ON PROCONGRUENCE CURVE COMPLEXES AND THEIR AUTOMORPHISMS

MARCO BOGGI, LOUIS FUNAR, AND PIERRE LOCHAK

Abstract. In this paper we start exploring the procongruence completions of three varieties of curve complexes attached to hyperbolic surfaces, as well as their automorphisms groups. The discrete counterparts of these objects, especially the curve complex and the so-called pants complex were defined long ago and have been the subject of numerous studies. Introducing some form of completions is natural and indeed necessary to lay the ground for a topological version of Grothendieck-Teichmüller theory. Based on previous work by the first author, we state and prove several basic results, among which reconstruction theorems in the discrete and complete settings, which give a graph theoretic characterizations of versions of the curve complex as well as a rigidity theorem for the complete pants complex, in sharp contrast with the case of the (complete) curve complex, whose automorphisms actually define a version of the Grothendieck-Teichmüller group, to be studied elsewhere (see [22]). We also prove an anabelian theorem pertaining to the moduli stacks of curves, one of the very few such results available in higher dimensions. We work all along with the procongruence completions – and for good reasons – recalling however that the so-called congruence conjecture predicts that this completion should coincide with the full profinite completion.

AMS Math Classification: 11R32, 14D22, 57M99.

1. Introduction

The primary goal of this paper is to start laying the foundations for a topological version of Grothendieck-Teichmüller theory and the goal of this short introduction is to provide some clues as to what this could mean ; and of course about the contents of the paper. For much more on the background landscape we refer once and for all to [23] and its references. Because numerous objects are involved we have gathered the main (essentially classical) definitions in a short Appendix which the reader is invited to consult when (s)he feels like it. We will also explicitly refer to it.

In a few words which will be considerably expanded below and possibly elsewhere, the situation can be described as follows. Let $S = S_{g,n}$ be a hyperbolic surface of finite type (cf. §A.1); it has (modular) dimension $d(S) = 3g - 3 + n$ which can be seen for instance as the (complex) dimension of the modular orbifold $\mathcal{M}(S)$ (cf. §A.2) or else as the maximal number of non intersecting simple closed curves lying on S, considered up to isotopy (these objects form a set which we denote $\mathcal{L}(S)$). Starting from $\mathcal{L}(S)$ one builds several (simplicial, non locally finite) complexes, especially the *curve complex* $C(S)$ (cf.§A.5), of dimension $d(S) - 1$, and the so-called two-dimensional *pants complex* $C_P(S)$ (cf. §A.7) of which it is enough to consider the 1-skeleton (the *pants graph*). The attached Teichmüller group (a.k.a. mapping class group) $\Gamma(S)$ (cf. §A.3) acts naturally on these objects ($\mathcal{L}(S)$, $C(S)$, $C_P(S)$).

The curve complex $C(S)$ was first constructed by W.J.Harvey in close analogy with buildings for reductive groups, from which the significance of its automorphisms was immediately recognized (see [23], Introduction, for a more detailed story and references). It was shown in the eighties, by N.V.Ivanov (cf. [18]) and J.L.Harer (cf. [13, 14]) independently, that the curve complex $C(S)$ has the homotopy type of a wedge of spheres, an important and fundationnal result. A few years later N.V.Ivanov proved (cf. [19] as well as [21]) that $C(S)$ is essentially rigid, the only automorphism not arising from the action of $\Gamma(S)$ being the mirror reflection (an orientation reversing automorphism of the underlying surface). This is embodied in the exact sequence (A 2) of §A.12. An important point is that it also enables one to control the automorphisms of the *group* $\Gamma(S)$, leading to the exact sequence (A 3), and indeed the automorphisms of any cofinite subgroup $\Gamma^\lambda(S) \subset \Gamma(S)$. The upshot is thus that both $C(S)$ and $\Gamma(S)$ are rigid with the mirror reflection as only non inner automorphism; in anticipation one can identify the reflection with complex conjugacy and consider that it generates the Galois group $Gal(\mathbb{C}/\mathbb{R}) \simeq \mathbb{Z}/2$.

The pants complex $C_P(S)$ was defined somewhat later and its automorphisms were considered relatively recently. D.Margalit showed (in [25]) that it is rigid as well, more precisely that one can replace $C(S)$ by $C_P(S)$ in the sequence (A 2), so that $Aut(C_P(S)) = Aut(C(S))$. This result will be reproved below (in §2) in a different way.

Now to completions; they were introduced in [3] in an effort to attack the congruence conjecture (cf. §A.10). Although this was actually not achieved in [3] (see the review of D.Abramovitch in MathSciNet for a careful and well-intended discussion), the idea of completing various geometric or in fact topological objects (cf. §A.11), primarily versions of the complexes of curves, appears as a deep and potentially fruitful one. Perhaps the main point or slogan of the present paper is that the automorphisms of the *completed* complexes have a lot to do with Grothendieck-Teichmüller theory (in all genera, not only genus 0) and the corresponding group. This is also the main theme of the manuscript [22] (2007, unpublished).

More specifically let $\hat{C}(S)$ and $\hat{C}_P(S)$ denote the respective profinite completions of the curves and pants complexes. Then $\hat{C}_P(S)$ remains rigid whereas $\hat{C}(S)$ acquires an enormous automorphism group, which is precisely (a somewhat sophisticated version of) the Grothendieck-Teichmüller group. These issues are discussed in detail in [22] but watertight proofs are missing there, for technical reasons which in some sense amount to the fact that one does not know how to prove (the highly plausible fact) that $\hat{C}(S)$ is isomorphic to the profinite completion $\hat{C}_G(S)$ of the group theoretic version $C_G(S)$ of the curve complex (cf. A.6).

Fortunately things become somewhat easier when working with the congruence completions. In terms of covers the congruence completion $\check{\Gamma}(S)$ describes the (orbifold unramified finite) covers of the modular orbifold $\mathcal{M}(S)$ *arising from covers of S itself*, which are obviously much more manageable. Whether these covers are cofinal or not is the question which the congruence conjecture purports to answer in a positive way. In any event here we take advantage of the results shown in particular in [4] to attack the questions in the framework of the congruence completions. Turning to the procongruence complexes $\check{C}(S)$ and $\check{C}_P(S)$ we prove that the latter one, namely the procongruence pants complex, remains rigid. That is we have a short exact sequence:

$$1 \to Inn(\check{\Gamma}(S)) \to Aut(\check{C}_P(S)) \to \mathbb{Z}/2 \to 1 \tag{1}$$

in which automorphisms are assumed to be continuous (this will be made precise in due time). One can again state that (with the mild exception of type $(1,2)$) $Out(\check{C}_P(S)) \simeq \mathbb{Z}/2 \simeq Gal(\mathbb{C}/\mathbb{R})$, just as in the discrete case, and the nontrivial outer automorphism comes again from orientation or complex conjugacy.

At first sight this may appear as a rather dull result: the procongruence pants complex is rigid, and this is also the case in the full profinite setting, modulo the congruence conjecture. In other words, rigidity survives completion in that case. So what? The point is that there are at least one surprise and one application in store. The surprise – if any – consists in the fact that the procongruence *curve* complex is *not* rigid. Far from it; indeed the outer automorphism group $Out(\check{C}(S))$ (for $d(S) > 3$, say) is enormous and can be taken as a higher genus version of the Grothendieck-Teichmüller group. In particular it is independent of S, that is of the type (g, n), and it naturally contains the absolute Galois group $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$. This and much more is elaborated in [22] (see also [23]) which however again does not contain watertight proofs inasmuch as the setting there is that of full profinite completion where certain tools are still lacking, in contrast with the case of the procongruence completion. The upshot is that the rigidity result shown in the present paper should pave the way for a thorough investigation of this new, topological version of Grothendieck-Teichmüller theory. It also affords a nice application in the form of a (weak) anabelian result for the moduli stacks of curves, which we develop in the last section of the paper. Roughly speaking, let S be as above, assume $d(S) > 3$ for simplicity (the low dimensional cases are well-understood) and regard the modular orbifold $\mathcal{M} = \mathcal{M}(S)$ (cf. §A.2) as a Deligne-Mumford stack over $\mathbb{Q}$. With this setting we show in particular that:

$$Out^*_{G_\mathbb{Q}}(\check{\Gamma}(S)) = Aut_\mathbb{Q}(\mathcal{M}(S)).$$

We will explain in the introduction to §8.2 below how this deserves to be called an anabelian result. Here let us just clarify a few ingredients. On the left-hand side the group $\check{\Gamma} = \check{\Gamma}(S)$ is the procongruence completion of the topological (orbifold) fundamental group $\Gamma = \Gamma(S)$ of $\mathcal{M} \otimes \mathbb{C}$, viewed as a complex orbifold (one should write $(\mathcal{M} \otimes \mathbb{C})^{an}$, where *an* denotes 'analytification'). Still on the left-hand side the star upperscript means that we confine attention to 'inertia perserving' automorphisms. This will be explained in more detail in due time (but see [23] for more context) and that decoration should be spurious: one conjectures that *all* automorphisms are inertia preserving. Finally the righ-hand side refers to the automorphisms of $\mathcal{M}$ as a $\mathbb{Q}$-stack. It is well-known however that one can replace $\mathbb{Q}$ with $\mathbb{C}$ and that the complex automorphisms are essentially the 'only possible ones', according to the classical Royden's theorem (and they form a small finite group). In a nutshell and in a more anabelian sounding fashion: the Galois invariant outer automorphisms of the procongruence completion of the orbifold fundamental group are given by the automorphisms of the stack itself. Moreover and modulo the congruence conjecture, $\check{\Gamma}(S)$ is nothing but the geometric fundamental group of the $\mathbb{Q}$-stack $\mathcal{M}(S)$.

2. Discrete complexes : rigidity and reconstruction

In this section we prepare the ground by recalling some rigidity results in the discrete setting in a fashion taylored to our needs (see §A.12 for a tightly compressed summary) and prove a reconstruction result which later on will be adapted to the procongruence setting; as a side benefit it provides another proof of the main result of [25], that is the rigidity of the discrete pants complex. To a hyperbolic surface S we associate in particular three *graphs*, namely the 1-skeleton $C^{(1)}(S)$ of the curve complex (cf. §A.5), the pants graph $C_P^{(1)}(S)$ (cf. §A.7) and the graph $C_*(S)$ (cf. §A.8). The definitions readily extend (cf. §A.9) to the case of not necessarily connected surfaces, with hyperbolic connected. These three graphs, and later on their respective completions, carry all the information we need. In some sense we are trying to pass from an essentially group theoretic framework, revolving around the Teichmüller group $\Gamma(S)$ (cf. §A.3), its completions and their cofinite subgroups to a *graph theoretic* one, based on the graphs above and later their completions, together with certain subgraphs.

2.1. **Rigidity of the discrete curves complex.** Basically this paragraph revolves around the two short exact sequences of §A.12. We start with the curve complex $C(S)$ and consider its group of simplicial automorphisms $Aut(C(S))$. There is a natural map $Mod(S) \to Aut(C(S))$ induced by letting a diffeomorphism act on loops (i.e. elements of $\mathcal{L}(S) = C^{(0)}(S)$; cf. §A.5), everything up to isotopy. The elements of the center of the left-hand group lie in the kernel of that map because they commute with twists, so there is an induced map $\theta : Inn(Mod(S))) \to Aut(C(S))$. Assume now that $C(S)$ is connected, that is $d(S) > 1$. Then it is not too difficult to show that θ is injective. The deep fundamental fact mentioned in the introduction and embodied by the sequence $(A\ 2)$ states that θ is also surjective for $(g, n) \neq (1, 2)$. This surjectivity, in item i) below, is due to N.V.Ivanov ([19]) and F.Luo ([21]):

Theorem 2.1. *Let S be a connected hyperbolic surface of type (g, n) with $d(S) > 1$. Then*
i) the natural map $\theta : Inn(Mod(S)) \to Aut(C(S))$ is an isomorphism except if $(g, n) = (1, 2)$, in which case it is injective but not surjective; in fact θ maps $Inn(Mod(S_{1,2}))$ onto the strict subgroup of the elements $Aut(C(S_{1,2}))$ which globally preserve the set of vertices representing nonseparating curves;
ii) $Aut(C^{(1)}(S)) = Aut(C(S))$.

Of course, if the type is different from $(1, 2)$ and $(2, 0)$, $Mod(S)$ is centerfree and θ provides an isomorphism between $Mod(S)$ and $Aut(C(S))$. Item ii) is easy but quite telling; it confirms that the pants complex and the pants graph (i.e. its 1-skeleton) have the same automorphisms. This fact will remain valid after completion. Here is a short proof. There is a natural map $Aut(C(S)) \to Aut(C^{(1)}(S))$ which is injective; indeed the restriction to the set of vertices is already injective. To prove surjectivity it is enough to give a graph theoretic characterization of the higher dimensional simplices of $C(S)$ and this is easily available: a moment contemplation will confirm that the k-dimensional simplices are in one-to-one correspondence with the *complete* subgraphs (a.k.a. *cliques*) of $C^{(1)}(S)$ with $k + 1$ vertices, i.e. subgraphs such that any two vertices are connected by an edge. This characterization proves ii). Note that to any simplicial complex one can associate the complex obtained by adding in all the cliques as simplices. Here $C(S)$ is a *flag complex*, that is, its simplices are exactly given by the cliques. This will also be the case of the other complexes we will meet (including in the profinite world) and it says that in fact all the information is contained is contained in a *graph*, namely the 1-skeleton of the relevant complex.

□

Remark 2.1. *The odd looking case of type $(1, 2)$ is actually easy to understand. It stems from the fact that $C(S_{1,2})$ and $C(S_{0,5})$ are isomorphic, whereas $\Gamma_{1,[2]}/Z(\Gamma_{1,[2]})$ maps into $\Gamma_{0,[5]}$ as a subgroup of index 5; indeed θ maps $Inn(Mod(S_{1,2}))$ injectively onto an index 5 subgroup of $Aut(C(S_{1,2}))$. See §A.4 and [22, 25] for a geometric discussion.*

N.V.Ivanov went on to show how to use the description of $Aut(C(S))$ afforded by Theorem 2.1 in order to study the action of $\Gamma(S)$ on Teichmüller space. He recovered in this way ([19]) the classical result of H.Royden about automorphisms of Teichmüller spaces:

Corollary 2.2. *If $d(S) > 1$, any complex automorphism of $\mathcal{T}(S)$ is induced by an element of $Mod(S)$.*

As N.V.Ivanov again showed, Theorem 2.1 also has immediate bearing on the automorphisms of the modular groups. Here we require one more definition, which will turn out to be of typical anabelian flavor:

Definition 2.3. An element of $Aut(\Gamma(S))$ is called *inertia preserving* if it (globally) preserves the set of cyclic subgroups generated by Dehn twists, that is maps a twist in $\Gamma(S)$ to a power of some other twist.

For a geometric discussion justifying this terminology we refer e.g. to [23]. In the present discrete setting we have the following

Theorem 2.4. *If $d(S) > 1$, all automorphisms of $\Gamma(S)$ are inertia preserving: $Aut^*(\Gamma(S)) = Aut(\Gamma(S))$.*

This result, which again is essentially due to N.V.Ivanov (cf. [18] and references therein) rests on a group theoretic characterization of twists inside $\Gamma(S)$. It is rarely stated independently or emphasized but we would like to stress it in view of the profinite or procongruence case; we also refer to [27] for a nice proof based on the notion of stable rank. This is because first we do not know how to prove the profinite or procongruence analog, which is unfortunate, and second because in the profinite setting this would feature a rather striking and precise analog of the so-called "local correspondence" in birational anabelian geometry (see [23] for more detail). Armed with Theorem 2.4 it is easy to use Theorem 2.1 in order to study the automorphisms of $\Gamma(S)$. Actually it turns out to be no more difficult to study morphisms between all the cofinite subgroups, (cf. [19], Theorem 2); we state this as:

Corollary 2.5. *Assume $d(S) > 1$ and $\Gamma = \Gamma(S)$ has trivial center; let $\Gamma_1, \Gamma_2 \subset \Gamma$ be two finite index subgroups. Then any isomorphism ϕ between Γ_1 and Γ_2 is induced by an element of $Mod(S)$, namely there exists $g \in Mod(S)$ such that $\phi(g_1) = g^{-1}g_1g$ for any $g_1 \in \Gamma_1$. In particular $Out(\Gamma(S)) \simeq \mathbb{Z}/2$.*

As usual one can study the two cases with nontrivial center, that is $(1,2)$ and $(2,0)$ in detail; see [27] for the latter one. This ends our review of the rigidity properties of the curves complex in the discrete setting, together with the group theoretic consequences. Before switching to the pants complex (or graph), we now introduce a kind of reconstruction technique for the various complexes.

2.2. **Reconstructing complexes and the rigidity of the pants graph.** In this paragraph we explore the local structure of our three complexes $C(S)$, $C_*(S)$ and $C_P(S)$ and show how to reconstruct them from local data. We especially focus on the three *graphs* obtained by retaining only the 1-skeleta of $C(S)$ and $C_P(S)$. As mentioned already we often abuse notation by writing $C_P(S)$ for the pants graph, bearing in mind that the full two-dimensional complex can be reconstructed from its 1-skeleton (cf. [25]). Trivially we have $C_P(S) \hookrightarrow C_*(S)$; it will turn out that this inclusion or rather its (equally trivial) analog after completion is of fundamental importance and lies in some sense at the very basis of a topological version of Grothendieck-Teichmüller theory. Note that (for $d(S) > 1$) $C_*(S)$ is the 1-skeleton of the dual of the simplicial complex $C(S)$. In terms of automorphisms $C_*(S)$ carries essentially the same information as $C(S)$ (see below for a precise statement) and it has been introduced essentially with a view to the above inclusion. Here we show (in the discrete setting) how to reconstruct the complexes from local data. Rigidity of the pants graph and *a fortiori* of the full complex will appear as an easy corollary. The proof of the reconstruction result (Theorem 2.10) is given in the next subsection.

Let us move to concrete and elementary notions. Given a surface S, a *subsurface* T is defined as $T = S \setminus \sigma$ where $\sigma \in C(S)$. We denote it S_σ; it is nothing but S cut or slit along the multicurve representing σ. In this definition the curves are defined as usual up to isotopy and one can choose a representative of the multicurve. One way to do this in a coherent way is to equip S with a (any) metric of constant negative curvature and use the (unique) geodesic representatives of the various multicurves. The metric plainly induces a metric with the same property on all the subsurfaces of S. There is a natural inclusion $C_*(S_\sigma) \subset C_*(S)$; in fact $C_*(S_\sigma)$ is the full subgraph of $C_*(S)$ whose vertices correspond to those pants decompositions of S which include σ (ditto for $C_P(S)$). For $\sigma \in C(S)$, we let $|\sigma|$ denote the number of curves which constitute σ. So $|\sigma| = dim(\sigma) + 1$ if $dim(\sigma)$ denotes the dimension of the simplex $\sigma \in C(S)$. The quantity $|\sigma|$ turns out to be more convenient in our context; in particular $d(S_\sigma) = d(S) - |\sigma|$. We include throughout the case of an empty cell (dimension -1): $S_\emptyset = S$. For example if σ is a maximal multicurve (pants decomposition), S_σ is a disjoint union of pants and $C_*(S_\sigma)$ is empty or reduced to a point (cf. §A.8) depending on convention. We call two simplices $\rho, \sigma \in C(S)$ *compatible* if the curves which compose ρ and σ do not intersect properly, that is they are either disjoint or coincide. Complex theoretically it means that ρ and σ lie in the closure of a common top dimensional simplex of $C(S)$. If ρ and σ are compatible, we define their unions and intersections $\rho \cup \sigma, \rho \cap \sigma \in C(S)$ in the obvious way. Then we clearly have:

Lemma 2.6. *If $\rho, \sigma \in C(S)$ are compatible simplices: $C_*(S_\rho) \cap C_*(S_\sigma) = C_*(S_{\rho\cup\sigma})$. If they are not compatible, this intersection is empty.* □

Here all graphs $C_*(S_\tau)$ ($\tau \in C(S)$) are considered as subgraphs of $C_*(S)$. This lemma has a number of equally obvious consequences. For instance $C_*(S_\rho) \subset C_*(S_\sigma)$ if and only if $\sigma \subset \rho$. Let us now return to the connections between C_* and C_P. The inclusion $C_P \subset C_*$ can be made more precise (cf. §A.8), given that two simplicial embeddings of F in $C_P(S)$ are either disjoint, or else intersect in a single vertex.

Lemma 2.7. *$C_*(S)$ is obtained from $C_P(S)$ by replacing every maximal copy of the Farey graph $F = C_P(S_{0,4}) = C_P(S_{1,1})$ inside $C_P(S)$ by a copy of the complete graph $G = C_*(S_{0,4}) = C_*(S_{1,1})$ associated to the vertices of the given Farey graph.* □

A maximal copy of F is a subgraph of $C_P(S)$ which is isomorphic to F and is not properly contained in another such subgraph. Note that the operation described in this lemma is *not* reversible; one cannot recognize $C_P(S)$ inside $C_*(S)$ without additional information and this may well be the seed of Grothendieck-Teichmüller theory. For the time being we note the following consequence in terms of automorphisms:

Lemma 2.8.

$$Aut(C_P(S)) \subset Aut(C_*(S))$$

Proof. An automorphism of $C_P(S)$ determines a permutation of the common vertex set $V(S)$ (cf. §A.9), which in turn defines an automorphism of $C_*(S)$ provided it is compatible with its edges. Lemma 2.7 and the fact that G is a complete graph ensure that this is always the case. □

So any automorphism of $C_P(S)$ determines an automorphism of $C_*(S)$ because both graphs share the same set of vertices and automorphisms of complexes are determined by their effect on the vertices. However *a priori* only certain automorphisms of $C_*(S)$ will preserve the additional structure given by the edges of $C_P(S)$, inducing an automorphism of this subgraph. In dimension 1, $Aut(G)$ is nothing but the permutation group on its vertices. Any automorphism of F determines a unique automorphism of G by looking at its effect on the vertices, but $Aut(F) \simeq PGL_2(\mathbb{Z})$ is certainly much smaller than $Aut(G)$. In the discrete case a kind of rigidification occurs for $d(S) > 1$ but this is *not* so after completion. Again this phenomenon lies at the very heart of Grothendieck-Teichmüller theory.

The (semi)local structure of C_* and C_P is not so mysterious. It is described in the following

Proposition 2.9. *Let $v \in V(S)$ be a vertex of $C_*(S)$ and $C_P(S)$, with $d(S) = k \geq 0$. Then v lies at the intersection of exactly k maximal copies of G (resp. F) in $C_*(S)$ (resp. $C_P(S)$). For any two copies G_i, G_j ($i \neq j$) one has $G_i \cap G_j = \{v\} \subset C_*(S)$ and two vertices $w_i \in G_i$, $w_j \in G_j$ with $w_i \neq v$, $w_j \neq v$ are* not *joined by an edge in $C_*(S)$.*

As for F, for any two copies F_i and F_j ($i \neq j$) we have $F_i \cap F_j = \{v\} \in C_P(S)$ and for any $w_i \in F_i$ such that v and w_i are connected by an edge, the vertices w_i and w_j are not connected by a finite chain in $C_P(S)$.

Proof. Let v be given as a pants decomposition $v = (\alpha_1, \ldots, \alpha_k)$. The main point here is that any triangle (complete graph on three vertices) of C_* or C_P is obtained by varying one of the α_i's keeping all the other curves α_j fixed. This in turn depends only on the already mentioned (and obvious) fact that two curves on a surface of dimension 1 always intersect. So we get k copies of G inside C_* which are indexed by the curves appearing in v. The rest of the statement and the transposition to C_P is easily verified.

Note that this shows that $d(S)$ can be read off (graph theoretically) from C_* or C_P. In fact it can be detected locally around any vertex v. To this end one can look for a star at v, namely a family $(w_i)_{i \in I}$ of vertices of $C_*(S)$ such that each w_i is connected to v by an edge and no two distinct w_i's are connected. Then $d(S)$ is the maximal possible number of such vertices i.e. the maximal cardinal of the index set I. Passing to $C_P(S)$, if $w_i, w_j \in F_i \subset C_P(S)$, then there is a finite chain connecting w_i and w_j in the link of v. Together with the last assertion of the statement, this shows that there are exactly $k = d(S)$ copies of F around v.

□

We now would like to reconstruct $C(S)$ from $C_*(S)$, hence also from $C_P(S)$ by Lemma 2.7. One way to do this is to set up a correspondence between the subgraphs of $C_*(S)$ which are graph theoretically isomorphic to some $C_*(S_\sigma)$ ($\sigma \in C(S)$) and the subsurfaces of S. This correspondence, to be later adapted to the complete setting, is interesting even in this relatively simple discrete case. A precise wording goes as follows:

Theorem 2.10. *Let $C \subset C_*(S)$ be a subgraph which is (abstractly) isomorphic to $C_*(\Sigma)$ for a certain surface Σ and is maximal with this property. Then there exists a unique $\sigma \in C(S)$ such that $C = C_*(S_\sigma)$.*

The proof is deferred to the next subsection. Here we list some fairly straightforward and important consequences. First one has:

Corollary 2.11. *$C(S)$ can be (graph theoretically) reconstructed from $C_*(S)$.*

Proof. Starting from $C_*(S)$ one builds a complex by considering subgraphs C as in the statement of the theorem, with the inclusion map as boundary operator. The result ensures that this simplicial complex is isomorphic to the curve complex $C(S)$. □

One then immediately gets:

Corollary 2.12. $Aut(C_*(S)) = Aut(C(S))$. □

Taking Lemma 2.8 into account, this shows that there is a natural injective map:

$$Aut(C_P(S)) \hookrightarrow Aut(C(S)),$$

from which by Theorem 2.1 we get the rigidity of the pants graph (*a fortiori* the pants complex) as

Theorem 2.13. *Let S be a hyperbolic surface of type (g, n) with $d(S) > 1$. Then the natural map*

$$\theta_P : Inn(Mod(S)) \to Aut(C_P(S))$$

is an isomorphism.

For the fact that here type $(1, 2)$ is no exception, see the last page of [25], of which we thus reproved the main result. We will see below (in §7) how this rigidity result (Theorem 2.13) does survive (procongruence) completion, in sharp and interesting constrast with item i) of Theorem 2.1)

2.3. **Proof of Theorem 2.10.** Let us start with some remarks and reductions. First we note that the word "maximal" is indeed necessary. For instance there are proper subgraphs of F (resp. G) which are isomorphic to F (resp. G). Second, implicit in the statement is the fact that any $C_*(S_\sigma) \subset C_*(S)$ does indeed answer the problem, namely it is maximal in its isomorphy class. Assume on the contrary that we have a nested sequence $C_*(S_\sigma) \subset C \subset C_*(S)$ where $d(S_\sigma) = k$, C is isomorphic to $C_*(S_\sigma)$ and the first inclusion is strict. Since C is connected, we can find a vertex $w \in C \setminus C_*(S_\sigma)$ which is connected by an edge to a vertex $v \in C_*(S_\sigma)$. Since S_σ has dimension k, we can find k vertices $w_i \in C_*(S_\sigma)$ as in the proof of Lemma 2.9 (with respect to v). But $w \in C$ is connected to v and it is easy to check that it is not connected to any of the w_i. In other words we have actually found $k + 1$ vertices which are connected to v and no two of which are connected, which contradicts the fact that C is isomorphic to $C_*(S_\sigma)$.

Having justified the statement, we can turn to the proof of Theorem 2.10, noticing first that uniqueness is clear: obviously $C_*(S_\sigma)$ coincides with $C_*(S_\tau)$ ($\sigma, \tau \in C(S)$) if and only if $\sigma = \tau$; this is also a very particular case of Lemma 2.6. From Lemma 2.9 we can now define $d(C) = d(\Sigma)$, which determines $|\sigma|$ (assuming the existence of σ) since $d(S_\sigma) = d(C) = d(S) - |\sigma|$. Next the result is true if $d(\Sigma) = 0$ because then $C_*(\Sigma)$ is just a point and so is C. Hence it does correspond to a vertex of $C_*(S)$, in other words to an actual pants decomposition of S. We will prove the result by induction on $k = d(\Sigma)$ but it is useful and enlightening to prove the case $k = 1$ directly. This is easy and essentially well-known in a different context. Much as in Lemma 2.9 the point is that any triangle inside $C_*(S)$ (or $C_P(S)$) determines a unique subsurface Σ with $d(\Sigma) = 1$. This sets up a one-to-one correspondence between subsurfaces of S of dimension 1 and maximal complete subgraphs of $C_*(S)$.

Now let $k > 1$, assume the result has been proved for $d(C) < k$ and consider a graph $C \subset C_*(S)$ as in the statement, with $d(C) = k$. We fix an isomorphism $C \xrightarrow{\sim} C_*(\Sigma)$. Changing notation slightly for convenience, we are looking for a subsurface $T \subset S$, defined by a cell of $C(S)$ and such that $C = C_*(T)$. Note that it may happen that the surfaces Σ and T (assuming the existence of the latter) are not of the same type because of the well-known exceptional low-dimensional isomorphisms between complexes of curves. One will have $C_*(\Sigma) \simeq C_*(T)$ and indeed, as a consequence of the result itself, $C(\Sigma) \simeq C(T)$, so for instance Σ could be of type $(0, 6)$ and T of type $(2, 0)$.

We may now consider subsurfaces of Σ and transfer the information to $C \subset C(S)$. Namely for any $\sigma \in C_*(\Sigma)$, we denote by $C_\sigma \subset C$ the subgraph corresponding to $C_*(\Sigma_\sigma)$ under the fixed isomorphism $C \simeq C_*(\Sigma)$. Actually, forgetting about this isomorphism, we just write $C_\sigma = C_*(\Sigma_\sigma) \subset C \subset C_*(S)$. By the induction hypothesis, for any $\sigma \in C(\Sigma)$, $\sigma \neq \emptyset$, there corresponds to C_σ a unique subsurface $S_{(\sigma)} \in S$. Beware of the fact that σ now runs over the cells of $C(\Sigma)$, not of $C(S)$, and this is the reason of the added brackets. In these terms we are trying to extend this correspondence to $\sigma = \emptyset$, i.e. find $T = S_{(\emptyset)}$.

In order to show the existence of T it is actually enough to show that there exists a k-dimensional subsurface of S, call it precisely T, such that any $S_{(\sigma)}$ with $\sigma \in C(\Sigma)$ not empty is contained in T. Indeed, the corresponding C_σ's form a covering of C. So assuming the existence of such a subsurface T, we find that $C \subset C_*(T)$; these two subgraphs being isomorphic and C being maximal by assumption, they coincide. In order to prove the existence of T, we can now restrict attention to the largest possible $S_{(\sigma)}$'s, i.e. to the case $|\sigma| = 1$, which simply means that σ consists of a single loop.

We are thus reduced to showing that there exists a k-dimensional subsurface $T \subset S$ such that, for any loop α on Σ, $S_{(\alpha)}$ is contained in T. Now $C(\Sigma)$ is connected because $k > 1$ and this can be used as follows. If α and β are two non intersecting curves on Σ, Σ_α and Σ_β are two subsurfaces of Σ of dimension

$k-1$ intersecting along the subsurface $\Sigma_{\alpha\cup\beta}$ of dimension $k-2$, where $\alpha\cup\beta$ is considered as a simplex of $C(\Sigma)$. Informally speaking for the time being, the union $S_{(\alpha)}\cup S_{(\beta)}$ has dimension k and this is the natural candidate for T. In other words the latter, if it exists, is determined by any two non intersecting loops of Σ. Returning to the formal proof, let γ and δ be two arbitrary loops on Σ. There exists a path in the 1-skeleton of $C(\Sigma)$ connecting γ to δ. It is given by a finite sequence $\gamma,\alpha_1,\ldots,\alpha_n,\delta$ of loops such that α_1 does not intersect γ, α_n does not intersect δ and for $1<i<n$, α_i does not intersect α_{i-1} and α_{i+1}. Using the existence of such a chain, we are reduced to the following situation. Let α, β, and γ be three loops on Σ such that $\alpha\cap\beta=\beta\cap\gamma=\emptyset$; there remains again to show that $S_{(\alpha)}$, $S_{(\beta)}$ and $S_{(\gamma)}$ are contained in a common k-dimensional subsurface T, and this will complete the proof of the result.

We can write $S_{(\alpha)}=S_\rho$, $S_{(\beta)}=S_\sigma$, $S_{(\gamma)}=S_\tau$, for certain simplices $\rho,\sigma,\tau\in C(S)$ with $|\rho|=|\sigma|=|\tau|=d(S)-k+1$. Moreover, because $\alpha\cap\beta=\emptyset$ (resp. $\beta\cap\gamma=\emptyset$) ρ and σ (resp. σ and τ) are compatible simplices. So we can consider $\rho\cap\sigma$ and $\sigma\cap\tau$, with $|\rho\cap\sigma|=|\sigma\cap\tau|=d(S)-k$. The corresponding surfaces $S_{\rho\cap\sigma}$ and $S_{\sigma\cap\tau}$ are both subsurfaces of S of dimension k. There remains only to show that they coincide: $S_{\rho\cap\sigma}=S_{\sigma\cap\tau}(=T)$. We argue much as above, when proving that a subcomplex of type $C_*(S_\sigma)\subset C_*(S)$ is maximal in its isomorphy class. The complexes $C_{\rho\cap\sigma}$ and $C_{\sigma\cap\tau}$ are two subcomplexes of dimension k inside C which is also of dimension k, and they are maximal such complexes, being attached to subsurfaces of S. This forces them to coincide – and in fact coincide with the whole of C. More formally, assume the contrary, that is $S_{\rho\cap\sigma}$ and $S_{\sigma\cap\tau}$ are distinct. Then, breaking the symmetry for a moment and relabeling if necessary, we can choose as above two vertices $v\in C_{\rho\cap\sigma}$ and $w\in C_{\sigma\cap\tau}\setminus C_{\rho\cap\sigma}$ which are connected by an edge. Then again pick a maximal family (w_i) of k vertices in $C_{\rho\cap\sigma}$ which are connected to v and are not mutually connected. Adding in the vertex w we get a family of $k+1$ vertices with the same properties, which contradicts the fact that $d(C)=k$ and completes the proof. □

3. Profinite complexes and the isomorphism theorem

In this section we introduce and study profinite completions of the simplicial complexes which have appeared above. We focus on the procongruence completion because crucial results are *not* available to-date for the *full* profinite completions, as will become clear below. General foundations pertaining to completions of "spaces", possibly equipped with group actions, are now available in a profinite context, thanks in particular to the work of G.Quick who has put these objects in the classical framework of model categories (see [33, 34] and references therein). However in our much more specific context we can and do rely on the more direct constructions of the first author (see [3, 4]). We then state and prove the crucial isomorphism result which very roughly speaking provides a bridge between group theoretic and complex or graph theoretic statements. We claim little novelty as to the framework and statements in this section, which are essentially borrowed from [4]. However some proofs in that paper (which itself uses [5] in a crucial way) are not so easy to decipher and it thus seemed useful to provide at times alternative proofs or at least sketches thereof, using a more concrete, if somewhat *ad hoc* approach. We have also added a short "guide for the perplexed" (§3.3) aiming at summarizing some of the main points of the theory, delineating a roadmap and pointing at a few serious bumps along the road.

3.1. **Completions etc.** Profinite complexes of curves were introduced in [3] ; the necessary constructions (and caveats) are summarized in [4], §3 to which we refer, especially concerning the congruence completions on which we focus hereafter. Minimal inputs appear in the Appendix below (§§A.10, 11). Starting as usual from a (connected) hyperbolic surface of finite type S and the attending Teichmüller group $\Gamma=\Gamma(S)$, one constructs in particular its (full) profinite completion $\hat\Gamma$ as well as its (pro)congruence completion $\check\Gamma$ (see §A.10). One then proceeds to show that the cofinite subgroups $\Gamma^{(m)}\subset\Gamma$ $(m>2)$ pertaining to the abelian levels $\mathcal{M}^{(m)}$ (see again §A.10 or [4] for much more) operate without inversion on the (discrete) curve complex $C(S)$. This implies that this Γ-simplicial complex $C(S)$ can be considered as a $\Gamma^{(m)}$-simplicial *set* for any $m>2$ (after numbering the vertices). Now by restricting to the congruence levels which dominate some such abelian level (that is the inverse system of congruence subgroups Γ^λ with $\Gamma^\lambda\subset\Gamma^{(m)}\subset\Gamma$ for some $m>2$) we define the *congruence completion* $\check C(S)$ which we can view as a $\check\Gamma$-simplicial profinite set, that is a simplicial object in the category of profinite sets, which moreover is equipped with an action of the congruence completion $\check\Gamma$. We refer again to [3, 4] for the necessary precisions. Roughly speaking this makes sense of the definition of the congruence completion as a pro-simplicial set defined by

$$\check C(S)_\bullet=\varprojlim_{\lambda\in\Lambda} C(S)_\bullet/\Gamma^\lambda$$

where Γ^λ runs over the congruence subgroups of Γ, indexed by the (countable) set Λ. We denote the finite quotients by $C^\lambda(S) = C(S)_\bullet/\Gamma^\lambda$. Note that one may and it is sometimes useful to restrict consideration to the normal or even characteristic subgroups Γ^λ since both types define cofinal inverse subsystems (because Γ is finitely generated). Note also that these completions are plainly defined "asymptotically", that is one can omit any subsequence of "large" subgroups. This is why for instance we may restrict to congruence subgroups which are contained in some subgroup $\Gamma^{(m)}$ $(m > 2)$.

So we regard $\check{C}(S)_\bullet$ as a simplicial object in the category of profinite sets, although below bullets are often omitted, while keeping in mind that we are indeed dealing with simplicial objects. There is a canonical *inclusion* $C(S) \hookrightarrow \check{C}(S)$ ([4], Prop. 3.3) with dense image and a natural continuous action of $\check{\Gamma}$ on $\check{C}(S)$.

In a similar fashion and for the same reasons we can define $\check{C}_P(S)$ as the inverse limit

$$\check{C}_P(S)_\bullet = \varprojlim_{\lambda\in\Lambda} C_P(S)_\bullet/\Gamma^\lambda$$

and regard it again as a simplicial object in the category of profinite sets. It is in fact a pro*graph*; the finite quotients are denoted $C_P^\lambda(S) = C(S)_\bullet/\Gamma^\lambda$. There is again a canonical inclusion $C_P(S) \hookrightarrow \check{C}_P(S)$ with dense image, which is equivariant for the Γ-action (resp. $\check{\Gamma}$-action) on $C_P(S)$ (resp. $\check{C}_P(S)$) and the inclusion $\Gamma(S) \hookrightarrow \check{\Gamma}(S)$. Finally, as in the discrete case, there is a one-to-one correspondence between the vertices of $\check{C}_P(S)$ and the simplices of $\check{C}(S)$ of maximal dimension $(= d(S) - 1)$. A deep additional information is contained in the *edges* of $\check{C}_P(S)$.

We now concentrate on alternative, more geometric and manageable descriptions of the congruence curves complex $\check{C} = \check{C}(S)$. More precisely we will shortly define the simplicial profinite complexes $\check{C}_{\mathcal{L}} = \check{C}_{\mathcal{L}}(S)$ and $\check{C}_{\mathcal{G}} = \check{C}_{\mathcal{G}}(S)$, denoted respectively $L(\hat{\pi})$ and $L'(\hat{\pi})$ in [4] ($\hat{\pi} = \hat{\pi}_1^{top}(S)$, the topological fundamental group of the surface S) to which we refer for more detail. An important result, stated and proved in the next subsection asserts that $\check{C}(S)$, $\check{C}_{\mathcal{L}}(S)$ and $\check{C}_{\mathcal{G}}(S)$ are isomorphic, so that we are indeed describing the *same* object from several standpoints. Typically, these three objects can be defined in the full profinite setting but the fact that they are isomorphic is not known.

In order to define $\check{C}_{\mathcal{L}}(S)$, where $\mathcal{L}$ stands for "loops" we first define its set of vertices $\hat{\mathcal{L}}(S) = \check{C}_{\mathcal{L}}(S)_0$, the set of unoriented proloops. Recall that in the discrete setting $\mathcal{L}(S) = C_{\mathcal{L}}(S)_0$ denotes the set of unoriented simple loops up to isotopy which moreover are not peripheral, that is do not bound a disc on S with a single puncture. We are looking for a completion which is *a priori* simpler and more manageable than the one afforded by $\hat{C}(S)$ in that it will involve only the fundamental group $\pi = \pi_1^{top}(S)$ and its completion, instead of the much more involved $\Gamma = \pi_1^{top}(\mathcal{M}(S))$, the topological or orbifold fundamental group of the moduli space of curves.

We proceed as follows (see again [4], §3). For a set X, let $\mathcal{P}(X)$ denote the set of *unordered* pairs of elements of X and for G a group, let $G/\sim$ denote the set of conjugacy classes in G. Now given $\gamma \in \pi$, denote by $\gamma^\pm$ the equivalence class of the pair (γ, γ^{-1}) in $\mathcal{P}(\pi)$ and by $[\gamma^\pm]$ the equivalence class of $\gamma^\pm$ in $\mathcal{P}(\pi/\sim)$. Note that the latter has a natural structure of profinite set. The point is that there is a natural *embedding* $\iota : \mathcal{L} \hookrightarrow \mathcal{P}(\pi/\sim)$. Indeed, given a loop $\ell \in \mathcal{L}$, it can be represented by an element $\gamma = \gamma(\ell) \in \pi$ and we define $\iota(\ell) = [\gamma^\pm]$, which is plainly independent of the choice of the representative γ of ℓ. Finally we define the set $\hat{\mathcal{L}} = \hat{\mathcal{L}}(S)$ of proloops on S as the closure of the image $\iota(\mathcal{L})$ inside $\mathcal{P}(\hat{\pi}/\sim)$, where we are using the nontrivial fact from combinatorial group theory (conjugacy separability for the group π) that the natural map $\mathcal{P}(\pi/\sim) \to \mathcal{P}(\hat{\pi}/\sim)$ is injective.

It is then easy to define, in much the same way, the simplicial complex $C_{\mathcal{L}}(S)$ (with $\mathcal{L} = C_{\mathcal{L}}(S)_0$) and its completion $\check{C}_{\mathcal{L}}(S)$ (with $\hat{\mathcal{L}} = \check{C}_{\mathcal{L}}(S)_0$)). For X a set and $k \geq 1$, define $\mathcal{P}_k(X)$ to be the set of unordered subsets of $\mathcal{P}(X)$ with k elements ($\mathcal{P} = \mathcal{P}_1$). Then we get a natural embedding $\iota_k : C(S)_k \hookrightarrow \mathcal{P}_{k+1}(\pi/\sim)$ ($\iota = \iota_0$) of the k-simplices of the discrete curve complex into the unordered sets of $k+1$ conjugacy classes of the group π modulo inversion. There remains only to define $\check{C}_{\mathcal{L}}(S)_k$ as the closure of the image and to organize the collection of profinite sets $(\check{C}_{\mathcal{L}}(S)_k)$ $(0 \leq k \leq d(S) - 1)$ into the simplicial complex $C_{\mathcal{L}}(S)_\bullet$, using the usual face and degeneracy operators (deleting and adding elements).

The last avatar $\check{C}_{\mathcal{G}}(S)$ of the congruence complex is actually easier to define. It is enough to define its sets of vertices $\check{C}_{\mathcal{G}}(S)_0$ and then proceed as above. Return to $\mathcal{L}(S)$; mapping a simple loop to the cyclic subgroup of Γ generated by the corresponding twist, we get a natural embedding $\mathcal{L} \hookrightarrow \mathcal{G}(\pi)/\sim$ where the right-hand side denotes the set of *cyclic* subgroups of π modulo conjugacy. Again we have a further natural injective map $\mathcal{G}(\pi/\sim) \hookrightarrow \mathcal{G}(\hat{\pi})/\sim$ and we denote by $\hat{\mathcal{G}}(S)$ the closure of the image of $\mathcal{L}$ in $\mathcal{G}(\hat{\pi})/\sim$ via the composite embedding. Equivalently we may consider the image $\mathcal{G}(S)$ of $\mathcal{L}$ in $\mathcal{G}(\pi)/\sim$ and then take its closure $\hat{\mathcal{G}}(S$ in $\mathcal{G}(\hat{\pi})/\sim$, corresponding to certain procyclic subgroups of $\hat{\pi}$, still up to conjugacy. Starting from $\hat{\mathcal{G}}(S) = \hat{C}_{\mathcal{G}}(S)_0$ we then build up the prosimplicial complex $\check{C}_{\mathcal{G}}(S)$ the same way we built $\check{C}_{\mathcal{L}}(S)$ out of $\hat{\mathcal{L}}(S)$.

The next subsection will be essentially devoted to showing that these three avatars of $\check{C}(S)$ (including $\check{C}(S)$ itself) are isomorphic. Here in closing we add a few simple but extremely useful remarks about this type of relatively new objects. First of all one should keep in mind that we are dealing with *compact* (totally disconnected) spaces. This means in particular that there is no "going to infinity". As a first extremely crude approximation $\hat{C}(S)$ or $\check{C}(S)$ differ as much from $C(S)$ as the ring $\mathbb{Z}_p$ of the p-adic integrs differs from $\mathbb{Z}$. Note for instance that Thurston's theory precisely starts from considerations connected with geometric intersection numbers, twists and ways of going to infinity, whether on Teichmüller space $\mathcal{T}(S)$ or on the curves complex $C(S)$. Nothing of the kind is available – nor even relevant – here. For much more on a dynamical viewpoint on these objects we refer to [23], §8.

Next we sketch a line of arguments which we will meet below more than once. Let $X(= X_\bullet)$ denote a discrete G-simplicial complex with G a finitely generated group. Assume the number of G-orbits in X is finite. Let G' be some completion of G and assume we have constructed a completion X' of X which is a G'-prosimplicial complex. In particular there is a natural morphism $\iota : X \to X'$ with dense image and X' enjoys the universal property that any morphism $\phi : X \to Z$ from X to a G'-prosimplicial complex Z factors through a unique $\phi' : X' \to Z$ i.e. $\phi = \phi' \circ \iota$. Moreover ι is equivariant for the G-action on X and G'-action on X'. Note that all the morphisms we consider are continuous for the natural topologies on their respective source and target.

In the situation above, there is at first a seemingly simple description of X' which goes as follows. Pick $k \geq 0$ and let X_k denote the k-skeleton of X ; by assumption one can decompose X_k into *disjoint* G-orbits enumerated by the *finite* set E_k:

$$X_k = \coprod_{\sigma \in E_k} G \cdot \sigma,$$

where the k-simplices σ are representatives in the orbits. Under these circumstances one can decompose the k-skeleton X'_k of X' as

$$X'_k = \coprod_{\sigma \in E_k} G' \cdot \iota(\sigma).$$

In other words it is covered by the G'-orbits of the images of the *same* simplices. Note that these orbits now may not be disjoint. In all the cases we will encounter X is residually finite, that is ι is injective, and we omit it from the notation. So X' is made of (not necessarily disjoint) G'-orbits and there are finitely many in every dimension. The one line proof of the above is both simple and instructive. Consider the right-hand side of the equality above: it is compact because so is G' and E_k is finite; it is dense because it contains $\iota(X)$. So it coincides with X'_k.

Finally let X and Y be as above and for simplicity assume they are both residually finite so that we identify X (resp. Y) with its image in X' (resp. Y'). Let $f : X \to Y$ be a simplicial morphism. It naturally determines a morphism $f' : X' \to Y'$ by the universality property of the completion. Moreover, if f is onto, so is f'. The proof is again one line : the image $f'(X')$ contains $f'(X) = f(X) = Y \subset Y'$ which is dense ; $f(X')$ being dense and compact (as the continuous image of a compact) in Y', it coincides with it.

3.2. **The isomorphism theorem.** Let us return to S and the three attending versions of the congruence complex, namely $\check{C}(S)$, $\check{C}_{\mathcal{L}}(S)$ and $\check{C}_{\mathcal{G}}(S)$. By the universality of the $\check{\Gamma}$-completion we have a sequence of (simplicial) maps :

$$\check{C}(S) \to \check{C}_{\mathcal{L}}(S) \to \check{C}_{\mathcal{G}}(S).$$

We can now apply (twice) the reasoning immediately above (end of §3.1) and conclude that both maps are surjective. Their *injectivity* constitutes one of the main statements in [4] :

Theorem 3.1 ([4], Theorem. 4.2)**.** *The natural maps $\check{C}(S) \to \check{C}_{\mathcal{L}}(S)$ and $\check{C}_{\mathcal{L}}(S) \to \check{C}_{\mathcal{G}}(S)$ are $\check{\Gamma}$-equivariant isomorphisms of prosimplicial sets.*

Sketch of proof. We will present a partial proof of this important result (using ideas from [4]), breaking it into three propositions. First it is clearly enough to show that the composition of the two maps is injective and one can actually restrict to showing that the map on the vertices, namely

$$\Phi : \check{C}(S)_0 \to \check{C}_{\mathcal{G}}(S)_0,$$

is injective, hence a bijection since it is known to be surjective. Recall that on the left

$$\check{C}(S)_0 = \check{\mathcal{L}} = \varprojlim_{\lambda \in \Lambda} \mathcal{L}/\Gamma^\lambda$$

where $\lambda \in \Lambda$ runs over the congruence subgroups of Γ. The right-hand side is given as the closure of the set of cyclic subgroups of π corresponding to elements of $\mathcal{L}(S)$ inside $\mathcal{G}(\hat{\pi})/\sim$, the set of procyclic subgroups of $\hat{\pi}$ ($\pi = \pi_1^{top}(S)$) modulo conjugacy. Both sides are naturally equipped with a $\check{\Gamma}$-action and the map Φ is equivariant and onto. The only moot point is injectivity, whose validity is equivalent to that of the statement of the theorem. We used the symbol $\check{\mathcal{L}}(S)$ because $\hat{\mathcal{L}}(S)$ has already been used for the set $\check{C}_{\mathcal{L}}(S)_0$ of vertices of $\check{C}_{\mathcal{L}}(S)$; *a posteriori* the theorem will confirm that $\check{\mathcal{L}}(S) = \hat{\mathcal{L}}(S)$.

Our first assertion reads:

Proposition 3.2. *The map Φ induces a bijection between the respective $\check{\Gamma}$-orbits of $\check{C}(S)_0$ and $\check{C}_{\mathcal{G}}(S)_0$.*

In fact these $\check{\Gamma}$-orbit have nothing mysterious. Indeed recall how curves and (Dehn) twists are related with the Γ-action in the discrete case. If $\alpha \in \mathcal{L}$ is a loop (i.e. an isotopy class of simple closed curves), τ_α the associated twist (we assume that the surface S has been given an orientation once and for all) and $g \in \Gamma$, then we have the familiar and elementary formula

$$\tau_{g\cdot\alpha} = g\tau_\alpha g^{-1}.$$

Anticipating (a lot) we remark that the Grothendieck-Teichmüller action can and will be seen essentially as a generalization of this formula to 'procurves' and 'protwists'. For the moment we recall that this provides a description of the Γ-orbits of the discrete complex $C(S) = C_{\mathcal{L}}(S) = C_{\mathcal{G}}(S)$ (with obvious definitions). Two loops α and β lie in the same Γ-orbit if and only if the topological types of the two slit surfaces $S_\alpha = S \setminus \alpha$ and $S_\beta = S \setminus \beta$ coincide. This is also the necessary and sufficient condition for the two associated twists τ_α and τ_β over α and β to be Γ-conjugate. The topological type of a twist τ_γ along a curve γ is defined as the type of S_γ, the surface S slit along γ, which we also refer to as the type of the curve γ itself.

Now any $\check{\Gamma}$-orbit in $\check{C}$ contains a *discrete* representative, i.e. a curve in $\mathcal{L}$ (see the end of §3.1). So the $\check{\Gamma}$-orbits of $\check{C}$ are enumerated, *with possible redundancies*, by the finitely many topological types of the slit surfaces S_α ($\alpha \in \mathcal{L}$), which also enumerate the irreducible components of the divisor at infinity of the stable compactification of $\mathcal{M}(S)$. The same is true of the $\check{\Gamma}$-orbits of $\check{C}_{\mathcal{G}}$, for the same reason. Since Φ is onto, this shows that Proposition 3.2 is a consequence of the following:

Proposition 3.3. *Given twists $\tau_\alpha, \tau_\beta \in \Gamma \subset \check{\Gamma}$ (with $\alpha, \beta \in \mathcal{L}$) two nontrivial powers τ_α^k, τ_β^ℓ ($k, \ell \in \hat{\mathbb{Z}} \setminus \{0\}$) are conjugate in $\check{\Gamma}$ if and only if $k = \ell$ and τ_α and τ_β (equivalently α and β) have the same topological type.*

Note that this will show that the topological type of a "protwist", or actually a power thereof is well-defined as the type of any discrete twist lying in the same $\check{\Gamma}$-orbit, a protwist being nothing but a $\check{\Gamma}$-conjugate of some *bona fide* discrete twist. We will henceforth often skip the prefix "pro" ("protwists", "procurves", etc.) when it should not lead to confusion. We also remark that it has long been known that the congruence levels separate the powers of a twist (or protwist for that matter). That is, given a twist τ, the natural map $\hat{\mathbb{Z}} \to \check{\Gamma}$ which sends $a \in \hat{\mathbb{Z}}$ to τ^a is injective. In other words the procyclic group $\langle\tau_\alpha\rangle$ generated by a twist τ_α is contained in $\check{\Gamma}$.

Granted Proposition 3.3 (see below for its proof) there remains to show what we state as:

Proposition 3.4. *For every $\alpha \in \mathcal{L} = C(S)_0 \subset \check{C}(S)_0$ the $\check{\Gamma}$-stabilizer $\check{\Gamma}_\alpha \subset \check{\Gamma}$ of α as an element of $\check{C}(S)_0$ coincides with the stabilizer of its image $\Phi(\alpha) \in \check{C}_{\mathcal{G}}(S)_0$.*

Here again one can reduce – as we did – the question to the stabilizer of a *discrete* curve by first acting with $\check{\Gamma}$. Moreover, by [3], Proposition 6.5, the $\check{\Gamma}$-stabilizer of α is the closure in $\check{\Gamma}$ of the stabilizer $\Gamma_\alpha \subset \Gamma$ of $\alpha \in C(S)_0$, that is viewed as an element of the discrete complex $C(S)$. Finally the discrete stabilizer Γ_α affords an elementary geometric description.

We have now reduced the proof of Theorem 3.1 to those of Propositions 3.3 and 3.4. We will present the first in detail, partly for its own sake, partly in order to illustrate certain techniques in a concrete, if somewhat *ad hoc* way. By contrast, we will essentially rely on [4] for the proof of Proposition 3.4.

Proof of Proposition 3.3. First let us clarify the (natural) definition of a profinite power. If – say – $g \in \check{\Gamma}$ and $k \in \hat{\mathbb{Z}}$, then $g^k \in \check{\Gamma}$ is defined explicitly as an inverse system. For a level $\lambda \in \Lambda$, let a_λ denote the order of the finite group Γ/Γ^λ. Then the λ-component of g^k reads ${g_\lambda}^{k_\lambda}$ where $g_\lambda \in \Gamma/\Gamma^\lambda$ is the λ-component of g and $k_\lambda \in \mathbb{Z}/a_\lambda$ is the a_λ-component of k (of course this definition is valid for any completion of any group).

Consider again τ_α and τ_β, where $\alpha, \beta \in \mathcal{L}(S)$. We need only prove the only if part of the statement: given two profinite powers τ_α^k and τ_β^ℓ ($k, \ell \in \hat{\mathbb{Z}} \setminus \{0\}$) there should exist a finite congruence quotient of Γ in which their images are *not* conjugate, if either α and β do not share a common type, or k and ℓ are different.

For $\alpha \in \mathcal{L}(S)$, let T_α denote the action in homology of τ_α. It is well-known that for any loop $\gamma \in \mathcal{L}(S)$ on the surface we have

$$T_\alpha[\gamma] = [\gamma] + \langle [\gamma], [\alpha] \rangle \ [\alpha]$$

where $[\gamma]$ denotes the homology class of the curve γ and $\langle \ , \ \rangle$ is the symplectic intersection form on S. Therefore T_α is either trivial, when the curve is separating (i.e. when $[\alpha] = 0$), or it can be represented by a nontrivial elementary matrix with one unit nonzero entry outside of the diagonal, when the curve α is nonseparating. Therefore the conjugacy classes of (powers of) twists along two curves, at least one of which is non separating, can be distinguished in any nontrivial congruence quotient of the integral symplectic group. We are thus reduced to the case where both α and β are separating, which we assume from now on.

Let $f : \tilde{S} \to S$ be a characteristic (finite unramified) cover associated to a finite index characteristic subgroup $K = \pi_1(\tilde{S}) \subset \pi = \pi_1(S)$, which can be identified with the image $f_*(\pi_1(\tilde{S}))$ by the map f_* induced at the level of fundamental groups. It is Galois with Galois group $G_K = \pi/K$. We would like to compute the action in homology of the lift of a twist to such a cover. If $\phi \in Aut(\pi)$ is an automorphism of π, its restriction to K determines an automorphism $\tilde{\phi} \in Aut(\tilde{\pi}))$ ($\tilde{\pi} = \pi_1(\tilde{S}) = K$), via the requirement of equivariance

$$f_* \circ \tilde{\phi} = \phi \circ f_*.$$

This also defines a way of lifting mapping classes $\varphi \in Out^+(\pi) = \Gamma = \Gamma(S)$ (where the superscript $+$ indicates the preservation of orientation) to $\tilde{\varphi} \in Out^+(\tilde{\pi}) = \tilde{\Gamma} = \Gamma(\tilde{S})$; the lift is well-defined up to the action of the Galois group G_K. Since mapping classes are determined by their action on the simple closed curves we derive that $\tilde{\varphi}$ is determined – again up to multiplication by an element of G_K – by the equivariance on these, that is the property that

$$f(\tilde{\varphi}(\tilde{\gamma})) = \varphi(f(\tilde{\gamma}))$$

(up to homotopy) for every $\gamma \in \mathcal{L}(S)$ and $\tilde{\gamma} \in \mathcal{L}(\tilde{S})$ with $\gamma = f_*(\tilde{\gamma})$.

We say that a loop $\tilde{\alpha} \in \mathcal{L}(\tilde{S})$ on $\tilde{S}$ is a lift of $\alpha \in \mathcal{L}(S)$ if it is a connected component of its preimage $f^{-1}(\alpha)$; any two lifts of any two equivalent curves (i.e. curves with the same type) are equivalent. We denote by α^n the curve obtained by traveling n times around α. If $\tilde{\alpha}$ is a lift of α, the restriction of f to $\tilde{\alpha}$ defines a finite covering of α of degree – say – $m(\alpha)$, which is independent of the choice of the lift, indeed only depends on the type (i.e. equivalence class) of α. In fact $m(\alpha)$ coincides with the order in $G_K = \pi/K$ of any element of π represented by the curve α, which can be seen as follows. Let $d(\alpha)$ denote this order; it is well-defined since the various elements representing α belong to a single conjugacy class of G_K. Then (the class of) $\alpha^{d(\alpha)}$ belongs to $K = f_*(\pi_1(\tilde{S}))$ and hence it can be lifted to a closed curve $\tilde{\alpha}$. Moreover $\tilde{\alpha}$ cannot be a power of some other curve $\tilde{\beta}$ ($\tilde{\alpha} = \tilde{\beta}^n$, $n > 1$) because if so the restriction of f to $\tilde{\beta}$ would cover α with degree $d(\alpha)/n < d(\alpha)$. Hence $\tilde{\alpha} \in \mathcal{L}(\tilde{S})$ and $m(\alpha) = d(\alpha)$.

We wish to describe a lift $\tilde{\tau}_\alpha$ of τ_α to $\tilde{\Gamma}$, or at least a suitable power of it. So let $\gamma \in \mathcal{L}(S)$, $\tilde{\gamma} \in \mathcal{L}(\tilde{S})$ a lift of γ. By the above

$$f(\tilde{\tau}_\alpha(\tilde{\gamma})) = (\tau_\alpha(\gamma))^{d(\gamma)}.$$

On the other hand it also holds that

$$f(\tau_{\tilde{\alpha}}(\tilde{\gamma})) = (\tau_\alpha^{d(\alpha)}(\gamma))^{d(\gamma)}$$

from which we conclude that

$$\tilde{\tau}_\alpha^{d(\alpha)} = \tau_{\tilde{\alpha}}.$$

We can now compute the action of $\tilde{\tau}_\alpha^{d(\alpha)}$ on the homology of $\tilde{S}$, which we denote $\tilde{T}_\alpha^{d(\alpha)}$. Indeed the action of any power is determined by

$$\tilde{T}_\alpha^{kd(\alpha)}[\tilde{\gamma}] = T_{\tilde{\alpha}}^k[\tilde{\gamma}] = [\tilde{\gamma}] + k\langle [\tilde{\gamma}], [\tilde{\alpha}] \rangle [\tilde{\alpha}]$$

with k an integer, $\tilde{\gamma} \in \mathcal{L}(\tilde{S})$ and the angle brackets denote the symplectic pairing on $\tilde{S}$ (here and below pairings will implicitly relate to the relevant surface). From the above we find in particular that

$$\tilde{T}_\alpha^{kd(\alpha)}[\tilde{\gamma}] = [\tilde{\gamma}] + k\langle [\tilde{\gamma}], [\tilde{\alpha}] \rangle [\tilde{\alpha}].$$

Raising this identity to the power $d(\beta))$ we find that

$$\tilde{T}_\alpha^{kd(\alpha)d(\beta)}[\tilde{\gamma}] = [\tilde{\gamma}] + kd(\beta)\langle [\tilde{\gamma}], [\tilde{\alpha}] \rangle [\tilde{\alpha}];$$

swapping (α, k) and (β, ℓ), this delivers

$$\tilde{T}_\beta^{\ell d(\alpha)d(\beta)}[\tilde{\gamma}] = [\tilde{\gamma}] + \ell d(\alpha)\langle [\tilde{\gamma}], [\tilde{\beta}] \rangle [\tilde{\beta}].$$

We now choose a basis of the integral homology group $H_1(\tilde{S}) = H_1(\tilde{S}, \mathbb{Z})$ as follows. First consider the curves $\tilde{\alpha}$ and $\tilde{\beta}$ along with their images by the deck transformation group G_K ; we then further adjoin simple closed curves to this set until we reach a maximal set of pairwise disjoint curves which is invariant under the action of G_K, so that no two curves are pairwise homotopic (and none is null homotopic).

Assume that the covering $\tilde{S}$ is such that the lifts $\tilde{\alpha}$ and $\tilde{\beta}$ of both original curves α and β are nonseparating on $\tilde{S}$. The validity of this crucial assumption will be discussed below. Granted this for the moment and using the basis of $H_1(\tilde{S})$ described above, the matrices corresponding to $\tilde{T}_\alpha^{kd(\alpha)d(\beta)}$, resp. $\tilde{T}_\beta^{\ell d(\alpha)d(\beta)}$ read

$$\begin{pmatrix} 1 & B_\alpha \\ 0 & 1 \end{pmatrix} \quad \text{resp.} \quad \begin{pmatrix} 1 & B_\beta \\ 0 & 1 \end{pmatrix}$$

where B_α (resp. B_β) is a diagonal matrix having $d = |G_K|$ nonzero entries equal to $kd(\alpha)$ (resp. $\ell d(\beta)$). Note that G_K acts via permutations on the basis of $H_1(\tilde{S})$.

Assume first that $k = \ell$ but α and β are not conjugate, and consider the principal congruence quotients of the integral symplectic group $Aut(H_1(\tilde{S}), \langle \ , \ \rangle)$ of the form $Aut(H_1(\tilde{S}, \mathbb{Z}/m\mathbb{Z}), \langle \ , \ \rangle)$ with m an integer.

We will show below that there exists an m such that the image of $\tilde{T}_\alpha^{kd(\alpha)d(\beta)}$ is not conjugate to any matrix in the G_K-orbit of $\tilde{T}_\beta^{kd(\alpha)d(\beta)}$, which will imply that the images of $\tilde{\tau}_\alpha^{kd(\alpha)d(\beta)} = \left(\tilde{\tau}_\alpha^k\right)^{d(\alpha)d(\beta)}$ and $\tilde{\tau}_\beta^{kd(\alpha)d(\beta)} = \left(\tilde{\tau}_\beta^k\right)^{d(\alpha)d(\beta)}$ are not conjugate in the image of Γ in $Aut(H_1(\tilde{S}; \mathbb{Z}/m\mathbb{Z}), \langle , \rangle)/G_K$. Finally it is known that the latter is a congruence quotient of Γ, which will complete the proof of the proposition in that case. The other case, when $k \neq \ell$ but α and β are conjugate, is easy (one can assume that $\alpha = \beta$).

In order to find a cover $\tilde{S} \to S$ as described above, it is enough to find a characteristic subgroup K such that $d(\alpha)$ and $d(\beta)$ are mutually prime. Indeed, picking then $m = d(\beta)$ above, the image of $\tilde{T}_\beta^{kd(\alpha)d(\beta)}$ will be the identity modulo m along with all its G_K-conjugates, whereas the image of $\tilde{T}_\alpha^{km(\alpha)m(\beta)}$ will be a nontrivial unipotent, provided the curve $\tilde{\alpha}$ is non separating on $\tilde{S}$. Summarizing the above, in order to complete the proof of the proposition there remains to find a characteristic cover $\tilde{S} \to S$ such that the lifts of α (and of β as well, in order to preserve symmetry) are nonseparating, whereas $d(\alpha)$ and $d(\beta)$, that is the respective orders of the lifts of α and β in the group of the cover, are coprime; here recall that the lifts of a given loop, i.e. the connected components of its preimage, are conjugate in the group of the cover.

The two requirements above are essentially independent. First note that for any cover, the lifts of the nonseparating loops are nonseparating. Next it turns out to be easy to exhibit covers of S in which the lifts of *all* the separating loops, hence *all* the loops, are nonseparating (see below). Furthermore the lifts of the separating loops are simple : $d(\alpha) = 1$ for any separating $\alpha \in \mathcal{L}(S)$. (Note that here we are actually dealing with conjugacy classes of loops since the elements of $\mathcal{L}(S)$ are not attached to a base point, but that does not affect the argument.) Then any further cover has the property that all the lifts are nonseparating and there remains to manufacture such a cover with $d(\alpha)$ and $d(\beta)$ coprime. We can thus break the remaining part of the proof of the proposition into two lemmas, the first of which reads:

Lemma 3.5. *For any integer $m \geq 1$, consider the cover $S^{(m)}$ corresponding to the invariant subgroup $\pi^{(m)}$ which is the kernel of the natural surjection*

$$p_{(m)} : \pi = \pi_1(S) \to H_1(S, \mathbb{Z}/m).$$

Then the lifts of all *the loops on S to $S^{(m)}$ are non separating and simple.*

Proof. Let $\alpha \in \mathcal{L}(S)$ be a loop on S. If α is non separating there is nothing to prove. If it is, then its image $[\alpha]$ in homology is trivial, and so in particular is its reduction in $H_1(S, \mathbb{Z}/m)$. In other words $\pi^{(m)}$ contains all the separating loops. So we find that $d(\alpha) = 1$ which, referring to the above, implies that the multiplicity $(= d(\alpha))$ of any connected component of the preimage $p_{(m)}^{-1}(\alpha)$ is also equal to 1. But this says that this preimage breaks into $d_m = |H_1(S, \mathbb{Z}/m)|$ non separating curves, whose union separates $S^{(m)}$.

□

Here are some additional remarks. First the covers $S^{(m)}$ are precisely those which are used when defining the abelian levels $\mathcal{M}(S)^{(m)}$ and the principal congruence subgroups $\Gamma(S)^{(m)} \subset \Gamma(S)$ (see e.g. [5], §1). Then $\pi^{(m)} = [\pi, \pi] \cdot \pi^m$ is a cofinite invariant subgroup of π and so is $K^{(m)} = [K, K] \cdot K^m$ for any cofinite invariant subgroup $K \subset \pi$. Lemma 3.10 in [5] (whose proof is much trickier) asserts that all the covers associated to such subgroups (under some mild additional conditions) have the property that the inverse image of a loop does not contain separating loops. Indeed it states much more which we refrain from detailing here. This provides a much larger sample of covers and constitutes the basis for the essential "linearization" of the tower of congruence subgroups of $\Gamma(S)$ (see §3.3 below). Finally and in a different vein, we note the tantalizing analogy between the breaking of the preimage of separating loops in a Galois cover and the completely split primes in a Galois field extension. We now turn to the second and concluding lemma namely:

Lemma 3.6. *Given $\alpha, \beta \in \mathcal{L}(S)$ there exists a finite, unramified, Galois cover of S such that $d(\alpha)$ and $d(\beta)$ are coprime.*

Proof. Fix two coprime numbers ℓ and m. Consider the quotient group

$$\Pi = \Pi^{(\ell,m)} = \pi_1(S)/\langle \alpha^\ell = 1, \beta^m = 1\rangle$$

This is the fundamental group of a 2-complex obtained by adding 2-cells along the relations. It splits as an amalgamated product $\Pi = \Pi_1 *_{\langle\alpha\rangle} \Pi_2 *_{\langle\beta\rangle} \Pi_3$ where:

$$\Pi_1 = \pi_1(S_1)/\langle \alpha^\ell = 1\rangle, \Pi_2 = \pi_1(S_2)/\langle \alpha^\ell = \beta^m = 1\rangle, \Pi_3 = \pi_1(S_3)/\langle \beta^m = 1\rangle.$$

Notice now that the Π_j's are fundamental groups of orbifolds, namely they are Fuchsian groups of nonzero genus. In particular they are conjugacy separable, hence they admit finite quotients in which α has order ℓ and β has order m. Pick such finite quotients Q_j of Π_j, so that Π surjects onto an amalgamated product $Q = Q_1 *_{\langle\alpha\rangle} Q_2 *_{\langle\beta\rangle} Q_3$. It is well-known that a graph of groups in which the vertex groups are finite, such as Q, is virtually free. Let now $\overline{Q}$ denote a finite quotient of Q such that $ker(Q \to \overline{Q})$ is free. Then the images of α and β have respective orders ℓ and m in $\overline{Q}$, since the kernel is torsionfree.

Consider next a finite index characteristic subgroup K of π contained in $ker(\pi \to \overline{Q})$, for instance the intersections of its images by all the conjugacy automorphisms. Then G_K surjects onto $\overline{Q}$ and in particular the orders of α and β in G_K are divisors of ℓ and m, respectively. In particular, these are coprime integers.

This completes the proof of the lemma, hence also of Proposition 3.3. □

As mentioned above we refer to [4] for the proof of Proposition 3.4, which will complete the proof of Theorem 3.1. In fact the core of the proof of Proposition 3.4, to be found at the very end of the proof of Theorem 4.2 in [4] (top of p.5200) consists in a direct application of the "linearization theorem" in [5], to which we return in the next subsection. In essence it does not differ so much from the proof of Proposition 3.3 presented above, which is in line with the proof of the linearization theorem.

Thanks to the isomorphism theorem we will henceforth often refer to *the* (pro)congruence curve complex, without explicitly distinguishing between its three versions, namely $\check{C}(S)$, $\check{C}_{\mathcal{L}}(S)$ and $\check{C}_{\mathcal{G}}(S)$. As a last item in this paragraph we mention a fairly direct consequence of Proposition 3.3, namely:

Proposition 3.7. *The $\check{\Gamma}(S)$-orbits of the simplices of the procongruence complex $\check{C}(S)$ are in one-to-one correspondence with the $\Gamma(S)$-orbits of the simplices of the discrete complex $C(S)$.*

Proof. It is enough to show that if two discrete $(k-1)$-simplices $\underline{\alpha} = \{\alpha_1, \ldots, \alpha_k\}$ and $\underline{\beta} = \{\beta_1, \ldots, \beta_k\}$, as viewed in $\check{C}(S)$, sit in the same $\check{\Gamma}$-orbit, then they actually belong to the same Γ-orbit. Proposition 3.3 takes care of the case of loops ($k = 1$) and then one proceeds by induction. Assuming $\underline{\alpha}$ and $\underline{\beta}$ are in the same $\check{\Gamma}$-orbit, Proposition 3.3 says there exists $g \in \Gamma$ such that $g(\alpha_1) = \beta_1$. After twisting by g we may thus assume that $\alpha_1 = \beta_1$. Now by assumption there exists $h \in \check{\Gamma}$ such that $h(\underline{\alpha}) = \underline{\beta}$ and h belongs to $\check{\Gamma}_{\alpha_1}$, the stabilizer of the loop α_1 in $\check{\Gamma}$. By [4] (Theorem 4.5) this stabilizer is naturally isomorphic to an extension of $\check{\Gamma}(S_{\alpha_1})$ by the procyclic group $\langle\tau_{\alpha_1}\rangle$ generated by the twist along α_1. Here S_{α_1} denotes as usual the surface S slit along the loop α_1 and note that we are using the fact that we consider precisely the procongruence completion (see [3], Proposition 6.6). Multiplying out by a (profinite) power of the twist τ_{α_1}, we are led to dealing with $(k-2)$-simplices on the surface S_{α_1}, where the assertion holds true by induction, which proceeds either on the dimension of the simplices or on the modular dimension of the underlying surface S. □

3.3. **Elucidation.** Before moving forward it may be desirable, indeed necessary, to elucidate the actual content of the above isomorphim result and its significance. The point is roughly that objects which are more or less clearly equivalent (isomorphic) in the discrete case, are definitely not obviously so after completion. Sometimes the equivalence requires a difficult proof and sometimes it simply does not hold true. So let us first briefly review the various objects connected with isotopy classes of simple closed curves (a.k.a. loops) on a connected *oriented* hyperbolic surface S. We will essentially confine ourselves to the case of a single loop, higher simplices are determined by their vertices.

Let us first summarize and review four constructions, starting from an *oriented* loop $\vec{\gamma}$ on S, where we may consider that $\vec{\gamma} \in \pi = \pi_1(S)$. Since π is constructed picking out a basepoint $P \in S$ this means that we choose a loop through P in the free isotopy class of $\vec{\gamma}$. Let $\gamma \in \mathcal{L}(S) = C(S)_0$ denote $\vec{\gamma}$ after forgetting the orientation.

Working again with the fundamental group π, specifying γ is equivalent to specifying a pair $\gamma^{\pm}$ of two oriented loops with opposite orientations. Passing to conjugacy classes in order to free the construction from the choice of a basepoint, we find that $\gamma \in \mathcal{L}$ leads to an unordered pair $[\gamma^{\pm}]$ of elements of $\pi/\sim$ which is now an element $C_{\mathcal{L}}(S)_0$.

That was so to speak on the *graph* theoretic side. Now from a *group* theoretic viewpoint, γ defines the cyclic subgroup $\langle\gamma\rangle \subset \pi$ it generates inside π ($\vec{\gamma}$ and $\vec{\gamma}^{-1}$ define the same subgroup). Considering the subgroup $\langle\gamma\rangle$ up to conjugacy in π leads to the definition of γ as an element of $C_{\mathcal{G}}(S)_0$. Slightly more generally, given any integer $k > 0$, one can consider the finite index cyclic subgroup $\langle\gamma^k\rangle \subset \pi$. This will prove useful below.

Finally one can pass to the Teichmüller group $\Gamma(S)$. Then γ defines the twist τ_γ along it (using the orientation of S) and again the cyclic group $\langle\tau_\gamma\rangle \subset \Gamma(S)$ or its finite index subgroups $\langle\tau_\gamma^k\rangle \subset \Gamma(S)$ $(k > 0)$.

So far so good in the *discrete* case. Part of the fundational work then consists in exploring what happens after completion. A main point is that one can complete either working directly with π, the fundamental group of the surface S, and thus its profinite completion $\hat{\pi}$, or with $\Gamma = \Gamma(S)$, the fundamental group of the moduli space $\mathcal{M}(S)$. It is clear *a priori* that these two forms of completions can be related only if one considers completions of Γ that are no finer than the congruence completion $\check{\Gamma}$, which records the covers of $\mathcal{M}(S)$ coming from covers of S. Recall that the congruence conjecture asserts that in fact $\check{\Gamma} = \hat{\Gamma}$. So in some sense the problem, from this foundational standpoint, consists in setting up a dictionary between these two kinds of completions, and also, in a slightly different but closely related fashion, between the graph theoretic and the group theoretic information.

Concretely, what are then the main tools and results ? We will list one essential tool and two foundational results, globally referring to [4, 5]. Let us give these threee statements names as it can help further reference as well as pointing to the core of the matter. The tool leads to a kind of *linearization* of the problem, replacing homotopy with homology. The first result is precisely the *isomorphism* theorem above (Theorem 3.1) ; the second one expresses a property we will refer to as *twist separability*. Let us now go into somewhat more detail.

The idea of "linearization" is fairly old and may be ascribed to E.Looijenga. It has actually been used in the proof of Proposition 3.3 above. A general expression of this principle is embodied by Corollary 7.8 in [4]. A proper statement is cumbersome and requires introducing a lot of notation, so let us content ourselves with the main idea, namely that given S as above, a loop $\gamma \in \mathcal{L}(S)$ is entirely determined by the projective set of the *homology* classes of its preimages on the (finite unramified) covers of S. Explicitly and with $\pi = \pi_1(S)$, let $K \subset \pi$ an invariant finite index subgroup (normal would be enough but invariant is forced when working with mapping class groups), let $G_K = \pi/K$ denote the quotient group, $p_K : S_K \to S$ the ensuing Galois cover with group G_K. For $\alpha \in \mathcal{L}(T)$ a loop on a surface T, let $[\alpha] \in H_1(T, \mathbb{Z})$ denote the associated integral homology class. Then given $\gamma \in \mathcal{L}(S)$, we can consider the projective system $([p_K^{-1}(\gamma)])_K$ of homology classes on S_K, where K runs through the cofinite invariant subgroups of π (for $K = \pi$, $S_K = S$ and we omit the mention of $p_\pi = id$). Roughly speaking, the theorem asserts that γ is entirely determined by the family of "linear" data $([p_K^{-1}(\gamma)])_K$.

What are the obvious obstacles which arise when trying to identify a loop via its homology class? In fact, a loop $\alpha \in \mathcal{L}(S)$ is trivial in homology, that is $[\alpha] = 0 \in H_1(S)$, if and only if α is separating. More generally, given non intersecting loops $\alpha, \beta \in \mathcal{L}(S)$, their homology classes coincide ($[\alpha] = [\beta]$) for the appropriate orientations if and only if they form a cut pair, that is their union separates the surface (the first case can be seen as the case $\beta = \emptyset$). This is why it is important to detect a large sample of covers $p_K : S_K \to S$ such that for any loop on S, more generally any simplex $\sigma \in C(S)$, the inverse image $p_K^{-1}(\sigma)$ does not contain separating curves nor cut pairs. This is provided by the important Lemma 3.10 in [5] (see above, after the proof of Proposition 3.3).

Passing to the first main result, it was already mentioned that it is embodied by the *isomorphism theorem* (Theorem 3.1) above. Here we simply insist again that its main thrust lies in connecting, on the one hand completion via the Teichmüller group $\Gamma(S)$ i.e. the fundamental group of the moduli space $\mathcal{M}(S)$, which is used when defining $\check{C}(S)$, on the other hand completion via the much simpler and more tractable fundamental group $\pi = \pi_1(S)$ of the surface S itself, which is used when defining both $\check{C}_{\mathcal{L}}(S)$ and $\check{C}_{\mathcal{G}}(S)$.

The second main result traces a fundamental link between (pro)curves and (pro)twists, that is between the graph theoretic and the group theoretic facets of the theory. This is Theorem 5.1 in [4], which can be stated more easily. Starting in the discrete setting we have (after orienting the surface S) a natural injective map $d : \mathcal{L}(S) \hookrightarrow \Gamma(S)$ which to a loop $\gamma \in \mathcal{L}(S)$ assigns the corresponding twist τ_γ. Given $k \in \mathbb{Z} \setminus \{0\}$ it can be generalized to $d_k : \gamma \mapsto \tau_\gamma^k$ ($d = d_1$), still an injective map between the same source and target.

As usual, upon completion the plot thickens and things become more interesting. From the injective map d and the natural embedding $\Gamma \hookrightarrow \check{\Gamma}$ we get a (still injective) map which we denote by the same name for simplicity $d : \mathcal{L} \hookrightarrow \check{\Gamma}$. By the universality of the progruence completion, this leads to a map

$$\hat{d} : \check{\mathcal{L}}(S) \to \check{\Gamma}(S)$$

which now may or may not be injective (this is precisely the moot point here) with, as above,

$$\check{\mathcal{L}}(S) = \check{C}(S)_0 = \varprojlim_{\lambda \in \Lambda} \mathcal{L}(S)/\Gamma^{\lambda}.$$

Finally the isomorphism theorem ensures that $\check{\mathcal{L}}(S) = \hat{\mathcal{L}}(S) = \check{C}_{\mathcal{L}}(S)_0$, namely the set of (pro)curves on S. This can be generalized in the obvious way to $\hat{d}_k : \check{\mathcal{L}}(S) \to \check{\Gamma}(S)$ for any $k \in \mathbb{Z} \setminus \{0\}$ and indeed jazzed up to $k \in \hat{\mathbb{Z}} \setminus \{0\}$, using the density of $\mathbb{Z}$ in $\hat{\mathbb{Z}}$. Note from a topological viewpoint that in the complete case we are always considering *continuous* maps between *compact* spaces.

We may now state the second fundamental result about twists separability we have been alluding to:

Theorem 3.8 ([4], Thm. 5.1)**.** *For any $k \in \hat{\mathbb{Z}} \setminus \{0\}$ the map*

$$\hat{d}_k : \check{\mathcal{L}}(S) \to \check{\Gamma}(S)$$

is injective.

In words : a (pro)curve can be detected via any (profinite) power of the associated twist. Indeed more is true, as can be gathered from the – difficult – proof of the above result (see [4], Remark 5.14). Let $\mathcal{D}^k \subset \Gamma(S)$ denote the set of k-th powers of twists ($k \in \mathbb{Z} \setminus \{0\}$) and let $\check{\mathcal{D}}^k \subset \check{\Gamma}(S)$ denote its closure in $\check{\Gamma}(S)$. Extend the definition of $\check{\mathcal{D}}^k$ to $k \in \hat{\mathbb{Z}} \setminus \{0\}$. Then one can show that the intersection $\check{\mathcal{D}}^k \cap \Gamma(S)$ is exactly $\mathcal{D}^k$ if $k \in \mathbb{Z}$ and is empty if not. This leads to the following striking corollary of the above theorem, or rather of its proof, which will be substantially strengthened below (see in particular Proposition 4.3):

Corollary 3.9. *Let $\alpha, \beta \in \hat{\mathcal{L}}(S)$ be two (pro)curves, $k, \ell \in \hat{\mathbb{Z}} \setminus \{0\}$ two nonzero (pro)integers, then the equality $\tau_\alpha^k = \tau_\beta^\ell$ holds if and only if $\alpha = \beta$ and $k = \ell$.*

We will next proceed to record some important consequences of these foundational results, before moving to the study of the *automorphisms* of our various simplicial complexes. Note again that these complexes are entirely determined by their 1-skeleta. So in some sense we are primarily interested in profinite *graphs*.

Remark 3.1. *The above is in some ways reminiscent of anabelian geometry and the exploration of the structure of the Galois groups of fields, especially fields of functions (including in positive characteristic). On this topic we refer for instance to* [36] *and in particular some basic phenomena summarized there in Proposition 1.5, as well as to the work of F.Pop (starting with* [32]*; see also §8.2 below).*

4. From Graphs to Groups and back : Centralizers and Normalizers of twists

Fixing as usual a connected oriented hyperbolic surface S, there is a natural action of $\check{\Gamma}(S)$ on the three isomorphic versions of the curve complex, namely $\check{C}(S)$, $\check{C}_{\mathcal{L}}(S)$ and $\check{C}_{\mathcal{G}}(S)$. This group also acts on powers of twists by conjugation. By Theorem 3.8, given a loop $\alpha \in \mathcal{L}(S)$, the centralizer $Z(\tau_\alpha)$ of τ_α in $\check{\Gamma}$ coincides with the stabilizer $\check{\Gamma}_\alpha$ of α for the action of $\check{\Gamma}$ on $\check{\mathcal{L}} = \hat{\mathcal{L}}$. This still holds true for τ_α^k with $\alpha \in \hat{\mathcal{L}}$ and $k \in \hat{\mathbb{Z}} \setminus \{0\}$. Now the stabilizer $\check{\Gamma}_\alpha$ admits a rather explicit description, and more generally so does $\check{\Gamma}_\sigma$, the stabilizer of a simplex $\sigma \in \check{C}(S)$ (equivalently $\check{C}_{\mathcal{L}}(S)$, $\check{C}_{\mathcal{G}}(S)$). The structure is identical to the one occurring in the discrete case and this can be vindicated relatively easily; see [4], Theorem 4.5 and references there. It is important to insist at this point that we are definitely using the pro*congruence* completion. The analogous description for the full profinite completion (as stated in [3]) remains unproved to-date.

Here we give a short and partial account of the descriptions of the centralizers and normalizers of twists as well as of the commutative subgroups of $\check{\Gamma}(S)$ generated by finite sets of commuting twists. We refer globally to [4, 7] for detailed results and proofs. The second reference improves on the first, addressing in particular the case of multitwists, that is products of powers of commuting twists. As is often the case the results are easily predictable from the discrete case, where direct geometric proofs are elementary. The proofs however are a different and much more involved matter. These results – and more – in the procongruence case essentially follow from Theorems 3.1 and 3.8, as well as the improvement of the latter in [7] (§§5,6). They are again *not* available to-date in the full profinite setting.

First an important connection between commuting twists and "nonintersecting procurves" is given by:

Theorem 4.1. *(cf.* [4]*, Corollary 6.4). Let $\alpha_1, \alpha_2, \ldots, \alpha_k \in \check{\mathcal{L}}(S)$ be proloops and $\tau_{\alpha_1}^{h_1}, \tau_{\alpha_2}^{h_2}, \ldots, \tau_{\alpha_k}^{h_k} \in \check{\Gamma}(S)$ denote nontrivial powers of the associated twists ($h_1, h_2, \ldots, h_k \in \hat{\mathbb{Z}} \setminus \{0\}$). Then the $\tau_{\alpha_i}^{h_i}$'s pairwise commute if and only if the α_i's span a simplex $\underline{\alpha} \in \check{C}_{\mathcal{L}}(S)$.*

Moreover the centralizer $Z_{\check{\Gamma}}(\tau_{\alpha_1}^{h_1}, \tau_{\alpha_2}^{h_2}, \ldots, \tau_{\alpha_k}^{h_k}) \subset \check{\Gamma}(S)$ of this family of powers of twists coincides, up to possible permutations of the curves, with the stabilizer $\check{\Gamma}_\alpha$ of the simplex $\underline{\alpha}$ for the action of $\check{\Gamma}(S)$ on $\check{C}_{\mathcal{L}}(S)$. □

Before the statement we used the phrase "nonintersecting procurves" with inverted commas. It shoud be understood that the fact that the curves $\alpha_1, \alpha_2, \ldots, \alpha_k$ span a simplex of $\check{C}_{\mathcal{L}}(S)$ (equivalently of $\check{C}(S)$) *defines* them as "nonintersecting". There is no direct definition available in a profinite context. The theorem above says that nonintersection may equivalently be characterized by the commutation of any set of nontrivial powers of the associated twists.

Normalizers of finite families of commuting twists are quite close to their centralizers. Just as in the discrete case, they only differ by a possible finite group of permutations. More precisely let again $\alpha_1, \alpha_2, \ldots, \alpha_k \in \check{\mathcal{L}}(S)$ span a $(k-1)$-simplex $\underline{\alpha} \in \check{C}_{\mathcal{L}}(S)$, let $\underline{h} = \{h_1, h_2, \ldots, h_k\}$ denote a k-tuple of nonzero profinite integers, and let $\tau_{\underline{\alpha}}^{\underline{h}} = \{\tau_{\alpha_1}^{h_1}, \tau_{\alpha_2}^{h_2}, \ldots, \tau_{\alpha_k}^{h_k}\}$ be the corresponding family of powers of twists. Finally, let $G_{\underline{\alpha},\underline{h}} \subset \check{\Gamma}(S)$ denote the closed free abelian group spanned by the components of $\tau_{\underline{\alpha}}^{\underline{h}}$. We will abbreviate this to $G_{\underline{\alpha}}$ if $h_i = 1$ for all $i = 1, \ldots, k$. With these pieces of notation we have:

Theorem 4.2. *(cf.* [4]*, Theorem 6.6). The normalizer $N_{\check{\Gamma}}(G_{\underline{\alpha},\underline{h}}) \subset \check{\Gamma}(S)$ coincides with the stabilizer $\check{\Gamma}_\alpha$ of the simplex $\underline{\alpha}$ for the action of $\check{\Gamma}(S)$ on $\check{C}_{\mathcal{L}}(S)$.* □

We refer the reader to [7], Corollary 6.2 for a strengthening of Theorems 4.1 and 4.2 to the analogous description of the centralizer and normalizer of a single multitwist, as a corollary of the following result. Let $\underline{\alpha} = \{\alpha_1, \alpha_2, \ldots, \alpha_k\} \in \check{C}(S)_{k-1}$ and $\underline{\beta} = \{\beta_1, \beta_2, \ldots, \beta_\ell\} \in \check{C}(S)_{\ell-1}$ be two simplices, $\underline{h} = \{h_1, h_2, \ldots, h_k\}$ and $\underline{i} = \{i_1, i_2, \ldots, i_\ell\}$ two sets of nonzero profinite integers. Then we have:

Theorem 4.3. *(cf.* [7]*, Theorem 6.1). The equality of the products*

$$\tau_{\alpha_1}^{h_1} \tau_{\alpha_2}^{h_2} \cdots \tau_{\alpha_k}^{h_k} = \tau_{\beta_1}^{i_1} \tau_{\beta_2}^{i_2} \cdots \tau_{\beta_\ell}^{i_\ell}$$

holds in $\check{\Gamma}(S)$ if and only if $k = l$, $\underline{\alpha} = \underline{\beta}$ and $\underline{h} = \underline{i}$. □

It is both telling and useful to rephrase the statements above in a more topological and intrinsic fashion. Recall that for a simplex $\underline{\alpha} \in \check{C}(S)(\simeq \check{C}_{\mathcal{L}}(S))$, we denote by $G_{\underline{\alpha}} \subset \check{\Gamma}(S)$ the closed abelian subgroup generated by the twists along the multicurves defining the vertices of $\underline{\alpha}$. We actually already proved the following

Proposition 4.4. *Let $\underline{\alpha}, \underline{\beta} \in \check{C}(S)$ be two simplices. Then:*
i) If $U \subset G_{\underline{\alpha}}$ is an open *subgroup of $G_{\underline{\alpha}}$, the normalizer $N_{\check{\Gamma}}(U) \subset \check{\Gamma}(S)$ coincides with the stabilizer $\check{\Gamma}_\alpha$ of $\underline{\alpha}$ for the action of $\check{\Gamma}(S)$ on $\check{C}(S)$. In particular $N_{\check{\Gamma}}(U) = N_{\check{\Gamma}}(G_{\underline{\alpha}})$. Moreover the centralizer $Z_{\check{\Gamma}}(U)$ has finite index in $N_{\check{\Gamma}}(G_{\underline{\alpha}})$, the latter being an extension of a finite permutation group by the former.*
ii) The intersection of the groups $G_{\underline{\alpha}}$ and $G_{\underline{\beta}}$ is given by

$$G_{\underline{\alpha}} \cap G_{\underline{\beta}} = G_{\underline{\alpha} \cap \underline{\beta}}.$$

In particular, $G_{\underline{\alpha}} \cap G_{\underline{\beta}}$ is open in $\underline{\alpha}$ (resp. $\underline{\beta}$) if and only if $\underline{\alpha} \subset \underline{\beta}$ (resp. $\underline{\alpha} \subset \underline{\beta}$). □

Given the above it is now easy and useful to manufacture yet another representation of the congruence curves complex, which we call $\check{C}_{\mathcal{T}}(S)$ ($\mathcal{T}$ for twist). Let us start from the discrete situation. Then we have the curve complex $C(S)$ and two incarnations or representations of it, $C_{\mathcal{L}}(S)$ and $C_{\mathcal{G}}(S)$, respectively by means of curves and conjugacy classes of cyclic subgroups of $\pi = \pi_1(S)$. All three are isomorphic and it is also fairly easy to prove the discrete version of Proposition 4.4 above. In particular, for a simplex $\underline{\alpha} \in C(S)$, $G_{\underline{\alpha}} \subset \Gamma(S)$ denotes the free abelian group generated by the commuting twists along the curves defining the vertices of α and in the statement of the discrete analog of Proposition 4.4 one should read "open" as "finite index". Let now $\mathcal{G}(\Gamma)$ denote the (discrete) poset of all the subgroups of $\Gamma = \Gamma(S)$. There is a natural map

$$C(S) \to \mathcal{G}(\Gamma)$$

which to a simplex $\underline{\alpha}$ associates the group $G_{\underline{\alpha}}$. It is injective by the discrete analog of ii) in Proposition 4.4 and the image has a natural structure of simplicial complex induced by that of $C(S)$. We call this image $C_{\mathcal{T}}(S)$; it is (tautologically) isomorphic to $C(S)$ and realizes this complex inside $\mathcal{G}(\Gamma)$, which is equipped with a natural action of Γ by conjugation.

We now return to the procongruence setting, adding in a useful refinement; namely we would like to work "virtually" in the sense of group theory, that is up to considering open subgroups. (This is of course doable, *mutatis mutandis*, in the discrete case as well.) The first observation is that the *closed* subgroups of a profinite group have a natural structure of profinite (po)set. Sticking to our specific case, with $\check{\Gamma} = \check{\Gamma}(S)$, we define $\mathcal{G}(\check{\Gamma})$, as the set of *closed* subgroups of $\check{\Gamma}$, which can be written as

$$\mathcal{G}(\check{\Gamma}) = \varprojlim_{\lambda\in\Lambda} \mathcal{G}(\Gamma/\Gamma^{\lambda}),$$

exhibiting it as a profinite set. Here Γ^{λ} runs through the normal congruence subgroups of Γ and $\mathcal{G}(\Gamma/\Gamma^{\lambda})$ denotes the *finite* set of the subgroups of the finite group Γ/Γ^{λ}. We also have an action of $\check{\Gamma}$ on $\mathcal{G}(\check{\Gamma})$ by conjugation, as well as a $(\Gamma - \check{\Gamma})$-equivariant map $\mathcal{G}(\Gamma) \to \mathcal{G}(\check{\Gamma})$ sending a subgroup of Γ to its closure in $\check{\Gamma}$.

On the other hand we define a weight function on procurves $w : \check{\mathcal{L}}(S) \to \mathbb{Z}_+^*$, with values in the strictly positive integers, requiring that it be $\check{\Gamma}$-invariant. Since the $\check{\Gamma}$-orbits of $\check{\mathcal{L}}(S)$ are in one-to-one correspondence with the types of (ordinary) loops on S, there only remains to assign an arbitrary (strictly positive) integer to each of the finitely many types. In a more geometric or modular way, this is tantamount to assigning such an integer to every irreducible component of the divisor at infinity of the stable compactification of $\mathcal{M}(S)$.

Given a weight function w we now consider the map $C(S) \to \mathcal{G}(\check{\Gamma})$ which sends a discrete simplex $\underline{\alpha} = \{\alpha_1, \ldots, \alpha_k\}$ to the closed abelian subgroup generated (as a $\hat{\mathbb{Z}}$-module) by the $\tau_{\alpha_i}^{w(\alpha_i)}$ $(i = 1, \ldots, k)$. We then take the closure of the image of $C(S)$ and call it $\check{C}_{\mathcal{T},w}(S) \subset \mathcal{G}(\check{\Gamma})$; it is equipped again with a structure of profinite simplicial complex and an action of $\check{\Gamma}(S)$. In a slightly more intrinsic fashion this amounts to considering the discrete weighted complex $C_{\mathcal{T},w}(S) \subset \mathcal{G}(\Gamma)$ as mentioned above and map it to $\mathcal{G}(\check{\Gamma})$ via the natural map $\mathcal{G}(\Gamma) \to \mathcal{G}(\check{\Gamma})$. The closure of the image is by definition $\check{C}_{\mathcal{T},w}(S)$. This being said, the following result and its easy proof should not come as a surprise (compare [4], Proposition 6.8):

Theorem 4.5. *Let w be a weight function as above. There is a natural $\check{\Gamma}$-equivariant isomorphism*

$$\check{C}(S) \simeq \check{C}_{\mathcal{L}}(S) \to \check{C}_{\mathcal{T},w}(S).$$

Moereover the images $\check{C}_{\mathcal{T},w}(S) \subset \mathcal{G}(\check{\Gamma})$ for varying w are naturally isomorphic.

Proof. The map in the statement exists by universality and is onto by the usual argument: $C(S)$ is dense in $\check{C}(S)$ which is compact so that its image is closed. It is injective by Proposition 4.4. Note that the weight function w admits a unique extension from $\mathcal{L}(S)$ to $\check{\mathcal{L}}(S)$ since any proloop belongs to the $\check{\Gamma}$-orbit of a discrete loop and the $\check{\Gamma}$-action is type preserving. Finally i) in Proposition 4.4 shows that we can work virtually. Namely consider the trivial weight function w_0, assigning weight 1 to every element of $\check{\mathcal{L}}(S)$ and write $\check{C}_{\mathcal{T}}(S) = \check{C}_{\mathcal{T},w_0}(S)$. Then there is a canonical isomorphism $\check{C}_{\mathcal{T},w}(S) \simeq \check{C}_{\mathcal{T}}$; it is defined simply by mapping every power $\tau_{\alpha_i}^{w(a_i)}$ $(i = 1 \ldots, k)$ to the twist τ_{α_i} itself. □

The introduction of the weight function simply provides an explicit basis of open subgroups of the groups $G_{\underline{\alpha}}$, just as with the groups $G_{\underline{\alpha},\underline{h}}$. One can again rephrase the above in a more intrinsic fashion as follows (compare [4], Theorem 6.9). Let $\check{\Gamma}^{\lambda} \subset \check{\Gamma}$ be a normal open subgroup of $\check{\Gamma}$, equivalently the closure in $\check{\Gamma}$ of a normal conguence subgroup $\Gamma^{\lambda} \subset \Gamma$. We can form the profinite set $\mathcal{G}(\check{\Gamma}^{\lambda})$ of the closed subgroups of $\check{\Gamma}^{\lambda}$ and there is a natural map $\mathcal{G}(\check{\Gamma}) \to \mathcal{G}(\check{\Gamma}^{\lambda})$ defined by mapping each subgroup $G \subset \check{\Gamma}$ to the intersection $G \cap \check{\Gamma}^{\lambda}$. Now consider the prosimplicial complex $\check{C}_{\mathcal{T}}(S) = \check{C}_{\mathcal{T},w_0}(S)$ as above, which on the vertices is defined simply by mapping any (pro)loop $\gamma \in \check{\mathcal{L}}(S)$ to the associated (pro)twist $\tau_{\gamma} \in \check{\Gamma}(S)$. Then the images of $\check{C}_{\mathcal{T}}(S)$ in $\mathcal{G}(\check{\Gamma}^{\lambda})$ for varying $\lambda \in \Lambda$ are naturally isomorphic; in other words, the map $\check{C}_{\mathcal{T}}(S) \to \mathcal{G}(\check{\Gamma}^{\lambda})$ is injective for every $\lambda \in \Lambda$. The proof amounts to a translation of the above.

Let us briefly summarize where we stand. Given a hyperbolic surface S, we first defined the congruence curve complex $\check{C}(S)$ by completing the usual discrete version $C(S)$, using the action of the Teichmüller group. We now have at our disposal three other realizations of $\check{C}(S)$, namely $\check{C}_{\mathcal{L}}(S)$ and $\check{C}_{\mathcal{G}}(S)$ which are both constructed by using the fundamental group $\pi = \pi_1(S)$ of the surface and finally $\check{C}_{\mathcal{T}}(S)$ which uses $\Gamma(S)$, the fundamental group of the moduli stack $\mathcal{M}(S)$. All four appear as the completion or closure of natural discrete versions; referring to Grothendieck's manuscript *Longue marche à travers la théorie de Galois*, they are equipped with natural "discretifications". They are also provided with a natural action of $\check{\Gamma}(S)$ extending the actions of $\Gamma(S)$ on the respective isomorphic discrete versions. All four are isomorphic; moreover the isomorphisms are $\check{\Gamma}(S)$-equivariant and "natural" in the sense that once more they extend the obvious or say, geometric isomorphisms between the discrete versions (they are also natural with respect to varying S). Finally there is a dictionary between the graph or complex theoretic side and the group theoretic side, again extending the elementary discrete, geometric dictionary.

5. Procongruence complexes : structure and reconstruction

This section revolves around three results. First we show that the complexes $\check{C}(S_{g,n})$ are *not* isomorphic for different values of the type (g,n) (the latter being well-defined thanks to the results of §3 above ; see especially Proposition 3.7) except for a few low dimensional exceptions which already occur in the discrete setting. Indeed our result parallels the analogous one in the discrete case although the proof significantly departs from the discrete one. We refer especially to [21] for the proof in the discrete setting, including some more references and background. Note that this already parallels a classical nonisomorphism result for Teichmüller spaces, due to D.B.Patterson (see also [9]). Next we elucidate the structure of the procongruence complex of curves $\check{C}(S)$, or rather its close cousins $\check{C}_*(S)$ and $\check{C}_P(S)$, in a fashion which again parallels the discrete setting as discussed in §2 above. Finally we prove a reconstruction result in the procongruence case, on the model of Theorem 2.10 above for the discrete complexes.

5.1. **Isomorphisms and non isomorphisms among the congruence curve complexes.** Our first result reads:

Theorem 5.1. *Let $S = S_{g,n}$ and $S' = S_{g',n'}$ be two connected hyperbolic surfaces of different types (g,n) and (g',n'). Then the procongruence complexes $\check{C}(S)$ and $\check{C}(S')$ are* not *isomorphic, except for the following exceptional cases: $\check{C}(S_{1,1}) \simeq \check{C}(S_{0,4})$, $\check{C}(S_{1,2}) \simeq \check{C}(S_{0,5})$ and $\check{C}(S_{2,0}) \simeq \check{C}(S_{0,6})$.*

Before going into the proof proper, let us dispose of the low dimensional exceptions. We mentioned the one dimensional cases for the sake of completeness only; the isomorphism is then tautological, provided that $C(S_{1,1})$ and $C(S_{0,4})$ are redefined properly, as explained in §2 (see also §§A.7, 8). The two and three dimensional cases stem directly from the exceptional discrete cases (see e.g. [21]). One has $C(S_{1,2}) \simeq C(S_{0,5})$. Then $\check{C}(S_{1,2})$ (resp. $\check{C}(S_{0,5})$) is the completion of that complex with respect to the action of $\Gamma_{1,[2]}$ (resp. $\Gamma_{0,[5]}$). However, $\Gamma_{1,[2]}$ acts via the quotient by its center $\Gamma_{1,[2]}/Z$ ($Z = Z(\Gamma_{1,[2]}) \simeq \mathbb{Z}/2$) and we have an inclusion $\Gamma_{1,[2]}/Z \subset \Gamma_{0,[5]}$ where $\Gamma_{1,[2]}/Z$ can be identified with the stabilizer of one of the 5 marked points, so has finite index $(= 5)$ in $\Gamma_{0,[5]}$. This implies that $\check{C}(S_{1,2}) \simeq \check{C}(S_{0,5})$. The last case is analogous.

In order to prove Theorem 5.1 we first of all have to drastically reduce the number of possible isomorphisms between two complexes $\check{C}(S_{g,n})$ and $\check{C}(S_{g',n'})$ for different types (g,n) and (g',n'). This we do by introducing two invariants. The first one is the dimension of the complex, $d_{g,n} = dim(C(S_{g,n})) = 3g-3+n$. It is indeed invariant under completion, since $C(S_{g,n})$ injects densely into its completion $\check{C}(S_{g,n})$. We will then introduce another invariant, or rather two closely connected ones, which will require some preliminary lemmas. This departs from the discrete setting, where the cohomological dimension of the complex $C(S_{g,n})$ provides a second invariant (see [21] and item ii) in Remark 5.1 below).

The first lemma-definition introduces a useful invariant in the discrete case, which will subsequently be shown to survive completion. Denote by $L_C(\sigma)$ the link of the simplex σ in a simplicial complex C. We define a *graph* $L^-_C(\sigma)$, the *dual link* of σ, as follows: the set of vertices is the same as that of $L_C(\sigma)$, and we add an edge joining two vertices in $L^-_C(\sigma)$ if and only if these are *not* joined by an edge in $L_C(\sigma)$. Also and following [21], we say that a simple loop on S (or rather an element of $\mathcal{L}(S)$) is of *boundary type* if it bounds a subsurface of type $(0,3)$. The following lemma is immediate, after recalling that $C(S_{0,3})$ is empty:

Lemma 5.2. *Let $\alpha \in \mathcal{L}(S)$ be a simple closed curve on S connected hyperbolic. The dual link $L^-_{C(S)}(\alpha)$ is nonempty if S is different from $S_{1,1}$ (α nonseparating) and $S_{0,4}$ (α separating and of boundary type). If nonempty, the dual link $L^-_{C(S)}(\alpha)$ is connected if and only if α is either nonseparating or of boundary type.*

□

The key property to be used in the sequel is a certain type of persistence upon completion. We start with a definition. For *any* simplicial complex C, we say that it is (finitely) *chain connected* if every pair of vertices can be joined by a finite chain of edges in C. Note that this only depends on the 1-skeleton of C, and indeed below we work only with *graphs*. Note also that this is a *combinatorial* rather than topological property and indeed part of the argument below is combinatorial, independent of the underlying profinite topology. Here is the main invariance lemma:

Lemma 5.3. *Let $\alpha \in \mathcal{L}(S)$ be a simple closed curve on S connected hyperbolic. Then the dual link $L^-_{\check{C}(S)}(\alpha)$ is chain connected if and only if the* discrete *dual link $L^-_{C(S)}(\alpha)$ is (chain) connected.*

Proof. Here we put the word "chain" between brackets in the discrete case because it is clear that $L^-_{C(S)}(\alpha)$ is chain connected if and only if it is connected for the usual topology.

Let us first express the fact that two procurves $\alpha, \beta \in \check{\mathcal{L}}(S)$ are disjoint, in a concrete, combinatorial way. They are represented by coherent systems $(\alpha_\lambda)_{\lambda\in\Lambda}$ and $(\beta_\lambda)_{\lambda\in\Lambda}$ where Λ denotes as usual the inverse system of the congruence levels and $\alpha_\lambda, \beta_\lambda \in \mathcal{L}(S)$. One can project α_λ (resp. β_λ) to $\alpha^\lambda \in \mathcal{L}^\lambda(S) = \mathcal{L}(S)/\Gamma^\lambda = (C(S)/\Gamma^\lambda)^{(0)} = C^\lambda(S)^{(0)}$ (idem β_λ). For λ large enough (Γ^λ small enough), the group $\Gamma^\lambda \subset \Gamma(S)$ acts simplicially on $C(S)$. So we can choose α_λ and β_λ to be adjacent in $C(S)$ (that is, connected by an edge), and the projections α^λ and β^λ will then be adjacent in the quotient complex $C^\lambda(S)$. We thus conclude that $\alpha, \beta \in \check{\mathcal{L}}(S)$ are adjacent, that is are joined by an edge in $\check{C}(S)^{(1)}$, if and only if they can be represented by coherent systems $\alpha_\lambda, \beta_\lambda \in \mathcal{L}(S)$ such that α_λ and β_λ are adjacent (that is, are disjoint) in $C(S)$ for λ large enough (they may coincide for a finite number of λ's). If one of the two curves is discrete, say $\beta \in \mathcal{L}(S)$, the same holds with $\beta_\lambda = \beta$ for every $\lambda \in \Lambda$. Conversely two curves $\alpha, \beta \in \check{\mathcal{L}}(S)$ are *not* adjacent in $\check{C}(S)$, or say have nontrivial intersection, if and only if *for every* representatives $(\alpha_\lambda)_{\lambda\in\Lambda}$ and $(\beta_\lambda)_{\lambda\in\Lambda}$, $\alpha_\lambda \in \mathcal{L}(S)$ and $\beta_\lambda \in \mathcal{L}(S)$ intersect non trivially for λ large enough.

Going back to the proof of the lemma, let us first assume that the dual link $L^-_{\check{C}(S)}(\alpha)$ is chain connected in the congruence completion. By the previous lemma we have to show that α is either nonseparating or of boundary type. Assume the contrary, that is α separating not of boundarytype; we will show that it leads to a contradiction. By our assumption there exist then two loops $\beta, \gamma \in \mathcal{L}(S)$ lying in different connected components of the slit surface S_α. They determine two vertices of $L^-_{\check{C}(S)}(\alpha)$ and since this complex is chain connected there exists a finite chain of (pro)loops $\zeta_i \in \check{\mathcal{L}}(S)$ $(j = 0, \ldots, k)$ connecting β and γ in $L^-_{\check{C}(S)}(\alpha)$: $\zeta_0 = \beta$, $\zeta_k = \gamma$. By definition this means that ζ_j is *not* adjacent to ζ_{j+1} for $j = 0, 1, \ldots, k-1$. On the other hand α is adjacent to ζ_j for all $j = 0, 1, \ldots, k$, so there are defining families $(\zeta_{j,\lambda})_{\lambda\in\Lambda}$ with $\zeta_{j,\lambda} \in \mathcal{L}(S)$ disjoint from $\alpha \in \mathcal{L}(S)$ for all j and for λ large enough. But now since ζ_j and ζ_{j+1} are *not* adjacent, we find that $\zeta_{j,\lambda}$ and $\zeta_{j+1,\lambda}$ intersect nontrivially for λ large enough. In particular for any such λ the chain $(\zeta_{i,\lambda})_{j\in(0,k)}$ is connected and joins the two connected components of S_α, a contradiction.

Conversely, assume that $L^-_{C(S)}(\alpha)$ is chain connected and let $\check{\beta}, \check{\gamma}$ be two vertices of $L^-_{\check{C}(S)}(\alpha)$. We want to prove that there is a finite path in $L^-_{\check{C}(S)}(\alpha)$ connecting them. If $\{\check{\beta}, \check{\gamma}\}$ is an edge of $L^-_{\check{C}(S)}(\alpha)$ there is nothing to prove. If not the three pairs of vertices of the triplets $\{\alpha, \check{\beta}, \check{\gamma}\}$ are edges of $\check{C}(S)$ and the latter being a flag complex, $\{\alpha, \check{\beta}, \check{\gamma}\}$ forms a triangle (a 2-simplex), that is it belongs to $\check{C}(S)^{(2)}$. Now there exists $g \in \check{\Gamma} = \check{\Gamma}(S)$ such that $g \cdot \{\alpha, \check{\beta}, \check{\gamma}\} = \{\alpha, \beta, \gamma\} \in C(S)^{(2)}$ for some loops $\beta, \gamma \in \mathcal{L}(S)$; indeed we may – and did – choose $g \in \check{\Gamma}_\alpha \subset \check{\Gamma}$, the stabilizer of α. By assumption we can now find a path between β and γ in $L^-_{C(S)}(\alpha)$ and pull it back via g^{-1} to a path between $\check{\beta}$ and $\check{\gamma}$ in $L^-_{\check{C}(S)}(\alpha)$, which completes the proof. Note that we have actually been using a basic property of the procongruence topology, namely that it is inherited by a subsurface obtained by cutting a given surface S along a multi curve. Here the closure $\bar{\Gamma}_\alpha$ in $\check{\Gamma}$ of the discrete stabilizer along the curve α is isomorphic to its congruence completion $\check{\Gamma}_\alpha$, that is the completion of the modular group $\Gamma(S_\alpha)$ of the surface slit along α. This important property will be further explicited and proved in general in Proposition 5.8 below.

□

Next we define two closely connected invariants and explicitly compute them in the discrete case before showing that they survive after completion. Given a connected hyperbolic surface S, we let $Sep(S)$ denote the *maximal number of pairwise disjoint separating curves not of boundary type* on the surface S. Here and below "disjoint curves" means as usual "disjoint elements of $\mathcal{L}(S)$", that is "free isotopy classes of simple closed curves with disjoint representatives". We also denote by $NSep(S)$ the *maximal number of disjoint curves which are either nonseparating or of boundary type* on S. It is fairly easy to compute these numbers explicitly. This is taken care of by the following counting lemma:

Lemma 5.4. *We have*

$$Sep(S_{g,n}) = \begin{cases} \max(n-5, 0), & \text{if } g = 0; \\ \max(n-2, 0), & \text{if } g = 1; \\ 2g + n - 3, & \text{if } g \geq 2. \end{cases}$$

$$NSep(S_{g,n}) = \begin{cases} \left[\frac{n}{2}\right], & \text{if } g = 0; \\ 3g + n - 3, & \text{if } g \geq 1. \end{cases}$$

Proof. Any maximal set of disjoint curves on $S_{0,n}$ has $n-3$ elements, among which at least two are of boundary type. Hence $Sep(S_{0,n}) \leq n - 5$. In order to check equality, define inductively a system of curves by starting with a boundary type curve α_1, adjoining iteratively a new curve α_i surrounding α_{i-1} and a new boundary component (or marked point).

A separating curves on $S_{1,n}$ bounds a copy of $S_{1,k}$, $k \leq n$. Thus there are at most $n-1$ disjoint such curves, among which at least one is of boundary type. This yields $Sep(S_{1,n}) \leq n-2$. Again equality is attained by an obvious variant of the system of curves constructed above in genus 0.

Note now that the maximal number of pairwise disjoint separating curves on $S_{g,n}$ is $2g+n-3$ for any (g,n), so that $Sep(S_{g,n}) \leq 2g+n-3$. For $g \geq 2$ it is immediate to construct a system with exactly this number of disjoint separating curves, none of which is of boundary type ; the equality follows.

Moving to the computation of $NSep(S_{g,n})$, if $g=0$, all curves are separating and one can arrange as many curves of boundary type on $S_{0,n}$ as there are pairs of boundary components (or marked points), namely at most $\left[\frac{n}{2}\right]$. For $g \geq 1$, a set of pairwise disjoint curves on $S_{g,n}$ has at most $3g-3+n$ elements ($= d_{g,n}+1$) so that $NSep(S_{g,n}) \leq 3g+n-3$. Since one can construct a system with this many *nonseparating* curves on $S_{g,n}$, equality holds true.

□

We now show the invariance of these numbers under isomorphisms of complexes. Here we need only consider (combinatorial) simplicial isomorphisms, without any topological requirement, that is continuity with respect to the natural profinite topology is not required in the procongruence (or possibly a more general profinite) setting, as will be the case in the next paragraph.

Lemma 5.5. *A simplicial automorphism $\phi : \check{C}(S) \to \check{C}(S')$ preserves both Sep and NSep, that is we have $Sep(S) = Sep(S')$ and $NSep(S) = NSep(S')$.*

Proof. Let $\underline{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_s) \in C(S)^{(s-1)}$ be a simplex with every loop $\alpha_i \in \mathcal{L}(S)$ separating and not of boundary type. We assume $\underline{\alpha}$ has maximal dimension, that is $s = Sep(S)$. Since ϕ is simplicial, $\phi(\underline{\alpha})$ is a simplex of $\check{C}(S')$. As usual there exists a discrete simplex in the orbit of $\phi(\underline{\alpha})$, that is a $g \in \check{\Gamma}(S')$ such that $g \cdot \phi(\underline{\alpha}) \in C(S')$; call this discrete simplex $\underline{\beta} = (\beta_1, \beta_2, \dots, \beta_s) \in C(S')^{(s-1)}$.

Since both ϕ and the $\check{\Gamma}(S')$-action are simplicial, for every $i = 1, 2, \dots, s$, the link $L_{\check{C}(S)}(\alpha_i)$ and dual link $L^-_{\check{C}(S)}(\alpha_i)$ are combinatorially isomorphic to $L_{\check{C}(S)}(\beta_i)$ and $L^-_{\check{C}(S)}(\beta_i)$ respectively.

Now since $L^-_{\check{C}(S)}(\alpha_i)$ is not chain connected we derive that $L^-_{\check{C}(S)}(\beta_i)$ is not chain connected either and hence by Lemma 5.3 the curves β_i are separating not of boundary type. This implies that $Sep(S) \leq Sep(S')$. The reverse inequality follows by symmetry, that is by considering the inverse isomorphism ϕ^{-1}.

The proof for $NSep$ is completely similar.

□

We are now finally in a position to complete the proof of Theorem 5.1. Assume there exists a simplicial isomorphism $\phi : \check{C}(S) \to \check{C}(S')$, where $S = S_{g,n}$ and $S' = S_{g',n'}$ are of types (g,n) and (g',n') respectively. It follows that $C(S)$ and $C(S')$ satisfy the equalities :

$$dim(S) = dim(S'), \quad Sep(S) = Sep(S'), \quad NSep(S) = NSep(S').$$

Straightforward bookkeeping using Lemma 5.4 shows that if the two types are different, the only possible isomorphisms occur for:

(1) $g=2, n \geq 0$, $g'=0, n'=n+6$ and $n = \left[\frac{n+6}{2}\right]$, so that $n=0$. In this case $(g,n) = (2,0)$ and $(g',n') = (0,6)$.
(2) $g=1, n \geq 1$, $g'=0, n'=n+3$ and $n = \left[\frac{n+3}{2}\right]$, so that $n \in \{2,3\}$. Then $(g,n) = (1,2)$, $(g',n') = (0,5)$ or $(g,n) = (1,3)$, $(g',n') = (0,6)$.
(3) $g=2, n \geq 0$ and $g'=1, n'=n+3$.

In the first two cases we have to exclude a possible isomorphism betwee $\check{C}(S_{1,3})$ and $\check{C}(S_{0,6})$. Assume such an isomorphism $\phi : \check{C}(S_{1,3}) \to \check{C}(S_{0,6})$ does exist. Let α and γ be nonseparating curves on $S_{1,3}$ and β of boundary type, such that $\{\alpha, \beta, \gamma\} \in C(S_{1,3})^{(2)}$ form a (discrete) triangle. Next pick $g \in \check{\Gamma}(S_{0,6})$ such that $\{\alpha', \beta', \gamma'\} = g \cdot \phi(\{\alpha, \beta, \gamma\}) \in C(S_{0,6})$ is a discrete triangle (both $\check{\Gamma}((S_{0,6})$ and ϕ act componentwise).

Since the dual links of α and $\alpha' = g \cdot \phi(\alpha)$ are isomorphic (ditto for β and γ), it follows that α', β' and γ' are disjoint loops of boundary type on $S_{0,6}$. There then exists a diffeomorphism of $S_{0,6}$ swapping α' and β' while fixing γ'. Let h denote its class in the extended maping class group of $S_{0,6}$) (it may not preserve the orientation) and let $H = \phi^{-1} g^{-1} h g \phi$; it is an automorphism of $\check{C}(S_{1,3})$, exchanging α and β while keeping γ fixed.

Let now δ be a loop disjoint from α and β, intersecting γ in two points such that δ separates $S_{1,3}$ into two components $S_{1,1}$ containing α and $S_{0,4}$ containing β. Lemma 5.3 shows that the dual link $L^-_{\check{C}(S_{1,3})}(\delta)$ is disconnected. Consider the preimage $\{\alpha, \beta, H^{-1}(\delta)\} \in \check{C}(S_{1,3})^{(2)}$ of the simplex $\{\beta, \alpha, \delta\} \in C(S_{1,3})^{(2)}$ via

H and let $f \in \check{\Gamma}(S_{1,3})$ be such that $f \cdot \{\alpha, \beta, H^{-1}(\delta)\} \in C(S_{1,3})^{(2)}$ is a discrete triangle. The curves α and β form a cut pair, dividing $S_{1,3}$ into two components which are copies of $S_{1,1}$ and $S_{0,4}$ respectively. One can take f in the stabilizer of α and β, indeed in a group of type $\hat{\Gamma}_{0,4} = \hat{\Gamma}(S_{0,4})$ (which is profree on two generators) because $f \cdot H^{-1}(\delta)$ lies in the copy of $S_{0,4}$. The curve $f \cdot H^{-1}(\delta)$ is either nonseparating as a curve on $S_{1,3}$ or of boundary type. In both cases its dual link in the completed complex, $L^{-}_{\check{C}(S_{1,3})}(f \cdot H^{-1}(\delta))$ is chain connected. On the other hand it should be isomorphic to $L^{-}_{\check{C}(S_{1,3})}(\delta)$, which is disconnected. This contradiction proves that $\check{C}(S_{1,3})$ and $\check{C}(S_{0,6})$ are not isomorphic; of course $\check{C}(S_{1,3})$ and $\check{C}(S_{2,0})$ are not isomorphic either, since $\check{C}(S_{0,6})$ and $\check{C}(S_{2,0})$ are indeed isomorphic.

Turning to the last case (3) and a putative isomorphism $\phi : \check{C}(S_{2,n}) \to \check{C}(S_{1,n+3})$ we consider $\phi(\alpha)$ where α is nonseparating on $S = S_{2,n}$. Its image is a vertex of $\check{C}(S')$ ($S' = S_{1,n+3}$) which can be mapped to a discrete loop by some $g \in \check{\Gamma}(S')$. This curve $\beta = g \cdot \phi(\alpha) \in \mathcal{L}(S')$ has a well-defined type when considered in $\check{\mathcal{L}}(S')$ by Proposition 3.3 and the links and dual links of α and β are isomorphic. Thus β is either separating or of boundary type. Moreover the links of α in $\check{C}(S_{2,n})$ and of β in $\check{C}(S_{1,n+3})$ should be isomorphic. But the first link is isomorphic to $\check{C}(S_{1,n+2})$ and the second one to either $\check{C}(S_{0,n+5})$, if β is nonseparating or to $\check{C}(S_{0,n+2})$ if β is of boundary type. Using what we did above in case (2) we conclude that β should be nonseparating and that $n \in \{0, 1\}$. Thus either $(g, n) = (2, 0)$, $(g', n') = (1, 3)$ or $(g, n) = (2, 1)$, $(g', n') = (1, 4)$. Finally, an isomorphism between$\check{C}(S_{1,4})$ and $\check{C}(S_{2,1})$ would send a nonseparating curve α on $S_{1,4}$ to a nonseparating curve on $S_{2,1}$ with isomorphic links. However these are isomorphic to $\check{C}(S_{0,6})$ and $\check{C}(S_{1,3})$, respectively and it was shown above (case (2)), that these complexes are not isomorphic. This completes the proof of the theorem.

□

Remark 5.1. *i) As can be readily checked Theorem 5.1 is actually valid, with the same proof, for any* residually finite completion *of the curve complexes $C(S_{g,n})$. Completing $C(S)$ with respect to a quotient $\Gamma(S)'$ of the full profinite completion $\hat{\Gamma}(S)$, it amounts to requiring (see* [3]*, Prop. 5.1) that $\Gamma(S)'$ be residually finite (i.e. the natural map $\Gamma(S) \to \Gamma(S)'$ should be* into*). In particular this is the case of any completion which is finer than the congruence completion.*
ii) By a famous result of Harer-Ivanov, the curve complex $C(S)$ is homotopically equivalent to a wedge of spheres of dimension $h(S)$. The value of $h(S_{g,n})$, which is also the cohomological dimension of $C(S_{g,n})$, is explicit (see e.g. [21]*) and provides a second invariant (after the dimension) in the discrete case. However in the complete case we could not use it for a reason which perhaps deserves to be mentioned. One knows that $H^q(C(S), \mathbb{Q})$ vanishes for all $q \neq 0, h$ ($h = h(S)$); the same is true of every finite quotient $H^q(C(S)/\Gamma^\lambda, \mathbb{Q})$ (Γ^λ normal cofinite in $\Gamma(S)$). But we were not able to show that $H^h(C(S)/\Gamma^\lambda, \mathbb{Q})$ does* not *vanish. In fact one would like to show this for some (any) value of λ and in particular one can take λ large enough (Γ^λ small enough) so that all the components of the boundary $\partial\mathcal{M}^\lambda(S)$ of the associated level are smooth. The question is whether the combinatorics of these components contributes to the cohomology of the simplicial variety $\partial\mathcal{M}^\lambda(S)$. Note that $H^h(C(S)/\Gamma^\lambda, \mathbb{Q})$ injects into the Hodge weight 0 part of the rational cohomology of $\partial\mathcal{M}^\lambda(S)$. Is $W^0H^h(\partial\mathcal{M}^\lambda(S), \mathbb{Q})$ nontrivial for some λ, in particular for $\Gamma^\lambda = \Gamma(S)$, $\mathcal{M}^\lambda(S) = \mathcal{M}(S)$) ?*

5.2. **Local structure and reconstruction of congruence graphs.** Our next objective is the profinite analog of Theorem 2.10, which is interesting for its own sake and will be used in the next section, much as was done in §2 in the discrete setting, in order to start exploring the continuous automorphisms of the congruence complexes. In this subsection we will deal almost exclusively with the *graphs* $C_*(S)$ and $C_P(S)$ and their congruence completions, as they carry most of the relevant information. We refer to §§A.5, 7, 8, 9 for the basic definitions. Note that we will *not* make use of the isomorphism theorem in what follows, and for good reasons since we have not shown any result of that type pertaining to these graphs. It could however be interesting to state and prove such results.

We start from a surface S which is *not* assumed to be connected but is such that each of its finitely many connected component S_i is hyperbolic ($S = \coprod_{i \in I} S_i$). We define $\Gamma(S) = \prod_{i \in I} \Gamma(S_i)$, the *colored* modular group, and let each $\Gamma(S_i)$ act naturally on $C_*(S_i)$ and $C_P(S_i)$ so as to extend definitions to the non connected situation (see also §A.9). Finally we deal with the congruence completions $\check{C}_*(S)$ and $\check{C}_P(S)$ by completing the modular groups $\Gamma(S_i)$ of the connected components. Note that possible permutations of the pieces have no effect on completions, since they generate a finite group; in other words, the colored modular group has finite index in the full modular group.

If S is connected with $d(S) = 0$ i.e. is a trinion (a.k.a a pair of pants), $\check{C}_*(S) = \check{C}_P(S)$ is empty or conventionally reduced to a point and coincides with its discrete version. If S is connected with $d(S) = 1$,

$\check{C}_*(S) = \check{G}$ and $\check{C}_P(S) = \check{F}$, where the completion can be taken with respect to the natural action of $\Gamma_{0,4}(\simeq F_2)$, which has finite index in $\Gamma_{1,1}$ (see §§A.7,8) Although for resaon of coherence we use the notation for the congruence completion, here it does not differ from the full profinite completion. Recall that more generally the congruence conjecture holds for types (g, n) with $g = 0, 1, 2$, and n arbitrary.

If $d(S) > 1$, $\check{C}_*(S)$ identifies with the 1-skeleton of the dual of $\check{C}(S)$ but we are aiming at a direct description, actually valid in all dimensions $d(S) \geq 0$. There is a natural action of $\check{\Gamma}(S)$ on $\check{C}_*(S)$ and $\check{C}_P(S)$ and as usual, one can describe their common set of vertices (denoted $\check{V}(S)$) as a *finite* disjoint union $\coprod_{v\in\mathcal{F}} \check{\Gamma}\cdot v$ of $\check{\Gamma}$-orbits of discrete vertices $v \in V(S)$. Each $v \in V(S)$ represents a discrete maximal multicurve (a.k.a. a pants decomposition) of S and the finite set $\mathcal{F}$ enumerate the types (Γ-orbits) of such deompositions. The set $\check{E}(S)$ of edges of $\check{C}_*(S)$ can be described as follows:

Lemma 5.6. *The vertices $v, w \in \check{C}_*(S)$ are joined by an edge if and only the corresponding maximal multicurves differ by exactly one component up to relabeling.*

Proof. The statement should be interpreted as follows. Write $v = (\alpha_1, \ldots, \alpha_k)$ (resp. $w = (\beta_1, \ldots, \beta_k)$) where the α_i's and β_j's are (pro)curves and $k = d(S)+1$). One could assume that either v or w corresponds to a discrete pants decomposition but that does not really help. The claim is that the condition for v and w to be joined by an edge in $\check{C}_*(S)$ is the exact analog of what happens in the discrete case.

The "if" part of the statement is clear and we have to show the "only if" part. In order to do this, let $v = \varprojlim_{\lambda\in\Lambda} v^\lambda$, $w = \varprojlim_{\lambda\in\Lambda} w^\lambda$ where $\lambda \in \Lambda$ belongs to the set of congruence levels (here we may assume S connected for simplicity) and $v^\lambda, w^\lambda \in C_*^\lambda = C_*(S)/\Gamma^\lambda$. We can write $v^\lambda = (\alpha_i^\lambda)$, $w^\lambda = (\beta_i^\lambda)$ where the α_i^λ and β_i^λ represent Γ^λ-orbits of curves (i.e. they lie in $\mathcal{L}(S)/\Gamma^\lambda$). Moreover since v and w are joined by an edge in $\check{C}_*$, there exist discrete pants decompositions (A_i^λ), (B_j^λ) in C_* which project to v^λ and w^λ repectively and are joined by an edge in C_*. So (A_i^λ) and (B_j^λ) differ by at most one curve, after relabeling. For any $\lambda \in \Lambda$ consider the label (in $\{1, \ldots, k\}$) of the curve in the family (A_i^λ) which does not occur in (B_j^λ) (if they coincide pick any label). This may depend also on the chosen lifts of v^λ and w^λ but that does not matter. Now consider a cofinal sequence in Λ and choose a label which occurs infinitely often in the above construction. One finds that v and w can indeed be represented by multi(pro)curves (α_i) and (β_i) which coincide except for the entry in v corresponding to that label.

□

We are heading toward a statement and proof of the procongruence analog of Theorem 2.10, which deals with the graph $\check{C}_*(S)$. We take up the setting and notation of the beginning or §2.2, starting with a hyperbolic surface S. We assume that all connected components of S have the same modular dimension, which we denote $d(S)$, and that $d(S) > 0$. For a multicurve $\sigma \in C(S)$, S_σ denotes, as in §2, the surface S slit along the multicurve σ. We will first show that given such a multicurve, there is a natural embedding of the *procongruence* curve graph $\check{C}_*(S_\sigma)$ into $\check{C}_*(S)$ and that it is equivariant with respect of the actions of the attending modular groups $\check{\Gamma}(S_\sigma)$ and $\check{\Gamma}(S)$. This in essence is not new but it does embody an *essential* property of the *proconguence* topology, which we summarize in the following geometric lemma:

Lemma 5.7. *Let S be as above, $\sigma \in C(S)$ a multicurve, S_σ the surface with boundary obtained by cutting S along σ. Then every unramified Galois cover of S_σ is dominated by a Galois cover of S.*

Proof. To be sure, the lemma asserts that one can find a Galois cover of S which restricts to a cover of the multicurve σ, does not permute the pieces of S_σ, and dominates the given cover of S_σ as a surface with boundary. The proof is in fact elementary. By an immediate induction one restricts to the case of a single curve $\sigma = \{\alpha\}$ and this case is dealt with in [3], Lemmas 6.7 (α non separating) and 6.8 (α separating).

□

We insist that this elementary and relatively easy lemma is nonetheless a key point. In essence it says that the procongruence topology transfers nicely when cutting along a multicurve, producing subsurfaces of the ambient surface; see Proposition 5.8 below. The analog in the full profinite case, where one has to work with covers of moduli stacks and not just surfaces is *not* known.

We will now proceed to state the proposition we need in order to make good sense of the reconstruction problem in the procongruence setting. The proof is again quite easy, given the lemma above. In fact we will state the proposition for $\Gamma(S)$, $C_*(S)$ and $C_P(S)$ simultaneouslys because these are the objects we have to deal with but it simply expresses again the fact that the topology induced – so to speak – on S_σ by the congruence topology attached to S coincides with the congruence topology attached to S_σ; this in turn is nothing but the content of Lemma 5.7 above. Put this way it is clear that it applies "functorially" and equivariantly to a lot of objects attached to surfaces (starting with $C(S)$, the curve complex itself), provided they display a nice behavior with respect to the operation of "cutting along multicurves". We refrain from giving a more abstract statement, but see Remark 5.2 below.

Given S and $\sigma \in C(S)$ we have natural embeddings: $\Gamma_\sigma \hookrightarrow \Gamma(S)$, $C_*(S_\sigma) \hookrightarrow C_*(S)$, $C_P(S_\sigma) \hookrightarrow C_P(S)$. In the first case Γ_σ denotes the stabilizer of σ in $\Gamma(S)$; for the two others see the beginning of §2.2 above. Using that the congruence completion is residually finite, we get corresponding embeddings $\Gamma_\sigma \hookrightarrow \check{\Gamma}(S)$, $C_*(S_\sigma) \hookrightarrow \check{C}_*(S)$ and $C_P(S_\sigma) \hookrightarrow \check{C}_P(S)$. This leads to continuous embeddings of the respective closures: $\bar{\Gamma}_\sigma \hookrightarrow \check{\Gamma}(S)$, $\bar{C}_*(S_\sigma) \hookrightarrow \check{C}_*(S)$ and $\bar{C}_P(S_\sigma) \hookrightarrow \check{C}_P(S)$. Note that by the universality of the congruence completion the discrete embeddings into the respective completions factors through the completions of the respective sources. In other words we also have (continuous) maps: $\check{\Gamma}_\sigma \to \check{\Gamma}(S)$, $\check{C}_*(S_\sigma) \to \check{C}_*(S)$ and $\check{C}_P(S_\sigma) \to \check{C}_P(S)$. These however are *not* known *a priori* to be injective. In fact, we *a priori* get *surjective* maps $\check{\Gamma}(S_\sigma) \to \bar{\Gamma}_\sigma$, $\check{C}_*(S_\sigma) \to \bar{C}_*(S_\sigma)$ and $\check{C}_P(S_\sigma) \to \bar{C}_P(S_\sigma)$ stemming from the fact that the induced topology from S is *a priori* a quotient (i.e. at most as fine) as the congruence topology attached to S_σ. Our next proposition asserts that these last maps are in fact isomorphisms:

Proposition 5.8. *Let S be as above, $\sigma \in C(S)$ be a multicurve, S_σ the surface S slit along σ. With the notation and construction as above, the resulting maps are all isomorphisms:*

$$\check{\Gamma}(S_\sigma) \xrightarrow{\sim} \bar{\Gamma}_\sigma, \quad \check{C}_*(S_\sigma) \xrightarrow{\sim} \bar{C}_*(S_\sigma), \quad \check{C}_P(S_\sigma) \xrightarrow{\sim} \bar{C}_P(S_\sigma).$$

Proof. As mentioned above the proposition is a fairly straightforward consequence of Lemma 5.7. One only needs to transfer the information from covers of surfaces to the analog on *congruence* levels of the associated moduli spaces $\mathcal{M}(S)$ and $\mathcal{M}(S_\alpha)$. This again is part of Lemmas 6.7, 6.8 in [3]. One could also use, in the same spirit, isomorphism results of the type $\check{C}(S) \simeq \check{C}_{\mathcal{L}}(S)$ (see Theorem 3.1), whose goal is precisely to transfer information from the congruence levels of the moduli space $\mathcal{M}(S)$ to covers of the surface S itself – and back. But we have not shown or even stated such general results outside of the case of the curve complex $C(S)$. □

So in the end we get continuous embeddings: $\check{\Gamma}_\sigma \hookrightarrow \check{\Gamma}(S)$, $\check{C}_*(S_\sigma) \hookrightarrow \check{C}_*(S)$ and $\check{C}_P(S_\sigma) \hookrightarrow \check{C}_P(S)$ with closed, hence compact images since the sources are compact. From the first isomorphism, namely $\bar{\Gamma}_\sigma \simeq \check{\Gamma}(S_\sigma)$, we conclude that in the last two cases the maps are equivariant with respect to the action of $\check{\Gamma}(S_\sigma)$ and $\check{\Gamma}(S)$ on the sources and targets respectively. We remark that in the above we have been a little sloppy about boundary curves, not always distinguishing very carefully between a surface with or without boundary. In fact we have left it to the reader to straighten out some details for her/himself.

Remark 5.2. *i) As anticipated in Grothendieck's* Esquisse*, there exists an underlying beautiful "dictionary" between objects of* a priori *very different natures, from topology to arithmetic through hyperbolic, conformal, complex or algebraic geometry. For instance one can – should – consider the completed stack $\bar{\mathcal{M}}(S)$, or more generally $\bar{\mathcal{M}}^\lambda(S)$ as a simplicial or stratified object where the strata are enumerated by $C(S)/\Gamma(S)$ (resp. $C(S)/\Gamma^\lambda(S)$). Note that formally the generic stratum $\mathcal{M}(S)$ corresponds to $S = S_\emptyset$, that is to $\emptyset \in C(S)^{(-1)}$. In essence, Proposition 5.8 says that the congruence completion respects this simplicial character of the stably completed moduli stacks of curves.*
ii) The analog of Proposition 5.8 is not known in the full profinite case and in fact its validity is equivalent *to that of the congruence subgroup conjecture (this is the case for several statements in §§3, 4, 5). Indeed, assuming it holds true one can prove the conjecture working by induction on the the modular dimension $d(S)$ and using equivariant spectral sequences as in* [3], *§6.*

We now turn to the reconstruction problem. We first note that one can view $\check{C}_P(S) \subset \check{C}_*(S)$ as a closed subgraph with the identical set $\check{V}(S)$ of vertices and a set $\check{E}_P(S) \subset \check{E}(S)$ of edges. Indeed consider the natural injections $C_P(S) \hookrightarrow C_*(S)$ and $C_*(S) \hookrightarrow \check{C}_*(S)$; by composition we get an equally natural injection $C_P(S) \hookrightarrow \check{C}_*(S)$; taking the closure of $C_P(S)$ inside $\check{C}_*(S)$ yields $\check{C}_P(S)$ as should be clear from the above. Alternatively the injection of $C_P(S)$ into $\check{C}_*(S)$ factors through $\check{C}_P(S)$ by universality and the resulting map $\check{C}_P(S) \to \check{C}_*(S)$ is injective. Yet it is not so easy to give a description of $\check{C}_P(S)$ inside $\check{C}_*(S)$ or equivalently of $\check{E}_P(S)$ as a subset of $\check{E}(S)$. We insist on that matter because it will turn out that, modulo reconstruction of the whole of $\check{C}(S)$ from the graph $\check{C}_*(S)$ (Corollary 5.14 below) and the rigidity of $\check{C}_P(S)$ (Theorem 7.1 below), we are getting quite close to the actual root of Grothendieck-Teichmüller theory in this profinite topological ("nonlinear") setting. In particular it will evolve (elsewhere) that the set of injective morphisms $j : \check{C}_P(S) \hookrightarrow \check{C}_*(S)$ is a close profinite analog of the variety of associators introduced by V.G.Drinfel'd in the prounipotent case. Here, however, we restrict attention to the "natural" injection or inclusion $\check{C}_P(S) \subset \check{C}_*(S)$, or equivalently $\check{E}_P(S) \subset \check{E}(S)$.

We are first aiming at a better understanding of the local structure of $\check{C}_*(S)$ and $\check{C}_P(S)$, more precisely at making sense and proving an analog of Lemma 2.7. We start with:

Lemma 5.9. *Let S be* connected *hyperbolic and let $\sigma \in C(S)$ be a multicurve which is* not *maximal. Then if $g \in \check{\Gamma}(S)$ stabilizes the subcomplex $\check{C}_*(S_\sigma)$ of $\check{C}_*(S)$, i.e. $g(\check{C}_*(S_\sigma)) = \check{C}_*(S_\sigma)$, it stabilizes σ, i.e. $g(\sigma) = \sigma$.*

Proof. Assume that $g(\sigma)$ is different from σ. We want to show that there exists a proloop $\check{\gamma} \in \check{\mathcal{L}}(S)$ which is disjoint from σ but not from $g(\sigma)$. First, there exists $\beta \in \mathcal{L}(S)$ contained in σ (i.e. β is a vertex of σ) such that $g^{-1}(\beta)$ is *not* a vertex of σ. By hypothesis, for any $\tau \in C_*(S_\sigma)$ we have $g(\tau) \in \check{C}_*(S_\sigma)$. In particular, if we consider a maximal multicurve $\tau \in C_*(S_\sigma)$ containing the curve β, we derive that the proloop $g(\beta) \subset g(\tau)$ must be disjoint from σ, namely that the simplex $(\sigma, g(\beta)) \in \check{C}(S)$. Further, there exists $h \in \check{\Gamma}(S)$ such that $h(\sigma, g(\beta))$ is a discrete simplex. We may choose h such that $h(\sigma) = \sigma$ and $hg(\beta) = \alpha \in \mathcal{L}(S)$ is then a simple closed curve disjoint from σ, so that $(\sigma, \alpha) \in C(S)$. The component of S_σ containing $\alpha \in \mathcal{L}(S_\sigma)$ cannot be of type $(0,3)$ since it contains the curve α. Thus S_σ contains some simple closed curve γ which intersects geometrically α. The proloop $\check{\gamma} = h^{-1}(\gamma)$ then satisfies the original requirement.

Observe now that any proloop $\check{\gamma}$ as above can be completed to a maximal multicurve $\tau \in \check{C}_*(S_\sigma)$, since discrete curves have this property and $h(\check{\gamma})$ is discrete. Now τ contains the curve $\check{\gamma}$ which intersects $g(\sigma)$, so that $g^{-1}(\tau)$ does not belong to $\check{C}_*(S_\sigma)$, contradicting the assumption that g stabilizes $\check{C}_*(S_\sigma)$. □

For any subsurface $\Sigma \subset S$ we may identify the (pro)graph $\check{C}_*(\Sigma)$ (resp. $\check{C}_P(\Sigma)$) with a subgraph of $\check{C}_*(S)$ (resp. $\check{C}_P(S)$). We can now proceed with:

Lemma 5.10. *Let Σ_1 and Σ_2 be two distinct subsurfaces of S of dimension 1. Then the intersection $\check{C}_P(\Sigma_1) \cap \check{C}_P(\Sigma_2)$ in $\check{C}_P(S)$ is either empty or consists of a single vertex.*

Proof. Let Σ_j be the connected component of S_{σ_j} of dimension 1, where the σ_j $(j = 1, 2)$ are codimension 1 simplices of $C(S)$. The vertices of $\check{C}_P(\Sigma_j) \subset \check{C}_P(S)$ are of the form $(\sigma_j, \check{\gamma}_j)$, where $\check{\gamma}_j \in \check{\mathcal{L}}(\Sigma_j)$ is a proloop on Σ_j. If the intersection $\check{C}_P(\Sigma_1) \cap \check{C}_P(\Sigma_2)$ is non-empty there exist two such vertices which coincide, i.e. such that $(\sigma_1, \check{\gamma}_1) = (\sigma_2, \check{\gamma}_2)$ as vertices of $\check{C}_P(S)$. Since Σ_1 and Σ_2 are distinct we can write $\sigma_j = (\sigma'_j, \delta_j)$ for simple closed curves $\delta_j \in \mathcal{L}(\Sigma_j)$ such that $\sigma'_1 = \sigma'_2$, $\check{\gamma}_1 = \delta_2$ and $\delta_1 = \check{\gamma}_2$. In particular $\check{\gamma}_j \in \mathcal{L}(\Sigma_j)$ is a (discrete) simple closed curve $(j = 1, 2)$ and for fixed Σ_j the common vertex is a unique discrete vertex. □

Given $\sigma \in C(S)$, we recall that S_σ denotes the surface S obtained by cutting S along the curves in σ and then crushing boundary circles to punctures. There is a natural injection $C_P(S_\sigma) \to C_P(S)$, which sends the pants decomposition τ of S_σ to the pants decomposition $\tau \cup \sigma$ of S. This construction extends to the completions, as follows. Let $\check{\sigma} \in \check{C}(S)$. There exists then $g \in \check{\Gamma}(S)$ and $\sigma \in C(S)$ a discrete simplex such that $g \cdot \sigma = \check{\sigma}$. We set then

$$\check{C}_P(S_{\check{\sigma}}) = g \cdot \check{C}_P(S_\sigma) \subset \check{C}_P(S)$$

This is well-defined and independent on the choices involved, as the topological type of $\check{\sigma}$ is well-defined. We will need the following properties of $\check{C}_P(S)$:

Lemma 5.11.
i) $\check{C}_P(S)$ is covered by the images of $\check{C}_P(S_{\check{\sigma}})$, where $\check{\sigma} \in \check{C}(S)$ is a simplex of codimension $cd(\check{\sigma}) = 1$:

$$\check{C}_P(S) = \bigcup_{\check{\sigma} \in \check{C}(S),\ cd(\check{\sigma})=1} \check{C}_P(S_\sigma).$$

ii) Given Σ a dimension 1 surface we construct the complete *prograph $\overline{C}_*(\Sigma)$ whose vertices are those of $\check{C}_P(\Sigma)$. We define $\overline{C}_*(S)$ as the quotient of the disjoint union $\bigsqcup \overline{C}_*(\Sigma)$, over all dimension 1 subsurfaces Σ of S, by the equivalence relation which identifies vertices $v \in \overline{C}_*(\Sigma_1)$ and $w \in \overline{C}_*(\Sigma_2)$ if their respective images under the natural embeddings $\check{C}_P(\Sigma_i) \hookrightarrow \check{C}_P(S)$ coincide. Then $\overline{C}_*(S)$ is isomorphic to $\check{C}_*(S)$.*
iii) Say that the simplices ρ and τ of $\check{C}(S)$ are compatible *if for every pair of vertices v and w of ρ and σ respectively, either $v = w$ or v and w are* not *joined by an edge in $\check{C}_P(S)$. Then if ρ and τ are compatible*

$$\check{C}_*(S_\rho) \cap \check{C}_*(S_\sigma) = \check{C}_*(S_{\rho \cup \sigma});$$

otherwise the intersection is empty.

Proof. The first statement follows from its discrete counterpart and Proposition 5.8. The second item is a consequence of Proposition 5.8 along with Lemma 5.10. Then Lemma 5.10 and Lemma 5.6 imply the last claim, which is the procongruence analog of Lemma 2.6. □

The procongruence analog of Lemma 2.7 is a straightforward consequence of this lemma, that is:

Proposition 5.12. *The graph $\check{C}_*(S)$ is obtained from $\check{C}_P(S)$ by replacing every maximal copy of $\hat{F}$ inside $\check{C}_P(S)$ by a copy of $\hat{G}$.* □

Let us now proceed towards the reconstruction theorem, starting however with a discussion about its proper statement and meaning in the complete setting. In the discrete case the natural action of $\Gamma(S)$ on $C(S)$ translates into an action of $\Gamma(S)$ on the graphs $C_*(S_\sigma)$, viewed as subgraphs of $C_*(S)$. For $g \in \Gamma(S)$, $\sigma \in C(S)$, we get the following equivariance formula:

$$g \cdot C_*(S_\sigma) = C_*(S_{g\cdot\sigma})$$

which also holds with C_* replaced by C_P. In particular the reconstruction principle of Corollary 2.11 respects the natural Γ-action.

In the procongruence case, using the natural action of $\check{\Gamma}(S)$ on $\check{C}(S)$, we find that $g \cdot \check{C}_*(S_\sigma)$ is a well-defined closed subgraph of $\check{C}_*(S)$ for $g \in \check{\Gamma}(S)$ and $\sigma \in C(S)$. At this point one is tempted to write down the same formula as above, replacing the objects with their respective congruence completions, that is $g \cdot \check{C}_*(S_\sigma) = \check{C}_*(S_{g\cdot\sigma})$ for any $g \in \check{\Gamma}(S)$, $\sigma \in \check{C}(S)$. In the general case however, that is for $g \notin \Gamma(S)$ *and* $\sigma \notin C(S)$, neither side is *a priori* well-defined. If we pick $\sigma \in C(S)$ a *discrete* simplex and $g \in \check{\Gamma}(S)$ arbitrary, then the right-hand side can be *defined* by the left-hand side. Then extend the definition to any $\sigma \in \check{C}(S)$ using as usual the fact that the $\check{\Gamma}(S)$-orbit of any simplex in $\check{C}(S)$ contains a discrete representative.

One thus gets a family $(\check{C}_*(S_\sigma))_{\sigma\in\check{C}(S)}$ of closed subgraphs of $\check{C}_*(S)$ which is indexed by the profinite simplicial set $\check{C}(S)$ and is equipped with a natural simplicial action of $\check{\Gamma}(S)$. These subgraphs are distinct for σ *not* maximal, that is $\check{C}_*(S_\sigma) = \check{C}_*(S_\tau)$ if and only if $\sigma = \tau \in \check{C}(S)$. In fact in order to vindicate this assertion, it is enough to show that for any discrete $\sigma \in C(S)$ and any $g \in \check{\Gamma}(S)$, $g \cdot \check{C}_*(S_\sigma) = \check{C}_*(S_\sigma)$ if and only if $g \cdot \sigma = \sigma$, which is Lemma 5.9 above. As in the discrete case, reconstructing $\check{C}(S)$ out of $\check{C}_*(S)$ consists in graph theoretically detecting or characterizing the family $(\check{C}(S_\sigma))_{\sigma\in\check{C}(S)}$, which can be made into a prosimplicial complex using the inclusion of curves as a boundary operator.

In what follows, for $\tau \in \check{C}(S)$, one can think of $\check{C}_*(S_\tau)$ via the defining formula $\check{C}_*(S_\tau) = g \cdot \check{C}_*(S_\sigma)$ for $\sigma \in C(S)$ discrete, $g \in \check{\Gamma}(S)$, $g \cdot \sigma = \tau$, thus avoiding making sense directly of the symbol S_τ, that is "S slit along the profinite simplex τ". Finally it may be worth pointing out the possible connection with what Grothendieck calls *discrétifications* in his *Longue Marche à travers la théorie de Galois* (§26). Roughly speaking and to be specific, given a finitely generated residually finite group G and its profinite completion $\hat{G}$ one can consider the set of its discretifications, that is of the dense injections $G \hookrightarrow \hat{G}$. This can be seen as a natural extension of the notion of integral lattice or integral structure in the linear setting. These discretifications will form a torsor under a group which is not easy to capture in general but may be worth keeping in mind. In an analogous way one can view the set of dense embeddings $C(S) \hookrightarrow \check{C}(S)$ as the set of integral structures on $\check{C}(S)$ and in our context the above seemingly formal definitions become more natural, since the group $\check{\Gamma}(S)$ will act naturally on these structures ("discretifications") as well.

We can now state the procongruence version of Theorem 2.10 as:

Theorem 5.13. *Let S be a connected hyperbolic surface, $C \subset \check{C}_*(S)$ a subgraph which is topologically isomorphic to $\check{C}_*(\Sigma)$ for a certain surface Σ and is maximal with this property. Then there exists a unique $\sigma \in \check{C}(S)$ such that $C = \check{C}_*(S_\sigma)$.*

Proof. If one wishes to stick to S_σ for discrete simplices $\sigma \in C(S)$, the assertion can be rephrased by saying that there exist $\sigma \in C(S)$ and $g \in \check{\Gamma}(S)$ such that $C = g \cdot \check{C}_*(S_\sigma)$. Two solutions (σ, g) and (σ', g') satisfy $g \cdot \sigma = g' \cdot \sigma' \in \check{C}(S)$. As in the discrete setting, the case $\sigma = \emptyset$ should be included and corresponds to the full complex $\check{C}_*(S)$.

With Lemmas 5.6 and 5.11 at our disposal, the proof proceeds along the lines of the proof in the discrete case. We need only show the existence part, uniqueness being clear, as in the discrete case. The first step consists in showing that a subgraph of the form indicated in the statement is maximal. To this end, one can consider a discrete $\sigma \in C(S)$ and prove that $\check{C}_*(S_\sigma)$ is maximal in its isomorphy class. The proof follows the one in the discrete case in §2.3.

Here and as in the discrete case again, it is more elegant (although not necessary) to include the case $d(\Sigma) = 0$, i.e. Σ a trinion, of type $(0,3)$, by declaring that the attending curve complex is reduced to a point rather than empty: $C(S_{0,3}) = C_*(S_{0,3}) = \{*\}$. The case of dimension 0 is then clear: the vertices of $\check{C}_*(S)$ correspond to maximal multicurves (not necessarily discrete). One can start induction from there, or treat the case $d(\Sigma) = 1$ independently, as in the discrete case. We do not do it in detail because the inductive argument applies to that case as well. Suffice it to say that it is still true that any triangle in $\check{C}_*(S)$ defines a unique subsurface of dimension 1, possibly after twisting by an element of $\check{\Gamma}(S)$.

Having disposed of the low-dimensional cases, we argue again by induction on $k = d(\Sigma)$. So we pick $k > 1$, assume the statement is true for $d(\Sigma) < k$ and fix an isomorphism $C \xrightarrow{\sim} \check{C}_*(\Sigma)$. For $\sigma \in C(S)$ we then define $C_\sigma \simeq \check{C}_*(\Sigma_\sigma) \subset \check{C}_*(S)$ as in the discrete case. This time the union of the C_σ's as σ runs over the nonempty simplices of $C(\Sigma)$ form a dense part of C, which is sufficient for the same argument as in the discrete case to go through. Namely in order to conclude the proof, it is enough to show that there exists a k-dimensional subsurface $T \subset S$ and an element $g \in \check{\Gamma}(S)$ such that for any (nonempty) $\sigma \in C(\Sigma)$, $C_\sigma \subset g \cdot \check{C}_*(T) \subset \check{C}_*(S)$.

We may again (as in the discrete case) restrict to $|\sigma| = 1$, i.e. to the discrete loops on Σ. To any such loop $\alpha \in \mathcal{L}(\Sigma)$ we can attach by induction a subsurface $S_{(\alpha)} \subset S$ of dimension $k-1$ and an element $g_\alpha \in \check{\Gamma}(S)$ such that $C_\alpha = g_\alpha \cdot \check{C}_*(S_{(\alpha)}) \subset \check{C}_*(S)$. As usual, having fixed an isomorphism $C \xrightarrow{\sim} \check{C}_*(\Sigma)$ we write an equality sign for the sake of simplicity.

Next we use, again as in the discrete case, the connectedness of $C(\Sigma)$ which is ensured by the assumption on the dimension ($k > 1$). So we have to study the following situation. We consider three discrete loops α, β, and γ on Σ such that $\alpha \cap \beta = \beta \cap \gamma = \emptyset$. We attach to them as above pairs $(g_\alpha, S_{(\alpha)} = S_\rho)$, $(g_\beta, S_{(\beta)} = S_\sigma)$ and $(g_\gamma, S_{(\gamma)} = S_\tau)$ for certain simplices $\rho, \sigma, \tau \in C(S)$ with $|\rho| = |\sigma| = |\tau| = d(S) - k + 1$. Moreover ρ and σ (resp. σ and τ) are compatible simplices.

As in the discrete case, the situation should be entirely determined by any two pairs of non intersecting curves on Σ, after which one can worry over a possible overdetermination. The reasoning below may appear more transparent if one recalls that a graph of the form $g \cdot \check{C}_*(S_\sigma)$ is actually determined by the profinite simplex $g \cdot \sigma$ and so depends on g only up to the subgroup of $\check{\Gamma}(S)$ fixing σ, which is nothing but the centralizer of the multitwist corresponding to σ. These centralizers are determined in §4 above. So let us first examine what happens when trying to paste the data for α and β. After twisting we may assume that $g_\alpha = 1$ and write $g_\beta = g \in \check{\Gamma}(S)$. Next we know that the intersection $C_\alpha \cap C_\beta$ has dimension $k-2$ and indeed is isomorphic to a twist of $\check{C}_*(\Sigma_{\alpha\cup\beta})$. This implies that $|\rho \cap \sigma| = d(S) - k$ and that g fixes $\varpi = \rho \cap \sigma$, that is $g \in Z_\varpi$. Writing $T = S_\varpi$ we find that $S_{(\alpha)} = S_\rho \subset T$. Moreover, because g fixes ϖ we can find $h \in \check{\Gamma}(T)$ such that $C_\beta = g \cdot \check{C}_*(S_\sigma) = h \cdot \check{C}_*(S_\sigma)$. But then, since $h \in \check{\Gamma}(T)$, $h \cdot \check{C}_*(S_\sigma) \subset \check{C}_*(T)$ and so we get the inclusion $C_\beta \subset \check{C}_*(T)$. Returning to our original notation, we found a k-dimensional subsurface $T \subset S$ such that $C_\alpha \subset g_\alpha \cdot \check{C}_*(T)$, $C_\beta \subset g_\beta \cdot \check{C}_*(T)$ and in fact $g_\beta = g_\alpha = g$. Proceeding in the same way with the pair (β, γ) we get a possibly different pair (g', T'). Now in order to compare T and T', we use again the fact that there is a large intersection, namely that $C_\beta \subset g \cdot \check{C}_*(T) \cap g' \cdot \check{C}_*(T')$. This implies that one can modify – say – g' in order to achieve $g = g'$ and then, because T, T' and Σ are all of dimension k, one shows as in the discrete case that $T = T'$.

□

We now draw a consequence of this recognition result, much as in the discrete case, before turning to the study of the automorphism groups of the congruence complexes. Indeed Theorem 5.13 yields the analog of Corollary 2.11:

Corollary 5.14. *For $d(S) > 1$, $\check{C}(S)$ can be reconstructed from $\check{C}_*(S)$.*

Proof. In fact, as mentioned above, one reconstructs $\check{C}(S)$ by considering the set of subgraphs of $\check{C}_*(S)$ satisfying the conditions stated in Theorem 5.13, making it into a prosimplicial complex by using inclusion and deletion of curves as the face and boundary operators respectively. The theorem ensures that the resulting complex is indeed isomorphic to $\check{C}(S)$.

□

As a last item in this section and a corollary of what has been done above, we return to the issue of the possible isomorphisms between complexes of different types:

Proposition 5.15. *Let $S = S_{g,n}$ and $S' = S_{g',n'}$ be connected hyperbolic surfaces of different types. Then:*
i) $C_*(S_{1,1}) \simeq C_*(S_{0,4})$, $C_*(S_{1,2}) \simeq C_*(S_{0,5})$, $C_*(S_{2,0}) \simeq C_*(S_{0,6})$ and there are no other isomorphims;

ii) same as i) above in the procongruence setting, that is with $C_*(S)$ replaced by $\check{C}_*(S)$ everywhere;
iii) $C_P(S_{1,1}) \simeq C_P(S_{0,4})$ and this is the only nontrivial isomorphism between discrete pants graphs;
iv) $\check{C}_P(S_{1,1}) \simeq \check{C}_P(S_{0,4})$ and there are no other nontrivial isomorphisms in the procongruence case except perhaps in modular dimensions 2 and 3, as in i) above.

Proof. Item i) holds true if we replace $C_*(S)$ by $C(S)$ (see e.g. [21]). For $d(S) = 1$ $C_*(S_{0,4} = C_*(S_{1,1} = G$, where G is the complete graph on the vertices of the Farey graph F (see §A.8). If $d(S) > 1$, $C_*(S)$ is the 1-skeleton of the dual of $C(S)$; conversely $C(S)$ can be reconstructed from $C_*(S)$ by Corollary 2.11. So the cases of isomorphisms for $C_*(S)$ and for $\check{C}(S)$ coincide.

The reasoning for ii) is identical, using Theorem 5.1 and Corollary 5.14.

For iii), that is concerning the discrete pants graph, Lemma 2.7 says that $C_*(S)$ can be reconstructed from $C_P(S)$ (but not vice versa!) so that the cases of possible isomorphisms for $C_P(S)$ are among the possibilities $C_*(S)$. In dimension 1 we do have $C_*(S_{1,1}) \simeq C_*(S_{0,4}) = F$. In order to rule out the other two possibilities it is enough to show that $C_P(S_{1,2})$ and $C_P(S_{0,5})$ are not isomorphic. In fact, assume there exists an isomorphism $\phi : C_P(S_{2,0}) \xrightarrow{\sim} C_P(S_{0,6})$. Let then $\alpha \in \mathcal{L}(S_{2,0})$ be a nonseparating loop; it is mapped to a loop $\alpha' = \phi(\alpha) \in \mathcal{L}(S_{0,6})$ which is of boundary type. Cutting the surfaces along α and α' respectively, we find that ϕ induces an isomorphism between $C_P(S_{1,2})$ and $C_P(S_{0,5})$. For a proof that such an isomorphism does not exist, see [35], §12.

To iv) we use Proposition 5.12 to conclude that $\check{C}_*(S)$ can be reconstructed from $\check{C}_P(S)$ (again, this is not reversible) and that the possible isomorphisms are *at most* those which obtain for $\check{C}_*(S)$. In dimension 1 we get $\check{C}_P(S_{1,1}) \simeq \check{C}_P(S_{0,4}) = \hat{F}$ but in dimensions 2 and 3, although the discrete graphs are *not* isomorphic, we cannot *a priori* rule out possible isomorphisms between their respective profinite analogs. Recall that the congruence conjecture holds for types (g, n) with $g = 0, 1, 2$ and n arbitrary, so that we may replace the procongruence by the (full) profinite completion as far as the low dimensional complexes mentioned above are concerned. By now it is easy (and we leave it to the reader) to transpose the reduction argument in the proof of item iii) to the profinite setting. So the only moot point that remains consists in showing that the completed complexes $\hat{C}_P(S_{1,2})$ and $\hat{C}_P(S_{0,5})$ are *not* isomorphic. Note that on top of the fact that the discrete complexes are not isomorphic, this statement is rather "obvious" from the viewpoint of Grothendieck-Teichmüller theory because owing to the two-level principle and the fact that $\hat{C}_P(S_{1,1})$ and $\hat{C}_P(S_{0,4})$ *are* indeed isomorphic, the discrepancy between $\hat{C}_P(S_{1,2})$ and $\hat{C}_P(S_{0,5})$ actually carries the whole difference between the genus 0 and the general case of Grothendieck-Teichmüller theory (see [15, 22]). □

6. Automorphisms of procongruence complexes

We now start investigating the continuous automorphisms of the procongruence complexes attached to a connected hyperbolic surface S, especially our three favorite complexes $\check{C}(S)$, $\check{C}_P(S)$ and $\check{C}_*(S)$. These are by definition equipped with the profinite topology, that is they are limits of finite complexes over the inverse system Λ indexing the principal congruence subgroups of the Teichmüller modular group $\Gamma(S)$. Note that $\check{C}_P(S)$ and $\check{C}_*(S)$ are defined as graphs and we will recall below how $\check{C}(S)$, being a flag complex, is entirely determined by its 1-skeleton. So the first striking fact is that we need actually deal only with profinite *graphs*, that is one-dimensional complexes.

Returning to the general case and working with the curve complex for definiteness, $\check{C}(S)$ is defined as the (inverse) limit over Λ of the finite complexes $C^\lambda(S) = C(S)/\Gamma^\lambda$. A continuous automorphism, which is also open since $\check{C}(S)$ is compact, is given by a system of compatible maps $C^\mu(S) \to C^\lambda(S)$ where λ runs over Λ and $\mu \in \Lambda$ $(\mu \geq \lambda)$. When varying $\lambda \in \Lambda$, a basis of neighborhoods of the identity in $Aut(\check{C}(S))$ is defined by those automorphisms which induce the natural projection. Note that these elementary neighborhoods are *not* subgroups. This defines the structure of $Aut(\check{C}(S))$ as a profinite group. The same applies to $\check{C}_P(S)$ and $\check{C}_*(S)$. We will further elucidate and use this notion of continuity in §7.2 below.

This section aims at proving some basic and fundamental properties of the automorphism groups attached to the three complexes above. The reconstruction theorem above (Theorem 5.13) will play a significant role; on the one hand and much as in the discrete case, it paves the way towards some basic results, demonstrating how much of the information about the original curve complex $\check{C}(S)$ can be transferred to $\check{C}_*(S)$, which we recall is nothing but the 1-skeleton of the dual of $\check{C}(S)$ (in modular dimension $d(S) > 1$). The gain is that we have a natural inclusion of profinite graphs $\check{C}_P(S) \hookrightarrow \check{C}_*(S)$ which actually summarizes the main part of the information we are interested in (see also above Proposition 5.12). Anticipating again, we remark that from the point of view of Grothendieck-Teichmüller theory, this reconstruction result demonstrates how the so-called "Teichmüller tower" is *not* really needed: at every level, that is for a given modular dimension $d(S)$, the corresponding congruence curve complexes contain all the information coming from the lower levels. Putting this together with the "two level principle" ("*principe des deux premiers étages*"), will imply, as will be shown elsewhere, that one needs only consider a single, given profinite graph $\check{C}(S)$, with S of large enough dimension and genus, in order to investigate the automorphism group of the whole "tower", a kind of very strong stability result.

6.1. **Basic results.** First we state and prove explicitly a proposition which has already been alluded to, namely:

Proposition 6.1. *For any hyperbolic surface S, $\check{C}(S)$ is a flag complex. As a consequence every automorphism of the 1-skeleton can be extended to an automorphism of the full complex:*

$$Aut(\check{C}(S)) = Aut(\check{C}(S)^{(1)}).$$

Proof. Recall that a flag complex is a simplicial complex such that every clique is a simplex. That is if $\sigma = (v_i)_{i\in I}$ is a finite set of vertices such that every pair of elements of I defines an edge, then σ is a simplex. This is obviously the case of the discrete complex $C(S)$ but the preservation of this property under completion is in general a delicate question. Fortunately here we can take advantage of Theorem 4.5 (which itself constitute a highly nontrivial result), say for the trivial weight function. It then asserts that $\check{C}(S) \xrightarrow{\sim} \check{C}_{\mathcal{T}}(S)$ where the isomorphism is defined by mapping a simplex $\underline{\alpha} = \{\alpha_1, \ldots, \alpha_k\} \in \check{C}(S)$ to the closed free abelian group $G_{\underline{\alpha}} = \langle \tau_{\alpha_1}, \ldots, \tau_{\alpha_k}\rangle \subset \check{\Gamma}(S)$ generated by the corresponding protwists. It is then clear that this group is abelian if and only the twists are *pairwise* commuting which translates into the fact that $\check{C}_{\mathcal{T}}(S) \subset \mathcal{G}(\check{\Gamma}(S))$ is a flag complex, hence $\check{C}(S)$ as well. The second assertion of the proposition is an immediate corollary of the first.

□

So we have reduced our problem to studying the automorphisms of the graphs $\check{C}^{(1)}(S)$, $\check{C}_*(S)$ and $\check{C}_P(S)$. Now the analog of Lemma 2.8 holds true for the congruence graphs thanks to Proposition 5.12 i.e. the procongruence analog of Lemma 2.7. So we get:

Proposition 6.2. *Given the injection $\check{C}_P(S) \hookrightarrow \check{C}(S)$, there is a natural injection*

$$Aut(\check{C}_P(S)) \hookrightarrow Aut(\check{C}_*(S)).$$

□

Note that we favored the word 'injection' over 'inclusion' because it has a more dynamical flavor and the set of (not necessarily natural) injections $\check{C}_P(S) \hookrightarrow \check{C}(S)$ will play a leading role in topological Grothendieck-Teichmüller theory, as already noticed. In particular and in sharp constrast to what happens in the discrete case, the injective map of the proposition is *very far* from being an isomorphism. The next section (§7) will be devoted to determining the first group, namely $Aut(\check{C}_P(S))$.

As a next step and thanks to the reconstruction theorem, more accurately Corollary 5.14, we find that the automorphism groups of $\check{C}(S)$, or equivalently of its 1-skeleton $\check{C}(S)^{(1)}$ and that of $\check{C}_*(S)$ coincide for $d(S) > 1$. (The cases $d(S) = 0, 1$ are well-known; besides the two 0-dimensional complexes occurring for $d(S) = 1$ are isomorphic: $\check{C}(S_{0,4}) \simeq \check{C}(S_{1,1})$). We record this piece of information as:

Proposition 6.3. *For $d(S) > 1$, $Aut(\check{C}(S)) \simeq Aut(\check{C}_*(S))$.*

□

Our next result will require substantially more work. Recall that the type of a proloop, that is an element of $\check{\mathcal{L}}(S) = \check{C}(S)^{(0)}$, is well-defined, and more generally so is the type of any simplex $\sigma \in \check{C}(S)$. An automorphism $\phi \in Aut(\check{C}(S))$ is *type preserving* if it maps every simplex to one of the same type. In other words ϕ is type preserving if it preserves the $\check{\Gamma}(S)$-orbits: $\phi(\sigma) \in \check{\Gamma}(S) \cdot \sigma$ for every $\sigma \in \check{C}(S)$. Before stating our next result, we remark that it does *not* use any notion of topology, dealing in principle with automorphisms which respect the simplicial structure, not necessarily the profinite topology. However the notion of type itself does require more structure; in fact it has not even been defined in the full profinite setting, and for good reasons. So we keep the notation $Aut(\check{C}(S))$, denoting *continuous* simplicial automorphisms of $\check{C}(S)$, although some statements do not require continuity. We now state:

Theorem 6.4. *Let S be a connected hyperbolic surface; if S is* not *of type* $(1, 2)$, *every simplicial automorphism of $\check{C}(S)$ is type preserving. If $S = S_{1,2}$, an element of $Aut(\check{C}(S))$ is type preserving if and only if it preserves the set of separating curves.*

Note that here we assumed S to be connected for simplicity only. The statement for arbitrary hyperbolic surfaces is only slightly more involved and the extension is obvious. Moreover the statement is empty for $d(S) = 0, 1$ and these cases have been included only formally. From now on we assume that $d(S) > 1$. Then for $d(S) = 2$, either $S = S_{0,5}$ or $S = S_{1,2}$, with the exceptional isomorphism $C(S_{0,5}) \simeq C(S_{1,2})$ and ditto for the respective congruence completions. This gives rise to the exception recorded in the statement.

We will break the bulk of the proof into two lemmas and then complete the proof of the theorem. First, except for type $(1, 2)$, simplicial automorphisms preserve the set of separating curves.

Lemma 6.5. *Let S be connected hyperbolic, $d(S) > 1$, S not of type $(1, 2)$. Every simplicial automorphism $\phi \in Aut(\check{C}(S))$ maps a separating curve $\check{\alpha} \in \check{\mathcal{L}}(S)$ to a separating curve $\phi(\check{\alpha})$.*

Proof. Suppose that $\check{\alpha}$ were nonseparating whereas $\phi(\check{\alpha})$ is separating. From Lemma 5.2 $\phi(\check{\alpha})$ must be of boundary type. There exist $g, h \in \check{\Gamma}(S)$ such that both $\alpha = g \cdot \check{\alpha}$ and $\varphi(\alpha) = h \cdot \phi(\check{\alpha})$ are discrete curves. Moreover $\varphi = h \cdot \phi \cdot g^{-1}$ is also an automorphism of $\check{C}(S)$. Then the links of the vertices α and $\varphi(\alpha)$ in $\check{C}(S)$, namely $\check{C}(S_\alpha)$ and $\check{C}(S_{\varphi(\alpha)})$, should be isomorphic. From our assumptions S_α is of type $(g-1, n+2)$ while $S_{\varphi(\alpha)}$ is of type $(g, n-1)$. Then Theorem 5.1 implies that $(g, n) \in \{(1, 2), (1, 3)\}$.

In order to get rid of the case $(g, n) = (1, 3)$ we closely follow the proof of ([21], Lemma 2.2). Extend α to a pants decomposition $\{\alpha, \beta, \gamma\}$, where β and γ are non-separating. Then $(\varphi(\alpha), \varphi(\beta), \varphi(\gamma))$ is a 2-simplex of $\check{C}(S)$ and hence there exists $k \in \check{\Gamma}(S)$ such that $k \cdot \varphi(\alpha) = \varphi(\alpha)$ and $k \cdot \varphi(\beta)$, $k \cdot \varphi(\gamma)$ are discrete curves which form a pants decomposition of S. Now $\varphi(\alpha)$ bounds a subsurface $S_{1,2}$. Then, $k \cdot \varphi(\beta)$ and $k \cdot \varphi(\gamma)$ are contained in the subsurface $S_{1,2}$ and hence one of them, say $k \cdot \varphi(\beta)$, must be non-separating. Choose a simple curve which must be of the form $\varphi(\check{\delta})$ in $S_{1,2}$ disjoint from $k \cdot \varphi(\beta)$; it bounds a subsurface $S_{1,1}$ of $S_{1,2}$. Then $\varphi(\check{\delta}) \subset S_{1,3}$ is a separating curve not of boundary type. By the proof of Lemma 5.5, $\check{\delta}$ is a proloop which is separating and not of boundary type on $S_{1,3}$. On the other hand $(\varphi(\alpha), \varphi(\check{\delta}), \varphi(\beta))$ is a 2-simplex of $\check{C}(S)$ and hence $(\alpha, \check{\delta}, \beta)$ is also a 2-simplex. There exists $m \in \check{\Gamma}(S)$ such that $m \cdot \alpha = \alpha$, $m \cdot \beta = \beta$ and $m \cdot \check{\delta} = \delta$ are discrete curves on S. Moreover, α and β are non-separating, δ is separating not of boundary type, while α, β and δ are pairwise disjoint. This is impossible and the claim follows.

□

Next it turns out that the requirement in the statement of the theorem concerning the exceptional case $S = S_{1,2}$ is actually general. In this lemma we will deal with curves (loops), that is elements of $\check{\mathcal{L}}(S)$. It will then be easy to generalize this to multicurves i.e. arbitrary simplices of $\check{C}(S)$. So for the moment we state:

Lemma 6.6. *For any hyperbolic surface S, a simplicial automorphism which preserves the sets of separating classes of curves also preserves the type of the curves.*

Proof. We may and do assume that $d(S) > 1$. The proof follows the lines of ([21], Lemma 2.3). Let $\phi \in Aut(\check{C}(S))$ mapping separating elements of $\check{\mathcal{L}}(S)$ to such. By lemma 5.2, ϕ preserves the set of proloops of boundary type. Let then $\check{\alpha} \in \check{\mathcal{L}}(S)$ be separating not of boundary type. As in the previous lemma, after left and right composition of ϕ with elements of $\check{\Gamma}(S)$ we can assume that both $\check{\alpha} = \alpha$ and $\phi(\alpha)$ are discrete curves. The slit surfaces have two connected components: $S_\alpha = S^1_\alpha \cup S^2_\alpha$ and $S_{\phi(\alpha)} = S^1_{\phi(\alpha)} \cup S^2_{\phi(\alpha)}$, none of them of type $(0, 3)$.

Now, ϕ induces an isomorphism between the dual links $\phi : L^-_{\check{C}(S)}(\alpha) \to L^-_{\check{C}(S)}(\alpha)$. The proof of Lemma 5.2 shows that $L^-_{\check{C}(S)}(\alpha)$ is not chain connected and in fact it has exactly two connected components. A connected component $L^{-,j}_{\check{C}(S)}(\alpha)$ of $L^-_{\check{C}(S)}(\alpha)$ consists of those vertices of $L^-_{\check{C}(S)}(\alpha)$ corresponding to the proloops $\check{\gamma}$ on one connected component S^j_α of S_α. As ϕ preserves chain connectedness, it must send a connected component of $L^{-,j}_{\check{C}(S)}(\alpha)$ isomorphically to a connected component of $L^{-,j}_{\check{C}(S)}(\phi(\alpha))$. Observe now that $L^{-,j}_{\check{C}(S)}(\phi(\alpha))$ is the dual of a profinite curve graph, as it has the same set of vertices as $\check{C}^{(1)}(S^j_\alpha)$ while two vertices are adjacent in $L^{-,j}_{\check{C}(S)}(\phi(\alpha))$ if and only if they are not adjacent in $\check{C}^{(1)}(S^j_\alpha)$.

In particular ϕ induces isomorphisms $\check{C}^{(1)}(S^j_\alpha) \to \check{C}^{(1)}(S^j_{\phi(\alpha)})$. These are flag complexes by Proposition 6.1 and we get isomorphisms $\check{C}(S^j_\alpha) \to \check{C}(S^j_{\phi(\alpha)})$. We can use now Theorem 5.1 to derive that either α and $\phi(\alpha)$ have the same topological type or else:

(1) $S^1_\alpha = S^2_{\phi(\alpha)} = S_{1,1}$, $S^2_\alpha = S^1_{\phi(\alpha)} = S_{0,4}$;
(2) $S^1_\alpha = S^2_{\phi(\alpha)} = S_{1,2}$, $S^2_\alpha = S^1_{\phi(\alpha)} = S_{0,5}$;
(3) $S^1_\alpha = S_{1,1}$, $S^1_{\phi(\alpha)} = S_{0,4}$, $S^1_\alpha = S_{0,5}$, $S^2_{\phi(\alpha)} = S_{1,2}$.

None of these cases can occur since an isomorphism $\check{C}(S^1_\alpha) \to \check{C}(S^1_{\phi(\alpha)})$ will necessarily send a nonseparating curve β to a separating one of $S^1_{\phi(\alpha)}$, hence of $S_{\phi(\alpha)}$. This would contradict Lemma 6.5. The claim follows. □

End of proof of Theorem 6.4. We use induction on the dimension of the simplex $\sigma \in \check{C}(S)$. Lemma 6.6 yields the claim when the dimension 0. Assume it holds true up to dimension $k-1$ and let σ be a k-dimensional simplex. After composing ϕ on the left and on the right ϕ by two elements of $\check{C}(S)$ we may assume that $\sigma = (\alpha_0, \alpha_1, \ldots, \alpha_k)$, $\phi(\sigma) = (\beta_0, \beta_1, \ldots, \beta_k)$ are both discrete simplexes. By the induction hypothesis $\sigma' = (\alpha_0, \alpha_1, \ldots, \alpha_{k-1})$ and $\phi(\sigma') = (\beta_0, \beta_1, \ldots, \beta_{k-1})$ lie in the same $\check{\Gamma}(S)$-orbit. Indeed by Proposition 3.7, they are in the same $\Gamma(S)$-orbit, namely they have the same topological type. Composing further with an element of $\Gamma(S)$ we may assume that σ' is fixed pointwise: $\phi(\alpha_i) = \alpha_i$, $i = (0, 1, \ldots, k-1)$.

Denote by the same letters the traces of the curves α_k and β_k on the surface $S_{\sigma'}$ hyperbolic of dimension $d(S)-k$ but not necessarily connected. However Lemma 6.6 still holds true in that case (for obvious reasons), hence α_k and β_k have the same topological type on $S_{\sigma'}$. An element of $\Gamma(S_{\sigma'})$ sending α_k to β_k lifts to a mapping class in $\Gamma_{\sigma'}(S) \subset \Gamma(S)$ which maps σ to $\phi(\sigma)$, proving the claim and completing the proof of Theorem 6.4. □

6.2. **Automorphisms of goups and complexes.** In this short subsection we make the connection between group automorphisms on the one hand, automorphisms of complexes on the other. The next statement will serve to emphasize how computing the automorphisms of curve complexes enables one to study, not only the automorphism groups $Aut(\check{\Gamma}(S))$ of the procongruence modular groups, but indeed the groups $Aut(\check{\Gamma}^\lambda(S))$ for all values of $\lambda \in \Lambda$, that is the automorphism groups of the open subgroups of the procongruence modular groups. In the discrete setting the analogous statement comes from Theorem 2.4 and leads, via Theoreme 2.1 to the statement of Corollary 2.5. In the procongruence (or profinite) setting, we first define *inertia preserving* automorphisms just as in the discrete case, namely:

Definition 6.7. An element of $Aut(\check{\Gamma}(S))$ is *inertia preserving* if it globally preserves the set of procyclic subgroups generated by Dehn twists, that is maps a twist in $\check{\Gamma}(S)$ to a profinite power of a twist.

We denote again with an upperscript the subgroup $Aut^*(\check{\Gamma}^\lambda(S)) \subset Aut(\check{\Gamma}^\lambda(S))$ of the inertia preserving automorphisms. Here however the analog of Theorem 2.4, asserting that *every* automorphism preserves inertia, although conjectured to hold true, is not available. We only remark that this statement stands in close analogy with the so-called local correspondence of anabelian geometry. So we will deal explicitly with the subgroup of inertia preserving automorphisms and we do indeed restrict attention to *auto*morphisms, as opposed to the more general isomorphisms appearing in Corollary 2.5. This is purely for the sake of simplicity. The extension to isomorphisms would be easily available. In this context we have:

Proposition 6.8. *For every hyperbolic S and every congruence level $\lambda \in \Lambda$ there is a natural morphism:*

$$\gamma_\lambda : Aut^*(\check{\Gamma}^\lambda(S)) \to Aut(\check{C}(S)).$$

This morphism is injective *if $\Gamma^\lambda(S)$ has trivial center, thus in particular if $\Gamma(S)$ itself has trivial center.*

Proof. Note that in the last assertion we refer to the centers of the discrete groups, which actually coincide with those of the completed ones (cf. [4], Corollary 6.2). Since we are working with colored modular groups, the only exceptions are the types $(1,1)$ and $(2,0)$, in which cases the center is of order 2, generated by the hyperelliptic involution. One should also pay attention to the levels such that $\Gamma^\lambda(S) \subset \Gamma(S)$ contains this involution. These cases could easily be treated in detail but we refrain to do so here.

The main remark and the main point in this proof consists in the fact that given $\phi \in Aut^*(\check{\Gamma}^\lambda(S))$ one can assign to every simplex $\underline{\alpha} \in \check{C}(S)$ an image in a coherent way, thereby defining $\gamma_\lambda(\phi) \in Aut(\check{C}(S))$. This is a direct consequence of Proposition 4.4. Let again $G_{\underline{\alpha}} \subset \check{\Gamma}(S)$ denote the commutative subgroup topologically generated by the (pro)twists along the (pro)curves attached to the vertices of $\underline{\alpha}$, and let $U^\lambda_{\underline{\alpha}} = G_{\underline{\alpha}} \cap \check{\Gamma}^\lambda(S)$. Then $U^\lambda_{\underline{\alpha}}$ is open in $G_{\underline{\alpha}}$ and by Proposition 4.4, for a simplex $\underline{\beta} \in \check{C}(S)$, the intersection $U^\lambda_{\underline{\alpha}} \cap U^\lambda_{\underline{\beta}}$ is open in $U^\lambda_{\underline{\alpha}}$ if and only if $\underline{\alpha} \subset \underline{\beta}$.

So given $\phi \in Aut^*(\check{\Gamma}^\lambda(S))$ it makes senses to define $\tilde{\phi} = \gamma_\lambda(\phi) \in Aut^*(\check{\Gamma}^\lambda(S))$ via the formula:

$$\phi(U^\lambda_\sigma) = U_{\tilde{\phi}(\sigma)},$$

which is valid for every simplex $\sigma \in \check{C}(S)$ and every congruence level $\lambda \in \Lambda$. One should pay attention to the exact meaning of this formula. Indeed on the right-hand side $U_{\tilde{\phi}(\sigma)}$ denotes a kind of "generic"' open subgroup of the group $G_{\tilde{\phi}(\sigma)}$. It is asserted, in accordance with the above, that there exists a *unique* simplex $\tilde{\phi}(\sigma) \in \check{C}(S)$ such that the left-hand side, namely $\phi(U_\sigma^\lambda)$, is open in $G_{\tilde{\phi}(\sigma)}$; this property *defines* $\tilde{\phi} = \gamma_\lambda(\phi)$.

We have thus defined a map γ_λ for every $\lambda \in \Lambda$. It is actually easy to see that this a coherent family with respect to the level λ. More precisely consider $\mu \geq \lambda$, so that $\Gamma^\mu(S) \subset \Gamma^\lambda(S)$ and assume that $\check{\Gamma}^\mu(S)$ is invariant (characteristic) in $\check{\Gamma}^\mu(S)$ (recall that these groups are topologically finitely generated, so that invariant subgroups are cofinal). Then there is a natural restriction map $\rho_{\lambda,\mu} : Aut^*(\check{\Gamma}^\lambda) \to Aut^*(\check{\Gamma}^\mu)$ and it is clear that $\gamma_\lambda = \gamma_\mu \circ \rho_{\lambda,\mu}$.

We finally address the issue of the injectivity of the map γ_λ. We use the natural action of $\check{\Gamma}(S)$ on $\check{C}(S)$, which defines an *injective* map $Inn(\check{\Gamma}^\lambda(S)) \hookrightarrow Aut(\check{C}(S))$. Moreover, for every $\lambda \in \Lambda$, $\phi \in Aut^*(\check{\Gamma}^\lambda(S))$, $g \in \check{\Gamma}^\lambda(S)$ and $\sigma \in \check{C}(S)$, we find that:

$$\phi(g)(\sigma) = \tilde{\phi} \circ g \circ \tilde{\phi}^{-1}(\sigma)$$

(with $\tilde{\phi} = \gamma_\lambda(\phi)$). If $\Gamma^\lambda(S)$ is centerfree, so is $\check{\Gamma}^\lambda(S)$ as mentioned above i.e. $\check{\Gamma}^\lambda(S) = Inn(\check{\Gamma}^\lambda(S)))$. Then if $\tilde{\phi} = id$ the formula above implies that $\phi(g)g^{-1} \in Z(\check{\Gamma}^\lambda(S))$ hence $\phi(g) = g$ for all $g \in \check{\Gamma}^\lambda(S)$; in other words $\phi = id$, proving injectivity and completing the proof.

□

6.3. **The arithmetic Galois action.** We remark now that the above makes it possible to define a *faithful* arithmetic Galois action on the completed curve complex. We will stick here to the basic and most important case, namely the action of $G_{\mathbb{Q}}$, the absolute Galois group of the field $\mathbb{Q}$, on the curve complex. Recall that for S hyperbolic connected, the (Deligne-Mumford) moduli stack $\mathcal{M}(S)$ is defined over $\mathbb{Q}$, hence a natural outer action $G_{\mathbb{Q}} \to Out(\hat{\Gamma}(S))$. Here the full profinite completion $\hat{\Gamma}(S)$ actually stands for the geometric étale fondamental group: $\hat{\Gamma}(S) = \pi_1(\mathcal{M}(S) \otimes \bar{\mathbb{Q}})$. Very little is known about this action but two pieces of information are quite relevant here. First it is inertia preserving, as initially showed by A.Grothendieck and J.Murre (see [24] for references and much more on this and related topics); second it is faithful for $d(S) > 0$ as a consequence of Belyi's theorem. It is easy to see that this action descends to the congruence quotient $\check{\Gamma}(S)$ and remains faithful, because in particular $\check{\Gamma}(S) = \hat{\Gamma}(S)$ for $d(S) \leq 5$. Moreover the outer action can be (non canonically) lifted to a bona fide action by picking a (possibly tangential) rational basepoint on the moduli stack $\mathcal{M}(S)$. All in all, after picking a rational basepoint we get a faithful inertia preserving action $G_{\mathbb{Q}} \hookrightarrow Aut^*(\check{\Gamma}(S))$ for $d(S) > 0$ (we again refer to [24] for much more background, references, etc.). By composing with the map γ ol Proposition 6.8 ($\gamma = \gamma_\lambda$ for λ the trivial level: $\Gamma^\lambda(S) = \Gamma(S)$) we get:

Proposition 6.9. *Let S be connected hyperbolic with $d(S) > 0$; then there is a map:*

$$G_{\mathbb{Q}} \hookrightarrow Aut^*(\check{C}(S))$$

which is injective and canonical up to composition with the action of $Inn(\check{\Gamma}(S))$ on $\check{C}(S)$.

For $d(S) > 1$ the same holds true for the graph $\check{C}_*(S)$.

Proof. The possible composition by an inner automorphism of $\check{\Gamma}(S)$ comes from the choice of a rational basepoint. In other words the proposition asserts the existence of a natural faithful outer action of $G_{\mathbb{Q}}$ on $\check{C}(S)$. Here the only thing which requires proof is the injectivity in the two cases (types $(1,1)$ and $(2,0)$) where $\Gamma(S)$ has nontrivial center. But the kernel of the map γ of Proposition 6.8 is then generated by an involution. Now any involution in $G_{\mathbb{Q}}$ is conjugate to complex conjugacy so that it is enough to check that the image of this latter element is not central; but this is clear since it corresponds to a reflection of the surface. We thus find that the image of $G_{\mathbb{Q}}$ in $Aut(\check{\Gamma}(S))$ does not intersect the kernel of γ, which completes the proof for the curve complex.

The last assertion comes either from the reconstruction theorem or from the fact that, for $d(S) > 1$, $\check{C}_*(S)$ identifies with the 1-skeleton of the dual of $\check{C}(S)$.

□

It should be stressed that we get a faithful action of the arithmetic Galois group on a profinite *space*, whereas it is more common to get an action on a profinite *group*, which itself arises as a cohomological or homotopical invariant of an underlying "classical" space. Actually, given – say – a geometrically connected scheme defined over $\mathbb{Q}$, one can make its étale covers into a p(r)oset by considering a (pro)point in the (pro)-universal cover, then let $G_{\mathbb{Q}}$ act on this proset, much as is done with "dessins d'enfants". These correspond to the type $(0,4)$ and simply give a "pictionary" of the finite (étale, that is here simply unramified) covers of $\mathcal{M}_{0,4}(\mathbb{C})$, alias $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$ alias $\mathbb{C} \setminus \{0, 1\}$; recall also the isomorphism $C(S_{0,4}) \simeq C(S_{1,1})$ and that these complexes are 0-dimensional. The resulting Galois action is indeed faithful as an easy corollary of

Belyi's theorem, but it is essentially no easier to study than the usual action on the geometric fundamental group. Here curve complexes retain a kind of homotopical information at infinity from the tower of covers of the moduli stacks in all dimensions and are much more amenable to a direct study.

The above Galois action readily extends to an action of the Grothendieck-Teichmüller group $\mathbb{I}\Gamma$ as defined in [15] (see also [31]) for background material) essentially by the very definition of this group which however we skip here, as it would lead us too far afield. For the sake of clarity, we record this explicitly as

Proposition 6.10. *Let S be connected hyperbolic with $d(S) > 0$; then there is a map:*

$$\mathbb{I}\Gamma \to Aut^*(\check{C}(S))$$

which is canonical up to composition with the action of $Inn(\check{\Gamma}(S))$ on $\check{C}(S)$ and is injective if $\Gamma(S)$ is centerfree. For $d(S) > 1$ the same holds true for the graph $\check{C}_*(S)$.

□

Note that here we cannot *a priori* exclude the existence of a nontrivial kernel in the two cases when $\Gamma(S)$ has nontrivial center. It may be useful to remind the reader that there is a nested sequence of profinite groups:

$$G_{\mathbb{Q}} \subset \mathbb{I}\Gamma \subset \widehat{GT} \subset Aut^*(\hat{F}_2),$$

where $\widehat{GT}$ is the original "genus 0" Grothendieck-Teichmüller group introduced by V.Drinfeld, $\mathbb{I}\Gamma$ is the version adapted to all genera constructed in [15] and [31], whereas $F_2 = \mathbb{Z} * \mathbb{Z}$ denotes the free group on 2 generators. For any S as in the proposition, there is also an injective map $\mathbb{I}\Gamma \to Aut^*(\check{\Gamma}(S))$ giving rise to a canonical injection $\mathbb{I}\Gamma \hookrightarrow Out^*(\check{\Gamma}(S))$. If S has genus 0 we can enlarge $\mathbb{I}\Gamma$ to $\widehat{GT}$ both here and in Proposition 6.10, that is both in the group and complex theoretic frameworks. Finally it is essential that both Propositions 6.9 and 6.10 are "badly" wrong for the pants complex $\check{C}_P(S)$, as will become clear in the next section. We also refer the reader to [22, 23] for much more on these and related topics.

7. Rigidity of the procongruence pants complex

7.1. **Main result.** We now turn to the study of the automorphism group of the procongruence pants graph $\check{C}_P(S)$, where S is hyperbolic; we assume S is connected for simplicity of exposition; all morphisms between profinite objects are assumed to be continuous (see above, beginning of §6). We will return to a more careful elucidation of the definitions in the next paragraph (§7.2). Actually the results in this section hardly depend on the type of completion, provided it is fine enough, in particular residually finite. So let us denote by a prime ($'$) a completion which sits between the procongruence and the full profinite one. In other words we pick an inverse system of levels (cofinite subgroups of $\Gamma = \Gamma(S)$) which contains the congruence system Λ and of course is contained in the full system M (see §A.10). The reader who is willing to make life simpler or lighter is welcome to elect Λ and stick to the congruence completion. Indeed for ease of notation, below we will refer to our fixed inverse system as Λ.

There are natural epimorphisms:

$$\hat{\Gamma} \twoheadrightarrow \Gamma' \twoheadrightarrow \check{\Gamma},$$

and ditto for the other completed objects. The group Γ' is residually finite (i.e. there is a natural embedding $\Gamma \hookrightarrow \Gamma'$) since $\check{\Gamma}$ is. Of course if the congruence conjecture holds true (which we do *not* assume here) all three completions coincide. We will also be interested in the respective centers of these profinite group. This is known only for the conguence completion (see [4]) : one has $Z(\check{\Gamma}) = Z(\Gamma)$ so that $Inn(\check{\Gamma}) = (Inn(\Gamma))^{\vee}$.

Now recall there is an action of $\Gamma(S)'$ on $C_P(S)'$, giving rise to an injective map:

$$Inn(\Gamma(S)') \hookrightarrow Aut(C_P(S)').$$

In complete parallel with what happens in the discrete setting but in sharp contrast with the case of the curve complex $C(S)'$, we will show that this map is almost surjective. Note that we get $(Inn(\Gamma(S))'$ on the left-hand side of the short sequence of the theorem, rather than $Inn(\Gamma(S)')$. As mentioned above these groups coincide in the procongruence case. We now state the main result of this section as

Theorem 7.1. *For every S connected hyperbolic there is a split short exact sequence:*

$$1 \to (Inn(\Gamma(S))' \to Aut(C_P(S)') \to \mathbb{Z}/2 \to 1.$$

In other words and more concisely :

$$Aut(C_P(S)') = \big(Aut(C_P(S))\big)'.$$

In other words $Out(C_P(S)') \simeq \mathbb{Z}/2$, like in the discrete case, and the sequence is split by complex conjugacy (see below for more detail). Before going to the proof proper, we start by exploring the low dimensional cases. In dimension 0, $S = S_{0,3}$, all complexes are empty and there is nothing to prove. If $d(S) = 1$, the type is $(0,4)$ or $(1,1)$, the congruence conjecture holds true and we are dealing with with $\hat{C}_P(S_{1,1}) \simeq \hat{C}_P(S_{0,4}) \simeq \hat{F}$, the profinite Farey tesselation. We should also recall that we identify the classical Farey *tesselation* with the – no less classical – Farey *graph*, which is dual to the 1-skeleton of F, with one vertex inside every triangle of the tesselation.

What about the attending modular groups? Here we should use the full, non colored groups, slightly abusing notation by writing $S_{g,n}$ rather than the more correct $S_{g,[n]}$, so that $\Gamma(S_{g,n}) = \Gamma_{g,[n]}$ (see §A.4). Let $Z = \langle \iota \rangle \simeq \mathbb{Z}/2$ be generated by the (hyper)elliptic involution ι. Then Z is the center of $\Gamma_{1,1}$, $\Gamma_{1,[2]}$ and Γ_2 (with their respective involutions); all the other groups $\Gamma_{g,[n]}$ have trivial center and so do the other completions $\check{\Gamma}_{g,[n]}$ (but we do not know whether e.g. $\hat{\Gamma}_3$ has trivial center).

Returning to dimension 1, $\Gamma_{0,[4]} \simeq \Gamma_{1,1}/Z \simeq PSL_2(\mathbb{Z})$ where we use the quotient of $\Gamma_{0,[4]}$ which acts effectively; for the stacky phenomenon involved here, see §8.1 below. So on the left-hand side of the exact sequence in the statement of the theorem we find the profinite completion of the ubiquitous group $PSL_2(\mathbb{Z})$. The one-dimensional case of the result can thus be stated as follows:

Proposition 7.2. *One has:*

$$Aut(\widehat{F}) \simeq (\widehat{PGL_2(\mathbb{Z})}).$$

The right-hand factor in the attending short exact sequence is generated by the class of the matrix $diag(-1,1)$. We will prove this proposition in §7.3 below, not only as a warmup but also because it contains a large part of the essence of the general higher dimensional case.

We close this subsection with a short review of the two and three dimensional cases, where the congruence conjecture has been vindicated. In dimension 2, $\Gamma_{1,[2]} = \Gamma_{1,2} \times Z$ (direct product) and $\Gamma_{1,[2]}/Z = \Gamma_{1,2} \subset \Gamma_{0,[5]}$ of index 5 and self-normalizing (see §A.4). This takes care of the left-hand side of the exact sequence in dimension 2. Recall that the pants graphs for types $(0,5)$ and $(1,2)$ are *not* isomorphic (cf. Prop. 5.15). Finally in dimension 3, the congruence conjecture is still valid, the Teichmller groups $\Gamma_{0,6}$ and $\Gamma_{1,3}$ have trivial centers and we just record the fact that $Z(\Gamma_2) \simeq \mathbb{Z}/2$ (generated by the hyperelliptic involution) with $\Gamma_2/Z \simeq \Gamma_{0,[6]}$. For all the other types (g,n) the centers of the discrete and procongruence groups are trivial: $Inn(\check{\Gamma}(S_{g,n})) = \check{\Gamma}_{g,[n]}$.

7.2. **Morphisms of pro-objects.** In this short paragraph we depart from our specific situation in order to solidify the fundations a bit, making contact with homotopy theory and recalling how pro-objects are handled. In view of the relative concreteness of our situation we need only a few basic inputs, to be found essentially in the Appendix to [2]. For more and a more modern approach in the framework of model categories, see [33, 34] and their references.

First recall some vocabulary: an inverse or projective limit ($\varprojlim$) is just a limit, an inductive one ($\varinjlim$) is a colimit, an inverse or projective system is a particular case of a cofiltering category. We use a generic Λ, which in our case concretely denotes any inverse system which is finer than the one defining the congruence completion. To make the connection with categories, simply declare that there is a (unique) morphism $\lambda \to \mu$ if and only if $\lambda \geq \mu$. Now indeed for every pair λ, μ of elements of Λ there exists $\nu \in \Lambda$ such that $\nu \to \lambda$ and $\nu \to \mu$. Next we consider pro-objects (rather than just limits) associated with the category of finite simplicial complexes (one could use simplicial sets instead, which are easier to deal with, but this is unimportant here). These are given as coherent collections $X = (X^\lambda)_{\lambda \in \Lambda}$, equivalently as maps $\Lambda \to X$, where the X^λ's are simplicial complexes, in our case $C^\lambda(S) = C(S)/\Gamma^\lambda(S)$, $C_P^\lambda(S) = C_P(S)/\Gamma^\lambda(S)$ or the like. These maps are made into functors by defining morphisms between two pro-objects X and Y:

$$Hom(X,Y) = colim_\mu \ lim_\lambda \ Hom(X^\lambda, Y^\mu).$$

Here beware of the fact that we are considering a cofiltering category Λ as the primary object, rather than the opposite filtering category Λ^o. So we get a contravariant functor as usual but limits and colimits are swapped. In any case this essentially amounts to describing the respective variances of "source" and "image" by chains of morphisms $\lambda' \to \lambda \to \mu \to \mu'$ (i.e. $\lambda' \geq \lambda \geq \mu \geq \mu'$ in the case of inverse systems). We remark that the homotopy theorist or the practitioner of tensor calculus could justly complain that we are not always careful enough, in terms of variance, in the use of sub- and superscripts . In particular it would be more sensible to write Γ_λ rather than Γ^λ throughout this paper but bad (including typographical) reasons have unfortunately prevailed.

The basic and useful result we need essentially says that by a clever reindexing we can bring a morphism of pro-objects to a more manageable form, namely

Proposition 7.3. *let $f : X \to Y$ be a map of pro-objects in a small category C, with indexing cofiltering category Λ ; then it can be represented, up to isomorphism, by an inverse system of maps $(\phi^\lambda : X^\lambda \to Y^\lambda)_{\lambda\in\Lambda}$. Moreover f is invertible if and only if there exists a system $(\psi^\lambda : X^\lambda \to Y^\lambda)_{\lambda\in\Lambda}$ with $\psi_\lambda \circ \phi_\lambda = id$.*

Proof. This is essentially Corollary 3.2 in the Appendix of [2]. Here "represent" has a precise technical meaning (see *loc. cit.*) which however can be safely ignored by the reader. This statement is an especially useful but rather particular case of Proposition 3.3 (*ibidem*; see also [SGA 4]). Note that the system of maps $(\phi^\lambda)_\lambda$ can be viewed as a pro-object in the category of maps in C, which helps unraveling the definition of $pro - C$, the procategory built from C. The addition on invertibility in the statement comes as a particular case of Scholie 3.5, *loc. cit.*. □

Let us come back to our favorite objects, namely $C(S)$, $C_*(S)$, $C_P(S)$ or variants like the arc complexes which are important ingredients of the theory - see [23] - although they do not appear in this paper. The definition of the topology on the attending completions ($\hat{C}(S)$ etc.), hence also of the continuity of maps, implies that continuous maps come from maps between the corresponding pro-objects ($(C^\lambda(S))_{\lambda\in\Lambda}$ etc.). Note that here everything started from the (near simplicial) action of a discrete group (essentially $\Gamma(S)$) on simplicial objects ; one can work in broader contexts but again we stick here to our relatively basic and concrete needs.

So in our case we have a pro-object $X = (X^\lambda)_{\imath\in\Lambda}$ with a limit $\hat{X}$ which is compact (and completely discontinuous). Then we consider an endomorphism ϕ of the pro-object X, corresponding to a continuous endomorphism of $\hat{X}$ (with the same name). We remark that the map ϕ is any case open since it is continuous, $\hat{X}$ is compact and every open set is closed. But anyway we assume that ϕ is an automorphism, so is invertible (hence indeed open !) with inverse ψ. Applying Proposition 7.3, we see that we can assume that we have two systems of maps $(\phi^\lambda, \psi_\lambda : X^\lambda \to Y^\lambda)_{\lambda\in\Lambda}$ between finite simplicial complexes, with $\psi_\lambda = \phi_\lambda^{-1}$ $(\psi_\lambda \circ \phi_\lambda = \phi_\lambda \circ \psi_\lambda = id)$.

After these preliminaries we may now return to our situation.

7.3. **The one-dimensional case.** In this subsection we detail the geometry of the one-dimensional case and prove Prop. 7.2 above. Since $C_P(S_{0,4}) = C_P(S_{1,1}) = F$ we may and will restrict attention to $S = S_{0,4}$. Then $\mathcal{M}_{0,4} \cong \mathbb{P}^1 \setminus \{0, 1, \infty\}$ which for ease of notation we denote $\mathcal{M}$ in this paragraph. Up to a stacky phenomenon which needs not concern us here (see §8.1 below or [24] for much more detail) the (rigidified) version of $\mathcal{M}_{0,[4]}$ is given by the quotient $\mathcal{M}_{0,4}/\mathcal{S}_3$. Here $\mathcal{S}_4$ permutes the 4 punctures and acts effectively via $\mathcal{S}_3 = \mathcal{S}_4/V$, where V is the Klein 4-group. This is summarized by the short exact sequence

$$1 \to \Gamma_{0,4} \to \Gamma_{0,[4]} \to \mathcal{S}_3 \to 1.$$

In terms of groups $\Gamma_{0,4} \simeq F_2$, the free group on two generators, and the sequence identifies with

$$1 \to \Gamma_{0,4} \cong F_2 \to PSL(2, \mathbb{Z}) \to PSL(2, \mathbb{Z}/2\mathbb{Z}) \cong \mathcal{S}_3 \to 1.$$

Making things again slightly simpler and more transparent we may and will work with levels, that is cofinite subgroups of $\Gamma_{0,[4]} \cong PSL(2, \mathbb{Z})$, which are contained in $\Gamma_{0,4} = F_2 = \ker(PSL(2, \mathbb{Z}) \to PSL(2, \mathbb{Z}/2\mathbb{Z}))$; note that this exhibits $\Gamma_{0,4}$ as a *principal* congruence subgroup. In fact recall that much more generally, for any S of genus g we denote by $\Gamma^{(m)}(S)$ the abelian level of order m of $\Gamma(S)$, namely $\ker(\Gamma(S) \to Sp(2g, \mathbb{Z}/m\mathbb{Z}))$ and by $\mathcal{M}^{(m)}$ the corresponding moduli space. Then $\Gamma_{0,4} = \Gamma^{(2)}_{0,[4]}$ and $\mathcal{M} = \mathcal{M}_{0,4} = \mathcal{M}^{(2)}_{0,[4]}$. In other words we work with covers of $\mathcal{M}$ and not only of $\mathcal{M}/\mathcal{S}_3$. Note also that this way we got rid of all the stacky (or obifold) phenomena, elementary as these may be in this very particular case.

Returning to a bit of geometry, we start again with $\mathcal{M} \cong \mathbb{P}^1 \setminus \{0, 1, \infty\}$. Then $\overline{\mathcal{M}} \simeq \mathbb{P}^1$ with boundary divisor $\partial\mathcal{M} = \{0, 1, \infty\}$. We denote the analytic version also $\mathcal{F} = \overline{\mathcal{M}}(\mathbb{C}) = \mathbb{P}^1(\mathbb{C})$ anticipating on the higher dimensional cases to be tackled below. Next $\mathcal{F} = \mathbb{P}^1(\mathbb{C})$ is naturally triangulated into two triangles, say black and white, by the two closed hemispheres, with vertices $\{0, 1, \infty\}$. The common boundary is the equator, i.e. the set $\overline{\mathcal{M}}(\mathbb{R})$ of the real points inside $\overline{\mathcal{M}}(\mathbb{C})$ or more algebraically the set of fixed points of the complex conjugacy, generating the Galois group $Gal(\mathbb{C}/\mathbb{R})$. Now the attending Teichmüller space $\mathcal{T}_{0,4} \simeq \mathcal{T}_{1,1}$ (see §A.2) is the Poincaré upper half-plane $\mathcal{H}$ or equivalently (i.e. up to a fractional transformation) the Klein disk $\mathcal{D}$, equipped with the respective actions of $PSL_2(\mathbb{Z})$. Lifting the ideal triangulation of $\mathcal{M}$ via the natural projection

$$\mathcal{T}_{0,4} = \mathcal{H} \to \mathbb{P}^1 \setminus \{0, 1, \infty\} = \mathcal{M}(= \mathcal{M}_{0,4})$$

we get the classical bicolored Farey tesselation F of the upper half-plane $\mathcal{H}$ or the disk $\mathcal{D}$. We also get the projective system of finite complexes $(\mathcal{F}^\lambda)_{\lambda\in\Lambda}$ defined by all the finite covers of $\mathcal{M}$. Here we should clarify notation a bit. First we use the name $\mathcal{F}^\lambda$, anticipating again on the higher dimensional cases; next we call the projective system Λ because anyway the congruence conjecture is true in this case ($\Lambda = M$ in the notation of §A.10); last we stick to the letter Λ in spite of the fact that we consider only the levels corresponding to covers of $\mathcal{M}$. The (stack or orbifold) quotient $\mathcal{M}_{0,[4]} \simeq \mathcal{M}/\mathcal{S}_3$ corresponds to just *one* triangle of the Farey tesselation, with appropriate identifications of all three vertices (resp. edges).

Proof of proposition 7.2. Let $\phi \in Aut(\hat{F})$, a continuous automorphism. It is given (see beginning of §6) by a compatible system of maps $\phi_{\lambda\mu} : F^\lambda \to F^\mu$, with $F^\lambda = F/\Gamma^\lambda \cong C_P(S)/\Gamma^\lambda$ (ditto for μ), writing $\Gamma = \Gamma(S_{0,4}) \cong F_2$. Here F^λ is given either as a graph or as an ideal triangulation of $\mathcal{M}^\lambda$, or else as an ordinary triangulation (tesselation) of the completion $\overline{\mathcal{M}}^\lambda$. Recall also the canonical projection $p_\lambda : \mathcal{M}^\lambda \to \mathcal{M}$, which is an unramified cover.

Translation into a a more algebraic language: $\mathcal{M}^\lambda = X^\lambda$ is an algebraic curve which is actually defined over $\overline{\mathbb{Q}}$, the projection $p_\lambda : X^\lambda \to X^0 = \mathbb{P}^1 \setminus \{0, 1, \infty\} = \mathcal{M}$ is an (algebraic) Belyi map (also defined over $\overline{\mathbb{Q}}$) determining an ideal triangulation of X^λ. It extends to a map (still p_λ) $\overline{\mathcal{M}}^\lambda = \overline{X}^\lambda \to \overline{X}^0 = \overline{\mathcal{M}}$ between the respective completions. As a piece of notation, we will consistently denote the "ground level" with a zero, writing $\Gamma^0 = \Gamma$, $\mathcal{M}^0 = \mathcal{M}$, $\overline{\mathcal{M}}^0 = \overline{\mathcal{M}}$, etc. The projective algebraic curve $\overline{X}^\lambda$ has singularities ("nodes") lying over the points 0, 1 and ∞ for the projection p_λ. Its normalization $\tilde{\overline{X}}^\lambda$ is a smooth, not necessarily connected, projective algebraic curve, also defined over $\overline{\mathbb{Q}}$ as well as every connected component. One can view the (dual of) the triangulation of $\overline{X}^\lambda$ as a "dessin d'enfant" drawn on the curve viewed as a topological surface and rigidifying the situation entirely. For this translation we refer to [37] and many other papers. Note that we are *not* using the astonishing part of Belyi's theorem, which asserts that *every* algebraic curve defined over $\overline{\mathbb{Q}}$ arises in this way. For instance the fact that X^λ is affine algebraic defined over $\overline{\mathbb{Q}}$ was known to A.Weil, as well as everything that is mentioned above (see [39] which introduces the notion of "descent" and contains all the necessary material – and more).

Recalling Proposition 7.3, we see that ϕ is actually determined (after reindexing and renaming) by a coherent system of *invertible* maps $\phi_\lambda : F^\lambda \to F^\lambda$. We may for a moment consider such a map as a differentiable and invertible morphism of $\mathcal{M}^\lambda$ to itself, well-defined up to isotopy. Every vertex, resp. edge, resp. face (triangle) of the tesselation is mapped to another one, which determines a diffeomorphism up to isotopy. Actually much more is true but is not needed here. To wit the map ϕ_λ *a priori* permutes the connected components of the normalization $\tilde{\overline{X}}^\lambda$ and on each of these components it will be either algebraic or "antialgebraic" i.e. algebraic after composing with the complex conjugacy (which is a well-defined involution), according to whether it preserves or inverts the orientation determined by the complex structure (this being a combinatorial invariant). But again this *a priori* knowledge is not really necessary.

Now for every $\lambda \in \Lambda$ there is a canonical projection $\mathcal{H} \to \mathcal{M}^\lambda$ and $\mathcal{H}$ is simply connected ; indeed it is the universal cover of $\mathcal{M}^\lambda$ for every $\lambda \in \Lambda$. So ϕ_λ lifts to $\tilde{\phi}_\lambda : \mathcal{H} \to \mathcal{H}$, preserving the Farey tesselation F of $\mathcal{H}$. The lift $\tilde{\phi}_\lambda$ is well-defined up to the left action of Γ^λ. This implies that $\tilde{\phi}_\lambda$ is (isotopic to) an element of $PGL_2(\mathbb{Z}) \cong Aut(F)$. It belongs to $PSL_2(\mathbb{Z})$ if and only if the original map ϕ_λ is orientation preserving which is now seen to be a *global* property, that is independent of the choice of a connected component of $\tilde{\overline{X}}^\lambda$. We thus get a compatible system of elements $\tilde{\phi}_\lambda \in PGL_2(\mathbb{Z})$, that is an element of $\widehat{PGL_2(\mathbb{Z})} \cong \widehat{PSL_2(\mathbb{Z})} \rtimes \mathbb{Z}/2$, which completes the proof of Prop. 7.2.

□

Returning for a moment to the more general maps $\phi_{\lambda\mu}$ $(\lambda, \mu \in \Lambda; \lambda \geq \mu)$ and from what we just showed these are either all holomorphic or all antiholomorphic. In fact, assuming the étale cover $\mathcal{M}^\lambda \to \mathcal{M}^\mu$ is Galois (recall this defines a cofinal subsystem of Λ), up to a possible twist by complex conjugacy, the map $\phi_{\lambda\mu}$ can be chosen as belonging to the Galois group of this cover.

Two remarks in closing : First this one-dimensional rigidity result rests in an essential way on the rigidity of the projective line, equivalently of the thrice punctured complex sphere. Namely there is one and only one conformal structure up to isotopy ($\mathcal{T}_{0,3} = \{*\}$) and every diffeomorphism is isotopic either to the identity or to complex conjugacy ($Mod(S_{0,3}) = \mathbb{Z}/2$). The depth of these by now old and basic results should thus not be underestimated (cf. Schönflies theorem). Second and for the more topologically inclined reader we note that part (not all) of what we have done as well as part of what we do below can be expressed in another language, namely that of the so-called *flat surfaces* (see e.g. [38] and references therein).

7.4. **Fulton curves and the pants complex.** In order to tackle the higher dimensional cases we need to add in some more geometry, as will be done in this and the next subsection. Here we relate $C_P(S)$ and its completed versions with an important geometric object. Starting from our usual connected hyperbolic surface S ($d(S) \geq 1$) we consider again the attached moduli space $\mathcal{M} = \mathcal{M}(S)$, viewed here as a complex orbifold (see §A.2), $\overline{\mathcal{M}}$ the stable (Bers-Deligne-Mumford) compactification of $\mathcal{M}$. The boundary divisor $\partial\mathcal{M} = \overline{\mathcal{M}} \setminus \mathcal{M}$ classifies Riemann surfaces with nodes of the same type as S, or one-dimensional proper complex D-M stack with quadratic singularities and finite groups of automorphisms, in a more algebraic language. Passing to a level structure $\lambda \in \Lambda$, that is a representable étale cover $\mathcal{M}^\lambda \to \mathcal{M}$, we then get a compactification $\overline{\mathcal{M}}^\lambda$ with divisor at infinity $\partial\mathcal{M}^\lambda = \overline{\mathcal{M}}^\lambda \setminus \mathcal{M}^\lambda$. We now define a curve, or rather a one-dimensional orbifold (D-M stack) $\mathcal{F}^\lambda$:

Definition 7.4. Let S be hyperbolic connected of modular dimension $d = d(S)$. The one-dimensional orbifold $\mathcal{F}(S) \subset \overline{\mathcal{M}}(S)$ is such that its (closed) points represent Riemann surfaces (curves) with *at least* $d-1$ nodes (quadratic singularities). For an arbitrary level $\lambda \in \Lambda$ we let $\mathcal{F}^\lambda(S) \subset \overline{\mathcal{M}}^\lambda(S)$ denote the preimage of $\mathcal{F}(S)$ via the canonical projection $\overline{\mathcal{M}}^\lambda(S) \to \overline{\mathcal{M}}(S)$.

In other words $\mathcal{F} = \mathcal{F}(S)$ is nothing but the *closure of the one-dimensional stratum* in the stable stratification of $\overline{\mathcal{M}} = \overline{\mathcal{M}}(S)$. A complex point of $\mathcal{F}$ represents an algebraic curve which is a stable graph of copies of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, save perhaps for an irreducible component of type $(0, 4)$ or $(1, 1)$. When $d(S) = 1$, that is $d - 1 = 0$, $\mathcal{F}$ coincides with $\overline{\mathcal{M}}$. As soon as $d(S) > 1$, $\mathcal{F}$ is contained in the boundary $\partial\mathcal{M}$ and more generally $\mathcal{F}^\lambda \subset \partial\mathcal{M}^\lambda$. The importance of this one-dimensional stratum in the stratification of $\overline{\mathcal{M}}$ was first recognized in connection with a conjecture formulated by W.Fulton, hence the notation (see e.g. [12]).

Each irreducible component of $\mathcal{F}$ is (isomorphic to) a moduli space of dimension 1 and can be triangulated as above into two triangles. Lifting that triangulation to the corresponding Teichmüller space $\mathcal{T} = \mathcal{T}(S)$ produces again a copy of the Farey tesselation F. It is bicolored and complex conjugacy permutes the colors of the triangles. On the other hand, for any level $\lambda \in \Lambda$, one gets a cover $\mathcal{F}^\lambda \to \mathcal{F}$, which ramifies at most over points representing curves with the maximal ($= d$) number of nodes (singularities), that is graphs of trinions. The triangulation of $\mathcal{F}$ thus lifts uniquely to $\mathcal{F}^\lambda$. Moreover, and this is where the connection between $\mathcal{F}(S)$ and $C_P(S)$ comes in, it is easily seen that $\mathcal{F}^\lambda(S)$ is naturally isomorphic to $C_P^\lambda(S) = C_P(S)/\Gamma^\lambda$, after identifying as usual the Farey graph with the corresponding tesselation. In slightly more detail, it is enough to show this for the trivial level, and then lift the result to every $\lambda \in \Lambda$. Moreover this is a local assertion, in the sense that we can fix $d-1$ curves, after which we are reduced to the one-dimensional situation of the last subsection. We record this as

Lemma 7.5. *For every $\lambda \in \Lambda$, $\mathcal{F}^\lambda$ is a compact stable orbifold curve (i.e. a complex one-dimensional proper D-M stack with nodal singularities and finite group of automorphisms). It is equipped with a natural bicolored tesselation, whose dual graph is isomorphic to $C_P^\lambda(S) = C_P(S)/\Gamma^\lambda$* □

Note that here again the tesselation and its dual graph carry the same information which is the seed of the transformation of *combinatorial* properties into *analytic* ones. As for completions, because of Proposition 5.8 and the validity of the congruence conjecture in dimension 1, the induced topology from $C_P(S)'$ on a one-dimensional piece is the full profinite topology. In other words, fix $\sigma \in C(S)$ a $(d-1)$-dimensional simplex (multicurve); then because here we allow permutations of points, $\Gamma(S_\sigma) \simeq PSL_2(\mathbb{Z})$, where $\Gamma(S_\sigma)$ denotes the Teichmüller group of S cut along σ, *modulo the boundary twists*. Inside the finite simplicial complex we thus find a copy of $F/(\Gamma^\lambda \cap \Gamma(S_\sigma))$ and as λ increases, the limit yields a copy of the completion $\hat{F}$ of the Farey graph inside $C_P(S)'$ for every $(d-1)$-dimensional simplex of $\sigma \in C(S)$, equipped with an action of $\widehat{PSL_2(\mathbb{Z})}$.

7.5. **Spherical blow-ups.** We will need one more geometrical ingredient. We fix again S of type (g, n) and assume that $d(S)(= 3g - 3 + n) > 1$. The point is to "thicken" the boundary $\partial\mathcal{M}^\lambda$ of a level structure $\mathcal{M}^\lambda$ and construct a real tubular neighborhood; the construction will also apply to the Teichmüller space $\mathcal{T}$. One could proceed topologically, more precisely simplicially, as in [14], Chapter 2 (and references therein). Here we adopt a more algebraic viewpoint. Details can be found in [3]].

The idea is childishly simple. Consider the real plane $\mathbb{R}^2$, mark a point, say the origin, and regard it as a puncture. Then dig a hole there, that is replace the point by a tiny disk. This gives a tubular neighborhood of the point with a circle ($\cong \mathbb{P}^1(\mathbb{R})$) as boundary. The topology remains unchanged (from a homotopical viewpoint). That is basically all we do, in all dimensions and a more algebraic language.

Definition 7.6. Let $Res_{\mathbb{C}/\mathbb{R}}$ denote the restriction-of-scalars functor from $\mathbb{C}$ to $\mathbb{R}$. We define the spherical (real oriented) blow-up$\widehat{\mathcal{M}}^\lambda$ of $\overline{\mathcal{M}}^\lambda$ along $\partial\mathcal{M}^\lambda$ as the real analytic stack with corners obtained after cutting along the exceptional divisor the ordinary blow-up of $Res_{\mathbb{C}/\mathbb{R}}\overline{\mathcal{M}}^\lambda$ along $Res_{\mathbb{C}/\mathbb{R}}\partial\mathcal{M}^\lambda$.

Recall that $\partial\mathcal{M}^\lambda$ is a divisor with normal crossings of the smooth compact complex orbifold $\overline{\mathcal{M}}^\lambda$. Moreover, for λ large enough (dominating suitable Looijenga levels; see [5]) $\overline{\mathcal{M}}^\lambda$ is a smooth projective variety, which can be assumed henceforth along with the fact that $\partial\mathcal{M}^\lambda$ has *strict* normal crossings, i.e. its irreducible components are smooth.

As a local model choose coordinates around $0 \in \mathbb{C}^n$, belonging to the divisor $\{z_1 \cdot z_2 \cdots z_k = 0\}$, $1 \leq k \leq n$. Then the polar coordinates map

$$r : (\mathbb{R}_{>0} \times S^1)^k \times \mathbb{C}^{n-k} \to (\mathbb{C}^*)^k \times \mathbb{C}^{n-k}$$

given by

$$r((r_1, \theta_1), \ldots, r_k, \theta_k), z_{k+1}, \ldots, z_n)) = (r_1 e^{i\theta_1}, \ldots, r_k e^{i\theta_k}, z_{k+1}, \ldots, z_n)$$

extends *ad literam* to a real analytic map

$$r : (\mathbb{R}_{\geq 0} \times S^1)^k \times \mathbb{C}^{n-k} \to \mathbb{C}^n$$

which defines the spherical blow-up of $\mathbb{C}^n$ along this divisor. As can be readily inferred from the explicit form of the map r, the restriction to a complex codimension k smooth stratum of the divisor is a fibration with fiber a k-dimensional torus $\mathbb{T}^k \cong (S^1)^k$.

Note that $\widehat{\mathcal{M}}^\lambda$ is a real analytic orbifold (or manifold for large λ) with *corners*, being locally a cartesian product of manifolds with boundary. Nevertheless the boundary singularities of the PL manifold $\widehat{\mathcal{M}}^\lambda$ has a canonical smooth structure. It is diffeomorphic to the complement $\overline{\mathcal{M}}^\lambda \setminus \mathcal{N}(\partial\mathcal{M}^\lambda)$ of a tubular neighborhood $\mathcal{N}(\partial\mathcal{M}^\lambda)$ of $\partial\mathcal{M}^\lambda$ in $\overline{\mathcal{M}}^\lambda$; in turn the "boundary" $\partial\widehat{\mathcal{M}}^\lambda = \widehat{\mathcal{M}}^\lambda \setminus \mathcal{M}^\lambda$ is diffeomorphic to the boundary of $\mathcal{N}(\partial\mathcal{M}^\lambda)$.

The local picture globalizes so that there is a natural retraction $r : \widehat{\mathcal{M}}^\lambda \to \overline{\mathcal{M}}^\lambda$, such that its restriction $r : r^{-1}(\mathcal{M}^\lambda) \to \mathcal{M}^\lambda$ is a real analytic isomorphism whereas $r|_{r^{-1}(\partial\mathcal{M}^\lambda)} : r^{-1}(\partial\mathcal{M}^\lambda) \to \partial\mathcal{M}^\lambda$ is a fibration with fiber $\mathbb{T}^k$ over every complex codimension k stratum of $\partial\mathcal{M}^\lambda$.

Observe now that the inclusion $\mathcal{M}^\lambda \hookrightarrow \overline{\mathcal{M}}^\lambda$ can be lifted to an embedding $\mathcal{M}^\lambda \hookrightarrow \widehat{\mathcal{M}}^\lambda$, which moreover is a homotopy equivalence. Since there is a cofinal sequence of congruence Galois levels λ for which $\partial\mathcal{M}^\lambda$ is strict normal crossing, we derive that for *every* level λ the inclusion $\mathcal{M}^\lambda \hookrightarrow \widehat{\mathcal{M}}^\lambda$ induces an isomorphism on the the *orbifold* fundamental groups. In particular the orbifold fundamental group of $\widehat{\mathcal{M}}$ is the group $\Gamma = \Gamma(S)$ (whereas $\overline{\mathcal{M}}$ has trivial fundamental group). All this however is not quite necessary for our purpose, since we can always confine attention to high enough levels λ.

Passing to Teichmüller space, denote by $\widehat{\mathcal{T}}$ the (orbifold) universal cover of $\widehat{\mathcal{M}}$. Then $\Gamma(S)$ acts with finite stabilizers on $\widehat{\mathcal{T}}$, with quotient $\widehat{\mathcal{M}}$. We refer to [3] for more details. The description of the space $\widehat{\mathcal{T}}$ (or a variant thereof) is actually at the origin of the introduction of the curve complex $C(S)$ by W. Harvey. Indeed $C(S)$ is nothing but the nerve of the covering of $\partial\widehat{\mathcal{T}}$ by its irreducible components. Since moreover these components and their mutual intersections are contractible, $\partial\widehat{\mathcal{T}}$ is homotopically equivalent to the geometric realization of $C(S)$.

A central object in the sequel is the preimage $\widehat{\mathcal{F}}^\lambda \subset \partial\widehat{\mathcal{M}}^\lambda$ of $\mathcal{F}^\lambda \subset \partial\mathcal{M}^\lambda$ via the natural retraction map $r : \widehat{\mathcal{M}}^\lambda \to \overline{\mathcal{M}}^\lambda$. We also let $\widehat{\mathcal{H}} \subset \partial\widehat{\mathcal{T}}$ denote the preimage of $\widehat{\mathcal{F}} \subset \partial\widehat{\mathcal{M}}$ via the universal covering map $\widehat{\mathcal{T}} \to \widehat{\mathcal{M}}$. For $S = S_{g,n}$ the natural map $r|_{\widehat{\mathcal{F}}^\lambda} : \widehat{\mathcal{F}}^\lambda \to \mathcal{F}^\lambda$ is generically, i.e. over the open one-dimensional stratum, a fibration with fiber a torus $\mathbb{T}^{3g-4+n}$.

Turning to a finite Galois level structure λ, Γ^λ acts on $\widehat{\mathcal{T}}$, the map $\partial\widehat{\mathcal{T}} \to \partial\widehat{\mathcal{T}}/\Gamma^\lambda$ is orbifold Galois unramified with Galois group Γ^λ and $\partial\widehat{\mathcal{T}}/\Gamma^\lambda$ is isomorphic to $\partial\widehat{\mathcal{M}}^\lambda$. If λ sits high enough (e.g. dominates an abelian level of level $m \geq 2$) the action of Γ^λ on $\widehat{\mathcal{T}}$ is free and the word "orbifold" becomes spurious. We collect this into the following:

Lemma 7.7. *$\widehat{\mathcal{T}}/\Gamma^\lambda$ is isomorphic to $\widehat{\mathcal{M}}^\lambda$ and $\partial\widehat{\mathcal{T}}/\Gamma^\lambda$ is isomorphic to $\partial\widehat{\mathcal{M}}^\lambda$.*

Let us briefly mention the connection between the Harvey compactification or bordification $\widehat{\mathcal{T}}$ as above and the classical Bers bordification $\overline{\mathcal{T}}$ of the Teichmüller space $\mathcal{T}$, that is the Teichmüller space of stable nodal curves. The action of $\Gamma(S)$ also extends to $\overline{\mathcal{T}}$ with quotient $\overline{\mathcal{T}}/\Gamma(S) = \overline{\mathcal{M}}$. The connection is given by a natural proper comparison map $\widehat{\mathcal{T}} \to \overline{\mathcal{T}}$ which restricts to a comparison map between the Harvey and Bers boundaries $\partial\widehat{\mathcal{T}} \to \partial\overline{\mathcal{T}}$ (see [3] for details).

The structure of the Harvey boundary $\partial\widehat{\mathcal{T}}(S_{g,n})$ is actually quite simple and intimately connected with the real analytic Fenchel-Nielsen coordinates. An irreducible component has the form $\widehat{\mathcal{T}}(S_{g-1,n+2}) \times \mathbb{R}$ or else it is a product $\widehat{\mathcal{T}}(S_{g_1,n_1}) \times \widehat{\mathcal{T}}(S_{g_2,n_2}) \times \mathbb{R}$, with $g_1+g_2=g, n_1+n_2=n$ or $\widehat{\mathcal{T}}(S_{g-1,n+2}) \times \mathbb{R}$.

On the other hand the components of the Bers boundary $\partial\overline{\mathcal{T}}$ are of the form $\overline{\mathcal{T}}(S_{g-1,n+2}) \times \mathbb{R}$ or else $\overline{\mathcal{T}}(S_{g_1,n_1}) \times \overline{\mathcal{T}}(S_{g_2,n_2}) \times \mathbb{R}$, with $g_1+g_2=g, n_1+n_2=n$. Generically the map $\partial\widehat{\mathcal{T}} \to \partial\overline{\mathcal{T}}$ is the retraction along the real axis $\mathbb{R}$ which covers a circle in the corresponding component of $\partial\widehat{\mathcal{M}}$.

We define $\mathcal{H}$ as the image of $\widehat{\mathcal{H}}$ in the Bers boundary $\partial\overline{\mathcal{T}}$ of $\overline{\mathcal{T}}$. When restricting the comparison map over the complex one-dimensional stratum $\mathcal{H}$ we are generically contracting a $(3g-4+n)$-dimensional real cone. Turning to a finite Galois level structure $\mathcal{M}^\lambda$ we find that, since $\partial\widehat{\mathcal{T}} \to \partial\widehat{\mathcal{M}}^\lambda$ is a Galois (orbifold unramified) cover with group Γ^λ, $C^\lambda(S) = C(S)/\Gamma^\lambda$ is the nerve of the covering of $\partial\widehat{\mathcal{M}}^\lambda$, or in fact of $\partial\mathcal{M}^\lambda$ itself, by its irreducible components.

7.6. **From automorphisms of the pants complex to lifting maps between Fulton curves.** Going back to the proof of Theorem 7.1, we start with an automorphism $\phi \in Aut(C_P(S)')$, assuming for definiteness that $d(S) > 1$ so that $\mathcal{F}(S) \subset \partial\mathcal{M}(S)$. We follow the one-dimensional case proof, with an additional complication stemming essentially from the fact that now $\mathcal{H} \subset \partial\overline{\mathcal{T}}$ is *not* simply connected.

Applying Proposition 7.3, ϕ can be represented as an inverse system of invertible simplicial maps (simplicial automorphisms)

$$\phi_\lambda : C_P^\lambda(S) \to C_P^\lambda(S)$$

between finite graphs ($\lambda \in \Lambda$). The first part of the proof is completely identical to its one-dimensional version, for the good and simple reason that the $\mathcal{F}^\lambda(S)$'s are one-dimensional whatever S. More precisely, we know (from Lemma 7.5) that $C_P^\lambda(S) = C_P(S)/\Gamma^\lambda$ is the dual graph of a natural bicolored tesselation of $\mathcal{F}^\lambda = \mathcal{F}^\lambda(S)$ and we are back to the situation of §7.3 above, with $\mathcal{F}^\lambda$ in lieu of $\overline{X}^\lambda$. Appealing again e.g. to [37], we find that this determines a complex structure on $\mathcal{F}^\lambda$ for every $\lambda \in \Lambda$ ($\mathcal{F}^0 = \mathcal{F}$) and in fact these are again projective algebraic curves defined over $\overline{\mathbb{Q}}$. Note that we may restrict to λ large enough (say dominating an abelian level of level > 2) so that $\mathcal{F}^\lambda$ is indeed a *bona fide* curve, as opposed to an orbifold curve (one-dimensional DM stack); one could accommodate the stack structure just as well. In fact this complex structure concides with the one inherited from the fact that $\mathcal{F}^\lambda \subset \partial\mathcal{M}^\lambda$ is nothing but the closure of the one-dimensional stratum of the stratified variety (more correctly DM stack) $\overline{\mathcal{M}}^\lambda$. To see this it is enough to consider the case of the ground level $\mathcal{F}^0 = \mathcal{F} \subset \partial\mathcal{M}$, the point being that Γ acts isometrically on the Teichmüller space $\mathcal{T}$ equipped with the Teichmüller metric. Returning to the curve $\mathcal{F}^\lambda$ we can proceed exactly as in §7.3 and arrive at the following statement:

Lemma 7.8. *Given a simplicial automorphism $\phi_\lambda : C_P^\lambda(S) \to C_P^\lambda(S)$, it determines a real analytic map (with the same name) $\phi_\lambda : \mathcal{F}^\lambda \to \mathcal{F}^\lambda$, which on every analytically irreducible component of the projective curve $\mathcal{F}^\lambda$ is either holomorphic or antiholomorphic.* □

We leave it to the reader to further enhance this statement. Basically all the objects are algebraic and defined over numberfields and so are the morphisms, *a priori* possibly after twisting by complex conjugacy. All this however is not needed below. In fact we will use much *less*. It would be enough for instance to promote the ϕ_λ given as simplicial automorphisms of graphs to – say – piecewise differentiable maps between the $\mathcal{F}^\lambda$ viewed as differentiable surfaces (with easily controlled singularities). This may serve to illustrate a phenomenon which very much struck Grothendieck and a handful of others in the early eighties, namely the natural rigidification of a situation, under certain circumstances, all the way from topology to arithmetic geometry.

The next and in fact last step of the proof, which does not occur in dimension 1, consists in lifting the maps ϕ_λ to (the boundary of) Teichmüller space. This is where the process of thickening the boundary, as described in the previous subsection, comes in. First for every level $\lambda \in \Lambda$ there is a canonical isomorphism $\widehat{\mathcal{F}}^\lambda \simeq \mathcal{F}^\lambda \times_{\mathcal{F}} \widehat{\mathcal{F}}$, from which it follows that there is a canonical lift of $\phi_{\lambda\mu}$ to a real analytic morphism $\hat{\phi}_\lambda : \widehat{\mathcal{F}}^\lambda \to \widehat{\mathcal{F}}^\lambda$. Moreover, the projection $\hat{q}_\lambda : \widehat{\mathcal{H}} \to \widehat{\mathcal{F}}^\lambda$ is an unramified cover with Galois group Γ^λ. Therefore, for any given Galois level λ, there is a short exact sequence:

$$1 \to \pi_1(\widehat{\mathcal{H}}) \to \pi_1(\widehat{\mathcal{F}}^\lambda) \to \Gamma^\lambda \to 1.$$

Here both $\widehat{\mathcal{H}}$ and $\widehat{\mathcal{F}}^\lambda$ are manifolds (or orbifolds) with corners and their fundamental group denotes the topological version or possibly Thurston's orbifold refinement, although we can actually assume that $\lambda \in \Lambda$ sits high enough for $\widehat{\mathcal{F}}^\lambda$ to be an ordinary manifold (surface). We sometimes skip this proviso in the sequel. Fixing $\lambda \in \Lambda$, the map $\hat{\phi}_\lambda : \widehat{\mathcal{F}}^\lambda \to \widehat{\mathcal{F}}^\lambda$ yields the composition $\hat{\phi}_\lambda \circ \hat{q}_\lambda : \widehat{\mathcal{H}} \to \widehat{\mathcal{F}}^\lambda$. The last point in the proof of Theorem 7.1 consists in ascertaining the following lifting property:

Proposition 7.9. *For every $\lambda \in \Lambda$ the map $\hat{\phi}_\lambda \circ \hat{q}_\lambda : \widehat{\mathcal{H}} \to \widehat{\mathcal{F}}^\lambda$ lifts to $\hat{h}_\lambda : \widehat{\mathcal{H}} \to \widehat{\mathcal{H}}$. Equivalently:*

$$\hat{\phi}_{\lambda *}(\pi_1(\widehat{\mathcal{H}})) \subseteq \pi_1(\widehat{\mathcal{H}}).$$

Proof. The equivalence stated in the proposition is clear since the projection $\hat{q}_\lambda$ is unramfied. Note that this came for free in the one-dimensional case since then $\mathcal{H}$ is the whole of Teichmüller space, to wit Poincaré upper half-plane, thus simply connected (in fact contractilble).

Return to the short exact sequence above for a (high enough) Galois level $\lambda \in \Lambda$, namely again:

$$1 \to \pi_1(\widehat{\mathcal{H}}) \to \pi_1(\widehat{\mathcal{F}}^\lambda) \to \Gamma^\lambda \to 1.$$

In particular $\pi_1(\widehat{\mathcal{H}}) \subseteq \pi_1(\widehat{\mathcal{F}}^\lambda)$ for every such λ; moreover, the groups $\pi_1(\widehat{\mathcal{F}}^\lambda)$ form a decreasing nested sequence, that is $\pi_1(\widehat{\mathcal{F}}^{\lambda'}) \subseteq \pi_1(\widehat{\mathcal{F}}^\lambda)$ for $\lambda' \geq \lambda$ two levels as above, because the natural projection $\widehat{\mathcal{F}}^{\lambda'} \to \widehat{\mathcal{F}}^\lambda$ is finite unramified (and Galois). So we find that

$$\pi_1(\widehat{\mathcal{H}}) \subseteq \bigcap_{\lambda \in \Lambda} \pi_1(\widehat{\mathcal{F}}^\lambda).$$

Let us denote the right-hand side by K. To show that equality prevails it is enough to go back once again to the short exact sequence and recall that $\cap_{\lambda\in\Lambda}\Gamma^\lambda = \{1\}$ because the completion Γ' is finer than the congruence completion, hence residually finite (the topology is Hausdorff; note that this has nothing to do with the congruence subgroup conjecture). So in fact $\pi_1(\widehat{\mathcal{H}}) = K$. On the other hand, since the $\pi_1(\widehat{\mathcal{F}}^\lambda)$'s form a nested sequence $\hat{\phi}_{\lambda *}(K) \subseteq K$, proving the proposition.

In fact again equality prevails since the inclusion holds true for the inverse map $\hat{\phi}_{\lambda *}^{-1}$ just as well, so that $\hat{\phi}_{\lambda *}(\pi_1(\widehat{\mathcal{H}})) = \pi_1(\widehat{\mathcal{H}})$. In short we have actually proved that

$$\pi_1(\widehat{\mathcal{H}}) = \bigcap_{\lambda \in \Lambda} \pi_1(\widehat{\mathcal{F}}^\lambda)$$

and

$$\hat{\phi}_{\lambda *}(\pi_1(\widehat{\mathcal{H}})) = \pi_1(\widehat{\mathcal{H}})$$

for every high enough Galois level $\lambda \in \Lambda$

□

7.7. **Proof of Theorem 7.1.** Recall that $\overline{\mathcal{T}}$ denotes the Bers bordification of the Teichmüller space $\mathcal{T}$ and that $\mathcal{H}$ is the image of $\widehat{\mathcal{H}}$ in the Bers boundary $\partial\overline{\mathcal{T}}$ of $\overline{\mathcal{T}}$. By Proposition 7.9 we get a real analytic map $\hat{h}_\lambda : \widehat{\mathcal{H}} \to \widehat{\mathcal{H}}$.

On the other hand, by construction the map $\hat{\phi}_\lambda : \widehat{\mathcal{F}}^\lambda \to \widehat{\mathcal{F}}^\mu$ respects the singular toric fibration $\widehat{\mathcal{F}}^\lambda \to \mathcal{F}^\lambda$. Then its lift $\hat{\phi}_\lambda$ to $\widehat{\mathcal{H}}$ also respects the singular fibration $\widehat{\mathcal{H}} \to \mathcal{H}$, as this property is of local nature. It therefore descends to a real analytic $h_\lambda : \mathcal{H} \to \mathcal{H}$, lifting the original $\phi_\lambda : \mathcal{F}^\lambda \to \mathcal{F}^\lambda$. Furthermore h_λ respects the natural tesselation of $\mathcal{H}$, a property which is again of local nature and can be readily checked on the restriction of h_λ to any copy of the Poincaré upper half-plane sitting inside $\mathcal{H}$, since h_λ lifts ϕ_λ.

All in all we find that h_λ determines an automorphism of the tesselation of $\mathcal{H}$, thus indeed an automorphism of the pants graph $C_P(S)$. We now appeal to Margalit's rigidity result ([25] or Theorem 2.13 above) asserting that such an automorphism is induced by an element of $Mod(S)$. By varying $\lambda \in \Lambda$ along the projective system defining the completion prime ($'$), we get the assertion of Theorem 7.1.

□

We end this paragraph with two remarks. First the significance of this rigidity result lies in its being the seed of a profinite version of Grothendieck-Teichmüller theory, much in the spirit of the *Esquisse*. One should insist again that the completed *curve* complex, say $\check{C}(S)$, is far from rigid. Indeed its automorphism group appears as a version of the Grothendieck-Teichmüller group and thus contains the Galois group $Gal(\mathbb{Q})$; it is independent of S for S "generic" enough.

This underlines the fact that we followed a slightly paradoxical strategy. In fact Theorem 7.1 is proved by finally appealing to the rigidity of the discrete pants complex $C_P(S)$. But in turn this result is shown by D. Margalit (in [M]) by appealing to the rigidity of the discrete *curve* complex ([19, 21] or Theorem 2.1 above). In other words, picking again the congruence completion for definiteness, we find that in general $Aut(C_P(S)) = Aut(C(S))$, $Aut(\check{C}_P(S)) = \big(Aut(C_P(S))\big)^{\vee}$ but $Aut(\check{C}(S))$ is wildly different from $\big(Aut(C(S))\big)^{\vee}$. This is actually the seed of a kind of "profinite deformation theory", which is a somewhat bizarre expression since deformation theory usually requires something like a linearization process and the introduction of "infinitesimal" objects. In genus 0 and for the *prounipotent* (not profinite) completion this is tightly connected to the "deformation of associativity", say in braided tensor categories (see the end of §7.3 above for a very first indication).

8. On some anabelian properties of the moduli stacks of curves

The main goal of this closing section is to demonstrate how the objects and techniques of this paper, more specifically the rigidity result of the last section, can serve to show properties with strong anabelian flavor about the moduli stacks of curves. We will first state and prove a relatively "raw" version which should make the essence of the matter clear. We then say a few words about the background and context of anabelian geometry; finally we will enlarge and refine the first result by adding in some necessary but easy details.

In this last section, we naturally adopt the language of algebraic geometry. In particular we consider the $\mathcal{M}_{g,[n]}$'s $(2g-2+n>0)$ as Deligne-Mumford stacks defined over $\mathbb{Q}$ (in fact over $\mathbb{Z}$ but we will not need this deep fact). The square brackets ($[n]$) indicate that the n marked points are *not* labeled (cf. §A.2) and there is a finite stack étale (orbifold unramified in the language of complex geometry) Galois cover $\mathcal{M}_{g,n} \to \mathcal{M}_{g,[n]}$ with group $\mathcal{S}_n$, the permutation group on n objects. One retrieves a model of the spaces used above via $\mathcal{M}(S_{g,n}) = (\mathcal{M}_{g,n} \otimes \mathbb{C})^{an}$ (idem replacing n with $[n]$), that is, $\mathcal{M}(S_{g,n})$ is (isomorphic to) the anaytification of the $\mathbb{C}$-stack $\mathcal{M}_{g,n} \otimes \mathbb{C}$.

8.1. **Galois invariant automorphisms of procongruence modular groups.** So we will start with a purely group theoretic statement, which however requires an anabelian result for its proof in the one-dimensional case. Let $\Gamma_{g,[n]} = \Gamma(S_{g,[n]})$ denote the full modular group, $\Gamma_{g,n} = \Gamma(S_{g,n})$ its colored subgroup, the first one being an extension of $\mathcal{S}_n$ by the second. Let $G_{\mathbb{Q}}$ denote as usual the Galois group of $\mathbb{Q}$. There is a natural outer action of $G_{\mathbb{Q}}$ on the full profinite completion $\hat{\Gamma}_{g,[n]}$ (because $\mathcal{M}_{g,[n]}$ is defined over $\mathbb{Q}$) which descends to the congruence completion $\check{\Gamma}_{g,[n]}$ (because it respects the principal congruence levels). The same holds true for the colored objects (replacing $[n]$ with n). We pick once and for all a $\mathbb{Q}$-rational tangential basepoint "at infinity", that is based on the boundary of the stable completion $\overline{\mathcal{M}}_{g,[n]}$, and use it to lift the outer action to a proper one. For $\sigma \in G_{\mathbb{Q}}$ we denote again by σ the induced element of $Aut^*(\hat{\Gamma}_{g,[n]})$ where we recorded in the upper script the already mentioned fact that the arithmetic Galois action is inertia preserving, that is preserves the set of profinite Dehn twists, or if one prefers, the set of conjugacy classes of (ordinary) Dehn twists. Again we may replace the full profinite completion by the procongruence one and possibly restrict to the colored group.

For two elements ϕ, ψ of a group Γ we denote as usual their commutator by $(\phi, \psi) = \phi\psi\phi^{-1}\psi^{-1} \in \Gamma$. We then make the following definition:

Definition 8.1. If Γ is a group equipped with an action of another group G, we say that $\phi \in Aut(\Gamma)$ is G-invariant if for any $\sigma \in G$, $(\sigma, \phi) \in Inn(\Gamma)$, that is ϕ commutes with the action of G modulo inner automorphisms. The subgroup of G-invariant automorphisms is denoted $Aut_G(\Gamma) \subset Aut(\Gamma)$.

In the case where the groups are equipped with a topology (here the profinite topology), all morphisms are naturally assumed to respect it, i.e. be continuous. Note that $Inn(\Gamma) \subset Aut_G(\Gamma)$ because the subgroup of inner automorphisms is normal in $Aut(\Gamma)$. Of course we are mostly interested in the case $\Gamma = \hat{\Gamma}_{g,[n]}$, $G = G_{\mathbb{Q}}$ and variants thereof. We will subsequently reinject some geometry but the statement of our basic result in its purely group theoretic formulation is now quite short, namely:

Theorem 8.2. *For every hyperbolic type* (g, n)*:*

$$Aut^*_{G_{\mathbb{Q}}}(\check{\Gamma}_{g,[n]}) = Inn(\check{\Gamma}_{g,[n]}).$$

Proof. Let us abbreviate $\Gamma_{g,[n]}$ to Γ and let $\phi \in Aut^*_{G_{\mathbb{Q}}}(\check{\Gamma})$ be a Galois invariant automorphism. Like any inertia preserving automorphism, it defines a continuous automorphism of $\check{C}(S)$ (see Proposition 6.8 or Proposition 6.9), so in particular of the proset of the vertices (0-simplices) of the graph $\check{C}_P(S)$, as these correspond to the top dimensional simplices of the curve complex $\check{C}(S)$. *If* we can prove that $\phi \in Aut(\check{C}_P(S))$ and is orientation preserving, then applying the rigidity theorem (Theorem 7.1) will complete the proof. To be complete one should note that type $(1,2)$ is no exception here. Indeed in order to show that ϕ is type preserving in that case, it is enough to show that it preserves the set of non separating (pro)curves. But ϕ comes from a *group* homomorphism and in $\check{\Gamma}_{1,[2]}$ the centralizers of separating and non separating twists are not isomorphic, which implies that indeed the induced automorphism on $\check{C}(S)$ is type preserving. So there remains to show that a Galois invariant automorphism of $\check{\Gamma}$ induces an automorphism of $\check{C}(S)$ which stabilizes the pants graph $\check{C}_P(S)$.

Let us first consider in some detail the one-dimensional case, that is (g,n) is either $(0,4)$ or $(1,1)$. First recall that $\mathcal{M}_{0,4} \simeq \mathbb{P}^1 \setminus \{0,1,\infty\}$, so that $\Gamma_{0,4}$ is free on two generators, whereas $\mathcal{M}_{0,[4]}$ is a gerbe, whose generic fibre is the constant gerbe associated with Klein's *Vierergruppe* $V \cong \mathbb{Z}/2 \times \mathbb{Z}/2$, over the stack quotient of $\mathcal{M}_{0,4}$ by the natural action of the symmetric group $\mathcal{S}_3$. One has $\pi_1^{orb}(\mathcal{M}_{0,4}/\mathcal{S}_3) \simeq PSL_2(\mathbb{Z})$ (orbifold π_1). This occurs because $\mathcal{S}_4$ does not act faithfully on $\mathcal{M}_{0,4}$ as a scheme or variety. So we have an exact sequence:

$$1 \to V \to \pi_1^{orb}(\mathcal{M}_{0,[4]} \to PSL_2(\mathbb{Z}) \to 1$$

and the right-hand group is itself given as an extension:

$$1 \to F_2 \to PSL_2(\mathbb{Z}) \to \mathcal{S}_3 \to 1$$

where $F_2 = \Gamma_{0,4} = \pi_1^{top}(\mathcal{M}_{0,4})$ denotes the free group on two generators (topological π_1; here $\pi_1^{top} = \pi_1^{orb}$).

As for $\mathcal{M}_{1,1}$, it identifies with the modular curve, with orbifold fundamental group $\Gamma_{1,1} \cong SL_2(\mathbb{Z})$. That is $\pi_1^{orb}(\mathcal{M}_{1,1})$ is given by an extension:

$$1 \to \mathbb{Z}/2 \to \pi_1^{orb}(\mathcal{M}_{1,1}) \to PSL_2(\mathbb{Z}) \to 1.$$

Now $\mathcal{M}_{0,4}$ covers both $\mathcal{M}_{0,[4]}$ and $\mathcal{M}_{1,1}$ where in the latter case, the 4 marked points identify with the Weierstrass points of the elliptic curve classified by a $\mathbb{C}$-point of $\mathcal{M}_{1,1}$. This shows, and it will appear more clearly below in a much more general setting (see §8.3), that we only need to treat the case of $\mathcal{M}_{0,4}$. The assertion can then be recast as (see again §8.3):

$$Out_{G_{\mathbb{Q}}}(\hat{\Gamma}_{0,4}) = \mathcal{S}_3,$$

where we used that $\check{\Gamma}_{0,4} = \hat{\Gamma}_{0,4}$ and $\mathcal{S}_3$ permutes the points 0, 1 and ∞, identifiying $\mathcal{M}_{0,4}$ with $\mathbb{P}^1 \setminus \{0,1,\infty\}$.

It is a truly remarquable fact that this is precisely the statement of perhaps the first anabelian result, as proved by H.Nakamura (first in the pro-ℓ, then in the profinite setting) around 1990. We refer the reader to [29] (and references therein) and will elaborate a bit on the subject in §8.2 below. So the one-dimensional case of our basic result is taken care of by the most basic (but important) anabelian result. In terms of complexes we know that $C_P(S_{1,1}) = C_P(S_{0,4}) = F$ (the Farey tesselation), that the respective congruence completions are isomorphic and they are again isomorphic to the full profinite completions. In a word there is juste one object here, namely $\hat{F}$, the profinite completion of F with respect to the action of $F_2 = \Gamma_{0,4}$ (recall that completions are insensitive to passing to cofinite subgroups). Moreover, from Theorem 7.1 or from a more direct proof, we find that $Aut(\hat{F}) = Inn(\hat{\Gamma}_{1,1})$, the latter group being isomorphic to the profinite completion of $PSL_2(\mathbb{Z})$. Recalling the exact sequence above, in order to prove the one-dimensional case of the theorem *without* appealing to (one-dimensional) anabelian geometry, one would indeed have to prove directly that a Galois invariant automorphism induces an automorphism of the profinite Farey tesselation. Clearly it would be extremely interesting to find an independent proof of this deep fact.

We now proceed to reduce the general case to the one-dimensional case. We fix the type (g,n) of S and assume that the modular dimension $d = d_{g,n} = 3g-3+n$ is > 1. Given $\phi \in Aut^*_{G_{\mathbb{Q}}}(\check{\Gamma}(S))$, it induces an automorphism of $\check{C}(S)$ (still denoted ϕ) and we have to show that for every edge $e \in \check{C}_P(S)^{(1)}$ with vertices v_0 and v_1, the images $\phi(v_0)$ and $\phi(v_1)$ are connected by an edge $e' = \phi(e) \in \check{C}_P(S)$. We first observe that this property is $\check{\Gamma}$-invariant for the natural action of $\check{\Gamma}$ on $\check{C}(S)$, which restricts to the action on $\check{C}_P(S)$: acting with $g \in \check{\Gamma}$ amounts to replacing ϕ with $g\phi g^{-1}$, regarding g as an element of $Aut(\check{C}(S))$. We may thus assume that v_0 is a discrete maximal multicurve, having $d-1$ curves in common with v_1. Let us denote by $\underline{\alpha} = \{\alpha_1, \ldots, \alpha_{d-1}\} \in C(S)^{(d-2)}$ this $(d-1)$-multicurve ($\underline{\alpha} \in C(S)^{(d-2)}$). We now apply ϕ; by type preservation (Theorem 6.4), $\phi(v_0)$ lies in the $\check{\Gamma}(S)$-orbit of v_0. Twisting again by an element of $\check{\Gamma}(S)$, we may thus assume that v_0 is fixed under ϕ, hence so is $\underline{\alpha}$ ($\phi(\underline{\alpha}) = \underline{\alpha}$).

The surface $S_{\underline{\alpha}}$, namely S cut along the multicurve $\underline{\alpha}$, is made of trinions (type $(0,3)$) plus exactly one connected component Σ of type $(0,4)$ or $(1,1)$. By Proposition 5.8, ϕ restricts to an automorphism $\bar{\phi}$ of $\check{C}(\Sigma) = \hat{C}(\Sigma)$. By the same Proposition and Theorem 4.5 in [4] (or Proposition 6.6 in [3]), the stabilizer $\Gamma_{\underline{\alpha}}$ of $\underline{\alpha}$ in $\check{\Gamma}(S)$ is isomorphic to an extension of $\check{\Gamma}(\Sigma) = \hat{\Gamma}(\Sigma)$ by the free proabelian group $G_{\underline{\alpha}}$ of rank $d-1$, generated by the twists along the components α_i of $\underline{\alpha}$.

Now the original automorphism $\phi \in Aut^*_{G_{\mathbb{Q}}}(\check{\Gamma}(S))$ restricts to an automorphism of the stabilizer $\Gamma_{\underline{\alpha}}$, which descends to a – still Galois invariant – automorphism of $\hat{\Gamma}(\Sigma)$. We denote this element by $\bar{\phi} \in Aut^*_{G_{\mathbb{Q}}}(\check{\Gamma}(\Sigma))$. It induces an automorphism with the same name, $\bar{\phi} \in Aut(\check{C}(\Sigma))$. Writing $v_0 = \{\underline{\alpha}, \beta_0\}$, $v_1 = \{\underline{\alpha}, \beta_1\}$, $\phi(v_0)$ and $\phi(v_1)$ are joined by an edge in $\check{C}_P(S)$ if and only if $\bar{\phi}(\beta_0) = \beta_0$ and $\bar{\phi}(\beta_1)$ are. That is we are indeed reduced to the one-dimensional case for the automorphism $\bar{\phi}$, which completes the proof. □

In closing, let us compile a short to-do and wish list of directions for possible generalizations or complaints:

a) We should explicit the connection with anabelian geometry and deepen the connection with the geometry of the moduli stacks. This will be briefly explained in the next paragraph.

b) The above can be generalized to larger basefields and to all Galois levels or, if one prefers, normal open subgroups of the modular groups. This is explained in §8.3 below.

c) One would like to replace the congruence completion with the full profinite one. This is the congruence subgroup conjecture, which is *not* proved below.

d) One would like to "remove the upper star", that is prove that all automorphisms, or perhaps just the Galois invariant ones, are *a priori* inertia preserving. Again this is reminiscent of the so-called local correspondence in anabelian geometry. This is *not* shown nor even addressed below.

8.2. **The anabelian context in a few words.** Anabelian geometry starts with a trivial remark of genius: the fundamental group is essentially the only non abelian (or nonlinear) invariant in classical algebraic topology. In the early eighties A.Grothendieck, having laid the ground for the most ambitious program of linearization of geometry, namely motives, embarked on a first foray into a genuinely nonlinear (or non abelian, or anabelian) world, coming back with an *esquisse* of two twin theories, Grothendieck-Teichmüller theory and anabelian geometry. The latter has to do with the fundamental group and the main slogan is to translate as much geometry as possible into (profinite) *group theoretic* statements. As for Grothendieck-Teichmüller theory, it revolves around the $\mathcal{M}_{g,n}$'s, which are (rational) $K(\pi, 1)$'s and stand at the forefront of algebraic geometry. The theory has been developped in several markedly different directions, with the profinite version (see in particular [15, 31]) being perhaps closest to the vision delineated in the *Esquisse*. In this paper we are attempting to prepare the ground for a topological version of Grothendieck-Teichmüller theory (see [22, 23]) in which, in a word, one is moving from (profinite) group theory to (profinite) simplicial complexes, indeed in large part to (profinite) *graph theoretic* statements.

Let us now focus on the very first inputs in anabelian geometry. We refer to [11, 17, 32, 29, 28, 26, 16] for a first serious exploration. Although largely arbitrary and not quite up to date, this list is arranged in rough chronological order and roughly increasing degree of precision and technicality. So start now with a basefield k, say of characteristic 0 although the case of positive characteristic is actually very interesting and has been studied deeply. One thus thinks of k as being just $\mathbb{Q}$ or a finite extension thereof or, following S.Mochizuki, a sub p-adic field (see §8.3 for the definition... or forget about it). For X a scheme (variety, Deligne-Mumford stack...) defined over k let $\bar{X} = X \otimes \bar{k}$, with $\bar{k}$ an algebraic closure of k, and write $\pi_1^{geom}(X) = \pi_1(\bar{X}) = \pi_1(X \otimes \bar{k})$ for the geometric (étale) fundamental group. After choosing an embedding $k \hookrightarrow \mathbb{C}$, $\pi_1^{geom}(X)$ is in fact (isomorphic to) the profinite completion of $\pi_1^{top}(X_{\mathbb{C}})$ the ordinary (topological) fundamental group of (the analytification of) the $\mathbb{C}$-scheme $X_{\mathbb{C}} = X \otimes \mathbb{C}$; for all this, see Grothendieck's [SGA1] as well as [30, 24] for the necessary adjustments in the case of Deligne-Mumford stacks (orbifolds in the complex category). Below we restrict attention to varieties and schemes for the sake of clarity but dealing with the $\mathcal{M}_{g,[n]}$'s does require addressing issues pertaining to orbifolds and Deligne-Mumford stacks.

The geometric fundamental group $\pi_1^{geom}(X)$ is equipped with a natural action coming from the augmentation $X \to Spec(k)$ expressing the fact that X is a k-scheme, simply because $\pi_1(Spec(k)) = Gal(\bar{k}/k) = G_k$ by definition (after picking a basepoint, i.e. an algebraic closure $\bar{k}$). Here we are concerned with *relative* anabelian geometry as envisioned in [11], which makes essential use of the natural augmentation map above, that is of the arithmetic Galois action. It was subsequently and somewhat surprisingly discovered that under certain circumstances one could forget about the Galois action, including in characteristic 0; higher dimensional birational anabelian geometry is the key phrase, F.Bogomolov and F.Pop the two "pioneers".

Now to a k-morphism $f : Y \to X$ of schemes of finite type over k, is functorially attached the outer G_k-equivariant homomorphism of the geometric fundamental groups

$$\pi_1^{geom}(f) : \pi_1(\bar{Y}) \to \pi_1(\bar{X}),$$

which is defined up to the group $Inn\big(\pi_1(\bar{X})\big)$ of inner automorphisms (because we do not specify basepoints, something which cannot be done functorially). Let $Hom_k^{dom}(Y, X)$ denote the set of dominant k-morphisms and $Hom^{op}_{G_k}\big(\pi_1(\bar{Y}),\pi_1(\bar{X})\big)^{ext}$ the set of outer G_k-equivariant (continuous) homomorphisms between their geometric fundamental groups, with open image.

Restricting attention to a category of sufficiently well-behaved k-schemes (say of finite type, normal, reduced and geometrically connected) and k-morphisms, one can define X to be *anabelian* if for every Y in the category, the geometric fundamental group functor π_1^{geom} establishes a bijection:

$$Hom^{dom}_k(Y,X) \xrightarrow{\sim} Hom^{op}_{G_k}\big(\pi_1(\bar{Y}),\pi_1(\bar{X})\big)^{ext}.$$

The first test for anabelianity is then the verification of the above property for $Y = X$ (see in particular [17]). In the only case in which anabelianity has been fully vindicated, i.e. hyperbolic curves, this test actually proved crucial. Let us remark that a dominant endomorphism of a hyperbolic curve is necessarily an automorphism (by Riemann-Hurwitz). The same holds true for an endomorphism of a level structure $\mathcal{M}^\lambda$ (by Royden theorem). Likewise, an open endomorphism of the fundamental group of a hyperbolic curve is necessarily an automorphism and the same holds true for the topological fundamental group of a level structure (by Ivanov's results in [19]), and presumably for the geometric algebraic fundamental group as well. In the case of hyperbolic curves, but it is reasonable to guess this holds for all anabelian varieties, the test on endomorphisms (i.e. $Y = X$) reads:

$$Aut_k(X) \xrightarrow{\sim} Out_{G_k}(\pi_1(\bar{X})).$$

Both for X a hyperbolic curve and a level structure $\mathcal{M}^\lambda$ over $\mathcal{M}_{g,[n]}$ (as a $\mathbb{Q}$ D-M stack) whose generic point has trivial automorphisms group, from the classical results quoted above (Hurwitz' for curves and Royden's for level structures) it follows that the map above is *injective*. The test for anabelianity thus asks whether or not it is surjective, that is whether or not every Galois equivariant outer automorphism of $\pi_1^{geom}(X)$ comes from a k-automorphism of X itself.

8.3. **On the weak anabelianity of the moduli stacks of curves.** Having provided a little more context, we now address the two items a) and b) of the list at the end of §8.1 above, starting with b). We take up the usual setting of §8.1 and we generalize Theorem 8.2 in two directions, by enlarging the basefield $\mathbb{Q}$ and by using Galois covers of the moduli stack $\mathcal{M}_{g,[n]}$. (Note that this is actually one and the same direction, from an arithmetic and geometric viewpoint respectively.)

So first we can replace $\mathbb{Q}$ by any field k over which the anabelian conjecture for hyperbolic curves is valid. Following S.Mochizuki ([26]) we can choose k to be a sub-p-adic field, that is a subfield of a *finitetely generated* extension of $\mathbb{Q}_p$ for some prime p. Especially noteworthy are the cases where k is a finite extension of $\mathbb{Q}$ or $\mathbb{Q}_p$. The latter is particularly interesting here since it corresponds to an especially "small" absolute Galois group G_k.

Next and writing $\mathcal{M} = \mathcal{M}_{g,[n]}$, $\Gamma = \Gamma_{g,[n]}$, we see $\mathcal{M}$ as a Deligne-Mumford stack defined over k, with geometric (étale) fundamental group $\hat{\Gamma} = \pi_1(\mathcal{M}\otimes\bar{k})$, with $\bar{k}$ an algebraic closure of k (if necessary, see e.g. [24] and references therein). For every Galois "level", that is every finite representable Galois cover $\mathcal{M}^\lambda \to \mathcal{M}$, denote as usual the geometric fundamental group by $\hat{\Gamma}^\lambda = \pi_1(\mathcal{M}\otimes\bar{k})$, which is open normal in $\hat{\Gamma}$ and the full profinite completion of Γ^λ, a cofinite normal subgroup of Γ. We denote by $G^\lambda = \hat{\Gamma}/\hat{\Gamma}^\lambda = \check{\Gamma}/\check{\Gamma}^\lambda = \Gamma/\Gamma^\lambda$ the finite geometric Galois group of the cover, that is the Galois group of the cover $\mathcal{M}^\lambda\otimes\bar{k} \to \mathcal{M}\otimes\bar{k}$.

Let us now focus again on the congruence completion and the natural representation $\check{\Gamma} \to Aut(\check{\Gamma}^\lambda)$ induced by restriction to $\check{\Gamma}^\lambda$ of the inner automorphisms. It factors through $Inn(\check{\Gamma}) \simeq \check{\Gamma}/Z(\check{\Gamma})$, whence a morphism $Inn(\check{\Gamma}) \to Aut(\check{\Gamma}^\lambda)$. This map is injective thanks to Corollary 6.2 in ([4]), which asserts that $Z(\check{\Gamma}) = Z_{\check{\Gamma}}(\check{\Gamma}^\lambda) = Z(\Gamma)$. We remark that the analogous statement in the full profinite case is still open. Here we may and do identify the image to $Inn(\check{\Gamma})$ for every $\lambda \in \Lambda$, so as to get a natural embedding $Inn(\check{\Gamma}) \hookrightarrow Aut^*(\check{\Gamma}^\lambda)$, after recalling that these automorphisms are indeed inertia preserving. Moreover, the induced representation $\check{\Gamma} \to Out(\check{\Gamma}^\lambda)$ factors through the natural outer action of the geometric Galois group $G^\lambda = \check{\Gamma}/\check{\Gamma}^\lambda$. Now if the field k is large enough so that $\mathcal{M}^\lambda$ and the geometric automorphisms of the cover $\mathcal{M}^\lambda \to \mathcal{M}$ are defined over k, then by definition these automorphisms will commute with the action of the arithmetic Galois group G_k; in other words we have natural embedding $Inn(\check{\Gamma}) \subset Aut^*_{G_k}(\check{\Gamma}^\lambda)$. The following theorem asserts that also the reverse inclusion holds:

Theorem 8.3. *Let $\mathcal{M}^\lambda$ be a Galois level structure over $\mathcal{M} = \mathcal{M}_{g,[n]}$ and k a sub-p-adic field of definition for $\mathcal{M}^\lambda$ and all its geometric automorphisms over $\mathcal{M}$; then:*

$$Aut^*_{G_k}(\check{\Gamma}^\lambda) = Inn(\check{\Gamma}).$$

One can rewrite this somewhat differently by considering the group $Inn(\check{\Gamma}^\lambda)$ of the inner automorphisms of $\check{\Gamma}^\lambda$, where in fact $Inn(\check{\Gamma}^\lambda) \simeq \check{\Gamma}^\lambda/Z(\Gamma))$. Then $Inn(\check{\Gamma}^\lambda) \subset Aut^*_{G_k}(\check{\Gamma}^\lambda)$ is a normal subgroup and writing $Out^*_{G_k}(\check{\Gamma}^\lambda) = Aut^*_{G_k}(\check{\Gamma}^\lambda)/Inn(\check{\Gamma}^\lambda)$ the theorem asserts that:

$$Out^*_{G_k}(\check{\Gamma}^\lambda) = G^\lambda,$$

the geometric Galois group of the cover $\mathcal{M}^\lambda \to \mathcal{M}$.

Theorem 8.2 corresponds of course to Theorem 8.3 when λ is the trivial level and $k = \mathbb{Q}$. In the course of the proof of Theorem 8.2 we also met a very particular case of Theorem 8.3 in dimension 1, when dealing with the case $\mathcal{M} = \mathcal{M}_{0,[4]}$, $\mathcal{M}^\lambda = \mathcal{M}_{0,4}$. There we had again $k = \mathbb{Q}$ because the action of $\mathcal{S}_3$ on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ is defined over $\mathbb{Q}$.

Proof. It follows step by step the proof of Theorem 8.2; we will only point out the differences. Starting from $\phi \in Aut^*_{G_k}(\check{\Gamma}^\lambda)$ one starts by applying Proposition 6.8. We stress that this is in fact an essential point: every inertia preserving automorphism of every open subgroup of $\check{\Gamma}(S)$ induces an automorphism of the congruence curve complex $\check{C}(S)$. Again this statement is not known to-date (2019) in the full profinite setting.

Let us examine the one-dimensional case. Then one replaces $\mathcal{M}_{0,[4]}$, or in fact $\mathcal{M}_{0,4} \simeq \mathbb{P}^1 \setminus \{0, 1, \infty\}$ (see the remarks after the proof) by some étale Galois cover $\pi : X \to \mathbb{P}^1 \setminus \{0, 1, \infty\}$. As basefield we can take any sub p-adic field contaning a *finite* extension of $\mathbb{Q}$ over which both X and the automorphisms of the cover π are defined. Then apply Mochizuki's anabelian result in [26]. If one is willing to stick to number fields (finite extensions of $\mathbb{Q}$) the result had been proved earlier (by A.Tamagawa), since in particular X is *affine* (see e.g. references in [26]). It is quite remarkable that here one applies the anabelian theorem to a curve X given by a Galois cover of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. Indeed recall that Belyi's theorem asserts that in fact *every* complete curve defined over a numberfield arises as the completion of a finite étale quotient of such a curve X (we take a quotient because we are working with *Galois* covers; it is not essential).

Finally the reduction to the one-dimensional case proceeds as in the proof of Theorem 8.2. With the same notation, let $p : \Gamma_{\underline{\alpha}} \to \hat{G}(\Sigma)$ denote the natural projection. Consider the projection $p(\Gamma_{\underline{\alpha}} \cap \check{\Gamma}^\lambda)$; the image is an open subgroup $\hat{\Gamma}^\mu \subset \hat{\Gamma}(\Sigma)$ and this time we get $\bar{\phi} \in Aut^*_k(\hat{\Gamma}(\Sigma))$ and we are back to the one-dimensional setting.

□

A few remarks may be in order. Passing from Theorem 8.2 to Theorem 8.3 we considered in essence étale covers of $\mathcal{M}_{g,[n]}$ and of the original basefield $\mathbb{Q}$ (more correctly $Spec(\mathbb{Q})$), or else normal open subgroups of $\check{\Gamma}_{g,[n]}$ and (not necessarily Galois) finite extensions of $\mathbb{Q}$. This yields Theorem 8.3 with a sufficiently large number field k as basefield. S.Mochizuki's result ensures that one can in fact considerably enlarge the basefield but the main point is that the results we get using curve complexes are essentially "virtual"; they are very robust with respect to passing to geometric *and* arithmetic étale covers. This is already apparent in the work of N.Ivanov (see in particular Corollary 2.5 above) as far as the geometric (or topological) side is concerned. Here for fixed $\lambda \in \Lambda$ the field k can be taken arbitrarily large, with Galois group G_k so to speak arbitrarily small if one restricts to the finite extensions of a given basefield. We may also take λ arbitrarily large, that is Γ^λ of arbitrary large index in $\Gamma_{g,[n]}$. In fact, when λ increases along the inverse system Λ, the basefield k has to increase since $k \supset k^\lambda$ where $k^\lambda \subset \bar{\mathbb{Q}}$ is a field of definition of $\mathcal{M}^\lambda$ together with its automorphisms.

We now essentially translate Theorem 8.3 into a geometric statement with anabelian flavor. However, to this end and in order to make use of the usual étale fundamental group, we have to assume the validity of the congruence subgroup conjecture. This is the last statement of the paper and the only one in which we do so, in order to get a neater and more natural (conditional) result. Here is the statement:

Theorem 8.4. *Let $\mathcal{M}^\lambda$ be a (stack) étale Galois cover of $\mathcal{M} = \mathcal{M}_{g,[n]}$, k a sub-p-adic field over which $\mathcal{M}^\lambda$ and the automorphisms of the cover $\mathcal{M}^\lambda/\mathcal{M}$ are defined. Let A^λ denote the automorphism group of the generic point of $\mathcal{M}^\lambda$ and let $\mathcal{M}^\lambda /\!\!/ A^\lambda$ denote the stack obtained by rigidifying $\mathcal{M}^\lambda$ with respect to A^λ.* Assume the validity of the congruence subgroup conjecture.

Then the geometric fundamental group functor induces an isomorphism:

$$Aut_k(\mathcal{M}^\lambda /\!\!/ A^\lambda) \xrightarrow{\sim} Out^*_{G_k}(\pi_1(\mathcal{M}^\lambda \otimes \bar{k}))$$

□

This is indeed a translation of Theorem 8.3, modulo the congruence subgroup conjecture which ensures that $\check{\Gamma} = \hat{\Gamma}(= \pi_1(\mathcal{M} \otimes \bar{k}))$ and thus idem for every level $\lambda \in \Lambda$. So there is nothing new to prove and we simply clarify the statement and translation.

First the automorphism group of the generic point of $\mathcal{M}^\lambda$ is trivial unless the type (g, n) is one of $\{(0, 4), (1, 1), (1, 2), (2, 0)\}$ and the corresponding level Γ^λ of $\Gamma_{g,[n]}$, in the first case, intersects non-trivially the Klein subgroup V of $\Gamma_{0,[4]}$ and, in all other cases, contains the (hyper)elliptic involution. So if A^λ is not trivial it is isomorphic to either $\mathbb{Z}/2$ or $\mathbb{Z}/2 \times \mathbb{Z}/2$. Note that the congruence subgroup conjecture has been vindicated in all these low dimensional cases. Now given an irreducible (1-)stack there is a canonical rigidifying procedure which removes the automorphisms (2-morphisms) at the generic point. We refer to [24] for much more detail but here not much is needed so that the reader may very well take this step for granted (see the examples below).

Assuming the validity of the congruence subgroup conjecture, let us identify the geometric fundamental groups of $\mathcal{M} = \mathcal{M}_{g,[n]}$ and of $\mathcal{M}^\lambda$ with $\check{\Gamma} = \check{\Gamma}_{g,[n]}$ and $\check{\Gamma}^\lambda$ respectively. By [4] Corollary 6.2, the only possible nontrivial element in the center of $\check{\Gamma}^\lambda$ is the (hyper)elliptic involution ι which then is also a generic automorphism of the stack $\mathcal{M}^\lambda$. Therefore, if $\mathcal{M}^\lambda$ has at most a central generic automorphism, it holds:

$$Out^*_{G_k}(\check{\Gamma}^\lambda) \cong Inn(\check{\Gamma})/Inn(\check{\Gamma}^\lambda) \cong \check{\Gamma}/(\check{\Gamma}^\lambda \cdot Z),$$

where $Z = Z(\Gamma)$ denotes the center of Γ, i.e. either $Z = \{1\}$ or $Z =< \iota >$. In case Z is not trivial, ι determines a nontrivial automorphism of the cover $\mathcal{M}^\lambda \to \mathcal{M}$ but a trivial one of the stack $\mathcal{M}^\lambda /\!\!/ A^\lambda$, where $A^\lambda = \Gamma^\lambda \cap < \iota >$. Therefore, in any case, it holds:

$$\check{\Gamma}/(\check{\Gamma}^\lambda \cdot Z) \cong Aut_k(\mathcal{M}^\lambda /\!\!/ A^\lambda)$$

and the claim of the theorem follows from Theorem 8.3.

There remains to consider the case $(g, n) = (0, 4)$. Since the center of $\hat{\Gamma}_{0,[4]}$ and of any of its open subgroups is trivial, we have:

$$Out^*_{G_k}(\check{\Gamma}^\lambda) \cong Inn(\hat{\Gamma}_{0,[4]})/Inn(\hat{\Gamma}^\lambda).$$

Let as above $V \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ denote the Klein subgroup of $\Gamma_{0,[4]}$, which is also identified with the group of generic automorphisms of the moduli stack $\mathcal{M}_{0,[4]}$. Let us remark that V is normally generated in $\Gamma_{0,[4]}$ (and then in $\hat{\Gamma}_{0,[4]}$) by any of its non-trivial elements. Therefore, two situations may occur: either $V = A^\lambda \subset \hat{\Gamma}^\lambda$ or $V \cap \hat{\Gamma}^\lambda = \{1\}$ and A^λ is trivial. Observe that in both cases the group V lies in the kernel of the natural representation $\hat{\Gamma}_{0,[4]} \to Out(\hat{\Gamma}^\lambda)$. Therefore, it holds:

$$Inn(\hat{\Gamma}_{0,[4]})/Inn(\hat{\Gamma}^\lambda) \simeq \check{\Gamma}_{0,[4]}/(\check{\Gamma}^\lambda \cdot V) \simeq Aut_k(\mathcal{M}^\lambda /\!\!/ A^\lambda)$$

and the assertion of the theorem follows again from Theorem 8.3.

Appendix A. Some definitions and known results

We have gathered here a number of definitions, most of which but not all are classical, and a number of results in the discrete setting, most of which but not all are used in the text. The point is simply to provide the reader with the basic notation and material, together with some more or less standard references.

A.1. A finite *type* is a pair (g, n) of non negative integers. Given a type, we let $S = S_{g,n}$ denote the – unique up to diffeomorphism – differentiable surface of genus g with n deleted points. We occasionally write $g(S)$ for the genus of S. The points can also be be considered as "holes", provided isotopies do not fix the boundary circles. A surface is of type (g, n) if it is diffeomorphic to $S_{g,n}$. The Euler characteristic of $S_{g,n}$ is $\chi(S) = 2 - 2g - n$; the surface is *hyperbolic* if $2g - 2 + n > 0$.

A.2. Attached to a surface S of type (g, n) are the *Teichmüller space* $\mathcal{T}(S)$ and *moduli space* $\mathcal{M}(S)$. We restrict henceforth to hyperbolic surfaces. The Teichmüller space $\mathcal{T}(S)$ is noncanonically identified with the standard Teichmüller space $\mathcal{T}_{g,n}$ associated with the given type. It has dimension $d(S) = d_{g,n} = 3g - 3 + n$, which we call the *modular dimension* of S or of the given type – we will often drop the adjective "modular". In turn $\mathcal{M}(S)$ is – again noncanonically – identified with $\mathcal{M}_{g,[n]}$, the moduli space of curves of the given type, with unlabelled marked points. We use brackets ($[n]$) when the points are unlabelled, that is are considered setwise. Note that to be consistent we should write $S_{g,[n]}$ rather than $S_{g,n}$ but we nevertheless retain the latter piece of notation for simplicity; also, $\mathcal{T}_{g,[n]} = \mathcal{T}_{g,n}$ because the definition of Teichmüller space involves a marking, so in particular the choice of generators of the fundamental group of the model surface.

A.3. We let $Mod(S) = \pi_0(Diff(S))$ denote the (extended) *mapping class group* of S, i.e. the group of isotopy classes of diffeomorphisms of S. The index 2 subgroup of orientation preserving isotopy classes is denoted $Mod^+(S)$. More generally an upper + will mean *orientation preservating.* We write $\Gamma(S) = Mod^+(S)$ and call it the *(Teichmüller) modular group.* It can be seen as the orbifold fundamental group of $\mathcal{M}(S)$ and as the Galois group of the orbifold unramified cover $\mathcal{T}(S)/\mathcal{M}(S)$. So we have the tautological exact sequence:

$$(A\ 1) \qquad 1 \to \Gamma(S) \to Mod(S) \to \mathbb{Z}/2 \to 1.$$

The group $\Gamma(S)$ is (noncanonically) isomorphic to $\Gamma_{g,[n]}$, defined as the fundamental group of the complex orbifold $\mathcal{M}_{g,[n]}$. The group $\Gamma_{g,[n]}$ is centerfree, except for 4 low-dimensional exceptions, i.e. types $(0,4)$, $(1,1)$, $(1,2)$ and $(2,0)$. In the first case the center is Klein's Vierergruppe ($\simeq \mathbb{Z}/2 \times \mathbb{Z}/2$); in the other three cases the center is isomorphic to $\mathbb{Z}/2$, generated by the (hyper)elliptic involution. We refer to any elementary text on the subject for more detail.

A.4. *Permutations* of points play a certain role in the theory. The moduli space of curves of genus g with n *ordered points* is denoted $\mathcal{M}_{g,n}$. The cover $\mathcal{M}_{g,n}/\mathcal{M}_{g,[n]}$ is finite, orbifold unramified (stack étale) and Galois with group $\mathcal{S}_n$, the permutation group on n symbols.

Let us detail one low dimensional example (or exception) which is mentioned in the text. The group $\Gamma_{1,[2]}$ has center Z isomorphic to $\mathbb{Z}/2$ as mentioned above. It is the *direct* product of that center and the corresponding ordered group: $\Gamma_{1,[2]} = \Gamma_{1,2} \times Z$. Moreover $\Gamma_{1,2} \subset \Gamma_{0,[5]}$ is the subgroup which corresponds to the permutations stabilizing the – say – fifth point. Geometrically speaking, to a genus 1 curve with 2 marked points one can associate 5 points, namely the 4 Weierstrass points plus the orbit of the two points under the elliptic involution; the two points can be indeed made to form an orbit, after a suitable translation. The 4 points can be permuted but the fifth one should be kept labeled under the action of the modular group, hence the above description. Finally, it is useful to note that $\Gamma_{1,2}$ is self-normalizing in $\Gamma_{0,[5]}$, so in particular not normal.

A.5. We now briefly summarize the definitions pertaining to various *curve complexes*, referring to any of the many references (e.g. [18, 19, 21] etc.) for more detail. It is remarkable that we will actually need only consider *graphs* (and prographs), that is complexes of dimension 1.

Given a surface S, hyperbolic and of finite type (see §A1), we let $\mathcal{L}(S)$ denote the set of isotopy classes of simple closed curves on S not isotopic to boundary curves (circles around the marked points). A *multicurve* is a set of non intersecting elements of $\mathcal{L}$ where non intersecting means that there exist representatives which do not intersect (see [10] or again any standard reference for detail).

The first complex $C(S)$ is the one originally defined by W.J.Harvey in the late sixties. A k-simplex of $C(S)$ is defined by a multicurve $\underline{\alpha} = (\alpha_0, \ldots, \alpha_k)$, so that the vertices of $C(S)$ correspond to elements of $\mathcal{L}(S)$. Boundary and face operators are defined by deletion and inclusion of curves respectively. This makes $C(S)$ into a (non locally finite) simplicial complex of dimension $d(S)-1$ where $d(S)$ is the modular dimension of S (see §A.2). We will write $C^{(k)}(S)$ for the k-dimensional skeleton of $C(S)$ and use a similar notation for the other complexes. Note that $\mathcal{L}(S) = C^{(0)}(S)$ is just the 0-skeleton (vertex set) of $C(S)$ but it is nonetheless useful to retain a specific piece of notation.

There is a natural action of $\Gamma(S)$ on $C(S)$ determined by saying that to $g \in \Gamma$ and a curve $\alpha \in \mathcal{L}$ one associates $g \cdot \alpha$, the image of the curve by g, everything up to isotopy.

A.6. Next we define $C_G(S)$, the *group theoretic complex.* It is useful essentially in the complete case (see below), so is included in the present discrete setting essentially to fix notation. Here all objects pertain to the discrete topology, so we add a superscript "disc". Let $\Gamma = \Gamma(S)$ and $\mathcal{G}^{disc}(\Gamma)$ denote the set of all subgroups of Γ. To every simplex (i.e. multicurve) $\sigma \in C(S)$ we assign the (discrete) free abelian group $G_\sigma^{disc} \in \mathcal{G}(\Gamma)$ spanned by the (Dehn) twists associated to σ. We then use the boundary and face operators as for $C(S)$ in order to make $\mathcal{G}^{disc}(\Gamma)$ into a simplicial complex, indeed a Boolean lattice.

In the discrete setting, $C_G(S)$ is (more or less trivially) isomorphic to $C(S)$ and we define a Γ-action on $C_G(S)$ so as to make the natural isomorphism equivariant. To $\alpha \in \mathcal{L}$, that is a vertex of $C(S)$, one thus assigns the cyclic group generated by τ_α, the twist along α. Note that at this point, we should and do fix an orientation for S. Then for $g \in \Gamma$ one has the well-known formula: $\tau_{g\cdot\alpha} = g\tau_\alpha g^{-1} \in \Gamma$. The right-hand side of this equality defines an action of Γ on $C_G(S)$ which makes the natural isomorphism between $C(S)$ and $C_G(S)$ Γ-equivariant.

A.7. We then come to the *pants complex* $C_P(S)$. It was briefly mentioned the appendix of the classical 1980 paper by A.Hatcher and W.Thurston (see [15] or [25]) and first studied in [15] where it is shown to be connected and simply connected for $d(S) > 2$. It is a two dimensional, not locally finite complex whose vertices are given by the pants decomposition (i.e. maximal multicurves) of S; these correspond to the simplices of highest dimension ($= d(S) - 1$) of $C(S)$. Given two vertices $\underline{\alpha}, \underline{\alpha}' \in C_P(S)$, they are connected by an edge if and only if $\underline{\alpha}$ and $\underline{\alpha}'$ have $d(S) - 1$ curves in common, so that up to relabelling (and of course isotopy) $\alpha_i = \alpha'_i$, $i = 1, \ldots, d(S) - 1$, whereas α_0 and α'_0 differ by an *elementary move*, which means the following. Cutting S along the α_i's, $i > 0$, there remains a surface Σ of modular dimension 1, so Σ is of type $(1, 1)$ or $(0, 4)$. Then α_0 and α'_0, which are supported on Σ, should intersect in a minimal way, that is they should have geometric intersection number 1 in the first case, and 2 in the second case (in the latter case their algebraic intersection number is 0). In the first case (genus 1), the edge (and move) is said to be of type S (for "simple", see [15]); in the second case (genus 0) of type A (for "associativity", see [15]). For $d(S) = 1$, the 1-skeleton of $C_P(S)$ is the Farey graph F.

We have thus defined the 1-skeleton $C_P^{(1)}(S)$ of $C_P(S)$ which, following [25], we call the *pants graph* of S. We will not give here the definition of the 2-cells of $C_P(S)$ (see [15] or [25]), as we will actually not use it. They describe certain relations between elementary moves, that is they can be considered as elementary homotopies; as mentioned above, pasting them in makes $C_P(S)$ simply connected for $d(S) > 2$ (cf. [15]). It is shown in [25] how to recover the full 2-dimensional pants complex from the pants graph. For $d(S) = 1$ the pants complex is the Farey tesselation, which we again denote F. We usually use only the pants graph, i.e. the 1-skeleton $C_P^{(1)}(S)$ of $C_P(S)$, which in order to simplify notation we will often simply denote $C_P(S)$.

A.8. We finally define the *graph* $C_*(S)$ which plays an important role in the complete case, while actually clarifying a number of issues even in the discrete case (see §2). The graph $C_*(S)$ shares the same set of vertices as $C_P(S)$, namely the maximal multicurves (a.k.a. pants decomposition) of S. The edges are defined simply by relaxing the minimal intersection condition in the definition of the edges of $C_P(S)$. In other words two vertices represented by maximal multicurves $\underline{\alpha} = (\alpha_i)_i$ and $\underline{\alpha}' = (\alpha'_i)_i$ ($i = 0, \ldots, d(S) - 1$) are joined by an edge if up to relabelling $\alpha_i = \alpha'_i$ for $i > 0$; then α_0 and α'_0 lie on a surface of type $(0, 4)$ or $(1, 1)$. So $C_P(S) \subset C_*(S)$ is a subgraph with the same set of vertices.

If S is connected (see however §A.9 below) of dimension 0, it is of type $(0, 3)$ (a *trinion* or pair of pants); by convention, $C_P(S) = C_*(S)$ is reduced to a point with no edge attached; note that usually one defines $C(S_{0,3}) = \emptyset$. If S is connected of dimension 1, it is of type $(0, 4)$ or $(1, 1)$. In both cases $C_P(S) = F$ coincides with the Farey graph. On the other hand, it is easily checked that $C_*(S)$ is the complete graph with the same vertices as F, which we denote by G: two simple closed curves on a surface of (modular) dimension 1 always intersect nontrivially. If $d(S) > 1$, $C_*(S)$ is nothing but the 1-skeleton of $C(S)^*$, the complex dual to $C(S)$. For this reason, when $d(S) = 1$, it becomes natural to define $C(S)$ as the dual of G, which is not the usual convention but seems to be the right one for our purposes.

A.9. It is useful to extend the definitions of the graphs $C_P(S)$ and $C_*(S)$ to *non connected surfaces*. The extension is rather trivial yet it shows that these two graphs are particularly well-behaved. The definitions are simply unchanged. We will write $V(S)$ for the set of vertices common to $C_*(S)$ and $C_P(S)$ (i.e. maximal multicurves), $E(S)$ (resp. $E_P(S)$) for the edges of $C_*(S)$ (resp. $C_P(S)$): $E_P(S) \subset E(S)$.

Let $S = S' \coprod S''$ be given as the disjoint sum of S' and S'', which themselves need not be connected. First note that modular dimension is additive: $d(S) = d(S') + d(S'')$. Then it is easy to describe $C_*(S)$ and $C_P(S)$ in terms of the graphs associated to S' and S''. For the vertices we get: $V(S) = V(S') \times V(S'')$; and for the edges of $C_*(S)$: $E(S) = E(S') \times V(S'') \coprod V(S') \times E(S'')$. Simply change E into E_P for the case of C_P. These prescriptions immediately generalize to an arbitrary number r of not necessarily connected pieces. If $S = \coprod_i S_i$, $d(S) = \Sigma_i d(S_i)$, $V(S) = \prod_i V(S_i)$ and $E(S) = \coprod_i V(S_1) \times \ldots \times E(S_i) \times \ldots \times \ldots V(S_r)$; replace again E with E_P when dealing with C_P.

A.10. We now come to *completions*, first of groups, then of the various simplicial complexes. Given S hyperbolic of finite type we start by indexing the inverse system of the cofinite (i.e. finite index) subgroups of $\Gamma = \Gamma(S) \simeq \Gamma_{g,[n]}$ by a set M, so that to any $\lambda \in M$ there correspond a subgroup Γ^λ and a cover $\mathcal{M}^\lambda/\mathcal{M}$ which we call a *level structure* following a traditional terminology in this context. For $\lambda, \mu \in M$ we write $\mu \geq \lambda$ if $\Gamma^\mu \subseteq \Gamma^\lambda$ i.e. if $\mathcal{M}^\mu$ is a covering of $\mathcal{M}^\lambda$, and we say that $\mathcal{M}^\mu$ (resp. Γ^μ) dominates $\mathcal{M}^\lambda$ (resp. Γ^λ).

For any subinverse system $\Lambda \subset M$ we get the corresponding completion of Γ as the limit :

$$\varprojlim_{\lambda \in \Lambda} \Gamma/\Gamma^\lambda.$$

The (full) profinite completion is obtained when $\Lambda = M$ and is denoted with a hat as usual:

$$\hat{\Gamma} = \varprojlim_{\lambda \in M} \Gamma/\Gamma^{\lambda}.$$

The analogous definition can be given for any group. Note that the groups we consider, "arising from geometry", are discrete and finitely generated. It implies that the system of all invariant (a.k.a. characteristic) subgroups is cofinal. That is for any $\lambda \in M$ one can find a (cofinite) invariant subgroup contained in Γ^{λ}.

The procongruence (or simply congruence) completion is specific of the situation at hand, selecting a particular subsystem Λ of cofinite subgroups of Γ. Denote by $\pi = \pi_1(S)$ the fundamental group of the surface S with respect to some basepoint and let $K \subset \pi$ be a characteristic subgroup of π. The elements $g \in \Gamma$ act on π (as 'mapping classes') up to inner automorphism, so there is a natural map: $\Gamma \to Out(\pi/K)$. We denote the kernel by $\Gamma^K \subset \Gamma$ and call it a *principal congruence subgroup*. It is normal and cofinite since π/K (and thus also $Out(\pi/K)$) is a finite group. A *congruence subgroup* of Γ is one which contains a principal subgroup. In particular, for $m \geq 2$ a positive integer, the abelian level $\mathcal{M}^{(m)}$ is defined by the subgroup $\Gamma^{(m)}$ which is the kernel of the natural map $\Gamma \to Sp_{2g}(\mathbb{Z}/m)$, that is $\Gamma^{(m)}$ is the group of diffeomorphisms of S (considered modulo isotopy) which fix the homology of the associated unmarked or compact surface modulo m.

The congruence completion, denoted $\check{\Gamma}$, is obtained by choosing for $\Lambda \subset M$ the system of all the congruence subgroups. We have a natural surjective map : $\hat{\Gamma} \to \check{\Gamma}$ and the congruence conjecture (first proposed by N.Ivanov) asserts that this is actually an isomorphism, which amounts to stating that the congruence subgroups form a cofinal system in M. If true and vindicated, that is *if* indeed $\hat{\Gamma} = \check{\Gamma}$, all the results of the present paper naturally come to hold true in the (full) profinite setting. For more on the congruence property, including from a homotopical viewpoint, and for references, we refer again to [23].

A.11. We now come to *profinite complexes* of curves. More generally, let $X_\bullet$ be a simplicial complex endowed with an action of $\Gamma = \Gamma(S)$. Then we can define its profinite completion as the inverse limit:

$$\hat{X}_\bullet = \varprojlim_{\lambda \in M} X_\bullet/\Gamma^{\lambda},$$

which we regard as a simplicial object in the category of profinite sets. The above definition would of course be valid for other groups than Γ and spaces X which are not necessarily simplicial complexes. However the action of Γ on X has to satisfy certain geometric conditions which in our cases are easily met (see [3], §5).

We apply the above to $\mathcal{L}(S)$, $C(S)$, $C_P(S)$ and $C_*(S)$, obtaining the respective (full profinite) completions $\hat{\mathcal{L}}(S)$, $\hat{C}(S)$, $\hat{C}_P(S)$ and $\hat{C}_*(S)$. We dropped the bullet subscript from the notation but stress that these are indeed simplicial objects. The profinite set $\hat{\mathcal{L}}(S)$ is thus the set of procurves and it is the set of vertices of $\hat{C}(S)$. The complexes $\hat{C}_P(S)$ and $\hat{C}_*(S)$ are in fact pro*graphs*. We will often drop the prefix "pro" for simplicity but it should definitely be emphasized that these profinite spaces are complicated objects, just like profinite groups and even more so; note that the group completion $\hat{\Gamma}$ is obtained via the above procedure by letting Γ act on itself by translation. We refer to [3] for basic properties of these profinite complexes of curves.

In the present paper however we almost only use the *congruence completion*, obtained by replacing as above (see A.10) the full system M by the substem Λ of the congruence subroups. This procedure delivers the respective congruence completions, namely $\check{\mathcal{L}}(S)$, $\check{C}(S)$, $\check{C}_P(S)$ and $\check{C}_*(S)$. The main reference is [4].

A.12. Our last item will deal very briefly with *automorphisms of discrete modular groups and curve complexes*. We refer to e.g. [19, 21] for more detailed statements and proofs. Our statements are geared towards the complete case and we have extracted what seems to be the significant minimum in that direction (more can be found in the body of the text). We let S be connected hyperbolic and of finite type; we assume that $d(S) > 1$ and S is not of type $(1, 2)$, that is S is of type $(0, 5)$ or $d(S) > 2$. This last assumption we make simply in order to avoid discussing well-known low-dimensional peculiarities (see [21]).

Then the automorphisms of the curve complex are described by the exact sequence:

$$1 \to Inn(\Gamma(S)) \to Aut(C(S)) \to \mathbb{Z}/2 \to 1, \tag{A 2}$$

where, in view of the profinite case, the group $\mathbb{Z}/2$ should be considered as generated by complex conjugacy, so isomorphic to the Galois group $Gal(\mathbb{C}/\mathbb{R})$. With our assumptions $Inn(\Gamma(S)) = \Gamma(S)$ except if S is of type $(2, 0)$, in which case the center has order 2. Yet it is best to think of the left-hand group as $Inn(\Gamma(S)) \subset Aut(\Gamma(S))$.

Denoting by $C^{(1)}(S)$ the 1-skeleton of $C(S)$, there is a natural injective map $Aut(C(S)) \to Aut(C^{(1)}(S))$ and this map is actually an isomorphism. This is an easy result, coming from a graph-theoretic characterization of the simplices of $C(S)$ inside the graph $C^{(1)}(S)$: they are in one-to-one correspondence with the finite *complete* subgraphs, so have to be preserved by any automorphism of the graph.

Using the sequence $(A\ 2)$ it is fairly easy to derive a description of the *group* automorphisms in the form of the following exact sequence:

$$(A\ 3) \qquad 1 \to Inn(\Gamma(S)) \to Aut(\Gamma(S)) \to \mathbb{Z}/2 \to 1.$$

In other words $Aut(\Gamma(S)) = Mod(S)$, $Out(\Gamma(S)) \simeq \mathbb{Z}/2$ and the only non inner automorphism is generated by a reflection of the surface, that is an orientation reversing involution of the surface S, alias complex conjugacy, the generator of $Gal(\mathbb{C}/\mathbb{R})$. Note that the existence of such a reflection shows that the three sequences $(A\ 1)$, $(A\ 2)$ and $(A\ 3)$ are split. Using $(A\ 2)$ again, it is fairly easy to extend the above to any finite index subgroup of $\Gamma(S)$ (cf. [19]), quite a substantial improvement. In fact $(A\ 3)$ remains valid if one replaces the middle group $\Gamma(S)$ by a normal finite index subgroup Γ^λ without changing the left and right hand groups.

Put somewhat differently, there is an *a priori* injective map $Aut(\Gamma(S)) \to Aut(C(S))$ and $(A\ 3)$ asserts it is an isomorphism. This is a close analog of a famous result of Tits which states hat under suitable assumptions, the automorphisms of the building of an algebraic group come from the automorphisms of the group itself.

References

[1] M.P.Anderson. Exactness properties of profinite completion functors. Topology **13** (1974), 229-239.

[2] M.Artin, B.Mazur. *Etale Homotopy.* Springer LN **100**, 1969.

[3] M.Boggi. Profinite Teichmüller theory. Math. Nachr. **279** (2006), 953-987.

[4] M.Boggi. On the procongruence completion of the Teichmüller modular groups. Trans. AMS **366** (2014), 5185-5221.

[5] M.Boggi. Galois coverings of moduli spaces of curves and loci of curves with symmetry. Geom. Dedicata **168** (2014), 113-142.

[6] M.Boggi, P.Zalesskii. Characterizing closed curves on Riemann surfaces via homology groups of coverings. Int. Math. Res. Not. IMRN 2017, 944-965.

[7] M.Boggi, P.Zalesskii. A restricted Magnus property for profinite surface groups. Trans. AMS **371** (2019), 729-753.

[8] X.Buff, J.Fehrenbach, P.Lochak, L.Schneps, P.Vogel. *Moduli spaces of curves, mapping class groups and field theory*, translated from the French by L.Schneps. SMF/AMS Texts and Monographs **9**, 2003.

[9] C.J.Earle, I.Kra. On isometries between Teichmüller spaces. Duke Math. J. **41** (1974), 583-591.

[10] A.Fathi, F.Laudenbach, V.Poénaru eds.*Travaux de Thurston sur les surfaces* (seconde édition), Astérisque **66-67**, SMF Publ., 1991.

[11] A. Grothendieck. Brief an G. Faltings. In *Geometric Galois Actions I*, edited by L. Schneps and P. Lochak, Lon. Math. Soc. LNS **242** (1997), 49-58.

[12] A.Gibney, S.Keel, I.Morrison. Towards the ample cone of $\overline{\mathcal{M}}_{g,n}$. J. Amer. Math. Soc. **15** (2002), 273–294.

[13] J.L.Harer, Stability of the homology of the mapping class groups of orientable surfaces, Annals of Mathematics **121** (1985), 215-249.

[14] J.L.Harer, The cohomology of the moduli spaces of curves, in *Theory of Moduli*, LN **1337** (1988), Springer, 138-221.

[15] A.Hatcher, P.Lochak, L.Schneps. On the Teichmüller tower of mapping class groups, J. reine und angew. Math. **521** (2000), 1–24.

[16] Y.Hoshi, S.Mochizuki. Topics surrounding the combinatorial anabelian geometry of hyperbolic curves I: inertia groups and profinite Dehn twists. Advanced Studies in Pure Mathematics **63** (2012), 659-811.

[17] Y.Ihara, H.Nakamura. Some illustrative examples for anabelian geometry in high dimensions. In *Geometric Galois Actions I*, edited by L. Schneps and P. Lochak, Lon. Math. Soc. LNS **242** (1997), 127-138.

[18] N.V.Ivanov. *Subgroups of Teichmüller modular groups.* Translations of Mathematical Monographs **115**, AMS Publ., 1992.

[19] N.V.Ivanov. Automorphisms of complexes of curves and of Teichmüller spaces. Internat. Math. Res. Notices **14** (1997), 651-666.

[20] M.Korkmaz. Automorphisms of complexes of curves on punctured spheres and on punctured tori. Topology Appl. **95** (1999), 85-111.

[21] F.Luo. Automorphisms of the complex of curves. Topology **39** (2000), 283-298.

[22] P.Lochak. Automorphism groups of profinite complexes of curves and the Grothendieck-Teichmüller group. Preprint, 2007.

[23] P.Lochak. Results and conjectures in profinite Teichmüller theory. Advanced Studies in Pure Mathematics **63** (2012), 263-335.

[24] P.Lochak, M.Vaquié. Groupe fondamental des champs algébriques, inertie et action galoisienne. Ann. Fac. Sci. Toulouse Math. **27** (2018), 199-264.

[25] D.Margalit. Automorphisms of the pants complex. Duke Math. J. **121** (2004), 457-479.

[26] S.Mochizuki. The local pro-p anabelian geometry of curves. Inventiones math. **138** (1999), 319-423.

[27] J.D.McCarthy. Automorphisms of surface mapping class groups: A recent theorem of N.Ivanov. Inventiones math. **84** (1986), 49-71.

[28] S.Mochizuki, H.Nakamura, A.Tamagawa. The Grothendieck conjecture on the fundamental group of algebraic curves. Sugaku Expositions **14** (2001), 31-53.

[29] H.Nakamura. Galois rigidity of pure sphere braid groups and profinite calculus. J. Math. Sci. Univ. Tokyo **1** (1994), 72-136.

[30] B.Noohi. Fundamental groups of algebraic stacks. J. Inst. Math. Jussieu **3** (2004), 69-103.

[31] H.Nakamura, L.Schneps. On a subgroup of the Grothendieck-Teichmüller group acting on the tower of profinite Teichmüller modular groups. Inventiones math. **141** (2000), 503-560.

[32] F.Pop. Glimpses of Grothendieck's Anabelian Geometry. In *Geometric Galois Actions I*, edited by L. Schneps and P. Lochak, Lon. Math. Soc. LNS **242** (1997), 113–126.

[33] G.Quick.Group action on profinite spaces. J. Pure Appl. Algebra **215** (2011), 1024-1039.

[34] G.Quick. Some remarks on profinite completion of spaces. Advanced Studies in Pure Mathematics **63** (2012), 413-448.

[35] K.J.Shackleton. Geodesic axes in the pants complex of the five-holed sphere. Topology and its Applications **158** (2011), 1640-1654.

[36] M.Saïdi, A.Tamagawa. On the Hom-form of Grothendieck's birational conjecture in positive characteristic. Algebra & Number Theory **5** (2011), 131-184.

[37] V.A.Voevodskii, G.Shabat. Equilateral triangulations of Riemann surfaces and curves over algebraic number fields. Soviet Math. Dokl. **39** (1989), 38-41.

[38] M.Troyanov. On the moduli space of singular Euclidean surfaces. In *Handbook of Teichmüller theory*, (A.Papadopoulos, Ed.) vol.1, 507–540, Eur. Math. Soc., 2007.

[39] A.Weil. The field of definition of a variety. American J. of Mathematics **78** (1956), 509-524.

Departamento de Matematicá , UFMG, Belo Horizonte - MG, Brasil
E-mail address: marco.boggi@gmail.com

CNRS et Institut Fourier, Laboratoire de Mathematiques UMR 5582, Université Grenoble Alpes, CS 40700, 38058 Grenoble, France
E-mail address: louis.funar@univ-grenoble-alpes.fr

CNRS et Centre de Mathématiques de Jussieu, Université Paris Pierre et Marie Curie 4, place Jussieu, F-75252 Paris cedex 05, France
E-mail address: pierre.lochak@imj-prg.fr