ASKING MATHEMATICAL QUESTIONS MATHEMATICALLY

John Mason
Open University
Milton Keynes, U.K.

ABSTRACT

In order to work with the inevitable didactic tension (the more clearly the teacher indicates the behaviour sought, the easier it is for students to display that behaviour without generating it from understanding) which is part of the *contrat didactique* described by Brousseau, it is useful to work with all six modes of interaction between teacher, student, and content: expounding, explaining, exploring, examining, expressing, and exercising. Posing students questions with the purposes or effects of focusing, testing or enquiring forms the basis for many of these modes, but like most interactions, questioning can become habitual and limited in scope. I conjecture that we are moved to pose questions as tasks, or to ask questions of students, when we ourselves experience a shift in the structure of our own attention, and this is why we often 'catch students out' with our tasks, and why we often get caught in a funnelling sequence of questions (in the sense of Bauersfeld). I offer the conjecture that questions arise, whether in planning or in the classroom, as a result of us experiencing a shift in our own attention; the purpose of our question is often, perhaps usually, to try to provoke a corresponding shift in our students. My claim is that by paying attention to the types of questions we ask, we can influence the development of students' awareness of mathematical thinking. I will look in detail at Probing Boundaries, Story Telling, and Probing Understanding.

La tension didactique (Plus le professeur indique clairement la conduite recherchée, plus les étudiants trouvent facile de la suivre sans la comprendre) est une partie du contrat didiactique décrit par Brousseau. Pour travailler avec la tension didactique inévitable, il est utile de travailler avec tous les six modes d'interaction entre le professeur, l'étudiant et le contenu: exposer, expliquer, explorer, examiner, exprimer, exercer. L'habileté de poser des questions aux étudiants afin qu'on puisse concentrer, examiner, ou s'informer, elle constitue la basse à beaucoup de ces modes; mais comme la plupart des interactions, l'habileté de questioner peut devient habituelle et limitée. Je prétend que si on fait attention aux genres des questions posées, on peut influencer le development de la conscience des opinions mathématiques des étudiants. Je propose la conjecture qu'un question nous se présente avant de ou pendant une classe, parce que notre attention est transformé soudain; le but de la question est souvent, peut être toujours, essayer provoquer un trnsformation correspondent pour nos étudiants. J'éxaminerai en détail à les frontières serrés, la narration des histoires mathématiques et la construction des problèmes fondamentales qui examine la comprehension.

INTRODUCTION

As background to my proposals, it is useful to review the notion of the *didactic tension* and to be explicit about one approach to counteracting that tension through making a distinction between *working-through* exercises, text, or even computer programs, and *working-on* them.

Didactic Tension

The didactic tension is endemic to teaching:

> the more clearly the teacher indicates the behaviour sought, the easier it is for students to display that behaviour without generating it from understanding

This tension is part of the *contrat didactique* (Brousseau 1984, 1997 p31, 41) between teacher and students, in which students agree to do (some of) what the teacher asks in the expectation that they will then learn. In a culture of box-ticking accountability this naturally leads to students minimising their energies by making the least effort, and expecting that somehow this will be sufficient for them to learn. They work through tasks rather than work on them: they do individual questions one after another with little or no attention paid to what is the same and what different between them. The same minimisation of energy even leads some teachers to treat the textbook as the teaching, seeing their task as making the work as pleasant as possible. Whitehead (1932) railed against this tendency:

> this evil path represented by a book or set of lectures which will practically enable a student to learn by heart all the questions likely to be asked ... culminates in a uniform examination [which] is so deadly (p7-8),

as did Dewey (1902 p22) and each generation before them. Student and teacher collude in expecting learning to happen almost independently or automatically, without the specific intention and engagement of the students. The result is that as teachers we have to work harder at getting students to work.

This theme is unintentionally present in a Thatcherite British Rail poster with a picture of the many tracks leading to a big London station:

On Monday morning British Rail
will mount an operation
four times bigger than D–Day.
(It's called getting you to work.)

The ambiguity between getting as transporting and getting as forcing typifies the extremes of the didactic tension. Put another way, the ambiguity between *to* as infinitive and as preposition plays itself out in a fundamental tension in motivation. One way of making use of the energy present in the tension, is to develop with students a way of working which distinguishes between

> *working through* exercises and *working on* exercises

The first describes the student who does a few questions, takes a break, does a bit more on the bus, copies a bit from a friend, and ends up with no overall sense of the exercises as examples of anything or what they are about. Contrast this with the student who in doing the exercises asks themself what is similar about the questions and what different, what it is about the context which enables the technique to work, what sorts of difficulties might the technique encounter in different situations, etc. That student is working-on the exercises.

The two states of *working-through* and *working-on* are completely different, and in particular they involve different energies. *Working-through* minimises effort through minimum involvement. It is unreflective and unmathematical. *Working-on* minimises effort mathematically, by trying to locate underlying structure and so reduce memory demands.

A teacher can try to invoke the energy of *working-through* by changing from the standard ‘do the odd numbered questions on page ...’ to such tasks as

> do enough of the problems on page ... so that you can write down an easy question of that type, a hard question of that type;
>
> write down the most general question of the type found on page ..., and show how to do that general question.
>
> What makes a question amenable to this technique?
>
> Construct a simple or easy, a peculiar or unusual, and a difficult or general example-question of this type.
>
> Construct a question and do it, which shows that you understand this topic.

Heuval-Panhuizen, Middleton, & Streefland (1995) analysed different versions of percentage questions which students generated. They found, as have I in other contexts, that students do not always correctly solve even their own ‘easy’ problems. The adjective *easy* refers to their view of the problem not to their success. Heuval-Panhuizen *et al.* analyse reasons for encountering difficulty in the problems proposed by the students. Such an analysis may be useful for research and for assessment development, as in the article, and for teacher appreciation of what students are experiencing. It could also be useful to students to see what each other considers difficult so they could help each other in some cases. The authors note that sometimes students create "really excellent problems" which teachers can then use for assessment in later years.

Notice that in tasks of the type suggested above, initiative is partially transferred to the student, for the number of questions they need to do depends on them. They actually have a meta-task, for their attention is directed out of the mere doing of the questions and into doing with a purpose, a higher order goal. To become skilful with a technique, means to reduce the amount of attention needed, so it is sensible to work on tasks with greater goals which can occupy attention, while provoking rehearsal of other techniques (Hewitt 1994).

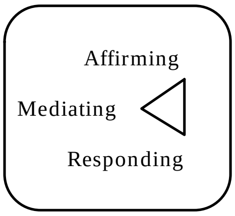
Looking at questions which students consider to be hard often reveals that their attention is not so much on the structure of the problem as on the kinds of numbers or calculations they have to carry out. For some a question with fractions, or surds, or letters is hard.

Of course the first time you set such a task, students will be put off and many will probably not understand what you mean. In order to be effective, such a gambit (Pimm 1987) needs to be become part of an ethos in the classroom in which mathematical thinking is praised, fostered, and sustained. Generality is at the heart of all mathematics, from recognising numbers as nouns rather than adjectives, through arithmetic techniques (either on numbers or on letters), to theorems and techniques at university. There are general theorems and general techniques. But there is evidence (Anthony 1994, Bills 1996) that students do not appreciate generality as much as their teachers expect. Indeed for Wittgenstein (1956, see also Glock 1996 p323-325) it posed a major philosophical problem. Consequently it is worthwhile being explicit about getting students into the habit of expressing generality (and checking it) themselves.

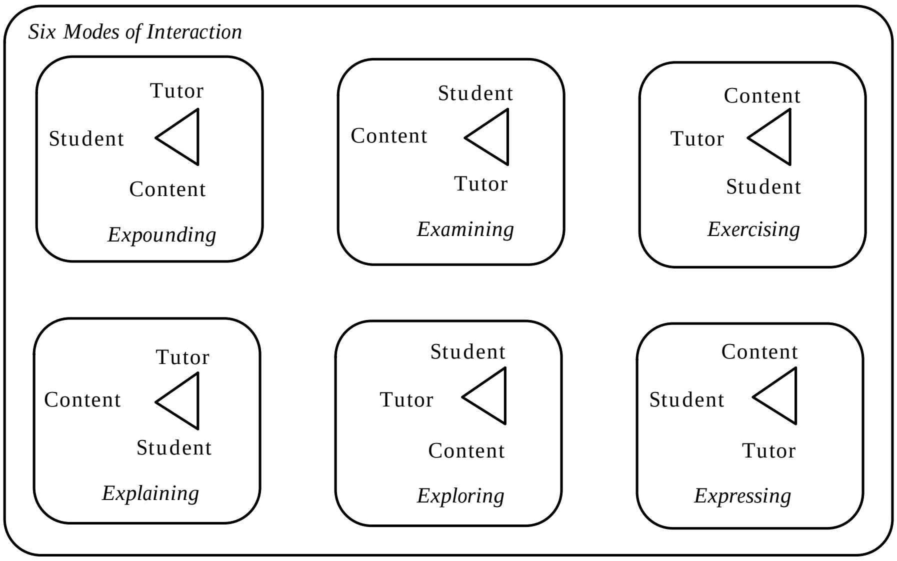
Hand in hand with generalising is of course specialising. Explicitly getting students to specialise from generalities, and to generalise from particular cases supports them in processes which are often left below the surface of awareness because they are so fundamental, so important. A useful exercise in this respect is to examine any textbook and see what proportion of the time the author does the generalising and the specialising for the student, what proportion of the time the student is explicitly encouraged to do it for themselves, and what proportion of the time the student is implicitly expected to do it for themselves.

SIX MODES OF INTERACTION

Interaction between expert and novice concerning mathematics brings together three impulses: teacher, student, and content, in an environment.



The compatibility of the environment with the modes of interaction is crucial, but here I shall concentrate on distinguishing the six different modes arising by allocating the forces present in an interaction, namely tutor, student, and content, to the three impulses in the six possible ways.



Each triangle can be 'read' using the formula 'the affirming acts upon or contacts the response, facilitated or enabled by the mediator'. You can try to apply this incantation to each triad, so as to contact what it is describing, but the triads will only have value when you can read and think with them rather than think about them. To do this requires making firm connections with your own experience.

As a set of six items, it is easy to generate the diagrams, but more difficult to make sense of the labels, which I have chosen for their homonymy, as I find them relatively easy to remember: they are known collectively as the six Ex's. However, some traditional meanings of these words have to be amplified or modified in order to capture the quality of the corresponding mode. There is considerable danger that the labels I have chosen will divert attention from the essence of the interaction type to a particular form, but this danger is inherent in all use of language and is not in my view sufficient reason to shirk labelling.

Expounding

Exposition has had bad press in recent years, in the push for exploratory, problem-solving approaches to learning mathematics. In English the word is associated with being authoritative and domineering, telling others what to do and how to do it, often without justification or purpose. I suggest that this is a tragic mis-construction of what exposition can be. After all, most interactions between people consist of one person telling the other something, whether through recounting an incident or passing on information.

There is more to expounding than simply talking to or at people. When exposition is 'talking at' students, it is empty. When exposition is an action, there is something almost palpable happening for both expounder and audience: the speaker is inspired, is in touch with the content in a significant way, and the audience is attentive and gripped because they can 'hear' what is being said. The presence of the audience (whether virtual, by virtue of expectation while preparing for a lecture, or actual) enables the expounder to get in touch with the content in a vigorous if not fresh way. The speaker is able to see more deeply or more broadly, and through their exposition entices students into this world of perception. Instead of tunnel vision in the midst of the pressure of the presence of the audience, the presence of the audience can have a liberating effect.

It is often said that in teaching one should 'start where the students are', though there are many ways to interpret this advice. One way is to arrange that students have had recent relevant experience so that they can relate to what is being said, so that recent and past experience is resonated and organised by what is said. This is one way in which telling can indeed be 'telling'. Of course I use the verb 'said' here metaphorically, to encompass different and multiple media.

Explaining

In this mode, the content is what brings the teacher into contact with the student. The initiative is still with the teacher, but the teacher tries to enter the world of the student, tries to make contact with their perceptions, their thought processes, their connections, based around the content. When this mode is operative, both asking and telling can be genuine and effective, because they relate directly to the student.

Unfortunately, 'to explain' is usually taken to mean 'to make plain', with a background metaphor of flat, dull, and seamless plane. An associated pedagogic assumption is that saying more, breaking down into smaller steps, and simplifying, provides ideal explanation from which students will learn more easily. This view induces the teacher to try to take full responsibility for acts which are necessarily the students' and not the teacher's, and fails to take account of the action which can take place when the teacher enters the world of the student.

Exploring

Here the student acts upon or encounters the content, mediated by the teacher. The student has to take the initiative, which makes this mode difficult to generate. I think Bruner had this in mind when he developed 'discovery learning', but the term was usurped by people who took it to mean that children had to re-discover everything for themselves. You cannot explore without making mistakes. Indeed, conjecturing is essential to exploration, by which I mean an atmosphere in which statements made are made with the intention of modifying them. The act of uttering them, and of having others hear them and respond to them, supports and enquiring, exploring action.

The importance of the teacher in this action is as mediator, doing for the students what they cannot yet do for themselves. Bruner (1986) described this as scaffolding, a term which goes in and out of favour, but whose essence lies in the fading away of the teacher's structuring so that students come to internalise it for themselves.

Creating materials to support exploration is far from easy. Published worksheets and texts are usually so caught in the *transposition didactique* (Chevellard 1985), the inevitable transformation of expert awareness into novice instruction, that such materials are usually taken by students to be exposition. The problem is that authors and publishers do not trust students. They want to retain control, but that limits the actions which they can support. They are frightened that students may not know what to do, so they provide structured assistance. Such exposition-structure is fine as long as students are weaned off it through internalising it into their own initiative. But if students depend on the text to structure their thinking (fill in this table, do these special cases, finds a formula which ...), then they become trained in dependency and fail to exercise their initiative.

Examining

Here the student verifies their own criteria against that of the expert, the teacher. Consequently they 'supplicate' to be examined (language which is still found in the regulations of some institutions. The content is what brings student and teacher together. It is what they share, and on which one is tested by the other. But it is not the mechanical testing of mass assessment systems; it is not a matter of going in and answering questions and then finding out how you did. It is the culmination of formal education in which the student shows that they have internalised not just the content, but the criteria by which understanding is judged. They know that they know and are confirmed in this knowing.

The word *examining* may be too limiting, because this mode of interaction also occurs when a student brings something to class (whether a question, object, or an idea) which the teacher chooses to follow up, since it is the content which provides the ground or means through which the student contacts the teacher. Most teachers are eager for genuine questions or other input from their students. I have found that it is necessary to work with students on their skills of questioning each other in a conjecturing atmosphere in order to release genuine questioning by them.

Expressing

That delightful moment when there is a welling up of energy and you cannot stop yourself from saying to someone, 'Look at this ...', is an example of expressing. Here it is the content which initiates, which takes over and acts upon the listener (most often the teacher, but not necessarily). The student finds themself as the medium of expression. It is well known that you really only learn a topic when you try to teach it to others. Contacting the energy in the content, and letting that come to the surface may shift you into expository or explanatory mode, but it starts with a taste of the mode of expressing. The difference between expressing and expounding is that in the first the audience is not expected to be transformed by the occasion, to learn something. Rather, they are assisting the content to come to the surface, to be expressed, so that it is more easily critiqued. This is part of a conjecturing atmosphere, in which those who are confident listen, and those who are uncertain use the presence. Often a mathematician will rush into a colleague's office, appear to ask a question while spouting some mathematics, and then rush out again saying 'thank you'. What was needed was the witnessing colleague to act upon in order to bring ideas to the surface.

Exercising

It is common to set students exercises, but not so common that they undertake exercises as an important action in their learning. If someone sets you some exercises (for example, a doctor tells you to take more exercise), it rarely lasts any time if it ever starts at all. If someone asks you to rehearse some behaviour, it is resisted, unless there is a genuine desire to carry out that rehearsal, to test things out. But if something inside you takes the initiative, then exercising can become an important routine. So too with more intellectual practise and rehearsal. The content initiates, affirms, and the teacher is the means (by structuring tasks) whereby the content acts upon the student. If the exercises are routine, there may only be routine interaction; if the exercises involve the student reflecting, generalising, setting their own ‘similar’ tasks (in particular, or generalised), the student is supported in shifting into examining mode, taking the initiative to show that they truly understand.

SEEKING BALANCE

No one mode is better than another. Rather, each constitutes an important action which contributes to learning. One of the virtues of distinguishing the six modes is that it helps identify over-stressing of some modes at the expense of others. For example, whereas ancient Greeks and Victorians developed exposition to a high art, problem solving, investigation, and group work have been lauded more recently. But these are only some of the possible actions, and it is ineffective to stress some at the expense of others.

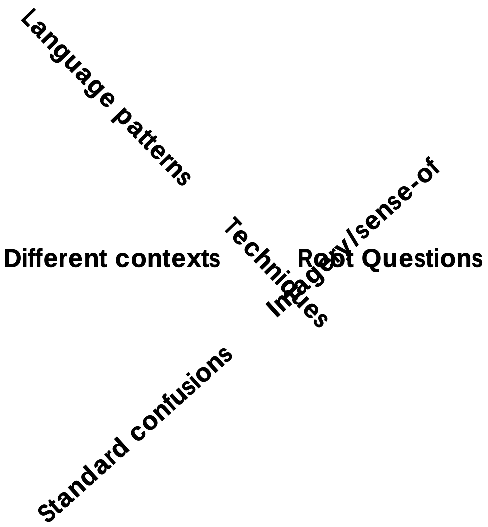
Awareness of the modes can also sensitise people to unintended transitions from one to another. For example, as soon as the teacher in explaining mode thinks they know where the student is ‘going wrong’ or is ‘missing’ an idea or a technique, there is a strong pull to jump into expository mode, to drag the student back into the teacher’s world. Such a switch of mode is fine, as long as it is intentional, judged to be appropriate through sensitivity to the situation, and not just an automatic switch of energies and focus of attention induced through what Mary Boole called ‘teacher lust’ (Tahta 1962).

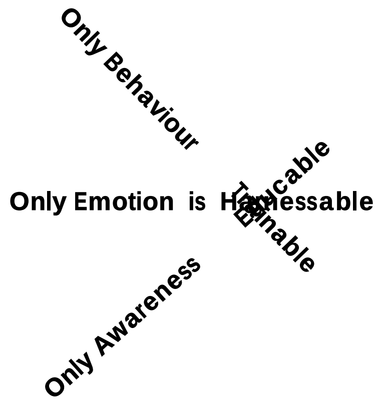
The modes are not even intended to be completely distinct. If you examine your experience you will find that there can be very rapid shifts from one to another, or even that elements of more than one seem to be present at the same time. Furthermore, some seem more stable than others (exposition is more stable than explaining or expressing, examining more stable than exploring).

Within each mode, there can be assertions and questions by student and by teacher. The distinctions between the modes lie not in who is speaking or listening, but on where the initiative, response, and mediation lies.

PREPARING TO TEACH A TOPIC

In the language of the 6 ex's, in a balanced learning environment students are given explorations to undertake and exercises to work-on. They are also assessed, ideally when the student wishes to verify their own internal criteria for understanding the topic. For exploration purposes, and for in depth assessment purposes, it is desirable to have a collection of problems which really probe student understanding and use of the techniques, the technical terms and associated language patterns, their knowings, their images and connections, and their appreciation of the source and uses of the topic. These aspects form a framework for 'preparing to teach a topic' (shown on the left; Griffin & Gates 1989) corresponding to three aspects of the psyche, behaviour, emotion and awareness.





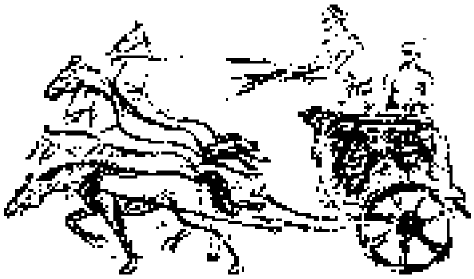
The root problems which generated the topic as a topic to teach, complete with behavioural aspects of techniques and technical terms, give access to energy of enquiry and problematique which can be reproduced in the classroom, and the different contexts in which the topic or the techniques can be applied provide a different form of energy. The standard confusions which students construct using their mathematical powers but mis-applied to what they happen to stress and ignore as the topic is presented. The images and sense-of are components of the concept-image (Tall & Vinner 1981) and connections with other topics, techniques, themes, processes, and heuristics.

The three strands are based on three intentionally enigmatic assertions, the first of which is due to Gattegno (1987):

> Only awareness is educable; Only behaviour is trainable; Only emotion is harnessable

Displayed as three interwoven threads, they are the three strands which make up modern psychology's description of the psyche (cognition, behaviour, affect), but the assertions themselves draw upon the ancient and more complex image of a person as a chariot (Mason 1995).

A person is likened to a horse, chariot, and driver. Later versions change the chariot into a carriage, and then a hansom cab, as appropriate to the times. The owner (originally a charioteer) hires a driver, who has responsibility for maintaining the chariot fabric and tackle, and for looking after the horses. If the chariot body is allowed to decay, if the reins and harness go mouldy or stiff, if the horses become hungry or mangy, or if the driver becomes dissolute, then the chariot cannot be used properly, and the owner will stay away. If the horses are not guided, they will follow each sudden movement, and graze at every opportunity. If the carriage is not maintained, it will creak and crack under stress. If the driver is not awake to where the chariot is headed, then the route will be missed, and there may be crashes.



The Katha Upanishad (III v3-4) offers one reading:

> Know thou the atman (self) as the owner of a chariot,
> The chariot as the body,
> Know thou the buddhi (intellect) as the chariot-driver,
> And mind as the reins.
> The senses, they say, are the horses;
> The objects of sense, what they range over.
> What then is experience?
> ‘Self, sense and mind conjoined’ the sage replies.

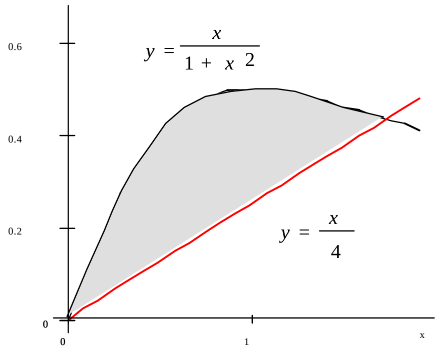
adapted from Zaehner (1966), p 176.

Reins and harness (mental imagery) enable the driver to guide and direct the horses (emotions, senses), while the shafts enable the horse’s energy to pull the chariot. The particular version shown offers five horses for the five senses.

The ancient origins of this image speak to an abiding awareness which, as long as it does not become mechanical, can inform teacher preparation for a topic.

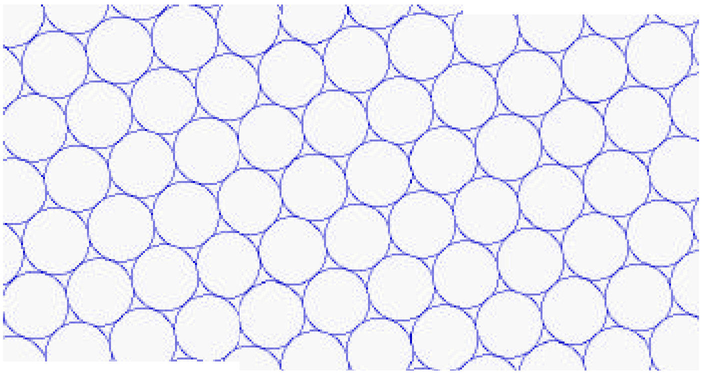
STRUCTURE OF ATTENTION

What are students attending to at any given moment? For example, some 17 year old mathematics students were assigned the following task for homework, but before leaving the class, they were asked to 'say what you see'.



Many of the students were transfixed by the $\sqrt{3}$, to the extent that they could not attend to the structure. Whereas an expert either uses the diagram to detect the difference of two integrals and then uses the minus sign to reconstruct the task in this form, or uses the minus sign to interpret the diagram. In either case, the expert has learned to suppress any misgivings about the number $\sqrt{3}$ and treat it simply as a symbol. They have not yet learned to control the placing of their attention, to alter what they stress and ignore.

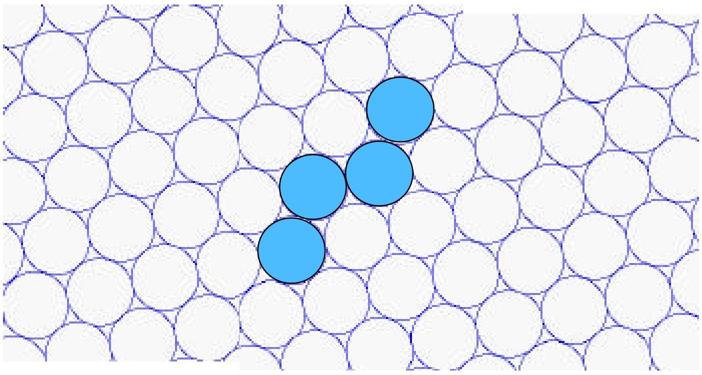
Another example is given by the following sequence of experiences. Look at the first picture and see it as just part of a uniform close packing of the infinite plane. Now see it as made up of layers, first in one direction, then another, then a third. Concentrate until you can see and change between the layers in different directions.



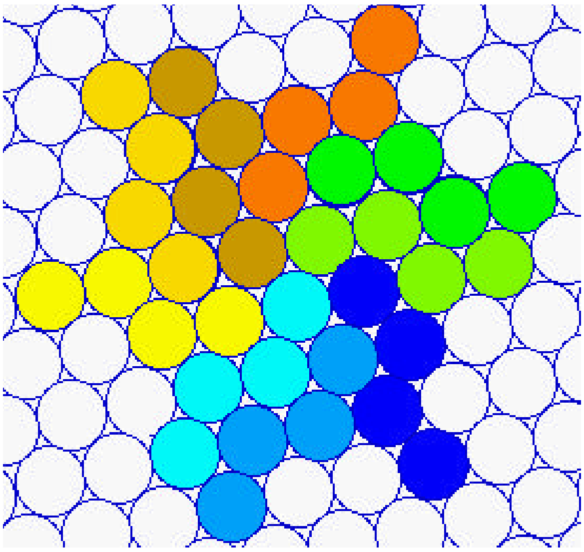
Now pick one circle and make it the centre, with a ring of circles around that, and another around that, on and on to fill out the plane. Your attention has changed in structure; your sense of the circles being filled out in rings is different from your sense of them filled out before. Pick two touching circles and see them as the centre of a ring of touching circles, and then an other ring and so on to fill out the plane. Now pick three circles mutually touching. See them as enveloped by a ring of touching circles, and then another ring, and so on. I suggest that you have to work at doing this, at holding the particular central circles still, and at getting a sense of the rings going out and out. But you can do it, and you are convinced that the whole plane is covered by such an arrangement.

In order to carry out those tasks, you had to shift the focus and locus of your attention, to stress (and consequently ignore) differently. To experience the generality you had to suspend your perception of the boundedness of the diagram, and 'see it' as extending indefinitely.

In the next diagram, see the shaded circles as a unit. Can you fill out the tessellation with copies of that unit?



Your sense of layers immediately permits you to see how to fill out the plane with this particular shape. Your attention is structured by your past experience and knowledge that layers tessellate. I refreshed that perception when I introduced the circles originally, and then shifted your attention away from layers to rings of circles surrounding a shape. You had to let go of the 'conditioning' of 'surrounding', and know-to think of layers. Of course your perceptual apparatus supports layers because our neuronal structure is stronger at seeing layers than at seeing 'surrounding'. Try this final experience. The next diagram shows the beginning of a filling-out of the circles using the one shape, but does it continue indefinitely? (The original is in colour.) Notice how much harder it is to see what is happening, to find structure in what is presented, and then to be sure.



Your recent experience seeing rings may have blocked you momentarily from seeing that this shape fills pairs of consecutive layers very easily, so of course it fills out all the circles. But can you make it the centre of a series of rings of such shapes? That is really difficult without drawings.

This experience is intended to provoke you into noticing how you have the power to stress (and consequently to ignore) selectively; that is, to alter the structure of your attention. Sometimes your attention was tightly focused on holding a shape, sometimes diffuse as you got a sense of filling out all the circles. Making yourself change the way you stress is one way to shift the structure of your attention, but there are many more. Experts can intentionally ignore (as they do by replacing chunks of symbols by something simpler in order to concentrate on form) as well as intentionally stress, but novices find this very difficult. To test this, put up a complicated expression involving functions, fractions, a surd or two, and the ask students what they see. Almost certainly they will mention the items they are not confident about. Whereas an expert can ignore what they are unsure of until they have learned more about it, novices tend to be frozen by the presence of uncomfortable items.

There is a great deal more to be said about the nature and structure of attention, but not room here. Suffice it to say however that considerations such as this have led me to some observations about attention (see also Mason & Davis 1989).

> Attention can be unifocal or multifocal (as when for example you use a diagram as the basis for 'adding' more elements to it or seeing it move or change in some way'). The focus of attention can be broad and fuzzy or narrow and sharp. Attention can also be single or split, as when you are simultaneously aware of (or rapidly alternate between) an image of some place you have been or would like to be, watching your fingers while typing on a machine, and aware that you are producing images on the screen in front of you. Attention can be centred on wholeness, on making distinctions, on actions (including making relationships and connections), on activity, on essence, and on the present moment. Finally, attention is framed by principles and slogans, perspectives and viewpoints.

I see the circles exercise as *generic*, because it struck me that when I suddenly see a different way to stress and ignore, I find myself wanting to ask others a question: do you see what I see? I also see the exercise as a *metaphor* for the way in which we are prompted into asking students questions: suddenly we see differently, and we are socialised into asking a question to provoke students to see what we are seeing.

The presence of technical terms in mathematics, as in other disciplines such as mathematics education, signal the fact that someone noticed their attention altered or shifted in some way, and found it convenient or important to label that shift. Thus *angle* signals an awareness of turning, and *integral* a process of limiting summation and area. This leads me to two conjectures:

> Every technical term signals a shift in the structure of attention
>
> For a student to use a technical term to express their own meaning, they have to experience a corresponding shift of their attention.

In other words, learning is more than 'doing what you are asked to do' as the *didactic tension* suggests. It requires attention to be restructured. Commonly this is accomplished through some sort of disturbance which students then make sense of by using their undoubted powers of construal. The question for us as teachers is whether we are provoking students into invoking those powers, or whether we often 'try to do the work for them', so training them into dependency on us and into a perception that mathematics is only about 'doing sums', as the English say.

What this means is that it is worth inspecting what it is that comes to mind as we contemplate a technical term, question, problem, or topic, and then work to provoke similar awarenesses to come to our students' minds. Questions are what we usually employ to do this focusing, although often the format is strictly an assertion ("See this", "Do that") when what we really mean is "Can you see this?" and "Can you do that?".

The shifts conjectures, like all those in this paper, are based on my observations and expressed in language I find comfortable but which is likely to be novel or uncomfortable for you. To be investigated and construed they need to be linked to your past experience, and then checked to see if they inform your future actions or help you make sense of experience.

QUESTIONING

The topic of this paper is questioning, putting forward the thesis that the style and nature of questions used influences the sense that students make of the subject matter, that the questions we ask are not always well prepared because we are stimulated to ask a question when we ourselves suddenly experience a shift of our own attention, and that they could be informed by working on questioning, informed by our sense of mathematics.

Briefly, by 'sense of mathematics', I mean the various themes and actions, heuristics and processes with which we are familiar, and the connections we are aware of between topics. Often those connections are via themes and processes. Example of themes include

Invariance Amidst Change

For example, the sum of the angles of a triangle is 180°, the sum is invariant over all triangles.

Doing & Undoing

For example, $\int_0^2 (1-x)dx = 0$; what similar integrals will also give 0? If multiplying numbers or polynomials is seen as 'doing', 'undoing' is factoring.

Freedom & Constraint

> For example, equations are constraints imposed on a set of free variables, as are geometrical construction tasks.

and examples of processes include

Imagining & Expressing

Specialising & Generalising

Conjecturing & Convincing

The more deeply we are imbued with these themes, with a rich collection of specific instances and generalised awareness arising from encountering and recognising them, the more likely they are to inform our questioning.

Some questions reflect mathematical thinking and mathematical structure. For example, 'characterise all those numbers which can be represented as one more than the product of four consecutive numbers' is typical of many questions in mathematics leading to important theorems. By contrast, many of the prompts used in classrooms are far from being questions in any sense of the word, with no resonance with mathematics what so ever. My approach to the didactic tension is not to try to avoid it, nor to construct teacher-proof scripts which will provoke students into learning (as I find this notion implausible), but rather to work on sensitising myself (and hence help others to sensitise themselves) so as to respond creatively in the moment with particular students, informed by prior analysis, by reflection and preflection, and by awareness. The matter of particular interest in this paper is the construction and handling of questions.

THREE FORMS OF ASKING

It has often been noted that question asking in an educational context seems bizarre when viewed by an outsider (Ainley 1987, Ginsburg 1996). For whereas questions are usually asked in order to find out the answer, teachers and teacherly adults ask questions to which they already know the answer to find out of children know the answer. Thus

> "What do we put here?"

when asked by an adult is more likely to be testing than seeking information. What is perhaps amazing, or salutary, is that children usually work out which kind of asking is taking place. By the time children become students at tertiary level they are well and truly enculturated into 'educational questions' and it takes some a very long time to recognise that they are required to think for themselves (Perry 1968, Copes 1982). Some never do.

In between the two types of questioning (testing and seeking) is a pattern of interaction in which the teacher asks a question and the students try to guess what is in the teacher's mind. As an observer, it is easy to characterise a teacher's questioning style as 'guess what is in my mind', but as a teacher it is much harder to recognise when it is happening, and even harder to do something about it.

Interrogating my own experience I have discovered that very often it is the response I get which informs me as to the purpose of my question. I am so caught up in the asking that I am unaware of what it is that drives me to ask and of the fact that I am expecting a particular answer. I am deeply dwelling in my perception and in what I am thinking. But when I hear the response, I immediately recognise, for example, that the response was not at all what I was looking for, was not what I was thinking, and so I become aware that I was looking for some particular response; or if the person is unclear or hesitant in response, that they may not be attending to the topic with the same stressing that I am; or if the person has mis-interpreted a genuine enquiry as a testing question.

I suggest that there are three forms of asking, depending on the purpose of the asker: focusing, testing, and enquiring. There is no overt or covert evaluation of these. Rather, the point is to be clearer when I am asking questions which mode I am in, and whether that is the mode I wish to be in.

Asking as Focusing

In the midst of an event, I suddenly become aware of a pattern, or a generality, or of a feature which my attention focuses upon. My attention has shifted, and I see things freshly. I want (I consider it might be valuable for) the student to focus on what I am seeing. So I ask a question. I could point out what I am attending to, but there is a commonplace in teaching that asking questions is better than telling. So I ask. But my questions are directed, focused on what I am thinking.

Heinrich Bauersfeld (1994) called the questioning which arises when a teacher sees something and tries to get students to see it by a series of indirect, but increasingly directed questions, *the funnelling effect*. It is sometimes effective, but if it becomes a habit, if there are no alternatives, then the teacher is trapped in a potential collusion with the students which is unlikely to enhance learning. John Holt (1964) describes a funnelling incident in which he eventually realised that the student had devised a strategy of minimal response until eventually the questions became trivial and required no effort to answer.

Funnelling can be avoided, but it has to be noticed; you have to catch yourself in the act, or better yet, before the act. If you discover that your questions are intended to funnel, to focus attention, there are some alternative strategies. You can point out what you are attending to, just as a parent might, on a walk, point out to a child birds, plants or buildings which strike them as being of possible interest to the child. You can change tack, or even change the mode of interaction, or even in some cases, change topic, so that you can approach it differently another time. Even in mid funnelling it is possible to break out by acknowledging the 'game' being played, and to tell the student what it is you want them to focus on.

Focusing questions can be used to good effect when they signal a change of focus of attention. For example, instead of always asking students for an example, or what they did last time, or what they will do next, I can ask, "What question am I going to ask you?". The purpose of such a question is to signal a metacognitive shift from current content to the sorts of questions I habitually ask, in order to encourage the student to ask themselves those questions. Hyabashi & Shigematsu (1988) coined the expression *the inner teacher* to refer to the monitor (Schoenfeld 1985, Mason *et al.* 1984), the personal witness (Mason & Davis 1989) which can be developed. This is one way to initiate fading, moving from directing attention, through indirectly prompting attention, to the student spontaneously attending for themselves (asking process-based questions, specialising, generalising, re-presenting in a different medium etc.). When the student takes over what the teacher has been doing, scaffolding can be said to have taken place, and the teacher can attend to more sophisticated awareness which again the student will be scaffolded-faded into doing for themself. This process of scaffolding and fading, of moving from directed through indirect prompts to spontaneous use by the student, gives new meaning to the adage

> Do for the students what they cannot yet do for themselves

for it is so tempting to 'move things along efficiently' by doing things for the students that they could in fact do for themselves. But if they do not learn to do them for themselves when the teacher is present, they are unlikely to know to do them for themselves when the teacher has gone.

Asking as Testing

The most common form of asking questions in an educational context is to test the comprehension of the students, to prompt them to rehearse their articulation of some ideas, to make connections and to gain facility. But facility means that attention required is minimised. In order to learn to use less attention (to knowing tables, factoring polynomials, knowing to work backwards or to use a diagram etc.), attention needs to be drawn away (Hewitt 1994). But working through a set of routine exercises is as likely to attract attention to the rapid completion of the task as it is to attract attention away, to prompt the student to let go of direct attention to detail. Primary teachers are very experienced in creating games and other situations in which children are called upon to exercise several skills in order to achieve an unrelated aim. By attracting attention away to some greater goal, which is the directing, guiding role that mental imagery plays in any case, the student can be induced to rehearse by letting go of attention rather than holding onto it. The injunction to 'do enough so that you know how to do this type of question' puts the onus on the student, and invites them to attend not to the particular question but to the general class of questions and to the corresponding techniques.

Asking as Enquiring

Genuine enquiry is an important state for children to recognise and internalise as socially valued. Consequently it is an important state for teachers to enact. But it is difficult to enquire genuinely about the answer to problems or tasks which have well known answers and have been used every year. However, it is possible to be genuinely interested in how students are thinking, in what they are attending to, in what they are stressing (and consequently ignoring). Thus it is almost always possible to ask genuine questions of students, to engage with them, and to display intelligent directed enquiry. For if students are never in the presence of genuine enquiry, but always in the presence of adults who know all the answers, then students are likely to form the impression that there is an enormous amount to know, and that adults already know it all, when what society wants (or claims to want) is that each individual learn to enquire, to weigh up, to analyse, to conjecture, and to draw and justify conclusions.

Students can be supported in learning to ask genuine questions of each other and of the teacher. At first, 'Can you say that again?', or 'What did you say?' may be taken as a reasonable question (student to student or student to teacher). But as a supportive ethos develops in which everyone takes what is said as a conjecture to be modified not a statement to be validated by the teacher, students can be called upon to 'say what they can say' in the expectation that someone will be able to help them complete it. Thus when students are eager to say 'Can I try to say that?' because they are not sure they have quite got it, instead of waiting to put their hand up when they think they have got it correct, asking becomes asking and not a cover for trying to hide ignorance or uncertainty.

POSING QUESTIONS MATHEMATICALLY

I have chosen three different domains for posing questions to students in such a way as to construct a mathematical environment: probing boundaries, story telling and probing essence.

Probing Boundaries

The questions and prompts offered here are taken from Watson & Mason (1998). They arise from considering some of the work of Zygfryd Dyrszlag (1984), a Polish mathematics educator. He produced a list of 63 questions (translated for us by Anna Sierpinska) which can be asked by teachers to promote and monitor development of students' concepts. Anne Watson and I found Dyrszlag's questions reflected our sense of mathematical thinking and mathematical structure, and so we were stimulated to try to locate some underlying themes and structure, in order to integrate his wealth of ideas into a useful collection for ourselves and for colleagues. Here are just a few based on the one theme of clarifying what properties make something an example but when modified make it a non-example or vice versa:

- *What must be added to, removed, or altered in order: to admit ... as well, to force ...* (e.g.. What must be altered in order to convert a definite integral into the area between the function and the x-axis?)
- *What can be added, removed, or altered without affecting ...* (e.g.. what features of a convergent series can be altered and it still converge?)
- *Tell me what is wrong with ...* (e.g.. what is wrong with the reasoning $\int_0^2 x(2-x)dx = 0$ because you evaluate the function as 0 at both limits.)
- *What needs to be changed so that ...?* (e.g.. ... so that $3x^2 + 2x + 1$ has real roots; so that it has a positive maximum at $x = -1/3$.)

Some questions are more directed at altering an aspect to see what effect it has:

What if ...? (e.g.. what happens to the roots of a quadratic if the coefficient of x changes at a constant rate?)

If this is the answer to a similar question, what was the question? (e.g.. if the derivative of a function is the function itself, what was the function? What if the derivative was the inverse of the function? If the derivative of $F(x)$ is $f(x)$, what function has $f(x) + 2x$ as its derivative?

Do .. in two (or more) ways. What is quickest, easiest, ...? (

Change ... in response to imposed constraints

Of what is this a special case? What happens in general? (e.g.. Rolle's theorem, Taylor's theorem)

Is it always, sometimes, never ...? (e.g.. ... the case that if a function has a derivative at a point it is continuous at all points nearby? ... that the centroid of a figure lies inside the figure? ... that $xf(x)$ is continuous at $x = 0$?)

Describe all possible ... as succinctly as possible. (e.g.. ... polynomials through (1, 0), (2, 0), and (3, 0))

Give me one or more examples of ... (e.g.. a function which is continuous at a point but fails to have a derivative there; ... a function which fails to be Lipschitz)

Describe, Demonstrate, Tell, Show, Choose, Draw, Find, Locate, an example of ...

Is ... an example of ...? (e.g.. ... (0.3 0.7,-0.7 0.3) is an example of a rotation matrix?)

What makes it an example? What is exemplary about ...? (e.g.. what makes the positive rationals an example of a group? What is exemplary about **Erreur!**as an example of an integral achieved by using integration by parts?

The point about these questions is that they reflect the thinking of mathematicians probing a colleague who is stuck on a problem or idea.

Story Telling,

Many authors (e.g. Bruner 1996) see narrative, or *story telling* as the core and essence of being human, and *learning to tell stories* as the essence of education. Feyerabend (1991 p141) suggested that

> "All you can do, if you really want to be truthful, is to tell a story"

while Maturana (1988 p39) goes so far as to suggest that

> "Reality is not an experience, it is an argument in an explanation".

In English the word *story* sometimes has the connotation of something made-up or even false. I use it here as another word for a *rendition* or *an account*. Thus, students may have stories

> for why particular topics are included in a course,
>
> for what each technique is used for,
>
> for the steps of that technique (what I call *incantations*) as well as for why the technique works,
>
> or what technical terms mean and for how they fit together (or don't).

In many cases those stories are rather impoverished. For example, their story about a given topic often defers to authority: "someone knows it is important", with "but I don't" *sotta voce* (Perry 1968). Authority is also used to justify why a technique works. Combining that with little or nothing in the way of incantation, the student is reduced to memorising the steps in order, and is in no position to modify or adapt if the situation demands it. In Maturana's terms, they are very far from the technique becoming part of their reality.

One example of a device that can be used is to provoke students into expressing their comprehension using technical terms from the topic.

Connecting Technical Terms

The students may be asked to offer some of the technical terms used in a topic, or a list may be provided. They are then asked to use those terms in a sentence of two in such a way as to show how the terms are connected, or how they are used in the topic. For example, in change of basis one might use terms such as

transition matrix	new basis	domain	standard basis
linear transformation	matrix	codomain	representation

Students would be encouraged to use the terms in a sentence or two which describes the process of changing basis. Students can be asked to write their sentences down, or to say them to a colleague. The colleague comments on the sentences, and together they refine them. Students can have recourse to the teacher, the sentences could be handed in or written on the board, or students encouraged to try them out to the whole class.

The technical terms can also be put on cards and used as a study device: the student selects some cards and tries to link them altogether in a meaningful sentence.

Some teachers express concern that in such an exercise students might not achieve a correct statement. But we know that students usually leave their classes with all manner of partial stories. By being explicit about sense-making students are more likely to engage more publicly in forging meaning and in getting assistance to check the meaning they have made. One of the aims of such story-telling work is to encourage students to ask a question by saying

"I can say some of it, but not all. If I start, can you help me when I get stuck please."

This is ever so much more productive than having students ask

"Could you explain please", or "Could you say that again please."

Exposing Incantations

Another context for story telling is with techniques. A tutor can carry out standard techniques in public (on a board or at an OHP), while at the same time exposing some inner incantations:

I always start with the inside bit ... then I take this over here ... now what do I want? ... Well, I know that ... so ... and so on.

At the very least it awakens some students to the fact that one might have an inner dialogue while performing a technique. If students do not try to take notes, but instead try to 'enter the screen' with the tutor going sufficiently quickly that the students cannot take notes, but can enter into the doing, they can get a sense of what it is like to have facility. These sessions we call 'technique bashing'. They last only 20 to 30 minutes, but are found to be very valuable by students who have been working on a topic for a while. It provides a change of energy, and an opportunity to see an expert make mistakes, check their work, and talk about the process.

PROBING ESSENCE

At the Open University we fill the text with tasks for students to carry out, because we have found that if they are to make sense of what we have to say, it helps if they have some recent and relevant experience on which to draw. So we look for tasks which are likely to bring students into contact with the fundamental features of the topic as starters, for tasks which will expose wrinkles and variations in mid topic, and tasks which probe understanding at the end or later.

Examples

Linear Algebra

Given a non-singular linear transformation from a space to itself, which matrices can represent that transformation? Given a non-singular square matrix, which linear transformations can that matrix represent?

Change of basis and finding a matrix to represent a linear transformation forms the core of most introductory linear algebra, and a real challenge to most students. Any student who can satisfactorily answer these questions really understands what is going on, for they have to have accommodated the apparent paradox inherent in the answer.

Elementary Group Theory

Let G be a finite group, and P the set of all subsets of G (the powers set of G). Which subsets S of P themselves form a group under the operation $A \bullet B = \{ab: a \text{ in } A, \text{ and } b \text{ in } B\}$?

Work on this task pretty much forces the student to try some examples to see what might be going on, and brings the student into contact with normal subgroups and cosets, thus probing the fundamental notions of elementary group theory.

Limits

A proper convergent sequence does not have its limit as a member of itself. A set L is the set of cluster points of a set X if L is the set of limit points of all proper convergent sequences in X. What subsets of the real line can be the set of cluster points of some subset of the reals.

Graphing

If a quadratic such as $x^2 - 6x + 1$ is translated upwards at constant speed, how fast do the roots approach each other and what happens when there are no longer any real roots? What if $x^2 + 2k\text{x} + k$ has k changing at constant speed?

Differentiability

What sorts of values do you expect the derivative of a function to have near a point at which it has derivative zero?

Show that x^2 Floor$(1/x^2)$ $(x \quad 0)$, 1 when $x = 0$ is differentiable at 0. Show also that the derivative can exist and be arbitrarily large, arbitrarily close to 0.

Differentiability

The function Mod (x) is differentiable everywhere except at 0. For each subset W of the integers, construct a function which is differentiable everywhere except at points in W. What sets can be the points of non-differentiability of an otherwise differentiable function?

Students have a great tendency to monster-bar, to dismiss a single counter-example as pathological and to carry on assuming that 'most' examples are 'healthy'.

Parameters

Find the set of points at which the curves $e^{\lambda x}$ have a tangent which passes through the origin.

Liz Bills in her thesis studied, among other things, the natural confusion many students have between parameters and variables. For example, one approach we have seen to this task obtains the right answer despite a confusion of particular and general as follows:

Slope of tangent is $y = mx$ because it goes through origin. So equation is $y = \lambda e^{\lambda x} x$ which meets $y = e^{\lambda x}$ at $\lambda e^{\lambda x i} x = e^{\lambda x}$ ie when $x = 1/\lambda$. Thus the point is $(1/\lambda, e)$ which lies on a straight line parallel to the x axis. Of course! All you have done is scaled the x axis for each different curve, so it is like recording the fact of all those scalings.

Principles

The principle at work seem to be to take a technical term, and to locate a general structure in which the technical term is involved or can be instantiated, and then seek to characterise corresponding structures, most probably connected to an ‘undoing’.

TENSIONS

There are two major tensions which pervade questioning: implictness and explicitness, and variation and consistency.

Implicitness & Explicitness

Some people feel that if you are too explicit about styles and types of questions (and heuristics and processes) with students, the students will pick them up only superficially, rather than internalising them for themselves. They prefer to be less explicit, so that students internalise the question styles, perhaps even without realising it. If however you are too subtle about styles and types of questions, many students may never notice the content and effect of the questions, and so not develop any coherent inner teacher (Hyabashi & Shigematsu 1988).

As an example, in the 1950’s it was common practice in Canadian schools to require students embarking on a geometry proof to write down and complete the phrases Given ..., To Prove ..., and Proof ... . Most students seemed to employ this as an empty form. They would fill it in, and only then start thinking, whereas the teachers saw it as an aid to focusing attention on what was relevant. In the 1980’s my colleagues and I started suggesting I Want ..., and I Know ... as useful questions to ask when you get stuck. These are no more effective than the earlier ones if they are used mechanically. but the difference is that rather than proposing these as a place to start, we worked on ways to sensitise students to have the advice come to mind when it was most needed. This is the difference between algorithmic procedures (Brousseau 1997 p40) and sensitive working with the psychology and epistemology of the students. As well as teaching a topic, the teacher evokes students’ powers and awarenesses, invokes their automated behaviours, and provokes their energies, by supporting the students in noticing opportunities to choose to act (Mason 1996).

The negative side of explicitness is that the questions can be used superficially, as part of the social code but without significant meaning. They can even become an object of derision if overplayed. The positive side is that without explicitness, most students who are fully engaged with the content will not even be aware of what question they are being asked much less of the action it provokes inside them; rather they will be fully absorbed in trying to answer the question, caught up in the provoked action and perhaps trying to work out what answer is expected.

Variation & Consistency

If every encounter with a teacher produces a fresh type of question, the student is unlikely to recognise any patterns or structure in interactions with the teacher. Thus they are unlikely to internalise those questions so that they generate them themselves. On the other hand, if the same questions are used over and over, students soon become dependent on the teacher asking the questions: often they will wait until asked.

A sequence of directed or focused questions which over time gradually become more general and more indirect as prompts until they disappear altogether can have the effect of transferring initiative from teacher to student. Wood, Bruner & Ross (1976) called this scaffolding, a notion which was developed by Bruner based on Vygotsky's notion of a *zone of proximal development*, and Brown, Collins, & Duguid (1989) included fading. The essential component is not the initial scaffolding but rather the subsequent fading.

SUMMARY

I have argued that asking questions of students is fraught with difficulties such as the funnelling trap, or 'guess what is in my mind'. Questions can be testing, but they can also be focusing and they can be part of genuine enquiry. The questions we ask of students can be significant in supporting the sense students make of mathematics and of how mathematics is done. The questions we ask are informed and triggered by our sense of mathematics, but even when a mathematical question is asked, we often only become aware of what we had in mind when we notice our reaction to students' response. I have suggested that by attending to the kinds of questions we are asking we can sensitise ourselves to the possibility of asking questions which more nearly reflect our own sense of mathematical structure. I have indicated the place that I see questions occupying in the larger scene of classroom interactions or modes of interaction between tutor, content and student, as well as within the epistemological aims of teaching.

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