Open problems in Grothendieck-Teichmüller theory

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§0. Introduction, definitions, notation

The present note is not intended in any way as an introduction to Grothendieck-Teichmüller theory. It is essentially a concentrated list of questions in and around this theory, most of which are open, although we have included some questions which are natural to ask but easy to answer, and a few others which were open but are now settled. In order for the reader to appreciate the relative depth, difficulty, and interest of these problems, and their position within the theory, some previous knowledge is required. We do give some important facts and definitions, but they are intended to remind the reader of relatively well-known elements of the theory, to give something of the flavor of the objects concerned and to make statements unambiguous. They are not sufficient to provide a deep understanding of the theory.

To describe the main idea of the theory in a few words (see e.g. [L2] for more and references), one takes a category $\mathcal{C}$ of geometric objects (of finite type) defined over a field k; these can be k-varieties, k-schemes, or k-algebraic stacks, and a collection of k-morphisms between them, for instance all k-morphisms of k-varieties. Let $\pi_1^{geom}(X)$ denote the geometric fundamental group of X, that is the algebraic fundamental group of $X \otimes \overline{k}$, where $\overline{k}$ denotes the algebraic (or separable) closure of k; it is a finitely generated profinite group. One can view π_1^{geom} as a functor from $\mathcal{C}$ to the category of finitely generated profinite groups with continous morphisms up to inner automorphisms. One then considers the (outer) automorphism group of this functor, say $\mathrm{Out}(\pi_1^{geom}(\mathcal{C}))$. Concretely speaking its elements consists of collections $(\phi_X)_X$ with $\phi_X \in \mathrm{Out}(\pi_1^{geom}(X))$, indexed by objects $X \in \mathcal{C}$, and compatible with morphisms. One usually has additional requirements, namely that the ϕ_X satisfy some Galois-style properties, like the preservation of conjugacy classes of inertia groups. Since there is a canonical outer action of the absolute Galois group $G_k = \mathrm{Gal}(\overline{k}/k)$ on $\pi_1^{geom}(X)$ for each X, and it is compatible with morphisms, one gets a natural homomorphism $G_k \to \mathrm{Out}(\pi_1^{geom}(\mathcal{C}))$, which is injective in all the interesting cases. If in fact it is an isomorphism, one thus in principle gets a geometric description of the arithmetic group G_k. In the case where $\mathcal{C}$ is the category of regular quasiprojective $\mathbb{Q}$-varieties with all $\mathbb{Q}$-morphisms between them, F. Pop has shown that $\mathrm{Out}(\pi_1^{geom}(\mathcal{C}))$ is indeed equal to $G_{\mathbb{Q}}$ (2002, unpublished; the result is actually more general and stronger).

The specificity of *Grothendieck-Teichmüller theory* is that Grothendieck suggested (in [G1] and [G2]) studying the category $\underline{\mathcal{M}}$ of moduli spaces of curves with marked points, all of which are viewed as algebraic stacks defined over $\mathbb{Q}$. One does not *a priori* consider all possible $\mathbb{Q}$-morphisms between them, but only a certain family of morphisms coming from topological operations on the topological curves themselves, such as erasing points, cutting out subsurfaces by simple closed loops, or quotienting by finite-order diffeomorphisms. All these operations on topological curves yield natural morphisms between the associated moduli spaces (which include the classical Knudsen morphisms); these in turn yield homomorphisms between their geometric fundamental groups, which are nothing other than the profinite completions of the mapping class groups studied in this volume.

Perhaps the most insightful remark of Grothendieck on this topic is that the (outer) automorphism group $\mathrm{Out}(\pi_1^{geom}(\underline{\mathcal{M}}))$ of this category can actually be described explicitly, essentially as elements of the free profinite group on two generators satisfying a small finite number of equations, the reason for this being that in fact only the moduli spaces of dimensions 1 and 2 are important, the automorphism group remaining unchanged when the higher dimensional ones are added to the category. It is not known whether Grothendieck actually wrote down the defining equations of this group, which in essence is the Grothendieck-Teichmüller group.

However, in the seminal paper [Dr], V.Drinfel'd gave the definition of a profinite group $\widehat{GT}$ which is (essentially) equal to $\mathrm{Out}(\pi_1^{geom}(\underline{\mathcal{M}}_0))$ where $\underline{\mathcal{M}}_0$ is the category of moduli spaces of genus zero curves with marked points, whose geometric fundamental groups are essentially profinite braid groups. The argument above shows that there is a homomorphism $G_{\mathbb{Q}} \rightarrow \widehat{GT}$, which is injective by Belyi's celebrated theorem, and one of the essential goals of Grothendieck-Teichmüller theory is to compare these two groups. Another, somewhat alternative goal is to refine the definition of $\widehat{GT}$ to discover the automorphism group of the category of moduli spaces of all genus equipped with "as many $\mathbb{Q}$-morphisms as possible". This has been realized when the morphisms are point-erasing and cutting along simple closed loops, partially realized when quotients by finite-order diffeomorphisms are added, and in other, somewhat more general situations (see §2). But it is always possible to display other $\mathbb{Q}$-morphisms respected by $G_{\mathbb{Q}}$ and ask if any version of $\widehat{GT}$ also respects them.

Considering weaker profinite versions (pro-ℓ, pronilpotent), as well as proalgebraic (pro-unipotent, Lie algebra) versions of Grothendieck-Teichmüller theory has yielded new results, new conjectures and most interestingly, new links with aspects of number theory not visible in the full profinite situation. The later sections of this article are devoted to these.

Let us mention a handful of references which will provide the newcomer with entry points into the subject. For inspiration, we recommend reading parts 2 and 3 of Grothendieck's *Esquisse d'un Programme* ([G1]). The papers [Dr] and [I1] (as well as [De], although in a different vein) are certainly foundational for the subject. They still make very interesting, perhaps indispensable reading. Introductions to most of the main themes of the *Esquisse* are contained in the articles of [GGA]. In particular, introductions to the Grothendieck-Teichmüller group can be found in the article [S2] of [GGA] (see also [LS1], [L2]). The original article [Dr] of Drinfel'd introducing $\widehat{GT}$ is filled with impressive insights, but the point of view of moduli spaces is hardly touched upon, whereas the geometry of these spaces (in all genera) became central in [HLS] and [NS]. They can help make the bridge with the subject matter of the present volume.

In the rest of this section we will list some of the main definitions and terms of notation. Some of the objects are not defined from scratch, so that the exposition is not completely self-contained, however they are meant to make the subsequent statements understandable and unambiguous.

We start with a short list of the main geometric objects, which are also the main objects of study in the present volume:

– $\mathcal{M}_{g,n}$ (resp. $\mathcal{M}_{g,[n]}$) denotes the moduli space of smooth curves of genus g with n ordered (resp. unordered) marked points. These spaces can be considered as analytic orbifolds or as algebraic stacks over $\mathbb{Z}$, *a fortiori* over $\mathbb{Q}$ or any field of characteristic 0. We will make it clear what version we have in mind according to the context.

– $\Gamma_{g,n} = \pi_1^{orb}(\mathcal{M}_{g,n})$ (resp. $\Gamma_{g,[n]} = \pi_1^{orb}(\mathcal{M}_{g,[n]})$ denotes the orbifold fundamental group of the above space, as a complex orbifold. These are nothing but the mapping class groups of topologists, also called (Teichmüller) modular groups in the algebro-geometric context.

– $\widehat{\Gamma}_{g,n}$ (resp. $\widehat{\Gamma}_{g,[n]}$) are the profinite completions of the above groups. They are the geometric fundamental groups of $\mathcal{M}_{g,n}$ and $\mathcal{M}_{g,[n]}$ respectively, i.e. the fundamental groups of these spaces as $\overline{\mathbb{Q}}$- or $\mathbb{C}$-stacks.

We now pass to the Grothendieck-Teichmüller group in some of its most important versions. Others will appear in the course of the text. We start with the full profinite version $\widehat{GT}$, already mentioned above. Note that the profinite completion contains the maximum amount of information compared to the other completions and versions considered here.

First note that $\mathcal{M}_{0,4} \simeq \mathbb{P}^1 \setminus \{0, 1, \infty\}$. Then identify the topological fundamental group of the latter space with the free group F_2 on two generators x and y. This (non-canonical) identification amounts to picking two loops around 0 and 1 which generate the fundamental group of $\mathbb{C} \setminus \{0, 1\}$. With this identification we also identify $\pi_1^{geom}(\mathcal{M}_{0,4})$ with the profinite completion $\widehat{F}_2$, and we get a monomorphism:

$$\mathrm{Out}(\pi_1^{geom}(\underline{\mathcal{M}})) \hookrightarrow \mathrm{Out}(\widehat{F}_2).$$

In order to get $\widehat{GT}$, as originally defined in [Dr], replace $\underline{\mathcal{M}}$ with $\underline{\mathcal{M}}_0$, that is, use only genus 0 moduli spaces and pick a (tangential) basepoint in order to replace outer by *bona fide* actions. Finally require that the action preserve conjugacy classes of inertia groups, as the Galois action does. This produces again a monomorphism:

$$\widehat{GT} \hookrightarrow \mathrm{Aut}^*(\widehat{F}_2),$$

where the upper star refers to this inertia preservation condition. Concretely speaking, an element of $\widehat{GT}$ is given as a pair $F = (\lambda, f)$ with $\lambda \in \widehat{\mathbb{Z}}^*$ (invertible elements of $\widehat{\mathbb{Z}}$) and $f \in \widehat{F}_2'$ (topological derived subgroup of $\widehat{F}_2$). The action on $\widehat{F}_2$ is defined by:

$$F(x) = x^\lambda, \qquad F(y) = f^{-1} y^\lambda f.$$

One requires that these formulas define an *automorphism*, that is an invertible morphism, and there is no effective way to ensure this. Finally and most importantly the pair (λ, f) has to satisfy the following three relations (for the geometric origin of these relations, we refer to the introductions quoted above):

(I) $f(x, y)f(y, x) = 1$;
(II) $f(x, y)x^m f(z, x)z^m f(y, z)y^m = 1$ where $xyz = 1$ and $m = (\lambda - 1)/2$;
(III) $f(x_{12}, x_{23})f(x_{34}, x_{45})f(x_{51}, x_{12})f(x_{23}, x_{34})f(x_{45}, x_{51}) = 1$,

where in (III) (the pentagonal relation), the $x_{i,i+1}$ are the standard generators of the group $\widehat{\Gamma}_{0,5}$. We should also explain how substitution of variables is intended; for any homomorphism of profinite groups $\phi : \widehat{F}_2 \to G$ mapping $x \mapsto a$ and $y \mapsto b$, we write $\phi(f) = f(a, b)$ for $f \in \widehat{F}_2$ (f itself is equal to $f(x, y)$).

Thus $\widehat{GT}$ is the subgroup of $\mathrm{Aut}^*(\widehat{F}_2)$ whose elements are pairs $F = (\lambda, f)$ acting as above and satisfying (I), (II) and (III). Note that these are usually refered to as "relations" although "equations" would be more correct: indeed, $\widehat{GT}$ is a subgroup, not a quotient of $\mathrm{Aut}^*(\widehat{F}_2)$. We also mention that (I) is actually a consequence of (III), as was noted by H. Furusho, but we keep (I) in the definition nevertheless because of its geometric meaning.

There is a natural map $\widehat{GT} \to \widehat{\mathbb{Z}}^*$ defined by $F = (\lambda, f) \mapsto \lambda$. It is surjective and the kernel is denoted $\widehat{GT}^1$, which is an important subgroup of $\widehat{GT}$, the analog of which shows up in the various versions of the Grothendieck-Teichmüller group considered below.

The group $\widehat{GT}$ is profinite, as it is a closed sugroup of $\mathrm{Aut}(\widehat{F}_2)$ and any automorphism group of a finitely generated profinite group G has itself a natural structure of profinite group. Indeed, characteristic open subgroups are cofinal in G and $\mathrm{Aut}(G)$ can be written as the inverse limit:

$$\mathrm{Aut}(G) = \varprojlim_N \mathrm{Im}(\mathrm{Aut}(G) \to \mathrm{Aut}(G/N)),$$

where N runs through the open characteristic subgroups of G. Here we did not mention topology, because by a recent and fundamental result ([NiSe]), *any* automorphism of G is actually continuous. Applying this result to $\widehat{F}_2$, it makes the definition of $\widehat{GT}$ purely algebraic.

Starting from the profinite group $\widehat{GT}$, one can define interesting quotients in standard ways. In particular, one can define $GT^{\rm nil}$, the maximal pronilpotent quotient of $\widehat{GT}$; it is in fact the direct product of the pro-ℓ quotients $GT^{(\ell)}$, when ℓ runs through the prime integers. However, we have very little control over these quotients. More accessible are the groups $GT_{(\ell)}$ which are defined exactly like $\widehat{GT}$ except that we take $(\lambda, f) \in (\mathbb{Z}_\ell^*, F_2^{(\ell)})$ (where $F_2^{(\ell)}$ is the pro-ℓ completion of F_2). Comparison of $GT^{(\ell)}$ with $GT_{(\ell)}$ is an open question which we will record explicitly below.

We now pass to the proalgebraic setting, which will be useful especially in the later sections. All the algebraic groups $\underline{G}$ that we encounter, including those which make their appearance in the last sections only, in connection with motives and multiple zeta values, will be of the following type. The group $\underline{G}$ is linear proalgebraic over a field of characteristic 0, usually $\mathbb{Q}$ (sometimes $\underline{G}$ can actually be regarded as a progroup-scheme over $\mathbb{Z}$). It is an extension of the multiplicative group $\mathbb{G}_m$ by its prounipotent radical $\underline{G}^1$; the usual equivalence between unipotent algebraic groups and their Lie algebras extends to the proalgebraic setting, so $\underline{G}^1$ is isomorphic to its Lie algebra ${\rm Lie}(G^1)$. Moreover the latter is equipped with an action of $\mathbb{G}_m$ coming from the definition of G as an extension, and this action provides a natural grading, so that we can also consider the graded version of that Lie algebra, which is more amenable to concrete computations.

In the case of the Grothendieck-Teichmüller group, we encounter the same phenomenon as with the profinite versions mentioned above. We could consider the pro-ℓ quotient $GT^{(\ell)}$ and construct from it a prounipotent completion (over $\mathbb{Q}_\ell$). But again this is not easily accessible. So following [Dr], one first defines the prounipotent (or Malčev) completion $\underline{F}_2$. Then one defines $\underline{GT}$ by describing its k-points for k a field of characteristic 0. These are given again by pairs (λ, f) satisfying the relations as above, but now with $(\lambda, f) \in k^* \times \underline{F}_2(k)$.

The prounipotent radical $\underline{GT}^1$ is then defined as above. The associated Lie algebra is denoted $\mathfrak{gt}$ and its graded version $\mathfrak{grt}$. The latter is an especially important object, allowing for quite explicit computations. It was first defined and studied by V. Drinfeld and Y. Ihara. It is naturally defined over $\mathbb{Q}$, although Ihara showed it can in fact be defined over $\mathbb{Z}$, and this integral structure leads to very interesting arithmetic problems which we do not address in this note (see [I4], [McCS]).

Let us give here the explicit definition of $\mathfrak{grt}$, obtained by linearizing and truncating the defining relations of the group. Namely, the graded Lie algebra $\mathfrak{grt}$ is generated as a $\mathbb{Q}$-vector space by the set of homogeneous Lie polynomials $f(x, y)$ in two variables satisfying:

(i) $f(x, y) + f(y, x) = 0$;
(ii) $f(x, y) + f(z, x) + f(y, z) = 0$ with $x + y + z = 0$;
(iii) $f(x_{12}, x_{23}) + f(x_{23}, x_{34}) + f(x_{34}, x_{45}) + f(x_{45}, x_{51}) + f(x_{51}, x_{12}) = 0$,
where the x_{ij} generate the Lie algebra of the pure sphere 5-strand braid group.

This finishes our survey of the main definitions. Other objects will occur in the text, especially in the later sections. We remark that we refrained from explicitly using in this note the variant $\underline{GRT}$ of $\underline{GT}$, although it is conceptually quite significant. We refer to [Dr] and especially to [F1,F2] for more information on this point. Finally we note that we will sometimes use the bare letters GT as an abbreviation for "Grothendieck-Teichmüller" or as a "generic" version of the Grothendieck-Teichmüller group, so that this is not to be considered as a piece of mathematical notation.

Acknowledgments: Many of the questions and problems listed below arise naturally and were raised recurrently and independently by various people. We warmly (albeit anonymously) thank them all for sharing their preoccupations with us through the years. It is a pleasure to thank B. Enriquez and I. Marin for their interest and for suggesting interesting questions. We included some of these below (see §1 and §7) although in a simplified and incomplete version in order to minimize the necessary background. We are also delighted to thank H.Nakamura and the anonymous referee for many useful corrections and remarks.

§1. Group theoretical questions on $\widehat{GT}$

The fundamental result concerning the group $\widehat{GT}$ is that there is an injective homomorphism

$$G_{\mathbb{Q}} \hookrightarrow \widehat{GT}.$$

In some sense this is built into the definitions, via Belyi's result (cf. [G1] p.4; "[...] à vrai dire elle [l'action] est fidèle déjà sur le premier 'étage' [...]"). Drinfel'd indicated this fact in his original article [Dr], and Ihara gave the first complete proof. Let us recall a basic minimum. In order to associate an element $F_\sigma = (\lambda_\sigma, f_\sigma) \in \widehat{GT}$ to $\sigma \in G_{\mathbb{Q}}$, recall that there is a canonical outer $G_{\mathbb{Q}}$-action on the geometric fundamental group of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ which is inertia preserving. Proceedings as in the introduction, we get an element $F_\sigma \in \mathrm{Aut}^*(\widehat{F}_2)$ acting on the generators as:

$$F_\sigma(x) = x_\sigma^\lambda, \qquad F_\sigma(y) = f_\sigma^{-1} y_\sigma^\lambda f_\sigma.$$

Considering the abelianization of $\widehat{F}_2$ (i.e. the effect on homology) shows that $\lambda_\sigma = \chi(\sigma)$ where $\chi : G_{\mathbb{Q}} \to \widehat{\mathbb{Z}}^*$ is the cyclotomic character. As for f_σ, it becomes uniquely determined if one requires it to lie in the derived subgroup of $\widehat{F}_2$ (this is also the reason behind this requirement in the definition of $\widehat{GT}$). Ihara then went on, using geometric arguments, to prove that every such F_σ satisfies relations (I), (II) and (III), thus defining a homomorphism $G_{\mathbb{Q}} \to \widehat{GT}$. Injectivity is an easy consequence of Belyi's theorem.

Comparison between $\widehat{GT}$ and $G_{\mathbb{Q}}$ is a main goal of Grothendieck-Teichmüller theory. This comparison can be examined from various topological, geometrical and arithmetic points of view, the most straightforward of which may be direct group theory – at least in terms of questions to ask, if not to answer. For any group-theoretical property satisfied by $G_{\mathbb{Q}}$, it is natural to ask if $\widehat{GT}$ possesses the same property. Ihara began asking such questions in the early 1990's; we give a brief list here:

1.1. Let $(\lambda, f) \in \widehat{\mathbb{Z}}^* \times \widehat{F}_2'$. Does $x \mapsto x^\lambda$, $y \mapsto f^{-1} y^\lambda f$ extend to an automorphism, or can it actually determine a non-invertible endomorphism? One can ask the same question when (λ, f) satisfies (I), (II), (III).

Note that this question pertains to the full profinite setting only. Invertibility is immediately detected in the pronilpotent or proalgebraic situation.

1.2. Is $\widehat{GT}^1$ the topological derived subgroup of $\widehat{GT}$? In other words, is the abelianization of $\widehat{GT}$ obtained, like that of $G_{\mathbb{Q}}$ (by the Kronecker-Weber theorem), by the map $(\lambda, f) \mapsto \lambda$ corresponding to taking the cyclotomic character ($\lambda_\sigma = \chi(\sigma)$ for $\sigma \in G_{\mathbb{Q}}$)?

1.3. Does a version of the Shafarevitch conjecture hold for $\widehat{GT}$: is $\widehat{GT}^1$ a free profinite group on a countable number of generators?

1.4. $\widehat{GT}$ contains an element $c = (-1, 1)$ which acts on $\widehat{F}_2 \simeq \pi_1^{geom}(\mathbb{P}^1 \setminus \{0, 1, \infty\})$ as complex conjugation i.e. via $c(x) = x^{-1}$, $c(y) = y^{-1}$. Is the normalizer of c in $\widehat{GT}$ generated by c itself, as it is in $G_{\mathbb{Q}}$?

This question is natural but not open. It was resolved in the affirmative in [LS2], using methods of Serre and a profinite Kurosh theorem to compute the non-commutative cohomology group $H^1(\widehat{F}_2, \langle c \rangle)$.

1.5. Compare $GT_{(\ell)}$, as defined in [Dr] and in the introduction with $GT^{(\ell)}$, the maximal pro-ℓ quotient of $\widehat{GT}$. Similar questions arise in the proalgebraic setting.

1.6. Can anything be said about the finite quotients of $\widehat{GT}$? Obviously all abelian groups arise as quotients, since $\widehat{\mathbb{Z}}^*$ is a quotient of $\widehat{GT}$. But what non-abelian groups arise?

1.7. One of the difficulties of inverse Galois theory is that it is easier to prove that a given finite group G is a quotient of $G_{\mathbb{Q}^{\rm ab}}$ than of $G_{\mathbb{Q}}$, and that given the first result, it is not at all obvious how to deduce the second. Part of the problem is due to the difficulty of studying the outer action of $\widehat{\mathbb{Z}}^*$ on $G_{\mathbb{Q}^{\rm ab}}$ explicitly. This outer action is given explicitly, however, by the expression for the outer action of $\widehat{\mathbb{Z}}^*$ on $\widehat{GT}^1$, which contains $G_{\mathbb{Q}^{\rm ab}}$. Can this fact contribute to descending Galois groups over $\mathbb{Q}^{\rm ab}(T)$ to Galois groups over $\mathbb{Q}(T)$?

We close this section with a few words on the linear representations of $\widehat{GT}$. The theme would in principle require a section by itself, but since unfortunately practically nothing is known on this topic, we can remain brief. In particular, no irreducible non-abelian linear representation of any version of GT has been constructed to date. In [I2] Ihara constructed the $\widehat{GT}$ analog of the Soulé characters, so in particular of the Kummer characters. Other versions of at least some extensions of the Kummer and Soulé characters appear in [NS], [M] and a few other places. Hence the first question:

1.8. Are the various definitions of these characters equivalent (inasmuch as they overlap)? Investigate multiplicativity properties which come for free in the Galois case and are far from obvious in the $\widehat{GT}$ extensions (see [I2], §1.10).

I. Marin, in the article [M], constructs representations of $\underline{GT}^1(\mathbb{Q}_\ell)$ into $PGL_N(\mathbb{Q}_\ell((h)))$ (formal Laurent series). We do not recall the construction here, as it is quite complicated, noting only that one starts from an ℓ-adic representation of an infinitesimal braid group together with a given associator and then uses the action of $\widehat{GT}^1$ on the associators in order to produce a representation, provided a certain rigidity condition is fulfilled. The basic construction actually works in more general cases than ℓ-adic representations and is quite natural. The most important question in this context is:

1.9. Can one obtain non-abelian (projective) linear representations of $\underline{GT}^1(\mathbb{Q}_\ell)$ and GT_ℓ in this way?

Recall that there is a natural morphism $G_{\mathbb{Q}} \to \underline{GT}(\mathbb{Q}_\ell)$ (whose image is conjectured to be Zariski dense; see §8 below). This method thus also produces Galois representations. As the author explains, these representations are "often" abelian, hence the question above. Yet, in the abelian case and restricting to the Galois image, one produces characters of the Galois group. This leads naturally to the following problem.

1.10. Analyse the characters produced in [M] in terms of the Soulé characters.

Finally, given the map above, one can always pose the following (too) general question:

1.11. To what extent can one extend "classical" ℓ-adic representations (Tate modules, more generally étale cohomology etc.) into $\underline{GT}(\mathbb{Q}_\ell)$-representations?

§2. Other versions of $\widehat{GT}$

Several "refined" versions of the Grothendieck-Teichmüller group have been defined, each with relations added in order to satisfy some geometric property that $G_{\mathbb{Q}}$ is already known to satisfy. We do not give the definitions here (refering instead to the original articles), but simply the properties satisfied by each of four of the most interesting of these "refined" groups. Each of the groups discussed here contains $G_{\mathbb{Q}}$ as a subgroup.

– The group $\Lambda \subset \widehat{GT}$ is obtained by adding a single new relation (R) to the defining relations of $\widehat{GT}$ coming from $\mathcal{M}_{1,2}$, and has the property that it acts on the mapping class groups $\widehat{\Gamma}_{g,n}$ in all genera and respects the basic point erasing and subsurface inclusion morphisms of the moduli spaces, that is basically the classical Knudsen morphisms. We refer to [HLS] for an early version with extra hypotheses, and to [NS] for the general case.

– The group $\mathbb{I}\Gamma \subset \Lambda \subset \widehat{GT}$ is obtained by adding two new relations (IV) and (III′) to $\widehat{GT}$, and not only has the same property as Λ (because (III) together with (IV) implies (R)), but also respects the exceptional morphism $\mathcal{M}_{0,4} \to \mathcal{M}_{1,1}$, as well as the usual degree 6 quotient morphism already respected by $\widehat{GT}$ (cf. [NS]).

– The group $\widehat{GS}$ is defined by adding two new relations to the definition of $\widehat{GT}$ corresponding to respecting morphisms mapping $\mathcal{M}_{0,4}$ to special loci of $\mathcal{M}_{0,5}$ and $\mathcal{M}_{0,6}$ corresponding to points in those moduli spaces having non-trivial automorphisms. These two relations imply (R), (IV) and (III′), so that $\widehat{GS} \subset \mathbb{I}\Gamma \subset \Lambda \subset \widehat{GT}$. We refer to [S4] for the initial version with extra hypotheses, and to [T] for the general case.

– The group $\widehat{GTK}$, defined by Ihara in [I3], gives a definition obtained by adding an infinite series of relations requiring that the maps $\mathbb{P}^1 \setminus \{0, \mu_n, \infty\} \to \mathbb{P}^1 \setminus \{0, 1, \infty\}$ be respected, both by the quotient map $z \mapsto z^n$ and by the inclusion map, be respected by the $\widehat{GT}$-action on the π_1's of these curves, just as they are respected by the $G_{\mathbb{Q}}$-action (since all these curves and maps are defined over $\mathbb{Q}$). It is the only version of $\widehat{GT}$ which requires maps to be respected which are not maps between moduli spaces. Another version $\widehat{GTA}$ is defined in [I2] and elements of comparison between these two versions are discussed in [I3].

Faced with this dangerous explosion of versions of the original object, the most natural and pressing question is surely:

2.1. Are these groups actually different from each other? Or are some of the new relations already implied by previous ones, in particular by the original relations defining $\widehat{GT}$?

The two-level or locality principle. One of the fundamental geometric properties of the all-genera Grothendieck-Teichmüller group $\mathbb{I}\Gamma$ (resp. of $\widehat{GT}$) is that it is defined by relations coming from requiring the $\mathbb{I}\Gamma$- (resp. $\widehat{GT}$-) action to respect certain morphisms between the moduli spaces (resp. the genus zero moduli spaces) of dimensions 1 and 2. These relations *imply* that the analogous morphisms between higher dimensional (resp. genus zero) moduli spaces are automatically respected. Indeed, relations (I) and (II) reflect the requirement that there exists a $\widehat{GT}$-action on $\widehat{\Gamma}_{0,4}$ and $\widehat{\Gamma}_{0,[4]}$, respecting the homomorphisms coming from the moduli space morphism $\mathcal{M}_{0,4} \to \mathcal{M}_{0,[4]}$, while relation (III) comes from the requirement that the $\widehat{GT}$-action on $\widehat{\Gamma}_{0,4}$ extends to $\widehat{\Gamma}_{0,5}$ respecting the inclusion map $\Gamma_{0,4} \hookrightarrow \Gamma_{0,5}$ corresponding to erasing the fifth point. Finally, relation (R) comes from requiring $\mathbb{I}\Gamma$ to act on $\widehat{\Gamma}_{1,2}$ in such a way that the morphism $\mathcal{M}_{0,5} \to \mathcal{M}_{1,2}$ is respected.

2.2. Does the group $\widehat{GS}$ satisfy a two-level principle? In other words, does the assumption that it respects the special homomorphisms between moduli spaces in the first two levels imply that it automatically respects higher dimensional special morphisms (either in genus zero or in general)?

This is a highly mysterious property. Indeed the automorphisms of the curves in the first two levels involve essentially the platonic primes 2, 3 and 5, and so does the definition of $\widehat{GT}$ itself, whereas any finite group can be realised as the automorphism group of a suitable smooth hyperbolic curve. Is this reflected in the first two levels? In the same vein, one might ask:

2.3. Can the groups $\widehat{GTK}$ or $\widehat{GTA}$ be defined by a finite number of relations? Do they also satisfy some two-level principle?

§3. Moduli spaces of curves, mapping class groups and GT

We include in this section some questions concerning important algebraic and geometric aspects of the moduli spaces of curves which, even though not directly connected with the Grothendieck-Teichmüller group, are nonetheless closely related to the general themes of Grothendieck-Teichmüller theory. We also remark that the connection between that theory and anabelian geometry is far from clear at the moment.

3.1. A first very ambitious question is: Are the moduli spaces *anabelian*?

This is intended in the original sense introduced by Grothendieck in his seminal letter to G.Faltings (reproduced in [GGA]). Namely one is asking whether $\mathcal{M}_{g,n}$, viewed as a $\mathbb{Q}$-stack is the only $K(\pi,1)$ space (up to isomorphism) with augmented arithmetic fundamental group isomorphic to $\pi_1(\mathcal{M}_{g,n}) \to G_{\mathbb{Q}}$. We refer to the contribution of F.Pop in [GGA] ([Pop]) for a detailed and categorical formulation for general schemes. If indeed one prefers to work with schemes, the same question can be asked about any finite Galois étale covering of $\mathcal{M}_{g,n}$ which is a scheme.

3.2. Anabelian varieties (or schemes or stacks) should be rigid (see [N1], [Sx] for detailed discussions). This leads to the following test for anabelianity ([IN]): is it true that

$$\mathrm{Out}_{G_{\mathbb{Q}}}(\widehat{\Gamma}_{g,n}) = \mathrm{Aut}(\mathcal{M}_{g,n,\overline{\mathbb{Q}}})(=\mathcal{S}_n)?$$

($\mathcal{S}_n$ is the permutation group on n objects.) This statement has been proved for $g=0$ ([N2] and references therein) and indeed in greater generality, replacing $\mathbb{Q}$ with any field finitely generated over $\mathbb{Q}$ (see also [IN], §4). This is actually one of the only known results in higher dimensional anabelian geometry.

We refer to [IN] for the formulation of other similar tests for abelianity; they are all equivalent *if and when* $\widehat{\Gamma}_{g,n}$ is centerfree. Note also that giving the outer action of $G_{\mathbb{Q}}$ on $\widehat{\Gamma}_{g,n}$ is equivalent to giving the augmented arithmetic fundamental group as in **3.1** above, if and only if again $\widehat{\Gamma}_{g,n}$ is centerfree.

3.3. There also arises the question of comparing $\mathrm{Out}^*(\widehat{\Gamma}_{g,n})$ and $\mathrm{Out}(\widehat{\Gamma}_{g,n})$. Are they by any chance equal? A weaker version of this question, restricted to the $G_{\mathbb{Q}}$-equivariant exterior automorphisms, is: Do $\mathrm{Out}_{G_{\mathbb{Q}}}(\widehat{\Gamma}_{g,n})$ and $\mathrm{Out}^*G_{\mathbb{Q}}(\widehat{\Gamma}_{g,n})$ coincide?

So another ambitious and perhaps optimistic question is: does *any* (outer) automorphism of $\widehat{\Gamma}_{g,n}$ preserve inertia at infinity (conjugacy classes of Dehn twists)? This is important as the first group is more amenable to study than the second a priori larger group. This leads to asking whether one can give a group theoretic characterization of Dehn twists inside the profinite completion $\widehat{\Gamma}_{g,n}$ analogous to the one obtained by N.Ivanov in the discrete case (see [Iv] §7.5 and references therein). Note that this is strongly reminiscent of the so-called 'local correspondence' in birational anabelian geometry ([Sz]).

3.4. Is $\Gamma_{g,n}$ a good group in the sense of Serre?

Recall ([Se] §2.6) that a discrete and residually finite group G is good if the injection into its profinite completion induces an isomorphism in cohomology with finite (equivalently, torsion) coefficients. The question for $\Gamma_{g,n}$ is classical (see the contribution of T.Oda in [GGA]) and the answer is affirmative with an easy proof for $g \le 2$. It is also easy to show that $\Gamma_{g,n}$ is good if $\Gamma_g = \Gamma_{g,0}$ is. So the problem actually arises only for Γ_g with $g \ge 3$. In [B], goodness is announced for H^k, $k \leq 4$, with a very interesting application of the first non-trivial case, namely $k = 2$ (cf. [Sx]).

The connection between goodness and anabelianity stems from the fact that good groups have many open subgroups, in the sense that any cohomology class can be made to vanish by restriction to a suitable open subgroup. Geometrically speaking, a $K(\pi, 1)$ scheme (stack) whose geometric fundamental group is good has many étale covers which should give rise to an interesting Galois action. So the idea is that a $K(\pi, 1)$ quasiprojective scheme (stack) whose geometric fundamental group is universally centerfree and good is a "good" candidate for being anabelian. A prominent "anti-example" is the moduli stack $\mathcal{A}_g$ of principally polarized abelian varieties, whose fundamental group is *not* good because of the congruence property for the symplectic group $Sp_{2g}(\mathbb{Z})$ ($g > 1$)). This stack has few étale covers in the sense that for example they are all defined over $\mathbb{Q}^{ab}$ (cf. [IN,§3]).

3.5. How does the Grothendieck-Teichmüller action (of the group $\mathbb{I}\Gamma$, for example) on $\widehat{\Gamma}_{g,[n]}$ behave with respect to the finite-order elements?

Here we mean particularly those elements which come from the discrete group $\Gamma_{g,[n]}$, in other words are realisable as automorphisms of algebraic curves. It is not known whether all finite-order elements of the profinite completion come from the discrete group; one can ask:

3.6. Is every finite order element in $\widehat{\Gamma}_{g,[n]}$ conjugate to one in $\Gamma_{g,[n]}$?

This is essentially the torsion counterpart of the question about Dehn twists raised in **3.3.** Proving that $\Gamma_{g,n}$ is good would be a big step towards answering this question (see [LS2]).

Return to the $G_{\mathbb{Q}}$ and $\widehat{GT}$ actions on $\widehat{\Gamma}_{g,[n]}$; in a more detailed fashion, **3.5** actually asks whether the action is "cyclotomic" on the torsion elements (arising from the discrete group), in the sense that we can assert that for $F = (\lambda, f) \in \widehat{GT}$ and a finite-order element $\gamma \in \widehat{\Gamma}_{g,[n]}$, we have $F(\gamma) \sim \gamma^{\lambda}$; here $\sim$ denotes conjugation in $\widehat{\Gamma}_{g,[n]}$. The answer to this question is known to be affirmative only for $g = 0$ and for a few low dimensional spaces such as $\mathcal{M}_{1,1}$, $\mathcal{M}_{1,2}$, $\mathcal{M}_2$ (cf. [S4]). In the other cases, it is conjectured but not known, even for $G_{\mathbb{Q}}$.

3.7. It is shown in [HS] that for each $n \geq 5$, we have:

$$\widehat{GT} \simeq \mathrm{Out}^*_{\mathcal{S}_n}(\widehat{\Gamma}_{0,n}).$$

The geometric significance of $\mathcal{S}_n$ here is as in **3.2** for $g = 0$. Does the analogous all-genera isomorphism:

$$\mathbb{I}\Gamma \simeq \mathrm{Out}^*_{\mathrm{Aut}(\mathcal{M}_{g,n})}(\widehat{\Gamma}_{g,n})$$

also hold? If not, is the right-hand group at least defined by a finite number of relations, possibly coming only from dimensions 1 and 2?

§4. Dessins d'enfants

The theory of dessins d'enfants and their various descriptions, topological, algebraic, combinatorial and others has been described in several articles (e.g. [S1]). Here, we use the definition of a dessin d'enfant as being equivalent to a *Belyi cover*, i.e. a finite cover:

$$\beta : X \to \mathbb{P}^1$$

of Riemann surfaces unramified outside 0, 1 and ∞. Belyi's famous theorem states that an algebraic curve over $\mathbb{C}$ is defined over $\overline{\mathbb{Q}}$ if and only if it can be realized as a Belyi cover (the 'only if' direction being the really new one). There is a natural Galois action on the set of dessins, as these are defined over $\overline{\mathbb{Q}}$. Since $\widehat{GT}$ by definition acts on $\pi_1^{geom}(\mathbb{P}^1 \setminus \{0, 1, \infty\}) \simeq \widehat{F}_2$, it also acts on the set of dessins, which are in one-to-one correspondence with conjugacy classes of finite index subgroups of $\widehat{F}_2$, and this $\widehat{GT}$-action extends the $G_{\mathbb{Q}}$-action. The first question is the basic and original one about dessins d'enfants:

4.1. Can one give a complete list of Galois invariants of dessins, i.e. enough combinatorial invariants of the Galois action on dessins to determine the Galois orbits?

The standard Galois invariants are such things as valencies (i.e. ramification indices over 0, 1, ∞), order of the Galois group of the Galois closure of the Belyi cover, the Galois group itself, in fact, various extensions of this group, etc. None of these invariants seems to be enough to actually distinguish Galois orbits of dessins, although putting them together astutely yields more than using them singly (cf. [W]). By a combinatorial invariant, we mean one which is computable combinatorially from the two permutations defining the dessins (see [S1]). Very few have been found to date (see however [Z]) and one is not even sure whether it is possible to distinguish Galois orbits via combinatorial invariants only.

A slightly weaker question would be to give a combinatorial method for determining the number field of moduli of the dessin (or even its degree). On this subject, R. Parker expressed a remarkable conjecture. To phrase this conjecture, we consider a dessin to be given by the equivalent data of a finite group G on two generators, say of order n, together with an explicit choice a and b for the two generators. The dessin can easily be reconstructed from this by injecting the group into S_n via its action on itself by right multiplication, which gives a and b as permutations whose cycle lengths describe the valency lists and whose cycles themselves give the cyclic order of edges around these valencies.

4.2. *Richard Parker's suggestion:* Let G be a finite group generated by two elements a and b, and consider the element P in the group ring $\mathbb{Q}[G \times G]$ given by $P = \sum_{g \in G}(gag^{-1}, gbg^{-1})$. Choosing the basis of pairs (g, h) for the vector space $\mathbb{Q}[G \times G]$, right multiplication by P gives an automorphism of the vector space which can be written as a matrix M_P. Could the Galois closure of the field of moduli K of the dessin associated to G, a, b be generated over $\mathbb{Q}$ by the eigenvalues of M_P?

This can be proven without difficulty for dessins with abelian or dihedral Galois groups. It would be a good exercise to complete the genus zero case, by dealing with the remaining cases, namely the automorphism groups of the five Platonic solids. This is easy to do for the smaller ones, and confirms the conjecture. The larger ones should be easy too, except that the computations soon become gigantic...

Another natural question is whether the $\widehat{GT}$-action on dessins transmutes into an action on curves, as follows.

4.3. Let X be an algebraic curve, and suppose we have two dessins on X, i.e. two different Belyi functions $\beta_1 : X \to \mathbb{P}^1$ and $\beta_2 : X \to \mathbb{P}^1$. Then the images of these two dessins under an element $\sigma \in G_{\mathbb{Q}}$ are both dessins on the curve $\sigma(X)$. As above there is also a $\widehat{GT}$-action on dessins. If $F = (\lambda, f) \in \widehat{GT}$, do the two new dessins $F(\beta_1)$ and $F(\beta_2)$ also lie on the same Riemann surface?

Let us define an action of $\widehat{GT}$ on $\overline{\mathbb{Q}}$ in the following, rather artificial manner. First choose a fundamental domain D for the natural action of $\mathcal{S}_3$ on $\mathbb{P}^1\mathbb{C} \setminus \{0, 1, \infty\}$ and represent any elliptic curve E by its Legendre form: $y^2 = x(x-1)(x-\lambda)$ with $\lambda \in D$. Then take the dessin β on E given by the Belyi polynomial produced by starting from the function x on E and applying Belyi's original algorithm. For each $F \in \widehat{GT}$ and each $j \in \overline{\mathbb{Q}}$, define $F(j)$ to be the j-invariant of the elliptic curve underlying the dessin $F(\beta)$.

Several questions arise from this construction. Apart from the obvious "Is this an automorphism of $\overline{\mathbb{Q}}$?" and the questions of whether the value of $F(j)$ depends on the choice of λ, the choice of dessin on E, and whether there is not some more canonical way of defining this action, here are some that appear more approachable.

4.4. Does this definition of the action of $F \in \widehat{GT}$ on $\overline{\mathbb{Q}}$ fix $\mathbb{Q}$?

The answer is certainly yes and probably not hard to show, but needs to be written down.

4.5. What can one compute in the case where λ is of low degree, for instance a square root of a rational number?

4.6. Can one show that the action is at least additive?

Dessins became very popular as topological, indeed combinatorial objects on which the profinite group $G_{\mathbb{Q}}$ acts faithfully. But it is possible to consider more general such objects; in particular, instead of looking at covers of $\mathbb{P}^1 - \{0, 1, \infty\}$, the next natural step would be to consider covers of $\mathbb{P}^1-$ 4 points, or of elliptic curves, so that the variation in the complex structure of the base enriches the arithmetic and geometric structures of the covers.

Using elliptic curves as a base yields the "origamis" (square tiled surfaces) studied in [L1]; they naturally sit inside the moduli spaces of curves, and thus are a priori directly connected with GT. We refer to [L1], [Mö], [Sc] as well as papers by G. Schmithüsen and F. Herrlich for this material.

One can ask the same questions about origamis that one asks about dessins: determine invariants, Galois orbits etc. Let us rather mention two specific problems which seem to be within reach. Since origamis are really higher dimensional versions of dessins, as they are topological surfaces and degenerate to dessins when approaching the boundary of the appropriate moduli space, the first question or problem is naturally:

4.7. Study, geometrically and arithmetically, the degeneration of origamis into dessins.

The second problem is related and even more specific. Very few non-trivial combinatorial Galois invariants have been constructed for dessins; one of these is Zapponi's invariant constructed in [Z]. For differentials there is also one and only one invariant, which is indeed a Galois invariant, namely the parity of the spin structure (see [KZ]). Both invariants are signs (they are $\mathbb{Z}/2\mathbb{Z}$-valued). The question is:

4.8. Express the parity of the spin structure for origamis in a combinatorial way. Using **4.7**, can one relate this invariant with Zapponi's invariant?

§5. Number theory and $\widehat{GT}$; a direct approach

The previous section already shows how the study of $\widehat{GT}$ can lead naturally to number theoretic questions, although number theory is not immediately visible in the definition of $\widehat{GT}$, even viewed as an automorphism group of fundamental groups of moduli spaces. The following Galois-theoretic question was formulated by Y. Ihara. It is a basic interrogation about the Galois action in the pro-ℓ setting, and indeed represents the analog of Belyi's result in that context.

5.1. Let M^* denote the fixed field of the kernel of the homomorphism:

$$G_{\mathbb{Q}} \to \mathrm{Out}^*(F_2^{(\ell)}).$$

Does M^* coincide with $M^{(\ell)}$, the maximal pro-ℓ extension of $\mathbb{Q}$ unramified outside ℓ?

The point is that Grothendieck's theory of the specialization of the fundamental group implies that $M^* \subset M^{(\ell)}$ (see [I1]); note that M^* is just the field of definition of the proscheme defined by the tower of ℓ-covers of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$.

One of the most frequently asked questions in the early days of $\widehat{GT}$ was the following: How can one see the primes, or the decomposition groups, or the Frobenius elements in $\widehat{GT}$? In other words, can one define local versions of $\widehat{GT}$, i.e. subgroups which would correspond in a natural way to the p-adic decomposition subgroups (defined up to conjugacy) $G_{\mathbb{Q}_p} = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ in $G_{\mathbb{Q}}$? We will denote such a subgroup GT_p (not to be confused with $GT_{(\ell)}$ in §§0,1). For the moment, we have two ways of defining such a GT_p at our disposal.

The first definition uses the action of $\widehat{GT}$ on $\overline{\mathbb{Q}}$ defined before **4.4** and so is rather *ad hoc.* Elements σ in a subgroup $G_{\mathbb{Q}_p}$ are characterized inside $G_{\mathbb{Q}}$ by the following property:

$$\sigma \in G_{\mathbb{Q}_p} \ \text{ if and only if } |j|_p \leq 1 \Rightarrow |\sigma(j)|_p \leq 1 \ \text{ for all } j \in \overline{\mathbb{Q}}.$$

So by analogy, we take GT_p to be the subgroup of $\widehat{GT}$ satisfying the same property.

For the second definition, we recall that Y. André defined (in [A1]) the temperate fundamental group $\pi_1^{temp}(X)$ of a p-adic manifold X, which injects naturally into the algebraic profinite fundamental group $\pi_1^{alg}(X)$. Considering $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ over $\mathbb{Q}_p$, we can define (as in [A1]) GT_p to be the subgroup of $\widehat{GT}$ which preserves the subgroup $\pi_1^{temp}(\mathbb{P}^1 \setminus \{0, 1, \infty\})$. André has shown that considering $G_{\mathbb{Q}}$ as a subgroup of $\widehat{GT}$, one has:

$$GT_p \cap G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p).$$

5.2. Do the two definitions above coincide?

5.3. Can one describe inertia subgroups of the subgroups GT_p corresponding to those of $G_{\mathbb{Q}_p}$? Can one characterize Frobenius elements? Note that it was already observed in [I1] that even in the Galois setting, it could be hard to recognize the Frobenius elements from their geometric action.

§6. *GT* and Mixed Tate Motives

If we restrict to the unipotent setting, the objects considered above are motivic, and moreover of mixed Tate type. In this relatively restricted context, the paradise of motives is a reality; in particular there exists a Tannakian category $MT(\mathbb{Z})$ of Mixed Tate motives over $\mathbb{Z}$. It is equipped with a canonical (Beilinson-de Rham) fiber functor and the associated fundamental group G^{mot} is an extension of the multiplicative group $\mathbb{G}_m$ by its prounipotent radical U^{mot}. The latter is graded by the $\mathbb{G}_m$ action, and its graded Lie algebra L^{mot} is free on one generator in each odd degree $2k+1$, $k \geq 1$. This landscape is described in particular in the work of A.B.Goncharov ([Go1,2] as well as many other papers by the same author), in [DG] and in [A2]. The motivic point of view suggests many more or less "standard" conjectures, of which we will state only a few in the last three sections, without entering into all the necessary technicalities (even in the statements). We will also give some slightly less standard, more directly GT-oriented problems.

One can define the motivic fundamental group $\pi = \pi_1^{\mathrm{mot}}(\mathbb{P}^1 \setminus \{0, 1, \infty\})$ of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ and its coordinate ring is an ind-object in $MT(\mathbb{Z})$ (see [DG] for details). Then one can consider the Tannakian subcategory $\langle \pi \rangle \subset MT(\mathbb{Z})$ generated by π, i.e. containing all the motives obtained from π by taking tensor products of π and its dual, direct sums and subquotients. A very strong conjecture asks:

6.1. Is the inclusion $\langle\pi\rangle \hookrightarrow MT(\mathbb{Z})$ an isomorphism? If G_π denotes the fundamental group of the Tannakian category $\langle\pi\rangle$, then there is a natural epimorphism $G^{\mathrm{mot}} \to G_\pi$, and this question is equivalent to asking whether it is an isomorphism.

In order to get closer to GT, it is natural to introduce the genus 0 moduli spaces $\mathcal{M}_{0,n}$ into the motivic landscape, together with their stable compactifications $\overline{\mathcal{M}}_{0,n}$. The motivic cohomology of these spaces belongs to $MT(\mathbb{Z})$, whereas that of higher genera moduli spaces does not. The motivic fundamental groups $\pi_1^{\mathrm{mot}}(\mathcal{M}_{0,n})$ are constructed in [DG] and also belong to $MT(\mathbb{Z})$. One can then propose the following task:

6.2. Consider the Tannakian subcategory of $MT(\mathbb{Z})$ generated by the fundamental groups of the $\mathcal{M}_{0,n}$ $(n \geq 4)$; how does its fundamental group compare with G_π and $\underline{GT}$?

There exists in fact a monomorphism $G_\pi \hookrightarrow \underline{GT}$ (see e.g. [A2]) and the point is:

6.3. Is the natural monomorphism $G_\pi \hookrightarrow \underline{GT}$ an isomorphism?

Putting **6.1** and **6.3** together underlines the fact that one can reasonably ask whether the three groups G^{mot}, G_π and $\underline{GT}$ coincide. In the next two sections, we present weaker and more specific versions of this type of expectation. We believe that having the higher dimensional genus 0 moduli spaces come into play should help to approach them, in conformity with the original spirit of Grothendieck-Teichmüller theory.

§7. The Hodge side: GT and multiple zeta values

Multiple Zeta Values (hereafter simply multizeta values) are real numbers defined using either infinite series or integrals, the latter representation being much more recent. They satisfy two very different looking families of relations, namely the quadratic or (regularized) double shuffle relations and the associator (or modular, or GT) relations. We write Z_w for the $\mathbb{Q}$-vector space spanned by the multizeta values of weight $w \geq 0$ (where we formally set $\zeta(0) = 1$ and $\zeta(1) = 0$), and $Z_\bullet$ for the $\mathbb{Q}$-algebra of the multizeta values, filtered by the weight (see any paper on the subject, including [Go1,2], [F1,F2], [A2] etc.). Thus, we have $Z_0 = \langle 1\rangle$ and $Z_1 = 0$, $Z_2 = \langle\zeta(2) = \pi^2/6\rangle$, $Z_3 = \langle\zeta(3)\rangle$, and then the dimensions grow quickly as multizeta values appear (see **7.3** below).

In order to capture some of the main algebraic and combinatorial properties of these real numbers without confronting intractable transcendence problems, three other filtered $\mathbb{Q}$-algebras have been introduced: $Z_\bullet^{DS}$, $Z_\bullet^{GT}$ and $Z_\bullet^{\mathrm{mot}}$. The first (resp. second) of these consists in taking formal multizeta symbols which satisfy only the double shuffle (resp. associator) relations, which are known to be satisfied by the genuine multizeta values and conjectured to form a complete set of algebraic relations between them. These two algebras are graded by the weight. Note that in papers written in French or by French speaking authors (e.g. [A2], [E], [R]), 'DS' reads 'DM' or 'DMR' ('Double mélange régularisé'). The algebra of motivic multizeta values $Z_\bullet^{\mathrm{mot}}$ constructed by Goncharov is more complicated to define: Goncharov has proved in [Go2] that it is a quotient of $Z_\bullet^{DS}$, but perhaps not strictly, and no further relations are explicitly known. It is known, however, that this algebra is also graded by the weight. By contrast we have:

7.1. Conjecture: The weight induces a grading on $Z_\bullet$.

In other words, it is not even known whether there are any linear relations between real multizeta values of different weights. This conjecture immediately implies the transcendence of all multizeta values (since a minimal polynomial would yield such a linear relation), in particular of the values $\zeta(2n+1)$ at odd positive integers of Riemann's zeta function, so it is expected to be extremely difficult.

On the subject of $Z_\bullet^{\rm mot}$, Goncharov has shown that it is naturally realizable as a subalgebra of the universal enveloping algebra $\mathcal{U}(L^{\rm mot})$ of the Lie algebra $L^{\rm mot}$ of the unipotent part $U^{\rm mot}$ of the motivic Galois group $G^{\rm mot}$ of $MT(\mathbb{Z})$ (cf. §6 above). Goncharov conjectures that in fact

$$Z_\bullet^{\rm mot} \xrightarrow{\sim} \mathcal{U}(L^{\rm mot})^\vee.$$

This leads to the following well-known conjecture:

7.2. The algebras $Z_\bullet/\pi^2 Z_\bullet$, $Z_\bullet^{DS}$, $Z_\bullet^{GT}$, $Z_\bullet^{\rm mot}$ and $\mathcal{U}(L^{\rm mot})^\vee$ are canonically isomorphic.

Equivalently one conjectures the isomorphism of the five algebras $Z_\bullet$, $\mathbb{Q}[\pi^2] \otimes_{\mathbb{Q}} Z_\bullet^{\rm mot}$, etc. Moreover, it is easily seen ([Go1]) that:

$$\mathbb{Q}[\pi^2] \otimes_{\mathbb{Q}} \mathcal{U}(L^{\rm mot})^\vee \simeq \mathcal{U}(L[s_2, s_3])^\vee,$$

where $L[s_2, s_3]$ is the free Lie algebra on two generators in weights 2 and 3; the dimensions of the graded parts of this algebra are given by the coefficients of the generating series $1/(1 - t^2 - t^3)$, which leads to the following dimension conjecture for all five algebras:

7.3. Dimension conjecture (D. Zagier): Letting d_w denote the dimension of Z_w (resp. DS, GT, mot) as a $\mathbb{Q}$-vector space, one has $d_w = d_{w-2} + d_{w-3}$ (with $d_0 = d_2 = 1$, $d_1 = 0$).

The upper bound for $Z_\bullet$, i.e. the fact that the actual dimension of Z_w is less than or equal to the conjectured one, was proved independently by A. Goncharov and T. Terasoma (see [A2] for references and a sketch of proof following Goncharov). The statement involves no transcendence, and the proof is motivic.

Leaving aside $Z_\bullet$ itself, one can explore the possible isomorphisms between the other algebras. For instance:

7.4. Can one find an explicit isomorphism between $Z_\bullet^{DS}$ and $Z_\bullet^{GT}$, i.e. an explicit way to obtain double shuffle relations from associator relations and vice versa?

This is in principle an algebraic or even combinatorial problem, which has proved difficult and enticing.

The original multizeta values $Z_\bullet$ can be seen as periods of motives of the category $\langle \pi \rangle \subset MT(\mathbb{Z})$ (cf. §6), where Goncharov has shown how to attach a 'framing' to a mixed Tate motive, which yields a complex-valued period (modulo periods of motives of lower weights). A weaker version of **6.1** is thus:

7.5. Do the multizeta values give all the periods of (framed) mixed Tate motives over $\mathbb{Z}$?

Recall that the genus 0 moduli spaces $\mathcal{M}_{0,n}$ and their stable completions $\overline{\mathcal{M}}_{0,n}$ provide important objects of $MT(\mathbb{Z})$. In fact, Goncharov and Manin ([GoM]) have shown that there are canonical framed mixed Tate motives associated to these spaces, whose periods are the multizeta values. This also provides an alternative but equivalent definition of $Z_\bullet^{\rm mot}$.

The above question can be rephrased purely analytically. There is a map:

$$p : \overline{\mathcal{M}}_{0,n+3} \to (\mathbb{P}^1)^n,$$

obtained by successive blowups (so it is a birational isomorphism). Consider the standard real n-simplex $\Delta = \{0 < t_1 < t_2 < \ldots < t_n < 1\} \subset (\mathbb{P}^1)^n$; the topological closure of the preimage $p^{-1}(\Delta)$ is the standard associahedron $K \subset \overline{\mathcal{M}}_{0,n+3}$. Relative periods are of the form $\int_K \omega$, where ω is a top dimensional logarithmic form; these can be explicitly determined. The multizeta values correspond to very particular such ω's, those having only factors of t_i or $1-t_i$ in the denominators. The analytic form of **7.5** now reads:

7.6. Are all such integrals given by $\mathbb{Q}$-linear combinations of multizeta values?

Of course not all such integrals converge, and one can dream up several different versions of the above, the simplest of which is to make the statement only for the convergent integrals (there is a nice geometric criterion for convergence).

The multizeta values satisfy the double shuffle relations and the associator relations. The associator relations in particular come directly from the geometry of the $\mathcal{M}_{0,n}$. Thus, from the perspective of Grothendieck-Teichmüller theory, it is very natural to ask:

7.7. Can one give generalized double shuffle and/or associator relations valid for all the relative periods of the $\overline{\mathcal{M}}_{0,n}$ and coming from the geometry of these spaces?

In other words the task consists in exploring the combinatorics of the relative periods of these spaces. This could be useful for attacking, but is logically independent of question **7.6**.

Similar or equivalent questions to the above are posed by several authors in the proalgebraic context. The algebras $Z_\bullet^{DS}$, $Z_\bullet^{GT}$ and $Z_\bullet^{\mathrm{mot}}$ are all commutative Hopf algebras which are universal enveloping algebras of Lie coalgebras; these results are due, in chronological order, to V. Drinfel'd and Y. Ihara for $Z_\bullet^{GT}$, A. Goncharov for $Z_\bullet^{\mathrm{mot}}$ and G. Racinet [R] for $Z_\bullet^{DS}$. Their spectra are three unipotent affine group schemes:

$$\underline{G}^{DS}, \quad \underline{GT}^1 \quad \text{and} \quad \underline{G}^{MZ}.$$

Thus, the isomorphism questions in **7.2**, apart from the original $Z_\bullet$, can be rephrased as:

7.8. Are the affine unipotent group schemes $\underline{G}^{DS}$, $\underline{GT}^1$, $\underline{G}^{MZ}$ and U^{mot} all isomorphic?

The two points of view explained here concern duals of universal enveloping algebras of Lie algebras and their spectra, affine unipotent group schemes. Let us now rephrase some of these ideas from the point of view of the Lie algebras/coalgebras themselves. This can be quite enlightening and leads to new results and connections. For instance, Goncharov has computed the coproduct on $Z_\bullet^{\mathrm{mot}}$ explicitly, and deduced the expression of the Lie cobracket on the vector space $Z_{>2}^{\mathrm{mot}}/(Z_{>0}^{\mathrm{mot}})^2$, making it into a Lie coalgebra. Generalizing this to a question about $Z_\bullet$ itself yields an equivalent but more striking and precise version of **7.5**:

7.9. Let $\mathfrak{nz}$ (for 'new zeta') be the $\mathbb{Q}$ vector space obtained by quotienting $Z_\bullet$ by the ideal generated by Z_0, Z_2 and $(Z_{>0})^2$ (or equivalently, quotienting $Z_{>2}$ by $(Z_{>0})^2$). Is there a surjection (isomorphism?) $(L^{\mathrm{mot}})^\vee \to \mathfrak{nz}$, thus defining a structure on $\mathfrak{nz}$ of a Lie coalgebra, whose dual would thus be (freely?) generated by one element in each odd rank?

The algebra Z^{GT} is related to the Lie algebra $\mathfrak{grt}$ (cf. §0) by the fact that Z^{GT} is the dual of the universal enveloping algebra of $\mathfrak{grt}$; equivalently, $Z_{>2}^{GT}/(Z_{>0}^{GT})^2$ is a Lie coalgebra dual to $\mathfrak{grt}$. Similarly, the double shuffle Lie algebra $\mathfrak{ds}$ is given as a vector space by $\big(Z_{>2}^{DS}/(Z_{>0}^{DS})^2\big)^\vee$; it is however also quite simple to define directly, cf. [R]. What Racinet actually proved is that this vector space is closed under the Poisson (alias Ihara) bracket, from which one deduces that the enveloping algebra and its dual are Hopf algebras, so that the spectrum $\underline{G}^{DS}$ of the dual is an affine group scheme.

In analogy with **7.9**, we have the following conjectures, the first of which was made by Ihara much before the second:

7.10. Conjecture: There are surjections, or better isomorphisms, from the free Lie algebra L^{mot}

on generators $s_3, s_5, s_7, \ldots$ to the Grothendieck-Teichmüller and double shuffle Lie algebras:

$$L^{\mathrm{mot}} \twoheadrightarrow \mathfrak{grt} \quad \text{and} \quad L^{\mathrm{mot}} \twoheadrightarrow \mathfrak{ds}.$$

In other words, $\mathfrak{grt}$ and $\mathfrak{ds}$ would themselves be (freely) generated by one generator in each odd rank ≥ 3. Note, however, that the obvious depth filtration (by the descending central series) on L^{mot} would not map to the natural depth filtration on elements of $\mathfrak{grt}$ and $\mathfrak{ds}$ under such an isomorphism. There should be a more subtle depth filtration on L^{mot}.

Of course, **7.4** can now be rephrased in the Lie algebra context as:

7.11. Are $\mathfrak{grt}$ and $\mathfrak{ds}$ isomorphic?

Computations have confirmed that are they isomorphic in low ranks; the candidate isomorphism would simply be given by $f(x,y) \mapsto f(x,-y)$ (cf. [R]).

Most of the objects defined above are associated with $\mathbb{P}^1 \setminus \{0,1,\infty\}$. A natural generalization, much studied by A. Goncharov, is to consider $\mathbb{P}^1 \setminus \{0, \mu_N, \infty\}$ for any positive integer N. Then the Lie algebras $\mathfrak{grt}$ and $\mathfrak{ds}$ (alias $\mathfrak{dmr}$), along with the other attending objects can be generalized, as was done in [E] and [R] respectively, in which the authors define analogs for any N, denoted $\mathfrak{grtmd}(N)$ and $\mathfrak{dmrd}(N)$ respectively. These generalizations prompt one to ask the following questions (suggested by B. Enriquez):

7.12. Do the elements exhibited in [E] generate $\mathfrak{grtmd}(N)$ and are there relations?

About the first question, recall from above that for $N = 1$ it is not known whether $\mathfrak{grt}$ is generated by one generator in each odd degree ≥ 3 (cf. **7.10**). About the second question and contrary to the case $N = 1$, freeness is not expected. Finally it is natural to try and compare $\mathfrak{grtmd}(N)$ and $\mathfrak{dmrd}(N)$, just as we compare $\mathfrak{grt}$ and $\mathfrak{ds}$, say in the following relatively weak form:

7.13. Does the inclusion $\mathfrak{grtmd}(N) \subset \mathfrak{dmrd}(N)$ hold true (at least for an odd prime N)?

§8. The Galois side; GT and $G_{\mathbb{Q}}$ once again

Recall that there is a canonical injection $G_{\mathbb{Q}} \hookrightarrow \widehat{GT}$. It is conceivable that the profinite group $\widehat{GT}$ (or its refinements) may be different from $G_{\mathbb{Q}}$, but some simpler quotient may be equal to the corresponding quotient of $G_{\mathbb{Q}}$. In particular, let GT^{nil} be the nilpotent quotient of $\widehat{GT}$, i.e. the inverse limit over the finite nilpotent quotients of $\widehat{GT}$.

8.1. Is GT^{nil} isomorphic to the nilpotent completion of $G_{\mathbb{Q}}$?

Equivalently, one can ask whether the maximal pro-ℓ quotient $GT^{(\ell)}$ of $\widehat{GT}$ is isomorphic to the maximal pro-ℓ quotient of $G_{\mathbb{Q}}$ for each prime ℓ?

For any prime ℓ, Deligne and Ihara constructed (independently) a graded $\mathbb{Q}_\ell$-Lie algebra from the action of $G_{\mathbb{Q}}$ on $\pi^{(\ell)} \simeq F_2^{(\ell)}$, the pro-$\ell$ fundamental group of $\mathbb{P}^1 \setminus \{0,1,\infty\}$. Let $\pi^{(\ell)}[m]$ denote the descending central series of $\pi^{(\ell)}$: $\pi_1^{(\ell)}[0] = \pi^{(\ell)}$, $\pi^{(\ell)}[m+1] = [\pi^{(\ell)}, \pi^{(\ell)}[m]]$, i.e. the subgroup topologically generated by the commutators. There is a filtration of $G_{\mathbb{Q}}$ defined by:

$$I_\ell^k G_{\mathbb{Q}} = \mathrm{Ker}\big(G_{\mathbb{Q}} \to \mathrm{Out}(\pi^{(\ell)}/(\pi^{(\ell)})[k]\big).$$

It is easy to see that $I_\ell^0 G_{\mathbb{Q}} = G_{\mathbb{Q}}$ and $I_\ell^1 G_{\mathbb{Q}} = G_{\mathbb{Q}^{\mathrm{ab}}}$; it is not too hard to see that also $I_\ell^2 G_{\mathbb{Q}} = G_{\mathbb{Q}^{\mathrm{ab}}}$ and in fact $I_\ell^3 G_{\mathbb{Q}} = G_{\mathbb{Q}^{\mathrm{ab}}}$. Set:

$$DI^{(\ell)} = \big(\mathrm{Gr}^{\bullet}_{I_\ell} G_{\mathbb{Q}}\big) \otimes \mathbb{Q}_\ell = \bigoplus_{k \geq 0} \big(I_\ell^k G_{\mathbb{Q}} / I_\ell^{k+1} G_{\mathbb{Q}}\big) \otimes \mathbb{Q}_\ell.$$

This graded vector space is naturally equipped with a Lie bracket coming from the commutator map $(\sigma, \tau) \mapsto \sigma\tau\sigma^{-1}\tau^{-1}$ on the group $G_{\mathbb{Q}}$.

8.2. Conjecture: (Y. Ihara) The $DI^{(\ell)}$ have a common $\mathbb{Q}$-structure, i.e. there exists a $\mathbb{Q}$-Lie algebra DI such that $DI^{(\ell)} = DI \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$.

8.3. Furthermore, $DI \simeq \mathfrak{grt}$, i.e. $DI^{(\ell)} \simeq \mathfrak{grt} \otimes \mathbb{Q}_\ell$ for each ℓ.

These two Lie algebras can be computed explicitly in low degree. They are equal at least up to degree 13. Conjecture **8.3** is stronger than **8.2**, but we have stated **8.2** separately because it might be more accessible. However, a third very natural question which crops up here, the analog of the same question given previously concerning $\mathfrak{nz}$ in **7.9** and $\mathfrak{grt}$ and $\mathfrak{ds}$ in **7.10**, is a theorem in the present case, thanks to a result of R.Hain and M.Matsumoto. Recall that L^{mot} is the motivic fundamental Lie algebra of $MT(\mathbb{Z})$, freely generated by one generator in each odd rank ≥ 3.

8.4. Theorem ([HM1,2]): There is a surjection $L^{\mathrm{mot}} \otimes \mathbb{Q}_\ell \to DI^{(\ell)}$ for each prime ℓ.

Let us return briefly to the proalgebraic setting. From the ℓ-adic realization we get, for any prime ℓ, a morphism $G_{\mathbb{Q}} \to G^{\mathrm{mot}}(\mathbb{Q}_\ell)$ whose image is Zariski dense ([Go1], [HM1,2]; see also [A2]). Composing with the natural surjection we find a morphism $G_{\mathbb{Q}} \to G_\pi(\mathbb{Q}_\ell)$, again with dense image. We may now compose with the monomorphism into $\underline{GT}$ of **6.3**, restrict to the prounipotent part of the image as this is what is really at stake, and arrive at a question already posed in [Dr].

8.5. Is the image of the map $G_{\mathbb{Q}^{ab}} \to \underline{GT}^1(\mathbb{Q}_\ell)$ Zariski dense?

Here one can in fact replace $G_{\mathbb{Q}^{ab}}$ with the Galois group of $\mathbb{Q}_\ell(\mu_{\ell^\infty})$. In [F2], Furusho also asks the proalgebraic analog of **8.2**, **8.3** (his Conjectures B and C) and proceeds to show that they are actually equivalent to their pro-ℓ versions (see his Proposition 4.3.3).

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