otimes C(\hat{G})$ is the unique unital $\ast$-homomorphism satisfying $\Delta|_{C(G)} = \Delta_G$ (the comultiplication on $C(G)$) and $\Delta(u_\gamma) = \sum_{r \in \gamma \cdot G} u_{\gamma r} v_{\gamma r} \otimes u_r$ for all $\gamma \in \Gamma$. It is also shown that the Haar state on $G$ is a trace and is given by the formula $h(u_\gamma F) = \delta_{\gamma,1} \int_G F d\nu$ for all $\gamma \in \Gamma$ and $F \in C(G)$.
3 Representation theory of bicrossed products

3.1 Classification of irreducible representations

In this section we classify the irreducible representations of a bicrossed product. Let \((G, G)\) be a matched pair of a discrete countable group \(\Gamma\) and a second countable compact group \(G\) with actions \(\alpha, \beta\).

For \(\gamma \in \Gamma\) we denote by \(G_\gamma := G_{\gamma, \gamma}\) the stabilizer of \(\gamma\) for the action \(\beta : \Gamma \curvearrowright G\). Note that \(G_\gamma\) is an open (hence closed) subgroup of \(G\), hence of finite index: its index is \([\gamma : G]\). We view \(C(G_\gamma) = v_\gamma C(G) \subset C(G)\) as a non-unital C*-subalgebra. Let us denote by \(\nu\) the Haar probability measure on \(G\) and note that \(\nu(G_\gamma) = \frac{1}{[\gamma : G]}\) so that the Haar probability measure \(\nu_\gamma\) on \(G_\gamma\) is given by \(\nu_\gamma(A) = |\gamma : G| \nu(A)\) for all Borel subset \(A\) of \(G_\gamma\).

For \(\gamma \in \Gamma\) we fix a section, still denoted \(\gamma, \gamma : \gamma \cdot G \rightarrow G\) of the canonical surjection \(G \rightarrow \gamma \cdot G : g \mapsto \gamma \cdot g\). This means that \(\gamma : \gamma \cdot G \rightarrow G\) is an injective map such that \(\gamma \cdot (r \cdot g) = r \cdot (\gamma \cdot g)\) for all \(r \in \gamma \cdot G\). We choose the section \(\gamma\) such that \(\gamma(\gamma) = 1\), for all \(\gamma \in \Gamma\). For \(r, s \in \gamma \cdot G\), we denote by \(\psi^G_{r,s}\) the \(\nu\)-preserving homeomorphism of \(G\) defined by \(\psi^G_{r,s}(g) = \gamma(r)g\gamma(s)^{-1}\). It follows from our choices that \(\psi^G_{r,s} = \text{id}\) for all \(\gamma \in \Gamma\). Moreover, for all \(g \in G\), one has \(\psi^G_{r,s} = \text{id}\) in \(G_\gamma\), if and only if \(g \in G_{r,s}\). It follows that \(\psi^G_{r,s}\) is an isomorphism and an homeomorphism from \(G_{r,s}\) to \(G_{r,s}\) intertwining the Haar probability measures.

Let \(u : G_{\gamma} \rightarrow U(H)\) be a unitary representation of \(G_{\gamma}\) and view \(u\) as a continuous function \(G_{\gamma} \rightarrow B(H)\) which is zero outside \(G_{\gamma}\) i.e. a partial isometry in \(B(H) \otimes C(G)\) such that \(u^* = u^* u = \text{id}_H \otimes v_{\gamma, \gamma}\). Define, for \(r, s \in \gamma \cdot G\), the partial isometry \(u_{r,s} := u \circ \psi^G_{r,s} := (g \mapsto u(\psi^G_{r,s}(g))) \in B(H) \otimes C(G)\) and note that \(u_{r,s}^*u_{r,s} = u_{r,s}u_{s,r,s} = \text{id}_H \otimes 1_{G_{r,s}}\). In the sequel we view \(u_{r,s} \in B(H) \otimes C(G)\) in \(B(H) \otimes C(G)\) and we define:

\[
\gamma(u) := \sum_{r,s \in G_{\gamma}} e_{r,s} \otimes (1 \otimes u_{r,s})u_{r,s} \in B(\ell^2(\gamma \cdot G)) \otimes B(H) \otimes C(G),
\]

where we recall that \(e_{r,s}\), for \(r, s \in \gamma \cdot G\), are the matrix units associated to the canonical orthonormal basis of \(\ell^2(\gamma \cdot G)\).

The irreducible unitary representations of \(G_{\gamma}\) are described as follows.

Theorem 3.1. The following holds.

1. For all \(\gamma \in \Gamma\) and \(u \in \text{Rep}(G_{\gamma})\) one has \(\gamma(u) \in \text{Rep}(G_{\gamma})\).
2. The character of \(\gamma(u)\) is \(\chi_{\gamma}(u) = \sum_{r,s \in G_{\gamma}} u_{r,s}^* \chi(u) \otimes \psi^G_{r,s}\).
3. For all \(\gamma \in \Gamma\) and \(u, w \in \text{Rep}(G_{\gamma})\) one has \(\dim(\text{Mor}_G(\gamma(u), \mu(w))) = \delta_{\gamma, \gamma} \delta_{\mu, \mu} \dim(\text{Mor}_G(u, w \circ \psi^G_{\gamma, \gamma}))\).
4. For all \(\gamma \in \Gamma\) and \(u \in \text{Rep}(G_{\gamma})\) one has \(\gamma(u) \simeq_{\gamma} \gamma^{-1}(\pi \circ \alpha_{\gamma-1})\) (which makes sense since \(\alpha_{\gamma-1} : G_{\gamma-1} \rightarrow G_{\gamma}\) is a group isomorphism and an homeomorphism).
5. \(\gamma(u)\) is irreducible if and only if \(u\) is irreducible. Moreover, for any irreducible unitary representation \(u\) of \(G_{\gamma}\) there exists \(\gamma \in \Gamma\) and \(v\) an irreducible representation of \(G_{\gamma}\) such that \(u \simeq_{\gamma} v\).

Proof. (1) Writing \(\gamma(u) = \sum_{r,s} e_{r,s} \otimes V_{r,s}\), where \(V_{r,s} := (1 \otimes u_{r,s})u_{r,s} \in B(H) \otimes C(G)\), it suffices to check that, for all \(r, s \in \gamma \cdot G\) one has \(\text{id} \otimes \Delta)(V_{r,s}) = \sum_{t \in G_{\gamma}} (v_{r,t} \otimes 1_{12(12,13)})\). We first claim that, for all \(r, s \in \gamma \cdot G\), \(\text{id} \otimes \Delta)(u_{r,s}) = \sum_{t \in G_{\gamma}} (u_{r,t} \otimes 1_{12(12,13)})\). To check our claim, first recall that, for all \(r, s \in \gamma \cdot G\) one has \(\psi^G_{r,s}(g) \in G_{\gamma}\) if and only if \(r \cdot g = s\). Let \(r, s \in \gamma \cdot G\) and \(g, h \in G\). For \(t = r \cdot g \cdot G \in \gamma \cdot G\) one has:

\[
u_{r,s}(gh) = u(\gamma(r)g\gamma(t)^{-1}\gamma(t)h\gamma(s)^{-1}) = u(\psi^G_{r,s}(g)\psi^G_{r,s}(h)) = \begin{cases} u_{r,t}(u_{r,s}(h)) & \text{if } r \cdot gh = s, \\ 0 & \text{otherwise}. \end{cases}
\]

Since we also have \(u_{t,s}(h) = 0\) whenever \(r \cdot gh \neq s\) we find, in both cases, that \(u_{r,s}(gh) = u_{r,t}(u_{r,s}(h))\). Now, for \(t \neq r \cdot g\) we have \(u_{r,t}(g) = 0\) so the following formulae holds for any \(r, s \in \gamma \cdot G\) and any \(g, h \in G\):

\[
v_{r,t}(g)u_{r,s}(gh) = u_{r,t}(g)u_{r,s}(h).
\]
Hence, for all $r, s, t \in \gamma \cdot G$, $(1 \otimes v_{r,t} \otimes 1)(\text{id} \otimes \Delta)(u_{r,s}) = (u_{r,t})_{12}(u_{t,s})_{13}$. Using this we find:

\[
\sum_{t \in \gamma \cdot G} (V_{r,t})_{12}(V_{t,s})_{13} = \sum_{t} (1 \otimes u_{r,v_{t}} \otimes 1)(u_{r,t})_{12}(1 \otimes 1 \otimes u_{1}v_{t}) (u_{t,s})_{13} = \sum_{t} (1 \otimes u_{r,v_{t}} \otimes u_{1}v_{t}) (u_{r,t})_{12}(u_{t,s})_{13} = \left(1 \otimes \left(\sum_{t} u_{r,v_{t}} \otimes u_{1}v_{t}\right)\right) (\text{id} \otimes \Delta)(u_{r,s}).
\]

Since $v_{r}$ is a unitary representation of $G$ and a magic unitary we also have:

\[
\Delta_{r,s}(u_{r,v_{r}}) = \sum_{t} (u_{r,v_{t}} \otimes u_{t}) (v_{t,s} \otimes v_{t,s}) = \sum_{t} u_{r,v_{t}} \otimes u_{t}v_{t}.
\]

This shows that $\gamma(u)$ is a representation of $G$. We now check that $\gamma(u)$ is unitary. As before, since for all $r, s \in \gamma \cdot G$ one has $\psi^{\gamma_{r,s}}_{s}(g) \in G_{\gamma}$ if and only if $r \cdot g = s$ and because $u$ is a unitary representation of $G_{\gamma}$, we have, for all $r, t \in \gamma \cdot G$, $(1 \otimes v_{r})u_{r,t}u_{r,t}^{*} = 1 \otimes v_{r}$. Hence,

\[
\sum_{t \in \gamma \cdot G} V_{r,t}V_{t,s}^{*} = \sum_{t} (1 \otimes u_{r})(1 \otimes v_{t})u_{r,t}u_{r,t}^{*}(1 \otimes u_{t}^{*}) = \sum_{t} (1 \otimes u_{r}) \left(\sum_{t} (1 \otimes v_{t})u_{r,t}u_{r,t}^{*}\right) (1 \otimes u_{t}^{*}) = \delta_{r,s}(1 \otimes u_{r}) \left(\sum_{t} (1 \otimes v_{t})\right) (1 \otimes u_{t}^{*}) = \delta_{r,s}.
\]

A similar computations shows that $\sum_{t \in \gamma \cdot G} V_{r,t}^{*}V_{t,s} = \delta_{r,s}$.

(2) The character of $\gamma(u)$ is given by

\[
\chi(\gamma(u)) = \sum_{r \in \gamma \cdot G} (\text{Tr} \otimes \text{id})(V_{r,r}) = \sum_{r} u_{r,v_{r}}(\text{Tr} \otimes \text{id})(u_{r,r}) = \sum_{r} u_{r,v_{r}} \chi(u) \circ \psi^{\gamma}_{r,r}.
\]

(3) Let $\gamma, \mu \in \Gamma$ and $u, w$ be representations of $G_{\gamma}$ and $G_{\mu}$ respectively. Since the Haar measure on $G$ is invariant under the action $\alpha$ and the homeomorphisms $\psi^{\gamma}_{r,r}$ and $\psi^{\mu}_{r,r}$, we find by 1,

\[
\dim(\text{Mor}(\gamma(u), \mu(w))) = h(\chi(\gamma(u))\chi(\mu(w))) = \sum_{r \in \gamma \cdot G, s \in \mu \cdot G} h(u_{r,s^{-1}} \alpha_{s}(v_{r,s} \chi(u) \circ \psi^{\gamma}_{r,s} \circ \psi^{\mu s}_{r,s} \circ \psi^{\gamma}_{s,s} \circ \psi^{\mu}_{s,s})).
\]

Now, note that $\psi^{\mu}_{r,r} \circ (\psi^{\gamma}_{r,r})^{-1} \circ (\psi^{\mu}_{s,s})^{-1} = \text{Ad}(h)$, where $h = \mu(r) \gamma(r)^{-1} \mu(\gamma)^{-1}$. Moreover it is clear that $\mu \cdot h = \mu$, so $h \in G_{\mu}$. Since the characters of finite dimensional unitary representation of a group $A$ are central functions i.e. invariant under $\text{Ad}(\lambda)$ for all $\lambda \in \Lambda$, we have $\chi(\mu) \circ \psi^{\mu}_{r,r} \circ (\psi^{\gamma}_{r,r})^{-1} \circ (\psi^{\mu}_{s,s})^{-1} = \chi(\mu) \circ \text{Ad}(h) = \chi(\mu)$. Hence:

\[
\dim(\text{Mor}(\gamma(u), \mu(w))) = \delta_{\gamma,\mu} \cdot \chi(u) \circ (\psi^{\mu}_{r,r})^{-1} \chi(\mu)d\nu = \delta_{\gamma,\mu} \int_{G_{\mu}} \chi(u) \circ (\psi^{\mu}_{r,r})^{-1} \chi(\mu)d\nu = \delta_{\gamma,\mu} \dim(\text{Mor}_{G_{\mu}})(u \circ (\psi^{\mu}_{r,r})^{-1}, w) = \delta_{\gamma,\mu} \int_{G_{\mu}} \chi(u) \circ \psi^{\mu}_{r,r} d\nu = \delta_{\gamma,\mu} \dim(\text{Mor}_{G_{\gamma}})(u, w \circ \psi^{\mu}_{r,r}).
\]
Using the discussion above we find, for all $\gamma \in \Gamma$ and $g \in G$, $(\gamma \cdot g)^{-1} = \gamma^{-1} \cdot \alpha_g(g)$. Hence $v_{r^{-1} \gamma^{-1}} \circ \alpha_g \equiv v_{\gamma} \gamma$ and $(\gamma \cdot G)^{-1} = \gamma^{-1} \cdot G$. In particular, $\alpha_g : G_r \to G_{r^{-1}}$ is an homeomorphism and, by the bicrossed product relations, one has, for all $g \in G_{\gamma}$ and $h \in G$, $\alpha_g(h) = \alpha_g(g)\alpha_g^{-1}(h) = \alpha_g(g)\alpha_g(h)$ so that $\alpha_g : G_{\gamma} \to G_{\gamma^{-1}}$ is also a group homomorphism.

For $r \in \gamma \cdot G$ one has $\gamma^{-1} \cdot \alpha_g(\gamma(r)) = (\gamma \cdot \gamma(r))^{-1} = r^{-1} = \gamma^{-1} \cdot (r^{-1})$. This implies that, for all $\gamma \in \Gamma$, there exists a map $\eta_\gamma : \gamma \cdot G \to G_{\gamma^{-1}}$ such that, for all $r \in \gamma \cdot G$, one has $\alpha_g(\gamma(r)) = \eta_\gamma(r)\gamma^{-1}(r^{-1})$.

Let now $r \in \gamma \cdot G$ and $g \in G_r$. One has, using the bicrossed product relations, that $e = \alpha_r(\gamma(r)\gamma(r)^{-1}) = \alpha_r(\gamma(r))\alpha_r(\gamma(r)^{-1})$, hence

$$
(\alpha_r \circ \psi_{r\gamma})(g) = \alpha_r(\gamma(r)\alpha_r(g)(\gamma(r)^{-1}) = \alpha_r(\gamma(r)\alpha_r(g)(\alpha_r(\gamma(r))))^{-1} = \eta_\gamma(r)(\psi_{r^{-1},r^{-1}} \circ \alpha_r(g)(\eta_\gamma(r))^{-1}.
$$

Hence, for all $\gamma \in \Gamma$, if $w \in \text{Rep}(G_{\gamma^{-1}})$, since $\chi(w) \in C(G_{\gamma^{-1}})$ is central we have

$$
\chi(w) \circ \alpha_r \circ \psi_{r\gamma}(g) = \chi(w) \circ \psi_{r^{-1},r^{-1}} \circ \alpha_r(g) \quad \text{for all } r \in \gamma \cdot G, g \in G_r.
$$

Using the discussion above we find,

$$
\chi(\gamma^{-1}(\pi \circ \alpha_{\gamma^{-1}})) = \sum_{r \in \gamma \cdot G} u_{r^{-1}} v_{r^{-1}} \chi(\pi) \circ \alpha_{\gamma^{-1}} \circ \psi_{r^{-1},r^{-1}}^{-1} = \sum_{r \in \gamma \cdot G} (\chi(\pi) \circ \alpha_{\gamma^{-1}} \circ \psi_{r^{-1},r^{-1}}^{-1} \circ \alpha_r)(v_{r^{-1}} \circ \alpha_r(u_{r^{-1}})) = \sum_{r \in \gamma \cdot G} \chi(\pi) \circ \psi_{r} \circ u_{r} = \sum_{r \in \gamma \cdot G} (\chi(\pi) \circ \psi_{r} \circ u_{r}) u_{r} = \chi(\gamma^{-1}(\pi)) \chi(\pi) \circ \alpha_{\gamma^{-1}} \circ \psi_{r^{-1},r^{-1}}^{-1} = \chi(\gamma^{-1}(\pi) \circ \alpha_{\gamma^{-1}})
$$

(5). By the general theory it suffices to show that the linear span $X$ of coefficients of representations of the form $\gamma(u)$, for $\gamma \in \Gamma$ and $u$ an irreducible unitary representation of $G_{\gamma}$, is a dense subset of $C(G)$. Note that, for all $\gamma \in \Gamma$, the relation $1 = \sum_{r \in \gamma \cdot G} v_{r^{-1}}$ implies that any function in $C(G)$ is a sum of continuous functions with support in $G_{\gamma,r} := \{g \in G : \gamma \cdot g = r\}$, for $r \in \gamma \cdot G$. Moreover, since $G_{\gamma,r} = (\psi_{r\gamma})^{-1}(G_{\gamma})$, any continuous function on $G$ with support in $G_{\gamma,r}$ is of the form $F \circ \psi_{r\gamma}^{-1}$, where $F \in C(G_r)$. Since the linear span of coefficients of irreducible unitary representation of $G_{\gamma}$ is dense in $C(G_{\gamma})$, it suffices to show that, for any $\gamma \in \Gamma$, for any irreducible unitary representation of $G_{\gamma}$, $u : G_{\gamma} \to U(H)$, any coefficient $u_{ij} \in C(G_{\gamma}) = v_{\gamma,r}C(G) \subset C(G)$ satisfies $u_{ij} \in X$. But this is obvious since one has

$$
u_{ij} u_{ij} = u_{ij} v_{\gamma,\gamma} u_{ij} = u_{ij} v_{\gamma,\gamma} u_{ij} \circ \psi_{\gamma,\gamma} = \gamma(u_{ij}) \in X.
$$

Finally, the fusion rules are described as follows.

Let $\gamma, \mu \in \Gamma$, $u : G_{\gamma} \to U(H_u), v : G_{\mu} \to U(H_v)$ by unitary representations of $G_{\gamma}$ and $G_{\mu}$ respectively. For any $r \in (\gamma \cdot G)(\mu \cdot G)$, we define the $r$-twisted tensor product of $u$ and $v$, denoted $u \otimes v$ as a unitary representation of $G_r$ on $K_r \otimes H_u \otimes H_v$, where

$$K_r := \text{Span}\{e_s \otimes e_t : s \in \gamma \cdot G \text{ and } t \in \mu \cdot G \text{ such that } st = r\} \subset l^2(\gamma \cdot G) \otimes l^2(\mu \cdot G).
$$

For $g \in G$, we define:

$$
(u \otimes v)(g) = \sum_{r,s,t \in \Gamma} e_{ss'} \otimes e_{tt'} \otimes v_{ss'}(\alpha_s(g))v_{tt'}(g)u(\psi_{ss'}^r(\alpha_s(g))) \otimes v(\psi_{tt'}^r(g)) \in U(K_r \otimes H_u \otimes H_v).
$$

Theorem 3.2. The following holds.

1. For all $\gamma, \mu \in \Gamma$, all $r \in (\gamma \cdot G)(\mu \cdot G)$ and all $u, v$ finite dimensional unitary representations of $G_{\gamma}, G_{\mu}$ respectively the element $u \otimes v$ is a unitary representation of $G_r$.  

5
2. The character of \( u \otimes v \) is
\[
\chi(u \otimes v) = \sum_{s,t \in \mathbb{C}G, r \in \mathbb{C}G} (v_s \otimes \alpha_t)v_{tt'}(\chi(u) \otimes \psi^\alpha_{s,t}(\alpha_t(\otimes)) \otimes v_{tt'-(s,t)}).
\]

3. For all \( \gamma_1, \gamma_2, \gamma_3 \in \Gamma \) and all \( u, v, w \) unitary representations of \( G_{\gamma_1}, G_{\gamma_2} \) and \( G_{\gamma_3} \) respectively, the number 
\[
\text{dim}(\text{Mor}_G(\gamma_1(u), \gamma_2(v) \otimes \gamma_3(w)))
\]

is equal to:
\[
\begin{cases} 
\frac{1}{|G_\gamma|} \sum_{r \in \mathbb{C}G} \text{dim}(\text{Mor}_G(\chi(r) \otimes v \otimes w)) \\
0 \quad \text{if } \gamma_1 \cdot G \cap (\gamma_2 \cdot G)(\gamma_3 \cdot G) \neq \emptyset,
\end{cases}
\]

otherwise.

Let us observe that, by the bicrossed product relations, we have, for all \( \gamma_1, \gamma_2, \gamma_3 \in \Gamma \),
\[
\gamma_1 \cdot G \cap (\gamma_2 \cdot G)(\gamma_3 \cdot G) \neq \emptyset \iff \gamma_1 \cdot G \subset (\gamma_2 \cdot G)(\gamma_3 \cdot G).
\]

**Proof.** (1). Put \( w = u \otimes v \) and let \( g, h \in G_r \). Then, \( w(gh) \) is equal to:
\[
\sum_{s,t \in \mathbb{C}G, \gamma \in \mathbb{C}G} e_{ss'} \otimes e_{tt'} \otimes v_{ss'}(\alpha_t(gh))v_{tt'}(\chi(u) \otimes \psi^\alpha_{s,s}(\alpha_t(gh))) \otimes v(\psi^\mu_{t,t'}(gh)).
\]

Since \( v_{tt'}(g) \neq 0 \) precisely when \( t \cdot g = y \), the factor \( v_{ss'}(\alpha_t(gh))v_{tt'}(\chi(u) \otimes \psi^\alpha_{s,s}(\alpha_t(gh))) \otimes v(\psi^\mu_{t,t'}(gh)) \) is equal to:
\[
\sum_{x \in \Gamma, \gamma \in \mathbb{C}G} v_{xx'}(\alpha_t(gh))v_{tt'}(\chi(u) \otimes \psi^\alpha_{x,x'}(\alpha_t(gh))) \otimes v(\psi^\mu_{t,t'}(gh))
\]
\[
= \sum_{x \in \Gamma, \gamma \in \mathbb{C}G} v_{xx'}(\alpha_t(gh))v_{tt'}(\chi(u) \otimes \psi^\alpha_{x,x'}(\alpha_t(gh))) \otimes v(\psi^\mu_{t,t'}(gh)).
\]

Moreover, since for all \( g \in G_r \), and all \( s, t \) such that \( st = r \), one has, whenever \( t \cdot g = y \) and \( s \cdot \alpha_t(g) = x \),
\[
xy = (s \cdot \alpha_t(g))(t \cdot g) = (st) \cdot g = r \cdot g = r,
\]

it follows that the only non-zero terms in the last sum are for \( x \in \gamma \cdot \Gamma \) and \( y \in \mu \cdot G \) such that \( xy = r \). By the properties of the matrix units we see immediately that \( w(gh) = w(g)w(h) \).

To end the proof of (1), it suffices to check that \( w(1) = 1 \), which is clear, and that \( w(g)^* = w(g^{-1}) \) for all \( g \in G_r \). So let \( g \in G_r \). One has:
\[
w(g)^* = \sum_{s,t \in \mathbb{C}G, \gamma \in \mathbb{C}G} e_{ss'} \otimes e_{tt'} \otimes v_{ss'}(\alpha_t(g))v_{tt'}(\chi(u) \otimes \psi^\alpha_{s,s}(\alpha_t(g))^{-1}) \otimes v(\psi^\mu_{t,t'}(g)^{-1}).
\]

Note that for all \( t, t' \in \Gamma \) and all \( g \in G_r \), one has \( v_{ss'}(g) = v_{ss'}(g^{-1}) \). Also, using the bicrossed product relations one finds that \( \alpha_t(g)^{-1} = \alpha_{-t}(g) \) for all \( r \in \Gamma \) and \( g \in G_r \). In particular, \( v_{ss'}(\alpha_t(g))v_{tt'}(g) = v_{ss'}(\alpha_t(g^{-1}))v_{tt'}(g^{-1}) \) and when \( t' \cdot g = t \), one has \( v_{ss'}^\gamma(\alpha_t(g))^{-1} = v_{ss'}^\gamma(\alpha_t(g^{-1})) \). It follows immediately that \( w(g)^* = w(g^{-1}) \).

(2) Is a direct computation.

(3) One has \( \text{dim}(\text{Mor}_G(\gamma_1(u), \gamma_2(v) \otimes \gamma_3(w))) = h(\gamma_1(u)^* \gamma_2(v) \chi(\gamma_3(w))) \) which is equal to:
\[
\sum_{r \in \mathbb{C}G, s \in \mathbb{C}G, t \in \mathbb{C}G} v_{ss,t}(\chi(u) \otimes \psi^\gamma_{r,r'})v_{tt'}(u_{s,s'}u_{s,s}v_{ss'}(v) \otimes \psi^\gamma_{s,t}v_{tt'}(\chi(w) \otimes \psi^\gamma_{t,t'}))
\]
\[
= \sum_{r, s, t} v_{ss,t}(\chi(u) \otimes \psi^\gamma_{r,r'})v_{tt'}(u_{s,s'}u_{s,s}v_{ss'}(v) \otimes \psi^\gamma_{s,t}v_{tt'}(\chi(w) \otimes \psi^\gamma_{t,t'}))
\]
\[
= \sum_{r \in \mathbb{C}G} \int_G v_{ss,t}(\chi(u) \otimes \psi^\gamma_{r,r'})v_{tt'}(u_{s,s'}u_{s,s}v_{ss'}(v) \otimes \psi^\gamma_{s,t}v_{tt'}(\chi(w) \otimes \psi^\gamma_{t,t'})) dv
\]
\[
= \sum_{r \in \mathbb{C}G} \frac{1}{|G_r|} \int_G v_{ss,t}(\chi(u) \otimes \psi^\gamma_{r,r'})v_{tt'}(u_{s,s'}u_{s,s}v_{ss'}(v) \otimes \psi^\gamma_{s,t}v_{tt'}(\chi(w) \otimes \psi^\gamma_{t,t'})) dv
\]
\[
= \frac{1}{|G_r|} \sum_{r \in \mathbb{C}G} \frac{1}{|G_r|} \sum_{s \in \mathbb{C}G} \text{dim}(\text{Mor}_G(u \otimes \psi^\gamma_{r,r'}, v \otimes w)).
\]

Note that, whenever \( \gamma_1 \cdot G \cap (\gamma_2 \cdot G)(\gamma_3 \cdot G) = \emptyset \), there is no non-zero terms in the sum above. \( \square \)
3.2 The induced representation

In this section, we explain how the induced representation maybe viewed as a particular twisted tensor product.

For \( \gamma \in \Gamma \) and \( u : G_\gamma \to \mathcal{U}(H) \) is a unitary representation of \( G_\gamma \) we define the induced representation:

\[
\text{Ind}^G_\gamma(u) := \varepsilon_{G_{\gamma^{-1}}} \otimes u : G \to \mathcal{U}(l^2(\gamma \cdot G) \otimes H); \quad g \mapsto \sum_{r,s \in G} e_{rs} \otimes v_{rs}(g) \psi_{\gamma}(g).
\]

It follows from Theorem 3.2 that \( \text{Ind}^G_\gamma(u) \) is indeed a unitary representation of \( G \). We collect some elementary and well known facts about this representation in the following Proposition. Note that, in property 3, we use the symbol \( \text{Res}^G_{\gamma_0}(u) \) for \( u \in \text{Rep}(G) \) to denote the restriction of \( u \) to a representation of \( G_\gamma \). Hence, property 3 motivates the name induced representation for the representation \( \text{Ind}^G_\gamma(u) \).

**Proposition 3.3.** The following holds.

1. For all \( \gamma \in \Gamma \) and all \( u \in \text{Rep}(G_\gamma) \) one has \( \chi(\text{Ind}^G_\gamma(u))(g) = \sum_{r,s \in G} v_{rs}(g) \chi(u)(\psi_{\gamma}(g)) \) for all \( g \in G \).

2. For all \( \gamma \in \Gamma \) and all \( u, v \in \text{Rep}(G_\gamma) \) one has \( u \preceq v \implies \text{Ind}^G_\gamma(u) \preceq \text{Ind}^G_\gamma(v) \).

3. For all \( \gamma \in \Gamma \), \( u \in \text{Rep}(G) \) and \( v \in \text{Rep}(G_\gamma) \) one has \( \dim(\text{Mor}_G(u, \text{Ind}^G_\gamma(v))) = \dim(\text{Mor}_G(\text{Res}^G_{\gamma_0}(u), v)) \).

**Proof.** (1). It is obvious, by definition of \( \text{Ind}^G_\gamma(u) \).

(2). If \( u \preceq v \) then \( \chi(u) = \chi(v) \). Hence, \( \chi(\text{Ind}^G_\gamma(u)) = \chi(\text{Ind}^G_\gamma(v)) \) by (1). So \( \text{Ind}^G_\gamma(u) \preceq \text{Ind}^G_\gamma(v) \).

(3). Let \( \gamma \in \Gamma \), \( u \in \text{Rep}(G) \), and \( v \in \text{Rep}(G_\gamma) \). One has,

\[
\dim(\text{Mor}_G(u, \text{Ind}^G_\gamma(v))) = \sum_{r \in G} \int_G \chi(\bar{\gamma}) v_{rr}(g) \omega_{\gamma}(g) \, d\nu = \frac{1}{|\gamma \cdot G|} \sum_{r \in G} \int_{G_\gamma} \chi(\bar{\gamma}) \omega_{\gamma}(g) \, d\nu.
\]

Since \( \psi_{\gamma} : G_\gamma \to G_\gamma \) is a Haar probability preserving homeomorphism we obtain

\[
\dim(\text{Mor}_G(\text{Res}^G_{\gamma_0}(u), v)) = \frac{1}{|\gamma \cdot G|} \sum_{r \in G} \int_{G_\gamma} \chi(\bar{\gamma}) \omega_{\gamma}(g) \, d\nu.
\]

Finally, since, for all \( g \in G \), \( \chi(\bar{\gamma}) \omega_{\gamma}(g)^{-1}(g) = \chi(\bar{\gamma})(g) \) (because \( \chi(\bar{\gamma}) \) is a central function on \( G \)) it follows that:

\[
\dim(\text{Mor}_G(u, \text{Ind}^G_\gamma(v))) = \dim(\text{Mor}_G(\text{Res}^G_{\gamma_0}(u), v)). \tag*{\Box}
\]

4 Length functions

Recall that given a compact quantum group \( \mathbb{G} \), a function \( l : \text{Irr}(\mathbb{H}) \to [0, \infty) \) is called a length function on \( \text{Irr}(\mathbb{H}) \) if \( l(\varepsilon) = 0 \), \( l(\rho) = l(z) \) and that \( l(x) \leq l(y) + l(z) \) whenever \( x \leq y \otimes z \). A length function on a discrete group \( \Lambda \) is a function \( l : \Lambda \to [0, \infty) \) such that \( l(1) = 0 \), \( l(r) = l(r^{-1}) \) and \( l(rs) \leq l(r) + l(s) \) for all \( r, s \in \Lambda \).

Let \( (\Gamma, G) \) be a matched pair with bicrossed product \( \mathbb{G} \). In view of the description of the irreducible representations of \( \mathbb{G} \), the fusion rules and the contragredient representation, it is clear that to get a length function on \( \text{Irr}(\mathbb{G}) \), we need a family of maps \( l_\gamma : \text{Irr}(G_\gamma) \to [0, \infty] \), for \( \gamma \in \Gamma \), satisfying the hypothesis of the following definition.

**Definition 4.1.** Let \( (\Gamma, G) \) be a matched pair, \( l : \text{Irr}(G) \to [0, \infty] \) and \( l_\Gamma : \Gamma \to [0, \infty] \) be length functions. The pair \((l, l_\Gamma)\) is matched if, for all \( \gamma \in \Gamma \), there exists a function \( l_\gamma : \text{Irr}(G_\gamma) \to [0, \infty] \) such that
(i) \( l_1 = l \) and \( l_r(\varepsilon_{G_r}) = \text{lr}(\gamma) \).

(ii) For any \( \gamma \in \Gamma \), \( r \in \gamma \cdot G \), and \( x \in \text{Irr}(G_r) \), we have \( l_r(x) = l_r([u^x \circ \psi_{r,r}]) \).

(iii) For any \( \gamma \in \Gamma \), \( x \in \text{Irr}(G_r) \), we have \( l_r(x) = l_{r-1}(\text{lr} \circ \alpha_{r-1}) \).

(iv) For any \( \gamma_1, \gamma_2, \gamma_3 \in \Gamma \), \( x \in \text{Irr}(G_{\gamma_1}), y \in \text{Irr}(G_{\gamma_2}), z \in \text{Irr}(G_{\gamma_3}) \), if \( \gamma_3 \in (\gamma_1 \cdot G)(\gamma_2 \cdot G) \), and

\[
\dim \text{Mor}_{G_r}(u^x \circ \psi_{r,r}, u^x \otimes_r u^y) \neq 0
\]

for some \( r \in \gamma_3 \cdot G \), then

\[
l_{\gamma_3}(z) \leq l_{\gamma_1}(x) + l_{\gamma_2}(y).\]

The next Proposition shows that our notion of matched pair for length functions is the good one, as expected.

**Proposition 4.2.** Let \((\Gamma, G)\) be a matched pair with bicrossed product \(G\).

1. If \( l \) is a length function on \( \text{Irr}(G) \) then the maps \( l : \text{Irr}(G) = \text{Irr}(G_1) \to [0, +\infty[ \) \( x \mapsto l([1(x)]) \) and \( l_r : \Gamma \to [0, +\infty[ \), \( x \mapsto l([\gamma(\varepsilon_{G_r})]) \) are length functions and the pair \((l_r, l)\) is matched.

2. If \( \text{lr} \) is any \( \beta \)-invariant length function on \( \Gamma \) then the map \( l'[\Gamma] : \text{Irr}(G_1) \to [0, +\infty[ \), \( \gamma \mapsto l'(\gamma) \) is a well defined length function on \( \text{Irr}(G) \).

3. If \((l_l, l_r)\) is a matched pair of length functions on \( (\Gamma, \text{Irr}(G)) \) then \( l_r \) is \( \beta \)-invariant and the maps \( l_l, l_r : \text{Irr}(G) \to [0, +\infty[ \), \( \gamma \mapsto l_l(\gamma)(u^x) \) := \( l_l(x) \) and \( l_r(\gamma)(u^x) \) := \( l_r(x) + l_r(\gamma) \) are well-defined length functions.

**Proof.** (1). Since \( 1(\varepsilon_{G_r}) \) is the trivial representation of \( G \) one has \( l_{\Gamma}(1) = 0 \). Let \( \gamma, \mu \in \Gamma \) and note that \( \gamma \mu \in (\gamma \cdot G)(\mu \cdot G) \).

Moreover,

\[
\dim(\text{Mor}(\varepsilon_{G_r}, \varepsilon_{G_r} \otimes \varepsilon_{G_r})) = \int_{G_{\gamma}} \chi(\varepsilon_{G_r}, \varepsilon_{G_r}) d\nu_{G_{\gamma}} = |\gamma \mu \cdot G| \sum_{s \in \Gamma \mu, t \in \Gamma \mu, st = \gamma \mu} \nu(\alpha_{r-1}(G_s) \cap G_t \cap G_{\gamma \mu})
\]

Hence, since \( \alpha_{r-1}(G_s) \cap G_t \cap G_{\gamma \mu} \) is open and non-empty (it contains 1) we deduce that

\[
\dim(\text{Mor}(\varepsilon_{G_r}, \varepsilon_{G_r} \otimes \varepsilon_{G_r})) > 0.
\]

So \( \varepsilon_{G_r} \subset \varepsilon_{G_r} \otimes \varepsilon_{G_r} \) which implies, by the fusion rules of \( G \), that \( (\gamma \mu)(\varepsilon_{G_{\gamma \mu}}) \subset (\gamma \varepsilon_{G_r}) \otimes (\mu \varepsilon_{G_r}) \). Hence, since \( l \) is a length function, \( l_{\Gamma}(\gamma \mu) = l((\gamma \mu)(\varepsilon_{G_{\gamma \mu}})) \leq l((\gamma \varepsilon_{G_r})) + l((\mu \varepsilon_{G_r})) = l_r(\gamma) + l_r(\mu) \). Finally, note that, for all \( \gamma \in \Gamma \), \( \gamma^{-1}(\varepsilon_{G_{\gamma^{-1}}}) \cong \varepsilon_{G_{\gamma}} \). Hence,

\[
l_{\Gamma}(\gamma^{-1}) = l((\gamma^{-1}(\varepsilon_{G_{\gamma^{-1}}})) = l((\varepsilon_{G_{\gamma}})) = l_{\Gamma}(\gamma).
\]

So \( l_r \) is a length function on \( \Gamma \). It is obvious that \( l_{\gamma l} = l_1 \) is a length function on \( \text{Irr}(G) \). It is also clear that the pair \((l_r, l)\) is matched. Indeed, define \( l_r : \text{Irr}(G_1) \to [0, +\infty[ \) by \( l_r(x) = l([\gamma(\varepsilon_{G_r})]) \). Since \( l \) is a length function on \( \text{Irr}(G) \) and by assertion 4 of Theorem \ref{correct_theorem_number} and Theorem \ref{correct_theorem_number}, it is clear that the family \( \{l_r : \gamma \in \Gamma\} \) satisfies the conditions of Definition \ref{correct_definition_number}.

(2). Since \( l_{\Gamma} \) is \( \beta \)-invariant, the map \( l' \) is well defined by point 3 of Theorem \ref{correct_theorem_number}.

It is clear that \( l'(\varepsilon_{G_r}) = 0 \) and, by point 4 and 5 of Theorem \ref{correct_theorem_number} and since \( l' \) is a length function we also have that \( l'(z) = l'(z') \) for all \( z \in \text{Irr}(G) \). Let now \( \gamma_1, \gamma_2, \gamma_3 \in \Gamma \), \( x \in \text{Irr}(G_{\gamma_1}), y \in \text{Irr}(G_{\gamma_2}) \) and \( z \in \text{Irr}(G_{\gamma_3}) \) be such that \( \gamma_1(u^x) \subset \gamma_2(u^y) \otimes \gamma_3(u^z) \) then, by point 3 in Theorem \ref{correct_theorem_number} there exists \( r \in \gamma_1 \cdot G \), \( s \in \gamma_2 \cdot G \) and \( t \in \gamma_3 \cdot G \) such that \( r = st \) and \( u^x \circ \psi_{r,r} \subset u^y \otimes u^z \). Then,

\[
l'(\gamma_1(u^x)) = l_{\Gamma}(\gamma_1) = l_{\Gamma}(r) \leq l_{\Gamma}(s) + l_{\Gamma}(t) = l_{\Gamma}(\gamma_2) + l_{\Gamma}(\gamma_3) = l'(\gamma_2(u^y)) + l'(\gamma_3(u^z)).
\]

8
Fourier transform as:

Lemma 5.1. can easily be deduced from the following lemma. Given a length function \( l \) and its "Sobolev 0-norm" by \( \| H \) also holds when we assume \( a \in c_c(\hat{H}) \) and all \( \gamma \in \Gamma \) and all \( r \in \gamma \cdot \Gamma \), \( l_\Gamma(\gamma) = l_\Gamma((e_{G_r} \circ \psi_\gamma)) = l_\Gamma(r) \). Hence, \( l_\Gamma \) is \( \beta \)-invariant.

By point 2, we get a length function \( l' \) on \( \text{Irr}(G) \). Now, it is clear from Definition \[\text{L1}\] the fusion rules and the adjoint representation of a bicrossed product (point 3 of Theorem \[\text{D1}\] and point 4 of Theorem \[\text{E1}\]) that \( l : \gamma(u^x) \mapsto l_\gamma(x) \) is a length function on \( \text{Irr}(G) \). Since \( \tilde{l} = l + l' \), \( \tilde{l} \) is also a length function on \( \text{Irr}(G) \).

\[\square\]

5 Rapid decay and polynomial growth

In this section we study property \((RD)\) and polynomial growth for bicrossed-products.

5.1 Generalities

We use the notion of property \((RD)\) developed in [BVZ14] and recall the definition below. Since we are only dealing with Kac algebras, we recall the definition of the Fourier transform and rapid decay only for Kac algebras.

Let \( \mathbb{H} \) be a compact quantum group. We use the notation \( l^\infty(\hat{\mathbb{H}}) := \bigoplus_{x \in \text{Irr}(\mathbb{H})} B(H_x) \) to denote the \( l^\infty \) direct sum. The \( c_0 \) direct sum is denoted by \( c_0(\hat{\mathbb{H}}) \subset l^\infty(\hat{\mathbb{H}}) \) and the algebraic direct sum is denoted by \( c_c(\hat{\mathbb{H}}) \subset c_0(\hat{\mathbb{H}}) \). An element \( a \in c_c(\hat{\mathbb{H}}) \) is said to have finite support and its finite support is denoted by \( \text{Supp}(a) := \{ x \in \text{Irr}(\mathbb{H}) : ap_x \neq 0 \} \), where \( p_x \), for \( x \in \text{Irr}(\mathbb{H}) \) denotes the central minimal projection of \( l^\infty(\hat{\mathbb{H}}) \) corresponding to the block \( B(H_x) \).

For a compact quantum group \( \mathbb{H} \) which is always supposed to be of Kac type, and \( a \in c_c(\hat{\mathbb{H}}) \) we define its Fourier transform as:

\[
\mathcal{F}_\mathbb{H}(a) = \sum_{x \in \text{Irr}(\mathbb{H})} \dim(x) (\text{Tr}_x \otimes \text{id}) (u^x (ap_x \otimes 1)) \in \text{Pol}(\mathbb{H}),
\]

and its "Sobolev 0-norm" by \( \| a \|^2_{\mathbb{H},0} = \sum_{x \in \text{Irr}(\mathbb{H})} \dim(x) \text{Tr}_x (a^*a)p_x \).

Given a length function \( l : \text{Irr}(\mathbb{H}) \to [0, \infty) \), consider the element \( L = \sum_{x \in \text{Irr}(\mathbb{H})} l(x)p_x \) which is affiliated to \( c_0(\hat{\mathbb{H}}) \). Let \( q_n \) denote the spectral projections of \( L \) associated to the interval \( [n, n+1) \).

The pair \((\hat{\mathbb{H}}, l)\) is said to have:

- **Polynomial growth** if there exists a polynomial \( P \in \mathbb{R}[X] \) such that for every \( k \in \mathbb{N} \) one has

\[
\sum_{x \in \text{Irr}(\mathbb{H}), k \leq l(x) < k+1} \dim(x)^2 \leq P(k)
\]

- **Property \((RD)\)** if there exists a polynomial \( P \in \mathbb{R}[X] \) such that for every \( k \in \mathbb{N} \) and \( a \in q_k c_c(\hat{\mathbb{H}}) \), we have

\[
\| \mathcal{F}(a) \|_{C(\mathbb{H})} \leq P(k) \| a \|_{\mathbb{H},0}.
\]

Finally, \( \hat{\mathbb{H}} \) is said to have polynomial growth (resp. property \((RD)\)) if there exists a length function \( l \) on \( \text{Irr}(\mathbb{H}) \) such that \((\hat{\mathbb{H}}, l)\) has polynomial growth (resp. property \((RD)\)).

It is known from [Ve07] that if \((\hat{\mathbb{H}}, l)\) has polynomial growth then \((\hat{\mathbb{H}}, l)\) has rapid decay and the converse also holds when we assume \( \mathbb{H} \) to be co-amenable. Moreover, it is shown also shown in [Ve07] that duals of compact connected real Lie groups have polynomial growth. The fact that polynomial growth implies \((RD)\) can easily be deduced from the following lemma.

**Lemma 5.1.** Let \( \mathbb{H} \) be a CQG, \( F \subset \text{Irr}(\mathbb{H}) \) a finite subset and \( a \in l^\infty(\hat{\mathbb{H}}) \) with \( ap_x = 0 \) for all \( x \notin F \). Then,

\[
\| \mathcal{F}_\mathbb{H}(a) \| \leq 2 \sqrt{\sum_{x \in F} \dim(x)^2} \| a \|_{\mathbb{H},0}.
\]

**Proof.** One can copy the proof of Proposition 4.2, assertion (a), in [BVZ14] or the proof of Proposition 4.4, assertion (ii), in [Ve07].

\[\square\]
5.2 Technicalities

Let \((\Gamma, G)\) be a matched pair with actions \((\alpha, \beta)\) and denote by \(G\) the bicrossed product.

Recall that \(\text{Irr}(G) = \cup_{\gamma \in \Gamma} \text{Irr}(G_i)\), where \(I \subset \Gamma\) is a complete set of representatives for \(\Gamma/G\). For \(\gamma \in I\) and \(x \in \text{Irr}(G_i)\), we denote by \(\gamma(x)\) the corresponding element in \(\text{Irr}(G)\). If a complete set of representatives of \(\text{Irr}(G_i), x \in \text{Irr}(G_i)\) is given by \(u^* \in \mathcal{B}(H_x) \otimes \mathcal{C}(G_\gamma)\) then a representative for \(\gamma(x)\) is given by

\[
u^\gamma(x) := \sum_{r,s \in \Gamma, G} e_{rs} \otimes (1 \otimes u_r v_{rs}) u \circ \psi_{r,s} \in \mathcal{B}(l^2(\gamma \cdot G)) \otimes \mathcal{C}(G).
\]

The lemma below gives a way of obtaining an element \(\tilde{a} \in \text{c}_c(\hat{G})\) from an \(a \in \text{c}_c(\hat{G}_\gamma)\) in a suitable way so that they are compatible with the Fourier transforms.

**Lemma 5.2.** Let \(\gamma \in \Gamma\) and \(a \in \text{c}_c(\hat{G}_\gamma)\). Define \(\tilde{a} \in \text{c}_c(\hat{G})\) by:

\[
\tilde{a}_y = \sum_{x \in \text{supp}(a) \text{ and } y \in \text{Ind}_G^G(x)} \frac{\dim(x)}{\dim(y)} \sum_{i=1}^{\dim(\text{Mor}(y, \text{Ind}_G^G(x)))} (S^y_i)^*(e_{\gamma \cdot \beta} \otimes a_p x) S^y_i,
\]

where \(S^y_i \in \text{Mor}(y, \text{Ind}_G^G(x))\) is a basis of isometries with pairwise orthogonal images. The following holds.

1. If \((l_r, l)\) is a matched pair of length functions on \((\Gamma, \text{Irr}(G))\) then, for all \(y \in \text{supp}(\tilde{a})\) one has

\[
l(y) \leq \max\{l_r(x) : x \in \text{supp}(a)\} + l_r(\gamma),
\]

where \((l_r)_\gamma \in \Gamma\) is any family of maps realizing the compatibility relations of Definition 4.1.

2. \(\mathcal{F}_{\gamma}(a) = v_{\gamma \gamma} \mathcal{F}_G(\tilde{a})\).

3. \(\|\tilde{a}\|_{G,0} \leq \|a\|_{G,0}\).

**Proof.** (1). Since any \(y \in \text{supp}(\tilde{a})\) is such that \(y \subset \text{Ind}_G^G(x) = \varepsilon_{G_{\gamma^{-1}}} \otimes x\) for some \(x \in \text{supp}(a)\), it follows that any \(y \in \text{supp}(\tilde{a})\) satisfies \(l(y) = l_1(y) \leq l_{\gamma^{-1}}(\varepsilon_{G_{\gamma^{-1}}}) + l_r(\gamma) = l_r(\gamma^{-1}) + l_r(\gamma) = l_r(\gamma) + l_r(x)\) for some \(x \in \text{supp}(a)\).

(2). One has:

\[
v_{\gamma \gamma} \mathcal{F}_G(\tilde{a}) = v_{\gamma \gamma} \sum_y \dim(y)(\text{Tr}_y \otimes \text{id})(u^y \tilde{a}_y \otimes 1)
\]

\[
= v_{\gamma \gamma} \sum_{x \in \text{supp}(a), y \subset \text{Ind}_G^G(x)} \frac{\dim(\text{Mor}(y, \text{Ind}_G^G(x)))}{\dim(y)} \sum_{i=1}^{\dim(\text{Mor}(y, \text{Ind}_G^G(x)))} \dim(x)(\text{Tr}_y \otimes \text{id})(u^y ((S^y_i)^*(e_{\gamma \cdot \beta} \otimes a_p x) S^y_i) \otimes 1)
\]

\[
= v_{\gamma \gamma} \sum_{x \in \text{supp}(a), y \subset \text{Ind}_G^G(x)} \dim(x)(\text{Tr}_y \otimes \text{id})((S^y_i)^* \otimes 1)\text{Ind}_G^G(x)(e_{\gamma \cdot \beta} \otimes a_p x \otimes 1)(S^y_i \otimes 1))
\]

\[
= v_{\gamma \gamma} \sum_{x \in \text{supp}(a), y \subset \text{Ind}_G^G(x)} \dim(x)(\text{Tr}_y \otimes \text{id})(\text{Ind}_G^G(x)(e_{\gamma \cdot \beta} \otimes a_p x \otimes 1)(S^y_i \otimes 1))
\]

\[
= v_{\gamma \gamma} \sum_{x \in \text{supp}(a)} \dim(x)(\text{Tr}_x \otimes \text{id})(u^x a_p x \otimes 1) = \mathcal{F}_{\gamma}(a).
\]

(3). One has:

\[
\|\tilde{a}\|_{G,0}^2 = \sum_y \dim(y)\text{Tr}_y(\tilde{a}^* \tilde{a}_y)
\]

10
Lemma 5.3. For all $\gamma$, $\|a\|^2_{G,0} \leq \sum_{x,y,i} \dim(x) \dim(y) \dim((S_{\gamma}^y)^*(e_{\gamma} \otimes a^* p_x) S_{\gamma}^y) = \sum_{x \in \text{supp}(a)} \dim(x) \dim(y) \dim((S_{\gamma}^y)^*(e_{\gamma} \otimes a^* p_x) S_{\gamma}^y) = \sum_{x \in \text{supp}(a)} \dim(x) \dim(y) \dim((S_{\gamma}^y)^*(e_{\gamma} \otimes a^* p_x) S_{\gamma}^y)

Since, for all $y, i$, $S_{\gamma}^y(S_{\gamma}^i)^*$ is a projection, one has $S_{\gamma}^y(S_{\gamma}^i)^* \leq 1$ hence,

$$\dim((S_{\gamma}^y)^*(e_{\gamma} \otimes a^* p_x) S_{\gamma}^y) \leq \dim((S_{\gamma}^y)^*(e_{\gamma} \otimes a^* p_x) S_{\gamma}^y).$$

Moreover, note that $y \in \text{Ind}_G^G(x)$ if and only if $\dim(\text{Mor}_G(\text{Res}_G^G(y,x))) \neq 0$. Since $x$ is irreducible, we find that $y \in \text{Ind}_G^G(x) \Leftrightarrow x \in \text{Res}_G^G(y)$. In particular, for any $y \in \text{Ind}_G^G(x)$ one has $\dim(x) \leq \dim(y)$.

Hence,

$$\|a\|^2_{G,0} \leq \sum_{x,y,i} \dim(x) \dim(y) \dim((S_{\gamma}^y)^*(e_{\gamma} \otimes a^* p_x) S_{\gamma}^y) = \sum_{x \in \text{supp}(a)} \dim(x) \dim(y) \dim((S_{\gamma}^y)^*(e_{\gamma} \otimes a^* p_x) S_{\gamma}^y) \leq \dim((S_{\gamma}^y)^*(e_{\gamma} \otimes a^* p_x) S_{\gamma}^y).$$

Lemma 5.3. Let $(l_\Gamma, l)$ be a matched pair of length functions on $(\Gamma, \text{Irr}(G))$. If $(\hat{G}, l)$ has polynomial growth, then there exists $C > 0$ and $N \in \mathbb{N}$ such that:

- $\|F_G(a)\| \leq C(k + 1)^N \|a\|_{G,0}$ for all $a \in c_c(\hat{G})$ with $\text{supp}(a) \subset \{x \in \text{Irr}(G) : l(x) < k + 1\}$.
- $|\gamma \cdot G| \dim(x) \leq C(l_\Gamma(\gamma) + l_\gamma(x) + 1)^N$ for all $\gamma \in \Gamma$, $x \in \text{Irr}(G)$.  
- For all $\gamma \in \Gamma$, $\sum_{x \in \text{Irr}(G), l(x) < k+1} \dim(x)^2 \leq C^2 (k + l_\Gamma(\gamma) + 1)^{2N}$.

Proof. Let $P \in \mathbb{R}[X]$ be such that $\sum_{x \in \text{Irr}(G), l(x) < k+1} \dim(x)^2 \leq P(k)$ for all $k \in \mathbb{N}$ and et $C_1 > 0$ and $N_1 \in \mathbb{N}$ be such that $P(k) \leq C_1(k + 1)^{N_1}$ for all $k \in \mathbb{N}$. By Lemma 5.1 one has, for all $a \in c_c(\hat{G})$, with $\text{supp}(a) \subset \{x \in \text{Irr}(G) : l(x) < k + 1\}$, $\|F_G(a)\| \leq 2 \sqrt{P(k)} \|a\|_{G,0} \leq \sqrt{C_1(k + 1)^{N_1}} \|a\|_{G,0}$. Now, suppose that $\text{supp}(a) \subset \{x \in \text{Irr}(G) : l(x) < k + 1\}$ so that $a \in q_k c_c(\hat{G})$, where $q_k = \sum_{j=0}^k p_j$ and $p_j = \sum_{x \in \text{Irr}(G), l(x) < k+1} \dim(x)^2 \leq C^2 (k + l_\Gamma(\gamma) + 1)^{2N}$. One has,

$$\|F_G(a)\| = \sum_{j=0}^k \|F_G(a p_j)\| \leq \sum_{j=0}^k \sqrt{C_1(j + 1)^{N_1}} \|a\|_{G,0} \leq \sqrt{C_1(k + 1)^{N_1}} \|a\|_{G,0}. \quad (5.1)$$

Now, let $\gamma \in \Gamma$ and $x \in \text{Irr}(G)$. By Proposition 5.3 one has:

$$|\gamma \cdot G| \dim(x) = \dim(\text{Ind}_G^G(x)) = \sum_{y \in \text{Irr}(G)} \dim(\text{Mor}_G(y, \text{Ind}_G^G(x))) \dim(y) = \sum_{y \in \text{Irr}(G), y \in \text{Ind}_G^G(x)} \dim(\text{Mor}_G(y, \text{Res}_G^G(y, x))) \dim(y).$$

Note that $\dim(\text{Mor}_G(y, \text{Res}_G^G(y, x))) \leq \dim(y)$ for all $x, y$. Moreover, since $\text{Ind}_G^G(x) \simeq c_{G_{\gamma-1}} \otimes x$ and the pair $(l_\Gamma, l)$ is matched, one has $\{y \in \text{Irr}(G), y \in \text{Ind}_G^G(x)\} \subset \{y \in \text{Irr}(G) : l(y) \leq l_\Gamma(\gamma) + l_\gamma(x)\}$. Hence,

$$|\gamma \cdot G| \dim(x) \leq \sum_{y \in \text{Irr}(G), l(y) < l_\Gamma(\gamma) + l_\gamma(x) + 1} \dim(y)^2 = \sum_{j=0}^{l_\Gamma(\gamma) + l_\gamma(x)} \sum_{y \in \text{Irr}(G), j \leq l(y) < j+1} \dim(y)^2.$$
where \( P \) polynomial growth, there exists a polynomial defined equation for all \( j \).

Proof. Let \( \gamma \in \Gamma \) and consider the matched pair of length functions \( p_F \in c_\ell(G) \) by \( p_F = \sum_{x \in F} P(x) \) and note that \( F_{\gamma}(p_F) = \sum_{x \in F} \dim(x) \chi(x) \) and \( \|a\|_{G,0}^2 = \sum_{x \in F} \dim(x)^2 \). Suppose that \( F \subset \{ x \in \text{Irr}(G) : l_r(x) < k + 1 \} \). Using Lemma 5.2 and the first part of the proof we find, since \( p_F \in c_\ell(G) \) with \( \text{Irr}(p_F) \subset \{ x \in \text{Irr}(G) : l_r(x) < l_r(\gamma) + k + 1 \} \),

\[
\left( \sum_{x \in F} \dim(x)^2 \right)^2 = \left( \sum_{x \in F} \dim(x) \chi(x)(1) \right)^2 \leq \sum_{x \in F} \dim(x) \chi(x) \leq C^2 (k + l_r(\gamma) + 1)^{2N} \sum_{x \in F} \dim(x)^2.
\]

Hence, for all nonempty finite subset \( F \subset \{ x \in \text{Irr}(G) : l_r(x) < k + 1 \} \) one has \( \sum_{x \in F} \dim(x)^2 \leq C^2 (k + l_r(\gamma) + 1)^{2N} \). The last assertion follows.

5.3 Polynomial growth for bicrossed product

We start with the following result.

**Theorem 5.4.** Suppose that that \( (\ell, l_r) \) is a matched pair of length functions on \( (\Gamma, G) \). If both \( (\ell, l_r) \) and \( (\widehat{\ell}, \ell_G) \) has polynomial growth then \( (\widehat{\ell}, \ell_G) \) have polynomial growth.

**Proof.** Let \( I \subset \Gamma \) be a complete set of representatives for \( \Gamma \) so that \( \text{Irr}(G) = \bigcup_{\gamma \in I} \text{Irr}(G_\gamma) \). Let \( k \geq 1 \) and define

\[
F_k := \{ z \in \text{Irr}(G) : \ell_k(z) < k \} \subset \bigcup_{\gamma \in I_k} T_{\gamma,k},
\]

where \( I_k := \{ \gamma \in \gamma : l_r(\gamma) < k + 1 \} \) and \( T_{\gamma,k} := \{ z \in \text{Irr}(G_\gamma) : l_r(z) < k + 1 \} \). Since \( (\ell, l_r) \) has polynomial growth, there exists a polynomial \( P \) such that, for all \( k \in \mathbb{N} \), \( |I_k| \leq P_k \). Moreover, since \( (\ell, l_r) \) has polynomial growth, we can apply Lemma 5.2 to get \( C > 0 \) and \( N \in \mathbb{N} \) such that, for all \( k \in \mathbb{N} \) and all \( \gamma \in I_k \), one has \( \sum_{x \in T_{\gamma,k}} \dim(x)^2 \leq C^2 (2k + 2)^{2N} \) and, \( |\cdot| G \leq |G| \text{dim}(\xi_G) \leq C^2 (2k + 3)^N \). Hence, for all \( k \geq 1 \),

\[
\sum_{z \in F_k} \dim(z)^2 = \sum_{\gamma \in I_k} |\ell'(\gamma)|^2 \sum_{z \in T_{\gamma,k}} \dim(z)^2 \leq C^4 (2k + 2)^{2N} \sum_{\gamma \in I_k} |\ell'(\gamma)|^2 \leq C^4 (2k + 2)^{2N} (2k + 3)^{2N} |I_k| \]

\[
\leq C^4 (2k + 2)^{2N} (2k + 3)^{2N} \ell_P(k).
\]

To complete the proof of Theorem B, we need the following Proposition.

**Proposition 5.5.** Assume that there exists a length function \( l \) on \( \text{Irr}(G) \) such that \( (\widehat{\ell}, l) \) has polynomial growth and consider the matched pair of length functions \( (\ell, l_G) \) associated to \( l \) given in Proposition 4.2. Then \( (\ell, l_r) \) and \( (\widehat{\ell}, \ell_G) \) both have polynomial growth.

**Proof.** Assume that \( (\widehat{\ell}, l) \) has polynomial growth. Since the map \( \text{Irr}(G) \to \text{Irr}(G), x \mapsto 1(x) \) is injective, dimension preserving and length preserving (by definition of \( l_G \)), it is clear that \( (\widehat{\ell}, \ell_G) \) has polynomial
proof. Let us show that $(\Gamma, l_\Gamma)$ also has polynomial growth. Let $P$ be a polynomial witnessing $(RD)$ for $(\widehat{G}, l)$ and $k \in \mathbb{N}$. Define $F_k := \{ \gamma \in \Gamma : k \leq l_\Gamma(\gamma) < k + 1 \}$. One has, for all $k \in \mathbb{N}$,

$$|F_k| = \sum_{k \leq l([\gamma(x)]) < k+1} 1 \leq \sum_{k \leq l([\gamma(x)]) < k+1} |\gamma \cdot G|^2 = \sum_{k \leq l([\gamma(x)]) < k+1} \dim([\gamma(x)G])^2$$

$$\leq \sum_{z \in \text{Irr}(\Gamma)} \dim(z)^2 \leq P(k).$$

5.4 Rapid decay for bicrossed product

Recall that $l^\infty(\widehat{G}) = \bigoplus_{\gamma \in \Gamma/G} \bigoplus_{x \in \text{Irr}(G)} \mathcal{B}(l^2(\gamma \cdot G) \otimes H_x)$. Let us denote by $p_{\gamma(x)}$ the central projection of $l^\infty(\widehat{G})$ corresponding to the block $\mathcal{B}(l^2(\gamma \cdot G) \otimes H_x)$ and define, for $\gamma \cdot G \in \Gamma/G$, the central projection :

$$p_\gamma := \sum_{x \in \text{Irr}(G)} p_{\gamma(x)} \in l^\infty(\widehat{G}).$$

Note that $p_\gamma l^\infty(\widehat{G}) = \bigoplus_{x \in \text{Irr}(G)} \mathcal{B}(l^2(\gamma \cdot G) \otimes H_x) \simeq \mathcal{B}(l^2(\gamma \cdot G) \otimes L(G))$, where $L(G_\gamma) = \bigoplus_{x \in \text{Irr}(G)} \mathcal{B}(H_x)$ is the group von-Neumann algebra of $G_\gamma$ (which is also the multiplier $C^*$-algebra of $C^*_r(G_\gamma) = \bigoplus_{x \in \text{Irr}(G)} \mathcal{B}(H_x)$).

Using this identification, we define $\pi_\gamma : c_0(\widehat{G}) \to \mathcal{B}(l^2(\gamma \cdot G)) \otimes C^*_r(G_\gamma) \subset c_0(\widehat{G})$ to be the $\ast$-homomorphism given by $\pi_\gamma(a) = ap_\gamma$, for all $a \in c_0(\widehat{G})$. We also write, for $a \in c_0(\widehat{G})$, $\omega_{x,e}^\gamma(a) = \sum_{r,s \in G} e_{rs} \otimes \pi_{x,r}^\gamma(a)$, where we recall that $(e_{rs})$ are the matrix units associated to the canonical orthonormal basis $(e_r)_{r \in \mathbb{N}}$ of $l^2(G)$ and $\pi_{x,r}^\gamma : c_0(\widehat{G}) \to C^*_r(G_\gamma)$ is the completely bounded map defined by $\pi_{x,r}^\gamma := (\omega_{x,e} \otimes \text{id}) \circ \pi_\gamma$ and $\omega_{x,e}^\gamma \in \mathcal{B}(l^2(\gamma \cdot G))$, $\omega_{x,e}^\gamma(T) = (Te_r,e_r)$.

We start with the following result.

Theorem 5.6. Let $(l_\Gamma, l_G)$ be a matched pair of length functions on $(\Gamma, \text{Irr}(G))$. Suppose that $(\widehat{G}, l_G)$ has polynomial growth and $(\Gamma, l_\Gamma)$ has (RD). Then $(\widehat{G}, l)$ has (RD).

Proof. Let $a \in c_0(\widehat{G})$ and write $a = \sum_{\gamma \in S} \sum_{x \in T} a p_{\gamma(x)}$, where $S \subset I$ and $T_\gamma \subset \text{Irr}(G)$ are finite subsets.

Claim. The following holds.

1. $F_G(a) = \sum_{\gamma \in S} |\gamma \cdot G| \left( \sum_{r,s \in G} u_{r,s} F_G, (\pi_{x,r}^\gamma(a)) \circ \psi_{x,s}^\gamma \right)$. 
2. $\|a\|_{2,0}^2 = \sum_{\gamma \in S} |\gamma \cdot G| \left( \sum_{r,s \in G} \|\pi_{x,r}^\gamma(a)\|_{2,0}^2 \right)$. 

Proof of the Claim. (1) A direct computation gives:

$$F_G(a) = \sum_{\gamma \in S} \sum_{x \in T_\gamma} |\gamma \cdot G| \dim(x) (\text{Tr}(\gamma \cdot G) \otimes H_x)(\gamma(u^x)^* a p_{\gamma(x)} \otimes 1)$$

$$= \sum_{\gamma \in S} \sum_{x \in T_\gamma} |\gamma \cdot G| \dim(x) \sum_{r,s \in G} u_{r,s} (\text{Tr}_x \otimes \text{id})(u^x \circ \psi_{r,s}^\gamma(a) p_{r,s}) \otimes 1$$

$$= \sum_{\gamma \in S} \sum_{r,s \in G} u_{r,s} F_G, (\pi_{x,r}^\gamma(a)) \circ \psi_{x,s}^\gamma.$$ 

(2) Since $\pi_\gamma$ is a $\ast$-homomorphism, we have $\pi_{x,r}^\gamma(a^*a) = \sum_{x \in G} \pi_{x,r}^\gamma(a)^* \pi_{x,r}^\gamma(a)$ hence,

$$\|a\|_{2,0}^2 = \sum_{\gamma \in S} \sum_{x \in T_\gamma} |\gamma \cdot G| \dim(x) \left( \sum_{r,s \in G} (\text{Tr}_x \otimes \text{id})(\pi_{x,r}^\gamma(a)^* \pi_{x,s}^\gamma(a)) \right)$$

$$= \sum_{\gamma \in S} \sum_{r,s \in G} \|\pi_{x,r}^\gamma(a)\|_{2,0}^2.$$
Let us now prove the theorem. Let \( b = \sum_{\gamma \in S'} \sum_{t \in \Gamma} u_t \psi_{t,\gamma} \in C(\Gamma) \), where \( F_\gamma \in C(\Gamma) \) and \( S' \subset I \) is a finite subset. For all \( r \in \Gamma \), we denote by \( \gamma_r \) the unique element in \( I \) such that \( \gamma_r \cdot G = r \cdot G \). We may re-order the sums and write:

\[
F_\gamma(a) = \sum_{r \in \Gamma} 1_{S \cdot G}(r) |r \cdot G| \left( \sum_{s \in \Gamma} u_{r,s} F_{\gamma,\gamma_r} \left( \pi_{s,r}^{\gamma_r}(a) \right) \circ \psi_{s,t}^{\gamma_r} \right) \quad \text{and} \quad b = \sum_{t \in \Gamma} u_t 1_{S \cdot G}(t) \left( \sum_{t' \in \Gamma} \psi_{t',t}^{\gamma_r} \right).
\]

Also, \( \|a\|_{L^2}^2 = \sum_{r \in \Gamma} 1_{S \cdot G}(r) |r \cdot G| \left( \sum_{s \in \Gamma} \|\pi_{s,t}^{\gamma_r}(a)\|_{\mathcal{H}}^2 \right) \). Then, \( F_\gamma(a) b \|_{L^2(\mathcal{C})} \) is equal to:

\[
\sum_{r,t \in \Gamma} u_{r,t} 1_{S \cdot G}(r) 1_{S \cdot G}(t) |r \cdot G| \left( \sum_{s \in \Gamma} u_{r,s} \alpha_t \pi_{s,t}^{\gamma_r}(a) \circ \psi_{s,t}^{\gamma_r} \circ \alpha_t \psi_{t',t}^{\gamma_r} \right)^2
\]

\[
= \sum_{r,t \in \Gamma} 1_{S \cdot G}(r) 1_{S \cdot G}(t) |r \cdot G| \left( \sum_{s \in \Gamma} \|\pi_{s,t}^{\gamma_r}(a)\|_{\mathcal{H}}^2 \right)^2
\]

\[
\leq \sum_{r \in \Gamma} 1_{S \cdot G}(r) 1_{S \cdot G}(t) \left( \sum_{s \in \Gamma} \|\pi_{s,t}^{\gamma_r}(a)\|_{\mathcal{H}}^2 \right)^2
\]

\[
= \| \psi \circ \psi \|^2_{L^2(\Gamma)},
\]

where \( \| \cdot \|_2 \) and \( \| \cdot \|_{\infty} \) denote respectively the L^2-norm and the supremum norm on \( C(\Gamma) \) and \( \psi, \phi : \Gamma \to \mathbb{R}_+ \) are finitely supported functions defined by:

\[
\psi(r) := 1_{S \cdot G}(r) |r \cdot G| \left( \sum_{s \in \Gamma} \|\pi_{s,t}^{\gamma_r}(a)\|_{\mathcal{H}}^2 \right)^2 \quad \text{and} \quad \phi(t) := 1_{S \cdot G}(t) \left( \sum_{t' \in \Gamma} \|\psi_{t',t}^{\gamma_r}\|_{\mathcal{H}}^2 \right)^2.
\]

Note that \( \| \phi \|^2_{L^2(\Gamma)} = \| b \|^2_{L^2(\mathcal{C})} \). Moreover, one has, since \( \psi_{r,s}^{\gamma_r} : G_{r,s} \to G_{\gamma} \) is an homeomorphism,

\[
\| \psi \|^2_{L^2(\Gamma)} = \sum_{r \in \Gamma} 1_{S \cdot G}(r) |r \cdot G|^2 \left( \sum_{s \in \Gamma} \|\pi_{s,t}^{\gamma_r}(a)\|_{\mathcal{H}}^2 \right)^2
\]

\[
\leq \sum_{r \in \Gamma} 1_{S \cdot G}(r) |r \cdot G|^3 \sum_{s \in \Gamma} \|\pi_{s,t}^{\gamma_r}(a)\|_{\mathcal{H}}^2 \left( \sum_{t' \in \Gamma} \|\psi_{s,t}^{\gamma_r}\|_{\mathcal{H}}^2 \right)^2
\]

\[
= \sum_{r \in \Gamma} 1_{S \cdot G}(r) |r \cdot G|^3 \left( \sum_{s \in \Gamma} \|\pi_{s,t}^{\gamma_r}(a)\|_{\mathcal{H}}^2 \right)^2.
\]

For \( k \in \mathbb{N} \) let \( p_k = \sum_{\gamma \in I \cdot x \in \text{Irr}(G_{\gamma}) : \kappa \leq x \gamma \} \sum_{l \in \mathbb{N}} \| p_{l\gamma} \|_{C^0(\hat{G}_{\gamma})} \), hence we must have \( S \subset \{ \gamma \in \Gamma : l_{\gamma}(x) < k + 1 \} \) and, for all \( \gamma \in S, T_{\gamma} \subset \{ x \in \text{Irr}(G_{\gamma}) : l_{\gamma}(x) < k + 1 \} \). Hence, for all \( \gamma \in S \) and all \( r, s \in \gamma \cdot G \) one has \( \pi_{s,t}^{\gamma_r}(a) \in q_k^{\gamma_r} \), where \( q_k^{\gamma_r} = \sum_{j=0}^{k} p_j^{\gamma_r} \).
Since \((\hat{G}, l_G)\) has polynomial growth, there exists \(C > 0\) and \(N \in \mathbb{N}\) satisfying the properties of Lemma \[5.3\]. In particular, one has, for all \(\gamma \in \Gamma\), \(|\gamma \cdot G| \leq C(2|\gamma| + 1)^N\). Moreover, since \(S \subset \{g \in \Gamma : l_{r}(g) < k + 1\}\) and \(l_{r}\) is \(\beta\)-invariant, it follows that \(S \cdot G \subset \{g \in \Gamma : l_{r}(g) < k + 1\}\). By Lemma \[5.2\] (and Lemma \[5.3\]) we deduce that:

\[
\|\psi\|_{L^2(\Gamma)}^2 \leq \sum_{r \in \Gamma} 1_{S \cdot G}(r) |r \cdot G|^3 \sum_{s \in r \cdot G} \left\|\varphi_{r, s} \cdot \mathcal{F}_G(\pi_{s, r}(a))\right\|^2 \leq \sum_{r \in \Gamma} 1_{S \cdot G}(r) |r \cdot G|^3 \sum_{s \in r \cdot G} \left\|\mathcal{F}_G(\pi_{s, r}(a))\right\|^2 \\
\leq \sum_{r \in \Gamma} 1_{S \cdot G}(r) |r \cdot G|^3 \sum_{s \in r \cdot G} C^2(k + l_{r}(\gamma) + 1)^{2N} \left\|\pi_{s, r}(a)\right\|_{G,0}^2 \\
\leq C^2(2k + 2)^{2N} \sum_{r \in \Gamma} 1_{S \cdot G}(r) |r \cdot G|^3 \sum_{s \in r \cdot G} \left\|\pi_{s, r}(a)\right\|_{G,\gamma,r}^2 \\
\leq C^4(2k + 3)^{4N} \sum_{r \in \Gamma} 1_{S \cdot G}(r) |r \cdot G| \sum_{s \in r \cdot G} \left\|\pi_{s, r}(a)\right\|_{G,\gamma,r}^2 = C^4(2k + 3)^{4N} \|a\|_{C,0}^2.
\]

Since \((\Gamma, l_{r})\) has \((R, D)\), let \(C_2 > 0\) and \(N_2 \in \mathbb{N}\) such that for all \(k \in \mathbb{N}\), for all function \(\xi\) on \(\Gamma\) supported on \(\{g \in \Gamma : l_{r}(g) < k + 1\}\), we have \(|\xi \ast \eta|_{L^2(\Gamma)} \leq C^2(k + 1)^{2N_2} \|\xi|_{L^2(\Gamma)}\|\eta|_{L^2(\Gamma)}\). Note that \(\psi\) is supported on \(S \cdot G\) and \(S \cdot G \subset \{g \in \Gamma : l_{r}(g) < k + 1\}\). Hence, it follows from the preceding computations that:

\[
\|\mathcal{F}_G(\hat{a})\|_{L^2(\Gamma)}^2 \leq \|\hat{\psi} \ast \hat{\eta}|_{L^2(\Gamma)} \leq C^2(k + 1)^{2N_2} \|\psi|_{L^2(\Gamma)}\|\eta|_{L^2(\Gamma)} \leq C^4(2k + 3)^{4N} C_2^2(k + 1)^{2N_2} \|a\|_{C,0}^2 \|\hat{a}\|_{C,0}^2.
\]

where \(P(X) = C^2 C_2^2 (2X + 3)^{2N} (X + 1)^{2N_2}\). It concludes the proof.

To complete the proof of Theorem A, we need the following Proposition.

**Proposition 5.7.** Assume that there exists a length function \(l\) on \(\text{Irr}(\mathbb{G})\) such that \((\hat{G}, l)\) has \((R, D)\) and consider the matched pair of length functions \((l_{r}, l_G)\) associated to \(l\) given in Proposition \[4.2\]. Then \((\Gamma, l_{r})\) has \((R, D)\) and \((\hat{G}, l_G)\) has polynomial growth.

**Proof.** Suppose that \((\hat{G}, l)\) has \((R, D)\). The fact that \((\hat{G}, l_G)\) has \((R, D)\) follows from the general theory (since \(C(\hat{G}) \subset C(\mathbb{G})\) intertwines the comultiplication and the associated injection \(\text{Irr}(G) \to \text{Irr}(\mathbb{G})\), actually given by \((x) \mapsto 1(x))\), preserves the length functions). Let us show that \((\Gamma, l_{r})\) has \((R, D)\). Let \(k \in \mathbb{N}\) and \(\xi : \Gamma \to \mathbb{C}\) be a finitely supported function with support in \(\{\gamma \in \Gamma : k \leq l_{r}(\gamma) < k + 1\}\). Define \(\hat{\xi} \in c_0(\hat{G})\) by \(\hat{\xi} = \sum_{\gamma \in \Gamma} \frac{1}{|\gamma|} \left(\sum_{r \in \gamma \cdot G} \xi(r) e_r\right) p_{\gamma}(1)\), where we recall \(e_{rs} \in B(P^{\gamma}(\gamma \cdot G))\) for \(r, s \in \gamma \cdot G\) are the matrix units associated to the canonical orthonormal basis. Then,

\[
\mathcal{F}_G(\hat{\xi}) = \sum_{\gamma \in \Gamma} \sum_{r \in \gamma \cdot G} \xi(r) (\mathcal{T}_{P^G}(\gamma \cdot G) \otimes \text{id})(a^{(1)}(e_r \otimes 1)) = \sum_{\gamma \in \Gamma} \sum_{r \in \gamma \cdot G} \xi(r) u_r v_r
\]

also,

\[
\|\hat{\xi}\|_{C,0}^2 = \sum_{\gamma \in \Gamma} |\gamma \cdot G| |\mathcal{T}_{P^G}(\gamma \cdot G)| \sum_{r \in \gamma \cdot G} \frac{\|\xi(r)\|^2}{|\gamma \cdot G|^2} = \sum_{\gamma \in \Gamma} \frac{1}{|\gamma \cdot G|^2} \sum_{r \in \gamma \cdot G} |\xi(r)|^2 \leq \sum_{\gamma \in \Gamma} \sum_{r \in \gamma \cdot G} |\xi(r)|^2 = \|\xi\|_{C,G}^2.
\]

Since \(\xi\) is supported in \(\{\gamma \in \Gamma : k \leq l_{r}(\gamma) < k + 1\}\) and \(l_{r}\) is \(\beta\)-invariant, it follows that \(\text{supp}(\hat{\xi}) \subset \{z \in \text{Irr}(\mathbb{G}) : k \leq l(z) < k + 1\}\). Hence, denoting by \(P\) a polynomial witnessing \((R, D)\) for \((\hat{G}, l)\), we have:

\[
\left\|\sum_{\gamma \in \Gamma} \sum_{r \in \gamma \cdot G} \xi(r) u_r v_r\right\| \leq P(k) \|\xi\|_{C}.
\]

Denote by \(\Psi\) the unital *-morphism \(\Psi : C(\mathbb{G}) = \Gamma \ltimes C(\Gamma) \to C^*_{\gamma}(\Gamma)\) such that \(\Psi(u_r F) = \lambda_r F(1)\) for all \(\gamma \in \Gamma\) and \(F \in C(\Gamma)\). Since \(\Psi\) has norm one, denoting by \(\lambda(\xi) \in C^*_{\gamma}(\Gamma)\) the convolution operator by \(\xi\), we have

\[
\|\lambda(\xi)\| = \left\|\sum_{\gamma \in \Gamma} \sum_{r \in \gamma \cdot G} \xi(r) \lambda_r\right\| = \|\Psi(\sum_{\gamma \in \Gamma} \sum_{r \in \gamma \cdot G} \xi(r) u_r v_r)\| \leq \left\|\sum_{\gamma \in \Gamma} \sum_{r \in \gamma \cdot G} \xi(r) u_r v_r\right\| \leq P(k) \|\xi\|_{C}.
\]

This concludes the proof."
References


Pierre FIMA
Univ Paris Diderot, Sorbonne Paris Cité, IMJ-PRG, UMR 7586, F-75013, Paris, France
Sorbonne Universités, UPMC Paris 06, UMR 7586, IMJ-PRG, F-75005, Paris, France
CNRS, UMR 7586, IMJ-PRG, F-75005, Paris, France
E-mail address: pierre.fima@imj-prg.fr

Hua WANG
Univ Paris Diderot, Sorbonne Paris Cité, IMJ-PRG, UMR 7586, F-75013, Paris, France
Sorbonne Universités, UPMC Paris 06, UMR 7586, IMJ-PRG, F-75005, Paris, France
CNRS, UMR 7586, IMJ-PRG, F-75005, Paris, France
E-mail address: hua.wang@imj-prg.fr