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AUTOMORPHISMS OF PROFINITE AND PROCONGRUENCE CURVE COMPLEXES AND THE GROTHENDIECK–TEICHMÜLLER GROUP

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This paper is devoted primarily to the identification of the automorphism group for the profinite (and/or procongruence) completion of the curve complex $C(S)$ attached to an orientable hyperbolic surface of finite type S. It can be regarded as a sequel to the paper: Algebra i Analiz, **35**, no. 3 (2023), 57–137, where the author explored in particular (see Theorem 7.1 there) the rigidity of the completed *pants* (or maximal multicurve) complex $C_P(S)$. Roughly speaking $\mathrm{Out}(\widehat{C}_P(S)) = \mathrm{Out}(C_P(S)) = \mathbb{Z}/2$, where the outer automorphism group Out refers to the quotient of the automorphism group by the conjugacy action of the completed (respectively, discrete) Teichmüller (or mapping class) group $\Gamma(S)$. Here by contrast, it will emerge that $\mathrm{Out}(\widehat{C}(S)) = \widehat{GT}$ (say, if S is a punctured sphere with $n > 4$ punctures), the profinite version of the Grothendieck–Teichmüller group. Recall also that in Galois terms the arithmetic Galois group $G_{\mathbb{Q}} = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ is contained in $\widehat{GT}$ whereas $\mathbb{Z}/2 = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$. In passing, the geometric or topological emergence and meaning of the Grothendieck–Teichmüller group itself will be sdisplayed, emphasis on its natural relationship with the deformation theory, possibly also with the string topology.

Introduction

The present paper can be viewed as a sequel to [16]. In particular we return to the consideration of the same objects, which also constitute the subject matter of several recent papers, essentially with the same notation (see [2, 3]). We will thus be content with terse reminders as we go, referring the reader in particular to the Appendix of [16] if need be. We have also kept the reference list relatively short and have privileged recent (in 2024) articles whose reference list is itself easily accessible.

Let $S = S_{g,n}$ denote a differentiable and orientable surface of genus $g = g(S)$ with $n = n(S)$ points removed (punctures); we always assume it to be hyperbolic, i.e., $\chi(S) = 2 - 2g - n < 0$. We will say that $S = S_{g,n}$ is of type (g, n).

Ключевые слова: moduli space, modular dimension, orientable surface, orientation, simplicial complex.

To the surface S we attach the following objects: $\Gamma(S)$, $\mathrm{Mod}(S)$, $\mathcal{M}(S)$, $\mathcal{T}(S)$, $C(S)$, and $C_P(S)$. Here $\Gamma(S) \simeq \Gamma_{g,[n]}$ is the mapping class (or Teichmüller) group, i.e., the group of diffeomorphisms of S preserving orientation modulo isotopy, possibly permuting the punctures: $\Gamma(S) = \pi_0(Diff^+(S))$. We abbreviate $\Gamma_{g,0}$ to Γ_g and also use $P\Gamma(S) \simeq \Gamma_{g,n}$ the *pure* mapping class group, whose elements do *not* permute the punctures. This is a normal subgroup of $\Gamma(S)$ with quotient $\mathcal{S}_n$, the permutation group on n objects. In turn $\mathrm{Mod}(S)$ denotes the *extended* mapping class group, whose elements are isotopy classes of diffeomorphisms that may fail to preserve orientation. It is an extension of $\mathbb{Z}/2 = \{\pm 1\}$ by $\Gamma(S)$, which is split by any orientation reversing involution (mirror) of S.

The next item, $\mathcal{M}(S)$, denotes the moduli space of S, which from a point of view of algebraic geometry appears as a Deligne–Mumford (1-)stack defined over $\mathbb{Z}$; its complex analytic avatar classifies complex structures on S. It has dimension $d = d(S) = 3g - 3 + n$, which we call the modular dimension of S; it can be viewed as the (complex) dimension of the orbifold $\mathcal{M}(S)$. One has $\pi_1^{\mathrm{top}}(\mathcal{M}(S)) = \Gamma(S)$ where π_1^{top} denotes the topological (more accurately, orbifold) fundamental group of the complex orbifold $\mathcal{M}(S)$, the analytification of the complex points of the corresponding stack. Then $\mathcal{T}(S)$ is the Teichmüller space of S, a distinctly complex analytic object, which one can view as the (orbifold) universal cover of $\mathcal{M}(S)$.

Next the curve complex $C(S)$ is a $(d(S) - 1)$-dimensional simplicial complex whose k-simplexes ($k \geqslant 0$) are defined by a set of $k + 1$ incompressible and nonintersecting loops, where a *loop* is a simple closed curve drawn on S, modulo isotopy; loops are nonintersecting if there are representatives with this property. A loop is said to be *essential* if it does not cut out a disk minus one point on S, i.e., is not isotopic to a small circle around a single puncture of S. The set of essential loops (vertices of $C(S)$) is denoted by $\mathcal{S}$. For $\sigma \in C(S)$, we let $|\sigma|$ denote the number of curves that make up σ, so that $|\sigma| = dim(\sigma) + 1$ if $dim(\sigma)$ denotes the dimension of $\sigma \in C(S)$. In terms of modular dimensions, $d(S_\sigma) = d(S) - |\sigma|$.

Finally the "pants" complex $C_P(S)$ is a two-dimensional simplicial complex (for $d(S) > 1$) whose vertices are the maximal multicurves determined by sets of $d(S)$ nonintersecting simple curves, corresponding to the facets of $C(S)$ of maximal dimension ($= d(S) - 1$). For the full definition we refer to the Appendix of [16] or to [23, 8]. The main theorem of [8] asserts that for $d(S) > 1$ the pants complex $C_P(S)$ is (connected and) simply connected, which provides the basis of the Teichmüller lego developped in that paper. In [16] we also emphasized the role of three (pro)graphs, which carry essentially all the information, both in the discrete and in the complete cases. These are the respective 1-skeleta of $C(S)$ and $C_P(S)$ and their completions as well as the graph $C_\star(S)$ (and its completion), which for $d(S) > 1$ is noneother than the 1-skeleton of the simplicial complex dual to $C(S)$.

After this very short introduction of the main actors, let us recall some important results that were obtained in the discrete case in the eighties and nineties. More detailed reminders can be found again in [16], where a large part of these results were reproved by using markedly different methods (see §2 there). First it was shown in the eighties, by N. V. Ivanov and J. L. Harer independently, that $C(S)$ has the homotopy type of a wedge of spheres, an important and fundational result (see [15] for a more detailed history and references). Then came results about the automorphism groups of these objects. Actually the first result of that type was obtained by J. L. Dyer and E. K. Grossman who showed (American J. of Maths., 1981) that $\mathrm{Out}(B_n) = \mathbb{Z}/2 = \{\pm 1\}$ for $n > 2$, where B_n denotes the plane braid group on n strands. We will use the term *rigidity* for objects (groups or simplicial complexes) whose (outer) automorphism group is reduced to the identity or the action of the obvious involution (mirror image). The reader will find in [25] a detailed topological analysis patterned after the work of N. V. Ivanov and leading to the identification of the outer automorphism group $\mathrm{Out}(\Gamma(S))$ for all types of (orientable hyperbolic) surfaces of genus $g > 1$. Surfaces of genus 0 and 1 were dealt with somewhat later (see, e.g., [14]).

A few years later (around 1985) N. V. Ivanov (cf. [13] and references therein) proved another important result, namely that $C(S)$ is rigid. His result was completed in low (modular) dimensions by works of M.Korkmaz and F. Luo (cf. [14, 18]). Finally D. Margalit showed in [23] that the pants complex is also rigid (again, these results and somewhat more are reproved in [16] from a different angle). All in all, assume that the surface S is such that $d(S) > 1$ and is not of type $(1, 2)$ or $(2, 0)$, two exceptional types which are well understood (see below). Then:

$$\mathrm{Out}(\Gamma(S)) = \mathrm{Out}(C(S)) = \mathrm{Out}(C_P(S)) = \mathbb{Z}/2.$$

This useful but perhaps arguable piece of notation *Out* for a simplicial complex acted on by the group $\Gamma(S)$ will be further elucidated at the beginning of §3. We could and perhaps should write $\mathrm{Aut}(C(S))/\Gamma(S)$ where the action of $\Gamma(S)$ on curves is the natural one.

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Below, before going to the results concerning completions of these groups and simplicial complexes, we will recall, still in a nutshell, some necessary pieces of information.

1. *Completions*. Here we recall *very succinctly and informally* the definition of the completions we are going to manipulate. Much more (necessary!) "details" can be found in particular in [1, 2, 3] as well as in [15, 16]. Let G be a finitely generated discrete group and let Λ denote an indexing of the inverse system of its cofinite (i.e., finite index) normal subgroups G^λ, $\lambda \in \Lambda$. The (full) profinite completion of G, denoted by $\widehat{G}$, is defined as the following limit:

$$\widehat{G} = \varprojlim_{\lambda \in \Lambda} G/G^\lambda.$$

For any $M \subset \Lambda$, a sub inverse system, we can form $\widehat{G}^M$ defined by the same limit as above, but restricted to the $G^\mu \subset G$ with $\mu \in M \subset \Lambda$ (in particular $\widehat{G} = \widehat{G}^\Lambda$). We get a canonical onto map $\widehat{G} \to \widehat{G}^M$.

Next assume that G acts on some geometric object C. Below we use the case of $G = \Gamma(S)$ acting on $C = C(S)$ or $C_P(S)$, which are simplicial complexes. At this point one definitely needs to be more precise about the action, as is done in the papers mentioned above. In particular it is known that the action of $\Gamma(S)$ is virtually simplicial. Modulo some geometric properties, which are satisfied in these cases, one then defines the completion $\widehat{C}$ precisely as above:

$$\widehat{C} = \varprojlim_{\lambda \in \Lambda} C/G^\lambda.$$

Again for $M \subset \Lambda$ a sub inverse system, one defines $\widehat{C}^M$ by restricting the limit to the subgroups G^μ, $\mu \in M$ ($\widehat{C} = \widehat{C}^\Lambda$); there is a natural onto map $\widehat{C} \to \widehat{C}^M$.

2. *Center triviality.* In this and the next item, we will recall the few low-dimensional exceptions that literally do not enjoy certain otherwise general properties. This handful of exceptions is well understood and these cases have been described in every detail. We will therefore exclude them from some statements, for simplicity and in order to focus on the main phenomena. To start with, let $Z(G)$ denote the center of a group G. Then $\Gamma(S)$ has trivial center, that is $Z(\Gamma(S)) = \{1\}$), except for S of type $(0,4)$, $(1,1)$, $(1,2)$ and $(2,0)$. In the first case the center is of order 4 (Klein's Vierergruppe); in the last three cases it is of order 2, generated by the (hyper)elliptic involution. This property was extended literally to the congruence completions (see below) of these groups ([1], §4.3). The general case of the full *profinite* completion however, is still open: it is not known whether or not $\widehat{\Gamma}_g$, $g > 2$, has trivial center.

3. *Curve complexes and types*: For two surfaces S and S', the respective curve complexes $C(S)$ and $C(S')$ are *not* isomorphic, except for three exceptional isomorphisms: $C(S_{0,4}) \simeq C(S_{1,1})$, $C(S_{0,5}) \simeq C(S_{1,2})$, $C(S_{0,6}) \simeq C(S_{2,0})$ ([18], §2.2; the proof involves the Harer–Ivanov identification of the homotopy type

of $C(S)$). Again this continues to hold true for the procongruence completions $\check{\Gamma}(S)$ (see below) of the groups and thus also of the complexes (see [3], §5 or [16], §5.1). Note that in all the exceptional cases, here and in the preceding item, the procongruence and profinite completions coincide (see below).

4. *The congruence completion.* There is a particular subsystem of cofinite subgroups of $\Gamma(S)$, which deserves to be singled out, namely that generated by the principal congruence subgroups. Given the surface S, let $\pi = \pi(S) = \pi_1^{\text{top}}(S)$, its topological fundamental group and write $\Gamma = \Gamma(S)$. One has $\pi \simeq \pi_{g,n}$ with a well-known presentation; this group is free if $n > 0$. For any cofinite invariant (also known as characteristic) subgroup K, let Γ^K denote the kernel of the natural map $\Gamma \to \text{Aut}(\pi/K)$ defined via the action of the group Γ on π, hence also on the finite group π/K. Recall that since π is finitely generated, these cofinite invariant subgroups form a cofinal system in π. The G^K's are called principal congruence subgroups. A congruence subgroup is a subgroup that contains a principal congruence one. The congruence subgroups form an inverse subsystem M of cofinite subgroups of $\Gamma = \Gamma(S)$; the associated completion $\widehat{G}^M$, defined as above, will be denoted by $\check{\Gamma}$ and is called the congruence completion. We will also use a check to denote the corresponding completions $\check{C}$, especially with $C = C(S)$, $C_P(S)$, or $C_\star(S)$. Note that $\check{\Gamma}(S)$ can also be regarded as the closure in $\widehat{\Gamma}(S)$ of the image of the embedding (universal monodromy): $\Gamma(S) \hookrightarrow \text{Aut}(\widehat{\pi}(S))$ where $\widehat{\pi}(S)$ is the profinite completion of $\pi(S)$. In turn, given a complex structure on S, the latter can then be viewed as a complex algebraic curve $S_{\mathbb{C}}$, so that $\widehat{\pi}(S) \simeq \pi_1^{et}(S_{\mathbb{C}})$ is its *étale* fundamental group. We refer to [1, 3] and references therein for much more detail, as well as to [15], §§6, 7, for a broader, especially homotopy theoretic perspective.

5. *The congruence subgroup conjecture.* As mentioned above, for every S there is a natural map $\widehat{\Gamma}(S) \to \check{\Gamma}(S)$, which is onto. The congruence subgroup conjecture, proposed by N.V.Ivanov, asserts that this map is an isomorphism. This property would have very serious consequences, especially on the study of the moduli stack $\mathcal{M}(S)$, basically reducing the study of its*étale* covers to those of the surface S itself or an algebraic model of it. The validity of the conjecture *a priori* depends on the type (g, n) of the surface S. Write $CSP(g, n)$ (Congruence Subgroup Property) for the conjecture for type (g, n). Then in fact it depends only on the genus $g = g(S)$, so we can abbreviate it to $CSP(g)$. Here the *if* part of the statement, say that $CSP(g, n)$ implies $CSP(g, n + 1)$, is relatively easy, whereas the *only if* part is somewhat more tricky (see, e.g., [15], §7). The congruence conjecture has been vindicated for $g = 0, 1, 2$. We refer the reader to [2], whose author proved the conjecture for $g = 2$, for much more detail and references. So to-date (2024) we know that $CSP(g)$ holds true

for $g = g(S) \leqslant 2$; the cases of $g > 2$ however are still open. In particular the congruence subgroup conjecture holds true for $d(S) < 6$. For $d(S) = 6$, the only case where it remains to be vindicated is type $(3, 0)$. We do not know (in 2024) whether or not the surjecttive map $\widehat{\Gamma}_3 \to \check{\Gamma}_3$ is an isomorphism.

6. *Inertia preservation.* Both the Galois group and the Grothendieck–Teichmüller group act on the profinite Teichmüller groups $\widehat{\Gamma}(S)$ *preserving inertia*, that is the conjugacy classes of the procyclic subgroups generated by twists, which topologically generate $\widehat{\Gamma}(S)$. The name refers to the fact that these groups embody the inertia attached to the components of the divisor at infinity for a stable (Deligne–Mumford) completion of the moduli stack $\mathcal{M}(S)$. For a more general and algebro-geometric view of this setting, we refer the reader to [21] and references therein. We will denote the subgroup $\mathrm{Aut}(\widehat{\Gamma}(S))$ of the continuous automorphisms of $\widehat{\Gamma}(S)$ preserving inertia by $\mathrm{Aut}^*(\widehat{\Gamma}(S))$. This condition was detailed in the present setting in [3], §7.1; variants as well as various implications were discussed in great detail in [2], §3. We especially recall that for any S there is a natural embedding

$$\mathrm{Aut}^*(\widehat{\Gamma}(S)) \hookrightarrow \mathrm{Aut}(\widehat{C}(S))$$

of the (continuous) inertia preserving automorphisms of the profinite Teichmüller group $\widehat{\Gamma}(S)$ into the group of (continuous) automorphisms of the profinite curve complex. Ditto for the procongruence objects $\check{\Gamma}(S)$ and $\check{C}(S)$.

The above literally applies to the discrete setting: $\mathrm{Aut}^*(\Gamma(S))$ *a priori* denotes the subgroup of $\mathrm{Aut}(\Gamma(S))$ of the automorphisms that permute the conjugacy classes of twists. However in the discrete case, it is known that this is in fact the case of *every* automorphism, that is: $\mathrm{Aut}^*(\Gamma(S)) = \mathrm{Aut}(\Gamma(S))$ for all S. Indeed the proof of this very nontrivial property occurs as a milestone in the (original) proof of the rigidity of the group $\Gamma(S)$ (see in particular [25]). This is why we did not mention it before. The situation in the completed case is far less clear. It was shown in [9] (in a slightly different language) that in fact $\mathrm{Aut}^*(\widehat{\Gamma}(S)) = \mathrm{Aut}(\widehat{\Gamma}(S))$ when $g(S) = 0$, to wit, every automorphism of the profinite Teichmüller group of a punctured sphere preserves inertia. For $g > 0$ the possible coincidence of these groups is still open (2024).

7. *Automorphisms of complexes vs group automorphisms.* Given the above, we see that the identification of the group $\mathrm{Aut}(\widehat{C}(S))$ implies more or less immediate corollaries concerning the group automorphisms, i.e., $\mathrm{Aut}^*(\widehat{\Gamma}(S))$. There is much more to it however, namely roughly speaking there is an embedding $\mathrm{Aut}^*(\widehat{\Gamma}^\lambda(S)) \hookrightarrow \mathrm{Aut}(\widehat{C}(S))$ for every open subgroup $\widehat{\Gamma}^\lambda(S)) \subset \widehat{\Gamma}(S)$. The exact statement and the proof can be found in [16], §6.2, to which we generally refer in connection with this item. It may however be worth underlining the crux

of the matter. The point is that given two inertia (and decomposition) groups attached respectively to twists τ_α and τ_β along the simple closed curves α and β, the intersection of these groups is open in either of them if and only if the curves coincide ($\alpha = \beta$). This in turn is reminiscent of, in fact essentially identical to a main tenet of anabelian geometry, pertaining to the so-called "local theory". It can be for instance spotted, albeit in a much more algebraic setting, in early work of A. Tamagawa, indeed already in the pioneering articles of J. Neukirch who was the first to propose and show that the inertia subgroups inside $G_{\mathbb{Q}}$ (which are the conjugacy classes of the Galois groups of the p-adic fields $\mathbb{Q}_p$ for all the primes p) can be reconstructed group theoretically from the profinite group $G_{\mathbb{Q}}$. That was in the late sixties, a good fifteen years before Alexandre Grothendieck conceived of "anabelian geometry" (including its very name). Among other things it lead to a proof of the rigidity of $G_{\mathbb{Q}}$ ($\mathrm{Out}(G_{\mathbb{Q}}) = \{1\}$).

So in fact, the knowledge of something about the automorphisms of the curve complex enables us to determine or at least bound the (inertia preserving) automorphism groups of all the open subgroups of the Teichmüller group. In the text we will not elaborate on this aspect as much as we could, avoiding to work out the necessary details (which for that matter are really "details") and focussing rather on general phenomena. Let us however in this vein quote Ivanov's Theorem 2 in [13]. This is of course the discrete case and N. V. Ivanov also did not treat some simpler low genus cases (for which see [14]). So fix S and assume that $g(S) > 1$ for simplicity. The main result in the discrete case asserts, as mentioned above, that $C(S)$ is rigid, that is $\mathrm{Out}(C(S)) = \mathbb{Z}/2$ or equivalently $\mathrm{Aut}(C(S)) = \mathrm{Mod}(S)$, the extended mapping class group.

Then we can state ([13], Theorem 2). Let Γ_1 and Γ_2 be two cofinite subgroups of $\Gamma = \Gamma(S)$; then any isomorphism $\Gamma_1 \to \Gamma_2$ is induced by a conjugacy ($x \mapsto gxg^{-1}$) by an element $g \in \mathrm{Mod}(S)$. In particular, for a cofinite subgroup $\Gamma' \subset \Gamma$, the outer automorphism group $\mathrm{Out}(\Gamma')$ is finite.

[Indeed if Γ' is normal cofinite in $\mathrm{Mod}(S)$ with trivial center, $\mathrm{Out}(\Gamma') = \mathrm{Mod}(S)/\Gamma'$.]

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This completes our brief tour of some of the notions which are useful to keep in mind. Now to quote one of the main results of the present paper in a case which is particularly clear, yet quite important and telling, fix $S = S_{0,n}$, a sphere with $n > 4$ punctures. We show that:

$$\mathrm{Out}(\widehat{C}(S)) = \widehat{GT},$$

where $\widehat{GT}$ denotes the profinite version of the Grothendieck–Teichmüller group. We will include a short reminder in due time but here we should recall that $\widehat{GT}$ ($\supset G_{\mathbb{Q}}$) is a profinite group, which is *not* (topologically) generated and is *not* the profinite completion of any (known) discrete group. Also, from the above we see that the full profinite and the procongruence topologies coincide in this case and that the center of $\widehat{\Gamma}(S) \simeq \widehat{\Gamma}_{0,[n]}$ (as well as that of $\widehat{\Gamma}_{0,n}$) is trivial. It is also interesting to note that $\widehat{C}(S_{1,2}) \simeq \widehat{C}(S_{0,5})$, $\widehat{C}(S_{2,0}) \simeq \widehat{C}(S_{0,6})$. Here again $\mathrm{Out}(\widehat{C}(S)) = \mathrm{Aut}(\widehat{C}(S))/\widehat{\Gamma}(S)$; the quotient on the right-hand side will be justified at the beginning of §3.

This result is in stark contrast with the rigidity of the pants complex, as shown in [16] which states that:

$$\mathrm{Out}(\widehat{C}_P(S)) = \mathbb{Z}/2.$$

It also contrasts with the discrete case, both complexes then being rigid ($\mathrm{Out}(C(S)) = \mathrm{Out}(C_P(S)) = \mathbb{Z}/2$).

Perhaps the real point of the present paper, beyond the precise results it contains, consists of looking again at the *raison d'être* of the Grothendieck–Teichmüller group and the attending theory, pursueing the track initiated in [16]. Let us thus close this introduction with a few general and historical words. One of the ways Grothendieck–Teichmüller theory emerges is through the above constrast between both the discrete setting and the curve *vs* pants complexes, as hopefully will become clearer below. Now, completion is the necessary gesture bridging the gap between topology proper and arithmetic geometry, much as is the case between the theory of the topological fundamental group and that of its algebraic (*étale*) counterpart, especially for normal schemes overs separately closed fields. Also the Grothendieck–Teichmüller theory surely originates from Grothendieck's famous *Esquisse d'un programme*, but it was also conceived by V. G. Drinfel'd in line with his groundbreaking work on quantum groups. In fact, in the early sixties already, M. Gerstenhaber was studying the problem of "deforming associativity". He discovered among other things that the infinitesimal deformations are classified by the 2nd (with the appropriate grading) Hochschild cohomology groups of associative algebras. In essence the Grothendieck–Teichmüller group is also directly concerned with this deformation problem. An element is determined by two parameters, λ and f in Drinfel'd notation which may fail to be optimal but has become traditional. By far the most important and mysterious is the second one, $f \in \widehat{F}_2'$, the derived subgroup of the profinite completion of the free group on two generators. Actually $\lambda \in \widehat{Z}^{\times}$ governs the deformation of commutativity and f that of associativity; moreover f determines λ up to sign. The conclusion is that the present paper is less concerned with the arithmetic or the Galois side of the Grothendieck–Teichmüller theory, stemming from the inclusion $G_{\mathbb{Q}} \subset \widehat{GT}$, with possible equality, than with its proximity with deformation theory (see especially the end of §4 below). One could actually also keep an eye on (a possible completion of) the string topology, in which the S^1 or circle action, much as in cyclic (co)cohomology, plays a role that is not unlike the action of $\lambda \in \widehat{Z}^{\times}$ (or $\mathbb{G}_m$ in terms of algebraic groups) in the Grothendieck–Teichmüller theory.

§1. Injectivity, induction and the two-level principle

We fix a surface S of type (g, n) and recall the notation $S_\sigma = S \setminus \sigma$ for S slit along a set of $|\sigma| = k+1$ mutually nonintersecting loops ($k \geqslant 0$), that is, a k-simplex of the curve complex $C(S)$. As mentioned above we have formally added the case of $|\sigma| = 0$, $S_\varnothing = S$. Note that $g(S_\sigma) \leqslant g(S)$ for every $\sigma \in C(S)$. A *subsurface* T of S is of the form $T = S_\sigma$ for some $\sigma \in C(S)$; a *piece* of a surface is a connected component of a subsurface. Such a piece $S' \subset S$ is incompressible, that is, each boundary circle of S' is an essential loop on S.

In this section we will use the procongruence completion, both for groups and simplicial complexes. Recall two main properties of this completion. First there are isomorphism results (essentially due to M.Boggi) between three possible definitions of the completions $\check{\Gamma}$ and $\check{C}(S)$ in particular (see [16], §3 and references therein). These in particular make it possible to speak, without ambiguity, of (procongruence) loops and twists. The other crucial property of the procongruence completion lies in the fact that the trace on S_σ of the procongruence completion $\check{\Gamma}(S)$ (respectively, $\check{C}(S)$) is none other than the procongruence completion $\check{\Gamma}(S_\sigma)$ (respectively, $\check{C}(S_\sigma)$). For a precise statement and proof, see [16], Proposition 5.8. The corresponding statement is not known for the (full) profinite completion when $g = g(S) > 2$ and is equivalent to the validity of the congruence conjecture, i.e., to $CSP(g)$ (see above, item 5 in the Introduction). Actually the statements and proofs in this section concerning completions literally apply to any profinite completion Γ' of $\Gamma = \Gamma(S)$ that is residually finite (i.e., the natural map $\Gamma \mapsto \Gamma'$ is into) and satisfies the above property, i.e., the trace of the $\Gamma'(S)$-topology on S_σ is $\Gamma'(S_\sigma)$. It also applies to the "noncompletion", i.e., to the discrete group $\Gamma(S)$ itself, thus providing, together with the next section, another proof of the rigidity of the complex $C(S)$.

Below we will as much as possible reduce the problem to the case when S has modular dimension 2 ($d(S) = 2$). That this is at all plausible stems from a remarkable intuition of Grothendieck in the *Esquisse*, embodied in the so-called two-level principle ("*principe des deux premiers étages*"), which already has several incarnations (see [15] and references therein). In particular, the very existence and definition (in [4]) of the Grothendieck–Teichmüller group finds its origin there.

Associated with a subsurface S_σ, for some $\sigma \in C(S)$, is a natural inclusion $C_*(S_\sigma) \subset C_*(S)$ of *graphs*: $C_*(S_\sigma)$ is the full subgraph of $C_*(S)$ whose vertices correspond to those pants decompositions of S which include σ. In particular if σ is a maximal multicurve (pants decomposition), S_σ is a disjoint union of *trinions* (pants) and $C_*(S_\sigma)$ is reduced to a point. A similar description clearly holds for the pants graph (i.e., the 1-skeleton of $C_P(S)$) of a subsurface. The necessary material and more can be found in [16], especially in its Appendix and §2. Here we recall that $C_*(S)$ and the graph $C_P(S)$ share a common set of vertices, namely the set of pants decompositions (maximal multicurves) of S. It is also useful to recall that any simplex of the completed complex $\check{C}(S)$ lies in the $\check{\Gamma}(S)$-orbit of a simplex of the discrete complex $C(S)$; the finitely many orbits under the action of $\check{\Gamma}(S)$ are in one-to-one correspondence with the $\Gamma(S)$-orbits in the discrete case. This makes it possible to speak of the *topological type* of a simplex of $\check{\sigma} \in \check{C}(S)$, defined as the type of S_σ where $\sigma \in C(S)$ is any discrete simplex in the $\check{\Gamma}(S)$-orbit of $\check{\sigma}$. Note that S_σ may fail to be connected so that its type is defined as the (unordered) finite list of the types of its connected components. We refer to §A.9 in [16] for a few easy remarks about nonconnected surfaces.

We say that two simplexes $\rho, \sigma \in C(S)$ are *compatible* if the curves that compose ρ and σ do not intersect properly, that is they either are disjoint or coincide. Complex theoretically, this means that ρ and σ lie in the closure of a common higher dimensional simplex of $C(S)$. If ρ and σ are compatible, we define their union and intersection $\rho \cup \sigma, \rho \cap \sigma \in C(S)$ in an obvious way. We can now rephrase our goal in this section as comparison between the groups $\mathrm{Aut}(\check{C}(S))$ and $\mathrm{Aut}(\check{C}(S_\sigma))$, for $\sigma \in C(S)$. We see that $d(S_\sigma) = d(S) - |\sigma|$ is the sum of the dimensions of the components and we will assume that there is at least one piece (connected component of S_σ) with dimension at least 2. This may sound a little awkward but is in fact completely in line with the overall philosophy: the complexes associated with surfaces of dimension 1 simply do not carry enough structure. More technically, this is also optimal: if for instance a surface of type $(0, 6)$ is cut into two pieces of type $(0, 4)$, the automorphism groups attached to the surface and the subsurface are wildly different. By induction we can restrict attention to the case when σ consists of only one curve, in which case we use the notation $\alpha \in \mathcal{S}$ rather than $\sigma \in C(S)$. Now if $d(S) > 2$ and S_α is connected, that is if α is nonseparating, we get $d(S_\alpha) > 1$ and we expect an injectivity statement of the sort stated in Theorem 1.2 below.

But if for instance $g(S) = 0$, all curves are separating. Complexes suggest what seems to be the right notion as follows.

Definition 1.1. Given a connected surface S, a curve $\alpha \in \mathcal{S}$ is said to be *complex theoretically nonseparating* (CTNS) if either it is nonseparating in the usual sense (S_α is connected) or one of the two components of S_α is a trinion (i.e., of type $(0,3)$).

The rationale for the above definition and the denomination is as follows. If α is CTNS, then either S_α is connected or $S_\alpha = S' \coprod S''$ with — say — S'' a trinion. Then it is not quite true that $C_*(S_\alpha) \simeq C_*(S')$ but it *is* true that these complexes and their profinite completions have the same automorphisms if $d(S) > 1$. In fact S'' simply adds one point which is connected to all vertices and is left fixed by any automorphism, provided $d(S) > 1$. So in both cases $\mathrm{Aut}(\check{C}(S_\alpha)) = \mathrm{Aut}(\check{C}(S'))$ is the automorphism group associated with a connected surface of one less dimension. Of course this holds true for the discrete groups as well. This can be extended to non connected surfaces in a fairly obvious way but we will not require such an extension.

So the notion of a CTNS curve seems to be the right one when we deal with the general case. It is also interesting to check that it is in accordance with the exceptional isomorphisms in (modular) dimensions 1, 2, and 3, namely: $C(S_{0,4}) \simeq C(S_{1,1})$, $C(S_{0,5}) \simeq C(S_{1,2})$, and $C(S_{0,6}) \simeq C(S_{2,0})$. In the one dimensional case all curves are CTNS, and so it is in dimension 2, including the *separating* curves for type $(1,2)$. Then in dimension 3, the separating curves for type $(2,0)$, which are the only non-CTNS curves, break the surface into two pieces of type $(1,1)$ and correspond to the only type in $(0,6)$ that is *not* CTNS, namely those curves which cut the surface into two pieces of type $(0,4)$. We are ready to state the first main result.

Theorem 1.2. *Let S be a connected hyperbolic surface of finite type with $d(S) > 1$ and let $\alpha \in \mathcal{S}(S)$ be a complex theoretically nonseparating curve. Let then $F \in \mathrm{Aut}(\check{C}(S))$ be an automorphism of the profinite curves complex fixing the curve α. If F restricts to the identity on $\check{C}(S_\alpha)$, then $F \in \langle \tau_\alpha \rangle$, the procyclic group generated by the twist along α.*

Here we view $\langle \tau_\alpha \rangle$ as a subgroup of $\mathrm{Aut}(\check{\Gamma}(S))$, hence also as a subgroup of $\mathrm{Aut}(\check{C}(S))$; note that even if $\check{\Gamma}(S)$ (equivalently $\Gamma(S)$) has nontrivial center, $\langle \tau_\alpha \rangle$ injects into $\mathrm{Aut}(\check{\Gamma}(S))$. In the statement itself we have — implicitly — used results from [1] (and references therein) and [16], identifying $\mathrm{Aut}(\check{C}(S))$ with $\mathrm{Aut}(\check{C}_*(S))$ and using that the closure of $C_*(S_\sigma)$ in $\check{C}_*(S)$ is isomorphic to $\check{C}_*(S_\sigma)$. This is Proposition 5.8 in [16] and the place where we need to use the congruence rather than the full profinite completion (see also at the beginning of the present section). We should also recall that it is known ([16], Theorem 6.4 or [3]) that automorphisms of $\check{C}(S)$ preserve the topological type of curves; this is true but empty in dimension 1 and it breaks only for S of type $(1,2)$, in the usual, mild and well-understood way. We may however leave this case aside anyway because of the isomorphism $C(S_{1,2}) \simeq C(S_{0,5})$. Then starting with an arbitrary F, $F(\alpha) = g \cdot \alpha$ for some $g \in \check{\Gamma}(S)$ because F is type preserving and replacing F by $F' = g^{-1} \circ F$, we cantest the statement on F'.

This is a typical injectivity result, whose proof will be complete only at the end of §2, and yet another embodiment of the two-level principle, perhaps more geometric or conceptually more satisfying than the existing ones. For instance, consider a *group* automorphism $F \in \mathrm{Aut}^*(\check{\Gamma}(S))$, let α be a CTNS curve and assume that F restricts to the identity on the centralizer of the corresponding twist τ_α. Then F induces an automorphism of the complex $\check{C}(S)$ and, applying the statement above, we find that F reduces to the conjugacy by a power of τ_α. This would work with an inertia preserving automorphism of any open subgroup of $\check{\Gamma}(S)$, if we use a finite power of τ_α lying in that subgroup (which of course always exists). Here we are — again implicitly — using results about centralizers of twists (see [16] , §4 and references therein).

Now let (C_d) denote the claim of Theorem 1.2 for $d(S) = d$. Note that (C_0) is empty, whereas (C_1) is *false*. It turns out that, contrary to what happens with most inductive proofs, settling the initial case (C_2) is precisely as difficult as completing the inductive step. In this section we prove the inductive step. The initial case (C_2) will be dealt with in the next section. In other words, for the time being we are interested in showing the following.

Proposition 1.3. *For $d > 2$, assertion (C_{d-1}) implies assertion (C_d).*

The proof uses a topological property, which we first proceed to state. For a connected surface S, let $C_{ctns}(S) \subset C^{(1)}(S)$ be the full subgraph of the 1-skeleton of $C(S)$ whose vertices are given by the CTNS curves. In other words, the vertices are all CTNS curves and two vertices are joined by an edge if the corresponding curves do not intersect. We will also consider a subgraph of $C_{ctns}(S)$, say $C^{\vee}_{ctns}(S)$, defined as follows. Recall that a pair of curves $\alpha, \beta \in \mathcal{S}$ is called a *cut pair* if neither of the two curves is separating but their union is. We let $C^{\vee}_{ctns}(S)$ be the subgraph of $C_{ctns}(S)$ obtained by removing all edges corresponding to cut pairs. We need the following statement.

Proposition 1.4. *The graph $C_{ctns}(S)$ is connected if $d(S) > 1$; its subgraph $C^{\vee}_{ctns}(S)$ is connected if $d(S) > 1$ and S is not of type $(1,2)$.*

Proof of Proposition 1.3**, granted Proposition** 1.4**.** We start with $F \in \mathrm{Aut}(\check{C}(S))$, where S is connected hyperbolic of finite type and of dimension

$d = d(S) > 2$. We assume that F fixes α and restricts to the identity on $\check{C}(S_\alpha)$ for a CTNS curve α. We write S'_α for the connected component of S_α that is of dimension of $d-1$; in particular, if α is nonseparating in the usual sense of the word, $S'_\alpha = S_\alpha$. Consider now β, a curve such that α and β are joined by an edge in $C^\vee_{ctns}(S)$, that is β is also CTNS with respect to S, it is disjoint from α, and (α, β) is not a cut pair. Then we can consider the subsurface $S_{\alpha,\beta}$ obtained by cutting S along α and β, which is the same as cutting S'_α along β because β cannot live on a trinion if there is any. Recalling that $d(S) > 2$, let $S'_{\alpha,\beta}$ be again the only connected component of $S_{\alpha,\beta}$ that is not a trinion. Now reverse the order of the operations, denoting S'_β the component of S_β, not a trinion. The surface S'_β has dimension $d-1$ and $S'_{\alpha,\beta}$ is noneother than S'_β slit along α; possibly a chopped off trinion is discarded as usual.

We wish to apply the induction hypothesis to the surface S'_β, of dimension $d-1$, and the curve α. First α is indeed CTNS with respect to S'_β. In order to check this, two cases are to be excluded, bearing in mind of course that α and β are disjoint and that α is CTNS with respect to the original surface S. First α could cut off a trinion from S but not from S'_β; however this is clearly impossible. Second α could be nonseparating for S but separating for S'_β; but then β has to be nonseparating for S and (α, β) would be a cut pair for S, a case which is excluded by assumption. So we can indeed apply the inductive hypothesis to the pair (S'_β, α) and we see that, after possibly composing by a (profinite) power of the twist along α, the automorphism F, which fixes β because $\beta \in C(S_\alpha)$, actually fixes $\widehat{C}(S_\beta)$; that is, it fixes all (pro)curves that are disjoint from β.

Next we may replace α by β and do as above, except that now we clearly do not have to correct F by a twist along β. Proceeding in this way and applying Proposition 1.4 we conclude that F restricts to the identity on $\widehat{C}(S_\gamma)$ for *any* CTNS curve $\gamma \in \mathcal{S}$. Finally let $\delta \in \mathcal{S}$ be a curve which is *not* CTNS; by considering the subsurface S_δ, one finds a CTNS curve γ disjoint from δ. As a result $\delta \in C(S_\gamma)$ and so is fixed by F. So F is the identity on $C(S)$ which is dense in $\check{C}(S)$, hence $F = 1$, which finishes the proof of the proposition, modulo Proposition 1.4. □

We now return to the:

Proof of Proposition 1.4. The statement is classical in the sense that quite a few complexes derived from $C(S)$ have been introduced and shown to be connected. Let us start with the low dimensional cases. If $d(S) = 1$, $C^\vee_{ctns}(S) = C_{ctns}(S)$ is the full complex $C(S)$, which is not connected. If S is of type $(0, 5)$, again $C^\vee_{ctns}(S_{0,5}) = C_{ctns}(S_{0,5}) = C(S_{0,5}) = \check{C}$ which is connected. When S

has type $(1,2)$, one still has the identity $C_{ctns}(S_{1,2}) = C(S_{1,2}) = \check{C}$ so that $C_{ctns}(S_{1,2})$ is connected. But $C^{\vee}_{ctns}(S_{1,2})$ is not, because one cannot join the two elements of a cut pair inside it.

Next, for $d(S) > 2$ we first reduce the issue to the first assertion of the proposition. This reduction amounts to showing that with this dimensionality assumption two elements of a cut pair *can* be joined inside $C^{\vee}_{ctns}(S)$. So let (α, β) be a cut pair. The subsurface $S_{\alpha,\beta}$ has two components $S'_{\alpha,\beta}$ and $S''_{\alpha,\beta}$, whose dimensions add up to $d(S) - 2 > 0$. So at least one of them, say $S'_{\alpha,\beta}$, has strictly positive dimension. If $S'_{\alpha,\beta}$ has strictly positive genus, take γ a nonseparating curve of $S'_{\alpha,\beta}$ and the path (α, γ, β) connects α to β inside $C^{\vee}_{ctns}(S)$. If on the other hand $S'_{\alpha,\beta}$ has genus 0, take γ on $S'_{\alpha,\beta}$ bounding a trinion not containing α and β. Again the path (α, γ, β) then connects α to β inside $C^{\vee}_{ctns}(S)$.

So the matter is now reduced to proving that the graph $C_{ctns}(S)$ is connected if $d(S) > 1$ (the case of $d(S) = 2$ was actually dealt with above). We will use a technique which in essence is fairly classical (see in particular [27]). One could however use other and in fact more powerful approaches; in particular, "Putman's trick", which provides a general way of showing the connectivity of complexes with group actions, does apply to the present situation (see [31]).

Curves cutting off trinions will be called *trinion curves* for short. We write as usual $i(\alpha, \beta)$ for the intersection number of the two isotopy classes of curves α and β and assume (often implicitly) that *tight* representatives have been chosen, that is actual curves whose geometric intersection number is exactly $i(\alpha, \beta)$. We use induction on $i(\alpha, \beta)$, the cases where $i(\alpha, \beta) = 0, 1$ being easy. So, given $\alpha, \beta \in C_{ctns}(S)$ with $i(\alpha, \beta) > 1$, we need only find $\gamma \in C_{ctns}(S)$ with $i(\alpha, \gamma) < i(\alpha, \beta)$ and $i(\beta, \gamma) < i(\alpha, \beta)$; we will actually achieve more. In order to construct γ, we proceed as follows. First we define a *wave* β' of β (w.r.t. α) as a segment of β between two intersection points, that is $\beta' \cap \alpha = \partial\beta$ (we borrow the terminology from [MS] but give it a looser meaning: we do not require any additional property on β'). Now start from α and a wave β' of β. Let α' and α'' be the two segments of α cut out by β' ($\alpha = \alpha' \cup \alpha''$). We may now find close translates of α' and β', say α'_ε and β'_ε such that one can tie them together into (the isotopy class of a simple closed curve) $\gamma = \alpha'_\varepsilon \cup \beta'_\varepsilon$ and one has either $i(\alpha, \gamma) = 0$ and $i(\beta, \gamma) \leqslant i(\alpha, \beta) - 2$ or $i(\alpha, \gamma) = 1$ and $i(\beta, \gamma) \leqslant i(\alpha, \beta) - 1$.

Next, this γ can be essential or not and it can be CTNS or not. If it is inessential, then we can replace α' by α'' in the above and get $\delta = \alpha''_\varepsilon \cup \beta'_\varepsilon$. If both γ and δ are inessential, then since α itself is essential, it has to be a trinion curve. In particular it is separating and we now choose another wave of β, not situated on the side of the trinion cut out by α. Since $d(S) > 1$, so that

S is not of type $(0,4)$, we can proceed as above and find an essential curve. We thus may and will assume that the original $\gamma = \alpha'_\varepsilon \cup \beta'_\varepsilon$ is essential. Now if γ is nonseparating or is a trinion curve, we are done. So we assume that γ is separating and not a trinion curve; let S' and S'' the two surfaces cut out by γ, labeled in such a way that the segment α'' is contained in S''. We can now select a wave β_1 of β located in S'. Indeed if no such wave exists, then $i(\alpha,\beta) = 2$ and since γ is not a trinion curve we can find a CTNS curve on S', which is thus disjoint of α and β and connects these two curves in $C_{ctns}(S)$. We now construct γ_1 essentially as above, taking into account the simplifying fact that γ is separating. The wave β_1 determines two complementary segments γ' and γ'' of γ. Consider one of them, say γ', and the union $\gamma' \cup \beta_1$. Construct a sufficiently small regular tubular neighborhood of the latter curve and let γ_1 be the connected component of the boundary contained in S'. If γ_1 is not essential, use γ'' instead of γ'; one of the two curves has to be essential because γ is not a trinion curve. Then we deduce that γ_1 is essential, disjoint from α ($i(\alpha,\gamma_1) = 0$) and that $i(\beta,\gamma_1) \leqslant i(\alpha,\beta) - 3$. This completes the proof of Proposition 1.4, hence also of Proposition 1.3. Clearly the above proof is algorithmic. In particular one gets an upper bound on the distance from α to β in $C_{ctns}(S)$ that is linear as a function of their intersection number $i(\alpha,\beta)$. $\square$

We have thus completed the inductive step of Theorem 1.2 but are yet to prove the two dimensional case, which will be dealt with in the next section. Assuming that statement for the time being, we will state and prove a consequence of Theorem 1.2, which geometrically embodies the uniqueness part of the two level principle. Although this is actually more of a corollary, we state it as a theorem, given its independent interest.

Theorem 1.5. *Let S be connected of finite type and let $F, F' \in \mathrm{Aut}(\check{C}(S))$ be two automorphisms. If F and F' coincide on a piece $T \subset S$ with $d(T) > 0$, they differ by an element of $\check{\Gamma}(S)$.*

Proof, granted Theorem 1.2**.** Considering $F'^{-1} \circ F$ we may assume that F' is the identity, that F restrict to the identity on T and we wish to show that $F \in \check{\Gamma}(S)$. If $d(S) = 0, 1$, there is nothing to prove. For $d(S) > 1$, we may for instance proceed by descending induction on the dimension of T, the statement being certainly true if $d(T) = d(S)$, that is $T = S$. The injection $i : T \hookrightarrow S$ induces a map $i_* : C(T) \to C(S)$ which is not necessarily a monomorphism (although it *is* on the vertices). Again the closure of the image $i_*(C(T))$ in $\check{C}(S)$ coincides with $i_*(\check{C}(T)$, the image of the completion of $C(T)$ via the completed map (still denoted by i_* for simplicity). Let us now complete the relative boundary $\partial T \setminus \partial S$, which is a cell of $C(S)$, into a pants decomposition

σ of S, that is a top dimensional cell. After twisting by an element of $\check{\Gamma}(S)$, we may assume that F fixes σ, because as mentioned above F preserves topological types. Consider now a trinion T' cut off by σ that is adjacent to T, that is has at least a boundary curve α in common. Then apply Theorem 1.2 to the connected surface $T \cup T'$; after possibly twisting along α, we get a new automorphism that restricts to the identity on $T \cup T'$ and differs from the original F by an element of $\check{\Gamma}(S)$. This completes the induction and thus the proof (modulo the completion of the proof of Theorem 1.2). □

§2. The two dimensional case : a pentagonal story

This section is devoted to the two dimensional case, which governs the local structure of the curve complexes and their automorphisms. Clearly for $d(S) = 1$ the isomorphic 0-dimensional complexes $C(S_{0,4}) \simeq C(S_{1,1})$ do not carry enough information to control the corresponding automorphism groups. The case of the pants graph is more interesting: here the isomorphism of the curve complexes induces an isomorphism of the pants graphs, which can be enhanced to the Farey tesselation F, the latter being rigid, and so is its completion $\widehat{F}$: $\mathrm{Out}(F) = \mathrm{Out}(\widehat{F}) = \mathbb{Z}/2$ (see [16], §7.5). That this is *not* the case for $d(S) = 2$, will play a role and be discussed in § 5 below.

We now turn to S with $d(S) = 2$: The congruence subgroup conjecture has been vindicated in this modular dimension, so the full profinite and the congruence topologies coincide. We will use the former in the notation. Next, because of the isomorphism $C(S_{0,5}) \simeq C(S_{1,2})$ and the corresponding isomorphism between the completed complexes, in this section we may and do assume that S has type $(0, 5)$, setting $\Gamma = \Gamma(S) \simeq \Gamma_{0,[5]}$. The complexes $C(S) = C$, $C_*(S) = C_*$, and $C_P(S) = C_P$ are graphs with $C_P \subset C_*$ and C_* dual to C. The same relations apply to the respective completions.

Recall that for any S, $\Gamma(S)$ acts on $C(S)$ via its natural action on the set of loops, alias simple closed curves, $\mathcal{S}(S)$. If $\alpha \in \mathcal{S}(S)$, $\tau_\alpha \in \Gamma(S)$ denotes the twist along α (after fixing an orientation of S) and $g \in \Gamma(S)$, these actions are related *via* $\tau_{g\cdot\alpha} = g\tau_\alpha g^{-1}$. It is remarkable and nontrivial that these notions make sense in the procongruence setting, with the same relation, that is, for $\alpha \in \check{\mathcal{S}}(S)$ and $g \in \check{\Gamma}(S)$ (see [16, 3] and references therein). Here again, since the profinite and procongruence topologies coincide, we may and will freely use this geometric language in the profinite topology..

Now in the discrete setting, we have the following *standard pentagon*, which we denote by ϖ_0 (see Figure 1). It consists of the curves α_i, $i \in \mathbb{Z}/5$, and we view it as a pentagon $\varpi_0 \subset C_P(S) \subset C_*(S)$, each vertex consisting in a pants decomposition. Explicitly and clockwise we have the following multicurves and edges:

$$(\alpha_1, \alpha_4) \to (\alpha_2, \alpha_4) \to (\alpha_2, \alpha_5) \to (\alpha_3, \alpha_5) \to (\alpha_3, \alpha_1) \to (\alpha_4, \alpha_1).$$

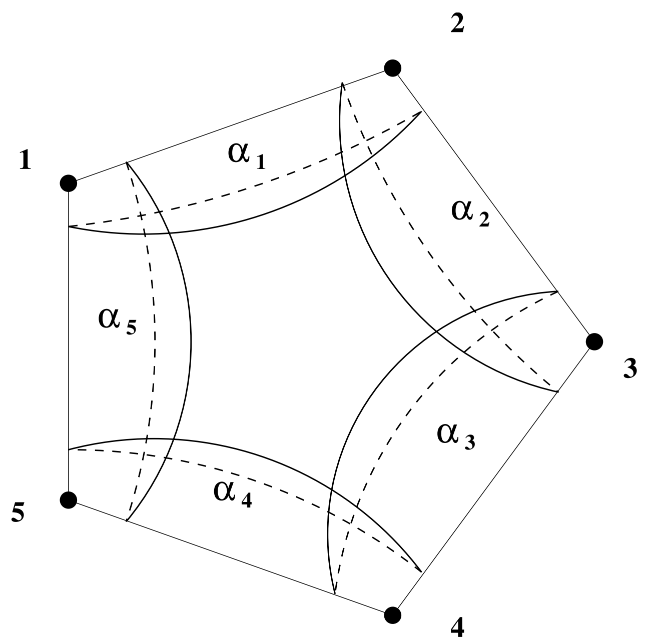

Figure 1

In the discrete case we say (see, e.g., [8]) that two (isotopy classes of) curves $\alpha, \beta \in \mathcal{S}$ *intersect minimally* if they intersect at two points, that is if their geometric intersection number is 2; since they are separating, their algebraic intersection is zero. It is not clear *a priori* how to *define* a pentagon in the profinite setting because one usually defines a discrete pentagon using minimal intersection; this is precisely what has to be avoided in the profinite case since the notion of geometric intersection cannot be extended to the completed setting, whereas the notion of algebraic intersection number readily extends, as given by the cup product on $H^1(S, \widehat{\mathbb{Z}})$. Here however it will emerge that the Grothendieck–Teichmüller group is actually closely related to the possible *deformations of the minimal intersection rule*. For the time being, still in the discrete setting, we have the following elementary an striking topological statement.

Lemma 2.1. ([18], *Lemma* 4.2) *Let S be of type* $(0, 5)$ *and let $\alpha_i \in \mathcal{S}$ ($i \in \mathbb{Z}/5$) be 5 loops on S such that α_i and α_{i+1} intersect for all $i \in \mathbb{Z}/5$ (i.e.,* *their geometric intersection number is nonzero*); *the following conditions are equivalent:*

i) α_i *and* α_{i+2} *are disjoint for all* $i \in \mathbb{Z}/5$;

ii) α_i *and* α_{i+1} *have minimal intersection for all* $i \in \mathbb{Z}/5$.

If the α_i satisfy i) and ii), they are are said to form a *pentagon*. Note that i) simply says that $P_i = (\alpha_i, \alpha_{i+2})$ is a pants decomposition for all $i \in \mathbb{Z}/5$, that is, P_i is a vertex of both complexes C_* and C_P, which indeed share the same set V of vertices. Now ii) says that P_i and P_{i+2} are joined by an edge in C_P: $(P_i, P_{i+2}) \in E_P$, with the use of the self-explanatory notation of [16] (see especially A.7 and A.8 there).

Since condition ii) does not extend to the profinite case, and for good reasons, we *define* profinite pentagons via condition i). Formally, this sounds as follows.

Definition 2.2. On a surface S of type $(0,5)$, five (pro)curves $\alpha_i \in \widehat{\mathcal{S}}$ $(i \in \mathbb{Z}/5)$ form a *pentagon* if α_i and α_{i+2} are disjoint for $i \in \mathbb{Z}/5$, that is, $P_i = (\alpha_i, \alpha_{i+2}) \in \widehat{V}$, the set of vertices of the completed pants graph $\widehat{C}_P$.

This definition is clearly graph theoretic; it also makes sense for a surface S of larger modular dimension and curves that sit on a subsurface of type $(0,5)$. There is another elementary topological statement which turns out to be quite useful.

Lemma 2.3. ([18], *Lemma* 4.3). *Let* S *be of type* $(0,5)$ *and let* $\alpha, \beta \in \mathcal{S}$ *be two loops on* S. *Then there exists at most one loop* $\gamma \in \mathcal{S}$ *that is disjoint from both* α *and* β.

This statement readily extends to the profinite case, which we record below.

Lemma 2.4. *A profinite version of Lemma* 2.3 *holds true*: *if* $\alpha, \beta \in \widehat{\mathcal{S}}$ *are* (*pro*)*loops on* S *of type* $(0,5)$, *there is at most one such loop* $\gamma \in \widehat{\mathcal{S}}$ *that is disjoint from both* α *and* β.

Proof. For once we will have recourse to the group theoretic framework, which makes that proof quite easy. It suffices indeed to recall the description of the intersection $Z(\tau_\alpha) \cap Z(\tau_\beta) \subset \widehat{\Gamma}$ for the centralizers of the twists on α and β in $\widehat{\Gamma}$. The description of the centralizers in the discrete case is elementary and extends to the procongruence, alias the profinite setting in our case ([16], §4 as well as [3] and references therein). It would be interesting to find a nice graph theoretic proof of the lemma. □

We observe that Lemma 2.4 implies that in a pentagon (γ_i), $i \in \mathbb{Z}/5$, $\gamma_i \in \widehat{\mathcal{S}}$, the loops γ_i and γ_{i+i} determine γ_{i+3} uniquely. Note also that in the discrete case, that is, if actually $\gamma_i \in \mathcal{S}$, (γ_i, γ_{i+2}) and $(\gamma_{i+1}, \gamma_{i+3})$ determine an edge of $C_P \subset C_*$; in the profinite case, they determine an edge of $\widehat{C}_*$.

At this point we have to break the exposition in order to recall some standard notation and elementary facts about ordinary braid groups and genus 0 mapping class groups. Everything we need — and beyond — is contained in [19]. In a nutshell and to fix notation, B_n denotes the plane Artin braid group on n strands ($n \geqslant 1$), generated by the elementary braids τ_i, $i = 1, \ldots, n-1$, representing the simple crossing of the ith strand in front of the $(i+1)$th one. The τ_i's satisfy the usual braid relations. It is sometimes useful to employ a redundant but symmetric system of generators: $\tau_{ij} = \tau_{j-1} \ldots \tau_{i+1} \tau_i \tau_{i+1}^{-1} \ldots \tau_{j-1}^{-1}$. Here $1 \leqslant i < j \leqslant n$ and one sets $\tau_{ji} = \tau_{ij}$, $\tau_{ii} = 1$; τ_{ij} corresponds to the simple crossings of the ith strand passing in front of the jth strands, the intermediate strands remaining in the background; see [19], Proposition 1 for more.

There is a natural surjection $B_n \to \mathcal{S}_n$ that maps τ_{ij} to the transposition $(i\ j)$. The kernel K_n is the pure plane braid group, generated by the x_{ij}'s with $x_{ij} = \tau_{ij}^2$. The following mutually commuting pure elements, namely $y_i = \tau_{i-1} \ldots \tau_2 \tau_1 \tau_2 \ldots \tau_{i-1}$ ($1 < i \leqslant n$; $y_1 = 1$) play an important role. In particular the center of B_n is free cyclic, generated by the product $\omega_n = y_1 y_2 \ldots y_n = (\tau_1 \ldots \tau_{n-1})^n$. Note that the y_i's, hence also ω_n, are pure and that ω_n also generates the center of K_n.

The sphere braid group $B_n(S^2)$ is naturally isomorphic to $B_n/\langle y_n \rangle$ where $\langle y_n \rangle$ denotes the normal closure of y_n in B_n. The image of the center ω_n generates the center of $B_n(S^2)$ and has order 2. Finally the modular group $\Gamma_{0,[n]}$ is determined as: $\Gamma_{0,[n]} \simeq B_n(S^2)/\langle \omega_n \rangle \simeq B_n/\langle y_n, \omega_n \rangle$. The elements τ_i are of course interpreted as twists exchanging points i and $i+1$ on a marked sphere and this generalizes to the τ_{ij}, exchanging points i and j. In [3] (§2.7 and §4.8) the τ_{ij}'s are dubbed *braid twists*; they also sometimes go under the name "half-twists". The full mapping class group $\Gamma_{0,[n]}$ is generated by the τ_i's, whereas the x_{ij}'s ($x_{ij} = \tau_{ij}^2$) generate the colored subgroup $\Gamma_{0,n}$. Elementary and useful properties of these objects are quite explicitly detailed in [19].

As for the first nontrivial case, the group B_3 is generated by two elements τ_1 and τ_2. Its center Z is free cyclic generated by $\omega_3 = (\tau_1 \tau_2 \tau_1)^2 = (\tau_1 \tau_2^2)^2 = (\tau_1 \tau_2)^3$. Then $B_3/Z \simeq PSL_2(\mathbb{Z})$. The pure braid group $K_3 \subset B_3$ is the direct product of the free group $F_2 = \langle x_{12}, x_{23} \rangle$ ($x_{12} = \tau_1^2$, $x_{23} = \tau_2^2$) and the center: $K_3 = \langle \tau_1^2, \tau_2^2 \rangle \times \langle (\tau_1 \tau_2 \tau_1)^2 \rangle$.

The mapping class group of the 5-punctured sphere, $\Gamma \simeq \Gamma_{0,[5]}$, and its pure subgroup $P\Gamma \simeq \Gamma_{0,5}$ are particularly relevant here. Referring the reader to Figure 1 above, we see that Γ is generated by the 5 braid twists τ_i, $i \in \mathbb{Z}/5$ where τ_i exchanges the points i and $i+1$ mod. 5. We write $a_i = x_{i,i+1} = \tau_i^2$, $i \in \mathbb{Z}/5$, for the twist along the loop α_i.

We will add some pieces of information to this terse list when needed but it is now time to return to considering an automorphism $F \in \mathrm{Aut}(\widehat{C}(S))$. The starting point is that F clearly has to take pentagons (in the sense of Definition 2.2) to pentagons. By requiring $F(\varpi_0)$ to be a pentagon, we *a priori* get 5 conditions on F, but we are interested in *outer* automorphisms, so we may and do normalize F by twisting by the action of $\widehat{\Gamma}$. First, after labeling the points, which are encoded in $\widehat{C}(S)$ or equivalently $\widehat{C}_*(S)$, we see that F induces a permutation of the five points, that is, an element of $\mathrm{Aut}(\mathcal{S}_5)$. Using the fact that $\mathrm{Out}(\mathcal{S}_5) = \{1\}$ we can twist by an element of $\widehat{\Gamma}$ in order to eliminate this automorphism and get a *pure* element of $\mathrm{Aut}(\widehat{C})$, that is, one which does not permute the points. Note that any loop separates S into a trinion and a piece of type $(0,4)$, defining a partition of the set of points into two subsets of 2 and 3 points respectively. An automorphism F is pure if and only if α and $F(\alpha)$ determine the same partition for any $\alpha \in \widehat{\mathcal{S}}$. As for a second normalization, we know that $\widehat{\Gamma}$ acts transitively on $\widehat{V}$, the set of pants decompositions (this is of course specific of type $(0,5)$). So twisting again by an element of $\widehat{\Gamma}$, we may assume that F fixes a pants decomposition, which for definiteness and to conform with almost standard conventions is chosen to be (α_1, α_4). So we may and do assume that F is pure, $F(\alpha_1) = \alpha_1$, and $F(\alpha_4) = \alpha_4$.

From there, one derives *a priori* formulas for the action of F on the α_i's. In order to write them out more explicitly, recall that the twist on the curve α_i of the standard pentagon ϖ_0 is denoted by $a_i = x_{i,i+1} \in \Gamma_{0,5}$, $i \in \Gamma_{9,5}$. Below we will often retain the notation $x_{i,i+1}$ simply because it has become traditional. It is also interesting to recall that these five elements $a_1 = x_{12}, a_2 = x_{23}, a_3 = x_{34}, a_4 = x_{45}, a_5 = x_{51}$ generate the group $\Gamma_{0,5}$. The next piece of well-known information is the explicit structure of the centralizers of twists in $\Gamma_{0,5}$. It suffices mention that the centralizer $Z(x_{12})$ of x_{12} in $\Gamma_{0,5}$ is given by $Z(x_{12}) = \langle x_{12}, x_{34}, x_{45}\rangle$ and that the centralizer of x_{12} in the completion $\check{\Gamma}_{0,5} = \widehat{\Gamma}_{0,5}$ is none other than the completion of $Z(x_{12})$. Moreover it coincides with the — *a priori* larger — normalizer of any open subgroup of the procyclic group $\langle x_{12}\rangle$. We have already mentioned these results, which in a more general context are essentially due to M. Boggi (see [16] and references therein).

We still need to recall a few bits and pieces of information and notation before proving what we are after, in Proposition 2.9 below. First about a certain "change of variables" in our case. Let $f \in \widehat{F}_2$ and specify two generators x and y of the discrete group F_2, writing $\widehat{F}_2$ as the completion of $F_2 = \langle x, y\rangle$. Then writing $f = f(x,y)$ we need to make sense of $f(a,b)$ for any two elements a, b of a profinite group G. To do this, simply observe by the universality of the profinite completion that there is a unique morphism $\widehat{F}_2 \to G$ mapping the ordered pair (x, y) to (a, b); write $f(a, b)$ for the image of $f = f(x, y)$. The desirable naturality properties are plain from this definition. Now recall again the well-known formula (in the discrete case) for the relationship between the action of an element $g \in \Gamma$ on a loop α and the associated twist τ_α, namely $\tau_{g\cdot\alpha} = g\tau_\alpha g^{-1}$. It is remarkable and quite nontrivial that this same formula continues to hold true for congruence completions, hence also for the (full) profinite ones when the congruence conjecture obtains; we refer again to [16] for detailed results (originally again largely due to M. Boggi), proofs, an references. By the invocation of profinite and procongruence complexes, it will turn out that we can essentially view the action of GT as an action on the loops of the completed curve complex. Again the action of λ (which is determined by f up to a sign) is reminiscent of the S^1 action in the string topology or a hypothetical completed version of it.

All this leads to the following formulas for the action of F on ϖ_0:

$$\begin{gathered} F(\alpha_1) = \alpha_1, \quad F(\alpha_2) = f \cdot \alpha_2, \quad F(\alpha_3) = g \cdot \alpha_3, \\ F(\alpha_4) = \alpha_4, \quad F(\alpha_5) = fh \cdot \alpha_5. \end{gathered} \tag{2.1}$$

In these formulas, f, g, and h are elements of $\widehat{\Gamma}_{0,5}$; more precisely we may assume that:

$$\begin{gathered} f = f(x_{12}, x_{23}) \in \langle x_{12}, x_{23} \rangle, \quad g = g(x_{34}, x_{45}) \in \langle x_{34}, x_{45} \rangle, \\ h = h(x_{45}, x_{51}) \in \langle x_{45}, x_{51} \rangle. \end{gathered} \tag{2.2}$$

Here we have used nothing except the structure of the centralizers of twists as recalled above, and the fact that $F(\varpi_0)$ is a pentagon. The elements f, g, h in $\widehat{F}_2$ are by no means unique but it will evolve that any of them determines the other two uniquely. We performed a first normalization: for instance f might depend on x_{45} but that does not alter the action so we may as well write f, g, and h as above. In terms of twists, this means for example that $F(a_2) = f a_2^\lambda f^{-1}$ ($a_2 = x_{23}$) for some $\lambda \in \widehat{\mathbb{Z}}^\times$, to be compared with the standard action of the Grothendieck–Teichmüller group (see, e.g., [19] or [30] and §4 below). We emphasize again that formulas (2.1) and (2.2) are forced upon us merely by requiring that $F(\varpi_0)$ be a pentagon and we note a slight discrepancy with the usual (Drinfel'd) notation. It is customary to let an $F \in \widehat{GT}$, with parameters $(\lambda, f = f(x, y))$, act on x and y according to $F(x) = x^\lambda$, $F(y) = f^{-1} y^\lambda f$. The variance of the action on loops and the associated twists would suggest to change f to f^{-1}, writing $F(y) = f y^\lambda f^{-1}$. Because of the 2-cycle relation ($f^{-1}(x, y) = f(y, x)$) this amounts to swapping the variables x and y. In any case we keep formulas (2.1) and (2.2) as they are, so that f will eventually identify with f^{-1} in Drinfel'd's original notation.

Below we will not refrain from using some group theoretic notions and results. It would be interesting to give completely geometric, complex theoretic proof of the results. Now for *every* triplet (f, g, h) of elements of $\widehat{\Gamma}$ as in formulas (2.2), 4 out of the 5 conditions for $F(\varpi_0)$ to be a pentagon are satisfied, that is all of them save for the disjointness of $F(\alpha_3)$ and $F(\alpha_5)$. In group theoretic terms this last condition is equivalent to the equation:

$$(gx_{34}g^{-1}, fhx_{51}h^{-1}f^{-1}) = 1, \tag{Π}$$

where $(a, b) = aba^{-1}b^{-1}$ denotes the commutator of a and b in $\widehat{\Gamma}$. Equation (Π) can thus be seen as the "fundamental equation", which should rigify everything. It is nonlinear and profinite, so *a priori* not easy to handle. We will actually get below a full classification of the pentagons in $\widehat{\Gamma}$ so in some sense of the solutions of (Π). But on the way we will need to investigate the special case where $f = 1$ (see Proposition 2.5 below), which we do using a group theoretic result.

For now we may normalize f, g, and h a little further, which eventually will bring us to exploring the notion of *adjacency* for pentagons. First, we may assume that f (respectively, g, h) has weight 0 with respect to x_{23} (respectively, x_{34}, x_{51}) by multiplying on the right by suitable (profinite) powers of these twists. Then we may twist F by powers of x_{12} and x_{45} in such a way that f and g will become *commutators*, that is have zero weight in both variables. This rigidifies the situation completely. Namely if we require that F fix the pants decomposition (α_1, α_4) and that f and g be commutators, there are no free parameters left. This is actually intrinsic and general (in any dimension): an automorphism that fixes a pants decomposition is determined up to a multitwist along that decomposition. It is precisely the role of tangential base points to rigidify by killing that possible twist: requiring that f and g be commutators amounts to fixing a kind of *topological tangential base point*; this is used and goes under various names in the literature, e.g., in an analytic setting in order to define Fenchel–Nielsen twists as real numbers (not only on the circle); see [NS] for the use of such a rigidification in the framework of Grothendieck–Teichmüller theory. Here we adopt a rather practical viewpoint on the matter, without spelling out the (known) underlying geometric picture.

We come back to our standard pentagon ϖ_0 and automorphism F. It is interesting to note first of all that h, if it exists, is determined by the pair (f, g) — assuming h is normalized to have weight 0 with respect to x_{51}. Indeed, this is a direct application of Lemma 2.4: $F(\alpha_5)$ is none other than the only curve that is disjoint from both $F(\alpha_2)$ and $F(\alpha_3)$. Next in the discrete case, *all* solutions of equation (Π) are obtained by twisting along the multicurve (α_1, α_4) we singled out. After the first normalization mentioned above, one gets: $f = x_{12}^m$, $g = x_{45}^n$ and then $h = x_{45}^n$. Here one can cross check that in this discrete toy situation, $F(\alpha_5)$ is indeed determined by f and g: the exponent n in h is the only one that ensures that $F(\alpha_5)$ commutes with $F(\alpha_3)$. This also yields a complete classification of pentagons in the discrete case and shows that they all belong to a single Γ-orbit. Simple as this may sound, it is key in showing the rigidity of the *discrete* curves complex $C(S)$ and thus Ivanov's original result.

In the procongruence (here profinite) setting, using a topological tangential base point, i.e., normalizing as described above, we get rid of the multitwists along (α_1, α_4), that is of the elements m, n, now in $\widehat{\mathbb{Z}}$. We will write $\widehat{F}_2' = \langle x_{12}, x_{23}\rangle'$ for the commutator subgroup of the free group generated by x_{12} and x_{23} and analogously for other such pairs and will show that setting $f = 1$ implies a kind of rigidity.

Proposition 2.5. *Consider the equation*

$$(gx_{34}g^{-1}, hx_{51}h^{-1}) = 1,$$

with $g \in \langle x_{34}, x_{45}\rangle'$ *and* $h \in \langle x_{45}, x_{51}\rangle$. *Then* $g = 1$ *and* $h \in \langle x_{51}\rangle$.

Since $g \in \langle x_{34}, x_{45}\rangle$ and $h \in \langle x_{45}, x_{51}\rangle$, this equation is in fact in the group (topologically) generated by x_{34}, x_{45} and x_{51} inside $\widehat{\Gamma}_{0,5}$, which is none other than the completion of the corresponding subgroup of $\Gamma_{0,5}$. The structure of the latter group is relatively simple: it is isomorphic to the group G with 3 generators x, y, and z and the only relation $(x, z) = 1$, stating that x and z commute. So $G \simeq \mathbb{Z}^2 * \mathbb{Z}$ and to get the isomorphism, simply map (x_{34}, x_{45}, x_{51}) to (x, y, z). For the sake of clarity, we thus restate Proposition 2.5 in the following equivalent form.

Lemma 2.6. *Let* $G = \langle x, y, z\rangle \simeq \mathbb{Z}^2 * \mathbb{Z}$ *be as above and let* $g \in \langle x, y\rangle' \simeq \widehat{F}_2'$, $h \in \langle y, z\rangle \simeq \widehat{F}_2$ *be two elements of* $\widehat{G}$ *satisfying the commutation relation* $(gxg^{-1}, hzh^{-1}) = 1$. *Then* $g = 1$ *and* $h \in \langle z\rangle$.

Proof. This is a direct application of Theorem 9.1.12 in [32]. By assumption, $(g^{-1}h)z(g^{-1}h)^{-1}$ centralizes x and the quoted result directly implies that $g^{-1}h \in \langle x, z\rangle$. So we can write: $h(y, z) = g(x, y)w(x, z)$ for some $w \in \langle x, z\rangle$. Moding out by the normal closure of y and using that g is a commutator, we deduce that in fact $w = w(z) \in \langle z\rangle$. We then see that g depends on y only and so that $g = 1$ because it is a commutator. This finishes the proof of the lemma, hence also of Proposition 2.5. □

Let us now exploit the above in order to finally complete the proof of Theorem 1.2. First we introduce an easy notion.

Definition 2.7. Two pentagons (in the sense of Definition 2.2) are said to be *adjacent* if they have an edge in common in $\check{C}_*(S)$.

This clearly makes sense and can be used on a surface S of any modular dimension. If ϖ and ϖ' lie on a surface of type $(0,5)$ and are adjacent, they share 3 curves in common, corresponding to two pants decompositions. By Lemma 2.4, it is equivalent to require that ϖ and ϖ' have 2 consecutive curves in common, or equivalently again two curves which intersect (that is which are not nonintersecting...). Proposition 2.5 can now be viewed as a description of the pentagons that are adjacent to the standard pentagon ϖ_0 on a surface of type $(0,5)$. Indeed, if ϖ is adjacent to ϖ_0, one may assume after relabeling that ϖ contains the curves α_1, α_2, and α_4. Then Proposition 2.5 indeed describes all pentagons that contain these curves: they are obtained by twisting ϖ_0 by a profinite power of x_{45}.

This notion of adjacency may look a little uninteresting at the first sight, but it will turn out to be surprisingly useful, as will become even clearer in the next section. Here is a first consequence of Proposition 2.5 which is made use of in the next section. Namely, on a surface S of type $(0,5)$, assume that ϖ and ϖ' are adjacent pentagons and that $\varpi \subset \widehat{C}_P(S)$; then also $\varpi' \subset \widehat{C}_P(S)$. In fact we can assume that $\varpi = \varpi_0$ is the standard pentagon and then this is an immediate consequence of Proposition 2.5; see Proposition 3.1 and Lemma 3.2 below for details and a generalization. Now we prove the following claim.

Lemma 2.8. *Take $F \in \mathrm{Aut}(\widehat{C}(S))$ (S of type $(0,5)$) fixing the standard pentagon ϖ_0, that is, $F(\alpha_i) = \alpha_i$ for all $i \in \mathbb{Z}/5$. Then F is either the identity or a reflection, that is, an orientation reversing involution.*

Proof. Indeed, consider any discrete pentagon $\varpi \subset C_P(S)$ that is adjacent to ϖ_0. Then F fixes the side common to ϖ_0 and ϖ, so $F(\varpi)$ is adjacent to ϖ_0 and by the remark above $F(\varpi) \subset \widehat{C}_P(S)$. Continuing in the same fashion, that is, considering a pentagon adjacent to ϖ, we see that $F(C_P(S)) \subset \widehat{C}_P(S)$, because any two pentagons of the discrete complex $C_P(S)$ can be connected by a finite sequence of mutually adjacent pentagons. So by density $F(\widehat{C}_P(S)) \subset \widehat{C}_P(S)$ and in fact $F(\widehat{C}_P(S)) = \widehat{C}_P(S)$ by changing F into F^{-1}. In other words, $F \in \mathrm{Aut}(\widehat{C}_P(S))$. Now using the rigidity of $\widehat{C}_P(S)$ ([16], Theorem 7.1), that is, $\mathrm{Aut}(\widehat{C}_P(S)) = \mathrm{Mod}(S)$, and the fact that F fixes ϖ_0, it is easy to see that F is indeed the identity or a reflection. □

We are now finally in a position to prove assertion (C_2) of §1, that is, the following statement.

Proposition 2.9. *The claim of Theorem 1.2 holds true for $d = 2$.*

Proof. Let $F \in \mathrm{Aut}(\widehat{C}(S))$, with $d(S) = 2$; we may assume that S is of type $(0, 5)$, that $F(\alpha) = \alpha$ for a curve $\alpha \in \mathcal{S}$, and that F restricts to the identity on $\widehat{C}(S_\alpha)$. We may further assume, still without loss of generality, that we are in the standard situation with, say, $\alpha = \alpha_4$. So F fixes α_1 and α_2 as well. Applying Proposition 2.5 we conclude that, possibly after twisting along α_4, F fixes ϖ_0 and then, by Lemma 2.8, that F is the identity, a reflection being excluded by the assumed triviality of F on $\widehat{C}(S_\alpha)$. □

This completes the proof of Theorem 1.2. In the next section, we will see some further applications of adjacency, combined with other simple geometric notions. Now, we add a few observations which we hope may help put the results in perspective. None of them will be used below so that they can be skipped without impairing further technical understanding.

Remarks 2.10.

1. Technically speaking, it should be noted that injectivity becomes much easier if one is *a priori* given *two* intersecting curves α and β such that $F \in \mathrm{Aut}(\check{C})$ restricts to the identity both on $\widehat{C}(S_\alpha)$ and $\widehat{C}(S_\beta)$ (compare [28], Lemma 3.2.2, for the group theoretic counterpart). The whole point here is to start from only one such curve.

2. Theorems 1.2 and 1.5 hold essentially *verbatim* both in the discrete, pro-ℓ, and pronilpotent settings. About the discrete case, see the next remark. In the pro-ℓ setting, one literally follows the proofs given in the present paper, using the pro-ℓ analogs of the results from [16, 3] and their references, which are easy variants. We will return more fully to the pro-ℓ setting at the end of §4, in the genus 0 case. The pronilpotent case is nothing but the "product" of the pro-ℓ cases for all the primes ℓ.

3. Starting with the discrete version of Theorem 1.5 (injectivity) and in order to prove the rigidity of the discrete complex, that is, $\mathrm{Aut}(C(S)) = \mathrm{Mod}(S)$ for $d(S) > 1$ except in two well-understood cases (types $(1, 2)$ and $(2, 0)$), it suffices to show that this is the case for S of type $(0, 5)$. But in the discrete case we have already observed that it is easy to classify all pentagons, which form a unique $\Gamma(S)$-orbit. So in order to reprove the classical (Ivanov–Korkmaz–Luo) rigidity result, it only remains to prove a discrete analog of Lemma 2.8. This in turn amounts to devising a direct elementary proof of the rigidity of the discrete pants graph $C_P(S)$, which looks like a feasible and perhaps interesting task.

4. Let us clarify the overall strategy somewhat more. We now have Theorem 1.2 and thus Theorem 1.5 at our disposal, that is an *injectivity* statement: two outer

automorphisms coincide if they coincide on a piece of modular dimension 1. Ideally speaking, we should then move to an *extension* result stating in substance that if $F \in \mathrm{Aut}(\check{C}(T))$ is an automorphism of $\check{C}(T)$, where T is a piece of a connected surface S with $d(T) > 1$, then it can uniquely be extended to an automorphism of the full complex $\check{C}(S)$. The real situation we will study in §§4, 5 is not quite as simple but almost. In the meantime we need also worry, among other things, about the *normality* of $\check{\Gamma}(S)$ in $\mathrm{Aut}(\check{C}(S))$, which we do below in the next section.

5. In [11, 12], Y. Ihara implemented a similar strategy, which in some sense comes from a crucial intuition of A. Grothendieck in his *Esquisse d'un programme*, in the simplest possible case, that is in a group or rather Lie theoretic context, for pronilpotent (i.e., not full profinite) completions of braid groups (i.e., genus 0). Curiously enough the injectivity result in [11] is the only one we know of in the literature which resembles Theorem 1.5. The author already noticed that his result is valid in the discrete case as well, providing a new proof of the Dyer–Grossman result on the rigidity of Artin's plane braid groups: $\mathrm{Out}(B_n) = \mathbb{Z}/2$, $n > 2$.

6. We finish with the rather obvious and somewhat "philosophical" remark that all the work to-date has been done in a group theoretic rather than complex theoretic setting. The latter is surely more geometric and in some sense more powerful as it encompasses the full system of the open subgroups of the Teichmüller modular groups. It is also often more flexible, as perhaps examplified above, although of course group theoretic techniques can at times be not only helpful but — at least to-date — hardly dispensed with.

§3. Transversality and normality

Such are the keywords of the present section, as will soon become clearer. Let us first briefly elucidate the pieces of notation $\mathrm{Out}(C(S))$ and its procongruence version, namely $\mathrm{Out}(\check{C}(S))$. Of course in the cases where the congruence conjecture has been vindicated we can use the profinite rather than the procongruence notation. The crux of the matter is again simply the relation $\tau_{g\cdot\alpha} = g\tau_\alpha g^{-1}$ between the action of the Teichmüller group $\Gamma(S)$ (or $\check{\Gamma}(S)$) on loops and the corresponding twists. As has been mentioned already, this holds true in the procongruence as well as in the discrete case. So fixing S and $g \in \check{\Gamma}(S)$, its action on (pro)loops corresponds to an *inner* action on the group $\check{\Gamma}(S)$ (recall that twists topologically generate $\check{\Gamma}(S)$). Moreover the action of the central elements is trivial on both side. In any case we assume that $d(S) > 1$ so that only $\check{\Gamma}_{1,2}$ and $\check{\Gamma}_2$ have nontrivial centers, both of order 2, generated by the (hyper)elliptic involution. Moreover $\check{C}(S_{1,2}) \simeq \check{C}(S_{0,5})$ and $\check{C}(S_{2,0}) \simeq \check{C}(S_{0,6})$. So we can work with the genus 0 versions and the center-free groups $\check{\Gamma}_{0,[5]}$ and $\check{\Gamma}_{0,[6]}$, writing $\mathrm{Out}(\check{C}(S)) = \mathrm{Aut}(\check{C}(S))/\check{\Gamma}(S)$ (ditto in the discrete case). But ... this is only if $\check{\Gamma}(S)$ is normal in $\mathrm{Aut}(\check{\Gamma}(S))$, which we are precisely undertaking to prove.

We first summarize part of the conclusions of the last section for the sake of clarity in the following proposition.

Proposition 3.1. *Let $\varpi \subset \widehat{C}_*(S)$ be a pentagon, with S of type $(0,5)$. The following conditions are equivalent*:
i) *ϖ is in the $\widehat{\Gamma}(S)$-orbit of the standard pentagon ϖ_0;*
ii) *ϖ is an inverse limit of pentagons in $C_P(S)$;*
iii) *ϖ is contained in $\widehat{C}_P(S)$;*
iv) *One side of ϖ is contained in $\widehat{C}_P(S)$.*

We make ii) and iii) completely explicit. Condition ii) means that there exists a sequence $(\varpi_\lambda)_{\lambda\in\Lambda}$ of pentagons in $C_P(S)$ such that ϖ is the inverse limit of the ϖ^λ's, where $\varpi^\lambda = \varpi_\lambda \quad mod\,\Gamma^\lambda$. We refer the reader to item 1 in the Introduction for the notation and the meaning of the set Λ, or to [16], §A.10, for somewhat more detail. Condition iii) means that the sides of ϖ lie in $\widehat{E}_P \subset \widehat{E}$, the (pro)set of edges of $\widehat{C}_P(S)$. See [16], §A.9, for detail on this hopefully self-explanatory notation and recall that $\widehat{C}_*(S)$ and $\widehat{C}_P(S)$ have the same set $\widehat{V}$ of vertices, namely the completion of the set of pants decompositions; because S is of type $(0,5)$, $\widehat{V}$ consists of only one $\widehat{\Gamma}$-orbit (with $\Gamma = \Gamma(S)$ as usual).

Proof of Proposition 3.1. The conditions are stated in order of apparent decreasing strength. So i) implies ii) which implies iii), which in turn implies iv) and we need only show that iv) actually implies i). So let $\varpi \subset \widehat{C}_*(S)$ with one side in $\widehat{C}_P(S)$. That side consists of 2 pants decompositions P and P'. We may assume that $P = (\alpha_1, \alpha_4)$, with α_1, α_4 as in the standard pentagon ϖ_0, and that $P' = (\alpha_1, \beta)$ for some $\beta \in \widehat{\mathcal{S}}$. Since $(P, P') \in \widehat{E}_P$, there exists a sequence $(\beta_\lambda) \subset \mathcal{S}$ such that β is the inverse limit of the β^λ, with $\beta^\lambda \in \mathcal{S}^\lambda = \mathcal{S}/\Gamma^\lambda$, $\beta^\lambda = \beta_\lambda \mod \Gamma^\lambda$. Now every pair $(\alpha_1, \beta_\lambda)$ lies, up to reflection, in the Γ-orbit of the pair (α_1, α_2). Actually, since β_λ is disjoint from α_4, this happens on the surface of type $(0,4)$ obtained by cutting S along α_4 and we are simply asserting that $SL_2(\mathbb{Z})$ acts transitively on the edges of the Farey tesselation. Since the sequence β^λ converges, we see that (α_1, β) lies in the $\widehat{\Gamma}$-orbit of (α_1, α_2); so twisting by an element of $\widehat{\Gamma}$ and possibly reflecting, we may assume that ϖ contains $P = (\alpha_1, \alpha_4)$ and $P' = (\alpha_2, \alpha_4)$. Proposition 2.5 then completes the proof. □

We already know that a pentagon adjacent to a pentagon contained in $\widehat{C}_P(S)$ lies itself in $\widehat{C}_P(S)$. Now consider an automorphism of $F \in \mathrm{Aut}(\widehat{C}(S))$. Clearly it preserves adjacency. If we take a pentagon $\varpi \subset F(\widehat{C}_P(S))$ and consider ϖ' adjacent to it, then $F^{-1}(\varpi)$ and $F^{-1}(\varpi')$ are adjacent as well, the first one being contained in $\widehat{C}_P(S)$. So $F^{-1}(\varpi')$ is contained in $\widehat{C}_P(S)$ and ϖ' is contained in $F(\widehat{C}_P)$. In other words we have shown the following.

Lemma 3.2. *Suppose that S is of type $(0,5)$, $F \in \mathrm{Aut}(\widehat{C}(S))$, $\varpi \subset F(\widehat{C}_P(S))$ is a pentagon, and ϖ' is adjacent to ϖ; then $\varpi' \subset F(\widehat{C}_P(S))$.*

Let now $F, F' \in \mathrm{Aut}(\widehat{C}(S))$ be two automorphisms and consider the images $F(\widehat{C}_P(S))$ and $F'(\widehat{C}_P(S))$. As usual only the edges are interesting here because the sets of vertices of these images both coincide with $\widehat{V}$. Under these circumstances, we have the following *transversality property*, which we can state for a general surface S and the procongruence completion.

Proposition 3.3. *Let S be hyperbolic of finite type with $d(S) > 1$, and S not of type $(1,2)$, let $F, F' \in \mathrm{Aut}(\check{C}(S))$ and assume that $F(\check{C}_P(S))$ and $F'(\check{C}_P(S))$ have at least one edge in common. Then these graphs coincide and, up to a possible reflection, $F' = F \circ g$ for some $g \in \check{\Gamma}(S)$.*

Proof. As usual we abbreviate $\Gamma(S)$, $C(S)$, and $C_P(S)$ respectively to Γ, C, and C_P. To the first assertion, precomposing by F'^{-1}, we may assume that $F' = 1$ and we have to show that if one of the edges of $F(\check{C}_P)$ is an edge of $\check{C}_P$, then $F(\check{C}_P) = \check{C}_P$. Assume that $e \in \check{E}_P$ is an edge of $F(\check{C}_P)$; then $e = (P, P')$ with P and P' two pants decompositions. Since F is type preserving ([16], Theorem 6.4 or [3], Theorem 5.3), P and $F(P)$ are in the same $\check{\Gamma}$-orbit. So twisting F by an element of $\check{\Gamma}$ we may assume that F fixes P. Then P and P' have $d-1$ curves in common ($d = d(S)$) and these $d-1$ curves will cut off a surface T of dimension 2 in S, so T is either of type $(0,5)$ or of type $(1,2)$. We may and do assume as usual that T is of type $(0,5)$, because the assertions are complex theoretic. Restricting attention to T, we are reduced to the case where S is itself of type $(0,5)$; then we can conclude that actually, up to reflection, $F \in \Gamma$. Indeed, any side of $\widehat{C}_P$ can be completed into a pentagon (see the proof of Proposition 3.1). So by Proposition 3.1, F maps a pentagon of $\widehat{C}_P$ into a pentagon of $\widehat{C}_P$. By Proposition 3.1 again, after twisting by an element of $\widehat{\Gamma}$, we obtain an automorphism that fixes the standard pentagon ϖ_0 and Lemma 2.8 finishes the proof.

As for the last part of the statement, if $F'(\check{C}_P) = F(\check{C}_P)$, then $F^{-1} \circ F'(\check{C}_P) = \check{C}_P$, so up to a possible reflection, $F^{-1} \circ F'$ coincides with the action of some element of $\check{\Gamma}$. Since we have not yet proved the normality of

$\check{\Gamma}$ in $\mathrm{Aut}(\check{C})$, using $F' \circ F^{-1}$ we could write just as well $F' = h \circ F$ for some $h \in \check{\Gamma}$. □

This statement can be seen as a strenghtening of Theorem 1.5, modulo the easy exception of type $(1,2)$ and so this transversality statement also features a kind of strong *injectivity* property.

As usual, the situation for type $(1,2)$ is easy to unravel. The proposition then breaks for the following reason — and no other. We have $\widehat{C}(S_{1,2}) \simeq \widehat{C}(S_{0,5})$, $\widehat{\Gamma}_{1,[2]}$ acts on $\widehat{C}(S_{1,2})$ via $\widehat{\Gamma}_{1,2} = \widehat{\Gamma}_{1,[2]}/Z$ and $\widehat{\Gamma}_{1,2} \subset \widehat{\Gamma}_{0,[5]}$, of index 5, is self-normalizing, in particular not normal, corresponding to the stabilizer of one of the 5 points. So $\widehat{\Gamma}_{0,[5]}$ acts on $\widehat{C}(S_{1,2}) \simeq \widehat{C}(S_{0,5})$ but an element that is not in $\widehat{\Gamma}_{1,2}$ clearly does not act via $\widehat{\Gamma}_{1,[2]}$.

We now come to the normality of $\check{\Gamma}(S)$ inside $\mathrm{Aut}(\check{C}(S))$.

Theorem 3.4. *Suppose that S is hyperbolic of finite type with $d(S) > 1$ and S is not of type $(1,2)$. Then $\check{\Gamma}(S)$ is a normal subgroup of $\mathrm{Aut}(\check{C}(S))$.*

Proof. Again the exceptional case $(1,2)$ is easy to understand. Here the failure of the statement boils down to the fact that $\Gamma_{1,2}$ is not normal in $\Gamma_{0,[5]}$ (see above the statement of this result).

First we observe that the statement is equivalent to the fact that for every $F \in \mathrm{Aut}(\check{C}(S))$, $\check{\Gamma}$ acts on $F(\check{C}_P(S))$, that is, the natural $\check{\Gamma}$-action on $\check{C}(S)$ (or $\check{C}_*(S)$) leaves $F(\check{C}_P(S))$ globally invariant. Indeed if $\check{\Gamma}$ is normal, then for $g \in \check{\Gamma}$, we have $F^{-1}gF = h \in \check{\Gamma}$, so that $F^{-1}gF(\check{C}_P) = h\check{C}_P = \check{C}_P$ and $gF(\check{C}_P) = F(\check{C}_P)$. Conversely, if the last claim is true, then $F^{-1}gF$ is an orientation preserving automorphism of $\check{C}_P$, so it is an element of $\check{\Gamma}$ and we are done.

So given $F \in \mathrm{Aut}(\check{C}(S))$ we need to show that $gF(\check{C}_P) = F(\check{C}_P)$ for every $g \in \check{\Gamma}$ and we can restrict attention to twists because they topologically generate $\check{\Gamma}$. Let $\tau = \tau_\alpha \in \widehat{\Gamma}$ be a twist along the loop α. It actually does not matter here whether α is a loop, i.e., a vertex of the discrete complex $C(S)$ or a proloop (a vertex of $\check{C}(S)$). Consider the inverse image $\beta = F^{-1}(\alpha)$. We can include β in a pentagon ϖ that is contained in $\check{C}_P(S)$ and in fact in $\check{C}_P(T)$ for T a 2 dimensional piece (connected surface) of S. We may assume, as in the proof of Proposition 3.3, that T is of type $(0,5)$, and then use, say, Proposition 3.1, which describes all the pentagons in $\widehat{C}_P(T)$. So $\varpi \subset \widehat{C}_P$ is a pentagon, and thus $F(\varpi) \subset F(\widehat{C}_P(T))$ is again a pentagon, which we may regard as belonging to $F(\check{C}_P)$; moreover α is one of the five curves it contains. We have simply shown that any curve α can be included in a pentagon inside $F(\widehat{C}_P)$.

Finally F and $F' = \tau \circ F$ give rise to two automorphisms of $\check{C}(S)$. Moreover, $F(\varpi)$ and$F'(\varpi)$ are adjacent pentagons by construction, the second being obtained from the first by a twist along one of its curves. Since $\varpi \subset \check{C}_P$, it remains to apply Proposition 3.3. □

⋆ ⋆ ⋆

Let us take stock of what we learned above, with a fresh start. Fix an orientable surface S of type (g, n) with strictly negative Euler characteristic ($-\chi(S) = 2g - 2 + n > 0$) and modular dimension $d(S) = 3g - 3 + n > 1$. Having proved that $\mathrm{Inn}(\check{\Gamma}(S))$ is normal in $\mathrm{Aut}(\check{C}(S))$, we can write the following short exact sequence, which *defines* the outer automorphism group $\mathcal{G}(S) = \mathrm{Out}(\check{C}(S))$:

$$1 \to \mathrm{Inn}(\check{\Gamma}(S)) \to \mathrm{Aut}(\check{C}(S)) \to \mathcal{G}(S) \to 1. \tag{3.1}$$

Here as above we are working with *continuous* automorphisms, which implies that $\mathrm{Aut}(\check{C}(S))$ is a profinite group; for details on the topologies involved see [15, 16] or [3] and its references. What can we say about the group $\mathcal{G}(S)$? We will first summarize the results that were obtained above, which pertain to a kind of generalized *injectivity*, addressing the question: when do two continuous automorphisms of $\check{C}(S)$ coincide, modulo reflection and the action of $\check{\Gamma}(S)$? Of course if the genus $g(S)$ is 0, 1, or 2, the CSP holds true, i.e., the procongruence and full profinite completions of the various objects coincide. In the next and last sections (§§4, 5) we will be busy with the *extension* problem: when and how to extend an automorphism from $\check{C}(T)$ to $\check{C}(S)$, where $T \hookrightarrow S$ is a piece of S. Note that here we are using the important property of the procongruence completion that it conforms to a stratification of $\mathcal{M}(S)$ (see [16], Proposition 5.8). This said, the extension problem was actually solved long ago, first in [8], then in [30] where additional details arising in the general case were worked out. The solution of the extension problem is essentially embodied by the Grothendieck–Teichmüller *lego*, as developed in those papers, and which for some reason has remained largely unknown and thus unused.

As for now, let $F \in \mathrm{Aut}(\check{C}(S))$ be a (continuous) automorphism. Summarizing what we have seen above, we may assert that F is determined up to reflection by the image $F(e)$ of a single arbitrary edge $e \in E_P$ of the pants graph $C_P(S)$. Moreover, if $F(e) \in \check{C}_P(S)$, then actually $F \in \check{\Gamma}(S)$, again modulo a possible reflection, that is, it belongs to the procongruence completion of the extended mapping class group $\mathrm{Mod}(S)$. These few sentences may well constitute the essential content of the present paper. The rest consists in some sense in an elucidation or an explicitation of these remarkable facts.

This geometric, complex theoretic picture demonstrates among other things how close we are to the original intuitions developed in [4], which lead to the introduction of the Grothendieck–Teichmüller group. That picture stems from profinite geometry, more particularly from the use of profinite complexes of curves. These carry a kind of homotopical information at infinity for the inverse system of finite covers of a given moduli stack $\mathcal{M}(S)$. This in itself can make it plausible that although they have no group structure *a priori*, their automorphism groups do enjoy amazing rigidity properties. In a way, these complexes enable one to work directly with covers rather than with fundamental groups, as far as moduli stacks of curves are concerned. In order to me more specific, let us introduce the following definition.

Definition 3.5. An oriented embedding is a $\check{\Gamma}(S)$-orbit for the natural left action of $\check{\Gamma}(S)$ on the set of injective maps $j : \check{C}_P(S) \hookrightarrow \check{C}_*(S)$. The set of oriented embeddings is denoted by $J(S)$.

In other words we identify two embeddings j and $j' = g \circ j$, $g \in \check{\Gamma}$. Let us now recall (see [3]) that $\check{C}_P(S)$ is equipped, contrary to $\check{C}_*(S)$ or $\check{C}(S)$, with a natural orientation, once the surface S has itself been given an orientation, which we assume once and for all. It is thus fairly natural to define the set of oriented embeddings by identifying two injective maps j_1 and j_2 as above if the composite map $j_1^{-1} \circ j_2$ lies in $\mathrm{Aut}^+(\widehat{C}_P(S))$, the group of oriented automorphisms of $\widehat{C}_P(S)$. Now assume that $d(S) > 1$, and that S is not of type $(1,2)$ or $(2,0)$ in order to avoid some easy niceties. Then the rigidity statement quoted above states that any such automorphism is induced by an element of the modular group: $\mathrm{Aut}^+(\check{C}_P(S)) \simeq Inn(\check{\Gamma}(S)) \simeq \check{\Gamma}(S)$. Next, there is a natural action of $\mathrm{Aut}(\widehat{C}(S))$ on $J(S)$ obtained by first identifying (see [16]) $\mathrm{Aut}(\check{C}(S))$ with $\mathrm{Aut}(\check{C}_*(S))$, the automorphism group of the pro*graph* $\check{C}_*(S)$, and then postcomposing: $\phi \cdot j = \phi \circ j$ for $j \in J$, $\phi \in \mathrm{Aut}(\widehat{C}_*(S))$. The action factors through the quotient $\mathrm{Aut}(\check{C}(S))/\check{\Gamma}(S) = \mathrm{Out}(\check{C}(S))$ where $\check{\Gamma}(S)$ acts effectively, by using the fact that $\check{\Gamma}(S))$ is normal in $\mathrm{Aut}(\widehat{C}(S))$. The upshot is that the natural action of $\mathcal{G}(S) = \mathrm{Out}(\check{C}(S))$ on $J(S)$ should be free and transitive. Indeed, we have the following claim.

Proposition 3.6. *If $d(S) > 1$, S is not of type $(1,2)$ or $(2,0)$, then $J(S)$ is a $\mathcal{G}(S)$-torsor.*

Proof. The group $\mathcal{G}(S)$ acts on $J(S)$ as explained above and the action is faithful by the definition of $J(S)$. It remains to show that it is also transitive, which is equivalent to showing that any embedding $j : \check{C}_P(S) \hookrightarrow \check{C}_*(S)$ defines a unique automorphism of $\check{C}_*(S)$. Since $\check{C}_P(S)$ and $\check{C}_*(S)$ have the same set $\check{V}$ of vertices, namely the completion of the set V of maximal multicurves (pants decompositions), j defines an automorphism of the vertices of $\check{C}_*(S)$ which has to extend to the edges of that prograph. The extension, if it exists, is unique, because $\check{C}_*(S)$ is a flag complex. There is at most one edge between two vertices. But then the problem is local and can be reduced to modular dimension 1, in which case it boils down to the easy assertion that an embedding $\widehat{F} \hookrightarrow \widehat{G}$, with F the Farey graph and the notation of [16] (§7.5, A.7, A.8), can be extended to an automorphism of the profinite complete graph on $\widehat{G}$. We are back to an extension problem, which has in fact long been solved and will be treated again below. As usual types $(1, 2)$ and $(2, 0)$ are the only mild exceptions. □

Now in fact there is a privileged point in $J(S)$, say j_0, namely the completion of the topological embedding $C_P(S) \hookrightarrow C_*(S)$. In other words the torsor $J(S)$ is "naturally" trivialized and for most purposes can be identified with the group $\mathcal{G}(S)$ itself, which in turn we will identify below with (versions of) the Grothendieck–Teichmüller group. How does this compare with the situation in [4], which led to the original definition of the Grothendieck–Teichmüller group? In [4], §4, the prounipotent genus 0 version of the Grothendieck–Teichmüller group $GT(k)$ (k a field of characteristic 0) appears via universal deformations of quasi-Hopf quasitriangular (or braided) universal enveloping algebras. It is a linear *proalgebraic* group; the full *profinite* version $\widehat{GT}$ is then introduced by analogy (the top of p. 846). The point we would like to make here is that we are in fact exploring a deformation theory in the (full) *profinite* geometric setting. How can one interpret J as a set of deformations?

Classically, if α and β are (isotopy classes of) simple closed curves (loops) on a surface, one says that they have *minimal intersection* if either they are supported on a subsurface of type $(1, 1)$ and intersect at one point only, or they are supported on a subsurface of type $(0, 4)$ and intersect at two points. This topological definition is the essential ingredient in the definition of the graph $C_P(S)$ ([8] or [16], §A.7). The graph $C_P(S)$ *and* its profinite completion are essentially rigid, and so is the *discrete* curve complex $C(S)$, whereas its completion $\widehat{C}(S)$ turns out to have lots of noninner automorphisms, parametrized by the profinite (or procongruence) *deformations of the minimal intersection rule*. The graphs $\widehat{C}_P(S)$ and $\widehat{C}_*(S)$ share the same set $\widehat{V}$ of vertices, which is none other than the completion of the set V of pants decompositions of the surface S. From the above we see that a kind of *transversality* property obtains: given two embeddings $j, j' \in J$, either their images coincide or they have *no* edge in common. An embedding is thus entirely specified by giving one of its edges, and such an edge deserves to be called a rule for minimal intersection. The topological rule recalled above corresponds to the topological embedding j_0.

What profinite simplicial curve complexes are doing is to enable one to devise a *profinite geometric deformation theory* in all genera. The torsor J corresponds to the set of associators (see [4], §§5, 6). However we are in the full profinite setting, which is far from the motivic, or here prounipotent proalglebraic setting. The topological (Betti) embedding j_0 plays the role of the "natural" associator represented by the open interval $(0, 1)$. Still on a rather hypothetical note, another way to put the above and the role of j_0 is to say that everywhere we are working with a preferred *discretification* (in the terminology of Grothendieck's *Longue marche à travers la théorie de Galois*, especially §26), which breaks a certain symmetry — as well as pointing to the fact that we are living in *our* restricted world and no other. To be more specific, consider G a discrete finitely generated group with full profinite completions $\widehat{G}$. For most groups, $\widehat{G}$ does *not* determine G, so that it makes sense to introduce the set of *discretifications* of the profinite group $\widehat{G}$, of which the original G is one, defined as the discrete subgroups of $\widehat{G}$ whose completions (or closure) coincide with $\widehat{G}$. These appear as kinds of nonlinear lattices in $\widehat{G}$, leading perhaps one to daydream of a nonlinear extension of the Hodge theory. Here we will be content to point out that singling out one particular, so to speak "physical", discretification does indeed break some kind of symmetry.

§4. The Grothendieck–Teichmüller group, genus zero

Let us go back to more mundane considerations. In this section, as the title indicates, we consider hyperbolic surfaces S of type $(0, n)$, that is, n-punctured spheres. We assume that $d(S) > 1$, or else $n > 4$. Then $\Gamma = \Gamma(S) \simeq \Gamma_{0,[n]}$ and we know that the CSP holds true. Our first task is to show the following.

Proposition 4.1.

$$\mathrm{Out}(\widehat{C}(S_{0,[5]})) = \widehat{GT}.$$

Before coming to the proof of the proposition however, which by now should not be too difficult, we have to start with a minimal reminder about the profinite version of the Grothendieck–Teichmüller group in genus 0, as originally introduced. We are not aiming here at much more than fixing notation and refer the reader to the following papers for background material (see [19, 10, 8, 30, 15, 16], and their many references). In some sense the present paper is aiming at introducing a different, much less group theoretic view of the Grothendieck–Teichmüller group in its various versions.

In order to sketch the landscape, let us start with the nested sequence of inclusions;

$$G_{\mathbb{Q}} \subset \widehat{GT} \subset \mathrm{Aut}^*(\widehat{F}_2),$$

where $G_{\mathbb{Q}}$ is the Galois group of $\mathbb{Q}$, $\widehat{GT}$ will be defined presently and $\mathrm{Aut}^*(\widehat{F}_2)$ is the profinite group of continuous inertia preserving automorphism of $\widehat{F}_2$, the full profinite completion of $F_2 = \langle x, y\rangle$, itself the free group on 2 generators x and y. Inertia preserving here means that the procyclic groups $\langle x\rangle$ and $\langle y\rangle$ are respectively mapped to conjugate groups by an element $F \in \mathrm{Aut}^*(\widehat{F}_2)$ and so is $\langle z\rangle$, with $xyz = 1$. We refer to [2] for a more general and detailed discussion, remarking only that "inertia" refers to the the procyclic inertia groups associated with the components of a divisor on a scheme (for much more, see [21] and its references). Twisting by inner automorphisms of $\widehat{F}_2$, one can normalize the elements of $\mathrm{Aut}^*(\widehat{F}_2)$ by requiring that the group $\langle x\rangle$ be globally fixed. Behind this normalization and in greater generality are such notions as tangential basepoints, splitting of certain short exact sequences, etc. but the long and the short is that, quite explicitly, the elements of $\mathrm{Aut}^*(\widehat{F}_2)$ we are interested in, including of course the elements of $\widehat{GT}$, are given as pairs $F = (\lambda, f)$ with $\lambda \in \widehat{\mathbb{Z}}^\times$ (the invertible elements of $\widehat{\mathbb{Z}}$) and $f \in \widehat{F}_2'$, the topological derived subgroup of $\widehat{F}_2$. The action of F on $\widehat{F}_2$ is defined by:

$$F(x) = x^\lambda, F(y) = f^{-1}y^\lambda f. \tag{4.1}$$

The first requirement for such an automorphism to belong to $\widehat{GT}$ is that it defines an *invertible* endomorphism, and there is no effective way to ensure invertibility in this profinite setting, in constrast to the pronilpotent one. Multiplication is given by composition in the automorphism group $\mathrm{Aut}(\widehat{F}_2)$, which leads to the following formula for the product of $F = (\lambda, f)$ and $F' = (\lambda', f')$:

$$F' \circ F = (\lambda\lambda', fF'(f)). \tag{4.2}$$

Then for an automorphism $F \in \mathrm{Aut}^*(\widehat{F}_2)$ to define an element of $\widehat{GT}$, the associated pair (λ, f) has to satisfy the following 3 relations (notation as in §2 above):

(I) (2-cycle) $f(x, y)f(y, x) = 1$;

(II) (3-cycle) $f(x, y)x^\mu f(z, x)z^\mu f(y, z)y^\mu = 1$ where $xyz = 1$ and $\mu = (\lambda - 1)/2$;

(III) (5-cycle) $f(x_{12}, x_{23})f(x_{34}, x_{45})f(x_{51}, x_{12})f(x_{23}, x_{34})f(x_{45,51}) = 1$.

Thus $\widehat{GT} \subset \mathrm{Aut}^*(\widehat{F}_2)$ is the subgroup whose elements are defined by pairs $F = (\lambda, f) \in \widehat{\mathbb{Z}}^* \times \widehat{F}_2'$ acting on $\widehat{F}_2$ as above and satisfying (I), (II), and (III).

Note that these are often refered to as "relations" although "equations" would be more correct, $\widehat{GT}$ being a closed subgroup, not a quotient of $\mathrm{Aut}^*(\widehat{F}_2)$.

Remark 4.1. It was noted long ago (by H. Furusho) that (I) is actually an easy consequence of (III). Indeed it suffices to erase the 5th point (or fill in the 5th puncture, or pull out the 5th strand, in various languages) to transform (III) into (I) because then x_{45} and x_{51} disappear (recall that f is a commutator), whereas $x_{34} = x_{12}$.

We nevertheless retain (I) in the definition of $\widehat{GT}$ because of its geometric meaning (see, e.g., [5]). Recall also that actually f determines λ up to a sign. Picking one identification of λ, $F^+ = (\lambda, f)$ and $F^- = (-\lambda, f)$ differ by mutiplication by $c = (-1, 1)$, which in complex or Galois terms is noneother than complex conjugacy ($F^- = F^+ \circ c$). This is readily inferred from the multiplication law, which simply reflects the composition of automorphisms. We add that there is no global uniform identification of λ, that is the projection $(\lambda, f) \mapsto f$ does not admit a global section.

Let us now turn to the proof.

Proof of Proposition 4.1**.** When $g(S) = 0$ there is a canonical injection $\widehat{GT} \hookrightarrow \mathrm{Out}(\widehat{C}(S))$, coming as the composition of two other natural injections. The first one reads $\widehat{GT} \hookrightarrow \mathrm{Out}^*(\widehat{\Gamma}(S))$, which is essentially by definition (or see [4, 19]). It will evolve that this injective map is actually an isomorphism but we are *not* using this here; rather Theorem 4.2 below will prove this isomorphism, with no extra assumption. Second there is a natural injection $\mathrm{Out}^*(\widehat{\Gamma}(S)) \hookrightarrow \mathrm{Out}(\widehat{C}(S))$ because $\mathrm{Out}^*(\widehat{\Gamma}(S))$ by definition permutes the conjugacy classes of the procyclic groups generated by twists, thus inducing an automorphism of the complex $\widehat{C}(S)$. Here we are actually using the validity of the congruence conjecture in genus 0 and the isomorphism results recalled and reproved in [16] (see also the original references there). Finally the map $\mathrm{Out}^*(\widehat{\Gamma}(S)) \to \mathrm{Out}(\widehat{C}(S))$ is injective because $\widehat{\Gamma}(S)$ is centerfree.

We now prove the opposite inclusion, that is $\mathrm{Out}(\widehat{C}(S)) \subset \widehat{GT}$. Given $F \in \mathrm{Out}(\widehat{C}(S))$, we singled out above the edge $a \in \widehat{E}_P$, that is the first side of the standard pentagon ϖ_0 (see Figure 1) in order to attach the element $f \in \langle x, y\rangle'$ to F. We have already noted the unfortunate discrepancy with the usual notation, coming from [4]: the element f in equations (2.1) and (2.2) will correspond to the parameter f^{-1} of the Grothendieck–Teichmüller group, simply because the Drinfel'd original notation goes against the variance of the respective actions of the Teichmüller group on curves and twists.

We could use any other edge $e \in \widehat{E}_P$ of the pants graph $\widehat{C}_P = \widehat{C}_P(S)$, and we get a map $f : \widehat{E}_P \to \widehat{F}_2'$ which, to $e \in \widehat{E}_P$, assigns an element $f_e \in \widehat{F}_2'$.

We wish to show that this map is constant, that is f_e is in fact independent of e. Now in our case $\widehat{\Gamma} = \widehat{\Gamma}(S)$ acts transitively on $\widehat{E}_P$. Indeed, any edge of $\widehat{C}_P$ can be completed into a pentagon and it remains to apply Proposition 3.1. So in order to show that f is a constant map, it suffices to show that it is $\widehat{\Gamma}$-invariant. Let again $F \in \mathrm{Aut}(\widehat{C}(S))$, $e, e' \in \widehat{E}_P$. There exists $g \in \widehat{\Gamma}$ with $g \cdot e = e'$; considering $g^{-1}Fg$, we are reduced to showing that F and $g^{-1}Fg$ define the same value of $f = f_e$. But this is clear because $\widehat{\Gamma}$ is normal in $\mathrm{Aut}(\widehat{C})$, so that $g^{-1}Fg = g^{-1}FgF^{-1}F = hF$ for some $h \in \widehat{\Gamma}$. More explicitly, this equivariance property goes as follows. Let $e = (\alpha, \beta)$ and $e' = (\alpha', \beta')$ be two edges of $\widehat{C}_P$, defined by the corresponding pairs of curves. Write $f_e = f$, $f_{e'} = f'$ and suppose $e' = g \cdot e$ with $g \in \widehat{\Gamma}$, so that $g\tau_\alpha g^{-1} = \tau_{g\cdot\alpha} = \tau_{\alpha'}$ and $g\tau_\beta g^{-1} = \tau_{\beta'}$. Then:

$$f'(\tau_{\alpha'}, \tau_{\beta'}) = gf(\tau_\alpha, \tau_\beta)g^{-1} = f(g\tau_\alpha g^{-1}, g\tau_\beta g^{-1}) = f(\tau_{\alpha'}, \tau_{\beta'}),$$

where the second identity is a formal algebraic fact.

It remains to show that $f \in \widehat{F}_2'$ coming from an element $F \in \mathrm{Aut}(\widehat{C}(S))$ satisfies relations (I), (II), (III) recalled above. In essence this is not really different from what we saw above as will be made even clearer after the proof. These relations essentially reflect the torsion of $\Gamma_{0,[5]} = \Gamma(S_{0,5})$ or better of the extended mapping class group $\mathrm{Mod}(S_{0,5})$ ($\Gamma_{0,[5]}$ has no 2-torsion). For further consequences of this remark, we refer to [20]. We have seen already how (I) is a consequence of (III) so does not need to be proved independently. We will nevertheless first give a proof of (I), which emphasizes the roles of reflections, but since this proof is not logically necessary we will remain a bit sketchy. Given the above, especially thanks to the equivariance property derived above, we are in any event essentially back to the usual group theoretic GT-setting so that the proofs very much follow the usual well-trodden path (see, e.g., [19]).

To (I): everything in relations (I) and (II) is local on a piece $S' \subset S \simeq S_{0,5}$ of type $(0,4)$ ($d(S') = 1$). Write as usual $x = x_{12}$, $y = x_{23}$, $f = f(x,y) \in \langle x, y\rangle'$. We can find a reflection r of the surface S ($r^2 = 1$) such that $r(x) = y$ and $r(y) = x$; of course $r \in \mathrm{Mod}(S)$ but r does not belong to $\Gamma(S) = \mathrm{Mod}^+(S)$. Then we compute that:

$$rFr(x) = fF(y) = f^{-1}(y,x)yf(y,x), \qquad rFr(y) = rF(x) = r(x) = y.$$

Here we use a group theoretic notation which should be taken *cum grano salis*. We should for instance use procyclic groups instead of generators, that is, replace x and y by $\langle x\rangle$ and $\langle y\rangle$. Next we show that rFr and F define the same f. In fact we can find another reflection r' such that $r'(x) = x^{-1}$ and $r'(y) = y^{-1}$ and $r'Fr'$ certainly defines the same element f as F. But $rr' \in \Gamma(S)$ and so $rFr = r'Fr'$ as elements of $\mathrm{Out}(\widehat{C})$. We can now twist rFr

by $f^{-1}(y,x)$ (that is, consider $\mathrm{Inn}(f^{-1}(y,x))rFr$), which does not alter it as an outer automorphism, and from the formulas above we readily deduce that it is also defined by $f'(x,y)=f^{-1}(y,x)$. This implies relation (I). We remark that in terms of finite groups, we have been playing around with the dihedral group of order 10, preserving the cyclic order on 5 points. Elements of $\Gamma(S)$ induce the cyclic subgroup of index 2 and the action of the reflections leads to relation (I).

We now pass to relations (II) and (III), actually starting with (III) which is a little easier to deal with at this point. Considering again the standard pentagon $\varpi_0\subset C_P$ whose vertices are the pants decompositions P_i, $i\in\mathbb{Z}/5$ (see Figure 1), let $F\in\mathrm{Aut}(\widehat{C}(S))$ and normalize it so that $F(P_1)=P_1$ ($P_1=(\alpha_1,\alpha_4)$). It acts on ϖ_0 via formulas (2.1) and we normalize it further so that f and g are commutators. We will freely use these topological tangential basepoints in what follows, using twists (that is, profinite multitwists) on pants decompositions. The group $\Gamma=\Gamma(S)\simeq\Gamma_{0,[5]}$ contains an element ρ of order 5, which is none other than the usual rotation of the same order. Explicitly $\rho=\tau_4\tau_3\tau_2\tau_1$ in terms of the standard generators of the Artin braid group B_5 (see §2), of which Γ is a quotient. Then ρ acts on the τ_i's ($i\in\mathbb{Z}/5$) by shifting indices: $\rho^{-1}\tau_i\rho=\tau_{i+1}$ and ditto for the twists $a_i=\tau_i^2=x_{i,i+1}$ along the loops α_i. In other words ρ rotates the pentagon ϖ_0 by $2\pi/5$.

Now let $F'=\mathrm{Inn}\big(f(x_{12},x_{23})x_{12}^a x_{45}^b\big)\circ F$ with $a,b\in\widehat{\mathbb{Z}}$. Using equivariance and injectivity (Theorem 1.2) we obtain that *there exist* a and b such that $F'=\rho^{-1}(F)=\rho F\rho^{-1}$. We continue in this way going around the pentagon, and find that $F=\mathrm{Inn}(\pi)F$ for a certain element $\pi\in\widehat{\Gamma}$ of the form:

$$\pi=f(x_{12},x_{23})x_{12}^{a_1}f(x_{34},x_{45})x_{34}^{a_3}f(x_{51},x_{12})x_{51}^{a_5}f(x_{23},x_{34})x_{23}^{a_2}f(x_{45},_{51})x_{45}^{a_4}.$$

Here we have used the fact that one element of the multitwist commutes with f (e.g., x_{45} commutes with $f(x_{12},x_{23})$) so that only one twist gets sandwiched between two f's. So we find that $\pi=1$ and this is precisely the usual pentagon (see relation (III)) except for these possible twists with exponents $a_i\in\widehat{\mathbb{Z}}$. But now the abelianization of $\widehat{\Gamma}$ is $\widehat{\Gamma}^{ab}\simeq\widehat{\mathbb{Z}}^5$ generated by the $x_{i,i+1}$'s and looking at the image of π in $\widehat{\Gamma}^{ab}$ we deduce that $a_i=0$ for all i, because f is a commutator, which completes the proof that f satisfies relation (III).

Relation (II) is proved in much the same way, essentially as in [5] or [19]. We will content ourselves with a sketch. Precisely as relation (I), it is fulfilled on $S'\subset S\simeq S_{0,5}$ of type $(0,4)$, cut out by α_4. We have an inclusion of the braid group on 3 strands B_3 into Γ; B_3 is generated by τ_1 and τ_2 with $\tau_1^2=x_{12}$, $\tau_2^2=x_{23}$. We recall from §2 that $\tau_{13}=\tau_2\tau_1\tau_2^{-1}$ and $x_{13}=\tau_{13}^2$. With this notation the so-called lantern relation on S' reads: $x_{12}x_{13}x_{23}=x_{45}$, which commutes of course with x_{12}, x_{13}, and x_{23}. There is now a triangle in $C_P(S)$, given by the pants decompositions $P_1 = (\alpha_1, \alpha_4)$, $P_2 = (\alpha_2, \alpha_4)$, and $P_3 = (\alpha_{13}, \alpha_4)$, x_{13} being the twist along the curve α_{13}, which we refrain from drawing (it is the "third curve" in the classical lantern). Instead of ρ, one then uses $\omega = \tau_1\tau_2$ which, regarded as an element of $\Gamma_{0,[5]}$, satisfies $\omega^3 = x_{12}x_{13}x_{23} = x_{45}$ and permutes x_{12}, x_{23}, and x_{13} cyclically. As noticed in [HLS], relation (II) actually holds true in any group generated by elements x, y, and z such that xyz is central. So we use of course the three twists $x = x_{12}$, $y = x_{23}$, and $z = x_{13}$ and can just as well set $x_{45} = 1$, collapsing the curve α_4 to a point. Proceeding as above one gets a triangular identity of the form:

$$f(x,y)x^a f(z,x)z^b f(y,z)y^c = 1,$$

for some $a, b, c \in \widehat{\mathbb{Z}}$. This time however, we are in the colored braid group on 3 strands, whose abelianization is generated by x, y, and z with the only relation $x + y + z = 0$. So looking at the abelianization of that relation we see that $a = b = c$ and call that common value μ, recovering relation (II). The value of μ is actually not well defined; it can be changed to $-(1+\mu)$ or in other words, setting $\lambda = 2\mu+1$ the sign of λ is still to be determined. One gets two elements of $\widehat{GT}$ differing from each other by a reflection, namely $(\pm\lambda, f)$, and one has simply to compare with the original F to determine which of them coincides with it. □

This shows Proposition 4.1 which, given the above, is essentially equivalent to the theorem below, in which $S_{0,5}$ is changed to $S_{0,n}$ for every $n > 4$. We insist however that from the viewpoint of the present paper, the crux of the matter does not lie so much in the three defining relations (or equations) of $\widehat{GT}$ as in the fact that the group $\mathrm{Out}(\widehat{C}(S))$ for $S = S_{0,n}$, $n > 4$, can be described purely in terms of pentagons as defined in Definition 2.2. Those are of the form $F(\varpi_0)$, $F \in \mathrm{Aut}(\widehat{C}(S))$, and F is entirely determined by the class of any of the edges modulo $\widehat{C}_P(S)$. In more precise terms, this is expressed by Proposition 3.6. Relations (I), (II), and (III), which *a priori* reflect the well-known Mac Lane's relations in braided categories, are direct consequences of this geometric, complex theoretic description.

It is interesting to keep in mind the action of $\widehat{GT}$ on $\widehat{\Gamma}_{0,[5]}$, which extends the defining action (4.1) on $\widehat{F}_2 = \widehat{\Gamma}_{0,4}$ (the formulas below appear in [19], Lemma 7). We will use the notation of the reminder below Lemma 2.4. Given $F = (\lambda, f) \in \widehat{GT}$, we write out its action on the twists $\tau_1 = \tau_{12}$, $\tau_2 = \tau_{23}$, $\tau_3 = \tau_{34}$, $\tau_4 = \tau_{45}$, and τ_{15} which generate $\widehat{\Gamma}_{0,[5]}$, namely:

$$F(\tau_1) = \tau_1^\lambda,$$

$$
\begin{aligned}
&F(\tau_2) = f(x_{23}, x_{12})\tau_2^{\lambda} f(x_{12}, x_{23}),\\
&F(\tau_3) = f(x_{34}, x_{45})\tau_3^{\lambda} f(x_{45}, x_{34}),\\
&F(\tau_4) = \tau_4^{\lambda},\\
&F(\tau_{15}) = f(x_{23}, x_{12}) f(x_{51}, x_{45})\tau_{15}^{\lambda} f(x_{45}, x_{51}) f(x_{12}, x_{23}).
\end{aligned}
\tag{4.3}
$$

We add that the a_i's ($i \in \mathbb{Z}/5$) generate the pure subgroup $\Gamma_{0,5}$, with $a_i = \tau_i^2 = x_{i,i+1}$ the twist along the loop α_i (see Figure 1). So, by the 2-cycle relation, these formulas imply, in fact are equivalent to (see Lemma 4.1 in [2]) the following:

$$
\begin{aligned}
&F(a_1) = a_1^{\lambda},\\
&F(a_2) = f^{-1}(a_1, a_2) a_2^{\lambda} f(a_1, a_2),\\
&F(a_3) = f(a_3, a_4) a_3^{\lambda} f^{-1}(a_3, a_4),\\
&F(a_4) = a_4^{\lambda},\\
&\quad(a_5) = \big(f(a_4, a_5) f(a_1, a_2)\big)^{-1} a_5^{\lambda} f(a_4, a_5) f(a_1, a_2)
\end{aligned}
$$

These formulas are to be compared with formulas (2.1) above, modulo the notational discrepancy explained after formulas (2.1), (2.2). As for the parameter $\lambda \in \widehat{\mathbb{Z}}^{\times}$, it has been mentioned already that it is determined up to a sign by f and can hardly be directly detected complex theoretically, because it is not visible on the curves themselves. Again this is reminiscent of the S^1-action in the string topology, after proper completion. Now, how do we make sense of these formulas? There are (at least) two ways, which are probably somehow equivalent. The first one is via the Grothendieck–Teichmüller lego, introduced in [8] and detailed, in the general case, in [30]. Somewhat more will be said about it below and in §5. The second way, which we will not try to justify completely although it would be quite feasible, consists of bringing back this action to that of $\widehat{GT}$ on the *groupoid* attached to $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, based at six standard real tangential basepoints (see, e.g., [5]).

First the action of F on a_1 and a_4 is a matter of normalization: F preserves the pants decomposition (α_1, a_4). Next the action on a_2 is precisely by definition of the parameter $f \in \widehat{F}_2'$. That leaves only $F(a_3)$ and $F(a_5)$ to be interpreted. Here the crux of the matter, which can be justified by going to the geometry of the stable completion of the moduli scheme $\mathcal{M}_{0,5} = \mathcal{M}(S_{0,5})$, consists of the fact that, since the decomposition (α_1, α_4) is preserved, we may let the points 1 and 2 (respectively, 4 and 5) shrink to one point, say 0 (respectively, ∞). That leaves us with only 3 points, which we may name 0, 1, and ∞, renumbering of the renaming point 3 as 1. Here 0, 1, and ∞ are not only a matter of names but represent the respective values on the complex projective line. The formulas above then simply match the action of $\widehat{GT}$ on the standard *groupoid* attached to $\mathbb{P}^1 \setminus \{0,1,\infty\}$. In particular $F(a_2)$ reflects the action on the straight path from 0 to 1 (Deligne's *dch*, "*droit chemin*", carrying unmistakable moral undertones...). Denoting this path by p, we have $F(p) = p \circ f$, which is the defining equation for f expressing the fact that it underlies a deformation of the associativity functor in a braided category. In turn $F(a_3)$ matches the action on the path from 1 to infinity, which can also be seen by, so to speak, inverting the poles of the sphere (hence the change from f to f^{-1}). As for the action on a_5, corresponding to the path from ∞ to the origin 0, it can be read off from the action on the path $p = dch$ by turning clockwise $((0,1,\infty) \mapsto (\infty,0,1))$. The lego of course delivers the same answer.

It is now high time to state a result in genus 0 and arbitrary n.

Theorem 4.2. *Let $S \simeq S_{0,n}$, $n > 4$, a sphere with n marked points, so $d(S) = n-3 > 1$ and $\widehat{\Gamma}(S) \simeq \widehat{\Gamma}_{0,[n]}$. Then:*

i) $\mathrm{Out}(\widehat{C}(S)) = \mathrm{Out}(\widehat{\Gamma}(S)) = \widehat{GT}$;

ii) *any automorphism of an open subgroup of $\widehat{\Gamma}(S)$ is induced by an automorphism of $\widehat{\Gamma}(S)$ up to the action of a finite group.*

Proof. This statement is actually a consequence of Proposition 4.1, modulo existing results (often from a long time ago!). Let $F \in \mathrm{Aut}(\widehat{C}(S))$; since it is type preserving, after composing by an element of $\widehat{\Gamma}(S)$ we may assume that F fixes a pants decomposition and consider the restriction of F to a piece $T \subset S$ of type $(0,5)$. By Proposition 4.1 that restriction is an element of $\widehat{GT}$, viewed as a subgroup of $\mathrm{Aut}(\widehat{C}(T))$, and by *injectivity* it determines F. In other words we get an injective map $\mathrm{Out}(\widehat{C}(S)) \hookrightarrow \widehat{GT}$. The surjectivity of this map is a matter of *extension* and this was shown long ago. Probably the best and most effective (explicit) way of showing this is via the use of the Grothendieck–Teichmüller lego, which is particularly simple in genus 0. In the language of [8] and [30] we encounter only A-moves, no S-moves. It all hinges on the main result of [8], which states that the 2-dimensional simplicial complex $C_P(S)$ is (connected and) *simply connected*. The lego enables one to explicitly compute the $\widehat{GT}$-action on any curve or multicurve. Using that $C(S)$ is dense in $\widehat{C}(S)$, we are done. In fact it suffices to compute the action on a finite number of curves whose associated twists generate $\Gamma(S)$ because of the second point of assertion i), namely $\mathrm{Out}(\widehat{C}(S)) = \mathrm{Out}(\widehat{\Gamma}(S))$, the proof of which is now straightforward. Indeed, there is an injection $\mathrm{Out}^*(\widehat{\Gamma}) \hookrightarrow \mathrm{Out}(\widehat{C}(S))$ and conversely $\widehat{GT} \subset \mathrm{Out}^*(\widehat{\Gamma})$; since $\mathrm{Out}(\widehat{C}(S)) = \widehat{GT}$ we conclude as in i). Finally, here and in ii), we use the results of [9] to the effect that in genus 0 all the automorphisms of $\widehat{\Gamma}(S)$ and its open subgroups are inertia preserving.

Item ii) comes essentially for free as in the discrete case (see [13]): The automorphisms of the curve complex control those of the open subgroups. Moreover in the discrete case these automorphisms are all inertia preserving (whatever the genus of S) and in the complete case this is true at least in genus 0. We also mention that by a classical result of N. Nikolov and D. Segal, as $\Gamma(S)$ is finitely generated, the open subgroups of $\widehat{\Gamma}(S)$ are precisely the subgroups of finite index (that is, any such subgroup is open). Taking up the notation of the introduction, let $\Gamma^\lambda \subset \Gamma(S)$ be a finite index subgroup. Again there is an injective map: $\mathrm{Aut}^*(\widehat{\Gamma}^\lambda(S)) = \mathrm{Aut}(\widehat{\Gamma}^\lambda(S)) \hookrightarrow \mathrm{Aut}(\widehat{C}(S))$. When combined with i), this proves ii). □

To be more explicit, if in particular Γ^λ is normal in $\widehat{\Gamma}(S)$, we get an exact sequence:

$$1 \to \widehat{\Gamma}(S) \to \mathrm{Aut}(\widehat{\Gamma}^\lambda) \to \mathcal{G}(S)^\lambda \to 1;$$

since $\widehat{\Gamma}(S) = \mathrm{Inn}(\widehat{\Gamma}(S))$, this can be rewritten as:

$$1 \to \Gamma(S)/\Gamma^\lambda \to \mathrm{Out}(\widehat{\Gamma}^\lambda) \to \mathcal{G}(S)^\lambda \to 1.$$

Both sequences are split and $\mathcal{G}(S)^\lambda \subset \mathcal{G}(S) = \widehat{GT}$ is a subgroup whose meaning is clear at least at the level of Galois groups. In Galois terms, Γ^λ corresponds to a level structure $\mathcal{M}^\lambda$, that is a finite Galois (stack unramified) cover of $\mathcal{M} = \mathcal{M}(S) \simeq \mathcal{M}_{0,[n]}$. To the cover $\mathcal{M}^\lambda/\mathcal{M}$ a field of moduli $K = K^\lambda$ is attached, which is also a field of definition because the cover is Galois. Then K is a finite extension of $\mathbb{Q}$ with Galois group G_K and there is an inclusion $G_K \subset \mathcal{G}(S)^\lambda$, which is natural up to inner automorphism.

We remark that assertion ii) seems hardly accessible to the group theoretic methods, which have been used hitherto in the profinite setting. It could be expanded and refined, again much as in the discrete case (see [13] and later articles on the subject), in particular by considering morphisms between open subgroups of $\widehat{\Gamma}(S)$ rather than merely automorphisms.

We also note, as a direct consequence, the following theorem which parallels the Dyer–Grossman rigidity statement in the discrete case, namely $\mathrm{Out}(B_n) = \mathbb{Z}/2$ for $n > 2$, and improves on the results of [19].

Theorem 4.3 ([26], Theorem B)). *$\mathrm{Out}(\widehat{B}_n/Z) = \widehat{GT}$ for $n > 3$.*

Proof. Here $Z = Z(\widehat{B}_n)$, the center of the plane Artin braid group on n strands. It is procyclic freely generated by ω_n in the notation of §2 above (see after the proof of Lemma 2.4 or [19]; see also [26]). Modulo the remarks below, this is a consequence of Theorem 4.2 and the following isomorphism and inclusions (see [19], Proposition A.4, and [26]):

$$\Gamma_{0,n+1} \simeq K_n/Z \subset B_n/Z \subset \Gamma_{0,[n+1]}$$

valid for all $n > 1$ with $Z = \langle \omega_n \rangle \simeq \mathbb{Z}$. This readily extends to the respective profinite completions. For $n > 3$, one has $\mathrm{Out}(\widehat{\Gamma}_{0,[n+1]}) = \widehat{GT}$ by i) of Theorem 4.2 and $\mathrm{Out}(\widehat{\Gamma}_{0,n+1})$ is an extension of $\mathcal{S}_{n+1}$ by $\widehat{GT}$, according to assertion ii) of the same theorem (see also [2] and references therein). Now in fact B_n/Z can be identified with the subgroup of $\Gamma_{0,[n+1]}$ that maps to the stabilizer of — say — the last point via the natural projection $\Gamma_{0,[n+1]} \to \mathcal{S}_{n+1}$. This stabilizer is isomorphic to $\mathcal{S}_n$ and self-normalizing in $\mathcal{S}_{n+1}$. We thus obtain $\mathrm{Out}(\widehat{B}_n/Z) = \widehat{GT}$. Two remarks are in order. First we again "removed the star" in these statements, thanks to the results in [25] in the discrete case and those in [9] in the complete case. Second, the procyclic center does add in a factor in the outer automorphism group, as shown recently in [26] (see Theorem B there). □

⋆ ⋆ ⋆

As a last topic in this section, we will briefly explain how the results of this section, more particularly Theorem 4.2 i), can be transposed to the pro-ℓ setting, where $\ell > 1$ is a prime integer, which we fix once and for all. We note that in [9] the full profinite and pro-ℓ cases were treated simultaneously, and that the setting and results of [1], a rather small part of which are used below, should make it possible to treat the case of arbitrary genus (especially $g > 2$) in a relative procongruence setting for general classes of finite groups, i.e., classes that are stable under extensions (see [1], Definition 1.1; this notion goes back to work of J-P.Serre in the early fifties). Yet in this paper and partly for the purpose of illustration we will restrict examination to the pro-ℓ setting (i.e., the class of finite ℓ-groups) and surfaces of genus 0, that is, n-pointed spheres, which is substantially simpler.

So we start again with a hyperbolic $S \simeq S_{0,n}$ $(n > 2)$. We define the pro-ℓ completion of $\Gamma_{0,n}$, denoted by $\Gamma_{0,n}^{(\ell)}$, as usual; it is the maximal pro-ℓ quotient of the full profinite completion $\widehat{\Gamma}_{0,n}$. Then we set $\Gamma_{0,[n]}^{(\ell)} = \mathcal{S}_n \ltimes \Gamma_{0,n}^{(\ell)}$, extending the semidirect structure in the discrete setting. Note that $\Gamma_{0,n}^{(\ell)}$ is a pro-ℓ group whereas $\Gamma_{0,[n]}^{(\ell)}$ is only virtually pro-ℓ. Both groups are residually finite, that is the natural maps $\Gamma_{0,n} \mapsto \Gamma_{0,n}^{(\ell)}$ and $\Gamma_{0,[n]} \mapsto \Gamma_{0,[n]}^{(\ell)}$ are injective. In fact it suffices to consider the first one and then the result comes by induction, if we recall that $\Gamma_{0,n}$ is an iterated extension of free groups.

There is however another possible definition of a pro-ℓ group attached to the pure mapping class group (see items 1 and 4 of the Introduction). Let $\pi = \pi_1^{\mathrm{top}}(S) \simeq \pi_{0,n}$ denote the topological fundamental group of S, which is a (discrete) free group on $n-1$ generators. There is a natural injective map $\Gamma \hookrightarrow \mathrm{Aut}(\pi)$, the topological universal monodromy, which descends to a map $\Gamma \to \mathrm{Aut}(\pi/K)$ for every cofinite characteristic subgroup K of π. Now restrict attention to those subgroups such that π/K is an ℓ-group, i.e., a group whose cardinal is a power of ℓ. The kernels of these maps for varying K will be called principal ℓ-congruence subgroups. An ℓ-congruence subgroup of Γ is a subgroup that contains a principal ℓ-congruence subgroup and the limit of these groups is the ℓ-congruence completion $\check{\Gamma}^{(\ell)}$. Equivalent definition: there is a natural map $\Gamma \to \mathrm{Aut}(\pi^{(\ell)})$ (where $\pi^{(\ell)}$ denotes the pro-ℓ completion of the free group π) extending the topological universal monodromy; the ℓ-congruence completion $\check{\Gamma}^{(\ell)}$ is the closure of the image of Γ. Note that the above can be repeated essentially word for word for any type (g,n); however, not suprisingly, the general situation turns out to be substantially more complicated (see [1]).

It is known that in general (for every type (g,n)) the ℓ-congruence completion $\check{\Gamma}^{(\ell)}(S)$ is virtually a pro-ℓ group. Next, it was shown by M.Asada that in genus 0 the ℓ-congruence conjecture holds true (see [1] §5.1) in the sense that $\check{\Gamma}^{(\ell)}(S)$ and $\Gamma^{(\ell)}(S)$ as defined above virtually coincide, that is contain isomorphic open (equivalently cofinite) subgroups. We now define the pro-ℓ completion $C^{(\ell)}(S)$ of the curve complex as in the Introduction, item 1, using $\check{\Gamma}^{(\ell)}(S)$ or $\Gamma^{(\ell)}(S)$ (i.e., the corresponding subinverse systems of finite quotients of $\Gamma(S)$) indifferently because, by the above, the resulting limits coincide. From now on we use and write $\Gamma^{(\ell)}(S) \simeq \Gamma^{(\ell)}_{0,[n]}$, defined as above.

It remains to define $GT^{(\ell)}$, the pro-ℓ version of $\widehat{GT}$, proceeding as in [4] (or [9], among other places). We simply copy word for word the definition of $\widehat{GT}$ but instead of elements $F = (\lambda, f) \in \widehat{\mathbb{Z}}^\times \times \widehat{F}_2'$ the elements of $GT^{(\ell)}$ are parametrized by pairs $(\lambda^{(\ell)}, f^{(\ell)}) \in \widehat{\mathbb{Z}}_\ell^\times \times F_2^{(\ell),}$. With an element $F = (\lambda, f) \in \widehat{GT}$ we can associate its projection $F^{(\ell)} = (\lambda^{(\ell)}, f^{(\ell)})$ where $\lambda^{(\ell)}$ and $f^{(\ell)}$ are the respective projections of λ and f. In particular to a Galois element $\sigma \in G_{\mathbb{Q}}$, the objects F_σ and $F_\sigma^{(\ell)}$ get attached. Beware of the fact that $GT^{(\ell)}$ is *not* the maximal pro-ℓ quotient of $\widehat{GT}$, not even a pro-ℓ group.

Now it was shown in [1] (see also [9]) that all the properties we have used in the proof of Proposition 4.1 and Theorem 4.2 still hold true in the pro-ℓ setting. In fact this is valid in much greater generality, namely for the relative progruence completion attached to a hyperbolic surface of general type and involving an arbitrary class of finite groups. Since in genus 0 the ℓ-congruence conjecture has been validated, we can freely use these properties in terms of pro-ℓ completion. In particular (see [1], Theorem 3.1) the pro-ℓ topology respects the stratification induced by cutting surfaces along multicurves, that is, the pro-ℓ version of Proposition 5.8 in [16] is valid. Here we have used the word "stratification" because in geometric terms it corresponds to the natural stratification of the divisor at infinity of the stable completions of the moduli stack $\mathcal{M}(S)$ and its *étale* covers (see [21]).

It remains only to copy word for word the proofs of Proposition 5.1 and Theorem 4.2 (or to treat both the full profinite and the pro-ℓ case simultaneously, as in [9]) to get the following statement.

Theorem 4.4. *Let ℓ be a prime integer, $S \simeq S_{0,n}$ an n-pointed sphere, $n > 4$; let $\Gamma^{(\ell)}(S) \simeq \Gamma^{(\ell)}_{0,[n]}$, $C^{(\ell)}(S)$, and $GT^{(\ell)}$ be as above. Then*:

$$\mathrm{Out}(C^{(\ell)}(S)) = \mathrm{Out}(\Gamma^{(\ell)}(S)) = GT^{(\ell)}.$$

We leave it to the interested reader to elaborate on the corollaries to this result, in the style of [13] and Theorem 4.2 ii) above, returning instead to the full profinite setting and moving to the case of surfaces of higher genus.

§5. The Grothendieck–Teichmüller group, strictly positive genus

We now move to the case of an arbitrary genus. In this section we will have to rely in part on the material and results developed in [8] and [30] to which we globally refer for more detail and geometric background. In that sense, what follows should not be regarded as self-contained because some notions and constructions would simply be too heavy to recall in complete detail. The proofs of the statements are of course complete, involving the results in the two above references and a few others.

In modular dimension 2, one finds types $(0,5)$ and $(1,2)$, which are of course closely related. In particular the corresponding curve graphs $C(S_{0,5})$ and $C(S_{1,2})$ are isomorphic but we will have to spell out this isomorphism here and the differences between these two cases are in fact responsible for the introduction of the subgroup $\mathbb{I}\Gamma \subset \widehat{GT}$, governing the situation in a strictly positive genus. In modular dimension 3, there are 3 types of surfaces, namely $(0,6)$, $(1,3)$, and $(2,0)$. The first and last one are again closely related (recall that $C(S_{2,0}) \simeq C(S_{0,6})$) but the middle one will turn out to represent the generic case; in particular $\mathbb{I}\Gamma = Out^*(\widehat{\Gamma}_{1,[3]})$. This universality is explored below and, from a somewhat different viewpoint, in the Appendix.

We will start with disposing of the two exceptional cases with nontrivial center, that is, types $(1,2)$ and $(2,0)$, from the point of view of group automorphisms, reminding the reader that the congruence conjecture has been vindicated in these cases. Of course there is nothing to do here concerning curve complexes because of the isomorphisms mentioned above. We thus state and prove the following claim, writing $\Gamma_2 = \Gamma_{2,0}$ (the content of this proposition appears also in [2] and other places).

Proposition 5.1.
i) $\mathrm{Out}^*(\widehat{\Gamma}_{1,[2]}/Z) = \mathrm{Out}^*(\widehat{\Gamma}_2/Z) = \widehat{GT}$;
ii) $\mathrm{Out}^*(\widehat{\Gamma}_{1,[2]}) = \mathrm{Out}^*(\widehat{\Gamma}_2) = \widehat{GT} \times \mathbb{Z}/2$.

Proof. Here $Z = \langle\iota\rangle \simeq \mathbb{Z}/2$ denotes the center of the group at hand, which in both cases is of order 2, generated by the (hyper)elliptic involution; for simplicity we denote the latter by the common letter ι. As to the first assertion, $\Gamma_{1,[2]} = \Gamma_{1,2} \times Z$ is the direct product of the pure group by its center and the same assertion is valid for the completed groups. So $\widehat{\Gamma}_{1,[2]}/Z = \widehat{\Gamma}_{1,2} \subset \widehat{\Gamma}_{0,[5]}$. Moreover and as already mentioned, $\widehat{\Gamma}_{1,2}$ is of index 5 and self-normalizing in $\widehat{\Gamma}_{0,[5]}$ (cf. [16], §A.4). Combining this with ii) of Proposition 4.1 (for $n = 5$) we get the first assertion. The case of $\widehat{\Gamma}_2$ is immediate, because $\widehat{\Gamma}_2/Z = \widehat{\Gamma}_{0,[6]}$, and it remains to apply Theorem 4.2.

Granted i), the proof of assertion ii) follows that of a similar statement in the discrete case. In particular, the case of Γ_2 was treated in detail in [25] (with $\mathrm{Out}(\Gamma_2/Z) = \mathbb{Z}/2$). The crux of the matter is to compute the abelianizations of the intervening groups. One has $\widehat{\Gamma}_2^{ab} = \Gamma_2^{ab} \simeq \mathbb{Z}/10$ and $\widehat{\Gamma}_{1,[2]}^{ab} = \Gamma_{1,[2]}^{ab} = \mathbb{Z}/2 \times \Gamma_{1,2}^{ab} \simeq \mathbb{Z}/2 \times \mathbb{Z}/12$. Here we have used that $\Gamma_{1,2}^{ab}$ is cyclic of order 12, which can be seen in several ways. For instance one can notice that $\Gamma_{1,2} \simeq B_4/Z$ and that B_4^{ab} is free cyclic, whereas the image of its center in the abelianization has order 12; this is because it is generated by the element ω_4 that is a product of 12 twists.

Using this information and proceeding as in [25], one gets ii). In both cases the exceptional outer automorphism, indeed a *bona fide* automorhism, is defined by mapping any twist τ to the product $\tau\iota$. This fixes the involution ι (so does the action of $\widehat{GT}$) and defines an automorphism which commutes with the action of $\widehat{GT}$ and has order 2. □

Type $(1,2)$ plays an important part in the geometric understanding of the situation and we thus need to examine it in more detail, first summarizing known and for a large part already mentioned facts. We denote by $\mathrm{Aut}^\sharp(\check{C}(S)) \subset \mathrm{Aut}(\check{C}(S))$ the subgroup of *type preserving* automorphisms, that is, those automorphisms such that a multicurve and its image have the same topological type. The following facts are proved in [16] and [3], partly adapting to the profinite or procongruence setting the treatment of [18] in the discrete case. First $F \in \mathrm{Aut}(\check{C}(S))$ is type preserving if (and of course only if) it preserves the set of separating curves. Second, if S is not of type $(1,2)$, we actually have $\mathrm{Aut}^\sharp(\check{C}(S)) = \mathrm{Aut}(\check{C}(S))$, this being true but empty for $d(S) = 0, 1$. This property has already been used above and we can replace $\check{C}(S)$ with $\widehat{C}(S)$ when the congruence conjecture has been vindicated. The proposition below summarizes information which follows from the above (see also the two references mentioned above).

Proposition 5.2. *Let $S \simeq S_{1,2}$ be a surface of type $(1,2)$, then*:

i) $\mathrm{Aut}^\sharp(\widehat{C}(S)) \subset \mathrm{Aut}(\widehat{C}(S))$ *has index* 5 *and $F \in \mathrm{Aut}(\widehat{C}(S))$ belongs to* $\mathrm{Aut}^\sharp(\widehat{C}(S))$ *if and only if* there exists *a separating curve whose image by F is separating*;

ii) $\mathrm{Aut}^\sharp(\widehat{C}(S))/\widehat{\Gamma}(S) = \widehat{GT}$;

iii) *let S be connected with $d(S) > 2$, $\alpha \in \check{\mathcal{S}}(S)$ a separating curve such that $S_\alpha = S' \coprod S''$ with S' of type* $(1,2)$; *let $F \in \mathrm{Aut}(\check{C}(S))$ fixing α, so that it induces $F' \in \mathrm{Aut}(\widehat{C}(S'))$; then F' is type preserving, i.e., $F' \in \mathrm{Aut}^\sharp(\widehat{C}(S'))$.*

Recall now the elementary topological construction of the isomorphism $C(S_{1,2}) \simeq C(S_{0,5})$. Namely, with the (topological) elliptic involution ι, we associate an unramified covering of degree 2, call it $p : S_{1,2} \to S_{0,5} \simeq S_{1,2}/\langle\iota\rangle$ where the points on $S_{0,5}$ are marked rather than deleted, so that p ramifies exacty at the (topological) Weierstrass points. We may and do assume that the two marked points on $S_{1,2}$ project to the 5th point on $S_{0,5}$. The projection p then looks like it is drawn in Figure 2.

Since ι is central in $\Gamma_{1,[2]}$, any loop $\alpha \in \mathcal{S}(S_{1,2})$ coincides with its image $\iota(\alpha)$ (proof: $\tau_{\iota(\alpha)} = \iota\tau_\alpha\iota = \tau_\alpha$). So with any loop $\alpha \in \mathcal{S}(S_{0,5})$ one can associate *one* (any) component of its preimage $p^{-1}(\alpha)$ and this yields a well-defined map $\mathcal{S}(S_{1,2}) \to \mathcal{S}(S_{0,5})$, which is actually an isomorphism. It will be a little more convenient to use the inverse map, which is readily extended to the full complexes in a simplicial way. We thus get an explicit isomorphism $\phi : C(S_{1,2}) \to C(S_{0,5})$.

It is interesting to remark that ϕ does *not* induce an isomorphism between the respective pants graphs $C_P(S_{1,2})$ and $C_P(S_{0,5})$ and these are in fact not isomorphic. This elementary topological fact, which will be further illustrated below, is in some sense responsible for the difference between types $(0,5)$ and $(1,2)$ and much more generally between genus 0 and a strictly positive genus. An important point is to distinguish between separating and nonseparating curves on $S_{1,2}$. This is in contrast with the case of $S_{1,1}$ and $S_{0,4}$ where this phenomenon does not arise, because there are no separating curves on $S_{1,1}$. In order to make these observations more precise we list the following easy facts, which however will not be used directly below: 1) if $\alpha, \beta \in \mathcal{S}(S_{1,2})$ intersect

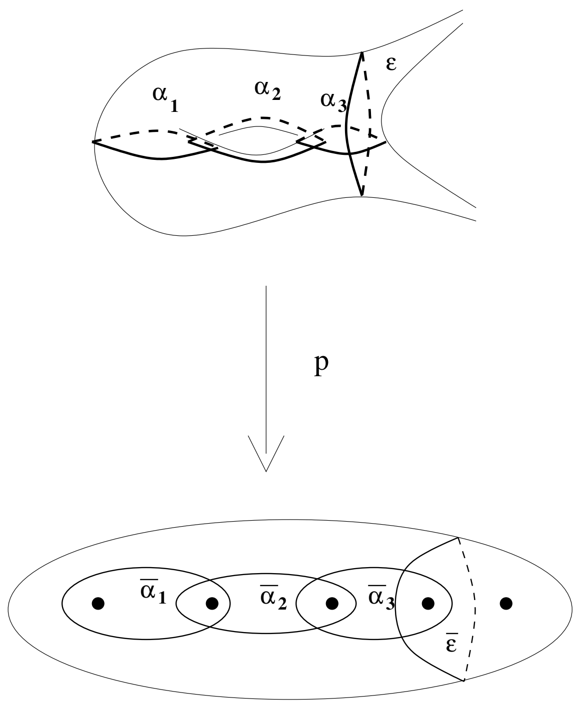

Figure 2

at exactly one point, they are both nonseparating; 2) if they intersect at 2 points, they are either both nonseparating or of opposite types (separating and nonseparating); in other words two, separating curves cannot intersect at two points and this is a specific feature of type $(1,2)$; 3) if $\alpha \in \mathcal{S}(S_{0,5})$, it determines a partition of $1,2,3,4,5$ into a pair and a triplet; then $p^{-1}(\alpha)$ is separating if and only if 5 belongs to the pair; 4) by the above, if $\alpha, \beta \in \mathcal{S}(S_{0,5})$ are joined by an edge in $C_P(S_{0,5})$ but their preimages $\alpha' = p^{-1}(\alpha)$ and $\beta' = p^{-1}(\beta)$ are separating (which is determined by using 3), then α' and β' are *not* joined by an edge in $C_P(S_{1,2})$.

In Figure 2 above, the twists a_i $(i = 1,2,3)$ along the curves α_i generate a copy of B_4 and ϕ induces the familiar injective map (with the same name) $\phi : \Gamma_{1,2} \simeq B_4/Z \hookrightarrow \Gamma_{0,[5]}$ defined by $\phi(a_i) = \tau_i$. We use bars to denote the image of a curve via the projection p. One should beware of the fact that $\bar{\alpha}_i$ does not go through the marked points; in fact $p^{-1}(\bar{\alpha}_i)$ consists of two copies of α_i, that is, $p^{-1}(\bar{\alpha}_i) = 2\alpha_i$, counting with multiplicity, which by the defintion of ϕ means that $\phi(\alpha_i) = \bar{\alpha}_i$. Note that the $\bar{\alpha}_i$'s are none other than the α_i's of Figure 1 in §2. Clearly, $\bar{\varepsilon}$ in the figure is precisely $\bar{\alpha}_4$ but we avoid that piece of notation, as α_4 does not really exist (see below). Here we do not discuss the loops involving, points 1 and 5, but they will play a role below (see Figures 4 and 5).

Consider again $F \in \mathrm{Aut}^\sharp(\widehat{C}(S_{1,2}))$, a type preserving automorphism. We can translate assertion ii) of Proposition 5.2 as follows; twisting by an element of $\widehat{\Gamma}_{1,[2]}$, and using that F is type preserving, we may assume that it fixes the pants decomposition (α_1, e). Then $F_1 = \phi \circ F \circ \phi^{-1}$ is an element of $\widehat{GT}$, say $F_1 = (\lambda, f)$, which acts on $\widehat{C}(S_{0,5})$ and also on $\widehat{\Gamma}_{0,[5]}$ in the "standard way", that is according to formulas (4.3). Viewed from upstairs, F acts on the τ_i's $(i = 1, 2, 3)$ as in those formulas, where $x_{i,i+1} = \tau_i^2 = a_i^2 = \bar{a}_i$ $(i = 1, 2, 3)$. Please pay attention to the exponents: τ_4 does *not* belong to the image of $\Gamma_{1,2}$ (it would permute marked points and Weierstrass points) but $\tau_4^2 = x_{45}$ does belong to that image. Explicitly, this is so by a form of the lantern relation: $x_{45} = (\tau_1\tau_2\tau_1)^2 = (\tau_1\tau_2)^3 = x_{12}x_{13}x_{23}$ $(x_{13} = \tau_1^{-1}x_{23}\tau_1)$. Finally, and this will turn out to be quite important, one can compute the image of $e \in \Gamma_{12}$ (the twist along ε) under ϕ and find that $\phi(e) = x_{45}^2$. This can be traced to the fact that the preimage $p^{-1}(\bar{\varepsilon})$ of the loop encircling the points 4 and 5 has only one component, namely e, counted with multiplicity 2, which the map ϕ does not directly take into account. It can also be seen from a well-known identity for a surface of genus 1 with one boundary component, namely the boundary twist e can be expressed as: $e = (\tau_1\tau_2\tau_1)^4 = (\tau_1\tau_2)^6$. In particular, we see that, of course, $F(e) = e^\lambda$.

We are now in a position to address the cases of strictly positive genus. Let us say that a surface S (hyperbolic of finite type) is *generic* or of *generic type* if $d(S) > 1$, $g(S) > 0$, and $Z(\Gamma(S)) = \{1\}$. Generic surfaces in that sense exist only for $d(S) > 2$ and the only generic type in modular dimension 3 is type $(1, 3)$, the other two types with the same dimension being $(0, 6)$ and $(2, 0)$. We will see in this section and in the Appendix that type $(1, 3)$ indeed deserves to be called generic. In that direction, we first note the following straightforward but suggestive topological lemma.

Lemma 5.3. *A surface S (hyperbolic of finite type) is of generic type (i.e., $d(S) > 1$, $g(S) > 0$, and $Z(\Gamma(S)) = \{1\}$) if and only if it contains a piece of type $(1, 3)$, i.e., there exists $\sigma \in C(S)$ such that the surface S_σ, i.e., S slit along σ, has a connected component of type $(1, 3)$.*

Let now S be of generic type and let $F \in \mathrm{Out}(\check{C}(S))$ be an outer automorphism of its procongruence curve complex, of which we pick a representative $F \in \mathrm{Aut}(\check{C}(S))$. We may find in S a piece T of type $(1, 3)$ and select $\sigma \in C(S)$ so that T is a component of S_σ. We then complete σ into a pants decomposition, i.e., select a top dimensional cell of $C(S)$ whose closure contains σ. Since

F is type preserving, we may then twist F by an element of $\check{\Gamma}(S)$ so that the resulting automorphism preserves that decomposition. So we may concentrate on T and simply assume that $S = T$, where the congruence conjecture holds true. Here we are actually using nontrivial facts from above and [16] (see also [3]) which have already been mentioned. Explicitly we first identify F with an element of $\mathrm{Aut}(\check{C}_*(S))$ using that the automorphism groups of $\check{C}(S)$ and of the prograph $\check{C}_*(S)$ are isomorphic. Then we make use of the fact that there is a natural injection $\check{C}_*(T) \hookrightarrow \check{C}_*(S)$ which is precisely the completion of the injection in the discrete case. Equivalently, the closure of $C_*(T) \subset C_*(S)$ inside $\check{C}_*(S)$ is indeed the procongruence completion $\check{C}_*(T)$, not a proper quotient thereof. Finally by the usual injectivity property, the restriction of F to $\mathrm{Aut}(\check{C}(T))$ completely determines $F \in \mathrm{Aut}(\check{C}(S))$. We will soon explain that conversely, any outer automorphism of $\check{C}(T) = \widehat{C}(T)$ extends an automorphism of $\check{C}(S)$. For the time being we take a look at the local situation, setting $S = T$.

The surface S is covered by a piece S_0 of type $(0, 5)$ obtained by cutting along the loop α_1 and a piece S_1 of type $(1, 2)$ obtained by cutting along δ' (see Figure 3 below). The intersection $S_0 \cap S_1$ is of type $(0, 4)$. One should be a little cautious at this point and below, concerning the differences between boundary curves and marked points and also about allowed permutations of points or boundary components. For the sake of simplicity and clarity, we will usually leave the routine justifications to the reader; one should remember that boundary twists are central in the appropriate modular groups and that the elements of $\widehat{GT}$, viewed as automorphisms, do not permute points.

Pictorially, the situation looks as follows:

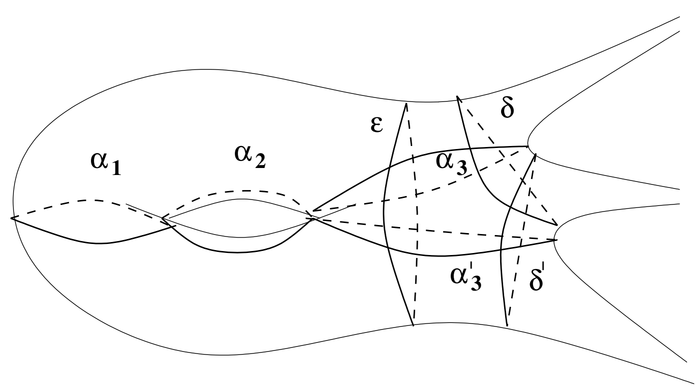

Figure 3

Starting again with $F \in \mathrm{Aut}(\widehat{C}(S))$ ($S \simeq S_{1,3}$), we may assume that it fixes the decomposition $(\alpha_1, \varepsilon, \delta')$. It thus determines two restrictions $r_0(F)$ and $r_1(F)$ to S_0 and S_1 respectively, in view of the same facts about localization that were recalled above. By assertion iii) of Proposition 5.2, $r_1(F)$ is type preserving. So by Theorem 4.2 and ii) of Proposition 5.2, we derive two elements $F_0, F_1 \in \widehat{GT}$, where F_0 is precisely the image of $r_0(F)$ in the outer automorphism group of S_0, and F_1 is constructed as detailed in the discussion below Figure 2, that is: $F_1 = \phi \circ r_1(F) \circ \phi^{-1}$.

Let us express the matching of the respective actions of F_0 and F_1 on the intersection $S_0 \cap S_1$. To this end, we introduce the following definition.

Definition 5.4. Let $F_0 = (\lambda_0, f_0)$ and $F_1 = (\lambda_1, f_1)$ be two elements of $\widehat{GT}$. They form a compatible pair if $\lambda_0 = \lambda_1$ and there exist $a, b, c \in \widehat{\mathbb{Z}}$ such that:

$$f_1(\tau_1^2, \tau_2^2) = \tau_2^{2a} f_0(\tau_1, \tau_2^4) \tau_1^{2b} (\tau_1 \tau_2^2)^{2c}. \tag{5.1}$$

Before returning to the geometric motivation behind this definition, we show that things are tighter than they may seem at first sight. In particular the values of a, b, and c are in fact determined by this identity. Computing them requires an adaptation of computations appearing in [30]. As mentioned already, λ extends the cyclotomic character on the Galois group. Similarly one can extend to $\widehat{GT}$ the additive Kummer characters $G_{\mathbb{Q}} \to \widehat{\mathbb{Z}}$. We will need only the case of $\rho_2 : G_{\mathbb{Q}} \to \widehat{\mathbb{Z}}$, the Kummer character at 2, which can be extended to a character of $\widehat{GT}$ in several equivalent ways: see [30], §5, including the closing remark of that section. If $F = (\lambda, f) \in \widehat{GT}$, we write $\rho_2(f)$, as the value of ρ_2 depends on f only. Then:

Proposition 5.5.

i) $\rho_2(f_0) = \rho_2(f_1)$.

ii) *Writing ρ for this common value, one has*: $a = -4\rho$, $b = -2\rho$, $c = 2\rho$.

Proof. We start with ii), for which it suffices to read [30] §§2,5 carefully; the arguments presented there literally apply in our case and produce the values: $a = -4\rho_2(f_0)$, $b = -2\rho_2(f_0)$, $c = 2\rho_2(f_0)$.

It remains to prove i), to which end we use a result from [29] which we first proceed to recall. There is a natural morphism $B_3 = \langle \tau_1, \tau_2 \rangle \to GL_2(\mathbb{Z})$ defined as usual by:

$$\tau_1 \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \tau_2 \mapsto \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

The same formulas define a map: $\widehat{B}_3 \to \widehat{GL_2(\mathbb{Z})} \to GL_2(\widehat{\mathbb{Z}})$, where the second map is the surjection to the modular completion. Given $F = (\lambda, f)$, we can consider the specialization $f(a,b) \in GL_2(\widehat{\mathbb{Z}})$ for any $a, b \in GL_2(\widehat{\mathbb{Z}})$. Then (4.2) in [29] asserts that:

$$f\Big(\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}, \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\Big) = \begin{pmatrix}1 & -4\rho_2(f)\\ 0 & 1\end{pmatrix}.$$

Note that the first argument of f is the image of τ_1^2. This is valid for any $F \in \widehat{GT}$, not necesssarily in $G_{\mathbb{Q}}$ (many more such formulas are available in that case). Now let $A_3 = \langle \tau_1, \tau_2^2 \rangle \subset B_3$ be the subgroup of the braids such that their third strand returns to its place, i.e., the preimage of the stablizer of 3 for the natural surjection $B_3 \to \mathcal{S}_3$. This subgroup has only one defining relation, namely that $(\tau_1\tau_2^2)^2$ is central. We now use a *different* specialization of equation (5.1), this time mapping $\widehat{A}_3$ to $GL_2(\widehat{\mathbb{Z}})$ according to:

$$\tau_1 \mapsto \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}, \qquad \tau_2^2 \mapsto \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}.$$

This is admissible because the image of $(\tau_1\tau_2^2)^2$ is central; in fact this element maps to the identity matrix. We now compare the two sides of this specialization of (5.1). By the result above, on the left-hand side we get an upper triangular matrix with 1's on the diagonal and $-4\rho_2(f_1)$ as the upper right entry. As for the right-hand side, since τ_2^4 maps to the identity and $f_0(\tau_1^2, \tau_2^2)$ is a commutator, that is has weight 0 in both arguments, the factor $f_0(\tau_1, \tau_2^4)\tau_1^{2b}$ maps to 1. So do the factors $\tau_2^{2a} = \tau_2^{-8\rho_2(f_0)} = (\tau_2^4)^{-2\rho_2(f_0)}$ and $(\tau_1\tau_2^2)^{2c}$, because τ_2^4 and $(\tau_1\tau_2^2)^2$ both map to 1. It remains only $\tau_1^{-4\rho_2(f_0)}$ which maps to the upper triangular matrix with 1's on the diagonal and $-4\rho_2(f_0)$ in the upper right corner. This proves the formula $\rho_2(f_0) = \rho_2(f_1)$. □

Let us return to automorphisms, namely to $F \in \mathrm{Aut}(\widehat{C}(S))$, giving rise to F_0 and F_1 as above. Although we started from an automorphism of $\widehat{C}(S)$, F_0 and F_1 can be viewed as group automorphisms. We write $F_0 = (\lambda_0, f_0)$, $F_1 = (\lambda_1, f_1)$ and examine what happens on $S_0 \cap S_1$. Both F_0 and F_1 induce an automorphism of the modular group of that surface, which is in fact pure, because F fixes α_1 and δ'. Collapsing these loops to points, we are left with two (inertia preserving) automorphisms of $\widehat{F}_2$, which must coincide as outer automorphisms, that is, in $\mathrm{Out}^*(\widehat{F}_2)$, because they both induce the original F. First it is plain that $\lambda_0 = \lambda_1 (= \lambda)$ and $F_0(e) = F_1(e) = e^\lambda$, recalling that F fixes ε.

It now only remains to recall the standard $\widehat{GT}$-action on $\widehat{C}(S_{0,5})$ in group theoretic terms, namely formulas (4.3), as well as the action on $\widehat{C}(S_{1,2})$, using the isomorphism ϕ. The crux of the matter consists of comparing the actions of F_0 and F_1 on α_3 (see Figure 3). On the one hand:

$$F_0(\alpha_3) = f_0(a_3, e)a_3^{\lambda} f_0(e, a_3),$$

because of the action of $\widehat{GT}$ on $\widehat{\Gamma}(S_0) \simeq \widehat{\Gamma}_{0,[5]}$ (boundary curves do not play any role here). This formula can for instance be deduced immediately looking at the pentagon $(\varepsilon, \alpha_3, \delta, \delta', \alpha_3')$ in S_0, or of course using the more general yoga associated with the notion of "lego"; see below. On the other hand $F_1(\tau_3)$ reads:

$$F_1(\tau_3) = f_1(x_{34}, x_{45})\tau_3^{\lambda} f_1(x_{45}, x_{34}),$$

after a careful translation of the notations as in the discussion below Figure 2 (in particular, here $a_3 = \tau_3$, not its square). As a result F_0 and F_1 coincide in $Out^*(\widehat{F}_2)$ if and only if $\lambda_0 = \lambda_1$ and

$$f_1(x_{34}, x_{45}) = x_{45}^a f_0(a_3, e) x_{34}^b \omega_3^c,$$

that is, using $a_3 = \tau_3$, $e = x_{45}^2$, if and only if they form a compatible pair of elements of $\widehat{GT}$.

We remark that the computation above is of course closely related to the one leading to the so-called relation (IV) (see [8, 30] and references therein) which was a motivating factor underlying the appearance and development of the "lego". However it is not identical. Here we are trying to "glue" two *a priori* distinct *outer* actions and thus do not require an identity $F_0 = F_1$, or else $f_0 = f_1$. This remark will be expanded below.

Proposition-Definition 5.6. *The set of compatible pairs forms a group* $\mathbb{I}\Gamma$*, which can be viewed as a subgroup of* $\widehat{GT}$ *by mapping* $(F_0, F_1) \in \mathbb{I}\Gamma$ *to* $F_0 \in \widehat{GT}$*.*

Proof. Using a fixed decomposition of $S_{1,3}$ as above, we see that $(F_0, F_1) \in \mathbb{I}\Gamma$ form a compatible pair if and only if they induce the same outer automorphism on the intersection, a condition which is clearly preserved under inversion and composition. So we get a group $\mathbb{I}\Gamma$ that is indeed a subgroup of $\widehat{GT}$ as indicated in the statement. Note that F_1 is obviously determined by F_0 according to (5.1), which we can view as the definition of f_1 (see Proposition 5.5 ii)). In other words, $F_0 = (\lambda, f_0) \in \widehat{GT}$ lies in $\mathbb{I}\Gamma$ if and only if $F_1 = (\lambda, f_1)$ is an element of $\widehat{GT}$, where f_1 is defined by the right-hand side of (5.1). Conversely, F_0 is also determined by F_1 because the occurence of two different F_0's for a given F_1 would contradict the injectivity result. □

The above definition and description may sound a little artificial at this point. Things will appear much more intrinsic below. The Appendix gives a rather different view of $\mathbb{I}\Gamma$ by means of an additional (somewhat implicit) relation in $\widehat{GT}$.

Starting with a compatible pair F_0, F_1 of elements of $\widehat{GT}$, we now describe an explicit procedure that constructs from it an automorphism $F \in \mathrm{Out}^*(\check{\Gamma}(S))$ and thus also $F \in \mathrm{Out}(\check{C}(S))$. To this end we basically follow the strategy of [8] and [30] (see in particular [8], Theorem 4 and [30], Proposition 8.1) which, as mentioned above, we will not however recall in full detail. The novelty here is that we start with *two* possibly distinct elements of $\widehat{GT}$ and the point is that the constructions presented in those papers generalize very smoothly in that direction. The (outer) automorphisms or actions we get as an output will be said to be *of lego type* and it will evolve that they describe in fact *all* the automorphisms of $\check{C}(S)$ and $\check{\Gamma}(S)$.

Let S be a hyperbolic orientable generic surface and $F_0 = (\lambda, f_0)$, $F_1 = (\lambda, f_1)$ a compatible pair of elements of $\widehat{GT}$. We also fix a pants decomposition $\mathcal{P} = (\alpha_1, \dots, \alpha_d)$, with $d = d(S)$, or rather an associated topological tangential basepoint. In the language of [30], this amounts to picking a quilt on $\mathcal{P}$, that is "add seams to the pants"; in terms of moduli spaces, $\mathcal{P}$ defines a maximally degenerate point on the boundary divisor $\partial\mathcal{M}(S)$ of the stable completion $\overline{\mathcal{M}}(S)$ of the moduli stack $\mathcal{M}(S)$ and one then picks a tangential basepoint at $\mathcal{P}$. We now wish to construct $F \in \mathrm{Aut}^*(\widehat{\Gamma}(S))$ from these data such that F coincides with F_0 and F_1 over the appropriate pieces.

First F is normalized so that $F(\alpha_i) = \alpha_i$ or equivalently $F(a_i) = a_i^{\lambda}$. Let now $\beta \in \mathcal{S}(S)$ be an arbitrary curve. There exists a finite sequence of pants decompositions, say $\mathcal{P} = \mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n = \mathcal{P}'$ such that β appears in the end decomposition $\mathcal{P}'$ and two consecutive decompositions $\mathcal{P}_k$ and $\mathcal{P}_{k+1}$ are joined by an edge in the pants complex $C_P(S)$. In other words, two consecutive elements of the chain differ by an elementary move, of type A or S (see [16], §A.7 and [8] for much more detail). Now we define $F(b)$ as $F(b) = t^{-1}b^{\lambda}t$ where $t = t(\mathcal{P}, \mathcal{P}')$ is a *transfer factor*. In turn t decomposes as $t = t_n \dots t_2 t_1$ where $t_k = t(\mathcal{P}_k, \mathcal{P}_{k+1})$ is an *elementary transfer factor*, describing the passage between two decompositions differing by an elementary move only. In terms of complexes the edges of the pants graph $C_P(S)$ are of two kinds, A (for associativity) and S (for simple), and we will define the respective elementary transfer factors associated with these two kinds of edges.

In order to complete the construction of F, it thus remains to define the elementary transfer factors, which we do as follows. Let $\mathcal{P}$ and $\mathcal{P}'$ be two decompositions differing by an elementary move. Up to relabeling we may assume that: $\mathcal{P} = (\alpha_1, \dots, \alpha_d)$, $\mathcal{P}' = (\alpha'_1, \dots, \alpha'_d)$, $d = d(S)$, $\alpha_i = \alpha'_i$ for $i > 1$, and (α_1, α'_1) sit either on a surface of genus 0 (type A) or of genus 1 (type S). In the former case $\mathcal{P}, \mathcal{P}'$ determine an A-move, in the second an S-move. We set:

$$t(\mathcal{P},\mathcal{P}') = f_0(a_1,a_1') \qquad \text{if }](\mathcal{P},\mathcal{P}') \text{ is of type } A,$$
$$t(\mathcal{P},\mathcal{P}') = (a_1')^{-8\rho} f_1(a_1^2,a_1'^2)a_1^{8\rho}(a_1a_1'a_1)^{\lambda-1} \qquad \text{if } (\mathcal{P},\mathcal{P}') \text{ is of type } S.$$

Here $\rho = \rho_2(f_0) = \rho_2(f_1)$ as above. Since $\lambda - 1 = 2\mu$ is even (in $\widehat{\mathbb{Z}}$), we can also write $(a_1a_1'a_1)^{2\mu}$ in the second line and $(a_1a_1'a_1)^2$ generates the center of the group (isomorphic to $\widehat{B}_3$) generated by a_1 and a_1'.

Remark 5.1. In [8], in order to focus on the basic features of the method, we treated the case where $\lambda = 1$, $\rho_2 = 0$, so that the factors in the elementary transfer S-factor above disappear (and $f_0 = f_1 = f$). This yields a group (denoted Λ there), which in particular contains the closed subgroup of $G_{\mathbb{Q}}$ that is the intersection of the kernels of the cyclotomic character and the Kummer character at 2: it is the Galois group of the extension of $\mathbb{Q}$ obtained by adjoining all roots of 1 and 2. The main purpose in [30] was to remove these restrictions, which entails fairly involved and cumbersome technicalities. In geometric terms, it sufficed in [8] to work with maximally degenerate basepoints (because $\lambda = 1$), whereas in [30] one had to take a closer look and keep track of the tangential basepoints. This was done by using a rigidifying structure, consisting of pants with seams (there dubbed "quilts"), as was done for instance when defining a Nielsen twist as a real number, not merely modulo 2π. As for ρ_2, the Kummer character at 2, it appears via the projection p from genus 1 to genus 0 (see Figure 2) or, putting it slightly differently, using the Legendre parametrization for elliptic curves. Although keeping track of these factors does make the analysis technically more involved, the guidelines remain very much the same, so that the newcomer might want to first take a look at the simpler and more restricted version of the construction appearing in the first paper before embarking on the reading of the second one.

In order for the above receipe to make sense at all, one needs to prove that the value of $F(b)$ does not depend on either the choice of $\mathcal{P}'$ containing β or on the sequence connecting $\mathcal{P}$ and $\mathcal{P}'$. This done (see below), one needs to show that this assignement of the value $F(b)$ for any twist defines an automorphism of $\check{\Gamma}(S)$. The first and crucial step, that is, the fact that $F(b)$ is well defined, relies essentially on the result from [8] (Theorem 2) asserting that the (full, two dimensional) pants complex $C_P(S)$ is simply connected. What one needs to show is thus that the definition of $F(b)$ is invariant under the elementary homotopies, that is when going around a face of the pants complex $C_P(S)$ (for the remainder of this section and at variance with the rest of the paper, $C_P(S)$ will by default refer to the full two-dimensional pants complex). There are four types of such faces, labeled respectively $(3A)$, $(3S)$, $(5A)$, and $(6AS)$ in [8] (see also [23] and [30]). As the names indicate, $(3A)$ (respectively, $(3S)$) determines a triangle of A-moves (respectively, S-moves) on a surface of type $(0,4)$ (resp. $(1,1)$); in turn $(5A)$ corresponds to the familiar pentagon on a surface of type $(0,5)$, giving rise to a pentagon of A-moves. Finally $(6AS)$ defines a hexagon on a surface of type $(1,2)$ and mixes A- and S-moves. This is a keypoint here.

Before coming back to it in detail, we briefly turn to the second step, namely how to show that this assignment of its image to every twist defines a morphism; that it is invertible is no problem. In other words, one has to check that these images satisfy the defining relations of $\Gamma(S)$ or rather $\check{\Gamma}(S)$, which amounts to the same (completion is right exact as a functor). At this point one uses the infinite presentation of $\Gamma(S)$ with all twists as generators, given in [6], refined in [17], and recalled in [8], Theorem 1 (see also [30], Theorem 9.2). As far as we are concerned here, the point is that the relations are all localized on surfaces of types $(0,4)$ and $(1,1)$, so involve only sequences of A-moves or S-moves but do not mix the two types.

The conclusion of the above analysis, at this stage, is that when starting from a pair (F_0, F_1) of elements of $\widehat{GT}$ (rather than from only one element as in [8] or [30]), a single point of difference lies in the analysis of the faces of type $(6AS)$. The rest can be copied quite literally, namely the verification of the good behavior of the definition with respect to faces of types $(3A)$, $(3S)$, and $(5AS)$ ([8], Lemma 6; [30], Proposition 8.1) and the fact that the algorithm described above, which we refer to as the Grothendieck–Teichmüller *lego* (a word used by Grothendieck in his *Esquisse*), defines an endomorphism, actually an automorphism of $\check{\Gamma}(S)$ ([8], §3, Step 2; [30], Proposition 9.1). Note that in these proofs one effectively uses the $\widehat{GT}$-relations for both F_0 and F_1 but *not* the fact that they form a compatible pair.

We come to the analysis of the cycles of type $(6AS)$, as shown in Figure 4 above, which is not especially difficult but is conceptually intriguing and hopefully revealing. Indeed this is in some sense the *only* place where the role of the *genus* is really visible. One way to put it is to recall that ϕ, as defined at the beginning of this section, provides an isomorphism between $C(S_{1,2})$ and $C(S_{0,5})$ and between the respective completions as well, but *not* between the pants complexes (or graphs) $C_P(S_{1,2})$ and $C_P(S_{0,5})$.

Now the projection onto $S_{0,5}$, obtained by quotienting by the elliptic involution as in Figure 2, looks like this:

The point is that the projection of the hexagon $(6AS)$ in $C_P(S_{1,2})$ decomposes into a pentagon (of type $(5A)$) and a triangle (of type $(3A)$) in $C_P(S_{0,5})$ as is schematically indicated in Figures 4 and 5. We used single lines for edges of type A and double lines for edges of type S in the pants graphs. This is an illustration of the phenomenon we already came across. More precisely,

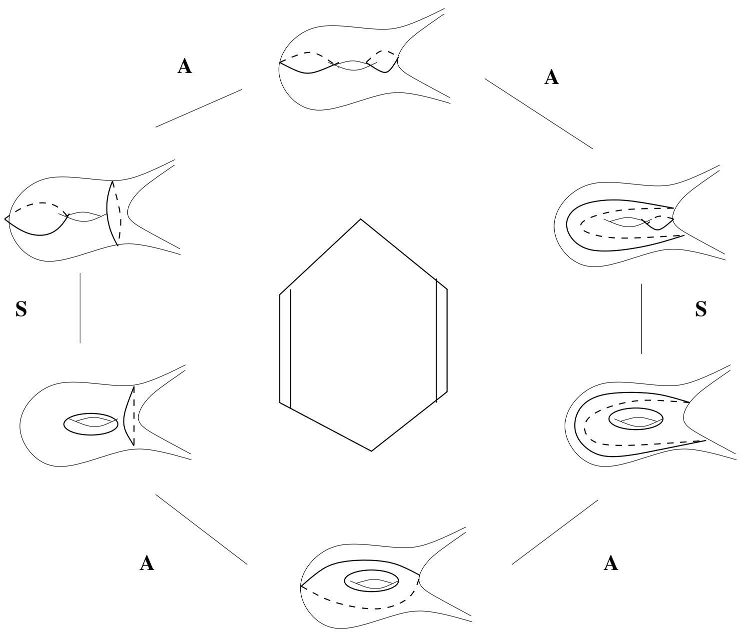

Figure 4

let $e = e_1$ (respectively, $e' = e_3$) be the separating curve appearing on the right-hand (respectively, left-hand) vertical side in Figure 4. These curves e and e' intersect at four points in $\mathcal{S}(S_{1,2})$ and so are *not* joined by an edge in $C_P(S_{1,2})$, whereas their images in $\mathcal{S}(S_{0,5})$ do give rise to an edge of the pants graph $C_P(S_{0,5})$. In order to get a closed circuit "upstairs", that is, in $C_P(S_{1,2})$, one thus has to insert, in a somewhat arbitrary way, an additional nonseparating curve e_2, as in the bottom picture of Figure 4. It is then easy to see that the images of these three curves e_1, e_2, e_3 form a triangle in $C_P(S_{0,5})$. We insist on these elementary topological considerations because in the end they seem to form the core of the difference of the analysis in genus 0 and that in a strictly positive genus.

Return to a compatible pair (F_0, F_1), that is, an element of $\mathbb{I}\Gamma$. Assume for the moment that $\lambda = 1$, $\rho = 0$; this is only for the sake of clarity and the appropriate factors can then be restored, following [30] (see below). Then one finds that $F(b)$ defined above does not change when we go along a circuit of type $(6AS)$ if and only if, in the notation of Figure 3, the following relation holds:

$$f_0(e_3, a_1) f_1(a_2^2, a_3^2) f_0(e_2, e_3) f_0(e_1, e_2) f_1(a_1^2, a_2^2) f_0(a_3, e_1) = 1. \qquad (R)$$

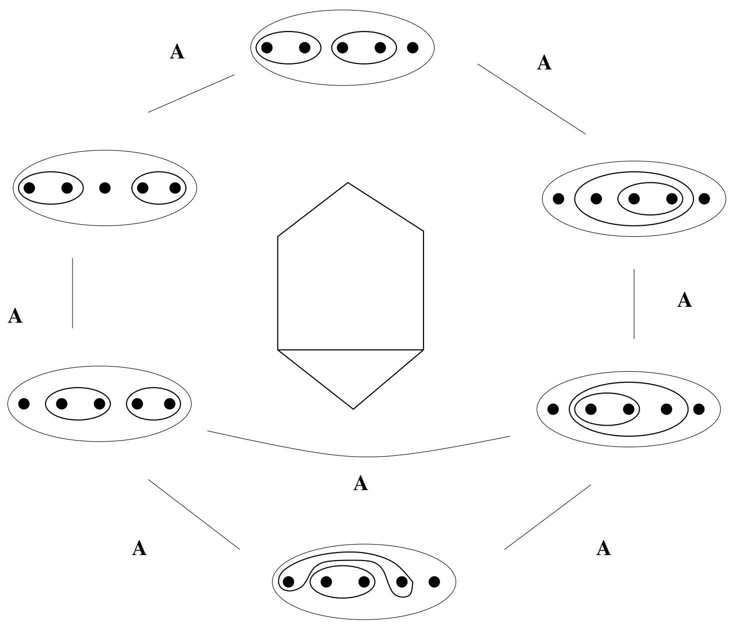

Figure 5

This is indeed a generalization of the relation introduced in [8] (see the display on the last page of that paper, p.23), except there, $f_0 = f_1 = f$; we return to that point below. It contains four factors of type f_0 (with no squares) corresponding to the four edges of type A (genus 0), and two factors of type f_1, with squares, corresponding to the two edges of type S (genus 1). Now we want to show that if (F_0, F_1) is a compatible pair (with the above restrictions on λ and ρ), relation (R) above is satisfied. Indeed, one transforms all A-terms in the relation using the compatibility condition (5.1) in Definition 5.4. For instance, $f_0(a_3, e_1) = f_1(x_{34}, x_{45})$. This done, one gets a 6-terms relations in $\widehat{\Gamma}_{0,5}$, which decomposes into a triangle and a pentagon; it holds true because $F_1 \in \widehat{GT}$, so satisfies relations (II) (triangle) and (III) (pentagon), which completes this sketch of the proof.

We do not detail it further because we can use [30] at this point, removing the restrictions on λ and ρ. One finds there a version (R') of (R), which is jazzed up to all elements of $\widehat{GT}$ (see [30], §1); we simply need to distinguish between the f_0 and f_1 factors (which is immediate) in order to get the analog of (R) above, as refined to *all* compatible pairs. The proof that compatible pairs satisfies (R') then literally follows the computation in [30], p. 543, whose geometric interpretation is essentially as above: the projection of the hexagon of mixed type ($6AS$) decomposes into a triangle ($3A$) and a pentagon ($5A$).

At this point we have actually shown the following.

Proposition 5.7. *Given a generic surface S, an element $F \in \mathbb{I}\Gamma$ (that is, a compatible pair (F_0, F_1) of elements of $\widehat{GT}$, with $F = F_0$), determines a well-defined element of $Out^*(\check{\Gamma}(S))$.*

Proof. This has been already proved, except that we started with a richer set of data, namely not only from S but also a piece of S of type $(1,3)$ and a pants decomposition of it (or of the whole of S). However, following again [8] and [30], or by injectivity and the fact that automorphisms are type preserving, the procedure described above produces an element of $Out^*(\check{\Gamma}(S)$, which is independent of these additional data. □

We can reinterpret this, highlighting the fact that the situation is more "natural" than it may still appear at this point. Let us first formally introduce the "lego".

Definition 5.8. Let S be a surface of generic type and let $F \in \mathrm{Aut}^*(\widehat{\Gamma}(S))$ be an automorphism. We say that it is of lego type if there exists a — necessarily compatible — pair (F_0, F_1) of elements of $\widehat{GT}$ such that F induces the outer automorphism defined by F_0 (respectively, F_1) on a piece of type $(0,4)$ (respectively, $(1,1)$). The action is of strict lego type if $F_0 = F_1$.

Here we did implicitly collapse the boundary curves of the pieces of types $(0,4)$ or $(1,1)$ and the action is precisely as in (4.1) where in the case of genus 1, $x = \tau_1^2$, $y = \tau_2^2$. Note that if $g(S) = 0$, any $F \in \widehat{GT}$ induces an action of strict lego type. If S is of generic type and an automorphism is of lego type, it is of strict lego type if and only if $F = F_0 = F_1$ and by Definition 5.4 and Proposition 5.5, that element $F = (\lambda, f) \in \widehat{GT}$ satisfies the following equation ($\rho = \rho_2(f)$):

$$f(\tau_1^2, \tau_2^2) = \tau_2^{-8\rho} f(\tau_1, \tau_2^4) \tau_1^{-4\rho} (\tau_1 \tau_2^2)^{4\rho}. \qquad (IV)$$

This is a translation of the fact that an action is strictly of lego type if it is of lego type and commutes with the elliptic involution. We will see below, and in fact have essentially already proved, that every automorphism is of lego type, but first we pause briefly in order to insert two observations.

Remark 5.2.

1. Relation (or equation) (IV) appeared in [30] and was discovered a little earlier by H. Nakamura, computing the action of the Galois group on the

Teichmüller modular (also known as mapping class) groups. Any element $\sigma \in G_{\mathbb{Q}}$ gives rise to $F_\sigma \in \widehat{GT}$ inducing an action of strict lego type. Relation (IV) for Galois elements can be proved in a fairly simple way, as was done in [30], Theorem 2.1. It is remarkable that the proof involves only $\mathbb{P}^1 \setminus \{0,1,\infty\}$ and order two covers of that space, which are also of genus 0. Abstractly speaking, granted that the Galois action is of lego type, it is clear that it should commute with the elliptic involution and thus be of strict lego type. This amounts to the observation that the generic elliptic involution, i.e., the order 2 automorphism of the generic point of the stack $\mathcal{M}_{1,1}$, is defined over $\mathbb{Q}$.

2. Using these facts on the action of $G_{\mathbb{Q}}$, we could in the proofs of the above statements (and below as well) restrict ourselves to considering compatible pairs $(F_0, F_1) \in \widehat{GT}$ with $\lambda = 1, \rho = 0$. This does simplify the matter (and the writing) quite a bit and can be achieved as follows. Any $\sigma \in G_{\mathbb{Q}}$ gives rise to an action of strict lego type, that is, the corresponding $F_\sigma = (\lambda_\sigma = \chi(\sigma), f_\sigma)$ satisfies relation (IV) above. On the other hand adding (all) roots of unity and (all) roots of 2 gives rise to disjoint extensions of $\mathbb{Q}$, so that given a pair $(\alpha, \beta) \in \widehat{\mathbb{Z}}^* \times \widehat{\mathbb{Z}}$, there exists $\sigma \in G_{\mathbb{Q}}$ with $\chi(\sigma) = \alpha$, $\rho_2(\sigma) = \beta$. We also recall that ρ_2 extends the character with the same name on $G_{\mathbb{Q}}$ $(\rho_2(\sigma) = \rho_2(f_\sigma))$ and that it is a cocyle on $\widehat{GT}$, that is $\rho_2(F \circ G) = \rho_2(F) + \lambda(F)\rho_2(G)$ for $F, G \in \widehat{GT}$ (see, e.g., [30], Corollary 5.2). Consider now $(F_0, F_1) \in \mathbb{I}\Gamma$ as above, and let $\sigma \in G_{\mathbb{Q}}$ such that $\chi(\sigma) = \lambda^{-1}$, $\rho_2(\sigma) = -\lambda^{-1}\rho$. Then $F' = (F'_0 = F_\sigma \circ F_0, F'_1 = F_\sigma \circ F_1) \in \mathbb{I}\Gamma$ satisfies $\lambda(F') = 1$, $\rho_2(F') = 0$ and it would suffice to prove, e.g., Proposition 5.7 for such elements. One can then set $\lambda = 1$, $\rho = 0$ both in the compatibility condition (5.1) of Definition 5.4 and in the definition of the transfer factors.

In [30], the computation on p.543 actually shows that (IV) implies (R'), recalling that (R') is like (R) above, only with $f_0 = f_1 = f$ and some correcting λ- and ρ-factors added ([30], §1). In fact, (R') is, almost by its very definition, a necessary and sufficient condition for an element of $\widehat{GT}$ to extend to strictly positive genus and give rise to an action of strict lego type. Now we can show the following fact.

Proposition 5.9. *Relations (R') and (IV) are equivalent.*

Proof. That (IV) implies (R') is known and was mentioned above. Conversely, if $F = (\lambda, f) \in \widehat{GT}$ satisfies (R'), one gets an action of strict lego type on $\widehat{\Gamma}_{1,[2]}$ from F by applying the above construction. It coincides with the original action of F on a piece of type $(1,1)$ (cutting along ε in Figure 2) and so by injectivity the two actions coincide everywhere (i.e., $F_0 = F_1$), which means that they coincide on the twist a_3 (see Figure 3). This translates into the fact that F satisfies (IV). □

We will denote by $\mathrm{I\!\Gamma}_s \subset \mathrm{I\!\Gamma}$ the subgroup of $\mathrm{I\!\Gamma}$ defined by the elements that induce an action of strict lego type. In other words $F \in \widehat{GT}$ belongs to $\mathrm{I\!\Gamma}_s$ if and only if it satisfies (IV), or equivalently (R'). Note that $\mathrm{I\!\Gamma}_s$ is indeed a subgroup of $\widehat{GT}$, as can be seen from its geometric definition and is proved directly in [30], § 6. We now gather the information in the next theorem.

Theorem 5.10. *Let S be of generic type, that is, $d(S)>1$, $g(S)>0$, $Z(\Gamma(S)) = \{1\}$. Equivalently S is of type (g,n) with $3g-3+n > 2$, $g > 0$, $(g,n) \neq (2,0)$; then*:

i) $\mathrm{Aut}^*(\check{\Gamma}(S)) = \mathrm{Aut}(\check{C}(S))$;

ii) $\mathrm{Out}^*(\check{\Gamma}(S)) = \mathrm{Out}(\widehat{C}(S)) = \mathrm{I\!\Gamma}$*; in particular, every automorphism of $\check{\Gamma}(S)$ is of lego type*;

iii) *every inertia preserving automorphism of every open subgroup of $\check{\Gamma}(S)$ is induced by an inertia preserving automorphism of the full group $\check{\Gamma}(S)$*;

iv) *in particular, if a subgroup of $\check{\Gamma}(S)$ is open and normal, then it is of the form $\check{\Gamma}^\lambda$ with Γ^λ a normal finite index subgroup of $\Gamma(S)$ and there is a* split *short exact sequence*:

$$1 \to \Gamma(S)/\Gamma^\lambda \to Out^*(\widehat{\Gamma}^\lambda) \to \mathrm{I\!\Gamma}^\lambda \to 1,$$

where $\Gamma(S)/\Gamma^\lambda$ is a finite group and $\mathrm{I\!\Gamma}^\lambda$ an open subgroup of $\mathrm{I\!\Gamma}$.

Proof. This is essentially a matter of gathering information from above. Assertion i) is stated explicitly because of its importance. The identity $\mathrm{Aut}^*(\check{\Gamma}(S)) = \mathrm{Aut}(\check{C}(S))$ is actually valid for $d(S) > 1$, $Z(\Gamma(S)) = \{1\}$, i.e., $3g - 3 + n > 1$, (g,n) not equal to $(1,2)$ or $(2,0)$. These exceptions are completely understood (cf. Proposition 5.1). When $g(S) < 3$, we have $\check{\Gamma}(S) = \widehat{\Gamma}(S)$ and $\check{C}(S) = \widehat{C}(S)$; when $g(S) = 0$, we have $\mathrm{Aut}^*(\check{\Gamma}(S)) = \mathrm{Aut}(\widehat{\Gamma}(S)$.

Assertion ii) has essentially been proved already. Let us quickly review the argument. Start from an element of $\mathrm{Out}(\widehat{C}(S))$ and select a representative $F \in \mathrm{Aut}(\widehat{C}(S))$. Choose a piece $T \subset S$ of type $(1,3)$ that is stable under F. This is possible after twisting by an element of $\check{\Gamma}(S)$ because F is type preserving and S is of generic type. Then pick a pants decomposition $\mathcal{P}$ adapted to T, that is, containing the boundary curves of T. Get two subsurfaces of T, say T_0 and T_1 of types $(0,5)$ and $(1,2)$ respectively. We may assume that they are determined by curves of $\mathcal{P}$. Then F induces automorphisms of the graphs $\widehat{C}(T_0)$ and $\widehat{C}(T_1)$, which are both isomorphic to $\widehat{C}(S_{0,5})$. Applying Proposition 4.1 (or Theorem 4.2) produces two elements F_0 and F_1 of $\widehat{GT}$, which form a compatible pair because F acts on the complex $\widehat{C}(T)$. So we get an element of $\mathbb{I}\Gamma$ that we can identify with F_0 (or with F itself). By Proposition 5.7, which involves constructions from [8] and [30], F induces an element of $\mathrm{Out}^*(\check{\Gamma}(S))$. This element in turn uniquely determines an automorphism of $\check{C}(S)$ (recall that $\check{\Gamma}(S)$ is centerfree). Finally injectivity garantees that we have gone full circle and recovered our original outer automorphism of $\check{C}(S)$, which completes the proof of ii). Note that *a posteriori* everything is of course intrinsic, that is, independent of the choices made in the course of the proof.

Assertions iii) and iv) are easy consequences and are proved exactly as in genus 0; see Theorem 4.2 and the explicitation below the proof of that result. The interpretation of $\mathbb{I}\Gamma^\lambda$ in the Galois case, that is, of the intersection $\mathbb{I}\Gamma^\lambda \cap G_{\mathbb{Q}}$, is also as discussed there. It can be added that here, and in genus 0 as well, we have used the fact that $\Gamma(S)$ is *universally* centerfree, i.e., every finite index subgroup has trivial center. Moreover, since $\Gamma(S)$ is finitely generated, the open subgroups of $\check{\Gamma}(S)$ are exactly its finite index subgroups. Again, here as well as in genus 0, one could address and classify more generally morphisms between open, not necessarily normal subgroups. □

So for a generic surface S,

$$\mathrm{Out}^*(\check{\Gamma}(S)) = \mathrm{Out}^*(\widehat{\Gamma}(S_{1,3})) = \mathbb{I}\Gamma.$$

Type $(1,3)$ is the only nonexceptional (generic) type in modular dimension 3, the other two being $(0,6)$ and $(2,0)$. This can be seen as copies of $S_{0,5}$ and $S_{1,2}$ (the two pieces in dimension 2) intersecting along a piece of dimension 1 (of type $(0,4)$). The Appendix provides a more symmetric view of the situation. At any rate, the above does illustrate again the two level principle, *cum grano salis* given the role played by type $(1,3)$. We also note that one has explicit formulas for the action of $\mathbb{I}\Gamma$ on generators of $\Gamma_{g,[n]}$. They are obtained by an easy modification of the formulas displayed in [30], §11, changing elementary transfer factors from f to f_0 or f_1 in an appropriate way, as we did above for relation (R).

⋆ ⋆ ⋆

In closing we add again a few remarks and questions. We have seen that any automorphism is of lego type (see Definition 5.8). Let $\mathbb{I}\Gamma_s \subset \mathbb{I}\Gamma$ be the subgroup of automorphism of *strict* lego type. An element of $\mathbb{I}\Gamma$ belongs to $\mathbb{I}\Gamma_s$ if and only if it satisfies the equivalent relations (R') and (IV) (cf. Proposition 5.9), equivalently if it commutes with the elliptic involution. Note that $\mathbb{I}\Gamma_s$ is contained in the group dentoted by $\mathbb{I}\Gamma$ in [30]. So we have a nested sequence:

$$G_{\mathbb{Q}} \subset \mathbb{I}\Gamma_s \subset \mathbb{I}\Gamma \subset \widehat{GT} \subset \mathrm{Out}^*(\widehat{F}_2)$$

and it may be interesting to speculate on the possible equalities occuring in this chain. Of course $\widehat{GT}$ is strictly smaller than $\mathrm{Out}^*(\widehat{F}_2)$, which is a very big group indeed. At the other end, comparing $G_{\mathbb{Q}}$ with $\mathbb{I}\Gamma_s$ or any reasonably explicit group "of geometric origin" poses a major challenge.

Let us briefly comment on the possible coincidences of the GT-like groups by simply stressing again their intrinsic meaning. First $\mathbb{I}\Gamma_s \overset{?}{=} \mathbb{I}\Gamma$ if and only if all automorphisms are *strictly* of lego type, or else commute with the elliptic involution. Put a little differently, if (F_0, F_1) is a compatible pair of elements of $\widehat{GT}$, then in fact $F_0 = F_1$. The possible identity $\mathbb{I}\Gamma \overset{?}{=} \widehat{GT}$ would be most interesting, as it would mean that "everything" already happens in genus 0. Recall that in terms of motives, that is, in the pro-ℓ (or pronilpotent, or prounipotent) setting, the situation (including multiple zeta values etc.) can be understood (largely conjecturally) at present only in genus 0, by means of the inconditional (not conjectural) mixed Tate motives, whereas the situation for $g > 2$, is very different, due to the appearance of a large Torelli group. Also one may venture to imagine that the identity $\mathbb{I}\Gamma \overset{?}{=} \widehat{GT}$ occurs only if in fact $\mathbb{I}\Gamma_s \overset{?}{=} \widehat{GT}$, that is, all elements of $\widehat{GT}$ would satisfy relation (IV). Recall that H. Furusho showed that the prounipotent version of the Grothendieck–Teichmüller group is defined by the sole 5-cycle relation; could this be true of the full profinite group $\widehat{GT}$? The identity $\mathbb{I}\Gamma_s \overset{?}{=} \widehat{GT}$, that is, showing that relation (IV) is actually redundant, would then reduce everything to pentagons, including in higher genus.

Another interesting question is to try and see whether one can "remove the star", that is determine whether for $d(S) > 1$ (but not for $d(S) = 1$), every group automorphism is inertia preserving, as shown in [9] (see also there references to previous works) in the case of genus 0, or more generally for configuration spaces. In other words, is it true that $\mathrm{Aut}^*(\widehat{\Gamma}(S)) \overset{?}{=} \mathrm{Aut}(\widehat{\Gamma}(S))$ for $d(S) > 1$? This is surely not merely a matter of technicality, leading rather to a hoard of questions and analogies. We will content ourselves with the barest remarks. On the one hand, this is true in the discrete setting, that is for $\Gamma(S)$, and comes in particular from the characterization of powers of twists inside $\Gamma(S)$ in purely group theoretic terms, a line of thought which can be traced to the work of N. Ivanov in the early 1980's. This is ultimately clear from the formula $\mathrm{Out}(\Gamma(S)) = \mathbb{Z}/2$ but it seems useful to isolate that statement, as was done in [25]. On the other hand, this can be seen as an analog at the level of fundamental groups and for the moduli stacks of curves, of the local correspondence in birational anabelian geometry, originally developed by (in very rough chronological order) J. Neukirch, F. Bogomolov, F. Pop, A. Tamagawa and others. Here the close analogy comes between the decomposition group of a rank 1 valuation inside the Galois group of a finitely generated field and the centralizer of a twist in the Teichmüller group. Now in order to characterize (powers of) twists inside Teichmüller groups, Thurston's theory seems hard to dispense with. In particular one can isolate the fact that pseudo-Anosov mapping classes are self-centralizing. This is relatively easy using Thurston's completion of te Teichmüller space (i.e., measured foliations), as first explained by N. Ivanov and written up in full detail by J. D. McCarthy in the early 1980's. So let us finish with a test question, still a far cry from what is actually desirable: let $g \in \Gamma(S)$ be a pseudo-Anosov diffeomorphism on an orientable hyperbolic surface S; is it true that the centralizer of g in the completion $\check{\Gamma}(S)$ is procyclic?

Appendix. On surfaces of type $(1,3)$

Here we add two observations with the goal of clarifying the "universal" character of type $(1,3)$. The first one has to do with the finite presentation of $\Gamma_{g,n}$ derived in [7] and the second one points to the relationship with generalized Artin braid groups. In the main body of the text (§5) we viewed $S_{1,3}$ as covered by two subsurfaces of type $(0,5)$ and $(1,2)$ respectively, intersecting along a piece of type $(0,4)$. Here is a more symmetric view of the situation:

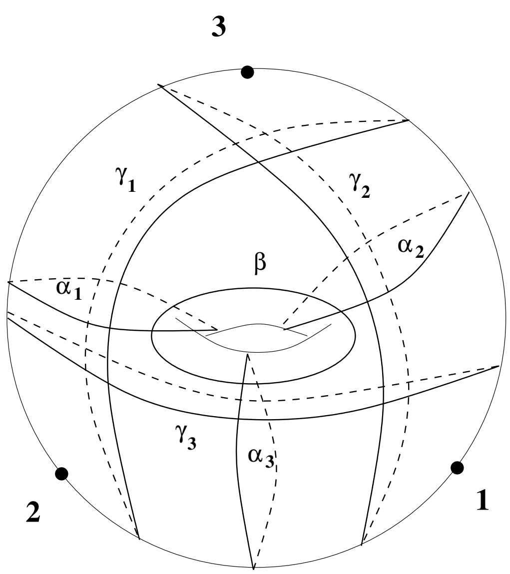

Figure 6

We use again the convention of Greek letters (α, β, γ, $\delta, \dots$) for loops and Latin letters (a, b, c, $d, \dots$) for the corresponding twists. We already know that it will make little difference to consider surfaces with boundary twists or points and to allow or not for permutations of points. So let us for a moment consider that we are working with boundary components (which cannot be permuted) and let δ_i ($i = 1, 2, 3$) denote a small loop encircling the point i (the δ_i's do not appear on Figure 6).

Let Γ_1^3 denote the modular group of genus 1 with three boundary components. By [Ge2], the group Γ_1^3 is generated by the a_i, d_i ($i = 1, 2, 3$) and b with three kinds of relations: 1) the obvious commutation relations for nonintersecting curves (in particular the d_i's are central), 2) the classical braid relations for each a_i with b ($a_i b a_i = b a_i b$), and 3) the *star relation*:

$$(a_1 a_2 a_3 b)^3 = d_1 d_2 d_3. \qquad (\star)$$

The main content of [7] (see also [6]) is that this is a general phenomenon. For any (g, n) one finds a similar collection of generating twists and the relations are of the three types above. Moreover, the star relation enables one to recover all classical relations supported on surfaces of modular dimensions 1 and 2 ([7], §2).

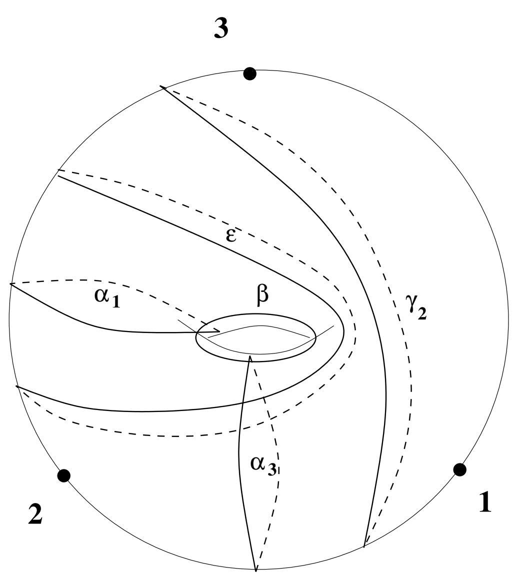

Figure 7

Consider for instance the surface of genus 1 with one marked point (labeled 2) and one boundary curve (γ_2; see Figure 7). Add a fictitious marked point (say between α_1 and the marked point 2) and apply ($\star$) to get: $(a_1^2 a_3 b)^3 = c_2$. This relation does not look very symmetric but can in fact readily be transformed into $(a_1 b a_3)^4 = (a_1 a_3 b)^4 = c_2$, using only braid relations. Similarly, one can consider the surface of genus 1 with one boundary component ε and recover in this way the relation $(a_1 b)^6 = e$, which was used above in §5 when gluing a subsurface of type $(0,5)$ obtained by cutting along α_1 with a subsurface of type $(1,2)$ obtained by cutting along γ_2.

Let now S_i (respectively, S'_i) be the subsurface of type $(0,5)$ (respectively, $(1,2)$) obtained by cutting along α_i (respectively, γ_i). Forgetting about boundary components (i.e., collapsing them to punctures), let $F \in \mathrm{Aut}^*(\widehat{\Gamma}_{1,3})$, fixing the pants decomposition $(\alpha_1, \alpha_2, \alpha_3)$. Restricting F to S_i we get an element of $\widehat{GT}$ and these three elements coincide; this way we get F_0. Similarly, after twisting, we can restrict the action to S'_i, retrieve again three elements of $\widehat{GT}$, which again coincide with an element F_1 and (F_0, F_1) form a compatible pair, so that $F_0 = F \in \mathbb{I}\Gamma$.

But we can also forget about F_1 and ask: when will F, identified with $F_0 \in \widehat{GT}$, extend to an automorphism of $\widehat{\Gamma}_{1,3}$ (equivalently of $\widehat{C}(S_{1,3})$)? Clearly the answer is: if and only if it preserves relation ($\star$), namely $(a_1 a_2 a_3 b)^3 = 1$. Here we already have $F(\alpha_i) = \alpha_i$, that is, $F(a_i) = a_i^\lambda$ for $i = 1,2,3$, and the only missing generator is b (if one adds boundary components, $F(d_i) = d_i^\lambda$). So we need only find — say — h such that $F(\beta) = h \cdot \beta$, or else $F(b) = h^{-1} b^\lambda h$. Putting this into ($\star$) we get an equation for h (which can perhaps be simplified; see [7], §2). Since F satisfies the rules of the lego ([8]), we can actually compute h using it; we leave it as an exercise to compute the transfer factor, that is, h, in this way. It takes three moves to extricate β from the α_i's, and of course f_1 (with $F_1 = (\lambda, f_1)$) appears in the answer.

We conclude from the above that [7] actually displays the universality of type $(1,3)$ in the form of the star relation ($\star$). In the text we pasted pieces of types $(0,5)$ and $(1,2)$ in order to define compatible pairs of elements of $\widehat{GT}$ and the group $\mathbb{I}\Gamma$. It can also be described, in a more symmetric way, by using ($\star$), which may be seen as a monodromy relation for the covering of $S_{1,3}$ by three subsurfaces S_i of type $(0,5)$.

We will give only very brief indications about the relationship with generalized Artin braid groups, which could lead to further developments. The Galois action on the completion of the braid group $A(E_7)$ associated with the root system (Dynkin diagram) E_7 was explicitly computed long ago and related to the Galois action on $\widehat{\Gamma}_3$, roughly at the same time when the Galois action on the $\widehat{\Gamma}_{g,n}$'s was also being explicitly computed (see [24] and its list of references). The relationship between generalized braid groups and mapping class groups was then made more general and explicit (see again [24]). We will content ourselves here with pointing to the suggestive isomorphism: $\Gamma_{1,3} \simeq A(D_4)/Z$, where $A(D_4)$ is the generalized braid group associated with the root system D_4 and Z denotes its center. The isomorphism is "natural" and suggests a notation for $A(D_4)$ which closely parallels our notation above for $\Gamma_{1,3}$. The center Z is free cyclic and in that notation, it is generated precisely by the element $(a_1a_2a_3b)^3$, which occurs in the star relation. Moreover, this is an instance of a quite general phenomenon (see [24]). Finally we mention that the Y-shaped diagram corresponding to D_4 is a central piece in the lego of Coxeter graphs giving rise to the generalized braid groups.

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