ELLIPTIC MULTIPLE ZETA VALUES AND THE ELLIPTIC DOUBLE SHUFFLE RELATIONS

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Abstract. We study the algebra $\mathcal{E}$ of elliptic multiple zeta values, which is an elliptic analog of the algebra of multiple zeta values. We identify a set of generators of $\mathcal{E}$, which satisfy a double shuffle type family of algebraic relations, similar to the double-shuffle relations for multiple zeta values. We prove that the elliptic double shuffle relations give all algebraic relations among elliptic multiple zeta values, if (a) the classical double shuffle relations give all algebraic relations among multiple zeta values and if (b) the elliptic double shuffle Lie algebra has a certain natural semi-direct product structure.

1. Introduction

The notion of elliptic multiple zeta value first made an explicit appearance in work of Enriquez [14] under the name "analogues elliptiques des nombres multizetas". These are *holomorphic functions*, depending on one complex variable τ in the Poincaré upper half plane $\mathfrak{H}$, and degenerate to multiple zeta values at the cusp $i\infty$. Although related, they should not be confused with the *numbers* also recently introduced under the denomination of "multiple modular values" [5]. Elliptic multiple zeta values are related to the elliptic Knizhnik–Zamolodchikov–Bernard (KZB) equation [7, 22], elliptic associators [13], multiple elliptic polylogarithms [6, 22] as well as universal mixed elliptic motives [17]. More recently, they have also found applications to computations in high energy physics [1].

Our main object of study in this paper is the $\mathbb{Q}$-algebra $\mathcal{E}$ of elliptic multiple zeta values, which has been studied previously in [2, 14, 24, 25]. This algebra has both a *geometric* and an *arithmetic* part: The geometric part $\mathcal{E}^{\mathrm{geom}}$ of $\mathcal{E}$ consists of certain linear combinations of iterated integrals of Eisenstein series for $\mathrm{SL}_2(\mathbb{Z})$ [5, 23], and is intimately connected with the bi-graded Lie algebra $\mathfrak{u}^{\mathrm{geom}}$ of the prounipotent radical of $\pi_1^{\mathrm{geom}}(MEM)$, where MEM denotes the Tannakian category of universal mixed elliptic motives [17]. More precisely, there is a monodromy representation of $\mathfrak{u}^{\mathrm{geom}}$ to the derivations of a free Lie algebra on two generators, whose image we denote by $\mathfrak{u}$. Our first result is then the

Theorem (Theorem 2.6 below)**.** *There is a natural isomorphism*

$$\mathcal{E}^{\mathrm{geom}} \cong \mathcal{U}(\mathfrak{u})^\vee,$$

where $\mathcal{U}(\mathfrak{u})^\vee$ is the graded dual of the universal enveloping algebra of $\mathfrak{u}$. In particular, $\mathcal{E}^{\mathrm{geom}}$ is a commutative, graded Hopf $\mathbb{Q}$-algebra.

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The proof of the theorem rests in large part on the $\mathbb{C}$-linear independence of iterated integrals of Eisenstein series, a result we prove along the way together with a form of *functional* independence for the same family of functions (Theorem 2.8). This functional independence is not put to use in the present paper but may be of independent interest, as the result can be viewed as a genus one analog of the linear independence of the classical polylogarithm functions [26].

Returning to the algebra $\mathcal{E}$ of elliptic multiple zeta values, we will see how its *arithmetic* part essentially coincides with the algebra $\mathcal{Z}$ spanned by the ordinary (genus zero) multiple zeta values. More precisely, we have the following

Theorem (Theorem 3.5 below)**.** *Let $\overline{\mathcal{E}} := \mathcal{E}/\langle 2\pi i\rangle$ be the quotient of $\mathcal{E}$ modulo the ideal generated by $2\pi i$. We have a canonical isomorphism*

$$\overline{\mathcal{E}}[2\pi i\tau] \cong \mathcal{E}^{\mathrm{geom}} \otimes_{\mathbb{Q}} \overline{\mathcal{Z}},$$

where $\overline{\mathcal{Z}} := \mathcal{Z}/\langle (2\pi i)^2\rangle$.

The proof systematically uses aspects of Ecalle's theory of moulds (cf. Section 4.1 for a short introduction as well as [12, 31] for more extensive treatments). The reason we work with the quotient $\overline{\mathcal{E}}$ of $\mathcal{E}$ modulo $2\pi i$ is because this makes it possible to apply some results from mould theory directly; they could probably be extended with some additional work to the full algebra $\mathcal{E}$. Combined with the isomorphism $\mathcal{E}^{\mathrm{geom}} \cong \mathcal{U}(\mathfrak{u})^{\vee}$, the theorem gives a complete description of the algebra of elliptic multiple zeta values (modulo $2\pi i$) in terms of multiple zeta values and special linear combinations of iterated Eisenstein integrals.

The last main result in this article concerns the algebraic relations satisfied by the elliptic multiple zeta values that we introduce, which are a system of generators of the algebra $\mathcal{E}$ different from the ones defined by Enriquez [14]. These algebraic relations form a family of elliptic double shuffle relations similar in nature to the well-known (extended) double shuffle relations for multiple zeta values [19]. Here, we recall (cf. [30]) that the double shuffle relations can be formulated conveniently as two functional equations satisfied by the generating series of multiple zeta values, Φ_{KZ} (also known as the Drinfeld associator [10]). In the language of mould theory [11, 31], the mould associated to Φ_{KZ} is *symmetral* and its swap is *symmetril.*

In order to obtain a similar result in the elliptic setting, we consider the generating series $E(\tau)$ of our elliptic multiple zeta values, which is related to Enriquez's elliptic KZB associator [13]; in particular its coefficients generate the same $\mathbb{Q}$-algebra $\mathcal{E}$, but unlike the elliptic KZB associator, it possesses a twofold symmetry that is very close to that of Φ_{KZ}, although surprisingly, somewhat simpler. We can describe this property quite easily on the Lie version $\mathfrak{e}(\tau)$ of $E(\tau)$ obtained by reducing the coefficients of $E(\tau)$ modulo $2\pi i$ and products; we show that $\mathfrak{e}(\tau)$ is Δ-bialternal, which means that it is the twist, by a very simple mould operator Δ, of a mould that is alternal with alternal swap. This rather simple symmetry may be somewhat surprising, since in the theory of multiple zeta values, the bialternality symmetry describes not the usual double shuffle Lie algebra [30], but instead its associated graded for the depth filtration [4].

Theorem (Theorem 4.3 below)**.** *The mould $\mathfrak{e}(\tau)$ is Δ-bialternal, i.e. the elliptic multiple zeta values satisfy the elliptic double shuffle relations modulo $2\pi i$.*

The proof of the theorem proceeds in several steps. First, we show that $\overline{E}(\tau)$ is equal to the image of a suitable element $ma(\Gamma(\Phi))$ under a certain automorphisms of moulds, where Γ denotes Enriquez's canonical section $\Gamma : GRT \to GRT_{\mathrm{ell}}$ from the (genus zero, graded) Grothendieck–Teichmüller group GRT to its elliptic analog GRT_{ell} [13], and $ma(\Gamma(\Phi))$ is the associated mould. This part of the proof relies on previous work by one of us [32]. In the second step, we use a deep result of Ecalle (cf. [31], Theorem 4.6.1) to the effect that $\Gamma(\Phi)$ (more precisely the associated mould), is Δ^*-bisymmetral, where Δ^* denotes the group version of Δ. To verify that this implies the Δ-bialternality of $\mathfrak{e}(\tau)$ is then a relatively straightforward exercise in mould calculus.

Of course, the most interesting question concerning any set of algebraic relations satisfied by a set of elements is whether those relations form a complete set, i.e. whether they are sufficient to generate all algebraic relations. We show that this is true in depth two (Proposition 4.5), and that in general, the elliptic double shuffle relations do give a complete set of algebraic relations between elliptic multiple zeta values modulo $2\pi i$, if we assume the following two conjectures:

a) The double shuffle relations generate all algebraic relations among the multiple zeta values modulo $2\pi i$
b) The elliptic double shuffle Lie algebra $\mathfrak{ds}_{\mathrm{ell}}$ [32] is isomorphic to a semidirect product $\mathfrak{ds}_{\mathrm{ell}} \cong \mathfrak{u} \rtimes \gamma(\mathfrak{ds})$, where $\mathfrak{ds}$ is the usual double shuffle Lie algebra and $\gamma : \mathfrak{grt} \to \mathfrak{grt}_{\mathrm{ell}}$ is the Lie version of Enriquez' section Γ.

Conjecture a) is a standard conjecture in multiple zeta value theory (cf. [19]). It would imply strong transcendence results, and therefore seems out of reach at the moment. Conjecture b), however, is purely algebraic, and may therefore be more tractable. It would follow for example from Enriquez' generation conjecture ([13], §10) together with the conjecture that $\mathfrak{grt}_{\mathrm{ell}} \cong \mathfrak{ds}_{\mathrm{ell}}$ (an elliptic version of Furusho's theorem [16]).

It should be mentioned that there has already been some work on algebraic relations, not between the elliptic multiple zeta values defined here, but between those defined by Enriquez, arising as the coefficients of his elliptic KZB associator. Those values were shown to satisfy a family of *Fay-shuffle relations* that was described in depth two in [2, 25] (where the term *length* instead of depth is used). It is proved in [25] that for elliptic multiple zeta values of depth two, the Fay-shuffle relations give a complete set of $\mathbb{Q}$-linear relations. The extension to all depths, as well as the precise relation between the Fay-shuffle and the elliptic double shuffle relations, will be the subject of a forthcoming paper.

The contents of this paper are organized as follows. In Section 2, we introduce the algebra $\mathcal{E}^{\mathrm{geom}}$ of *geometric elliptic multiple zeta values*, and describe their relation to iterated integrals of Eisenstein series and to the Lie algebra $\mathfrak{u}$ of special derivations. A crucial result is the linear independence of iterated Eisenstein integrals, which is proved in Section 2. In Section 3, we introduce elliptic multiple zeta values using the elliptic generating series $E(\tau)$, and prove the first two theorems above. In Section 4, we study the elliptic double shuffle relations between elliptic multiple zeta values, and give evidence for the completeness of this system of relations. We also study a second type of algebraic relations, called push-neutrality relations, which are related to the Fay-shuffle relations. The necessary background about moulds is briefly summarized in Section 4.1.

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2. Geometric Elliptic Multiple Zeta Values

In the first two sections, we respectively recall the definition of a certain Lie algebra $\mathfrak{u}$ of derivations [28, 34] and of iterated integrals of Eisenstein series [5, 23]. In §2.3, we introduce the algebra of geometric elliptic multiple zeta values, and prove that it is isomorphic to the graded dual of the universal enveloping algebra of $\mathfrak{u}$. The crucial step is a linear independence result for iterated integrals of Eisenstein series, which we prove (in slightly greater generality than needed) in §2.4.

2.1. **A family of special derivations.**

We begin by fixing our notation. For a $\mathbb{Q}$-algebra A, let $\mathfrak{f}_2(A) = \mathrm{Lie}_A[\![x_1, y_1]\!]$ be the *completed* (with respect to the descending central series) free Lie algebra over A on two generators x_1, y_1 with Lie bracket $[\cdot,\cdot]$. Its (topological) universal enveloping algebra will be denoted by $\mathcal{U}(\mathfrak{f}_2)_A$, and $F_2(A) := \exp(\mathfrak{f}_2(A)) \subset \mathcal{U}(\mathfrak{f}_2)_A$ is the set of exponentials of Lie series. Note that $\mathcal{U}(\mathfrak{f}_2)_A$ is canonically isomorphic to $A\langle\langle x_1, y_1\rangle\rangle$, the A-algebra of formal power series in non-commuting variables x_1, y_1. Moreover, $\mathcal{U}(\mathfrak{f}_2)_A$ is a complete Hopf A-algebra, whose (completed) coproduct Δ is uniquely determined by $\Delta(w) = w \otimes 1 + 1 \otimes w$, for $w \in \{x_1, y_1\}$. The group $F_2(A)$ can also be characterized as the set of group-like elements of $\mathcal{U}(\mathfrak{f}_2)_A$. Likewise, the Lie algebra $\mathfrak{f}_2(A) \subset \mathcal{U}(\mathfrak{f}_2)_A$ is precisely the subset of Lie-like (or primitive) elements. If $A = \mathbb{Q}$, we will write $\mathfrak{f}_2$ instead of $\mathfrak{f}_2(\mathbb{Q})$ and likewise $\mathcal{U}(\mathfrak{f}_2)$ and F_2 instead of $\mathcal{U}(\mathfrak{f}_2)_A$ and $F_2(A)$. Now let $\mathrm{Der}(\mathfrak{f}_2)$ denote the Lie algebra of derivations of $\mathfrak{f}_2$, and define $\mathrm{Der}^0(\mathfrak{f}_2)$ as the subalgebra of those $D \in \mathrm{Der}(\mathfrak{f}_2)$ which (i) annihilate the bracket $[x_1, y_1]$:

$$D([x_1, y_1]) = 0$$

and (ii) are such that $D(y_1)$ contains no linear term in x_1. Since $\mathfrak{f}_2$ is free, the commutator of y_1 is $\mathbb{Q} \cdot y_1$, from which it follows easily that every derivation $D \in \mathrm{Der}^0(\mathfrak{f}_2)$ is uniquely determined by its value on x_1. Similarly, the only non-zero derivation $D \in \mathrm{Der}^0(\mathfrak{f}_2)$ which annihilates y_1 is the derivation ε_0 defined by $x_1 \mapsto y_1$, $y_1 \mapsto 0$.

We next recall the definition of a family of derivations, which was first considered in [34], also played an important role in [7], and was studied in detail in [28].

Definition 2.1. For $k \geq 0$, define a derivation $\varepsilon_{2k} \in \mathrm{Der}^0(\mathfrak{f}_2)$ by

$$\varepsilon_{2n}(x_1) = \mathrm{ad}(x_1)^{2n}(y_1),$$

and denote by

$$\mathfrak{u} = \mathrm{Lie}(\varepsilon_{2n}; n \geq 0) \subset \mathrm{Der}^0(\mathfrak{f}_2)$$

the Lie subalgebra generated by the ε_{2n}.

Note that $\varepsilon_2 = -\mathrm{ad}([x_1, y_1])$, and thus ε_2 is central in $\mathfrak{u}$.

We also define a Lie subalgebra $\mathfrak{u}' \subset \mathfrak{u}$ as the kernel of the canonical projection $\mathfrak{u} \to \mathbb{Q}\varepsilon_0$. Equivalently,

$$\mathfrak{u}' = \mathrm{Lie}(\mathrm{ad}^k(\varepsilon_0)(\varepsilon_{2n}); n \geq 1,\ k \geq 0).$$

As seen above, every ε_{2k} is uniquely determined by its value on x_1, while ε_0 is the only non-zero derivation $D \in \mathfrak{u}$, which annihilates y_1. From this, we get

Proposition 2.2. *The $\mathbb{Q}$-linear evaluation maps*

$$v_{x_1} : \mathfrak{u} \to \mathfrak{f}_2, \quad D \mapsto D(x_1),$$
$$v_{y_1} : \mathfrak{u}' \to \mathfrak{f}_2, \quad D \mapsto D(y_1),$$

are injective.

For the applications to elliptic multiple zeta values, it will be more natural to scale the derivations ε_{2k} as follows:

$$\widetilde{\varepsilon}_{2k} := \begin{cases} \frac{2}{(2k-2)!}\varepsilon_{2k} & k > 0 \\ -\varepsilon_0 & k = 0. \end{cases}$$

In this way, $\widetilde{\varepsilon}_{2k}$ is the image of the Eisenstein generator $\mathbf{e}_{2k}$ under the monodromy representation $\mathfrak{u}^{\mathrm{geom}} \to \mathrm{Der}^0(\mathfrak{f}_2)$ (cf. [17], Theorem 22.3).

2.2. **Iterated Eisenstein Integrals.** In a sense to be made precise below, the derivation ε_{2k} naturally corresponds to integrals of Hecke-normalized Eisenstein series of weight $2k$ (for $\mathrm{SL}_2(\mathbb{Z})$), whereas commutators of ε_{2k} correspond to *iterated integrals of Eisenstein series.* These are special cases of *iterated Shimura integrals* (or *iterated Eichler integrals*) of modular forms introduced by Manin [23], and later generalized by Brown [5].[1]

For $k \geq 0$, let $G_{2k}(q)$ be the Hecke-normalized Eisenstein series, defined by $G_0(q) := -1$ and for $k \geq 1$

$$G_{2k}(q) = -\frac{B_{2k}}{4k} + \sum_{n\geq 1} \sigma_{2k-1}(n)q^n, \quad q = e^{2\pi i \tau}$$

Here, $\sigma_\ell(n) = \sum_{d|n} d^\ell$ denotes the ℓ-th divisor function, and the B_{2k} are the Bernoulli numbers defined by

$$\frac{z}{e^z - 1} = 1 - \frac{z}{2} + \sum_{n\geq 1} B_{2n}\frac{z^{2n}}{(2n)!}.$$

Via the exponential map $\exp : \mathfrak{H} \to D^*$, $\tau \mapsto q = \exp(2\pi i\tau)$, from the upper half-plane to the punctured unit disc

$$D^* = \{q \in \mathbb{C}, 0 < |q| < 1\},$$

we may consider G_{2k} as a function of either variable q or τ, and we shall do so according to context.

Next, we define iterated integrals of Eisenstein series. More generally, if $f(q) = \sum_{n=0}^\infty a_n q^n$ is such that $a_0 = 0$, (e.g. f is a cusp form), then the definition of the indefinite integral $\int_\tau^{i\infty} f(\tau_1)\mathrm{d}\tau_1$ poses no problem, as by definition f vanishes at $i\infty$. This is not the case for the Eisenstein series G_{2k}, and consequently $\int_\tau^{i\infty} G_{2k}(\tau_1)\mathrm{d}\tau_1$ diverges. It can be regularized by setting, for $k \geq 1$,

$$\int_\tau^{i\infty} G_{2k}(\tau_1)\mathrm{d}\tau_1 := \int_\tau^{i\infty} \Big[G_{2k}(\tau_1) - G_{2k}^\infty\Big]\mathrm{d}\tau_1 - \int_0^\tau G_{2k}^\infty \mathrm{d}\tau_1,$$

where $G_{2k}^\infty = -\frac{B_{2k}}{4k}$ is the constant term in the Fourier expansion of G_{2k} (if $k = 0$, a similar method works). Note that the integral of G_{2k} so defined satisfies the differential equation $\mathrm{d}f(\tau) = -G_{2k}(\tau)\mathrm{d}\tau$. The definition of regularized iterated integrals of Eisenstein series in [5], which is a special case of Deligne's tangential base point regularization ([8], §15) generalizes this construction, and runs as follows.

[1] To be precise, Manin defined iterated Shimura integrals of cusp forms between base points on the upper half-plane (possibly cusps), and the extension to Eisenstein series (which requires a regularization procedure) is due to Brown.

Let $W = \mathbb{C}[\![q]\!]^{<1}$ be the $\mathbb{C}$-algebra of formal power series, which converge on $D = \{q \in \mathbb{C} \,|\, |q| < 1\}$. We may decompose $W = W^0 \oplus W^\infty$ with $W^0 = q\mathbb{C}[\![q]\!]$ and $W^\infty = \mathbb{C}$. For a power series $f \in W$, define f^0 to be its image in W^0 under the natural projection, and define $f^\infty \in W^\infty$ likewise. For example, in the case of the Eisenstein series $G_{2k}(q)$ with $k > 0$, we have

$$G_{2k}^\infty = -\frac{B_{2k}}{4k}, \quad G_{2k}^0(q) = \sum_{n\geq 1} \sigma_{2k-1}(n) q^n.$$

We denote by $T^c(W)$ the *shuffle algebra* on the $\mathbb{C}$-vector space W. As a $\mathbb{C}$-vector space, $T^c(W)$ is simply the graded (for the length of tensors) dual of the tensor algebra $T(W) = \bigoplus_{n\geq 0} W^{\otimes n}$. It is customary to write down elements of the dual space $(W^{\otimes n})^\vee$ using bar notation $[f_1|, \ldots, |f_n]$. Moreover, $T^c(W)$ is naturally a commutative $\mathbb{C}$-algebra, whose product is the shuffle product $\sqcup\!\sqcup$, defined by

$$[f_1|\ldots|f_r] \sqcup\!\sqcup [f_{r+1}|\ldots|f_{r+s}] = \sum_{\sigma\in\Sigma_{r,s}} f_{\sigma^{-1}(1)} \ldots f_{\sigma^{-1}(r+s)},$$

where $\Sigma_{r,s}$ denotes the set of permutations σ on $\{1, \ldots, r+s\}$, such that σ is strictly increasing on both $\{1, \ldots r\}$ and on $\{r+1, \ldots, r+s\}$.

Now define a map $R : T^c(W) \to T^c(W)$ by the formula

$$R[f_1|\ldots|f_n] = \sum_{i=0}^{n} (-1)^{n-i} [f_1|\ldots|f_i] \sqcup\!\sqcup [f_n^\infty|\ldots|f_{i+1}^\infty].$$

Following [5], eq. (4.11), we can now make the

Definition 2.3. Given $f_1, \ldots, f_n \in W$ as above, their *regularized iterated integral* is defined as

$$I(f_1, \ldots, f_n; \tau) := (2\pi i)^n \sum_{i=0}^{n} \int_\tau^{i\infty} R[f_1|\ldots|f_i]_{\mathrm{d}\tau} \int_\tau^0 [f_{i+1}^\infty|\ldots|f_n^\infty]_{\mathrm{d}\tau},$$

where

$$\int_a^b [f_1|\ldots|f_n]_{\mathrm{d}\tau} := \int\cdots\int_{a\leq\tau_1\leq\ldots\leq\tau_n\leq b} f_1(\tau_1)\ldots f_n(\tau_n)\mathrm{d}\tau_1\ldots\mathrm{d}\tau_n.$$

Remark 2.4. The reason for the $(2\pi i)^n$-prefactor is to preserve the rationality of the Fourier coefficients. More precisely, if $f_1, \ldots, f_n$ have rational coefficients (i.e. $f_i \in W_\mathbb{Q} := \mathbb{Q}[\![q]\!]^{<1}$), then $I(f_1, \ldots, f_n; \tau) \in W_\mathbb{Q}[\log(q)]$, where $\log(q) := 2\pi i\tau$.

As is the case for usual iterated integrals ([18], Sect. 2), regularized iterated integrals satisfy the differential equation

$$\frac{\partial}{\partial\tau}\Big|_{\tau=\tau_0} I(f_1, \ldots, f_n; \tau) = -f_1(\tau_0) I(f_2, \ldots, f_n; \tau_0), \tag{2.1}$$

as well as the shuffle product formula

$$I(f_1, \ldots, f_r; \tau) I(f_{r+1}, \ldots, f_{r+s}; \tau) = \sum_{\sigma\in\Sigma_{r,s}} I(f_{\sigma(1)}, \ldots, f_{\sigma(r+s)}; \tau). \tag{2.2}$$

The only case of interest for us will be when $f_1, \ldots, f_n$ are given by Eisenstein series $G_{2k_1}, \ldots, G_{2k_n}$. In this case, we set

$$\mathcal{G}_{\underline{k}}(\tau) := I(G_{2k_1}, \ldots, G_{2k_n}; \tau),$$

where $\underline{k} = (k_1, \ldots, k_n)$ and likewise denote by

$$\mathcal{I}^{\mathrm{Eis}} := \mathrm{Span}_{\mathbb{Q}}\{\mathcal{G}_{\underline{k}}(\tau)\} \subset \mathcal{O}(\mathfrak{H})$$

the $\mathbb{Q}$-span of all iterated Eisenstein integrals $\mathcal{G}_{\underline{k}}(\tau)$ for all multi-indices $\underline{k}$ (including $\mathcal{G}_{\emptyset} := 1$ for the empty index). Note that $\mathcal{I}^{\mathrm{Eis}}$ is a $\mathbb{Q}$-subalgebra of $\mathcal{O}(\mathfrak{H})$ by (2.2), and that it contains $\mathbb{Q}[2\pi i\tau]$ as a subalgebra, since $\mathcal{G}_0(\tau) = 2\pi i\tau$.

2.3. **The τ-evolution equation and the algebra of geometric elliptic multiple zeta values.** We now put together the special derivations $\widetilde{\varepsilon}_{2k}$ and the iterated Eisenstein integrals into a single, formal series

$$g(\tau) := \sum_{\underline{k}} \mathcal{G}_{\underline{k}}(\tau)\widetilde{\varepsilon}_{\underline{k}}, \tag{2.3}$$

where the sum is over all multi-indices $\underline{k} \in \mathbb{Z}^n_{\geq 0}$, for all n, and for $\underline{k} = (k_1, \ldots, k_n)$, we define $\widetilde{\varepsilon}_{\underline{k}} := \widetilde{\varepsilon}_{2k_1} \circ \ldots \circ \widetilde{\varepsilon}_{2k_n} \in \mathcal{U}(\mathfrak{u})$, the universal enveloping algebra of $\mathfrak{u}$. From (2.1), it is clear that $g(\tau)$ satisfies the differential equation

$$\frac{1}{2\pi i}\frac{\partial}{\partial\tau} g(\tau) = -\Big(\sum_{k\geq 0} G_{2k}(\tau)\widetilde{\varepsilon}_{2k}\Big) g(\tau),$$

and it follows that $g(\tau)$ is group-like, i.e. it is the exponential $g(\tau) = \exp(r(\tau))$ of a Lie series $r(\tau) \in \widehat{\mathfrak{u}} \otimes_{\mathbb{Q}} \mathcal{I}^{\mathrm{Eis}}$ (here $\widehat{\mathfrak{u}}$ is the graded completion of $\mathfrak{u}$, and $\otimes$ denotes the completed tensor product).

Definition 2.5. Define the $\mathbb{Q}$-algebra $\mathcal{E}^{\mathrm{geom}}$ of geometric elliptic multiple zeta values to be the $\mathbb{Q}$-algebra generated by the coefficients of $r(\tau) \cdot x_1$.

Equivalently, $\mathcal{E}^{\mathrm{geom}}$ is equal to the $\mathbb{Q}$-vector space linearly spanned by the coefficients of the series $g(\tau) \cdot e^{x_1}$, because the coefficients of each of the power series $r(\tau) \cdot x_1$ and $g(\tau) \cdot e^{x_1}$ can be written as algebraic expressions in the coefficients of the other. Also, note that since every derivation in $\mathfrak{u}$ is uniquely determined by its value on x_1, the $\mathbb{Q}$-algebra $\mathcal{E}^{\mathrm{geom}}$ is also the same as the $\mathbb{Q}$-algebra spanned by the coefficients of $g(\tau)$, viewed as a series in the monomials $\widetilde{\varepsilon}_{2k_1} \circ \ldots \circ \widetilde{\varepsilon}_{2k_n}$.

We can now state the main result of §2.

Theorem 2.6. *For every $\mathbb{Q}$-subalgebra $A \subset \mathbb{C}$, there is an isomorphism*

$$\mathcal{U}(\mathfrak{u})^{\vee} \otimes_{\mathbb{Q}} A \cong \mathcal{E}^{\mathrm{geom}} \otimes_{\mathbb{Q}} A$$

of A-algebras. In particular, $\mathcal{E}^{\mathrm{geom}}$ is a commutative, graded Hopf algebra in a natural way.

Proof. The main ingredient in the proof will be to show that the iterated Eisenstein integrals $\mathcal{G}_{\underline{k}}(\tau)$ are linearly independent over $\mathbb{C}$, as functions in τ. More precisely, by Corollary 2.9, proved in the next section, there is a natural isomorphism

$$\mathcal{I}^{\mathrm{Eis}} \otimes_{\mathbb{Q}} A \cong T^c(V_{\mathrm{Eis}}) \otimes_{\mathbb{Q}} A,$$

where $T^c(V_{\mathrm{Eis}})$ is the shuffle algebra on the $\mathbb{Q}$-vector space V_{Eis} spanned by all Eisenstein series G_{2k}, $k \geq 0$.

Assuming Corollary 2.9 for the moment, the proof of Theorem 2.6 proceeds as follows. Since the tensor algebra $T(V_{\mathrm{Eis}})$ is freely generated by one element in every even degree $2k \geq 0$, we get a canonical surjection $T(V_{\mathrm{Eis}}) \to \mathcal{U}(\mathfrak{u})$ of $\mathbb{Q}$-algebras, which induces by duality an injection

$$\iota : \mathcal{U}(\mathfrak{u})^{\vee} \hookrightarrow T^c(V_{\mathrm{Eis}}) \cong \mathcal{I}^{\mathrm{Eis}}.$$

On the other hand, choosing a (homogeneous) linear basis $\mathcal{B}$ of $\mathcal{U}(\mathfrak{u})$, the element $g(\tau)$ naturally defines a map

$$\widetilde{\iota}: \mathcal{U}(\mathfrak{u})^\vee \hookrightarrow \mathcal{I}^{\mathrm{Eis}}$$
$$b^\vee \mapsto b^\vee(g(\tau)),$$

where $b^\vee \in \mathcal{B}^\vee$ are the dual basis elements. Clearly, the image of $\widetilde{\iota}$ does not depend on the choice of basis, and equals $\mathcal{E}^{\mathrm{geom}}$ by definition. On the other hand, it is easy to see that the maps $\iota, \widetilde{\iota}: \mathcal{U}(\mathfrak{u})^\vee \to \mathcal{I}^{\mathrm{Eis}}$ are equal, whence the result for $A = \mathbb{Q}$, and the general case follows simply by extension of scalars. Finally, it is well-known that the universal enveloping algebra of any graded Lie algebra has a natural structure of a (cocommutative) graded Hopf algebra, thus $\mathcal{U}(\mathfrak{u})^\vee$ is naturally a (commutative) graded Hopf algebra. □

2.4. **Linear independence.** In this subsection, we complete the proof of Theorem 2.6 by proving that the family of iterated Eisenstein integrals is linearly independent over $\mathbb{C}$, and that as a consequence $\mathcal{I}^{\mathrm{Eis}} \otimes_{\mathbb{Q}} \mathbb{C} \cong T^c(V_{\mathrm{Eis}}) \otimes_{\mathbb{Q}} \mathbb{C}$ as $\mathbb{C}$-algebras. In fact, the linear independence statement we prove is more general, and shows that iterated Eisenstein integrals are linearly independent over a certain function field in one variable.

We will use the following general linear independence result.

Theorem 2.7 ([9]). *Let $(\mathcal{A}, \mathrm{d})$ be a differential algebra over a field k of characteristic zero, whose ring of constants $\ker(\mathrm{d})$ is precisely equal to k. Let $\mathcal{C}$ be a differential subfield of $\mathcal{A}$ (i.e. a subfield such that $\mathrm{d}\mathcal{C} \subset \mathcal{C}$), X any set with associated free monoid X^*. Suppose that $S \in \mathcal{A}\langle\langle X\rangle\rangle$ is a solution to the differential equation*

$$\mathrm{d}S = M \cdot S,$$

where $M = \sum_{x\in X} u_x x \in \mathcal{C}\langle\langle X\rangle\rangle$ is a homogeneous series of degree 1, with initial condition $S_1 = 1$, where S_1 denotes the coefficient of the empty word in the series S. The following are equivalent:

(i) *The family of coefficients $(S_w)_{w\in X^*}$ of S is linearly independent over $\mathcal{C}$.*
(ii) *The family $\{u_x\}_{x\in X}$ is linearly independent over k, and we have*

$$\mathrm{d}\mathcal{C} \cap \mathrm{Span}_k(\{u_x\}_{x\in X}) = \{0\}. \tag{2.4}$$

Using this theorem, we can now prove linear independence of iterated Eisenstein integrals.

Theorem 2.8. *The family $\{\mathcal{G}_{\underline{k}}(\tau)\}$ is linearly independent over $\mathrm{Frac}(\mathbb{Z}[\![q]\!])$.*

Proof. We will apply Theorem 2.7 with the following parameters:

- $k = \mathbb{Q}$, $\mathcal{A} = \mathbb{Q}[\log(q)](\!(q)\!)$ with differential $\mathrm{d} = q\frac{\partial}{\partial q}$, and $\mathcal{C} = \mathrm{Frac}(\mathbb{Z}[\![q]\!])$ (the latter is a differential field by the quotient rule for derivatives)
- $X = \{a_{2k}\}_{k\geq 0}$, $u_{a_{2k}} = -G_{2k}(q)$, hence

$$M(q) = -\sum_{k\geq 0} G_{2k}(q) a_{2k}.$$

With these conventions, it follows from (2.1) that the formal series

$$1 + \int_q^0 [M]_{\mathrm{d}\log q} + \int_q^0 [M|M]_{\mathrm{d}\log q} + \ldots \in \mathcal{O}(\mathfrak{H})\langle\langle X\rangle\rangle,$$

with the iterated integrals regularized as in Section 2.2, is a solution of the differential equation $\mathrm{d}S = M \cdot S$, with $S_1 = 1$. Consequently, the coefficient of the word $w = a_{2k_1} \dots a_{2k_n}$ in S is equal to $\mathcal{G}(2k_1, \dots, 2k_n; \tau)$. Moreover, since the $\mathbb{Q}$-linear independence of the Eisenstein series is well-known (cf. e.g. [33], VII.3.2), it remains to verify (2.4) in our situation.

To this end, assume that there exist $\alpha_{2k} \in \mathbb{Q}$, all but finitely many of which are equal to zero, such that

$$\sum_{k \geq 0} \alpha_{2k} G_{2k}(q) \in \mathrm{d}\mathcal{C}. \tag{2.5}$$

Clearing denominators, we may assume that $\alpha_{2k} \in \mathbb{Z}$. Furthermore, from the definition of $\mathrm{d} = q\frac{\partial}{\partial q}$, one sees that the image $\mathrm{d}\mathcal{C}$ of the differential operator d does not contain any constant except for zero. Therefore, the coefficient of the trivial word 1 in (2.5) vanishes; in other words

$$\sum_{k \geq 0} \alpha_{2k} G_{2k}(q) = \sum_{k \geq 1} \alpha_{2k} E_{2k}^0(q) \in q\mathbb{Q}[\![q]\!].$$

Now the differential d is invertible on $q\mathbb{Q}[\![q]\!]$, and inverting d is the same as integrating. Hence (2.5) is equivalent to

$$\sum_{k \geq 1} \alpha_{2k} \mathcal{G}_{2k}^0(\tau) \in \mathcal{C}, \quad \mathcal{G}_{2k}^0(\tau) := \int_q^0 E_{2k}^0(q_1) \frac{\mathrm{d}q_1}{q_1}. \tag{2.6}$$

But this is absurd, unless all the α_{2k} vanish, as we shall see now. Indeed, if $f \in \mathcal{C} = \mathrm{Frac}(\mathbb{Z}[\![q]\!])$, then there exists $m \in \mathbb{Z} \setminus \{0\}$ such that $f \in \mathbb{Z}[m^{-1}](\!(q)\!)$. This follows from the well-known inversion formula for power series. On the other hand, the coefficient of q^p in $\mathcal{G}_{2k}^0(\tau)$, for p a prime number, is given by

$$\frac{\sigma_{2k-1}(p)}{p} = \frac{p^{2k-1}+1}{p} \equiv \frac{1}{p} \mod \mathbb{Z}.$$

Thus, we must have $\frac{1}{p}\sum_{k \geq 1} \alpha_{2k} \in \mathbb{Z}[m^{-1}]$, for every prime number p, in particular $\sum_{k \geq 1} \alpha_{2k}$ is divisible by infinitely many primes (namely, at least all the primes which don't divide m), which implies $\sum_{k \geq 1} \alpha_{2k} = 0$.

Now assume that k_1 is the smallest positive, even integer with the property that $\alpha_{k_1} \neq 0$. Consider the coefficient of $q^{p^{k_1}}$ in $\mathcal{G}_{2k}^0(\tau)$, which is equal to

$$\frac{\sigma_{2k-1}(p^{k_1})}{p^{k_1}} = \frac{1}{p^{k_1}} \sum_{j=0}^{k_1} p^{j(2k-1)} \equiv \begin{cases} \frac{1}{p^{k_1}} \mod \mathbb{Z} & \text{if } 2k > k_1 \\ \frac{1}{p^{k_1}} + \frac{1}{p} \mod \mathbb{Z} & \text{if } 2k = k_1. \end{cases}$$

By (2.6), we have $\frac{\alpha_{k_1}}{p} + \frac{1}{p^{k_1}} \sum_{k \geq 1} \alpha_{2k} \in \mathbb{Z}[m^{-1}]$, and by what we have seen before, $\sum_{k \geq 1} \alpha_{2k} = 0$. Hence $\frac{\alpha_{k_1}}{p} \in \mathbb{Z}[m^{-1}]$, for every prime number p, which again implies $\alpha_{k_1} = 0$, in contradiction to our assumption $\alpha_{k_1} \neq 0$. Therefore, in (2.6), we must have $\alpha_{2k} = 0$ for all $k \geq 1$ and (2.4) is verified. □

Corollary 2.9. *The iterated Eisenstein integrals $\mathcal{G}_{\underline{k}}(\tau)$ are $\mathbb{C}$-linearly independent, and for every $\mathbb{Q}$-subalgebra $A \subset \mathbb{C}$, we have a natural isomorphism of A-algebras*

$$\psi_A : T^c(V_{\mathrm{Eis}}) \otimes_{\mathbb{Q}} A \to \mathcal{I}^{\mathrm{Eis}} \otimes_{\mathbb{Q}} A$$
$$[G_{2k_1}| \dots |G_{2k_n}] \mapsto \mathcal{G}_{\underline{k}}(\tau),$$

where $\underline{k} = (k_1, \dots, k_n)$ and $V_{\mathrm{Eis}} = \mathrm{Span}_{\mathbb{Q}}\{G_{2k}(\tau) \,|\, k \geq 0\} \subset \mathcal{O}(\mathfrak{H})$.

Proof. Since $\mathbb{Q} \subset \mathrm{Frac}(\mathbb{Z}[\![q]\!])$, Theorem 2.8 shows in particular that the $\mathcal{G}_{\underline{k}}$ are linearly independent over $\mathbb{Q}$. Since the Eisenstein series G_{2k} have coefficients in $\mathbb{Q}$, it follows from the definition that $\mathcal{G}_{\underline{k}} \in \mathbb{Q}(\!(q)\!)[\log(q)]$, and elements of $W_{\mathbb{Q}}[\log(q)] = \mathbb{Q}(\!(q)\!)[\log(q)]$ are linearly independent over $\mathbb{Q}$, if and only they are so over $\mathbb{C}$.

For the second statement, it is clear that ψ_A is a homomorphism of $\mathbb{Q}$-algebras (since both sides are endowed with the shuffle product) and that it is surjective. The injectivity of ψ_A is just the A-linear independence of iterated Eisenstein integrals. □

Corollary 2.10. *We have $\mathcal{I}^{\mathrm{Eis}} \cap \mathbb{C} = \mathbb{Q}$ and $\mathcal{E}^{\mathrm{geom}} \cap \mathbb{C} = \mathbb{Q}$. In particular, the $\mathbb{Q}$-subalgebra of $\mathcal{O}(\mathfrak{H})$ generated by $\mathcal{I}^{\mathrm{Eis}}$ and $\mathbb{C}$ is canonically isomorphic to $\mathcal{I}^{\mathrm{Eis}} \otimes_{\mathbb{Q}} \mathbb{C}$.*

Proof. If some linear combination of the $\mathcal{G}_{\underline{k}}$ with coefficients in $\mathbb{Q}$ were equal to $c \in \mathbb{C}$, then since $\mathcal{G}_{\emptyset} = 1$, this would give a linear relation

$$-c\mathcal{G}_{\emptyset} + \sum_{\underline{k}} a_{\underline{k}} \mathcal{G}_{\underline{k}} = 0,$$

so by Theorem 2.8 we must have $c = a_{\emptyset}$, i.e. $c \in \mathbb{Q}$. The second statement follows from the first, since by definition of $\mathcal{E}^{\mathrm{geom}}$, it lies inside $\mathcal{I}^{\mathrm{Eis}}$. □

3. The generating series of elliptic multiple zeta values

In the first part of this section we will recall the definition of the elliptic associator defined by B. Enriquez and use it to define a power series $E \in F_2(\mathcal{Z})$; we then set $E(\tau) = g(\tau) \cdot E$, where $g(\tau)$ is the automorphism studied in the previous section. We call $E(\tau)$ the *elliptic generating series*, and its coefficients the *elliptic multizeta values*, or elliptic multiple zeta values. We define $\mathcal{E}$ to be the $\mathbb{Q}$-algebra generated by the elliptic multiple zeta values. This algebra is essentially the same as the one generated by the coefficients of the elliptic associator, but the elliptic multizeta values themselves are different from those coefficients (which are called elliptic analogs of multizeta values by Enriquez).

In the remainder of the section, we work modulo $2\pi i$. In particular, we consider the power series $\overline{\Phi}_{KZ}$ and $\overline{E}$ which are obtained from Φ_{KZ} and E by reducing the coefficients from $\mathcal{Z}$ to $\overline{\mathcal{Z}} = \mathcal{Z}/\langle(2\pi i)^2\rangle$.

In §3.2, we give an expression for $\overline{E}$ which relates it explicitly to the Drinfel'd associator $\overline{\Phi}_{KZ}$. In §3.3 we use this expression for $\overline{E}$ to prove the equality

$$\overline{\mathcal{E}}[2\pi i\tau] = \mathcal{E}^{\mathrm{geom}} \otimes_{\mathbb{Q}} \overline{\mathcal{Z}}.$$

These two results will allow us to compute the algebraic relations satisfied by the elliptic multizeta values, as well as algebraic relations satisfied by Enriquez' elliptic analogs of multizeta values, which are the coefficients of the elliptic associator (always modulo $2\pi i$). Because these results necessitate a very brief introduction to mould theory, we introduce them in §4.

3.1. **Definition of the elliptic generating series $E(\tau)$.** Throughout this section, we use the following change of variables: $a = y_1$ and $b = x_1$. This change of variables will be applied to all the expressions in x_1, y_1 encountered in the previous section, such as $g(\tau) \cdot y_1$, and we will also express other quantities studied by B. Enriquez in terms of a and b, in particular the elliptic associator. The purpose of this change of variables is for the application of mould theory in §4.

Let Ass_μ denote the set of genus zero associators $\Phi \in F_2(\mathbb{C})$ such that the coefficient of ab in Φ is equal to $\mu^2/24$ [10]. We will use the same elements t_{01}, t_{02}, t_{12} as in [13], but rewritten in the variables a, b:

$$t_{01} = Ber_b(-a), \quad t_{02} = Ber_{-b}(a), \quad t_{12} = [a, b], \tag{3.1}$$

where

$$Ber_x(y) = \frac{\mathrm{ad}(x)}{e^{\mathrm{ad}(x)} - 1}(y),$$

so that $t_{01} + t_{02} + t_{12} = 0$. Recall that Enriquez showed that a section from Ass_μ to the set of elliptic associators is given by mapping $\Phi \in Ass_\mu$ to the elliptic associator (μ, Φ, A, B) defined by

$$A = \Phi(t_{01}, t_{12}) e^{\mu\, t_{01}} \Phi(t_{01}, t_{12})^{-1}$$
$$B = e^{\mu\, t_{12}/2} \Phi(t_{02}, t_{12}) e^{b} \Phi(t_{01}, t_{12})^{-1}$$

(this is denoted (μ, Φ, A_+, A_-) in [13]).

In this section we take $\mu = 2\pi i$, so $\mu^2/24 = -\zeta(2)$, and consider Φ_{KZ}, the Drinfeld associator, whose coefficients are the (shuffle-regularized) multiple zeta values [15]. The Lie algebra $\mathfrak{f}_2 = \mathrm{Lie}[\![a, b]\!]$ is topologically generated by a and b, but since the operator Ber_b is invertible, we have

$$a = -Ber_b^{-1}(t_{01}) = \Big(\frac{e^{\mathrm{ad}(b)} - 1}{\mathrm{ad}(b)}\Big)(-t_{01}), \tag{3.2}$$

so that we can just as well take t_{01} and b as generators. Similarly, we can take $e^{t_{01}}$ and e^b as generators of the group $F_2 = F_2(\mathbb{Q}) = \exp(\mathfrak{f}_2)$, which is a priori generated by e^a and e^b.

Let us define an automorphism σ of $F_2(\mathcal{Z})$, where $\mathcal{Z}$ is the $\mathbb{Q}$-algebra of multiple zeta values, by

$$\sigma(e^{t_{01}}) = \Phi_{KZ}(t_{01}, t_{12}) e^{t_{01}} \Phi_{KZ}(t_{01}, t_{12})^{-1}$$
$$\sigma(e^{b}) \ = e^{\pi i t_{12}} \Phi_{KZ}(t_{02}, t_{12}) e^{b} \Phi_{KZ}(t_{01}, t_{12})^{-1}.$$

We set

$$E = 1 - a + \sigma(a), \quad C = \exp(E - 1).$$

The automorphism σ extends to an automorphism of the completed enveloping algebra $\mathcal{U}(\mathfrak{f}_2)$, and restricts to an automorphism of $\mathfrak{f}_2$. Thus the power series $\sigma(a)$ is Lie-like, so $E - 1$ is Lie-like. Thus, by Lazard elimination, it can be expressed in the variables a and $c_i = \mathrm{ad}(a)^{i-1}(b)$, $i \geq 1$. From now on, we expand all group-like and Lie-like power series in these variables, and when we refer to the *coefficients* of such power series, we intend the coefficients of the power series in these variables. (This language is adapted to mould theory and will be useful in §4.) Up to degree 4, the explicit expansion of E is given by

$$E = 1 - \frac{i\pi}{2} c_3 + \frac{\pi^2}{6} c_4 + \frac{i\pi}{12}[c_1, c_3].$$

We now recall the automorphism

$$g(\tau) = \sum_{\underline{k}} \mathcal{G}_{\underline{k}}(\tau) \widetilde{\varepsilon}_{\underline{k}}$$

defined in the previous section, and consider it as an automorphism of the group $F_2(\mathcal{E}^{\mathrm{geom}} \otimes_{\mathbb{Q}} \mathcal{Z})$. Acting on a, we find

$$g(\tau) \cdot a = a - \frac{1}{2\pi i}\mathcal{G}_2(\tau)\,\mathrm{ad}(a)^2(b) + \frac{3}{(2\pi i)^2}\mathcal{G}_{0,2}(\tau)\,\mathrm{ad}(b)^2(a) + \cdots$$

In [13], Enriquez studied the elliptic associator

$$\big(2\pi i, \Phi_{KZ}, A(\tau), B(\tau)\big) \tag{3.3}$$

where

$$A(\tau) = g(\tau) \cdot A, \quad B(\tau) = g(\tau) \cdot B.$$

In analogy with this, we set

$$E(\tau) = g(\tau) \cdot E = g(\tau)\big(1 - a + \sigma(a)\big), \quad C(\tau) = \exp\big(E(\tau) - 1\big).$$

As above, $g(\tau)$ extends to an automorphism of the universal enveloping algebra, so in particular it preserves the Lie algebra $\mathfrak{f}_2 \otimes_{\mathbb{Q}} (\mathcal{E}^{\mathrm{geom}} \otimes_{\mathbb{Q}} \mathcal{Z})$. Thus $E(\tau) - 1$ is Lie-like, and $C(\tau)$ is group-like.

Definition 3.1. The Lie-like power series $E(\tau) - 1$ is called the *elliptic generating series*, and its coefficients are the *elliptic multiple zeta values* or elliptic multizeta values. For $\underline{k} = (k_1, \ldots, k_r)$ we write $E(\underline{k})$ for the coefficient in $E(\tau) - 1$ of the monomial $c_{k_1} \cdots c_{k_r}$. The $\mathbb{Q}$-algebra generated by the elliptic multiple zeta values $E(\underline{k})$ is denoted $\mathcal{E}$.

We can use $C(\tau)$ to obtain a vector space basis for $\mathcal{E}$.

Lemma 3.2. *The underlying vector space of $\mathcal{E}$ is spanned by the coefficients of $C(\tau)$.*

Proof. Let $\mathcal{E}'$ denote the $\mathbb{Q}$-vector space generated by the coefficients of $C(\tau)$. Then $\mathcal{E}'$ is in fact a $\mathbb{Q}$-algebra, because $C(\tau)$ is a group-like power series so that the product of two of its coefficients can be written as a linear combination of such by using the (multiplicative) shuffle relations. Since $E(\tau) = 1 + \log(C(\tau))$, we see that the coefficients of $E(\tau)$ can be expressed as algebraic and thus linear combinations of the coefficients of $C(\tau)$, so that $\mathcal{E} \subset \mathcal{E}'$. Conversely, since $C(\tau) = \exp\big(E(\tau) - 1\big)$, the coefficients of $C(\tau)$ are algebraic combinations of those of $E(\tau)$, and therefore lie in $\mathcal{E}$, so $\mathcal{E}' \subset \mathcal{E}$, which completes the proof. □

3.2. **An expression for E modulo $2\pi i$.** From now until the end of this section, we work modulo $2\pi i$, in the sense that if a series has coefficients in $\mathcal{Z}$, we reduce these coefficients to the quotient $\overline{\mathcal{Z}}$ of $\mathcal{Z}$ modulo the idea generated by $(2\pi i)^2$, or equivalently, by $\zeta(2)$. We use overlining to denote the reduced objects. The goal of the section is to obtain an expression for $\overline{E}$ that relates it directly to the reduced Drinfeld associator $\overline{\Phi}_{KZ}$.

In order to approach this result, we will move from the Lie algebra of derivations over to power series in a and b by using the map given by evaluation at a. This is important because it allows us to compare derivations with power series in a and b such as $\overline{\Phi}_{KZ}$.

Let v_a denote the linear map given by evaluation at a, i.e.

$$\begin{aligned} v_a : \mathrm{Der}^0(\mathfrak{f}_2) &\to \mathfrak{f}_2 \\ D &\mapsto D(a). \end{aligned} \tag{3.4}$$

Let the push-operator be defined to cyclically permute the powers of a between the letters b in a monomial:

$$push(a^{k_0}b\cdots ba^{k_r}) = a^{k_r}ba^{k_0}b\cdots a^{k_{r-1}}, \tag{3.5}$$

extended to polynomials and power series by linearity. A power series is said to be *push-invariant* if $push(p)=p$. It is shown in [32] that the restriction of v_a to the Lie subalgebra generated by $\mathrm{Der}^0(\mathfrak{f}_2)\setminus\mathbb{Q}\varepsilon_0$ is an injective linear map whose image is equal to the space of push-invariant Lie series $\mathfrak{f}_2^{\mathrm{push}}\subset\mathfrak{f}_2$. The map v_a transports the Lie bracket and exponential from $\mathrm{Der}^0(\mathfrak{f}_2)$ to $\mathfrak{f}_2^{\mathrm{push}}$ as follows:

$$\langle D(a), D'(a)\rangle = [D,D'](a), \quad \exp_a\big(D(a)\big) = 1+\sum_{n\geq 1}\frac{1}{n!}D^n(a) \tag{3.6}$$

We have the useful identity

$$\exp(D)\cdot a = a + D(a) + \frac{1}{2}D^2(a)+\cdots = a-1+\exp_a(D(a)). \tag{3.7}$$

Let $\mathfrak{grt}_{\mathrm{ell}}$ be the elliptic Grothendieck-Teichmüller Lie algebra defined by B. Enriquez in [13]. Not surprisingly, this Lie algebra will be an essential tool in proving our results. Let us recall some of the basic facts concerning it. Firstly, Enriquez showed that there is a natural Lie morphism $\mathfrak{grt}_{\mathrm{ell}}\to\mathrm{Der}^0(\mathfrak{f}_2)$. It was further shown in [32] that this map is injective. We will identify $\mathfrak{grt}_{\mathrm{ell}}$ with its image in $\mathrm{Der}^0(\mathfrak{f}_2)$.

Enriquez also proved the following results. There is a canonical surjection $\mathfrak{grt}_{\mathrm{ell}}\to\mathfrak{grt}$. Let $\mathfrak{r}_{\mathrm{ell}}$ denote the kernel; then it is easy to see that $\mathfrak{u}\subset\mathfrak{r}_{\mathrm{ell}}$. Finally, Enriquez gave a section $\gamma:\mathfrak{grt}\to\mathfrak{grt}_{\mathrm{ell}}$ of the canonical surjection, and $\mathfrak{grt}_{\mathrm{ell}}$ has the form of a semi-direct product

$$\mathfrak{grt}_{\mathrm{ell}}\cong\mathfrak{r}_{\mathrm{ell}}\rtimes\gamma(\mathfrak{grt}).$$

We write γ_a for the composition map $v_a\circ\gamma$, so that

$$\gamma_a:\mathfrak{grt}\to\mathfrak{f}_2^{\mathrm{push}}. \tag{3.8}$$

Let $\exp^{\odot}$ denote the ("twisted Magnus") exponential map $\exp^{\odot}:\mathfrak{grt}\to GRT$. Then we have the commutative diagram

$$\begin{array}{ccccccc}
\mathrm{Der}^*(\mathrm{Lie}[[x,y]]) & \leftarrow & \mathfrak{grt} & \stackrel{\gamma}{\to} & \mathfrak{grt}_{\mathrm{ell}} & \stackrel{v_a}{\longrightarrow} & \mathfrak{f}_2^{\mathrm{push}}\\
\exp\downarrow & & \exp^{\odot}\downarrow & & exp\downarrow & & \downarrow\exp_a\\
\mathrm{Aut}^*(\mathrm{Lie}[[x,y]]) & \leftarrow & GRT & \stackrel{\Gamma}{\to} & GRT_{\mathrm{ell}} & \stackrel{1-a+v_a}{\longrightarrow} & F_2,
\end{array}$$

where Γ is the group homomorphism that makes the middle square commmute. The upper map $\mathfrak{grt}\to\mathrm{Der}^*(\mathrm{Lie}[[x,y]])$ in the left-hand square is the map that takes a Lie element $\psi\in\mathfrak{f}_2$ to the associated *Ihara derivation* D_ψ defined by

$$D_\psi(x)=0, \qquad D_\psi(y)=[\psi(x,y),y]. \tag{3.9}$$

Ihara [20, 21] studied these derivations in detail, and in particular, he showed that if $\Psi=\exp^{\odot}(\psi)$ and A_Ψ denotes the automorphism $\exp(D_\psi)$ of $\mathcal{U}(\mathrm{Lie}[[x,y]])$, then

$$A_\Psi(x)=x, \qquad A_\Psi(y)=\Psi\ y\ \Psi^{-1}. \tag{3.10}$$

The lower horizontal map of the left-hand square is thus given by $\Psi\mapsto A_\Psi$. In analogy with γ_a, we set $\Gamma_a=v_a\circ\Gamma$.

We can now state the main result of this subsection.

Theorem 3.3. *Let $\overline{E}$ be obtained from E by reducing the coefficients from $\overline{\mathcal{Z}}$ to $\mathcal{Z}/\langle(2\pi i)^2\rangle$. Then*

$$\overline{E} = \Gamma_a(\overline{\Phi}_{KZ}).$$

Proof. Let $\psi \in \mathfrak{grt}$, and let $\Psi = \exp^{\odot}(\psi) \in GRT$. Then $\gamma(\psi) \in \mathfrak{grt}_{\mathrm{ell}} \subset \mathrm{Der}^0(\mathfrak{f}_2)$ and $\Gamma(\Psi) = \exp\big(\gamma(\psi)\big) \in GRT_{\mathrm{ell}} \subset \mathrm{Aut}^0(\mathfrak{f}_2)$. The proof is based on a result from [13], Lemma-Definition 4.6, which states that the automorphism $\Gamma(\Psi)$ acts as follows:

$$\begin{aligned} \Gamma(\Psi)(t_{01}) &= \Psi(t_{01},t_{12})t_{01}\Psi(t_{01},t_{12})^{-1} \\ \Gamma(\Psi)(b) &= \log\big(\Psi(t_{02},t_{12})e^b\Psi(t_{01},t_{12})^{-1}\big), \end{aligned} \tag{3.11}$$

where t_{01} is as in (3.1). Recall from (3.2) that we can take t_{01} and b as generators of $\mathfrak{f}_2$.

Recall that $\overline{\Phi}_{KZ} \in GRT \otimes_{\mathbb{Q}} \overline{\mathcal{Z}}$. (This is the reason for which we work mod $2\pi i$, since the term $-\zeta(2)[x,y]$ in Φ_{KZ} means that it does not lie in GRT, preventing us from taking advantage of the results on $\mathfrak{grt}_{\mathrm{ell}}$.) Set $\phi_{KZ} = \log^{\odot}(\overline{\Phi}_{KZ})$, so that $\phi_{KZ} \in \mathfrak{grt}\otimes_{\mathbb{Q}}\overline{\mathcal{Z}}$. Let $\overline{\sigma}$ be the automorphism of $F_2(\overline{\mathcal{Z}})$ obtained from σ by reducing modulo $2\pi i$, i.e.

$$\begin{aligned} \overline{\sigma}(e^{t_{01}}) &= \overline{\Phi}_{KZ}(t_{01},t_{12})e^{t_{01}}\overline{\Phi}_{KZ}(t_{01},t_{12})^{-1} \\ \overline{\sigma}(e^b) &= \overline{\Phi}_{KZ}(t_{02},t_{12})e^b\overline{\Phi}_{KZ}(t_{01},t_{12})^{-1}. \end{aligned}$$

Comparing with the values of $\Gamma(\overline{\Phi}_{KZ})$ from (3.11) on the generators t_{01}, b of $\mathfrak{f}_2$, we find that $\overline{\sigma} = \Gamma(\overline{\Phi}_{KZ})$, so $\log(\overline{\sigma}) = \gamma(\phi_{KZ})$. Evaluating on a, we have

$$\log(\overline{\sigma})(a) = v_a\big(\gamma(\phi_{KZ})\big) = \gamma_a(\phi_{KZ}),$$

so by (3.7), we have

$$\overline{\sigma}(a) = a - 1 + \exp_a\big(\gamma_a(\phi_{KZ})\big) = a - 1 + \Gamma_a(\overline{\Phi}_{KZ}).$$

Since $E = 1 - a + \sigma(a)$, we have

$$\overline{E} = 1 - a + \overline{\sigma}(a) = \Gamma_a(\overline{\Phi}_{KZ}),$$

which concludes the proof. □

Corollary 3.4. *The $\mathbb{Q}$-algebra generated by the coefficients of $\overline{E}$ is all of $\overline{\mathcal{Z}}$.*

Proof. As remarked earlier (Lemma 3.2), the $\mathbb{Q}$-algebra linearly spanned by the coefficients of a group-like power series is equal to that multiplicatively generated by the coefficients of its log. Therefore in particular, since the coefficients of $\overline{\Phi}_{KZ}$ linearly span $\overline{\mathcal{Z}}$, the coefficients of ϕ_{KZ} multiplicatively generate the same ring. Similarly, the $\mathbb{Q}$-algebra generated by the coefficients of $\gamma(\phi_{KZ})$ (written in a basis of $\mathfrak{grt}$, say) is the same as the one linearly spanned by the coefficients of $\overline{E} = \Gamma(\overline{\Phi}_{KZ})$. But since the section map $\gamma_a : \mathfrak{grt} \to \mathfrak{f}_2^{\mathrm{push}}$ is injective and defined over $\mathbb{Q}$, it maps a linear basis of $\mathfrak{grt}$ to linearly independent elements of $\mathfrak{f}_2^{\mathrm{push}}$ with the same coefficients, so the coefficients of $\gamma_a(\phi_{KZ})$ again generate the same $\mathbb{Q}$-algebra as those of ϕ_{KZ}, which is $\overline{\mathcal{Z}}$. □

3.3. **Structure of the $\mathbb{Q}$-algebra $\overline{\mathcal{E}}$.** Since $E(\tau) = g(\tau)\cdot E$, the $\mathbb{Q}$-algebra $\mathcal{E}$ generated by the coefficients of $E(\tau)$ is contained in the $\mathbb{Q}$-algebra generated by $\mathcal{E}^{\mathrm{geom}}$ (the ring generated by the coefficients of $g(\tau)$) together with the multizeta algebra $\mathcal{Z}$ (generated by the coefficients of E). Thanks to Corollary 2.10, the algebra generated by these two rings is equal to their tensor product over $\mathbb{Q}$. Thus, working modulo $2\pi i$, the algebra generated by $\mathcal{E}^{\mathrm{geom}}$ and $\overline{\mathcal{Z}}$ is also equal to their tensor product. The main result of this subsection is the following comparison of the $\mathbb{Q}$-algebra $\overline{\mathcal{E}}$ generated by the coefficients of $\overline{E}(\tau)$ with $\mathcal{E}^{\mathrm{geom}}\otimes_{\mathbb{Q}}\overline{\mathcal{Z}}$.

Theorem 3.5. *We have* $\overline{\mathcal{E}}[2\pi i\tau]\cong\mathcal{E}^{\mathrm{geom}}\otimes_{\mathbb{Q}}\overline{\mathcal{Z}}$.

Proof. Let v_a denote the evaluation map introduced in (3.4). The exponential $\exp_a\bigl(\mathfrak{f}_2^{\mathrm{push}}\bigr)$ forms a group under the group law given by the Campbell-Hausdorff formula

$$\exp_a(f)*\exp_a(g)=\exp_a\bigl(ch\langle\, ,\,\rangle(f,g)\bigr).$$

The automorphism $A=\exp(D)$ acts on the group $\exp_a(\mathfrak{f}_2^{\mathrm{push}})$ via this multiplication, i.e.

$$A\bigl(\exp_a(f)\bigr)=\exp(D)\cdot\exp_a(f)=\exp_a\bigl(D(a)\bigr)*\exp_a(f). \tag{3.12}$$

We will use the multiplication law (3.12) to express the action of the automorphism $g(\tau)$ defined in (2.3) on $\overline{E}$.

We will need to use a linear basis of $\mathfrak{u}$ that is adapted to the depth grading. Recall that $\mathfrak{u}=\langle\varepsilon_0\rangle\oplus\mathfrak{u}'$. Let $u_0=\varepsilon_0$. For each $r\geq 1$, let $\mathfrak{u}'_r$ denote the subspace of derivations $D\in\mathfrak{u}'$ such that $v_a(D)$ is of homogeneous b-degree r. Let u_i, $i\geq 1$ denote a linear basis for $\mathfrak{u}'$ that is depth-graded, in the sense that each basis element u_i lies in some $\mathfrak{u}'_r$. Let $V=v_a(\mathfrak{u})$ and $V'=v_a(\mathfrak{u}')$, and for each $r\geq 1$, let $V'_r=v_a(\mathfrak{u}'_r)$. The images $v_i=v_a(u_i)$ with $u_i\in\mathfrak{u}'_r$ form a basis of $V'_r\subset\mathfrak{f}_2^{\mathrm{push}}$, since v_a is injective on $\mathfrak{u}'$ by Proposition 2.2. The u_i for $i\geq 0$ form a basis for $\mathfrak{u}$.

Let $r(\tau)=\log\bigl(g(\tau)\bigr)$. Since $r(\tau)\in\mathfrak{u}$, we can expand it in the basis u_i. We write

$$r(\tau)=\sum_{i\geq 0}r_iu_i. \tag{3.13}$$

Each coefficient r_i is an algebraic (so given the shuffle product, linear) expression in the $\mathcal{G}_{\underline{k}}$, and together they generate $\mathcal{E}^{\mathrm{geom}}\cong\mathcal{U}(\mathfrak{u})^\vee$. Also, note that $r_0=\mathcal{G}_0=2\pi i\tau$.

Let $r_a(\tau)=v_a\bigl(r(\tau)\bigr)=r(\tau)\cdot a$. Then because $\varepsilon_0(a)=0$, we can write

$$r_a(\tau)=\sum_{i\geq 1}r_iv_i\in V'\otimes_{\mathbb{Q}}\mathcal{E}_0^{\mathrm{geom}},$$

where $\mathcal{E}_0^{\mathrm{geom}}$ be the subring of $\mathcal{E}^{\mathrm{geom}}$ generated by the coefficents r_i, $i\geq 1$. We note that $\mathcal{E}_0^{\mathrm{geom}}\cong\mathcal{U}(\mathfrak{u}')^\vee$, viewing $\mathfrak{u}'$ as the vector space quotient of $\mathfrak{u}$ by ε_0.

We saw above that the ring $\overline{\mathcal{E}}$ lies in $\mathcal{E}^{\mathrm{geom}}\otimes_{\mathbb{Q}}\overline{\mathcal{Z}}$. The $\mathbb{Q}$-algebra $\mathcal{E}^{\mathrm{geom}}$ is generated by $r_0=2\pi i\tau$ and $\mathcal{E}_0^{\mathrm{geom}}$, so in order to prove that $\overline{\mathcal{E}}[2\pi i\tau]$ is equal to the full tensor product $\mathcal{E}^{\mathrm{geom}}\otimes_{\mathbb{Q}}\overline{\mathcal{Z}}$, it will suffice to prove separately that $\overline{\mathcal{E}}\supset\overline{\mathcal{Z}}$ and $\overline{\mathcal{E}}\supset\mathcal{E}_0^{\mathrm{geom}}$.

Let us write $\mathfrak{nz}$ for the vector space of *new multizeta values* obtained by taking the vector space quotient of $\overline{\mathcal{Z}}$ by the vector subspace spanned by $\mathbb{Q}$ and by the ideal of $\overline{\mathcal{Z}}$ generated by products z_1z_2 of elements $z_1,z_2\in\overline{\mathcal{Z}}\setminus\mathbb{Q}$.

Let $\mathcal{MZ}$ denote the $\mathbb{Q}$-algebra of *motivic multizeta values* defined by Goncharov (in which $\zeta^{\mathfrak{m}}(2)=0$), which is graded for the weight. Let $\mathfrak{nmz}$ denote the quotient of the space $\mathcal{MZ}_{>0}$ of positive weight elements by products. We have the sequence of inclusions

$$\mathfrak{nz}^\vee \subset \mathfrak{nmz}^\vee \subset \mathfrak{grt}, \tag{3.14}$$

where the first is the dual injection arising from the surjection $\mathcal{MZ} \to \overline{\mathcal{Z}}$ and the second is the dual injection arising from the fact that Goncharov's motivic multizeta values satisfy the associator relations. Note that these are all subspaces of $\mathfrak{f}_2$.

The Lie series ϕ_{KZ} lies in the vector space $\mathfrak{nz}^\vee \otimes_{\mathbb{Q}} \overline{\mathcal{Z}}$, but by (3.14), it can also be considered as lying in the larger vector spaces $\mathfrak{nmz}^\vee \otimes_{\mathbb{Q}} \overline{\mathcal{Z}}$ or $\mathfrak{grt} \otimes_{\mathbb{Q}} \overline{\mathcal{Z}}$. In particular, since it lies in $\mathfrak{grt}$, we can apply Enriquez' section to this element, giving the derivation $\gamma(\phi_{KZ})$ studied §3.2, and the Lie series $\gamma_a(\phi_{KZ}) = v_a\big(\gamma(\phi_{KZ})\big)$. Set

$$\mathfrak{e} = \gamma_a(\phi_{KZ}).$$

From Theorem 3.3, we have $\overline{E} = \Gamma_a(\overline{\Phi}_{KZ})$, i.e. $\overline{E} = \exp_a(\mathfrak{e})$. Using this and (3.12), we can compute

$$\begin{aligned}\overline{E}(\tau) &= g(\tau)\cdot \overline{E} = \exp\big(r(\tau)\big)\cdot \overline{E} = \exp\big(r(\tau)\big)\cdot \exp_a(\mathfrak{e})\\ &= \exp_a\big(v_a\big(r(\tau)\big)\big) * \exp_a(\mathfrak{e}) = \exp_a\big(r_a(\tau)\big) * \exp_a(\mathfrak{e})\\ &= \exp_a\big(ch_{\langle\,,\,\rangle}\big(r_a(\tau),\mathfrak{e}\big)\big).\end{aligned}$$

Set $\mathfrak{e}(\tau) = \log_a\big(\overline{E}(\tau)\big)$, so

$$\mathfrak{e}(\tau) = ch_{\langle\,,\,\rangle}\big(r_a(\tau),\mathfrak{e}\big) = r_a(\tau) + \mathfrak{e} + \frac{1}{2}\big\langle r_a(\tau),\mathfrak{e}\big\rangle + \cdots,$$

which we write as

$$\mathfrak{e}(\tau) = \mathfrak{e} + r_a(\tau) + s(\tau),$$

where $s(\tau)$ is the sum of all the bracketed terms. As always, the coefficients of $\mathfrak{e}(\tau)$ multiplicatively generate the same $\mathbb{Q}$-algebra as that spanned linearly by the coefficients of $\overline{E}(\tau)$, namely $\overline{\mathcal{E}}$. We will show that the ring of coefficients of $\mathfrak{e}(\tau)$ contains both $\overline{\mathcal{Z}}$ and $\mathcal{E}_0^{\mathrm{geom}}$.

It follows from Brown's result in [3] that the Lie algebra $\mathfrak{nmz}^\vee$ is identified with the fundamental Lie algebra of the category of mixed Tate motives over $\mathbb{Z}$, which is free on one generator in each odd weight ≥ 3. In [17], a category of mixed elliptic motives is defined, and it is shown that the fundamental Lie algebra of that category has a monodromy representation in $\mathrm{Der}^0(\mathfrak{f}_2)$ whose image Π is isomorphic to a semi-direct product $\Pi \cong V \rtimes \mathfrak{nmz}^\vee$. In particular, $\mathfrak{nmz}^\vee$ normalizes V, and therefore the bracket of an element of V (such as $r_a(\tau)$) with an element of $\mathfrak{nmz}^\vee$ (such as $\mathfrak{e}$) will lie in V, and so the entire bracketed term $s(\tau)$ lies in $V \otimes_{\mathbb{Q}} \overline{\mathcal{E}}$. Also $r_a(\tau)$ lies in $V \otimes_{\mathbb{Q}} \overline{\mathcal{E}}$, so since $\mathfrak{e} \in \mathfrak{nmz}^\vee \otimes_{\mathbb{Q}} \overline{\mathcal{E}}$, we have

$$\mathfrak{e}(\tau) \in \Pi \otimes_{\mathbb{Q}} \overline{\mathcal{E}}.$$

Let us choose a linear basis of elements z_i of $\mathfrak{nmz}^\vee$. Then the z_i and the v_i form a basis of Π. If we write $\mathfrak{e}(\tau)$ in this (or any) basis, then the coefficients of $\mathfrak{e}(\tau)$ in that basis generate $\overline{\mathcal{E}}$. In particular, the coefficient of z_i in $\mathfrak{e}(\tau)$ is equal to the coefficient of z_i in $\mathfrak{e}$, since V and $\mathfrak{nmz}^\vee$ form a direct sum of vector spaces. Thus these coefficients for all z_i generate $\overline{\mathcal{Z}}$, which proves that $\overline{\mathcal{E}} \supset \overline{\mathcal{Z}}$.

It remains to prove that $\overline{\mathcal{E}} \supset \mathcal{E}_0^{\mathrm{geom}}$, which is a priori the ring generated by the coefficients of $r_a(\tau)$ written in the basis of V given by the v_i. In $\mathfrak{e}(\tau)$, however, the coefficient of v_i is a sum $r_i + s_i$, where s_i is the coefficient of v_i in $s(\tau)$. We will prove that $\overline{\mathcal{E}} \supset \mathcal{E}_0^{\rm geom}$ by showing by induction on the depth that $\overline{\mathcal{E}}$ contains each individual coefficient r_i.

For the base case $r = 1$, the depth 1 part of $r_a(\tau)+s(\tau)$ comes entirely from $r_a(\tau)$, since the sum $s(\tau)$ of bracketed terms has no depth 1 part. Thus, the coefficients r_i of basis elements $v_i \in V_1$ occur as coefficients of $r_a(\tau) + s(\tau)$, and therefore they lie in $\overline{\mathcal{E}}$.

Now fix $r > 1$ and assume that that all the r_j that are the coefficients in $r_a(\tau)$ of basis elements $v_j \in V_s$ with $s < r$ lie in $\overline{\mathcal{E}}$, and consider a basis element $v_i \in V_r$. Its coefficient in $r_a(\tau) + s(\tau)$ is $r_i + s_i$. But since $s(\tau)$ is a sum of brackets, the coefficient s_i is an algebraic expression in elements of $\overline{\mathcal{Z}}$ and coefficients r_j of $r_a(\tau)$ corresponding to basis elements v_j of depth $< r$. Thus by the induction hypothesis together with the inclusion $\overline{\mathcal{Z}} \subset \overline{\mathcal{E}}$, we have $s_i \in \overline{\mathcal{E}}$, and thus $r_i \in \overline{\mathcal{E}}$. This shows that all the coefficients r_i of $r_a(\tau)$ lie in $\overline{\mathcal{E}}$, and thus $\overline{\mathcal{E}}_0^{\rm geom} \subset \overline{\mathcal{E}}$ as desired. This concludes the proof. □

4. The elliptic double shuffle and push-neutrality relations

In this section we use mould theory to explore and compare algebraic relations between the elliptic multizeta values (coefficients of $\overline{E}(\tau)$), and algebraic relations between Enriquez' elliptic analogs of multizeta values.

Our main result on elliptic multizeta values arises as a corollary of the preceding theorem and the main result of [32]. We show that $\overline{E}(\tau)$ satisfies a certain double family of algebraic relations called the *elliptic double shuffle relations*, related to the familiar double shuffle properties of Φ_{KZ}. In fact, they bear a close relation to the linearized double shuffle relations studied for example in [4]. We show that if one assumes certain standard conjectures in multiple zeta and Grothendieck-Teichmüller theory, the elliptic double shuffle relations can be expected to form a *complete* set of algebraic relations for the elliptic multiple zeta values mod $2\pi i$. We investigate these relations in detail in depth 2.

In §4.3 we turn our attention to the power series $A(\tau)$ that forms part of Enriquez' elliptic KZB associator [13]. Since we want to work modulo $2\pi i$ and $A(\tau) \equiv 0$ mod $2\pi i$, we first define a power series $\mathfrak{a}(\tau)$ that is closely related to $A(\tau)$ but not trivial mod $2\pi i$. The goal of the section is to display a double family of relations satisfied by $\mathfrak{a}(\tau)$. The first is just the usual shuffle, but the second is very different from the second shuffle relation satisfied by $\overline{E}(\tau)$; we call it the family of *push-neutrality relations* (of *Fay relations*). We show that these are related to the Fay-shuffle relations studied in [25].

4.1. **A very brief introduction to moulds.**

We recall some notions from Ecalle's theory of moulds [11, 12] that we will need in order to study algebraic relations between elliptic multiple zeta values. Besides the original references, a more detailed introduction to moulds can be found in [31].

4.1.1. *Moulds and bialternality.* In this article, we use the term 'mould' to refer only to rational-function valued moulds with coefficients in $\mathbb{Q}$. Thus, a mould is a family of functions
$$\{P(u_1, \ldots, u_r) \mid r \geq 0\}$$
with $P(u_1, \ldots, u_r) \in \mathbb{Q}(u_1, \ldots, u_r)$. In particular $P(\emptyset)$ is a constant. The *depth* r part of a mould is the function $P(u_1, \ldots, u_r)$ in r variables. By defining addition

and scalar multiplication of moulds in the obvious way, i.e. depth by depth, moulds form a $\mathbb{Q}$-vector space that we call $Moulds$. We write $Moulds_{pol}$ for the subspace of polynomial-valued moulds. The vector space ARI is the subspace of $Moulds$ consisting of moulds P with constant term $A(\emptyset) = 0$, and ARI_{pol} is again the subspace of polynomial-valued moulds in ARI.

The standard mould multiplication mu is given by

$$mu(P,Q)(u_1,\ldots,u_r) = \sum_{i=0}^{r} P(u_1,\ldots,u_i)Q(u_{i+1},\ldots,u_r). \tag{4.1}$$

For simplicity, we write $P\,Q = mu(P,Q)$. This multiplication defines a Lie algebra structure on ARI with Lie bracket lu defined by $lu(P,Q) = mu(P,Q) - mu(Q,P)$.

We now introduce four operators on moulds. The Δ-operator on moulds is defined as follows: if $P \in ARI$, then

$$\Delta(P)(u_1,\ldots,u_r) = u_1\cdots u_r(u_1+\cdots+u_r)P(u_1,\ldots,u_r). \tag{4.2}$$

The dar-operator is defined by

$$dar(P)(u_1,\ldots,u_r) = u_1\cdots u_r\,P(u_1,\ldots,u_r). \tag{4.3}$$

The $push$-operator is defined by

$$push(B)(u_1,\ldots,u_r) = B(u_2,\ldots,u_r,-u_1-\cdots-u_r). \tag{4.4}$$

Finally, the $swap$ operator is defined by

$$swap(A)(v_1,\ldots,v_r) = A(v_r,v_{r-1}-v_r,\ldots,v_1-v_2). \tag{4.5}$$

Here the use of the alphabet $v_1,v_2,\ldots$ instead of $u_1,\ldots,u_r$ is purely a convenient way to distinguish a mould from its swap.

The main property on moulds that we will need to consider is *alternality*. A mould P is said to be *alternal* if for all $r>1$ and for $1\le i\le [r/2]$, we have

$$\sum_{\mathbf{u}\in sh((u_1,\ldots,u_i),(u_{i+1},\ldots,u_r))} P(\mathbf{u}) = 0, \tag{4.6}$$

where the set of r-tuples $sh\big((u_1,\ldots,u_i),(u_{i+1},\ldots,u_r)\big)$ is the set

$$\big\{(u_{\sigma^{-1}(1)},\ldots,u_{\sigma^{-1}(r)})\,\big|\,\sigma\in S_r \text{ such that } \sigma(1)<\cdots<\sigma(i),\ \ \sigma(i{+}1)<\cdots<\sigma(r)\big\}.$$

The mould $swap(A)$ is alternal if it satisfies the same property (4.6) in the variables v_i.

We write ARI^{al} for the space of alternal moulds in ARI, and $ARI^{al/al}$ for the space of moulds which are alternal and whose swap is also alternal. We also consider moulds which are alternal and whose swap is alternal up to addition of a constant-valued mould. The space of these moulds is denoted ARI^{al*al} and we call them *bialternal*.

We say that a mould P is Δ-bialternal if $\Delta^{-1}(P)$ is bialternal, and we write $ARI^{\Delta\text{-}al*al}$ for the space of such moulds.

4.1.2. *From power series to moulds.* Let $c_i = \mathrm{ad}(a)^{i-1}(b)$ for $i\ge 1$ as in §3.1. Let the depth of a monomial $c_{i_1}\cdots c_{i_r}$ be the number r of c_i in the monomial; the depth forms a grading on the formal power series ring $\mathbb{Q}\langle\langle C\rangle\rangle = \mathbb{Q}\langle\langle c_1,c_2,\ldots\rangle\rangle$ on the free variables c_i. Similarly, we write $L[\![C]\!] = \mathrm{Lie}[\![c_1,c_2,\ldots]\!]$ for the corresponding free Lie algebra. By Lazard elimination, we have an isomorphism

$$\mathbb{Q}a\oplus L[\![C]\!]\cong \mathfrak{f}_2 = \mathrm{Lie}[\![a,b]\!].$$

Following Écalle, let ma denote the standard vector space isomorphism from $\mathbb{Q}\langle\langle C\rangle\rangle$ to the space $(Moulds)^{pol}$ defined by

$$\begin{aligned} ma : \mathbb{Q}\langle\langle C\rangle\rangle &\xrightarrow{\sim} (Moulds)^{pol} \\ c_{k_1}\cdots c_{k_r} &\mapsto (-1)^{k_1+\cdots+k_r-r} u_1^{k_1-1}\cdots u_r^{k_r-1} \end{aligned} \tag{4.7}$$

on monomials, extended by linearity to all power series.

It is well-known that $p \in \mathbb{Q}\langle\langle C\rangle\rangle$ satisfies the shuffle relations if and only if p is a Lie series, i.e. $p \in \mathrm{Lie}[\![C]\!]$. The alternality property on moulds is analogous to these shuffle relations, that is a series $p \in \mathbb{Q}\langle\langle C\rangle\rangle$ satisfies the shuffle relations if and only if $ma(p)$ is alternal (see e.g. [31], §2.3 and Lemma 3.4.1]). Writing ARI^{al} for the subspace of alternal moulds and ARI^{al}_{pol} for the subspace of alternal polynomial-valued moulds, this shows that the map ma restricts to a Lie algebra isomorphism

$$ma : \mathrm{Lie}[\![C]\!] \xrightarrow{ma} ARI^{al}_{lu,pol}.$$

Finally, we recall that for any mould $P \in ARI$, Écalle defines a derivation $arit(P)$ of the Lie algebra ARI_{lu}. We do not need to recall the definition of $arit$ here (but it is given in §4.4 below where we prove a technical lemma). For now it is enough to know that when restricted to polynomial-valued moulds, it is related to the Ihara derivations (3.9) via the morphism ma:

$$ma\big(D_f(g)\big) = -arit\big(ma(f)\big)\cdot ma(f).$$

For each $P \in ARI$, we also define the derivation

$$arat(P) = -arit(P) + \mathrm{ad}(P), \tag{4.8}$$

where $\mathrm{ad}(P)\cdot Q = lu(P, Q)$.

4.1.3. *Reminders on the elliptic double shuffle Lie algebra* $\mathfrak{ds}_{\mathrm{ell}}$. We end this subsection by recalling the definition and a few facts about the elliptic double shuffle Lie algebra $\mathfrak{ds}_{\mathrm{ell}}$ from [32].

Definition 4.1. The *elliptic double shuffle Lie algebra* $\mathfrak{ds}_{\mathrm{ell}}$ is the subspace of $\mathfrak{f}_2$ such that

$$ma\big(\mathfrak{ds}_{\mathrm{ell}}\big) = ARI^{\Delta\text{-}al*al}_{pol},$$

i.e. $\mathfrak{ds}_{\mathrm{ell}}$ consists of the Lie power series $f \in \mathfrak{f}_2$ such that $ma(f)$ is Δ-bialternal.

The following results are shown in [32].

Proposition 4.2. *The space* $\mathfrak{ds}_{\mathrm{ell}}$ *satisfies the following properties.*

(i) $\mathfrak{ds}_{\mathrm{ell}} \subset \mathfrak{f}_2^{\mathrm{push}}$, *where* $\mathfrak{f}_2^{\mathrm{push}}$ *has been defined in Section 3.2;*

(ii) $\mathfrak{ds}_{\mathrm{ell}}$ *is a Lie algebra under the bracket* $\langle\, ,\, \rangle$ *on* $\mathfrak{f}_2^{\mathrm{push}}$ *defined in* (3.6).

(iii) There is a Lie algebra inclusion

$$\widetilde{\mathfrak{grt}}_{\mathrm{ell}} \subset \mathfrak{ds}_{\mathrm{ell}},$$

where $\widetilde{\mathfrak{grt}}_{\mathrm{ell}}$ *is the Lie subalgebra of* $\mathfrak{grt}_{\mathrm{ell}}$ *generated by* $\gamma(\mathfrak{grt})$ *and* $\mathfrak{u}$.

4.2. **The elliptic double shuffle relations.** We can now give the elliptic double shuffle property of $\overline{E}(\tau)$. It is in fact phrased more directly as a property on $\mathfrak{e}(\tau) = \log_a\big(\overline{E}(\tau)\big)$, or rather, on the mould version of this power series

$$\mathfrak{e}_m(\tau) = ma\big(\mathfrak{e}(\tau)\big).$$

Theorem 4.3. *The mould $\mathfrak{e}_m(\tau)$ is Δ-bialternal, i.e. $\Delta^{-1}\big(\mathfrak{e}_m(\tau)\big)$ is a bialternal mould.*

Proof. We saw in the proof of Theorem 3.5 that $\mathfrak{e}(\tau) = \mathfrak{e} + r_a(\tau) + s(\tau)$ where $\mathfrak{e} \in \gamma(\mathfrak{grt}) \otimes_{\mathbb{Q}} \overline{\mathcal{E}}$ and $r_a(\tau) + s(\tau) \in \mathfrak{u} \otimes_{\mathbb{Q}} \overline{\mathcal{E}}$. Therefore, $\mathfrak{e}(\tau) \in \widetilde{\mathfrak{grt}}_{\rm ell}$ by the definition of $\widetilde{\mathfrak{grt}}_\ell$, and since $\widetilde{\mathfrak{grt}}_{\rm ell} \subset \mathfrak{ds}_{\rm ell}$ by Proposition 4.2 (iii), we also have $\mathfrak{e}(\tau) \in \mathfrak{ds}_{\rm ell} \otimes_{\mathbb{Q}} \overline{\mathcal{E}}$. But this is equivalent to

$$\mathfrak{e}_m(\tau) \in ARI_{pol}^{\Delta\text{-}al*al}.$$

□

We conjecture that the elliptic double shuffle relations form a complete set of algebraic relations between the elliptic multiple zeta values modulo $2\pi i$. This statement really breaks down into two statements, one concerning the arithmetic part $\overline{\mathcal{Z}}$ of $\overline{\mathcal{E}}$ and the other the geometric part $\mathcal{U}(\mathfrak{u})^\vee$. We show that indeed, the result follows from two conjectures: the first one a standard conjecture from multizeta theory, and the second a similar conjecture from elliptic multizeta theory. Due to the fact that it is much easier to work in the geometric situation than the arithmetic situation (as there are no problems of transcendence), we are actually able to prove that the elliptic double shuffle relations are complete in depth 2, without any recourse to conjectures (see Proposition 4.5).

The first conjecture amounts to the inclusions in (3.14) being all isomorphisms as well as the standard conjecture that the inclusion $\mathfrak{grt} \subset \mathfrak{ds}$ (proved by Furusho in [16]) is actually also an isomorphism. We simply state the conjecture

Conjecture 1: $\mathfrak{nz}^\vee \cong \mathfrak{ds}.$

This is equivalent to conjecturing that the double shuffle relations suffice to generate all the algebraic relations satisfied by multizeta values [19].

The second conjecture amounts to the existence of a canonical semi-direct product structure on the elliptic double shuffle Lie algebra $\mathfrak{ds}_{\rm ell}$. This is inspired by Enriquez result that the elliptic Grothendieck–Teichmüller Lie algebra $\mathfrak{grt}_{\rm ell}$ is isomorphic to a semi-direct product $\mathfrak{r}_{\rm ell} \rtimes \gamma(\mathfrak{grt})$ where $\mathfrak{r}_{\rm ell}$ is a certain Lie ideal of $\mathfrak{grt}_{\rm ell}$ containing $\mathfrak{u}$. Analogously, we have

Conjecture 2: $\mathfrak{u} \rtimes \gamma(\mathfrak{ds}) \cong \mathfrak{ds}_{\rm ell}.$

This conjecture is closely related to Enriquez' "generation conjecture" for $\mathfrak{grt}_{\rm ell}$ [13], namely that $\mathfrak{u} \cong \mathfrak{r}_{\rm ell}$. If Enriquez' conjecture were true, then the left hand side of our Conjecture 2 would be isomorphic to $\mathfrak{grt}_{\rm ell}$, and Conjecture 2 would reduce to showing that $\mathfrak{grt}_{\rm ell} \cong \mathfrak{ds}_{\rm ell}$ (the elliptic analog of Furusho's theorem [16]).

One can also merge Conjectures 1 and 2 into a single conjecture, thereby extending (3.14) to the elliptic setting. The elliptic analog of the motivic space $\mathfrak{nmz}^\vee$ is the elliptic motivic fundamental Lie algebra, which is conjecturally isomorphic to its image $\Pi = V \rtimes \mathfrak{nmz}^\vee$ in the derivation algebra ${\rm Der}^0(\mathfrak{f}_2)$ (cf. the proof of Theorem 3.5). Then we get inclusions

$$V \rtimes \mathfrak{nz}^\vee \subset V \rtimes \mathfrak{nmz}^\vee \cong \Pi \subset \widetilde{\mathfrak{grt}}_{\rm ell}, \tag{4.9}$$

which conjecturally are all equalities. Note that the first equality would also follow from Conjecture 1 above.

Proposition 4.4. *If Conjectures 1 and 2 are true, then the elliptic double shuffle relations generate all algebraic relations between elliptic multizeta values.*

Proof. By Conjecture 1, we would have $\overline{\mathcal{Z}} \cong \mathcal{U}(\mathfrak{ds})^\vee$, so since $\mathcal{E}^{\rm geom} \cong \mathcal{U}(\mathfrak{u})^\vee \cong \mathcal{U}(V)^\vee$ by Theorem 2.6, we would have

$$\overline{\mathcal{E}}[2\pi i\tau] \cong \mathcal{U}(V)^\vee \otimes_{\mathbb{Q}} \mathcal{U}(\mathfrak{ds})^\vee.$$

It is known that the underlying vector space of the universal enveloping algebra $\mathcal{U}(R \rtimes L)$ of a semi-direct product of Lie algebras $R \rtimes L$ is the space $\mathcal{U}(R) \otimes_{\mathbb{Q}} \mathcal{U}(L)$; in fact $\mathcal{U}(R \rtimes L)$ is a Hopf algebra equipped with the smash product ([27]) and with the standard coproduct for which elements of $R \rtimes L$ are primitive. The dual $\mathcal{U}(R \rtimes L)^\vee$ has underlying $\mathbb{Q}$-algebra $\mathcal{U}(R)^\vee \otimes_{\mathbb{Q}} \mathcal{U}(L)^\vee$ (and is equipped with the smash coproduct).

Thus by Conjecture 2, we would have the isomorphism of $\mathbb{Q}$-algebras

$$\overline{\mathcal{E}}[2\pi i\tau] \cong \mathcal{U}(\mathfrak{u})^\vee \otimes_{\mathbb{Q}} \mathcal{U}(\mathfrak{ds})^\vee \cong \mathcal{U}(\mathfrak{ds}_{\rm ell})^\vee.$$

Now, for any Lie algebra $\mathfrak{g}$ defined over $\mathbb{Q}$ and any $\mathbb{Q}$-algebra R, if f is an element of $\mathfrak{g} \otimes_{\mathbb{Q}} R$, then the subring of R generated by the coefficients of f (in a linear basis of $\mathfrak{g}$) generate a subring of R which is necessarily isomorphic to a quotient of $\mathcal{U}(\mathfrak{g})^\vee$; in other words, the coefficients of f satisfy relations that are imposed by the fact that f lies in the Lie algebra $\mathfrak{g}$, and possibly others. If this quotient is actually isomorphic to $\mathcal{U}(\mathfrak{g})^\vee$, this signifies that the coefficients do not satisfy any further algebraic relations than those imposed on them by the fact that f lies in $\mathfrak{g}$.

In our case, we have $\mathfrak{e}(\tau) \in \mathfrak{ds}_{\rm ell} \otimes_{\mathbb{Q}} \overline{\mathcal{E}}$, and the coefficients of $\mathfrak{e}(\tau)$, together with $2\pi i\tau$, generate $\overline{\mathcal{E}}[2\pi i\tau]$, which by the conjectures is isomorphic to $\mathcal{U}(\mathfrak{ds}_{\rm ell})^\vee$, implying that the coefficients of $\mathfrak{e}(\tau)$ do not satisfy any other algebraic relations than those imposed by the fact that $\mathfrak{e}(\tau)$ lies in $\mathfrak{ds}_{\rm ell}$, i.e. is Δ-bialternal. □

Explicit elliptic double shuffle relations. Let us take a closer look at what the Δ-bialternality properties are. The first property is that $\mathfrak{e}_m(\tau)$ is Δ-alternal, i.e. that $\Delta^{-1}(\mathfrak{e}_m(\tau))$ is alternal. But Δ trivially preserves alternality, so this is equivalent to saying that $\mathfrak{e}_m(\tau)$ is alternal, i.e. that for each $r > 1$,

$$(EDS.1) \qquad \sum_{u \in sh\big((u_1,\ldots,u_k),(u_{k+1},\ldots,u_r)\big)} \mathfrak{e}_m(\tau)(u) = 0$$

for $1 \leq k \leq [r/2]$. This condition is equivalent to the statement that the power series $\mathfrak{e}(\tau)$ is a Lie series.

The new relations on $\mathfrak{e}_m(\tau)$ are the second set, which say that up to adding on a constant-valued mould, the swap of the mould $\Delta^{-1}\big(\mathfrak{e}_m(\tau)\big)$ is also alternal, where the swap-operator is defined in (4.5). This alternality is given by the equalities for $r > 1$

$$(EDS.2) \qquad \sum_{v \in sh\big((v_1,\ldots,v_k),(v_{k+1},\ldots,v_r)\big)} swap\big(\Delta^{-1}\mathfrak{e}_m(\tau)\big)(v) = 0$$

for $1 \leq k \leq [r/2]$.

The swapped mould is given explicitly by

$$swap\big(\Delta^{-1}\mathfrak{e}_m(\tau)\big) = \frac{1}{v_1(v_1-v_2)\cdots(v_{r-1}-v_r)v_r}\,\mathfrak{e}_m(\tau)(v_r, v_{r-1}-v_r, \ldots, v_1-v_2).$$

Thus the alternality conditions in (EDS.2) are all sums of rational functions with denominators that are products of terms of the form v_i and $(v_i - v_j)$, which sum to zero. Therefore, by multiplying through by the common denominator

$$v_1\cdots v_r \prod_{i>j}(v_i-v_j),$$

the second elliptic shuffle equation can be expressed as a family of polynomial conditions on the mould $swap(\mathfrak{e}_m(\tau))$.

Elliptic double shuffle relations in depth 2. Let us work this out explicitly in depth 2. The usual alternality condition reduces to

$$(EDS.1\text{-depth 2}) \qquad \mathfrak{e}_m(\tau)(u_1,u_2) + \mathfrak{e}_m(\tau)(u_2,u_1) = 0.$$

The swap alternality condition reads

$$\frac{1}{v_1(v_1-v_2)v_2} swap(\mathfrak{e}_m(\tau))(v_1,v_2) + \frac{1}{v_1(v_2-v_1)v_2} swap(\mathfrak{e}_m(\tau))(v_2,v_1) = 0,$$

which, clearing denominators, reduces simply to

$$swap(\mathfrak{e}_m(\tau))(v_1,v_2) - swap(\mathfrak{e}_m(\tau))(v_2,v_1) = 0.$$

Since $swap(\mathfrak{e}_m(\tau))(v_1,v_2) = e_m(v_2, v_1-v_2)$, this is given by the relation

$$\mathfrak{e}_m(\tau)(v_2, v_1-v_2) = \mathfrak{e}_m(\tau)(v_1, v_2-v_1)$$

directly on $\mathfrak{e}_m(\tau)$. Applying the depth 2 swap operator from $\overline{ARI}$ to ARI (given by $v_1 \mapsto u_1+u_2$, $v_2 \mapsto u_1$), we transform this relation into

$$\mathfrak{e}_m(\tau)(u_1,u_2) = \mathfrak{e}_m(\tau)(u_1+u_2, -u_2).$$

Finally, $\mathfrak{e}_m(\tau)$ is of odd degree, so by the depth 2 version of (EDS.1), we have $\mathfrak{e}_m(\tau)(-u_2,-u_1) = \mathfrak{e}_m(\tau)(u_1,u_2)$, which gives

$$(EDS.2\text{-depth 2}) \qquad \mathfrak{e}_m(\tau)(u_1,u_2) = \mathfrak{e}_m(\tau)(u_2, -u_1-u_2).$$

Note that this is nothing other than $\mathfrak{e}_m(\tau)(u_1,u_2) = push\big(\mathfrak{e}_m(\tau)\big)(u_1,u_2)$ where the push-operator is defined in (4.4). Thus in depth 2, the Δ-bialternality conditions correspond to alternality and push-invariance of $\mathfrak{e}_m(\tau)$ (which in turn correspond to the fact that $\mathfrak{e}(\tau)$ is a Lie series that is push-invariant in depth 2 in the sense of power series, as in (3.5)). This simple reformulation is special to depth 2; the Δ-bialternal property does not lend itself so easily to a direct expression as a property of $\mathfrak{e}(\tau)$ in higher depths.

We end this subsection by showing that the conjecture that the Δ-bialternal relations are sufficient holds in depth 2.

Proposition 4.5. *The relations (EDS.1) and (EDS.2) in odd degrees are the only relations satisfied by $\mathfrak{e}_m(\tau)$ in depth 2.*

Proof. We can prove this result without recourse to any conjectures, essentially because depth 2 is too small to contain any of the arithmetic part of $\mathfrak{e}_m(\tau)$ (we qualify this statement below), and the geometric part $V = v_a(\mathfrak{u})$ is well-understood in depth two. We know that $\mathfrak{e}(\tau) \in \mathfrak{ds}_{\rm ell} \subset \mathfrak{f}_2^{push}$. The graded dimensions of $\mathfrak{f}_2$ in depth 2 are given by

$$dim(\mathfrak{f}_2^{\rm push})^2_n = \left\lfloor \frac{n-5}{6} \right\rfloor + 1. \tag{4.10}$$

Now the depth two part of $\mathfrak{ds}_{\rm ell} \supset V \rtimes \gamma(\mathfrak{nz}^\vee)$ is contained in the depth two part of V, since $\gamma(\mathfrak{nz}^\vee)$ is of depth ≥ 3. Thus

$$\dim \big(\mathfrak{ds}_{\rm ell}\big)^2_n = \dim V^2_n = \begin{cases} \lfloor \frac{n-5}{6} \rfloor + 1 & \text{if } n \text{ is odd} \geq 5 \\ 0 & \text{otherwise.} \end{cases} \tag{4.11}$$

Indeed, the last equality follows from the fact that in depth 2, V is spanned by the $[\varepsilon_{2j}, \varepsilon_{2k}](a)$ with $j < k$, $j,k \neq 1$, which are all of odd weight, and the fact that, as shown in [28], the only relations between these $\lfloor \frac{n-3}{4} \rfloor$ brackets come from period polynomials, whose number is given by $\lfloor \frac{n-7}{4} \rfloor - \lfloor \frac{n-5}{6} \rfloor$. Thus $V^2 = \mathfrak{ds}^2_{\rm ell} = (\mathfrak{f}_2^{push})^2$, so the Lie relation (EDS.1) and the push-invariance relation (EDS.2) suffice to characterize elements of $\mathfrak{ds}_{\rm ell}$ in depth 2. □

Depth 2 elements of $\mathfrak{ds}_{\rm ell}$ in low weights:

- in weight 5,

$$ma\big([\varepsilon_0, \varepsilon_4](a)\big) = 2u_1^3 + 3u_1^2u_2 - 3u_1u_2^2 - 2u_2^3.$$

- in weight 7,

$$ma\big([\varepsilon_0, \varepsilon_6](a)\big) = 2u_1^5 + 5u_1^4u_2 + 2u_1^3u_2^2 - 2u_1^2u_2^3 - 5u_1u_2^4 - 2u_2^5.$$

- in weight 9,

$$ma\big([\varepsilon_0, \varepsilon_8](a)\big) = 2u_1^7 + 7u_1^6u_2 + 9u_1^5u_2^2 + 5u_1^4u_2^3 - 5u_1^3u_2^4 - 9u_1^2u_2^5 - 7u_1u_2^6 2u_2^7.$$

- in weight 11,

$$\begin{aligned} ma\big([\varepsilon_0, \varepsilon_{10}](a)\big) &= 8u_1^9 + 36u_1^8u_2 + 74u_1^7u_2^2 + 91u_1^6u_2^3 + 41u_1^5u_2^4 - 41u_1^4u_2^5 \\ &\quad -91u_1^3u_2^6 - 74u_1^2u_2^7 - 36u_1u_2^8 - 8u_2^9 \\ ma\big([\varepsilon_4, \varepsilon_6](a)\big) &= -2u_1^7u_2^2 - 7u_1^6u_2^3 - 5u_1^5u_2^4 + 5u_1^4u_2^5 + 7u_1^3u_2^6 + 2u_1^2u_2^7. \end{aligned}$$

4.3. **The elliptic associator and the push-neutrality relations mod $2\pi i$.** Let $\big(A(\tau), B(\tau)\big)$ be the elliptic associator recalled in (3.3); in particular, $A(\tau)$ is given explicitly by $g(\tau)\cdot A$, where

$$A = \Phi_{KZ}(t_{01}, t_{12}) e^{2\pi i t_{01}} \Phi_{KZ}(t_{01}, t_{12})^{-1}.$$

In this subsection, we will investigate relations modulo $2\pi i$ satisfied by the power series $A(\tau)$.

The coefficients of $A(\tau)$ are the numbers called *elliptic analogs of multizeta values* (up to the powers of $2\pi i$ produced by the variable change above). The ring generated by the coefficients of $A(\tau)$ is closely related to the ring $\mathcal{E}$. However, there is an obvious difference due to the fact that the coefficients of $A(\tau)$ are all divisible by $2\pi i$, i.e. $A \equiv 1 \bmod 2\pi i$.

In this subsection we want to work modulo $2\pi i$, so we cannot use $A(\tau)$ as is. We start by defining a modified version of $A(\tau)$ whose reduction mod $2\pi i$ is not trivial.

Definition 4.6. Let $\mathfrak{a}$ be the power series with coefficients in $\overline{\mathcal{Z}}$ given by

$$\mathfrak{a} = \frac{1}{2\pi i}\log(A) \bmod 2\pi i = \overline{\Phi}_{KZ}(t_{01},t_{12})t_{01}\overline{\Phi}_{KZ}(t_{01},t_{12})^{-1},$$

and let $\mathfrak{a}(\tau) = g(\tau)\cdot\mathfrak{a}$, where $g(\tau)$ was defined in (2.3).

It follows from [24], Theorem 5.4.2, that the coefficients of the power series $\mathfrak{a}$ generate all of $\overline{\mathcal{Z}}$, so by the same argument as in the proof of Theorem 3.5, the coefficients of $\mathfrak{a}(\tau)$ together with $\mathcal{G}_0 = 2\pi i\tau$ generate all of $\mathcal{E}^{\mathrm{geom}}\otimes_{\mathbb{Q}}\overline{\mathcal{Z}}$. Therefore the coefficients of $\mathfrak{a}$ provide us with a new set of generators for the ring $\mathcal{E}^{\mathrm{geom}}\otimes_{\mathbb{Q}}\overline{\mathcal{Z}}$, different from the set studied in §3, given by the coefficients of $\mathfrak{e}(\tau)$ together with $\mathcal{G}_0$. Recall that the relations satisfied by the latter set are the Δ-bialternality relations given in Theorem 4.3.

The purpose of this subsection is to give a double family of relations satisfied by the coefficients of $\mathfrak{a}(\tau)$. The first one is the usual family of *alternality* relations and the second is the family of *push-neutrality* relations. These relations are related (mod $2\pi i$) to the *Fay-shuffle relations* introduced in [25], and studied explicitly in depth 2. We show that modulo $2\pi i$ our relations are the same as the Fay-shuffle relations. We also show that even in depth 2 and mod $2\pi i$, the alternality and push-neutrality relations are strictly weaker than the full set of algebraic relations that must be satisfied by the elliptic analogs of mzv's, whereas the Δ-bialternality is conjecturally complete.

We will give our relations in terms of mould theory (but see Corollary 4.10 for a translation into power series terms at the end). For this we recall the *push* and *dar*-operators defined in (4.4) and (4.3). We will say that a mould B is *push-neutral* if

$$B(u_1,\ldots,u_r) + push(B)(u_1,\ldots,u_r) + \cdots + push^r(B)(u_1,\ldots,u_r) = 0 \qquad (4.12)$$

for all $r \geq 1$, where *push* denotes the push-operator on moulds defined in (4.4).

Theorem 4.7. *Let $\mathfrak{a}_m(\tau) = ma\big(\mathfrak{a}(\tau)\big)$. Then $\mathfrak{a}_m(\tau)$ is alternal and $dar^{-1}\big(\mathfrak{a}_m(\tau)\big)$ is push-neutral in depth $r > 1$.*

Proof. Recall the derivation *arat* defined in (4.8). For any $P \in ARI$, set

$$\mathrm{Darit}(P) = dar\circ arat\big(\Delta^{-1}(P)\big)\circ dar^{-1}. \qquad (4.13)$$

It is shown in [32] that the map

$$\begin{aligned}\mathrm{Der}^0(\mathfrak{f}_2) &\hookrightarrow \mathrm{Der}(ARI_{lu})\\ D &\mapsto \mathrm{Darit}\big(ma(v_a(D))\big)\end{aligned} \qquad (4.14)$$

is an injective Lie morphism, so that we have

$$ma\big(D(f)\big) = \mathrm{Darit}\big(ma(v_a(D))\big)\cdot ma(f). \qquad (4.15)$$

Let $\mathfrak{a}_m = ma(\mathfrak{a})$, $\mathfrak{a}_m(\tau) = ma\big(\mathfrak{a}(\tau)\big)$, and $r_m(\tau) = ma\big(r_a(\tau)\big)$. Under the map (4.14), we have $r(\tau)\mapsto \mathrm{Darit}\big(r_m(\tau)\big)$, so

$$ma\big(r(\tau)\cdot\mathfrak{a}\big) = \mathrm{Darit}\big(r_m(\tau)\big)\cdot\mathfrak{a}_m.$$

Since

$$\mathfrak{a}(\tau) = g(\tau)\cdot\mathfrak{a} = \sum_{n\geq 0}\frac{1}{n!}r(\tau)^n\cdot\mathfrak{a}, \qquad (4.16)$$

we have

$$\mathfrak{a}_m(\tau) = \sum_{n\geq 0} \frac{1}{n!}\mathrm{Darit}\big(r_m(\tau)\big)^n \cdot \mathfrak{a}_m. \tag{4.17}$$

Let $\overline{\sigma}$ denote the automorphism of $\mathfrak{f}_2$ defined in §3.2. We have

$$\mathfrak{a} = \overline{\sigma}(t_{01}).$$

Recall from §3.2 that $\overline{\sigma} = \gamma(\phi_{KZ})$, where $\phi_{KZ} = \log_a\big(\overline{\Phi}_{KZ}\big)$.

The derivation $\gamma(\phi_{KZ})$ lies in $\mathrm{Der}^0(\mathfrak{f}_2)$, so $\gamma(\phi_{KZ}) \cdot t_{01} \in \mathfrak{f}_2$; thus $\mathfrak{a}$ is a Lie series. Since $r(\tau) \in \mathrm{Der}^0(\mathfrak{f}_2)$, we have $r(\tau)^n \cdot \mathfrak{a} \in \mathfrak{f}_2$ for all $n \geq 1$, so by (4.16), $\mathfrak{a}(\tau) = g(\tau) \cdot \mathfrak{a} \in \mathfrak{f}_2$, which means that $\mathfrak{a}_m(\tau)$ is alternal. This settles the first property of $\mathfrak{a}_m(\tau)$ stated in the theorem.

Let us consider the second property. Since $\gamma(\phi_{KZ}) \in \mathrm{Der}^0(\mathfrak{f}_2)$, it annihilates t_{12}. Therefore, setting $t'_{01} = t_{01} + \frac{1}{2}t_{12}$, we have

$$\mathfrak{a} = \gamma(\phi_{KZ}) \cdot t_{01} = \gamma(\phi_{KZ}) \cdot t'_{01}. \tag{4.18}$$

Set $T'_{01} = ma(t'_{01})$, and set

$$\mathfrak{z} = ma\Big(v_a\big(\gamma(\phi_{KZ})\big)\Big) = ma\big(\gamma_a(\phi_{KZ})\big).$$

Then by (4.15), the equality (4.18) translates into moulds as

$$\mathfrak{a}_m = \mathrm{Darit}(\mathfrak{z}) \cdot T'_{01}.$$

To complete the proof of the second property, we will use the following lemma, whose proof is deferred to the final subsection of this paper.

Lemma 4.8. *Let $A \in ARI$. If A is push-neutral, then $arat(P) \cdot A$ is push-neutral for all $P \in ARI$. If $dar^{-1}A$ is push-neutral, then $dar^{-1} \cdot \mathrm{Darit}(P) \cdot A$ is push-neutral for all $P \in ARI$.*

It is easy to see that if A is a push-invariant mould, then $dar^{-1}A$ is push-neutral, since

$$dar^{-1}(A)(u_1,\ldots,u_r) + push\big(dar^{-1}(A)\big)(u_1,\ldots,u_r) + \cdots + push^r\big(dar^{-1}(A)\big)(u_1,\ldots,u_r)$$

$$= \left(\frac{1}{u_1\cdots u_r} + \frac{1}{u_2\cdots u_0} + \cdots + \frac{1}{u_0u_1\cdots u_{r-1}}\right)A(u_1,\ldots,u_r)$$

$$= \left(\frac{u_0+u_1+\cdots+u_r}{u_0u_1\cdots u_r}\right)A(u_1,\ldots,u_r) = 0,$$

where $u_0 = -u_1 - \cdots - u_r$. By Proposition 4.9 below, $dar^{-1}T'_{01}$ is push-neutral and by Lemma 4.8, so is

$$dar^{-1}\mathfrak{a}_m = dar^{-1} \cdot \mathrm{Darit}(\mathfrak{z}) \cdot T'_{01}.$$

To show that $dar^{-1}\mathfrak{a}_m(\tau)$ is push-neutral we use the same lemma again. Since $dar^{-1}\mathfrak{a}_m$ is push-neutral, so is $dar^{-1} \cdot \mathrm{Darit}\big(r_m(\tau)\big) \cdot \mathfrak{a}_m$, and then successively, so is $dar^{-1} \cdot \mathrm{Darit}\big(r_m(\tau)\big)^n \cdot \mathfrak{a}_m$ for all $n \geq 1$. Thus $dar^{-1}\mathfrak{a}_m(\tau)$ is push-neutral by (4.17). This proves the theorem. □

The following proposition was used in the proof of Theorem 4.7.

Proposition 4.9. *The mould*

$$ma([t'_{01},a]) = -\sum_{n=2}^{\infty}\frac{B_n}{n!}ma([\mathrm{ad}^n(b)(a),a]) \tag{4.19}$$

is push-neutral.

Proof. It is enough to show the push-neutrality of $f_n := ma([\mathrm{ad}^n(b)(a),a])$ for all $n\geq 2$ separately. Using the definition of ma (cf. Section 4.1), we see that

$$ma(\mathrm{ad}^n(b)(a)) = -\sum_{k=1}^{n}(-1)^{n-k}\binom{n-1}{k-1}u_k \in \mathbb{Q}[u_1,\dots,u_n]. \tag{4.20}$$

Now in depth n, the operator $\mathrm{ad}(a)$ on $\mathbb{Q}\langle\langle C\rangle\rangle$ corresponds to multiplication by $-(u_1+\ldots+u_n)$. Consequently,

$$\begin{aligned} ma([\mathrm{ad}^n(b)(a),a]) &= -ma([a,\mathrm{ad}^n(b)(a)]) \\ &= -(u_1+\ldots+u_n)\sum_{k=1}^{n}(-1)^{n-k}\binom{n-1}{k-1}u_k \\ &= -\sum_{j,k=1}^{n}(-1)^{n-k}\binom{n-1}{k-1}u_ju_k. \end{aligned} \tag{4.21}$$

On the other hand, by the definition of the push-operator (4.4), we have $push(f_n) = -\sum_{j,k=1}^{n}(-1)^{n-k}\binom{n-1}{k-1}u_{j+1}u_{k+1}$, where the indices are to be taken mod n (so that $u_{k+n}=u_k$). Using the elementary fact that $\sum_{k=1}^{n}(-1)^{n-k}\binom{n-1}{k-1}=0$ for $n\geq 2$, it is now clear that

$$\sum_{i=0}^{n-1} push^i(f_n) = 0, \tag{4.22}$$

i.e. f_n is push-neutral for all $n\geq 2$, as was to be shown. □

We end this subsection by studying these relations more explicitly in depth 2 and comparing them with the elliptic double shuffle relations on $\mathfrak{e}_m(\tau)$. The alternality relation is of course the same:

$$(FS.1) \qquad \mathfrak{a}_m(\tau)(u_1,u_2)+\mathfrak{a}_m(\tau)(u_2,u_1)=0.$$

The push-neutrality relation in depth 2 is given by

$$(FS.2) \qquad \frac{1}{u_1u_2}\mathfrak{a}_m(\tau)(u_1,u_2)+\frac{1}{u_2u_0}\mathfrak{a}_m(\tau)(u_2,u_0)+\frac{1}{u_0u_1}\mathfrak{a}_m(\tau)(u_0,u_1)=0$$

where $u_0=-u_1-u_2$. Multiplying by the common denominator $u_0u_1u_2$ yields the polynomial relation

$$u_0\mathfrak{a}_m(\tau)(u_1,u_2)+u_1\mathfrak{a}_m(\tau)(u_2,u_0)+u_2\mathfrak{a}_m(\tau)(u_0,u_1)=0.$$

It was shown in [25] that the dimension of the space of polynomials in u_1,u_2 of odd degree d satisfying (FS.1) and (FS.2) is given by $\left\lfloor\frac{d}{3}\right\rfloor+1$. In terms of the weight $n=d+2$ of the corresponding polynomials in $\mathfrak{f}_2$, this is

$$\left\lfloor\frac{n-2}{3}\right\rfloor+1.$$

In weight 5, for example, there are two independent such polynomials:

$$u_1^2u_2-u_1u_2^2 \quad\text{and}\quad u_1^3-u_2^3.$$

In weight 7, there are again two independent polynomials, given by

$$u_1^4u_2 - u_1u_2^4 \quad \text{and} \quad u_1^5 + u_1^3u_2^2 - u_1^2u_2^3 - u_2^5.$$

In weight 9, the space is three-dimensional, given by

$$\begin{aligned}
&u_1^7 - 2u_1^4u_2^3 + 2u_1^3u_2^4 - u_2^7\\
&u_1^6u_2 - u_1u_2^6\\
&u_1^5u_2^2 + u_1^4u_2^3 - u_1^3u_2^4 - u_1^2u_2^5.
\end{aligned}$$

Finally, we work out the case of weight 11, where the dimension is four:

$$\begin{aligned}
&u_1^9 + 3u_1^5u_2^4 - u_1^4u_2^5 - u_2^9\\
&u_1^8u_2 - u_1u_2^8\\
&u_1^7u_2^2 - u_1^5u_2^4 + u_1^4u_2^5 - u_1^2u_2^7\\
&u_1^6u_2^3 + u_1^5u_2^4 - u_1^4u_2^5 - u_1^3u_2^6
\end{aligned}$$

Observe that these dimensions are significantly bigger than those given by the elliptic double shuffle equations (EDS.1) and (EDS.2) in depth 2. This is explained by the fact that the vector space generated by the coefficients of $\mathfrak{a}_m(\tau)$ in a given weight and depth is not equal to the one generated by the analogous coefficients of $\mathfrak{e}_m(\tau)$.

Under the conjecture $\overline{\mathcal{Z}} \cong \mathcal{U}(\mathfrak{grt})^\vee$, the $\mathbb{Q}$-algebra $\overline{\mathcal{E}}$ is isomorphic to $\mathcal{U}(\mathfrak{grt}_{\mathrm{ell}})^\vee$, and thus inherits a natural bigrading dual to that of $\mathfrak{grt}_{\mathrm{ell}}$. Together with products of elements of $\overline{\mathcal{E}}$ of smaller depth and weight (including $\mathcal{G}_0$), the coefficients of $\mathfrak{e}_m(\tau)$ in a given weight n and depth d span the bigraded part $\overline{\mathcal{E}}_n^d$, whereas those of $\mathfrak{a}_m(\tau)$ do not.

For example, in weight 5 and depth 2, the coefficients of $\mathfrak{e}_m(\tau)$ generate the 1-dimensional space $\langle 2\mathcal{G}_{0,4} + \mathcal{G}_0\mathcal{G}_4\rangle$. The bigraded subspace $\overline{\mathcal{E}}_5^2$ is spanned by $\mathcal{G}_2^2$, $\mathcal{G}_0\mathcal{G}_4$ and $\mathcal{G}_{0,4}$, but it is also spanned by the two products $\mathcal{G}_2^2$ and $\mathcal{G}_0\mathcal{G}_4$ and the single coefficient $2\mathcal{G}_{0,4} + \mathcal{G}_0\mathcal{G}_4$ of $\mathfrak{e}_m(\tau)$ in weight 5 and depth 2.

The weight 5, depth 2 coefficients of $\mathfrak{a}_m(\tau)$, however, do not lie in $\overline{\mathcal{E}}_5^2$. They span the 2-dimensional subspace $\langle -\frac{1}{12}\mathcal{G}_0\mathcal{G}_2 + \frac{3}{2}\mathcal{G}_0\mathcal{G}_4 + 3\mathcal{G}_{0,4} - \frac{1}{360}\mathcal{G}_0^2 + \frac{1}{2}\mathcal{G}_2^2, \frac{1}{240}\mathcal{G}_0^2 - 2\mathcal{G}_{0,4} - \mathcal{G}_0\mathcal{G}_4\rangle$ of $\overline{\mathcal{E}}$.

We end this subsection with a power series statement of the alternality and push-neutrality relations on $\mathfrak{a}_m(\tau)$.

Corollary 4.10. *The power series $\mathfrak{A} = [a, \mathfrak{a}(\tau)]$ is push-neutral in the sense that, if $\mathfrak{A}^r$ denotes the depth r part of $\mathfrak{A}$ for $r > 1$, then*

$$A^r + push(A^r) + \cdots + push^r(A^r) = 0$$

where push denotes the push-operator on power series defined in (3.5).

Proof. By Theorem 4.7, the mould $dar^{-1}\mathfrak{a}_m(\tau)$ is push-neutral. Consider the operator

$$-\Delta(A)(u_1, \dots, u_r) = u_1 \cdots u_r(-u_1 - \dots - u_r)A(u_1, \dots, u_r).$$

Since the factor $u_1 \dots u_r(-u_1 - \dots - u_r)$ is push-invariant, the mould $-\Delta(A)$ is push-neutral if A is. Therefore in particular $-\Delta\big(dar^{-1}\mathfrak{a}_m(\tau)\big)$ is push-neutral. But this mould is given by

$$\begin{aligned}
-\Delta\big(dar^{-1}\mathfrak{a}_m(\tau)\big)(u_1, \dots, u_r) &= -(u_1 + \cdots + u_r)\,\mathfrak{a}_m(\tau)(u_1, \dots, u_r)\\
&= ma\big([a, \mathfrak{a}(\tau)]\big)(u_1, \dots, u_r),
\end{aligned}$$

where the last equality is a standard identity (see Appendix A of [29] or (3.3.1) of [31]). Therefore the mould $ma([a,\mathfrak{a}(\tau)])$ is a push-neutral mould, i.e. $[a,\mathfrak{a}(\tau)]$ is push-neutral as a power series. □

4.4. **Proof of Lemma 4.8.** In order to prove this lemma, we need to have recourse to the complete formula for the action of *arat*. We first recall Écalle's formula for *arit* (cf. [12] or [31]), which is given as

$$\big(arit(P)\cdot A\big)(w)=\sum_{\substack{w=abc\\ c\neq\emptyset}}A(a\lceil c)P(b)-\sum_{\substack{w=abc\\ a\neq\emptyset}}A(a\rceil c)P(b),$$

where if the word $u=(u_1,\ldots,u_r)$ is decomposed into three chunks as $u=abc$, $a=(u_1,\ldots,u_i)$, $b=(u_{i+1},\ldots,u_{i+j})$, $c=(u_{i+j+1},\ldots,u_r)$, then we use Écalle's notation

$$\begin{aligned}a\rceil&=(u_1,\ldots,u_{i-1},u_i+u_{i+1}+\cdots+u_{i+j})\\ \lceil c&=(u_{i+1}+\cdots+u_{i+j+1},u_{i+j+2},\ldots,u_r).\end{aligned}$$

Moreover

$$\mathrm{ad}(P)\cdot A=mu(P,A)-mu(A,P)$$

where mu is the mould multiplication defined in (4.1); these correspond precisely to the 'missing' terms $a=\emptyset$ and $c=\emptyset$, so that $arat(P)\cdot A$ actually has the simpler expression

$$\big(arat(P)\cdot A\big)(w)=\sum_{w=abc}\big(A(a\lceil c)P(b)-A(a\rceil c)P(b)\big).\qquad(4.23)$$

Now let A be push-neutral, and let $P\in ARI$. We need to show that (4.23) is push-neutral. In fact we will show that the two terms

$$\sum_{w=abc}A(a\lceil c)P(b)\quad\text{and}\quad\sum_{w=abc}A(a\rceil c)P(b)\qquad(4.24)$$

of (4.23) are separately push-neutral.

Because the push-neutrality relations take place in fixed depth, we may assume that A is concentrated in depth s and P in depth t, with $s+t=r$. We will prove the push-neurality of the first term in (4.24); the proof for the second term is completely analogous.

Therefore the decompositions $w=abc$ we need to consider are those of the form

$$w=abc=(u_1,\ldots,u_i)(u_{i+1},\ldots,u_{i+t})(u_{i+t+1},\ldots,u_r),$$

and we can rewrite the first term of (4.24) as

$$\sum_{i=0}^{r-t}A(u_1,\ldots,u_i,u_{i+1}+\cdots+u_{i+t+1},u_{i+t+2},\ldots,u_r)P(u_{i+1},\ldots,u_{i+t}).$$

The k-th power of the push-operator acts by $u_i\mapsto u_{i-k}$, with indices considered modulo $(r+1)$. The push-neutrality condition thus reads

$$\sum_{k=0}^{r}\sum_{i=0}^{r-t}A(u_{1-k},\ldots,u_{i-1-k},u_{i-k},u_{i+1-k}+\cdots+u_{i+t+1-k},u_{i+t+2-k},\ldots,u_{r-k})$$
$$\cdot P(u_{i+1-k},\ldots,u_{i+t-k})=0.$$

We will show that the coefficients of each term $P(u_{m+1},\ldots,u_{m+t})$ sums to zero due to the push-neutrality of A. In fact it is enough to show that the coefficient of $P(u_1,\dots,u_t)$ sums to zero, as all the other terms are obtained from this one by applying powers of the push-operator.

The terms containing $P(u_1,\dots,u_t)$ are those for which the index $k=i$, so that $k\in\{0,\dots,r-t=s\}$, and we must show that the sum

$$\sum_{k=0}^{s} A(u_{r-k+2},\dots,u_r,u_0,u_1+\cdots+u_{t+1},u_{t+2},\dots,u_{r-k})$$

vanishes, where $u_0=-u_1-\cdots-u_r$ and we have shifted some of the indices modulo $(r+1)$ in order to make them positive. Note now that

$$u_1+\cdots+u_{t+1}=-u_0-u_{t+2}-\cdots+u_r.$$

As a result the last sum runs over the $(s+1)$ cyclic permutations of $u_{t+2},\dots,u_r,u_0$ and $-u_{t+2}-\cdots-u_r-u_0$, so it is equal to the sum over the push_s-orbit of just one term, say the one with $k=s$, i.e. to

$$\sum_{k=0}^{s} A(u_{t+2},\dots,u_r,u_0),$$

which indeed vanishes since A is push-neutral. This concludes the proof of Lemma 4.8. □

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