Long-time stability of near-integrable Hamiltonian systems

1. Some context and setup:

We shall be interested in the long but *finite* time behaviour of the action variables p of a near-integrable Hamiltonian system, with Hamiltonian:

$$H(p,q) = h(p) + \epsilon f(p,q), \quad (p,q) \in \mathbb{R}^n \times \mathbb{T}^n,$$

where $\mathbb{T}^n$ is the n-torus ($\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$), and (p,q) are action-angle variables attached to the integrable Hamiltonian h (they are essentially uniquely defined). The problem of investigating the flow of Hamiltonians like H, with ϵ a small parameter, is the subject matter of Hamiltonanian perturbation theory. Here we shall be interested in *classical* perturbation theory (which started at least with the works of Lagrange and Laplace) whose purpose is to study the stability properties of the action variables p, which in celestial mechanics, under certain conditions, involve the semi-axes of the near-elliptic trajectories. Thus, looking at the long-time stability properties of these variables includes in particular the study of the stability of planetary systems, like our solar system. The basic reason why the action variables can be stable at all is that the p-component $\partial H/\partial q$ of the Hamiltonian vector field has zero average on the torus, so that the averaged system for the action variables reduces to $dp/dt = 0$. Note that the information on the stability of the action variables can then be fed back into the equation and yields information on the behaviour of the angles. We can now formulate more precisely the kind of results to be discussed; namely letting $(p(0), q(0))$ be some initial conditions for the motion governed by H, we want to show that:

$$||p(t) - p(0)|| \leq \mathcal{C}(\epsilon), \quad \text{for} \quad |t| \leq \mathcal{T}(\epsilon). \qquad (*)$$

Here $||\ .\ ||$ denotes the – say euclidean – norm of a vector in $\mathbb{R}^n$, and $\mathcal{C}(\epsilon)$ and $\mathcal{T}(\epsilon)$ are functions of ϵ which tend to 0 (resp. infinity) as ϵ tends to 0. In order to make this statement precise, and also to have a chance that it hold true, one has to tell what the Hamiltonian H should be like, i.e. what kind of a) smoothness properties are assumed for H, b) what is the integrable part h like; c) where the initial conditions $(p(0), q(0))$ are picked up in phase space; and of course d) choose some functions $\mathcal{C}$ and $\mathcal{T}$. This we shall do in the next section, leaving out of course many interesting cases. Let us add two remarks here; first we have chosen to look at *autonomous* Hamiltonian *flows*, but other contexts are in principle, if not always in practice, easy to accomodate, for instance non-autonomous flows and symplectic maps. Second and perhaps more fundamentally, these results should be contrasted with *geometric* perturbation theory, whose scope is to detect geometric objects invariant under the flow, two important examples being of course periodic orbits; studied in great details by Poincaré, and tori (KAM theory); although periodic orbits may appear as just one-dimensional tori, they actually enjoy many interesting specific properties.

Once a geometric object has been detected – say a torus – one has often *ipso facto* proved a result like $(*)$ for initial conditions lying on this object (say a KAM torus) and $\mathcal{T} = \infty$, i.e. for *any* time. So results like $(*)$ are in fact basically meaningful *outside* the objects detected by geometric perturbation theory, and the two kind of results are thus somehow complementary.

2. Results and the general Nekhoroshev theorem:

The easiest case to deal with is the linear one, that is when one takes $h(p) = \omega \cdot p$, where ω is a fixed n-vector and the dot denotes ordinary dot product. In more physical terms, one is interested in the perturbation of harmonic oscillators whose fequencies are given by the components of ω. Assuming that f is an analytic functions, stability depends primarily on the arithmetic properties of ω. Assume in particular that ω is nonresonant, i.e. that $\omega \cdot k \neq 0$ for any nonzero $k \in \mathbb{Z}^n$. Then $(*)$ will take place for any intial condition $(p(0), q(0))$ over any polynomially long time, i.e. with $\mathcal{T}(\epsilon) = c\epsilon^N$ for any N and c a constant (all such "generic" – ϵ-independent – constants will be denoted c); for $\mathcal{C}$, one may take $c\sqrt{\epsilon}$, but these two constants c do depend on N and the one in the expression of $\mathcal{T}$ increases dramatically with N. The proof of this result is quite simple and purely algebraic. It consists in a step-by-step construction of the linearizing transformation; the resulting series is usually called a Birkhoff series; in fact, basically the same construction is used when dealing with the neighbourhood of an elliptic fixed point, which was the case originally considered by Birkhoff. The nonresonance of ω garantees the existence of the series, which in turn immediately yields $(*)$ at any polynomial order, as stated above. This situation has been the object of an enormous amount of studies, and helped developing the general theory of normal forms.

At the extreme opposite is the fully nonlinear case with which we will be dealing from now on. Fully nonlinear also means fully anisochronous from the physicist's viewpoint, that is the frecquency $\omega(p) = \partial h/\partial p$, which is a constant ω in the linear case, does depend on p, and effectively so. The easiest way to make the latter precise is to require that the determinant of the matrix $A(p) = \nabla^2 h(p) = \partial\omega(p)/\partial p$ (i.e. the determinant of the Hessian matrix of h) be nonzero, in other words that the frequency map $p \to \omega(p)$ be locally invertible. This is in particular good enough for KAM theory. For long-time stability, one in fact needs more. By far the most common condition, both in terms of theory and examples effectively encountered is the convexity condition, i.e. assume that the –symmetric– matrix $A(p)$ is sign definite (say positive, because if not one may reverse the sense of time). This include in particular most forms of kinetic energies, even more particularly the simplest one, $h(p) = \frac{1}{2}p^2$, for which the Hessian matrix is of course just the identity. We return more specifically to the convex case in section 3, but here we shall state Nekhoroshev's theorem in its general form, as it was stated in [N1] and proved in [N 2,3] (very roughly speaking, [N2] contains the important ideas and [N3] the – sometimes very tedious – computations). It concerns the case when h is *steep*, a condition devised by Nekhoroshev in the context of singularity theory. We unfortunately have to refer the reader to [N 2,3] for the rather involved definition. It is a general condition, of which convexity is in principle a very special case, although steep nonconvex Hamiltonians are not often encountered in practice. For instance, given n (the number of degrees of freedoms of the problem), the minimal degree of a polynomial in n variables which is steep and nonconvex grows fairly rapidly with n. Steepness was introduced by Nekhoroshev in a C^∞ setting but his theorem uses only the analytic case, and steepness in the analytic case was nicely characterized by Illyashenko in [I]. Suffice it to say that it can be checked algebraically on a jet of sufficiently high – but not a priori bounded – order of a function.

Now assume that H is analytic (i.e. both h and f are) and h is steep; then Nekhoroshev proved that $(*)$ holds for any initial condition $(p(0), q(0))$, with $\mathcal{T}(\epsilon) = c\exp(c/\epsilon^a)$ and $\mathcal{C}(\epsilon) = c\epsilon^b$. This is loosely refered to as "stability over exponentially long times". Here analyticity cannot be dispensed with and is ultimately responsible for the occurence of the exponential function; the very general geometric condition of steepness would also be very difficult to weaken. The *Nekhoroshev exponents* a, b lie between 0 and 1 and depend on geometric parameters called steepness exponents, which are principle computable from h. The more important is a, governing the stability time. In the most important case, namely for convex h, Nekhoroshev found an upper bound $a \leq c/n^2$. The above result is all the more impressive that apparently *no* such stability estimate had ever been derived before, including in a restricted case and over much shorter timescales. The easiest case would be for instance to take a convex unperturbed part h and investigate stability over a time on the order of $1/\epsilon$; this of course does not require analyticity. It sounds like a surprising fact that to all appearances such a statement had never been proved before. One may perhaps say that after two centuries of perturbation theory and some twenty years after Kolmogorov's work on invariant tori, Nekhoroshev directly proved the best possible result (see below for some more on this) setting in some sense the limits of what is probably the oldest physico-mathematical theory.

One more word on the linear case: assume that $h(p) = \omega \cdot p$, and that H is analytic (i.e. f is analytic); assume moreover that the frequency vector ω satisfies a diophantine condition, that is one has: $\omega \cdot k \geq \gamma\, |k|^{-\tau}$ for all nonzero $k \in \mathbb{Z}^n$ and some positive constant γ and τ (of necessity $\tau \geq n-1$). Then one has an exponential stability estimate, namely $(*)$ holds with the same functions as in Nekhoroshev's theorem, and the exponent a depends only on τ; in fact one can take $a = \frac{1}{\tau}$ and this is probably – generically – optimal. One should however beware of the fact that this result, whose proof is in principle much simpler than that of the stability estimates in the *nonlinear* case, is also of a fairly different nature; we return to this briefly in section 3.

3. Stability for quasiconvex systems and optimality:

In the general steep case, the exponents a and b are connected with the so-called steepness exponents, but the whole situation is far from being clear. The optimal values of the exponents, in particular that of a which governs the maximal possible speed of Arnold diffusion, is not known. The situation in the convex case is much better. Here convex means again that the Hessian matrix of h is sign-definite. A minor generalisation is to require only quasiconvexity, namely that the level sets of h be (strictly) convex. Then one proves that one can take $a = \frac{1}{2n}$ (n being the number of degrees of freedom) and this value is most likely to be optimal. Moreover its meaning is made clear from the proof (we refer to [L1]; see also [L2] and [P]). But convex systems also enjoy a very specific property which draws a clear *qualitative* distinction from the rest of steep systems; it is the long-time stability of resonances (see [L1]; connections with previous more physics oriented works are presented there and in [L2]). It can be stated as follows: let $\mathcal{M}$ be a submodule of $\mathbb{Z}^n$, of rank m; let $S_{\mathcal{M}}$ be the corresponding resonance surface in action space, namely the set of p such that $\omega(p) \cdot k = 0$ for $k \in \mathcal{M}$; this defines a surface of dimension d with $d + m = n$ (m being the "multiplicity" of the resonance). Then the following local stability results holds: let $(p(0), q(0))$ be some initial conditions and assume that $p(0)$ is close to the surface $S_{\mathcal{M}}$, more precisely that it is $O(\sqrt{\epsilon})$ close. Then again $(*)$ holds with the Nekhoroshev functions for $\mathcal{C}$ and $\mathcal{T}$, but now we can take $a = \frac{1}{2d}$, i.e. we may replace n by d in the time exponent. As a particular case, one can take $d = 1$ (i.e. $m = n - 1$), corresponding to the unperturbed periodic orbits, or rather tori filled with periodic orbits. In that case, we get that the neighbourhood of these tori is stable over times on the order of $\exp(c/\sqrt{\epsilon})$. This particular case turns out to be quite important for proving the general result (see below). The stability of resonances has important physical and mathematical consequences, as for instance the application of these stability estimates to systems of very large (or even infinite) dimension, in which case the global estimate $a = \frac{1}{2n}$ is useless. We shall not discuss here the interesting phenomena connected with the value of b; only note that one has $b \leq \frac{1}{2}$ (in physical parlance, the width of the resonances is on the order of $\sqrt{\epsilon}$) and that one may take $b = a = \frac{1}{2n}$. It is unknown whether the pair $a = \frac{1}{2n}$, $b = \frac{1}{2}$ holds uniformly in phase space. Given the discussion of this point in [L1,2], the weak guess of the redactor is that this is not the case.

4. A few words on the proofs:

Apart from the concept of steepness as a very weak nondegeneracy condition, Nekhoroshev essentially introduced two ideas, one analytic, and one geometric. The analytic idea can be briefly stated as that of doing normal theory to a high order, this order being itself a function of the perturbation parameter ϵ. The easiest, yet relevant case occurs whith one-phase averaging, where no small divisors formally show up. It can also be applied to yield the stability estimate in the linear case, as stated above, with a diophantine condition on the frequency. In essence, this is the observation that the normalizing series are of Gevrey type. Only this analytic (or even algebraic) property is needed to prove the stability property in the linear case and this is why the latter should perhaps be called a Gevrey-type estimates. The second idea is geometric and consists in a construction which we cannot really summarize here. Nekhoroshev constructs local resonant normal forms for the system governed by H but it is far from clear a priori how to glue these patches together, in order to get a global stability estimate. Here we mean *any* estimate, including over a comparatively short time (say $O(1/\epsilon)$) and in fact, as noted in section 1, such estimates just did not exist prior to Nekhoroshev's work. The patching procedure is rather delicate but the crux of the matter is to show that a given trajectory will evolve so as to reach "less and less" resonant regions, until it gets trapped in a fully nonresonant region, where the normal form precludes any motion for a long time, just as in the nonresonant linear case. The proof is contained in [N2,3], and the general ideas are nicely stated in [N1,2]. The convex case was reworked in [BGG] which gave the first easy to read account of the work of Nekhoroshev and emphasized its relevance. In the work of Nekhoroshev and in [BGG], the exponent a in the convex case was estimated as $O(1/n^2)$ and the proofs were too inticate to guess what the optimal value could be (this is still the case in the general steep case); moreover the convex case appeared as the simplest – and most important – case, but only as technically easier to handle and not as qualitatively different, as explained above. Also the results concerned only global estimates in phase space (independently of the initial conditions). These proofs (and also [P]) use the classical tools of perturbation theory, including in particular Fourier series and estimates on the small divisors which occur.

The approach in [L1,2], which lead to the results briefly recalled in section 2 is completely different. One main idea is to introduce *simultaneous* approximation in Hamiltonian perturbation theory, whereas small divisors embody *linear* approximation. This reduced the analytic part of the proof to *one*-phase averaging, eliminating the use of Fourier series altogether. All that is needed is the case noted in section 2 of the stability of the neighbourhood of periodic orbits ($d = 1$ above) which amounts to a Gevrey 1 property of the normalizing series, in which no small divisors appear. The "geometric" part in turn is reduced to an application of the elementary Dirichlet theorem of approximation theory, which also yields a hoard of local variants – including the stability of resonances stated above; the use of more refined approximation theorems leads to some more intricate and detailed statements. So although this yields optimal statement for convex systems, explains their specifity etc. it actually bypasses the traditional ingredients of perturbation theory (largely shared by KAM theory), via the introduction of simultaneous approximation. It was then shown in [P] that one could improve the "classical" proof (i.e. that of Nekhoroshev himself) to the point of recovering most of the results of [L1,2], but it remains cumbersome and one has better know the result in advance in order to be able to setup the estimates properly. Also, this "classical" way does not make it at all clear why these estimates are actually very likely to be optimal; the story of the optimality however, is another one, having to do with "Arnold diffusion". Indeed, global instability switches on precisely when estimates like $(*)$ really break and action variables effectively start drifting by some finite amount (i.e. quantities which do not tend to 0 together with ϵ).

4. References:

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