Fragments of nonlinear Grothendieck-Teichmüller theory

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After giving a short survey of the theory as it exists to-date, we discuss the specifity of what we call the nonlinear version of Grothendieck-Teichmüller theory, in contrast with the pronilpotent motivic approach.

1 Introduction

This note contains no new result; its main goal is to detail the meaning of the adjective 'nonlinear' occurring in the title, which we do in section 3. Moreover, as this same title indicates, the first section (supplemented by the Appendix) can hopefully be seen as providing a very short and incomplete introduction to the existing part of the theory. As for the short second section it has been included for the sake of pointing out a few basic phenomena and can perhaps be useful to the newcomer as it lists the main entry points into the recent literature. In connection with this we note once and for all that we have considered that the existing extremely efficient electronic data bases make it both hopeless and useless to aim at any kind of exhaustivity. Relevant references can nowadays easily be retrieved by using just the name of an author or an appropriate keyword.

Grothendieck-Teichmüller theory was conceived or dreamt of by A.Grothendieck in his *Esquisse d'un programme* (now available in [GGA]), following his *Longue marche à travers la théorie de Galois*. A few seminal papers, especially [D], [I1] and [De], started giving flesh to the vision. The theory is still very much in flux and there are several possible frameworks or versions which are connected in particular by various kinds of 'linearization' processes. This is in part what we will discuss in section 3, from a biased and prospective viewpoint. Clearly, linearizing a situation is at present a natural, indeed almost irresistible mathematical inclination, and the linear toolbox is much more varied, sophisticated and powerful than the nonlinear one. So it is not only unavoidable but also necessary and fruitful. But it does leave aside part of the richness of the situation, as we will try to suggest by recalling and assembling concrete and fairly elementary facts.

The adjective 'linear' should clearly here be taken in a rather extended acception, with partial synonyms such as 'abelian', '(pro)nilpotent', '(pro)solvable' or even 'motivic'. In turn 'nonlinear' can translate in several ways, such as 'profinite' or 'anabelian'. As a partial illustration of this dichotomy we quote the following sentence from the introduction of [De]: "Cette relation étroite avec la cohomologie indique que l'étude du π_1 rendu nilpotent est loin du rêve 'anabélien' de A.Grothendieck. Elle permet par contre de s'appuyer sur sa philosophie des motifs". We will see that on the nonlinear side, there may be a – as yet modest and very much incomplete – new landscape which is slowly emerging, perhaps in partial conformity with the vision outlined in the *Esquisse*.

2 A Short Reminder and A Reading Guide

We will be concerned here with what we call the nonlinear version of Grothendieck-Teichmüller theory, obviously as it is available to-date (2003), which means at a very incomplete stage of development. Apart from the *Esquisse* and the seminal papers already mentioned in the introduction we refer the reader to [Sc1], [LS1], the introduction to [HLS] and references therein for very imperfect and partial accounts. Clearly there does not and cannot exist at present a satisfactory survey of a largely unchartered landscape.

Getting started in a nutshell, skipping the necessary motivating questions: We first consider the collection of the fine moduli spaces of curves $\mathcal{M}_{g,n}$ for varying finite hyperbolic type (g, n), together with their stable completions $\overline{\mathcal{M}}_{g,n}$. These objects were constructed in [DM] as algebraic stacks over $\mathbf{Z}$ but we confine attention to the generic fiber and view them as separated regular Deligne-Mumford $\mathbf{Q}$-stacks.

The reader who finds this introit a little abrupt can go for instance to [IM], [PS], [DM] and the many other texts which introduce these objects in a more or less analytic or algebraic fashion. Here is a short *da capo*, in order to fix more notation and make the path from topology into arithmetic geometry perhaps more transparent. One can construct the space $\mathcal{M}_{g,n}$ ($g \geq 0$, $n \geq 0$, $2g-2+n > 0$) as a complex orbifold of dimension $3g-3+n$ classifying Riemann surfaces of genus g with n marked points, and then compactify it by adding in Riemann surfaces with nodes. The orbifold fundamental group of $\mathcal{M}_{g,n}$ coincides with the mapping class group of the topologists: $\pi_1^{orb}(\mathcal{M}_{g,n}) = \Gamma_{g,n}^{top}$. On the algebraic side, as mentioned above, one constructs $\mathcal{M}_{g,n}$ as a $\mathbf{Q}$-stack which can be completed by gluing a divisor at infinity parameterizing the stable curves. The theory of the fundamental group for stacks parallels, but with interesting differences, Grothendieck's classical (SGA 1) theory for schemes (see [No] and [LV]); for the specific case of the moduli stacks of curves, which are classifying spaces, we also refer to the contribution of T.Oda in [GGA], which uses simplicial techniques. In particular, if $\mathcal{X}$ is a geometrically connected separated Deligne-Mumford stack over $\mathbf{Q}$, one can consider its geometric (stack) fundamental group $\pi_1^{geom}(\mathcal{X}) = \pi_1(\mathcal{X} \otimes \overline{\mathbf{Q}})$. After fixing an embedding $\overline{\mathbf{Q}} \subset \mathbf{C}$, an extended version of Lefschetz principle implies that $\pi_1^{geom}(\mathcal{X})$ is canonically isomorphic to the profinite completion of $\pi_1^{orb}(\mathcal{X}^{an})$, where $\mathcal{X}^{an}$ is the complex orbifold obtained by analytification of $\mathcal{X} \otimes \mathbf{C}$. Here and above, π_1^{orb} is the fundamental group defined and studied by W.Thurston. In particular $\pi_1^{geom}(\mathcal{M}_{g,n}) = \Gamma_{g,n}$, where $\Gamma_{g,n}$ denotes the *profinite* Teichmüller modular group. Note that in the absence of superscript groups will be profinite by default.

The $\mathcal{M}_{g,n}$ and their stable completions fit together into a category or modular tower which we denote by $\underline{\mathcal{M}}$. This means that one can define smooth morphisms of geometric origin between the $\mathcal{M}_{g,n}$'s. There are actually several (existing or not yet existing) versions, the simplest and most classical being derived from the stable stratification. Recall that this is the stratification of $\overline{\mathcal{M}}_{g,n}$ such that each stratum is labelled by a stable graph, where edges symbolize normal crossings (nodes) and vertices correspond to regular curves of lower genera, such that the total arithmetic genus is g. The open stratum $\mathcal{M}_{g,n}$ has a graph reduced to a point and all strata are isomorphic, up to finite morphisms, to products of $\mathcal{M}_{g',n'}$'s. This gives rise to 'natural' morphisms, also called Knudsen or 'clutching' morphisms (see the papers by F.Knudsen). By their very definition these morphisms 'live at infinity', that is they are essentially (up to products and finite morphisms) closed immersions of moduli spaces of lower dimensions into the divisor at infinity $\overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n}$. One can also add in the point erasing morphisms, that is the fibration $\mathcal{M}_{g,n+1} \to \mathcal{M}_{g,n}$ defining the universal curve over $\mathcal{M}_{g,n}$ and the corresponding universal monodromy map. Finally one can take the action of the permutation groups into account and replace the $\mathcal{M}_{g,n}$ by the $\mathcal{M}_{g,[n]}$ or variants thereof. Here $\mathcal{M}_{g,[n]}$ denotes the moduli space of curves of genus g with n unlabelled marked points; there is a natural Galois covering $\mathcal{M}_{g,n} \to \mathcal{M}_{g,[n]}$ which is orbifold unramified of group $\mathcal{S}_n$ (the permutation group on n objects) and gives rise to yet another set of morphisms.

The long and the short is that the above defines, modulo a lot of 'technical details' a version of $\underline{\mathcal{M}}$ which is actually fairly simple minded for at least two reasons: a) the morphisms live at infinity, b) they are defined over $\mathbf{Q}$ (not only $\overline{\mathbf{Q}}$), essentially because they feature algebraic counterparts of familiar topological gestures: pasting, pinching, erasing etc. It seems fair to say however that this is the only version of the modular tower containing all the $\mathcal{M}_{g,n}$'s which has really been studied and understood in this context, that is in relation with the Grothendieck-Teichmüller group (see below). It gives rise to what we call the (Teichmüller) lego at infinity, for which we refer to [HLS] and [NS] from the viewpoint of Grothendieck-Teichmüller theory and to [FG], [BK] and references therein from other perspectives. In the case of genus 0 as initiated in [D], it is sometimes referred to as the 'geometry of associativity' because it was Drinfeld's groundbreaking idea to, so to speak not take associativity for granted (keywords: quasi-Hopf algebras, braided categories, McLane coherence relations, universal scattering matrix, Yang-Baxter equations, gravity operad etc. etc.). Last but not least, we note that this is *not* the version of $\underline{\mathcal{M}}$ and the hypothetical attending lego which Grothendieck seems to have in mind in the *Esquisse*, where he refers explicitly to curves with automorphisms, to which we will briefly return below.

Not much is known to-date beyond this version of the modular tower which again a) lives at infinity, b) is defined over $\mathbf{Q}$. In fact a) and b) are far from independent and it may well be that it is essentially the largest possible tower which is entirely defined over $\mathbf{Q}$. At any rate and for the time being, having built a more or less expensive version of $\underline{\mathcal{M}}$ over $\mathbf{Q}$, one applies the geometric fundamental group functor π_1^{geom}, from the category $(\mathbf{Q} - Stacks)$ of $\mathbf{Q}$-stacks with $\mathbf{Q}$-morphisms to the category $(Grps)$ of finitely generated profinite groups with continuous homomorphisms. In other words if $\mathcal{X}$ is a $\mathbf{Q}$-stack (say, nice: separated, D-M, quasicompact, geometrically connected), one sets as above $\pi_1^{geom}(\mathcal{X}) = \pi_1(\mathcal{X} \otimes \overline{\mathbf{Q}})$, which is a finitely generated profinite group. One can then indeed regard π_1^{geom} as a covariant functor from $(\mathbf{Q} - Stacks)$ to $(Grps)$. Letting this functor act on our modular tower, we get the Teichmüller tower $\underline{\Gamma} = \pi_1^{geom}(\underline{\mathcal{M}}) = \pi_1(\underline{\mathcal{M}} \otimes \overline{\mathbf{Q}})$, which is thus no more and no less than the collection of the (profinite) Teichmüller modular groups $\Gamma_{g,n}$ with varying type (g, n) and morphisms coming functorially from the morphisms in $\underline{\mathcal{M}}$.

The next step consists in considering the group $Out(\pi_1^{geom})$, that is the outer automorphism group of the functor π_1^{geom}, meaning the group of automorphisms modulo inner automorphisms on the right-hand side, *i.e.* in $(Grps)$. Let $G_{\mathbf{Q}}$ denote as usual the absolute Galois group of $\mathbf{Q}$. By a fairly easy extension to stacks of Grothendieck's short exact sequence (SGA1, §IX.6), we get a morphism $G_{\mathbf{Q}} \to Out(\pi_1^{geom})$. This map is injective, *i.e.* the outer Galois action is faithful as soon as we consider a big enough version of $(\mathbf{Q} - Stacks)$. We were a little fuzzy above as to which geometric objects we wish to include in $(\mathbf{Q} - Stacks)$ partly because Belyi's theorem (recalled in the Appendix) immediately implies a much more drastic assertion: as soon as $(\mathbf{Q} - Stacks)$ contains the single object $\mathbf{P}^1_{\mathbf{Q}} \setminus \{0, 1, \infty\}$, the action is faithful, that is the above map from $G_{\mathbf{Q}}$ to $Out(\pi_1^{geom})$ is an injection. To see this, just apply theorem A1 to the set of all elliptic curves defined over numberfields, that is with j-invariants in $\overline{\mathbf{Q}}$. From now on we will sometimes write $\mathbf{P}^* = \mathbf{P}^1 \setminus \{0, 1, \infty\}$ for brevity. Note that the action of $G_{\mathbf{Q}}$ is faithful when restricted to other kinds of coverings of $\mathbf{P}^*$, for instance on trees (see the article of L.Schneps in [DE] as well as later papers by L.Zapponi). The fact that the 'enormous' and complicated profinite group $G_{\mathbf{Q}}$ acts naturally and faithfully on various types of seemingly simple looking topological objects has been realized only in the last two decades and has by now come up in a variety of situations.

Before going back to our central geometric object, namely the modular tower, we mention an important recent and as yet unpublished result of F.Pop. We have just seen that the map $G_{\mathbf{Q}} \to Out(\pi_1^{geom})$ is injective. In fact, not only is the above map an injection, but under rather mild conditions it is an isomorphism! This means the following: consider $\mathcal{C} \subset (\mathbf{Q} - Stacks)$ a full subcategory and the restriction of π_1^{geom} to $\mathcal{C}$. We still get a natural map $G_{\mathbf{Q}} \to Out(\pi_1^{geom}(\mathcal{C}))$ which is still an injection under very mild conditions (*e.g.* if $\mathcal{C}$ contains $\mathbf{P}^*$). Moreover, and somewhat informally, as $\mathcal{C}$ gets smaller, the target gets larger. Pop's result says that actually, even for a rather 'small' sample of geometric objects $\mathcal{C}$, the above map is an isomorphism. Here is a sample statement: Define $\mathcal{C}$ to be the category whose objects are quasi-projective varieties defined over $\mathbf{Q}$ which are the complements in the projective plane of (not necessarily irreducible) curves; the morphisms are the dominant $\mathbf{Q}$-morphisms. With this definition of the geometric category $\mathcal{C}$ we have (F.Pop, 2000): *The natural map:* $G_{\mathbf{Q}} \to Out(\pi_1^{geom}(\mathcal{C}))$ *is an isomorphism.*

This is a striking and beautiful result, the first in this field which connects a purely arithmetic object (on the left-hand side) to a purely geometric object (on the right-hand side). Only we note for further reference that nothing is known about the right-hand side, apart from being proved to be isomorphic to the left-hand side.

Now let us go back to the modular tower $\underline{\mathcal{M}}$ and consider the group $Out(\pi_1^{geom}(\mathcal{M})) = Out(\underline{\Gamma})$ where the outer automorphisms are equivariant with respect to the morphisms in the Teichmüller tower $\underline{\Gamma}$. One has $G_{\mathbf{Q}} \hookrightarrow Out(\underline{\Gamma})$ since $\mathbf{P}^* \simeq \mathcal{M}_{0,4}$ appears as an object of $\underline{\mathcal{M}}$. In the colorful words of the *Esquisse*: 'L'action est déjà fid{graveele au premier étage'. Moreover the action of $G_{\mathbf{Q}}$ enjoys a well-known property: it preserves inertia groups and the inertia groups in $\Gamma_{g,n}$ associated to the components of the divisor at infinity of the stable completion of $\mathcal{M}_{g,n}$ are nothing but Dehn twists. Concrete conclusion: the action of $G_{\mathbf{Q}}$ on the $\Gamma_{g,n}$'s maps Dehn twists to conjugates of powers of themselves; in other words they preserve the conjugacy classes of the procyclic groups they generate. Here we are talking about the inertia 'at infinity' associated to the classical situation where a scheme X can be written as $X = \overline{X} \setminus D$ with $\overline{X}$ proper and D a divisor with (strict) normal crossings. This is discussed in general in [GM]; the adaptation to stacks is not completely banal but this is *not* by itself a typical stacky phenomenon. In fact other types of inertia appear for stacks (see below for references) which have just started being explored in this context. For the moment and in view of the above, we define $\mathbb{\Gamma} = Out^*(\underline{\Gamma}) \subset Out(\underline{\Gamma})$ to be the subgroup of inertia preserving outer automorphisms in that sense. This is by definition the *Grothendieck-Teichmüller group*, at least in its present, all genera, profinite version, and for the version of the modular tower $\underline{\mathcal{M}}$ outlined above.

As a first concrete approach, and in order to find 'coordinates' for $\mathbb{\Gamma}$, one notes that $\mathbb{\Gamma} = Out^*(\underline{\Gamma}) \subset Out^*(\pi_1^{geom}(\mathcal{M}_{0,4}))$. Then $\mathcal{M}_{0,4} \simeq \mathbf{P}^*$ (here over $\mathbf{Q}$) and $\pi_1^{geom}(\mathbf{P}^*) = \pi_1(\mathbf{P}^* \otimes \mathbf{C}) = \pi_1(\mathbf{C} \setminus \{0,1\}) = \widehat{F}_2$, the profinite completion of the free group on two generators, since obviously $\pi_1^{top}(\mathbf{C} \setminus \{0,1\}) \simeq F_2$. We remark that the importance of $\mathbf{P}^*$ was already mentioned in SGA 1 ('le problème des trois points') where Grothendieck notices that there was (and apparently still is, after four decades) no other way to compute its geometric fundamental group (say over $\overline{\mathbf{Q}}$) than to use Lefschetz principle as above ($\pi_1(\mathbf{C} \setminus \{0,1\})$ is the profinite completion of $\pi_1^{top}(\mathbf{C}\setminus\{0,1\})$), thus reducing oneself to elementary topology. In other words there is still no algebraic way of computing –say– $\pi_1(\overline{\mathbf{Q}}\setminus\{0,1\})$... At any rate proceeding as above we get $\mathbb{I}\Gamma \subset Out^*(\widehat{F}_2)$ and it is easy to see that the latter group can be parametrized by pairs $F=(\lambda,f)\subset \widehat{\mathbf{Z}}^*\times\widehat{F}_2'$ where $\widehat{F}_2'$ denotes the derived subgroup of $\widehat{F}_2$ (see any paper on the subject, starting with [I1] and [D]). If $F_2=\langle x,y\rangle$, that is x and y topologically generate $\widehat{F}_2$, the action of $F=(\lambda,f)$ is given explicitly by: $F(x)=x^\lambda$, $F(y)=f^{-1}y^\lambda f$. We also note that the outer automorphism groups we are considering, in particular $Out^*(\widehat{F}_2)$ and the Grothendieck-Teichmüller group $\mathbb{I}\Gamma$ are naturally endowed with the profinite topology because they are automorphism groups of topologically finitely generated groups; one uses here the fact that in such a group, characteristic sugbroups form a cofinal sequence.

Now the really amazing and central point, foreshadowed in the *Esquisse*, is that $\mathbb{I}\Gamma$ defined as above is in a certain sense explicitly computable: it is given as a subgroup of $Out^*(\widehat{F}_2)$ by a small number of relations (say four) which translate into equations on the pair (λ,f). In fact $\mathbb{I}\Gamma$ has been computed in [HLS] and [NS], adding one, perhaps not independent relation to the genus 0 version introduced in [D]. Note that the term 'relation' which is commonly used here should not be misleading; $\mathbb{I}\Gamma$ is given as a subgroup, not a quotient of $Out^*(\widehat{F}_2)$. We refer the reader to the articles quoted above for (much) more on this, but it may be good to mention Grothendieck's 'principe des deux premiers étages' ('two levels principle') at this point. It says that the generators of $\mathbb{I}\Gamma$ can be found at the first level of $\underline{\mathcal{M}}$ (moduli spaces of dimension 1) and the relations at the first and second level (moduli spaces of dimensions 1 and 2). Indeed we just found the generators by looking at $\mathbf{P}^*$, that is $\mathcal{M}_{0,4}$ and the first level consists of $\mathcal{M}_{0,4}$ and $\mathcal{M}_{1,1}$ (moduli of elliptic curves) which moreover are tightly related. We also notice that this principle (discussed from a geometric viewpoint in [L1]) reminds one of the way one computes the fundamental group of a cellular complex in topology.

Essentially by definition there is a natural inclusion $G_{\mathbf{Q}}\subset\mathbb{I}\Gamma$; see the contribution of Y.Ihara in [DE] for a proof from the viewpoint of algebraic geometry. Whether or not this inclusion is strict is a main driving question of this young field. Note that the situation is in some sense opposite to that of F.Pop's result quoted above. Throwing in more, or say different objects he proves the remarkable isomorphism: $G_{\mathbf{Q}}\simeq Out(\pi_1^{geom}(\mathcal{C}))$, giving in principle a geometric characterization of the arithmetic Galois group $G_{\mathbf{Q}}$. Yet as mentioned above we know nothing concrete about the right-hand-side, so it does not immediately help study the left-hand-side. On the other hand, using the Teichmüller tower as above, we get that $G_{\mathbf{Q}}\subset\mathbb{I}\Gamma=Out^*(\underline{\Gamma})$. Here in some sense we are able to 'compute' the right-hand side, but we do not know whether or not the inclusion is an isomorphism.

Although it took quite some time to complete the picture sketched above, it should still be considered as a rather primitive stage of the theory. True we have used moduli spaces of curves of *all finite types*, especially all genera, and we have used the full *profinite* completions, two important positive features. But we have used essentially only the structure of the modular tower *at infinity*. So we get what can be called a (Grothendieck-Teichmüller) *lego at infinity* or parabolic lego to take up the terminology of the Thurston-Bers classification of diffeomorphisms. From this point of view, one can ask for a different and probably much more subtle sort of lego, connected in particular with the automorphisms of curves (so that it could be termed *elliptic* lego) which is actually the only one mentioned in the *Esquisse* and would encode a lot more arithmetic than the one at infinity. In particular, one has to enrich the modular tower in a drastic way, probably throwing in objects and morphisms which are defined over $\overline{\mathbf{Q}}$, not only over $\mathbf{Q}$. Each such morphism is actually defined over a finite extension K of $\mathbf{Q}$ and leads to an equivariant action of G_K, that is an open subgroup of $G_{\mathbf{Q}}$ which however effectively depends on the particular morphism one is looking at. This is elementary Galois theory, but the counterpart is lacking at present on the Grothendieck-Teichmüller side. Since the exploration of these tracks is hardly beginning we prefer to stop here and refer the reader to recent texts which contain first results and try to isolate relevant features and objects (see in particular [L2], [LV], [LNS], [Sc2,3]).

Remark: The Grothendieck-Teichmüller action is studied in [HLS] and [NS] in the context of the modular tower $\underline{\mathcal{M}}$ and the attending Teichmüller tower $\underline{\Gamma} = \pi_1^{geom}(\underline{\mathcal{M}})$ in the version described at the beginning of this section, that is 'at infinity' and 'over $\mathbf{Q}$'. This is the only case to-date where a reasonably complete picture has been obtained outside of genus 0. These articles may not however look so user friendly and it might be helpful to the potential reader to get a few general clues as to the techniques. A main point is that 'complexes of curves' as introduced by W.Harvey in the sixties and actively studied by topologists since then describe the structure at infinity of moduli spaces of curves (see [Ha] for an introduction). In genus 0, they translate into the much simpler McLane's coherence relations for braided categories (see the contribution of L.Schneps in [PS] for a careful description of this translation; it is somehow implicit in [D]). A version of these complexes, introduced by A.Hatcher and W.Thurston in a classical paper, is the main tool of [HLS]. In this way, one captures the structure at infinity of the moduli spaces based on the so-called maximally degenerate points but not the more refined attending tangential base points. Algebraic translation (restricted to the genus 0 situation) in terms of braided categories: one cannot modify the commutativity operator (the universal R-matrix) but concentrates on the 'geometry of associativity', as pioneered in [D]. Galois translation: one must leave aside the cyclotomic action. Another difficulty which is left aside in [HLS] is connected with an important map at the first level of the modular tower (already mentioned in [D] for the same reasons): it maps an elliptic curve viewed as a twofold cover of the projective line to the projective line with the four ramification (*i.e.* Weierstrass) points marked. This implies a change in the uniformizing parameter at the origin of the elliptic curve. In turn this change of uniformizing parameter results in the occurrence of the Kummer character ρ_2 at 2 in the Galois action, as can be inferred from the asymptotics of the Legendre modular function. In particular one needs to devise a Grothendieck-Teichmüller analog of ρ_2 in order to pass from the Galois to the Grothendieck-Teichmüller action. Putting these two restrictions together one finds that [HLS] deals with a subgroup $\Lambda \subset \mathbb{I}\Gamma$ which contains the subgroup of the elements of $G_{\mathbf{Q}}$ with $\chi = 1$ (χ = cyclotomic character), $\rho_2 = 0$.

In [NS], the strategy of [HLS] is refined in two ways, in order to get the action of the full Galois group and the corresponding version $\mathbb{I}\Gamma$ of the Grothendieck-Teichmüller group: First one needs to take the cyclotomic action into account, which necessitates passing from maximally degenerate points to the more refined tangential base points. Topological translation: one has to rigidify, adding 'seams' to the 'pants', which leads to objects which in [NS] are called 'quilts' and appear elsewhere under various names in topological contexts (*cf.* [FG]). Second [NS] contains a definition and study of the analog of ρ_2. Putting these improvements together, one recovers the full $\mathbb{I}\Gamma$ action on the Teichmüller tower $\underline{\Gamma}$.

3 Glimpses of the motivic theory

Much of what can be called the linearized part of the theory has now been cast in the framework of mixed Tate motives. This is also where the genus 0 prounipotent Grothendieck-Teichmüller group, which is actually the version which was originally introduced by V.Drinfeld, connects with multiple zeta values. The few words below are meant to motivate the reader for further exploration and to point out a few general features in view of the discussion in the next section.

An important starting point was an insight of P.Deligne and Z.Wojtkowiak in the early eighties about the monodromy of polylogarithms; this is how mixed Tate motives were brought to bear on the subject in [De] (*cf.* the quotation in the introduction). At that time the category of MT motives was still conjectural and those were regarded in [De] as "systems of realizations", essentially the Hodge and ℓ-adic realizations. When k is a numberfield the abelian category $MT(k)$ of MT motives over k was constructed in the early nineties by M.Levine ([Le]) and in the late nineties it was pointed out by A.B.Goncharov that one could easily from there define the full subcategory $MT(\mathcal{O}_k(S))$ of MT motives which are unramified (in a suitable sense) outside the set S of primes in the ring $\mathcal{O}_k$ of integers of k. These constructions have been used especially for $k = \mathbf{Q}$ and $S = \emptyset$ ($\mathcal{O}_k(S) = \mathbf{Z}$). A recent detailed account can be found in [DG]. Among the objects of $MT(\mathbf{Z})$, one finds in particular the moduli spaces $\mathcal{M}_{0,n}$ ($n \geq 3$) of genus 0 curves, and among their relative periods all the multiple zeta values ([GMa]) occur, and may even give *all* the periods of $MT(\mathbf{Z})$. In fact the $\mathcal{M}_{0,n}$'s conjecturally generate the abelian category $MT(\mathbf{Z})$. Another and indeed more classical way of retrieving the multiple zeta values is via iterated integrals, or what amounts to the same $\pi_1^{uni}(\mathbf{P}^*_{\mathbf{C}})$, the prounipotent completion of the fundamental group of $\mathbf{P}^*_{\mathbf{C}}$. That the latter group can be viewed as an object of $MT(\mathbf{Q})$ is explained in detail in [DG], amplifying [De] and previous work by A.B.Goncharov and others.

Once the category $MT(B)$ of MT motives over a base scheme B has been constructed, one gets (unconditional, *i.e.* free from conjectures!) access to a paradise where things indeed go smoothly and are often reduced to linear algebra, modulo categorical work. Note that as first explained in detail by M.Levine, the construction of $MT(B)$ as an abelian category with all the desired properties rests in an essential way on the truth of the Beilison-Soulé vanishing conjecture for B, which gives control on the extension classes $Ext^1(\mathbf{Q}(n), \mathbf{Q}(m))$ of the pure objects $\mathbf{Q}(n)$. In particular that conjecture is now a theorem for $B = \mathcal{O}_k(S)$, making possible the unconditional construction of $MT(\mathcal{O}_k(S))$. Let us briefly review some of the salient points of that 'paradise' of MT motives, now a reality, with a view on the rough 'nonlinear' world to come.

The main point is that for very general reasons the situation is essentially governed by *prounipotent* groups. Let us consider $MT(k)$ with k a number-field so that the category is known to exist; the general pattern is true over any basis B, provided of course that $MT(B)$ has been constructed. Objects $M \in MT(k)$ come equipped with a filtration W, because they are constructed by iterated extensions of the pure objects $\mathbf{Q}(n)$ and one has the vanishing condition $Ext^1(\mathbf{Q}(n), \mathbf{Q}(m)) = 0$ for $n \geq m$. Taking the graduation Gr^W associated to this weight filtration, by the very definition of Tate objects, the graduate pieces are sums of copies of pure objects $\mathbf{Q}(n)$ of the appropriate weight. For compatibility reasons with Hodge theory, or say an appropriate interpretation of Cauchy formula, one assigns $\mathbf{Q}(1)$ the weight -2, so that $\mathbf{Q}(n)$ has weight $-2n$ and for any $M \in MT(k)$ and $n \in \mathbf{Z}$, $Gr^W_{-2n}(M)$ is a sum of copies of $\mathbf{Q}(n)$, which one can write explicitly as $Gr^W_{-2n}(M) = \mathbf{Q}(n) \otimes Hom(\mathbf{Q}(n), Gr^W_{-2n}(M)) = \mathbf{Q}(n) \otimes \omega_n(M)$, where the latter equality defines ω_n. Denoting by ω_* the direct sum of the ω_n one gets a functor $M \to \omega_*(M)$ from $MT(k)$ to the category of graded vector spaces, which respects the tensor product. Finally there is a natural action of the multiplicative group $\mathbf{G}_m$ on the pure objects, with $\mathbf{Q}(1)$ giving the standard (identity) representation, hence $\mathbf{Q}(n)$ the n-th power $(\lambda \mapsto \lambda^n)$. Note that the category of finite dimensional graded vector spaces is equivalent to that of finite dimensional representations of $\mathbf{G}_m$ (here one can consider $\mathbf{Q}$-vector spaces and regard $\mathbf{G}_m$ as a group scheme over $\mathbf{Q}$). By forgetting the graduation we pass from ω_* to ω, which is a $\otimes$-functor from $MT(k)$ to the category of finite dimensional vector spaces, exhibiting $MT(k)$ as a Tannakian category with fiber functor ω. So by the Tannakian dictionary, $MT(k)$ is equivalent to the category of the representations of a proalgebraic k-group $G_\omega = \underline{Aut}^{\otimes}(\omega)$. There is a natural surjection $G_\omega \to \mathbf{G}_m$ where $\mathbf{G}_m$ acts on $\mathbf{Q}(1) = \mathbf{Q}$ and in fact on all pure objects (sums of copies of $\mathbf{Q}(n)$). The kernel is a group U_ω which is *prounipotent*. Indeed by definition it respects the filtration W, or rather its image via ω and it acts trivially on the associated graduation Gr^W, since it acts trivially on pure objects.

Temporary conclusion: $MT(k)$ is equivalent to the category of the finite dimensional representations of a linear proalgebraic group G_ω, which is an extension of $\mathbf{G}_m$ by a prounipotent group U_ω. This is an extremely general result, which rests solely on the vanishing of the appropriate *Ext*-groups, leading so to speak to triangular matrices. Further, the functor which to a prounipotent algebraic group associates its Lie algebra is an equivalence of categories. This is explained in many places; apart from the papers we quoted already, a nice place is the Appendix of the classical paper by D.Quillen on *Rational Homotopy Theory.* Gain: the Lie algebra is easier to deal with for computations and it is explicitly an affine space, *i.e.* the spectrum of a polynomial ring.

Things are even better because of the vanishing of the higher *Ext*-groups, which is part of the Beilinson-Soulé conjecture. The vanishing of Ext^2 of pure objects is actually enough to conclude that U_ω is *free.* Moreover, generators are provided by suitably lifting bases of the vector spaces $Ext^1(\mathbf{Q}(0), \mathbf{Q}(n))$ for varying $n \geq 1$. We refer to *e.g.* [HM] or [DG] for detail and will briefly return to this fact in §3.1 below. As a result, the category $MT(k)$ (as well as $MT(\mathcal{O}_k(S))$) is equivalent to that of the linear representations of an algebraic group $G = G_\omega$ which is an extension of $\mathbf{G}_m$ by a *free prounipotent* group $U = U_\omega$. We insist that this important structure result is a consequence of very general principles which it may be useful to repeat in a few words: First there is a natural filtration which is derived from the existence of the weight filtration on the Hodge realisation and the fact that the latter is fully faithful. Next the naturality of the filtration implies that it is respected by the action of the motivic galois group. Moreover the very definition of a MT category shows that the pure objects, hence the graded pieces of the graduation associated to the filtration, are copies of $\mathbf{Q}(n)$, so that the Galois group of the pure motives is isomorphic to $\mathbf{G}_m$. Finally, in order to construct the category $MT(B)$, one uses the truth of the Beilinson-Soulé conjecture for the given base scheme B, which immediately implies the freeness of the unipotent radical U of the Galois group.

It is not our purpose here to detail the conjectures and results which are more or less tightly connected with the motivic viewpoint. Fortunately this has been done recently in several complementary ways. We refer to [A] for a careful analysis of the conjectures and their intricate connections from a motivic viewpoint (see also [HM]). A detailed and prospective discussion of the Galois picture can be found in [I2] and [F] discusses various aspects of the MZV's in this light. In particular the Galois and Hodge-de Rham aspects of the prounipotent version of the Grothendieck-Teichmüller group appear quite clearly in [I2] and [F], along with many references. The group $\underline{GT}$ introduced by V.Drinfeld in [D] is again an extension of $\mathbf{G}_m$ by a pronipotent group $\underline{GT}^1$ and in fact, by concatenating several conjectures together, one is led to predict that it should be isomorphic with the Galois group of $MT(\mathbf{Z})$, the two groups being viewed as proalgebraic groups over $\mathbf{Q}$. This would in particular unravel the structure of $\underline{GT}$ since as explained above the Galois group of $MT(\mathbf{Z})$ is an extension of $\mathbf{G}_m$

by a free prounipotent group whose generators are determined by the (known) structure of $K(\mathbf{Z}) \otimes \mathbf{Q}$, the nontorsion part of the K-theory of $\mathbf{Z}$. Such an identification would have enormous consequences of all kinds, both on the Galois and on the Hodge side, including for instance on the transcendence properties of the MZV's. Apart from the papers already quoted we refer to several recent papers by A.B.Goncharov for much more on this and related themes.

4 Nonlinear?

Indeed how can this adjective be understood in this context? We will discuss a series of topics coming from various regions of mathematics which are all relevant in the context of Grothendieck-Teichmüller theory and will hopefully provide a somewhat coherent and perhaps tantalizing picture. Let us also make it clear again that the 'linear' (in one way or another) techniques are at present more sophisticated and in some sense more powerful than the nonlinear ones, a statement which after all may also apply to such fields as dynamical systems or mathematical physics in general.

4.1 Profinite versus pronilpotent:

A first and basic contrast lies between the types of profinite groups mentioned in the title. In this subsection we briefly take up this profinite setting which was implicitly used in Section 1 and will return to the proalgebraic setting, already mentioned in Section 2, in the next item. These two subsections thus deal with various topologies on groups. One of our main objects of study is of course the study of arithmetic actions on geometric fundamental groups, that is of maps of type $G_k \to Out(\pi_1^{geom}(X))$ with X a scheme (stack) over the field k (replace *Out* by *Aut* in the pointed case). In practice some groups appear naturally endowed with the profinite topology; main class of examples: the arithmetic Galois groups G_k. Others can be seen naturally as completions of discrete groups; main class of examples: the geometric fundamental groups $\pi_1^{geom}(X)$. Here, under rather general assumptions on X, $\pi_1^{geom}(X)$ is the completion of the discrete group $\pi_1^{top}(X^{an})$ where X^{an} is an analytification of X. Recall that 1) this depends on the choice of an embedding of the groundfield k into $\mathbf{C}$, 2) again under rather general assumptions (*e.g.* X quasi-projective) the discrete group $\pi_1^{top}(X^{an})$ is finitely generated.

For the sake of clarity in §3.1 and §3.2 we will often use the letter Γ to denote a discrete finitely generated group and the letter G for profinite groups. It is useful to bear in mind the elementary fact that if $G = \widehat{\Gamma}$ is the profinite completion of Γ, Γ carries more information than G since G can be reconstructed from Γ but not *vice versa*: the topological fundamental group carries more information than the geometric fundamental group. In his *Longue Marche*, Grothendieck suggests that given G, say topologically finitely generated, it would be interesting to study the set of its discretifications, *i.e.* the set of discrete finitely generated $\Gamma \subset G$ such that $\widehat{\Gamma} = G$. Such a Γ he calls a *discretification* of G, which is the nonlinear analog of an integral structure (say on a vector space). Given the importance of the integral lattices in Hodge theory, this seems like a natural object to study although nothing substantial seems to exist at present on the issue.

Let us recall some very basic features of profinite topologies before returning to Galois actions, referring the reader to [RZ] for a recent thorough treatment with references. A profinite group G is an inverse limit of finite groups, which thus appear as finite quotients of G. If Γ is a discrete group, its profinite completion $G = \widehat{\Gamma}$ is defined as the inverse limit of the system (Γ/N) where N runs over the normal subgroups of finite index of Γ. If Γ is finitely generated, the sequence of characteristic subgroups N is cofinal so that one can restrict to these in the inverse system defining G. The pronilpotent quotient G^{nil} of G is obtained by restricting to the subsystem of nilpotent finite groups appearing in the definition of G. A first important feature is that primes do not interact in G^{nil}, so that the adelic picture is enough to deal with the pronilpotent world. This rests on the following elementary (but not so easy!) lemma: Any nilpotent finite group is the direct product of its Sylow subgroups. One thus gets that any pronilpotent group is a direct product of pro-p groups: the situation can be decomposed along the primes and these do not interact. Starting from a discrete Γ, its pro-p completion $G^{(p)}$ for p a given prime is of course defined as the inverse limit of the Γ/N with Γ/N a finite p-group and its pronilpotent completion is the direct product of the $G^{(p)}$'s. It may be useful to add a word about the connection with the descending central series. Given a discrete group Γ one defines as usual the (lower) central series $(\Gamma(n))_{n\geq 1}$ by: $\Gamma(1) = \Gamma, \Gamma(n{+}1) = (\Gamma, \Gamma(n))$, the group generated by the commutators of elements of Γ and $\Gamma(n)$. Let $\widetilde{\Gamma}$ be the inverse limit of the quotients $\Gamma/\Gamma(n)$. This group is a kind of partial pronilpotent completion; it is however in general not complete, and this is why we use the letter Γ. One can prove that the natural map $\widetilde{\Gamma} \to G^{nil}$ (G^{nil} the pronilpotent completion of Γ) is injective. In particular, Γ is nilpotent residually finite, *i.e.* injects into G^{nil}, if and only if the natural map $\Gamma \to \widetilde{\Gamma}$ is injective. This is the case for finitely generated free groups and for Teichmüller modular groups. These groups also inject into their pro-p completions for any odd prime p.

Profinite groups and actions associated to them seem very difficult to study in general. For instance, letting again $F_2 = \mathbf{Z} \star \mathbf{Z}$ denote the free group on two generators, $\widehat{F}_2$ its profinite completion, its automorphism group $Aut(\widehat{F}_2)$ (which contains $G_{\mathbf{Q}}$; see Section 1) seems rather intractable as such (contrast with the discrete case: $Out(F_2) \simeq GL_2(\mathbf{Z})$). Essentially nothing is known about that group. For instance in the profinite case it is extremely difficult, usually impossible to-date, to determine whether an explicitly given endomorphism is invertible; this is much more doable in the pronilpotent setting. In general, pronilpotent and prounipotent groups are much more amenable to computations for several reasons, so that it is tempting and useful to study nilpotent quotients first. Of course that entails a big loss of information: in terms of Galois groups, one can only hope to capture nilpotent finite extensions. We already mentioned that one is essentially reduced to studying the pro-p situation for varying prime p. The next big simplification comes from the fact that in the pro-p (or pronilpotent) world, there is no difference between projectivity and freeness; see for instance [Se1], §I.4, for the basic properties in that direction. In particular, if G is a pro-p group, not only can one test freeness cohomologically, but it is enough to do so with $\mathbf{Z}/p$ coefficients and trivial action: G is free if and only if $H^2(G, \mathbf{Z}/p) = 0$, in which case $H^1(G, \mathbf{Z}/p)$ describes generators (*cf. op. cit.*). This is exactly the strategy which we have seen in action above (*cf.* Section 2) in terms of mixed Tate motives, under slightly different circumstances. Compare with the full profinite situation and especially the famous Shafarevic conjecture; Shafarevic himself proved that $G_{\mathbf{Q}^{ab}}$ is projective but it is a long standing conjecture that this group is profree. A third feature of the nilpotent framework is that one can use Lie algebras and the attending toolbox, in particular derivations, the infinitesimal form of automorphisms. We will detail this linearization for Galois actions and refer to [NH] for more on pro-p groups.

So we return to the action $G_k \to Out(\pi_1^{geom}(X))$, where $G = \pi_1^{geom}(X)$ is topologically finitely generated and can be viewed as the profinite completion of $\Gamma = \pi_1^{top}(X^{an})$. The following construction was first explored by Y.Ihara in [I1]; see [I2] for a first-hand survey and references. These studies concern the case $k = \mathbf{Q}$, $X = \mathbf{P}^*$ which remains of central interest; the information in other cases, especially higher dimensional ones, remains much more fragmentary and much less detailed although much of the 'general nonsense' adapts with no problem. Because G is topologically finitely generated, $Aut(G)$ is naturally endowed with a profinite topology; because G^{nil} is a characteristic quotient, and indeed so are the pro-p quotients $G^{(p)}$, one may study the pro-p actions $\phi = \phi_p : G_k \to G^{(p)}$. An important theorem of Grothendieck implies that the latter morphism factors through the group $Gal(M/k)$ where M is the maximal extension of k unramified outside the primes lying over p. More precisely this is true when X is complete or a curve *e.g.* $\mathbf{P}^*$; the case of a regular higher dimensional noncomplete X is somewhat more subtle and there does not seem to exist a good account of it in the literature at present. So if M^* denotes the fixed field of the kernel of ϕ, one has $M^* \subset M$ where we drop the dependence on p from the notation.

Next one forms the derived sequence $(G(n))_{n\geq 1}$ and the quotient $(G/G(n))_{n\geq 1}$. Here $G = G^{(p)}$ is now a pro-p group, *e.g.* the pro-p completion $F_2^{(p)}$ of F_2 in the case $X = \mathbf{P}^*$. Let $\phi_n : G_k \to Out(G/G(n))$ denote the corresponding actions, which exist because $G(n)$ is characteristic in G. Letting M_n^* denote the fixed field of $Ker(\phi_n)$, we get a nested sequence: $k \subset M_1^* \subset \ldots \subset M^* = \bigcap_{n\geq 1} M_n^* \subset M$, and a corresponding filtration of $\mathcal{G} = Gal(M^*/k)$ by the $\mathcal{G}_n = Gal(M_n^*/k)$. The corresponding graduation is given by $gr^m(\mathcal{G}) = \mathcal{G}_n/\mathcal{G}_{n+1} = Gal(M_n/M_{n-1})$, $gr(\mathcal{G}) = \bigoplus_n gr^n(\mathcal{G})$. We finally got to the object which is most amenable to a detailed study. In particular $gr(\mathcal{G})$ is a graded Lie algebra over $\mathbf{Z}_p$ with the bracket inherited from the commutator in the group $\mathcal{G}$, using that $(\mathcal{G}_m, \mathcal{G}_n) \subset \mathcal{G}_{m+n}$ to ensure that it is well-defined. In the case $X = \mathbf{P}^*/\mathbf{Q}$, it was shown in [I1] that each $gr^n(\mathcal{G})$ is a free $\mathbf{Z}_p$-module of finite rank. Adding two conjectures to this tantalizing picture, it is believed that the rank of $gr^n(\mathcal{G})$ is independent of p (an assertion of motivic flavor) and that $M^* = M$. These results (resp. conjectures) are valid (resp. tantalizing) for more general X. At this point we refer the reader to [I2] and references therein for further action: we have just reached the foot of this as yet largely unexplored range of mountains. The connection with the Grothendieck-Teichmüller group, that is a pro-p, genus 0 version of it, is summarized in Lecture II of [I2]. This process also has a lot in common with the motivic approach of section 2; indeed, after it has been rephrased in terms of linear representations and proalgebraic groups, it may in large part be viewed as the Galois side of the motivic picture, as already largely foreshadowed in [D]; see [F] for a careful description of the picture.

To summarize, through this linearization process we have gained the possibility of using a lot of powerful tools, sometimes leading to concrete computations: Lie algebras, cohomology of pro-p groups etc. It also makes contact with the motivic picture, thus (in the case of $\mathbf{P}^*$) with multiple zeta values etc. On the other hand on the Galois side one has to restrict (at best) to the Galois group of the maximal extension unramified outside p (of course that should provide enough work for several generations...) and the corresponding quotient on the Grothendieck-Teichmüller side. We finally remark that the importance of the geometry of the modular tower may well be one of the deepest intuitions in the *Esquisse*. Here it enters only (at least at present) through the structure of the tower of pro-p braid groups (see [I2], Lecture II), that is through the geometry at infinity of the genus 0 case (see also the contribution of P.L. and L.Schneps in [DE]).

4.2 Group actions versus linear representations:

Taking things somewhat in the reverse order, a Tannakian category is equivalent to the category of the linear representations of a proalgebraic group, namely the fundamental group of the given category, after fixing a fiber functor. Grothendieck's classical (SGA 1) theory of the fundamental group uses a kind of nonlinear version: instead of linear representations, it establishes a dictionary between the category (Cov/X) of finite étale covers of a scheme X (or stack; see [No]) and the category of finite sets with $\pi_1(X)$-action. Although one can in principle linearize by passing from the action of a group on a finite set to the associated linear permutation representation this is not an innocuous operation.

Turning again to Grothendieck-Teichmüller theory, it deals first and foremost with the action of arithmetic Galois groups on geometric fundamental groups in conjunction – and here lies the charm of the situation – with the geometry of the modular tower $\underline{\mathcal{M}}$. In particular one starts from the collection of stacks

$\mathcal{M}_{g,n}$ and their discrete fundamental groups $\Gamma_{g,n}^{top}$, which are finitely generated and residually finite discrete groups. It is of course quite natural to study the linear representations of these groups, associated monodromy operators etc. These and other topics are the subject matters of topological quantum field theory and related subjects; we refer to [BK] for a survey with references. In particular conformal blocks in conformal field theory provide a linear form of lego, as first pointed out and explored in the classical 1989 paper of G.Moore and N.Seiberg. Of course this is only a form of the lego at infinity, to take up the terminology of Section 1 above. There also exists for instance an extended form of the Tannakian formalism for *braided* tensor categories. The point we want to make however in this subsection is that, when it comes to arithmetic fundamental groups and in particular Galois actions, it is quite hard to use linear representations, again beyond the prounipotent (motivic) setting. This can be seen as a variant of what has been pointed out above and one way of phrasing it more precisely is as follows.

Let Γ be a discrete finitely generated and say residually finite group in order to fix ideas. By definition the proalgebraic completion (or envelope) of Γ over the field k (assume $char(k) = 0$) is the k-proalgebraic group G^{alg} whose category of finite dimensional linear representations over k coincides with that of Γ. In other words any morphism $\Gamma \to GL_n(k)$ (for some n) factors through G^{alg}. There is a proalgebraic form of the Levi decomposition which asserts that G^{alg} is a split extension of the proreductive group G^{red} by its prounipotent radical G^{uni}, all groups being linear proalgebraic over k. The group G^{red} (resp. G^{uni}) can also be seen as the proreductive (resp. prounipotent) envelope of Γ and can be defined directly in the same way as G^{alg}, by considering reductive (resp. unipotent) linear representations of Γ. The prounipotent part G^{uni} can be fairly well understood; the point here is that G^{red} often looks intractable, or at least not easily amenable to computations.

This last sentence can be illustrated by means of a theorem of J.Tits (see [Ti]) which implies the following: Let G be a semisimple algebraic group over k; then there exists a countable infinite set F inside $G(k)$ such that any two elements of F generate a Zariski dense free subgroup of $G(k)$. Now consider the case $\Gamma = F_2$; by the above result one finds among the quotients of its reductive envelope F_2^{red} a countably infinite number of copies of any linear semisimple algebraic group over k. It does seem hard to use such an object in a concrete way. The situation does not improve of course if one considers the proalgebraic completion of a profinite group such as $G_{\mathbf{Q}}$, although this group $G_{\mathbf{Q}}^{alg}$ occurs almost by definition if one studies Galois representations and is connected with the motivic Galois group of $\mathbf{Q}$.

The (needless to say temporary and provisional) conclusion seems to be that when studying of geometric Galois actions, one can either work in a profinite setting, or if one goes to the proalgebraic setting, one almost inevitably has to stick to a prounipotent or almost prounipotent framework. Here 'almost prounipotent' refers primarily to a series of works by R.Hain (see [HM] and references therein) in which the author(s) investigate(s) extensions of reductive *algebraic* (as opposed to proalgebraic) groups by prounipotent groups. The case where the reductive quotient is simply the multiplicative group $\mathbf{G}_m$ is of particular interest and occurs naturally in various situations, as examplified in Section 2. It seems however difficult to tackle situations which involve *pro*reductive envelopes of 'really' nonabelian groups in an essential way.

Finally we note that in the pronilpotent and prounipotent cases, the dictionary between profinite and proalgebraic settings is well established. We refer in particular to [De], §9 and references therein. Let us just state some bare facts in a basic but representative situation; the proofs are essentially formal. Let Γ be a nilpotent torsionfree finitely generated group, $G = \widehat{\Gamma}$ its profinite (equivalently pronilpotent) completion. On the algebraic side, one constructs the $\mathbf{Q}$-prounipotent completion $G^{uni} = G^{alg}$ of Γ over $\mathbf{Q}$. Let ℓ be a prime and consider $G^{uni}(\mathbf{Q}_\ell)$, the group of $\mathbf{Q}_\ell$-points of G^{uni}. Then there is a natural injection $\Gamma \hookrightarrow G^{uni}(\mathbf{Q}_\ell)$, the closure of the image of Γ is compact open and it is isomorphic to the pro-ℓ completion $G^{(\ell)}$ of Γ, that is the pro-ℓ part of G, which is the direct product of the $G^{(\ell)}$'s. This statement provides the basic connection between the pronilpotent and prounipotent objects.

4.3 Good groups versus rigid ones:

Let Γ be a finitely generated and residually finite discrete group, $G = \widehat{\Gamma}$ its profinite completion and $j : \Gamma \hookrightarrow G$ the natural inclusion. Let M be a finite abelian group (in other words a finite $\mathbf{Z}$-module). The inclusion j induces a map on the cohomology with constant coefficients in M (that is with trivial Γ and G actions) $j^* : H^*(G, M) \to H^*(\Gamma, M)$. Following J-P.Serre ([Se1], §2.6) the group Γ is called *good* if j^* is an isomorphism for any finite M. By taking direct limits on the coefficients, this holds true for any torsion abelian group M.

Consider the special case where $\Gamma = \pi_1^{top}(X)$ and X is a classifying space, that is a $K(\Gamma, 1)$ and is defined over a field embedded in $\mathbf{C}$, so that $G = \pi_1^{geom}(X)$. There is a natural isomorphism $H^*(X, M) \simeq H^*(\Gamma, M)$, where $X = X(\mathbf{C})$ is viewed as a complex variety. If j^* is surjective, any cohomology class in $c \in H^*(X, M)$ comes from a class in $H^*(G, M)$; since G is profinite any such class vanishes when restricted to some open subgroup. This translates into the fact that there exists a finite étale cover $\pi : Y \to X$ (here $X = X^{geom} = X \otimes \overline{k}$), such that $\pi^*(c) = 0$. We thus find that X has many finite étale covers in the sense that given any cohomology class one can find a cover such that the pullback to it of the given class vanishes. The above considerations *could* apply to the moduli spaces of curves $\mathcal{M}_{g,n}$ with some qualifications; namely they are $\mathbf{Q}$-stacks and thus orbifolds when viewed analytically. As a consequence, in the above one should consider only such M whose elements have orders prime to the cardinals of the automorphism groups, which here are the just the automorphism groups of the curves of type (g, n), whose order is bounded by the classical Hurwitz bound. However the main stumbling block here is that it is unknown whether or not the mapping class groups $\Gamma_{g,n}^{top}$ are good for $g > 2$ (see note added in proof). We return to this important point below but note for now the connection between goodness and the abundance of étale covers. This was actually a motivating geometric property and can be used (SGA 4, §XI.4.6) in order to show that the étale topology is reasonable in the sense that any point on a scheme has a basis of acyclic neighborhoods.

Let us go on exploring goodness a little further and give some basic examples. We refer to [N2] for the precise statements and proofs. First goodness behaves well under extension: start from an exact sequence of discrete groups:

$$1 \to K \to E \to H \to 1. \tag{$*$}$$

Assume that H and K are good and K is of type FP (it has a finite projective resolution); then the sequence $(\widehat{*})$, obtained from $(*)$ by profinite completing each group is exact and the extension group E is good too. Since the completion functor is right exact, only the left-hand injectivity is in question in the first assertion; the second assertion is a direct consequence of the Hochschild-Serre spectral sequence computing the cohomology of E. In particular a group Γ is good if it has a good subgroup of finite index (use the above and the fact that finite groups are good); in geometric parlance goodness can be thought of up to finite étale covers. This also implies that we can accommodate non trivial continuous actions in the cohomology, that is the case where Γ acts on M via a finite quotient and so its completion $G = \widehat{\Gamma}$ acts continuously on M. So in fact if Γ is good $H^*(G, M) \simeq H^*(\Gamma, M)$ for any torsion module M acted on by Γ via a finite quotient. Next say that Γ is k-good if $H^k(G, M) \simeq H^k(\Gamma, M)$ for M as above (equivalently all M finite and trivial action). It is important to note that all (FP_∞ discrete) groups are 1-good; in fact a 1-cocycle for the trivial action is just a morphism and is determined by its values on the dense subgroup Γ of G. In dealing with the sequence $(*)$ and its term-by-term completion $(\widehat{*})$ we actually had to require only that H and K be 2-good. As is often the case with cohomology, this condition on the H^2-groups is the most often encountered in practice.

Examples: Finite groups are good; (finitely generated)free groups are good because they are of cohomological dimension 1; braid groups and genus 0 mapping class groups are good because they are constructed as successive extensions of free and finite groups; $SL_2(\mathbf{Z})$ is good because it has a free subgroup of finite index. Well-known and important *conjecture*: the mapping class groups (or Teichmüller modular groups) Γ_g^{top} are good for all g. Note that $\Gamma_{1,1} \simeq SL_2(\mathbf{Z})$ is good. Easy exercices: Γ_2 is good and if Γ_g is good, the pointed group $\Gamma_{g,n}$ is good for any $n \geq 0$ (one could also accommodate permutations of the marked points). As mentioned above, for classifying spaces, goodness does in some sense measure the abundance of étale covers, which is also connected with the richness of the Galois action and the attending anabelian phenomena. This is dramatically illustrated by the difference between curves and abelian varieties in this respect and this may well be the main point of the present subsection. To see this we first introduce a main class of counterexamples: higher rank arithmetic groups are *not* good in general. This is a consequence of the rigidity of such groups as we illustrate on examples leaving it to the reader to generalize. Specifically $SL_n(\mathbf{Z})$ (resp. $Sp_{2g}(\mathbf{Z})$) is not good for $n > 2$ (resp. $g > 1$).

A proof for $SL_n(\mathbf{Z})$ goes as follows and is easily adapted to the case of $Sp_{2g}(\mathbf{Z})$. The first and crucial step consists in using the (Mennicke-Baas-Milnor-Serre) congruence subgroup property which translates as: $\widehat{SL_n(\mathbf{Z})} \simeq SL_n(\widehat{\mathbf{Z}})$; here and below $n > 2$. Next and essentially by definition, $SL_n(\widehat{\mathbf{Z}}) \simeq \prod_p SL_n(\mathbf{Z}_p)$ where the product runs over the prime integers. Finally $SL_n(\mathbf{Z}_p)$ has a nontrivial center whenever the n-th roots of unity belong to $\mathbf{Z}_p$, that is when p is congruent to 1 modulo n, which by Dirichlet theorem happens infinitely often. Conclusion: the profinite completion of $SL_n(\mathbf{Z})$ has an infinite torsion center and thus does not have virtually finite cohomological dimension. By contrast the discrete group $SL_n(\mathbf{Z})$ is virtually torsionfree and does have virtually finite cohomological dimension. Hence it cannot be good.

It is plain from the above that we used first and foremost the fact that $SL_n(\mathbf{Z})$ or $Sp_{2g}(\mathbf{Z})$ has relatively 'few' open subgroups. Let us push this a little further, contrasting curves (the nonlinear side...) and abelian varieties (see also the contribution of Y.Ihara and H.Nakamura in [GGA]). On the side of $\mathcal{M}_g$ we have a classifying stack with conjecturally good topological fundamental group and many étale covers. In fact another topological conjecture, perhaps less widely believed than the goodness conjecture, predicts that a cofinal sequence is given by the *geometric* congruence subgroups, whose construction we do not recall here. In the case of $\mathcal{A}_g$ ($g > 1$), the classifying stack of principally polarized Abelian varieties, the topological fundamental group $Sp_{2g}(\mathbf{Z})$ is not good as indicated above, there are few étale covers and the associated action of the arithmetic Galois group does not carry much information. To wit, any finite étale cover of $\mathcal{A}_g$ is actually defined over $\mathbf{Q}^{ab}$ (the maximal abelian extension of $\mathbf{Q}$) so that the action of $G_{\mathbf{Q}}$ actually factors through $\mathrm{Gal}(\mathbf{Q}^{ab}/\mathbf{Q}) \simeq \widehat{\mathbf{Z}}^\times$ and reduces to the cyclotomic character. By contrast, the moduli spaces of curves are conjecturally anabelian, which at any rate involves the fact that the Galois action is quite 'rich' (*cf. loc. cit* for a discussion of anabelianity' in the higher dimensional case). Note however that the Modular and the Abelian towers share the first level and that the arithmetic Galois action is already faithful there (*i.e.* on $\widehat{SL_2(\mathbf{Z})}$).

The similarities and contrasts between mapping class groups and linear arithmetic groups have provided a rich theme in the literature since the late sixties and the above adds to the list of contrasts. We will not elaborate on the very embryonic theme of higher dimensional anabelian geometry but would like to mention that this phenomenon can also be partly connected (not only analogically) with all sorts of analytic and topological rigidity properties. We refer to [N1] for very interesting results in this direction, which involve goodness, Artin good neighborhoods etc. Finally we mention the similarity with the purely topological long-standing Borel conjecture which asserts that a *closed* manifold which is a $K(G,1)$ is determined up to homeomorphism by that property (see [Fa]; note that it implies Poincaré's conjecture). One can wonder for instance whether a **Q**-stack which is the classifying space of a finitely generated good and universally centerfree group is anabelian. Of course such a general and not even completely precise statement should not be taken too seriously but it may help making expectations a little more precise.

4.4 Amalgamation versus extension:

Extensions of groups, modules or more generally of objects in abelian categories are one of the main tools which enable one to analyze a situation in terms of the associated 'simple' objects. For instance nilpotent (or solvable) finite groups are obtained by definition as successive extensions of finite abelian groups, which are easily classified. As another example the simple objects of a Tate category, that is the pure Tate motives, are by definition also completely classified and easy to list and analyze, including from the Galois viewpoint. The same holds true in principle for pure Hodge structures etc.

Moving back to our favorite moduli stacks and their geometric fundamental groups, one finds that the genus 0 situation can be described in terms of extensions, although this may not always be the most revealing description. This rests on the fibration $\mathcal{M}_{0,n+1} \to \mathcal{M}_{0,n}$ ($n > 2$) obtained by erasing a given labelled marked point and which can be viewed as the universal geometric monodromy fibration, since $\mathcal{M}_{0,n+1} \simeq \mathcal{C}_{0,n}$ is the universal curve over $\mathcal{M}_{0,n}$ (which is a scheme since curves with labelled marked points have no automorphisms. So the situation is really simple here; no stacky niceties involved). Now one can consider the homotopy exact sequence associated with that fibration and recall that $\mathcal{M}_{0,n}$ is a classifying space, so that all higher homotopy groups vanish. As a result, one finds that $\Gamma_{0,n+1}$ is an extension of $\Gamma_{0,n}$ by the fundamental group of the fiber, which is nothing but F_{n-1}, the free group on $n-1$ generators. This unravels $\Gamma_{0,n}$ as a successive extension of free groups and $\Gamma_{0,[n]}$ is nothing but an extension of the permutation group $\mathcal{S}_n$ by $\Gamma_{0,n}$.

Note that already here, free groups that is amalgamations of copies of $\mathbf{Z}$ have occurred. In fact F_{n-1} came in as the fundamental group of the sphere with n points removed, which can be computed by induction using Van Kampen theorem, thus leading to amalgamation. In this subsection we would like to stress the importance of amalgamation in general as a way of describing the group theoretic Teichmüller tower from its constituents. Amalgamation occurs in terms of fundamental groups through various forms of the Van Kampen theorem, just as naturally as extensions occur in homological algebra through the various types of long exact and spectral sequences. Dealing with amalgamation and in order of increasing generality, one finds free products, amalgams of groups (and HNN extensions), the Bass-Serre theory of the fundamental groups of graphs of groups (the standard reference being [Se2]), and finally complexes of groups of which we will say but a few words at the end of this paragraph. We are especially interested here in comparing the discrete (or say abstract, that is paying no attention to topology) and the complete (or profinite) theory, the latter being actually fairly recent and still fairly incomplete.

The first piece of relevant information is that any mapping class group $\Gamma^{top}_{g,[n]}$ can be written as the fundamental group of a finite graph of groups involving only mapping class groups of strictly smaller modular dimensions ($d(g,n) = dim(\mathcal{M}_{g,[n]}) = 3g-3+n$), provided $d(g,n) > 2$. This can be seen as a reflection of Grothendieck's 'two levels principle' (see [L1]) and should perhaps be written up in detail. It goes roughly as follows (there are several possible variants): Consider the topological surface $S_{g,n}$ of finite hyperbolic type (g,n). Associated to it is the complex $A = A_{g,n}$ based on the so-called cut-systems, that is multicurves $\underline{\gamma}$ which are such that the result of cutting $S_{g,n}$ open along $\underline{\gamma}$ produces a sphere with holes. Such a $\underline{\gamma}$ is a union of g simple loops, so one gets a (not locally finite) complex of dimension $g-1$ (see [Ha] for detail and references), which is intimately connected with Schottky uniformization. Now $\Gamma^{top}_{g,n}$ acts on $A_{g,n}$ and it is easy to see that one can take one simplex of the first subdivision of $A_{g,n}$ as a fundamental domain. The main topological result here is that $A_{g,n}$ is connected and simply connected for $g > 2$ ([Ha], Theorem 6.4; the full result is actually more precise). This makes it possible to apply classical results (especially [So]) in order to write a presentation of $\Gamma^{top}_{g,n}$ as the fundamental group of a graph of groups. The groups which appear are the mapping class groups of surfaces obtained from $S_{g,n}$ by cutting along part of a cut-system, plus finite groups coming from symmetries of graphs and permutations of points. Concerning the latter one should in fact introduce the partially colored groups $\Gamma^{top}_{g,\sigma}$ where $\sigma \subset \mathcal{S}_n$ records the sugroup of the 'allowed' permutations. One recovers $\Gamma^{top}_{g,[n]}$ by choosing $\sigma = \mathcal{S}_n$. All these groups are extensions of finite groups by $\Gamma^{top}_{g,n}$ and here the real sticky point consists in understanding the bare groups Γ_g ($n = 0$) for $g \geq 3$. The $\Gamma^{top}_{g,\sigma}$ simply occur naturally as vertex groups of the graphs, which it is a good exercise to write down explicitly in the case of Γ_3 and Γ_4. We insist that at a geometric level, this construction again reflects the geometry at infinity of the $\mathcal{M}_{g,n}$, giving rise to the stable stratification etc.

Now what happens upon completion? Here we will discuss the relevant case (for us) of a finite graph and completion of the vertex and edge groups (see [ZM] and references therein). One can also treat the case of profinite graphs, where surprises do occur (see a later paper by the authors of [ZM]). So let C be a finite graph of groups, with vertex (resp. edge) groups G_v, $v \in V = V(C)$ (resp. G_e, $e \in E = E(C)$), and write $G = \pi_1(C)$ for its fundamental group where we do not specify the choice of a base point, *e.g.* a maximal tree of the underlying combinatorial graph, which plays no interesting role here. Here the G_v, G_e and G are finitely generated discrete groups and we assume that the structure morphisms $j_e : G_e \to G_v$ are injective; it is then part of the theory that the natural morphisms $j_v : G_v \to G$ are also injective. Consider now the term-by-term completed graph $\widehat{C}$, obtained by profinite completion of the vertex and edge groups, as well as the maps j_e connecting them (one can also use other sorts of completions, along admissible classes of finite groups). So $\widehat{C}$ is defined by a finite projective system of maps $\hat{\jmath}_e : \hat{G}_e \to \hat{G}_v$. The fundamental group $\overline{G} = \pi_1(\widehat{C})$ exists, with structure morphisms $\overline{\jmath}_v : \hat{G}_v \to \overline{G}$. In particular one can view $\overline{G}$ as the limit of the finite system defined by $\widehat{C}$, that is any system of compatible maps from $\widehat{C}$ to a profinite group H factors through $\overline{G}$. But several very serious difficulties now arise: there is no guarantee that either the maps $\hat{\jmath}_e$ or the maps $\overline{\jmath}_v$ are injective (even assuming as we do that the j_e are injective); $\overline{G}$ is a quotient of the full completion $\widehat{G}$ but there is no guarantee that it coincides with it. In fact one can give a fairly explicit description of the topology of $\overline{G}$: It is obtained by declaring open the subgroups H of G such that all the preimages $j_v^{-1}(G) \subset G_v$ have finite index in the G_v's. The standing condition 3.6.A. in [ZM] consists in assuming precisely that the $\hat{\jmath}_e$ and $\overline{\jmath}_v$ are injective. Let us assume that it holds true and on top of it that $\overline{G} \simeq \widehat{G}$; we will discuss below the meaning of these assumptions in the case of mapping class groups. In this ideal situation, there is a lot that can be said. In particular, if the constituent groups G_e and G_v are good, then G is good (P.L. and V.Sergiescu; unpublished). In order to prove this, one compares a discrete and completed long exact sequence, much as one does with the relevant Hochschild-Serre spectral sequences in order to show that goodness behaves well under extension (see §3.3 above). Here the relevant long exact cohomology sequence is obtained in the completed case from the short exact sequence of complexes in [ZM] (3.8, bottom row of the diagram) which is the analog of the discrete sequence first emphasized by I.M.Chiswell.

So in particular, given the above, *if* the assumptions above hold true, one easily derives the goodness of the groups $\Gamma_{g,n}^{top}$ by induction on the modular dimension. More generally it would be a very significant step in trying to unravel the structure of the profinite Teichmüller modular groups $\Gamma_{g,[n]}$. We will now examine how these assumptions translate in our case into a nice and apparently difficult geometric problem which it may be interesting to state in some detail. We outlined above how the discrete modular groups (mapping class groups) can be viewed as fundamental groups of finite graphs built out of modular groups of strictly inferior dimensions and finite groups. Moreover the injective structure maps j_e and j_v corresponding to edge and vertex groups are basically all of the same type: Letting as usual $S_{g,n}$ denote the topological surface of type (g,n) and $\underline{\gamma}$ a non disconnecting multicurve (*i.e.* a sub-cut-system), split $S_{g,n}$ along $\underline{\gamma}$ and consider the injection of the modular group of the split surface into the original one. All in all, after using induction on the number of curves in a multicurve and a few simple manipulations on extensions involving finite groups etc. one shows that the only serious question is as follows. Let γ be a (possibly separating) simple closed curve on the topological surface $S_{g,n}$, and let

Z_γ be the centralizer in the discrete modular group of the Dehn twist τ_γ along γ. We thus have an inclusion $j : Z_\gamma \hookrightarrow \Gamma^{top}_{g,n}$, which gives rise after completion to a map $\hat{\jmath} : \widehat{Z}_\gamma \hookrightarrow \Gamma_{g,n}$. The question is simply: Is $\hat{\jmath}$ injective? We emphasize that modulo the results mentioned above a positive answer would vindicate all the assumptions about completed graphs, would imply the goodness of the discrete modular groups etc. Let us dwell a little more on the topological and geometric meanings of that question, by way of illustration of some of the ideas we discussed in the past items. From a topological viewpoint, we find first that the centralizer Z_γ is an extension of the modular group of the split surface (with the holes collapsed to marked points) by the cyclic group generated by the twist along γ. In other words we have a short exact sequence:

$$1 \to \mathbf{Z} \to Z_\gamma \to \Gamma^{top}_{g-1,n+2} \to 1. \qquad (*)$$

Note that the action of $\Gamma^{top}_{g-1,n+2}$ on $\mathbf{Z} = \langle \tau_\gamma \rangle$ is the monodromy action and that the extension class $c \in H^2(\Gamma^{top}_{g-1,n+2}, \mathbf{Z})$ corresponding to $(*)$ has a clear geometric meaning. Indeed, moving to a more geometric and modular picture, consider the complex moduli space $\mathcal{M}_{g,n}(\mathbf{C})$, its completion $\overline{\mathcal{M}}_{g,n}(\mathbf{C})$ and $Z = Z(\gamma)$ the component of the divisor at infinity corresponding to γ (for instance there is one component defined by all non separating γ's). Let Z^ε be a sufficiently thin tubular neighborhood of Z in $\overline{\mathcal{M}}_{g,n}(\mathbf{C})$ and consider the intersection $Z^\varepsilon \setminus Z$ of Z^ε with $\mathcal{M}_{g,n}(\mathbf{C})$. Then $Z_\gamma \simeq \pi_1^{top}(Z^\varepsilon \setminus Z)$. Equivalently, take the conormal line bundle of the divisor Z and the associated circle bundle SZ; then again $Z_\gamma \simeq\simeq \pi_1^{top}(SZ)$. This gives a description of the class c of $(*)$ as follows: Consider the Chern class $c_1 \in H^2(Z, \mathbf{Z})$ of the conormal line bundle of Z; there is a natural injective map $H^2(\pi_1^{top}(Z), \mathbf{Z}) \hookrightarrow H^2(Z, \mathbf{Z})$ $(\pi_1^{top}(Z) \simeq \Gamma^{top}_{g-1,n+2})$ and c is the preimage of c_1. For a more algebraic view of this situation we refer to [GM] and for more general stacky situations to [LV]. Roughly speaking the completion of Z_γ is nothing but the geometric tame fundamental group of the completion of $\overline{\mathcal{M}}_{g,n}$ along Z and can be viewed as the decomposition group associated to the ideal defined by Z. Back to the original question on the injectivity of the completed map $\hat{\jmath}$. After a little contemplation it can be rephrased geometrically as follows: Let Y be an arbitrary unramified covering of the deleted tubular neighborhood $Z^\varepsilon \setminus Z$; does there exist an (orbifold) unramified covering of $\mathcal{M}_{g,n}(\mathbf{C})$ whose restriction to $Z^\varepsilon \setminus Z$ dominates Y? This can also be phrased in algebro-geometric terms, using a more algebraic version of tubular neighborhoods. The point is that we find again that a main difficulty consists in producing 'many' finite étale covers. And again goodness measures and would result from the abundance of such covers. And again the above question brings back to the possibility of moving inward from infinity on the moduli spaces. This is why apart from being interesting in itself it seems to weave once more the themes we have kept mentioning in this and the last item. We leave it to the reader to contrast the above with what happens on $\mathcal{A}_g$ for principally polarized varieties.

The above discussion deals with *graphs* of groups and completion. The next step in terms of considering amalgams of (pro)finite projective systems of groups consists in looking at *complexes* of groups, roughly speaking replacing graphs by higher dimensional CW complexes. We refer to [H], [C] and further papers by these and other authors. They consider only the discrete case and there does not seem to exist at present papers dealing with completion in this context (even for a finite two dimensional complex, say a triangle of groups). It may be that using the special loci in moduli spaces, that is the loci corresponding to curves with nontrivial automorphism groups, mapping class groups can be viewed as fundamental groups of complexes of groups in which only finite groups occur. Passing to the completed groups would then not involve any change in the constituents but might involve difficult questions on profinite spaces (see [S]). This circle of ideas, which perhaps deserves further investigation can be seen as the group theoretic facet of the notions briefly reviewed in §3.6 below.

4.5 All genera versus genus 0:

We will be brief on this contrast, as it has already been mentioned several times. The modular tower in genus 0 is clearly an extremely interesting object, which by now has appeared recurrently under several guises. The strong focus on this object can in large part be ascribed to V.Drinfeld and his series of groundbreaking articles culminating in [D]. Genus 0 objects are now legion as for instance quasi-Hopf algebras, braided categories and several types of operands (rooted trees and gravity operands). In some sense they describe the 'geometry of associativity' unearthed in the original papers of V.Drinfeld, and which categorically speaking is attached to the familiar McLane coherence relations (and Yang-Baxter equations). The theory of the multiple zeta values is also purely genus 0, at least to-date.

Moreover from the viewpoint of Grothendieck-Teichmüller theory, if one takes the automorphisms of curves into account, as one should, the genus 0 setting in some sense tells the whole story: the outer automorphisms of the geometric fundamental group respecting both the inertia at infinity and the automorphism groups in genus 0 coincides with $\mathbb{I}\Gamma$, the Grothendieck-Teichmüller group at infinity in all genera; we refer to [Sc3] for a precise statement and proof. In the next item, we will return to the emerging duality between automorphisms of curves and stable degeneration. Finally we have seen already that from a group theoretic viewpoint the genus 0 objects are also significantly simpler. For instance all groups occurring (braid groups, mapping class groups with or without permutations) are good for simple reasons and generally speaking, although they are far from well-understood (*e.g.* from a representation theoretic viewpoint), plane and sphere braid groups are much more amenable to computation than higher genera Teichmüller modular groups.

Yet it seems that one should really consider the whole modular tower $\underline{\mathcal{M}}$, as is strongly suggested in Grothendieck's *Esquisse*, with all its geometric complexity and richness, not only that coming from the stable stratification (see also §3.6 below). And one should also take stacks seriously, that is not kill automorphisms too quickly by adding level structures. In essence, by considering curves as one dimensional Deligne-Mumford stacks, allowing for coverings, quotients by automorphisms, stable fibrations, point adding and erasing morphisms and the modular counterpart of these operations, one arrives at a structure governed by an enriched version of the modular tower which looks like a natural environment for Grothendieck-Teichmüller theory, and in which it can for instance make contact with Thurston's topological vision (see [L2] for a preliminary exploration along these lines).

4.6 Stack inertia versus inertia at infinity:

As already mentioned several times, the current version of the modular tower $\underline{\mathcal{M}}$ essentially takes care of the structure at infinity (and partially of the universal geometric monodromy, via adding marked points). This can be phrased in many ways: it contains the Knudsen morphisms, it deals with the stable stratification, it is based on multicurves and pants decompositions, it stresses Dehn twists that is parabolic (reducible) diffeomorphisms of topological surfaces (in terms of the Thurston-Bers classification) or, in more algebraic parlance generators of the procyclic inertia subgroups associated with the irreducible components of the divisors at infinity of the moduli stacks of curves.

On the other hand, the loci in the moduli spaces representing curves with nontrivial automorphisms have attracted attention quite early; see [Co] for a nice modern reference with classical flavor. In the mid seventies H.Popp noticed that they build up stratifications of the moduli stacks in the sense of Zariski, reflecting the singularities of the associated coarse moduli spaces viewed as complex varieties. This viewpoint can be extended to general separated Deligne-Mumford stacks (see [LM], Remarque 11.5.2, [No] and [LV]), using the basic fact that geometric points of stacks come along with a group of automorphisms. Moreover, using the notion of inertia, one can give a unified treatment of the case of a divisor at infinity ([GM]) and of these loci with nontrivial automorphisms.

So one gets two stratifications of the completed moduli stacks of curves. The stable stratification ('at infinity') is defined over $\mathbf{Q}$, whereas the other one, *via* automorphisms ('stack inertia') is not. Indeed it carries enormous arithmetic information which remains largely mysterious at present. In many respects the stable stratification is more familiar: it embodies, with a stacky grain of salt, the familiar setting of a regular quasiprojective scheme X which can be written as $X = \overline{X} \setminus Z$ with $\overline{X}$ regular projective and Z a divisor with strict normal crossings. So modulo some stacky nuances, one is in the framework of [GM], with each component of the divisor at infinity Z corresponding to a conjugacy class of free procyclic groups $\widehat{\mathbf{Z}}$ with Tate twist 1: a topological generator up to conjugacy consists of a small loop 'around' the given divisor and the action of the arithmetic Galois group is that of the cyclotomic character on the roots of unity. The stratification *via* loci with nontrivial automorphisms (henceforth 'special loci' for short) is of a much more stacky nature; in group theoretic terms it is not about just (pro)cyclic groups but at the topological level reflects the boolean lattice of finite groups contained in the mapping class groups $\Gamma^{top}_{g,[n]}$. Here one should recall that $\Gamma^{top}_{g,[n]}$ is virtually torsionfree but apart from a few exception (see [L1] for detail and references) it is also generated by its torsion elements.

The contrast between these two stratifications deserves to figure in this list of nonlinear-versus-linear items partly because the stratification at infinity is intimately connected with (mixed) Hodge theory and has a distinct unipotent flavor, which can also be called parabolic in dynamical terms, thinking again of the Dehn twists in terms of diffeomorphisms of surfaces. This is however surely not the only relevant feature here and it seems that much remains to be unearthed. We will add a few remarks about the (discrete and profinite) group theoretic aspects and our embryonic knowledge of the Galois and Grothendieck-Teichmüller actions.

The Grothendieck-Teichmüller group has been defined as the outer automorphism group of the modular tower respecting the inertia groups at infinity. In the topological setting a result of N.Ivanov asserts that the maximal free abelian groups inside the mapping class group $\Gamma^{top}_{g,n}$ are precisely the groups generated by the Dehn twists on the curves of a pants decomposition of the model surface of type (g, n). Is there an analog in the profinite setting? Note that one should allow here for conjugates of the groups mentioned above, which in the discrete case are of the same type, not so in the profinite case. Such results would help build up a group theoretic characterization of inertia at infinity. But again our current knowledge on these matters is poor; for instance it is likely but at present unknown (for $g > 2$) whether $\Gamma_{g,n}$ is actually centerfree (this is now a consequence of goodness; see note added in proof); as a piece of warning recall the discussion for $SL_n(\mathbf{Z})$ in §3.3 above.

This theme 'free abelian groups and completion' which is connected with inertia at infinity can be paralleled under the heading 'finite subgroups and completion'. Here again, for $g > 2$, what happens in terms of finite subgroups upon profinite completion of the modular groups is unknown. The finite subgroups of the (discrete) mapping class groups are exactly the automorphism groups of Riemann surfaces and they survive completion (*i.e.* map injectively into the completed groups) because the modular groups are residually finite. But is it true that any finite subgroup of the $\Gamma_{g,n}$ arises in this way, *i.e.* is conjugate to the image of a finite subgroup of $\Gamma^{top}_{g,n}$ (again $SL_n(\mathbf{Z})$, $n > 2$ does not satisfy the analogous statement)? In other words is it true that the torsion of the profinite group $\Gamma_{g,n}$ 'essentially' comes from its discrete part? This looks both interesting and hard...

It is interesting because one might want to study the modular groups using the lattice of their finite subgroups and such tools as Farrell cohomology (see

[Br], [S] and [LS2]) or again complexes of groups as in §3.4. In fact even in the discrete case, and even maybe in genus 0, that is for the $\Gamma_{0,[n]}^{top}$, little seems to have been done, as alluded to in the *Esquisse*. Note that we know that the mapping class group $\Gamma_{0,[n]}$ is generated by its torsion, that it is virtually torsionfree ($\Gamma_{0,n}$ is torsionfree), that it is centerfree, that the corresponding discrete group is good etc. All this information is lacking in higher genera, and for good reasons. But it would still be interesting to better understand these groups in terms of their torsion. The baby example is $PSL_2(\mathbf{Z})$ or $SL_2(\mathbf{Z})$: $PSL_2(\mathbf{Z})$ contains a subgroup of finite index which is torsionfree and indeed free on two generators; it is just the principal subgroup of order 2, $P\Gamma(2) \simeq F_2 = \mathbf{Z} * \mathbf{Z}$, parameterizing elliptic curves rigidified by a level structure which 'kills' automorphisms (and not taking the generic involution into account as one in effect should). On the other hand $PSL_2(\mathbf{Z}) = \mathbf{Z}/2 * \mathbf{Z}/3$, reflecting the structures of the elliptic curves with automorphisms and, more accurately $SL_2(\mathbf{Z})$ is the amalgamation of $\mathbf{Z}/4$ and $\mathbf{Z}/6$ over the involution $\mathbf{Z}/2$. There does not seem to exist at present a careful description, using say complexes of groups, of the analogous situation in higher dimensions, that is for the spaces $\mathcal{M}_{0,[n]}$ ($n > 4$) and the attending discrete and profinite modular groups (see [Sc2] for the elementary description of the automorphism groups in those cases). In principle one would like to have available descriptions of the discrete and profinite modular groups by means of –say– complexes of groups which use both kinds of stratifications, the stable one and the one associated with automorphism groups, in the style of the above presentations of $P\Gamma(2)$ and $PSL_2(\mathbf{Z})$ (*cf.* §3.4 above).

Such descriptions of the profinite modular groups, which seem quite hard to obtain in higher genera, would be relevant to the Galois and Grothendieck-Teichmüller viewpoint. At present, in the Galois case one can prove general stack theoretic results which for instance imply that for any hyperbolic type (g, n), any element of finite order in $\Gamma_{g,n}$ (or $\Gamma_{g,[n]}$) is mapped to a conjugate of itself by an open subgroup of $G_{\mathbf{Q}}$ (see [LV] for detail and more along these lines). The analogous result is known for the Grothendieck-Teichmüller group in genus 0; see [Sc3] for this and other results on the Grothendieck-Teichmüller action on finite elements. Finally we mention [LNS] where the Galois action on some specific finite order elements in genus 0 is studied in detail.

In closing, we note that in the above we have concentrated on the special loci, or loci of curves with nontrivial automorphisms. There are actually many other arithmetic loci inside the moduli stacks which are interesting from the Galois and Grothendieck-Teichmüller viewpoints, such as arithmetic geodesic curves and arithmetic eigenloci; these are actually used in a Galois context in [LNS]. We refer to [L2] for a general geometric study and more references.

Appendix: Belyi's theorem and 'dessins d'enfant'

'Dessins d'enfants' (at least the phrase) were introduced by Grothendieck in section 3 of his *Esquisse*; they are in line with Grothendieck-Teichmüller theory in the sense that they have to do with curves and somehow also with moduli spaces thereof. They offer the archetype of a situation where arithmetic is (unexpectedly?) encoded in topology. Note that other examples, starting with 'origamis' are discussed in [L2]. In any case it is no wonder that this struck Grothendieck, and no chance that in the *Esquisse*, dessins are discussed after and as a kind of simple illustration of the first mathematical and most 'burning' section devoted to 'Grothendieck-Teichmüller theory'. We refer to [DE] for material on dessins; to the best of our knowledge [Z] is one of the very rare places, if not the only one, where the connection between dessins and moduli spaces of curves is effectively used.

A feature of the situation that was most striking for Grothendieck is embodied in Belyi's theorem, of which we give a brief discussion, partly because it is relevant to the above and partly because it may be useful to emphasize the hyperbolic side of it, which is less commonly mentioned than the conformal viewpoint. As appears from the *Esquisse* and Grothendieck's correspondence, he had somehow anticipated the result but without being completely convinced of its validity nor of course being able to prove it. The fact is that around 1978 G.Belyi gave a surprisingly simple *necessary* and sufficient condition for a complex curve to be defined over a numberfield. We refer to the contribution of L.Schneps in [DE] for the proof and the original reference. The statement and its amazingly simple proof should for instance be compared with the 'amazingly complicated' and roundabout proof of the Shimura-Tanyama-Weil conjecture which in analytic terms provides a very similar characterization of the *elliptic* curves which are defined over $\mathbf{Q}$ (not just $\overline{\mathbf{Q}}$). The original statement goes as follows:

Theorem A1 (G.Belyi, 1978): *A smooth complex curve X can be defined over a numberfield if and only if there exists a finite set $S \subset X$ of points of X such that $X' = X \setminus S$ can be realized as a finite unramified cover of $\mathbf{P}^1 \setminus \{0, 1, \infty\}$.*

The 'if' part was actually known to A.Weil in the mid-fifties and was quite elementary for Grothendieck; see the contribution of J.Wolfart in [GGA] for a detailed discussion and proof in the spirit of SGA 1, Corollaire X.1.8. The 'only if' part became central in Grothendieck's vision which partly materialized under the name 'Grothendieck-Teichmüller theory'.

Paraphrasing the 'only if' part, the first message is that all smooth curves defined over numberfields look like $\mathbf{P}^1 \setminus \{0, 1, \infty\}$, in the sense that they have a Zariski open dense set which can be realised as a finite étale covering of it. Such a cover is defined by a regular function $\beta : X' \to \mathbf{P}^1 \setminus \{0, 1, \infty\}$, nowadays called a Belyi function. Equivalently a Belyi function is given purely analytically as a meromorphic function on the Riemann surface X which is ramified at most over the three points 0, 1 and ∞ of the Riemann sphere. Note also that the curve X' is always affine (S is not empty) and that the points which have to be deleted are $\overline{\mathbf{Q}}$-points of X.

Because of the rigidity of $\mathbf{P}^1 \setminus \{0, 1, \infty\}$, namely the fact that its complex structure is unique and thus determined by the topology, an unramified finite covering can be given topologically and is completely encoded in a combinatorial graph looking just as candid as a 'dessin d'enfant' and yet carrying enough information to define for instance a numberfield (say its field of moduli). In terms of the associated Belyi function β, the dessin is nothing but the preimage of the segment $(0, 1)$, viewed up to homotopy on the curve X considered as a topological surface. So dessins d'enfant parametrize unramified coverings of $\mathbf{P}^1 \setminus \{0, 1, \infty\}$ and can be thought of as a pictionary of the geometric fundamental group of $\mathbf{P}^1 \setminus \{0, 1, \infty\}$, namely $\widehat{F}_2$, or almost equivalently of the all important $\widehat{SL_2(\mathbf{Z})}$; see again §3 of the *Esquisse* for an inspired praise of this group.

In connection with the ubiquity of $\mathbf{P}^1 \setminus \{0, 1, \infty\}$, we record the following suggestive

Proposition A2: $\mathbf{P}^1_{\mathbf{Z}} \setminus \{0, 1, \infty\} = \mathbf{Z} \setminus \{0, 1\}$ *is the only smooth (marked) hyperbolic curve over* $\mathbf{Z}$.

For the proof, note first that if a marked curve of genus 0 has at least 4 marked points, then two of them must collide modulo some prime, which takes care of the genus 0 case. In genus 1, there must be at least 1 marked point to ensure hyperbolicity, so we get an elliptic curve, and it cannot have good reduction everywhere. In higher genera we can ignore marked points and use the fact (due to J.-M.Fontaine) that there does not exist an Abelian variety over $\mathbf{Z}$, as would be given by the Jacobian of a smooth curve.

We also notice, and this is a crucial remark of Grothendieck, that $\mathbf{P}^1 \setminus \{0, 1, \infty\}$ is both a rigid hyperbolic curve and the first nontrivial moduli space ($\mathbf{P}^1 \setminus \{0, 1, \infty\} \simeq \mathcal{M}_{0,4}$) so that it lies at the crossroad of curves and their moduli, a bit as a nonlinear analog of the fact that elliptic curves provide a connection between curves and Abelian varieties. In any case, this elementary fact certainly has many deep consequences and is most relevant in the perspective of Grothendieck-Teichmüller theory.

Let us now move to the hyperbolic side of Belyi's result, refering in particular to [CIW]. First recall a classical notion: if G is a discrete or profinite group and H and H' are two subgroups of G, they are said to be *commensurate* if their intersection $H \cap H'$ has finite index in both H and H'. They are *commensurable* if there exist g and g' in G such that the conjugate subgroups gHg^{-1} and $g'H'g'^{-1}$ are commensurate; of course one can always take – say – $g' = 1$, replacing g with $g'^{-1}g$, but we favored a symmetric definition. These are obviously equivalence relations and the notion of commensurability makes sense for conjugacy classes of subgroups. We can now state the following:

Theorem A3 (Unramified hyperbolic version of A1): *A smooth complex curve X can be defined over a numberfield if and only if there exists a finite set $S \subset X$ such that the affine curve $X' = X \setminus S$ is uniformized by a Fuchsian group $\Gamma \subset PSL_2(\mathbf{R})$ (i.e. $X' \simeq \Gamma \backslash \mathcal{H}$) with Γ commensurable to $PSL_2(\mathbf{Z})$.*

Next and last we revisit the same situation in an orbifold (or stacky) fashion, paying attention to ramification. Here triangle groups, which are almost never arithmetic, play a role which is completely parallel to that of the arithmetic groups in the last statement.

Let $\Delta_{p,q,r} \subset PSL_2(\mathbf{R})$ be the Fuchsian triangle group defined by the triple of positive (finite or infinite) integers (p, q, r). We assume hyperbolicity as usual, namely that $p^{-1} + q^{-1} + r^{-1} < 1$. As an abstract group $\Delta_{p,q,r}$ is generated by three generators x, y, z with relations $xyz = 1$, $x^p = y^q = z^r = 1$. Triangle groups are rigid objects in the sense that all Fuchsian groups inside $PSL_2(\mathbf{R})$ with this abstract presentation are conjugate. So $\Delta_{p,q,r}$ is actually well-defined up to conjugacy. We have insisted on this classical rigidity property because it represents the orbifold counterpart of what happens with $\mathbf{P}^1 \setminus \{0, 1, \infty\}$, whose topological fundamental group is the free group on two generators $F_2 = \Delta_{\infty,\infty,\infty}$. These groups and the attending curves or rather orbifolds play a role in organizing the moduli spaces of curves which is very much an extension of that of $\mathbf{P}^1 \setminus \{0, 1, \infty\}$. In fact they show up in connection with the so-called curves with many automorphisms, which represent the 0-dimensional strata of the stratification of the moduli stacks by the automorphism groups; see *e.g.* [L2], [LV] and [Sc2] for more on this and further references.

Let us record an arithmetic statement in those terms, in order to illustrate how curves defined over numberfields arise as quotients of the upper half-plane via the proper and discontinuous but not necessarily free action of a group which has finite index in a possibly cocompact Fuchsian triangle group.

Proposition A4: *Let X be a smooth complex curve which can be written as a quotient $X \simeq \Gamma \backslash \mathcal{H}$, where $\Gamma \subset PSL_2(\mathbf{R})$ is commensurable with a Fuchsian triangular group. Then X can be defined over a numberfield.*

Here X may be projective or affine and the action is isometric, in particular proper and discontinuous, but not necessarily free. It does not make much sense to state an 'only if' part (a hyperbolic ramified version of Belyi theorem), simply because by puncturing one may always get back to the situation of theorem A3, that is choose $p = q = r = \infty$. In order to prove the proposition one notes that by puncturing X at the points lying over 0, 1 and ∞, we get a curve which is dominated by a finite étale cover of $\mathbf{P}^1 \setminus \{0, 1, \infty\}$. This cover X' can be written as $X' \simeq \Gamma' \backslash \mathcal{H}$, where here the action is free, Γ' uniformizes X' and $\Gamma' \simeq \pi_1(X')$ (topological fundamental group). We insist on the fact that Γ' is commensurable with, indeed is a finite index subgroup of $PSL_2(\mathbf{Z})$, whereas Γ in general is *not* commensurable with that group. So 'puncturing', that is in hyperbolic terms converting elliptic elements in a Fuchsian group into parabolic ones, is a very deep operation but somehow it does not alter the fact of being defined over a numberfield.

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