The universal Ptolemy-Teichmüller groupoid

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Abstract

We define the *universal Ptolemy-Teichmüller groupoid*, a generalization of Penner's universal Ptolemy groupoid, on which the Grothendieck-Teichmüller group – and thus also the absolute Galois group – acts naturally as automorphism group. The essential new ingredient added to the definition of the universal Ptolemy groupoid is the *profinite local group* of pure ribbon braids of each tesselation.

§0. Introduction

The goal of this article is to give a completion (by braids) of Penner's *Ptolemy group* G such that there is a natural action of the Grothendieck-Teichmüller group (and a fortiori, the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$) on it. This work was motivated on the one hand by the deep relation of the Ptolemy group – shown to be isomorphic to Richard Thompson's group and to the group of piecewise $\mathrm{PSL}_2(\mathbb{Z})$-transformations of the circle – and mapping class groups and the geometry of moduli spaces in general, most visibly in genus zero, and on the other by the presence of the remarkable pentagonal relation, stimulating the natural reflex of the authors to associate every pentagon appearing in nature to that of the Grothendieck-Teichmüller group $\widehat{GT}$.

The difficulties in defining a $\widehat{GT}$-action on G were the following. Firstly, the profinite version of $\widehat{GT}$ which interests us (mainly by virtue of the fact that it contains the Galois group) acts on profinite groups, whereas via its isomorphism with Richard Thompson's group, G is known to be simple, and therefore its profinite completion is trivial. Furthermore, G contains no braids and $\widehat{GT}$ naturally seems to introduce them into every situation where it appears. The undertaking therefore framed itself as follows: instead of restricting attention to G, is it possible to extend G by braids, in such a way that it is possible to define a $\widehat{GT}$-action on a profinite version of the extension, in such a way that the pentagonal relation of G reflects that of $\widehat{GT}$? The answer turned out to be nearly yes, namely in order to succeed it was necessary to use not the group, but the groupoid interpretation of G, in which its elements are considered as morphisms between marked tesselations (this groupoid is known as the Ptolemy groupoid), and to *relax the condition of the Ptolemy groupoid stating that the group of morphisms from any marked tesselation to itself is trivial to a condition stating that the group of morphisms from any marked tesselation to itself is isomorphic to a certain ribbon braid group.* It is the profinite completion of the ensuing braid-groupoid which admits a $\widehat{GT}$-action.

In §1, we give the definition and presentation of the Ptolemy group, and its interpretation as a groupoid whose objects are marked tesselations of the Poincaré disk. In §2, we recall the definitions and important properties of braid and mapping class groups, and their generalizations to ribbon braid and mapping class groups, which will be the braid groups used to extend the Ptolemy groupoid to the Ptolemy-Teichmüller groupoid. §3 is devoted to the actual construction of the Ptolemy-Teichmüller groupoid $\mathcal{P}_\infty$, with a "physical" interpretation of the new, added groups of non-trivial morphisms from a tesselation to itself via braids of ribbons viewed as hanging from the intervals; at the end of the section we define the profinite completion of the groupoid $\hat{\mathcal{P}}_\infty$ by simply taking the profinite completions of each of the local groups, while preserving the set of objects (i.e. marked tesselations) and the basic (Ptolemy) morphisms from one to another. §4 contains the main theorem of the article (theorem 4) explicitly a $\widehat{GT}$-action on the universal profinite Ptolemy-Teichmüller groupoid $\hat{\mathcal{P}}_\infty$; the role of the two pentagons appears in lemma 5. Finally, in §5 we give a very brief discussion of the relation between the situation considered here and the geometry and arithmetic of genus zero moduli spaces.

This article was motivated by the idea of discovering a link between number theory and Penner's universal Ptolemy groupoid, an idea suggested to us by Bob Penner who had himself had conversation with Dennis Sullivan, and immediately seized upon by us because of the distant echo of the pentagonal defining relation of $\widehat{GT}$ which could be heard (upon listening carefully) when considering Penner's sequence of ten moves giving a fundamental defining relation of the Ptolemy group. We recall with great pleasure the enthusiasm of our early discussions with him about the possible links between those relations; later communal discussions with Dennis Sullivan and Vlad Sergiescu were both enlightening and stimulating. We warmly thank all three.

§1. The universal Ptolemy groupoid

A groupoid is a category all of whose morphisms are isomorphisms. We begin by giving some basic definitions leading to the definition of the universal Ptolemy groupoid from [P1, 4.1] (see also [P2]).

Identify the Poincaré upper half-plane with the Poincaré disk via the transformation $(z-i)/(z+i)$. Traditionally, points on the Poincaré disk are labeled by the corresponding upper half-plane, so that for instance the points -1, $-i$, 1 and i on the unit circle in $\mathbb{C}$ are labeled 0, 1, ∞ and -1, whereas the central point $0 \in \mathbb{C}$ is labeled i. In particular, $\mathbb{P}^1\mathbb{R}$ is wrapped once around the boundary of the disk, the rational numbers of course lie densely in it. We will be particularly concerned with these rational numbers.

Let a *marked tesselation* be a maximal (i.e. triangulating) tesselation of the Poincaré disk such that its vertices lie on the set of rational numbers on the boundary, equipped with a directed oriented edge. The *standard* marked tesselation is the dyadic tesselation T^* with the marked edge from 0 to ∞:

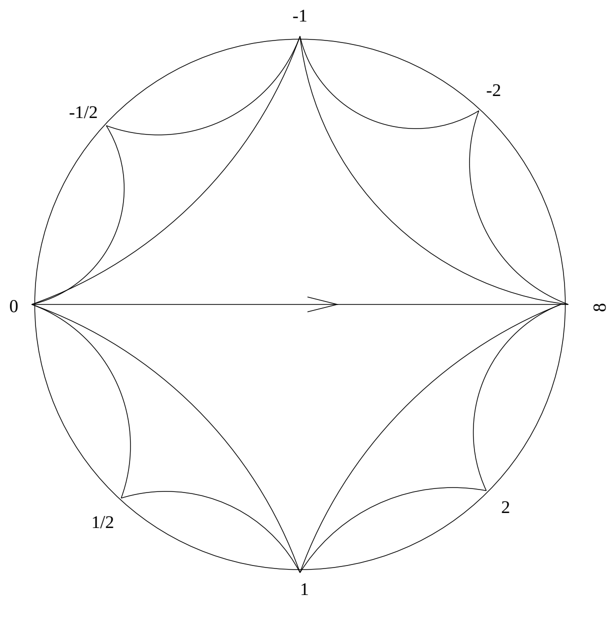

Figure 1. The standard dyadic tesselation T^* with its oriented edge

The *elementary move* on the oriented edge of a tesselation changes it from one diagonal of the unique quadrilateral containing it to the other by turning it counterclockwise; it is of order 4.

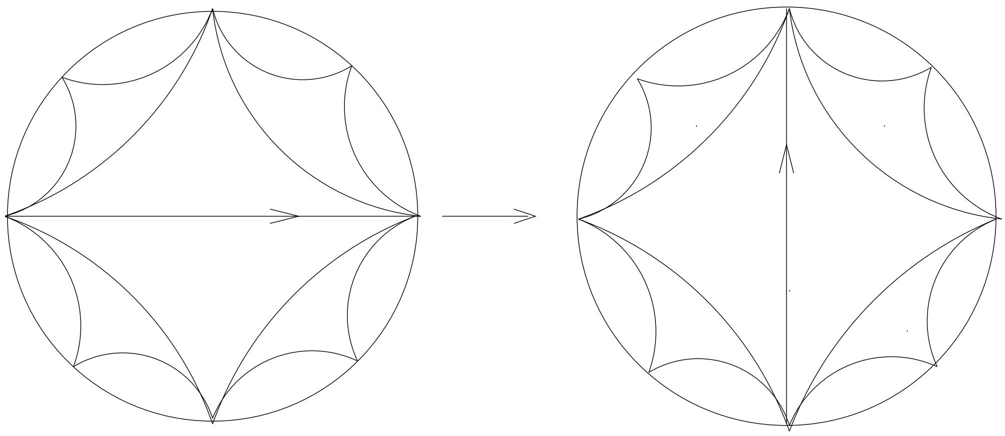

Figure 2. The elementary move on the oriented edge

An *arrow-moving* move on a marked tesselation is an operation on the tesselation which moves the oriented edge to another edge without changing the tesselation itself.

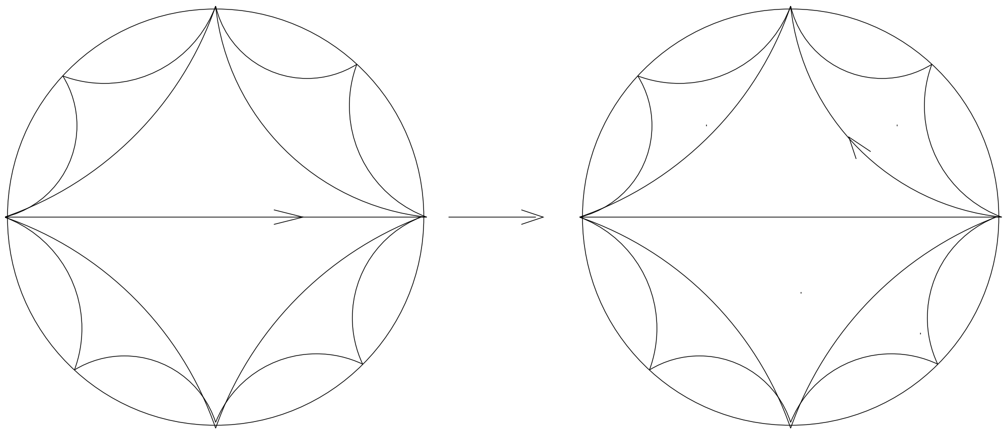

Figure 3. An arrow-moving move

Definition: The *universal Ptolemy groupoid* is the groupoid defined as follows:

Objects: The marked tesselations;

Morphisms: Finite sequences (or chains) of elementary moves on the oriented edge and arrow-moving moves;

Relations: The only morphism from an object to itself is the trivial one.

Remark: The condition that the local groups of morphisms (i.e. groups of morphisms from an object to itself) are trivial implies that if T_1 and T_2 are two marked tesselations, then there is a unique morphism in the groupoid from T_1 to T_2.

On any marked tesselation T, let α denote the elementary move on its oriented edge, as shown for the standard marked tesselation in Figure 2. Let the triangle to the "left" of the oriented edge of T denote the triangle lying to our left if we imagine ourselves to be lying face down on the tesselation along the oriented edge, with our head in the direction indicated by the arrow, and let β denote the move which sends the arrow counterclockwise to the next edge of the triangle to the left of the oriented edge, as in Figure 3. As noted by Penner, the universal Ptolemy groupoid can be given a group structure, simply because if we write a chain of α's and β's, it can be considered as a morphism on any given tesselation: at each point in the chain, the tesselation being acted on is the one resulting from application of all the previous moves. In particular, given a starting tesselation, a word in α and β uniquely determines a morphism in the groupoid, and conversely, every morphism in the groupoid can be given as a starting tesselation and a word in α and β. Moreover, clearly, if a word in α and β gives the trivial morphism from some tesselation to itself, then the same word will give the trivial morphisms from every tesselation in the Ptolemy groupoid to itself. We want to find exactly which words these are, namely to determine the relations in the group generated by α and β induced by the condition that the group of local morphisms of a tesselation is trivial. In other words, we need to determine the chains of α's and β's which bring a marked tesselation to itself.

Theorem 1. *All words in α and β taking a given tesselation to itself are generated by the following words (where square brackets denotes the commutator):*

$$\alpha^4, \quad \beta^3, \quad (\alpha\beta)^5, \quad [\beta\alpha\beta, \alpha^2\beta\alpha\beta\alpha^2], \quad [\beta\alpha\beta, \alpha^2\beta\alpha^2\beta\alpha\beta\alpha^2\beta^2\alpha^2].$$

Proof. In the contribution to this volume by M. Imbert, it is proved that Penner's group G is isomorphic to Richard Thompson's group. Therefore, it suffices to show that the group, let us call it G, defined by generators α and β and relations as in the statement of the theorem is isomorphic to Thompson's group. Let us give a presentation of Thompson's group which can be found on page 2 of Thompson's unpublished notes [T] (and under a different but recognizable notation, [CFP], lemma 5.2.). It is given by three generators, R, D and c_1, and six relations, namely: $[R^{-1}D, RDR^{-1}] = 1$, $[R^{-1}D, R^2DR^{-2}] = 1$, $c_1 = Dc_1R^{-1}D$, $RDR^{-1}Dc_1R^{-1} = D^2c_1R^{-2}D$, $Rc_1 = (Dc_1R^{-1})^2$ and $c_1^3 = 1$. We define a homomorphism ϕ from Thompson's group to G by setting $\phi(R) = \alpha^2\beta^2$, $\phi(D) = \alpha^3\beta$ and $\phi(c_1) = \beta$. We need to check first that ϕ is really a homomorphism, and second that it is invertible, so an isomorphism. To see that it is a homomorphism it suffices to compute the images of both sides of the six defining relations by ϕ. All are easily seen to hold in G. Indeed, the first two relations are the analogous commutator relations in G, and $c_1^3 = 1$ is sent by ϕ to $\beta^3 = 1$. The two sides of the relation $c_1 = Dc_1R^{-1}D$ are sent to β and $\alpha^3\beta\beta\beta\alpha^2\alpha^3\beta = \beta$ and the two sides of $Rc_1 = (Dc_1R^{-1})^2$ are sent to α^2 and $(\alpha^3\beta\beta\beta\alpha^2)^2 = \alpha^2$. Finally, rewriting the remaining relation as $D^2c_1R^{-2}DRc_1^{-1}D^{-1}RD^{-1}R^{-1} = 1$, we see that the left-hand side is sent by ϕ to

$$\alpha^3\beta\alpha^3\beta\cdot\beta\cdot\beta\alpha^2\beta\alpha^2\cdot\alpha^3\beta\cdot\alpha^2\beta^2\cdot\beta^2\cdot\beta^2\alpha\cdot\alpha^2\beta^2\cdot\beta^2\alpha\cdot\beta\alpha^2$$
$$= \alpha^2(\alpha\beta\alpha\beta\alpha\beta\alpha\beta\alpha\beta)\alpha^2 = \alpha^2(\alpha\beta)^5\alpha^2 = 1.$$

This shows that ϕ is a homomorphism; to show it is an isomorphism, it suffices to define $\phi^{-1}(\alpha) = c_1D^{-1}$ and $\phi^{-1}(\beta) = c_1$ (indeed, the relation $c_1 = Dc_1R^{-1}D$ shows that Thompson's group is generated by the two elements c_1 and D). ♢

Remark. The generators of this group can be identified with the corresponding moves on marked tesselations in the Ptolemy groupoid. Indeed, the words α^4 and β^3 clearly bring a marked tesselation back to itself and are therefore trivial; similarly $(\alpha\beta)^5 = 1$ corresponds exactly to Penner's trivial sequence of 10 moves (cf. (c) on p. 179 of [P1]; note that $(\alpha\beta)$ simultaneously moves the two diagonals of a pentagon whereas Penner moves one at a time). Finally, the two commutation relations in G imply that that elementary moves on the diagonals of two neighboring quadrilaterals commute, and elementary moves on the diagonals of two quadrilaterals separated only by a triangle; it is a remarkable fact that the commutation of all pairs of elementary moves taking place in disjoint quadrilaterals (Penner's relation (d) on p. 179 of [P]) are consequences of the relations in G.

§2. Ribbon braids

Let us recall the definitions of the Artin ribbon braid groups and the ribbon mapping class groups. First recall the definitions of the usual Artin braid and mapping class groups. The Artin braid group B_n for $n \geq 1$ is generated by $\sigma_1, \ldots, \sigma_{n-1}$ with the relations

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ for } 1 \leq i \leq n-2 \text{ and } \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j| \geq 2.$$

There is a natural surjection $\rho : B_n \to S_n$ for $n \geq 1$ which induces a natural surjection $\rho : M(0,n) \to S_n$ (for $n = 4$ we have the surjection ρ :

$B_3/Z \to S_3$). The kernel of ρ (denoted by K_n in B_n and $K(0,n)$ in $M(0,n)$) is known as the *pure* braid group resp. mapping class group. Both K_n and $K(0,n)$ are generated by the elements $x_{ij} := \sigma_{j-1}\cdots\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}\cdots\sigma_{j-1}^{-1}$ for $1 \le i < j \le n$. The center of B_n and of K_n is cyclic generated by the element $\omega_n = x_{12}x_{13}x_{23}\cdots x_{1n}\cdots x_{n-1,n}$. The mapping class group $M(0,n)$ (resp. the pure mapping class group $K(0,n)$) is the quotient of B_n (resp. K_n) by the following relations:

(i) $\omega_n = 1$;
(ii) $x_{i,i+1}x_{i,i+2}\cdots x_{i,n}x_{i,1}\cdots x_{i,i-1} = 1$; for $1 \le i \le n$, where the indices are considered in $\mathbb{Z}/n\mathbb{Z}$.

The *Artin ribbon braid group* B_n^* is a semi-direct product

$$B_n^* \simeq \mathbb{Z}^n \rtimes B_n;$$

it is generated by generators $\sigma_1,\ldots,\sigma_{n-1}$ of the B_n factor and $t_1,\ldots,t_n$ (all commuting) of the $\mathbb{Z}^n$ factor, with the "semi-direct" relations given by:

$$\begin{cases} \sigma_i t_i \sigma_i^{-1} = t_{i+1} \text{ for } 1 \le i \le n-1 \\ \sigma_i t_{i+1} \sigma_i^{-1} = t_i \text{ for } 1 \le i \le n-1 \\ (\sigma_i, t_j) = 1 \text{ for } 1 \le i \le n-1, \quad j \ne i, i+1 \end{cases}.$$

This group is visualized like the usual braid groups, except that the strands are replaced by flat ribbons, so that a twist on any one of them is non-trivial. The subgroup of B_n^* consisting of "flat" braids of the ribbons (i.e. without twists on the ribbons, cf. Figure 4) is canonically isomorphic to B_n, and from now on we identify B_n and its generators σ_i with this subgroup of B_n^*.

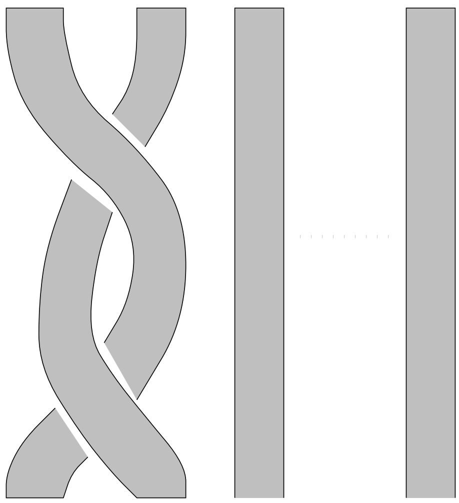

Figure 4. The flat braid x_{12}

We identify $\mathbb{Z}^n$ with the abelian subgroup of B_n^* generated by a full (2π) twist t_i on each of the n ribbons (cf. Figure 5).

Figure 5. A full twist on a ribbon

This visualization corresponds to the definition of B_n^* as a semi-direct product $\mathbb{Z}^n \rtimes B_n$ given above.

Since the twists t_i commute with pure braids, the pure ribbon braid subgroup K_n^* of B_n^* is just a direct product $\mathbb{Z}^n \times K_n$. Let us define the ribbon mapping class group $M^*(0,n)$ to be B_n^* modulo the following relations. Firstly, the center of B_n^* is generated by the element which also generates the center of K_n, namely $\omega_n = x_{12}x_{13}x_{23}\cdots x_{1n}\cdots x_{n-1,n}$, together with the subgroup $\mathbb{Z}^n$. The first relation we quotient out by is

(i′) $\omega_n = \prod_i t_i$.

Now, instead of using the usual sphere relations by which we quotient B_n to obtain $M(0,n)$, we use the *ribbon sphere relations*:

(ii′) $x_{i,i+1}x_{i,i+2}\cdots x_{i,n}x_{1,i}\cdots x_{i,i-1} = t_i^2$.

The quotient of B_n^* by the relations in (i′) and (ii′) is the *ribbon mapping class group* $M^*(0,n)$.

The surjection of B_n onto S_n extends to B_n^* by sending the subgroup $\mathbb{Z}^n$ to 1, and the kernel of this surjection is the *pure ribbon braid group* K_n^*. The surjection passes to $M^*(0,n)$ and its kernel in this group is denoted by $K^*(0,n)$. Attention: although the group B_n^* is a semi-direct product of B_n with $\mathbb{Z}^n$ and the subgroup K_n^* is a direct product of K_n with $\mathbb{Z}^n$, analogous statements do not hold for $M^*(0,n)$ or $K^*(0,n)$; although these groups are extensions of $M(0,n)$ and $K(0,n)$ respectively by $\mathbb{Z}^n$, the extensions are not split. We refer to [MS], Appendix B for a detailed discussion of these groups in a similar but more geometric context.

Let us give some admirable properties of the ribbon groups.

(1) It is well-known that removing any strand gives a surjection from K_n onto K_{n-1}, which induces a surjection from $K(0,n)$ onto $K(0,n-1)$. There are analogous natural surjections from K_n^* into K_{n-1}^* and from $K^*(0,n)$ into $K^*(0,n-1)$ obtained by removing one ribbon. Removing several ribbons thus gives surjections from $K^*(0,n)$ onto $K^*(0,m)$ for $m < n$.

(2) The braid obtained by holding two adjacent ribbons i and $i+1$ firmly by their bottom ends and twisting them one full turn is equal to $t_i t_{i+1} x_{i,i+1}$. We denote this ribbon braid by $t_{i,i+1}$. It is the same as the full twist on the single "wide" ribbon obtained by sewing the two adjacent ribbons together. The expression for the simultaneous full twist of several adjacent ribbons can easily be deduced from this one by induction.

(3) There are natural injections K_m into K_n for $m < n$ given by dividing up the n strands into m adjacent packets (each of which can consist of one or more strands) called $A_1, \ldots, A_m$; the subgroup of K_n generated by the "flat" braids x_{A_i,A_j} (as in Figure 4, considering each packet as a ribbon) is isomorphic to K_m. Analogously, there are natural injections of K_m^* into K_n^* for any division of the n ribbons into m adjacent packets. Each packet, considered as adjacent ribbons sewn together, forms a "wide ribbon", and the group K_m^* of braids on these wide ribbons is naturally a subgroup of K_n^*.

Now, there is no such natural injection for the pure mapping class groups $K(0,n)$, because the twist on a packet of strands is non-trivial whereas the twist on a single strand is trivial. This problem is eliminated for the ribbon mapping groups where the ribbons and the wide ribbons behave in the same way with respect to twists. Therefore we have such injections for the pure ribbon mapping class groups $K^*(0,m) \hookrightarrow K^*(0,n)$; *this point represents the major advantage of the use of ribbon braid groups with respect to ordinary braid groups.*

§3. The universal Ptolemy-Teichmüller groupoid

The universal Ptolemy-Teichmüller groupoid $\mathcal{P}_\infty$ is a generalization of the universal Ptolemy groupoid in the sense that we add morphisms from a given marked tesselation to itself. The objects of $\mathcal{P}_\infty$ are those of the universal Ptolemy groupoid, namely marked braid tesselations; what we do here is to relax the condition stating that the local groups are all trivial to a condition defining the local groups as certain ribbon braid groups.

To explain exactly what is going on, we visualize a marked tesselation T a little differently; we assume that there is a ribbon hanging from the interval on the circle delimited by each edge of the tesselation. To be precise, each edge of the tesselation actually divides the circle into *two* intervals, and we want to choose only one of them; we choose to hang the ribbon from the interval lying entirely on one side of the oriented edge. This makes sense for every edge except the oriented one, to which we associate the interval lying to the left of it in the sense explained earlier. Note that since the ribbons are determined by the edges of T, assuming their presence does not add anything to T; *the point of adding them is that the non-trivial morphisms from T to itself which we are going to introduce correspond exactly to braiding them.* Before continuing, we note that if one considers the infinite trivalent tree *dual* to the tesselation, then it comes to the same thing to attach a strand to each of its "ends" (the rationals) and consider the set of strands in a given interval as forming a ribbon, and this in turn is equivalent to attaching a strand to each vertex of the tree (uniquely associated to a rational). This idea, due to Greenberg and Sergiescu (cf. [GS]) was one of the starting points of this article.

Consider thus from now on each marked tesselation T to come equipped with its ribbons. Each ribbon is automatically associated to an *interval* on the circle (delimited by an edge of T, a fortiori by two numbers in $\mathbb{P}^1\mathbb{Q}$). Two ribbons of T are said to be *disjoint* if their intervals are disjoint except for at most one common endpoint. They are said to be *neighbors* if their associated intervals are disjoint except for exactly one common endpoint. Two ribbons of T are said to be *adjacent* if their intervals are delimited by two sides of a triangle of the tesselation; thus, adjacent ribbons are of course neighbors and neighbors are disjoint, but the converses are not necessarily true.

Two neighboring ribbons of a given marked tesselation can always be made into adjacent ribbons of another tesselation which can be obtained from the first by a finite number of elementary moves, successively reducing to zero the (finite) number of edges coming out of the common endpoint of the two ribbons and lying inside the smallest polygon of the tesselation having as two of its edges those associated to the ribbons. On the left-hand side of Figure 6, we show two neighboring ribbons; the smallest polygon of the tesselation containing having their corresponding edges as edges is a quadrilateral and there is just one edge coming out of the common endpoint of the two ribbons and lying inside it (namely, its diagonal). Thus, after the elementary move on that diagonal, the two ribbons become adjacent (right-hand side), associated to two sides of the triangle A.

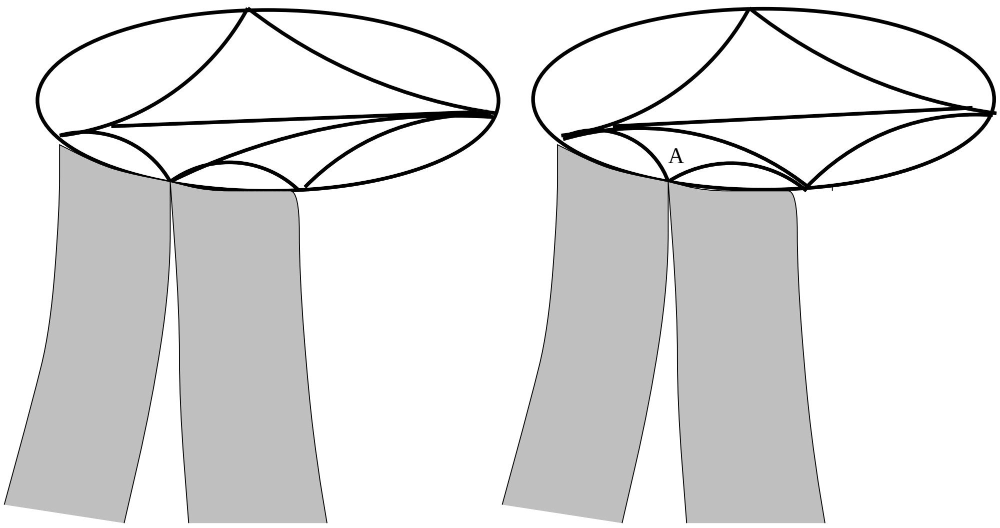

Figure 6. Adjacent and neighboring ribbons of a tesselation

Definition: Let A and B be disjoint ribbons of a given marked tesselation T. Let t_A^T denote the full twist on A, oriented as in Figure 5, and let x_{AB}^T denote the flat braid of A and B shown in Figure 7, where the ribbon on the left-hand side passes in front of the right-hand one (whether the observer stands inside or outside the tesselation).

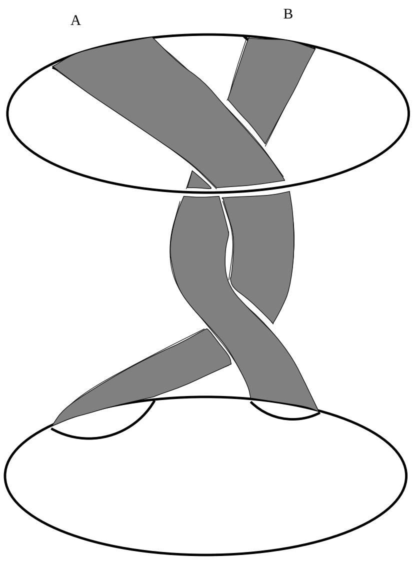

Figure 7. The braid x_{AB}^T

The *local group K^T at a marked tesselation T* is a group of morphisms from T to itself, essentially given by braiding the ribbons associated to intervals of T; it is defined as follows.

Definition. Let K^T be generated by the flat braids x^T_{AB} for all pairs of disjoint ribbons A and B of T and by the full twists t^T_A on each of these ribbons. Define the set of relations in the group K^T to be the set of relations coming from the finite polygons of T containing the oriented edge, as follows. Let S be such a polygon, say with n sides, and let $A_1,\ldots,A_n$ be the n ribbons associated to the intervals determined by the sides; they are pairwise disjoint (which would not be the case if the polygon S did not contain the oriented edge of T and thus lay entirely on one side of it). Let $K^T(S)$ denote the subgroup of K^T generated by the twists $t^T_{A_i}$ and the flat braids $x^T_{A_i,A_j}$. Then the relations of K^T are generated by all the relations induced by the assertion: *For every S, the group $K^T(S)$ is isomorphic to the pure ribbon mapping class group $K^*(0,n)$.*

Definition: The *universal Ptolemy-Teichmüller groupoid $\mathcal{P}_\infty$* is defined as follows:

Objects: Marked tesselations.

Morphisms: They are of two types: firstly, those of the universal Ptolemy groupoid, which act on marked tesselations as usual, and secondly, the groups $\mathrm{Hom}(T,T)\simeq K^T$ of morphisms from each marked tesselation to itself.

Relations: The full set of relations in the universal Ptolemy-Teichmüller groupoid $\mathcal{P}_\infty$ is given by:

(i) those of the universal Ptolemy groupoid (any sequence of elementary moves leading from a tesselation to itself is equal to 1);

(ii) those of the local ribbon braid groups;

(iii) commutativity of these two types of morphisms as in equation (1) below.

The universal Ptolemy-Teichmüller groupoid contains the universal Ptolemy groupoid as a subgroupoid, because of (i). Let us explain (ii) and (iii) further. Firstly, from now on we use the term *interval* to denote an interval of the circle delimited by two rationals, and an *interval of* T or equivalently, a *ribbon of* T to denote the interval delimited by an edge of the tesselation T, as before (the one on the opposite side from the oriented edge of T). Not every interval of the circle is an interval of T, of course, but every interval is the union of a finite number of intervals of T; we call such an interval a *wide interval* or a *wide ribbon* of T (obtained by sewing together a finite number of neighboring ribbons of T). Thus every interval of the circle is associated to a ribbon or a wide ribbon of T.

Let A and B denote two disjoint intervals (recall that "disjoint" intervals may have one endpoint in common), equipped with ribbons. Let the braid x_{AB} denote the usual flat braid (as in Figure 7); this twist can be applied to ribbons corresponding to any two disjoint intervals of the circle, without needing to refer to a specific tesselation. However, fixing a tesselation T, we see that the ribbons associated to the intervals A and B are either ribbons or wide ribbons of T, which implies that the braid x_{AB} actually lies in K^T for all T. We write x^T_{AB} when we want to consider the braid x_{AB} *as an element of* K^T.

Recall that given marked tesselations T' and T, there is a unique morphism γ in the universal Ptolemy groupoid from T' to T. This gives rise to a canonical isomorphism between $K^{T'}$ and K^T via $K^{T'} = \gamma^{-1}K^T\gamma$. For all pairs of disjoint intervals A and B of the circle, this isomorphism has the property that

$$x^{T'}_{AB} = \gamma^{-1}x^T_{AB}\gamma \tag{1}$$

in the universal Ptolemy-Teichmüller groupoid. This is what is meant by the commutation of the morphisms of the universal Ptolemy groupoid with braids in (iii) above.

Proposition 2. *Let T be a marked tesselation.*

(i) A set of generators for the group K^T is given by the set of braids x^T_{AB} for all pairs of ribbons A and B corresponding to disjoint intervals of the circle, and the twists t^T_A on the wide ribbons of T corresponding to all intervals of the circle. The set of relations associated to this set of generators is independent of T.

(ii) A smaller set of generators for K^T is given by the twists t^T_A and x^T_{AB} where A and B are disjoint ribbons of T. A set of relations for K^T associated to this set of generators is given by the relations between the generators of each $K^T(S)$ for finite polygons S of T.

(iii) Another set of generators for K^T is given by the set of twists t^T_A on

all ribbons of T and x^T_{AB} where A and B are pairs of adjacent or neighboring wide ribbons of T. Indeed, if S a finite polygon of T containing the oriented edge and the ribbons of S are the ribbons of T associated to the edges of S, then each subgroup $K^T(S)$ is generated by the twists on ribbons of S and the x^T_{AB} where A and B are adjacent or neighboring wide ribbons of S.

Proof. (i) and (ii) are immediate consequences of the definition of K^T. For (iii), we start by showing the statement for $K^T(S) \subset K^T$. By definition, this subgroup is isomorphic to $K^*(0,n)$ where n is the number of edges of the polygon S.

A set of generators for the pure Artin braid group K_n is given by the elements $x_{i\cdots j,j+1\cdots k}$ for all $1 \leq i \leq j < k \leq n$; this braid denotes the flat braid of the "neighboring packets" of strands numbered $i \cdots j$ and $j+1 \cdots k$, which can be considered as ribbons; it looks like the one in Figure 4, with one or several strands in place of the ribbons. To see that this set really generates, it suffices to write each of the usual generators x_{ij} of K_n in terms of these, which can be done via the formula

$$x_{ij} = x_{i\cdots j-1,j} x^{-1}_{i+1\cdots j-1,j}$$

(draw the picture!) If $j = i+1$, the usual x_{ij} is itself a twist of neighboring packets, which consist of one strand each.

The flat braids of neighboring packets $x_{i\cdots j,j+1\cdots k}$ also generate the quotient $K(0,n)$ of K_n, so adding in the full twists on ribbons, we have a set of generators for $K^*(0,n)$. Since this group is isomorphic to $K^T(S)$, this proves the statement of (iii) for the groups $K^T(S)$. It follows immediately for K^T since the group K^T is generated by the subgroups $K^T(S)$ as S runs over all the finite polygons of T containing the oriented edge of T. $\diamondsuit$

Let us now describe the "profinite completion" of the universal Ptolemy-Teichmüller groupoid $\mathcal{P}_\infty$. We need the following:

Lemma 3. *(i) If S and R are finite polygons containing the oriented edge of a given marked tesselation T, and S lies inside R, then the subgroup $K^T(S) \subset K^T$ is contained in $K^T(R)$.*

(ii) For $n \geq 4$, let S_n denote the 2^n-gon in the standard marked dyadic tesselation T^* obtained from S_4, the basic quadrilateral containing the oriented edge 0∞, by successively dividing every edge into two. Then

$$K^{T^*} = \bigcup_{n\geq 2} K^{T^*}(S_n),$$

where this union of groups is given by the natural inclusion of $K^{T^*}(S_n)$ *into* $K^{T^*}(S_{n+1})$ *induced by the inclusion of* S_n *in* S_{n+1}*, as in (i).*

(iii) Let $\hat{K}^{T^*}(S_n)$ *denote the profinite completion of* $K^{T^*}(S_n)$ *for* $n \geq 2$*. Then the inclusion of* S_n *into* S_{n+1} *induces a natural inclusion of* $\hat{K}^{T^*}(S_n)$ *into* $\hat{K}^{T^*}(S_{n+1})$*.*

Proof. Part (i) follows from the existence of injections $K^*(0,m) \to K^*(0,n)$ for $m < n$, sending ribbons in $K^*(0,m)$ to wide ribbons in $K^*(0,n)$, cf. property (3) in §2. Part (ii) is a corollary of this, since every polygon of T^* lies inside S_n for some sufficiently large n. Finally, (iii) is a consequence of the fact that like all braid and mapping class groups, the $K^{T^*}(S_n)$ inject into their profinite completions. ♢

Let us define the *profinite local group* at T^* by

$$\hat{K}^{T^*} := \bigcup_{n\geq 2} \hat{K}^{T^*}(S_n).$$

For any pair of disjoint intervals A, B of the circle, there exists n such that $x_{AB}^{T^*}$ lies in $K^{T^*}(S_n)$, since every rational is a vertex of some S_n. Since $K^{T^*}(S_n)$ injects into its profinite completion, the braid $x_{AB}^{T^*}$ lies in $\hat{K}^{T^*}(S_n)$. The group $\hat{K}^{T^*}$ is topologically described by the same set of generators and relations as K^{T^*}. We define the *profinite local group* at any marked tesselation T to be the one obtained from $\hat{K}^{T^*}$ as in equation (1), i.e. by conjugating by the unique morphism γ in the universal Ptolemy groupoid which takes T^* to T.

Definition: Let the profinite completion $\hat{\mathcal{P}}_\infty$ of $\mathcal{P}_\infty$ be the groupoid defined as follows:

Objects: Marked tesselations T.

Morphisms: All the morphisms of the universal Ptolemy groupoid, together with the local groups $\mathrm{Hom}(T,T) = \hat{K}_T^*$ at each T.

Relations: (i) those of the universal Ptolemy groupoid;

(ii) those of the local profinite braid groups;

(iii) commutativity of these two types of morphisms as in equation (1).

§4. $\widehat{GT}$ and the automorphism group of $\hat{\mathcal{P}}_\infty$

Definition: The automorphism group $\mathrm{Aut}^0(\hat{\mathcal{P}}_\infty)$ of the completed Ptolemy groupoid $\hat{\mathcal{P}}_\infty$ is defined to be the set of automorphisms of the groupoid $\hat{\mathcal{P}}_\infty$ which act trivially on the set of objects.

The goal of this article is to show that $\hat{\mathcal{P}}_\infty$, considered as a completion of the universal Ptolemy groupoid, has the property that the Grothendieck-Teichmüller groupoid lies in its automoprhism group $\mathrm{Aut}^0(\hat{\mathcal{P}}_\infty)$ (cf. §0). To prove this, we begin by recalling the definition of $\widehat{GT}$ (cf. the survey [S] for references and details).

Definition: Let $\underline{\widehat{GT}}$ be the monoid of pairs $(\lambda, f) \in \hat{\mathbb{Z}}^* \times \hat{F}_2'$, satisfying the three following relations, the first two of which take place in the profinite completion $\hat{F}_2$ of the free group on two generators F_2, and the third in the profinite completion $\hat{K}(0,5)$ of the pure mapping class group:

(I) $f(y,x)f(x,y) = 1$;

(II) $f(z,x)z^m f(y,z)y^m f(x,y)x^m = 1$, where $m = \frac{1}{2}(\lambda - 1)$ and $z = (xy)^{-1}$;

(III) $f(x_{34}, x_{45})f(x_{51}, x_{12})f(x_{23}, x_{34})f(x_{45}, x_{51})f(x_{12}, x_{23}) = 1$.

Under a suitable multiplication law, this set forms a monoid. The group $\widehat{GT}$ is defined to be the group of invertible elements of the monoid $\underline{\widehat{GT}}$. Drinfel'd and Ihara showed that the group $\widehat{GT}$ contains the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ as a subgroup in a natural way (again, cf. [S] for all relevant references). In particular, whenever $\widehat{GT}$ acts on an object, this object becomes equipped with a Galois action, indicating a – sometimes quite unexpected – link with number theory, which was one of the main motivations behind this article.

Theorem 4. *There is an injection* $\widehat{GT} \hookrightarrow \mathrm{Aut}^0(\hat{\mathcal{P}}_\infty)$.

Proof. Let $F = (\lambda, f) \in \widehat{GT}$. Then we let F act trivially on the set of objects of $\hat{\mathcal{P}}_\infty$. The proof is outlined as follows: first we define the action of F to be trivial on arrow-moving morphisms, next we give its definition on the morphisms of the universal Ptolemy groupoid (considered as a subgroupoid of $\hat{\mathcal{P}}_\infty$) and prove in proposition 5 that the relations of the universal Ptolemy groupoid are respected, and finally we define the action on a set of generators of the profinite local group $\hat{K}^T$ – using lemma 5 to show that the action on the generators is well-defined – and prove in lemma 6 that the action extends to an automorphism of $\hat{K}^T$. This takes care of two of the three types of morphisms in $\hat{\mathcal{P}}_\infty$; to conclude, we show that the commutation relations of equation (1) are respected.

Let us proceed to the definition of the action of F on the morphisms of $\hat{\mathcal{P}}_\infty$.

Arrow-moving morphisms. F acts trivially on these.

Elementary morphisms. The oriented edge of a marked tesselation T determines two pairs of adjacent edges forming a quadrilateral called the *basic quadrilateral of* T; we call the ribbons hanging from the intervals delimited by these four edges X_T, Y_T, Z_T and W_T respectively, going around counterclockwise from the point of the arrow.

Let α_T denote the elementary move on the oriented edge of T. We set

$$F(\alpha_T) = \alpha_T \cdot f(x^T_{X_T Y_T}, x^T_{Y_T Z_T}). \tag{2}$$

The profinite braid $f(x^T_{X_T Y_T}, x^T_{Y_T Z_T})$ is a morphism from T to itself, lying in $\hat{K}^T$, so $F(\alpha_T)$ is a morphism of $\hat{\mathcal{P}}_\infty$.

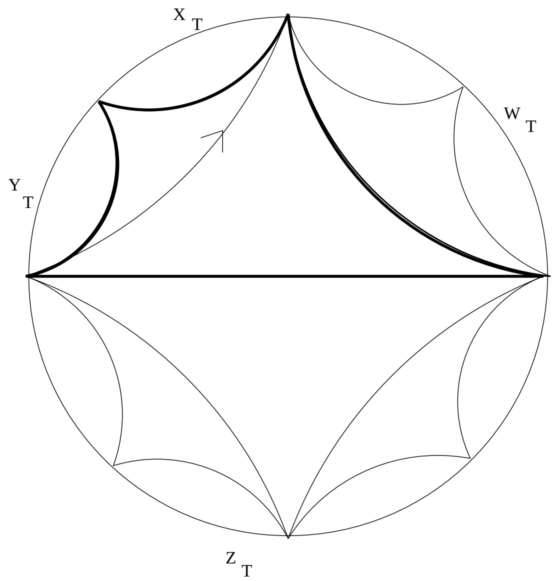

Figure 8. The basic quadrilateral of a marked tesselation

Lemma 5. *This action of $\widehat{GT}$ respects all the relations of the universal Ptolemy groupoid.*

Remark. The lemma can be restated by saying that every element of $\widehat{GT}$ is a groupoid-isorphism from the universal Ptolemy subgroupoid of $\hat{\mathcal{P}}_\infty$ to its image. This shows that $\widehat{GT}$ respects the first of the three types of relation in $\hat{\mathcal{P}}_\infty$.

Proof. The point is to check that if $F = (\lambda, f)$ is an element of $\widehat{GT}$, then its action on any two finite sequences of elementary moves leading from T to T' is the same, since in the universal Ptolemy groupoid two such sequences give the same morphism. It is equivalent to check that F respects the relations in the groupoid, and these relations were given explicitly in theorem 1. So we simply need to compute the action of F on the five relations in α and β given in the statement of theorem 1. Here we index the moves according to the tesselation they are acting on, so that the group can be recovered from the groupoid by dropping all indices.

The relation $\beta^3 = 1$ is trivially satisfied since F fixes β. It is also clear that F respects the two commutation relations since they involve braids on disjoint ribbons (or at worst, one braid is made of ribbons all of which are contained in a single ribbon of the other braid) and such braids commute in the local braid groups. As for the pentagon relation $(\alpha\beta)^5 = 1$, or equivalently $(\beta\alpha)^5 = 1$, we have $F(\beta\alpha) = \beta\alpha f(x_{XY}, x_{YZ})$, and five repeated applications of this map, together with the use of equation (1) to push all the factors of $\beta\alpha$ to the left, leave us with exactly the famous pentagon relation (III) defining $\widehat{GT}$, equal to 1.

Let us check the remaining relation, $\alpha^4 = 1$. To start with, fix a marked tesselation T_0, so that α_{T_0} and β_{T_0} are the moves shown in figures 2 and 3. Then by equation (2), $F(\alpha_{T_0}) = \alpha_{T_0}\, f(x^{T_0}_{X_{T_0}Y_{T_0}}, x^{T_0}_{Y_{T_0}Z_{T_0}})$. Let T_1, T_2 and T_3 denote the tesselations obtained from T_0 via α_{T_0}, $\alpha_{T_1}\alpha_{T_0}$ and $\alpha_{T_2}\alpha_{T_1}\alpha_{T_0}$ (i.e. α, α^2 and α^3 in the group), and write $\alpha_i = \alpha_{T_i}$, so that

$$\alpha_3\alpha_2\alpha_1\alpha_0 = 1.$$

Note that for the four tesselations T_i, we have

$$x^{T_i}_{X_iY_i} = x^{T_i}_{Z_iW_i} \text{ and } x^{T_i}_{Y_iZ_i} = x^{T_i}_{W_iX_i},$$

where X_i, Y_i, Z_i and W_i are the ribbons attached to the four intervals of the each tesselation T_i shown in figure 8. Applying F to this relation, we obtain

$$\begin{aligned}
&F(\alpha^4) = F(\alpha_3\alpha_2\alpha_1\alpha_0) = \\
&\alpha_3 f(x^{T_3}_{W_3X_3}, x^{T_3}_{X_3Y_3})\alpha_2 f(x^{T_2}_{Z_2W_2}, x^{T_2}_{W_2X_2})\alpha_1 f(x^{T_1}_{Y_1Z_1}, x^{T_1}_{Z_1W_1})\alpha_0 f(x^{T_0}_{X_0Y_0}, x^{T_0}_{Y_0Z_0}) \\
&= \alpha_3\alpha_2\alpha_1\alpha_0 f(x^{T_0}_{Y_0Z_0}, x^{T_0}_{X_0Y_0}) f(x^{T_0}_{X_0Y_0}, x^{T_0}_{Y_0Z_0}) f(x^{T_0}_{Y_0Z_0}, x^{T_0}_{X_0Y_0}) f(x^{T_0}_{X_0Y_0}, x^{T_0}_{Y_0Z_0}) \\
&= \alpha_3\alpha_2\alpha_1\alpha_0 = \alpha^4 = 1,
\end{aligned}$$

since f satisfies $f(x, y) = f(y, x)^{-1}$ by relation (I) of $\widehat{GT}$. This takes care of the relation $\alpha^4 = 1$. ♢

Braids: Let us define the action of $\widehat{GT}$ on the local groups $\hat{K}^T$. We begin by defining an action on a set of generators of $\hat{K}^T$. By (iii) of proposition 2, a set of (topological) generators of $\hat{K}^T$ is given by the braids x^T_{AB} where A and B are neighboring wide ribbons of T, i.e. (finite) unions of neighboring ribbons corresponding to neighboring edges of a finite polygon of T containing the oriented edge. An example is shown in figure 9.

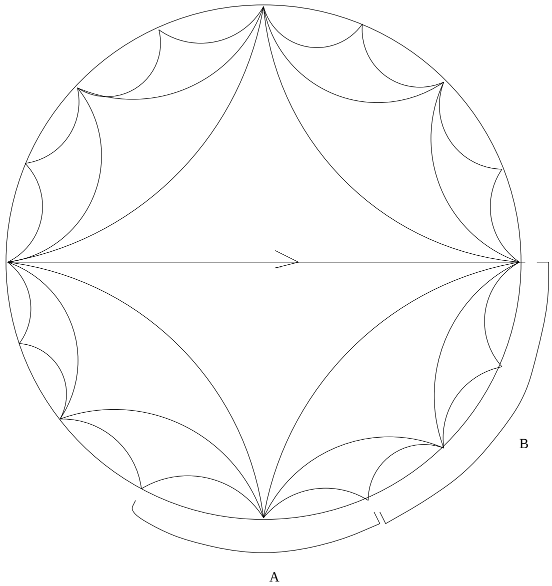

Figure 9. Neighboring intervals corresponding to wide ribbons of T

To define the action of F on all the x^T_{AB} for neighboring wide ribbons A and B of T, it suffices to define it only on the x^T_{AB} for *adjacent* (possibly wide) ribbons A and B (recall that adjacent ribbons are ribbons attached to intervals corresponding to two edges of a triangle of T) for all tesselations T; we can then extend it to pairs of neighboring wide ribbons by equation (1) and the fact that we know the action of F on morphisms of the universal Ptolemy groupoid. This works as follows. Firstly, if A and B are adjacent (possibly wide) ribbons of T we set

$$F(x^T_{AB}) = (x^T_{AB})^\lambda. \tag{3}$$

If A and B are neighboring (possibly wide) ribbons of T, i.e. (unions of) intervals corresponding to edges of a finite polygon S of T, then we change T into another tesselation T' such that the intervals A and B are two edges of a triangle of T', via a *finite number of elementary moves on T, all taking place inside S.* By writing them down explicitly and using (2) and the action of $\widehat{GT}$ on elementary moves, we can compute the explicit expression for $F(x^T_{AB})$ as follows. Choose a finite sequence of marked tesselations $T^* = T_0, \ldots, T_r$ such that

(1) for $i > 0$, T_i is obtained from T_{i-1} by one elementary morphism g_i on some edge lying inside S (not an edge of S);

(2) each T_i contains the polygon S and is identical to T^* outside of S;

(3) the ribbons corresponding to the intervals A and B are adjacent ribbons of T_r, i.e. the intervals A and B are two sides of a triangle of T_r.

Let $\gamma = g_r \circ g_{r-1} \circ \cdots g_1$; then no matter what choices we make for $T_1, \ldots, T_r$ and $g_1, \ldots, g_r$, γ is the *unique* morphism of the universal Ptolemy groupoid taking T^* to T_r. By equation (1), we have $\gamma^{-1} x^{T_r}_{AB} \gamma = X^{T^*}_{AB}$. By repeated applications of equations (1) and (2), we find an element $\eta \in K^*(0,s) \subset K^*_{T^*}$ such that $F(\gamma) = \gamma\eta$; the fact that η is well-defined is a consequence of lemma 5. Thus, the action of F on $X^{T^*}_{AB}$ when A and B are neighboring wide ribbons is given by

$$F(x^T_{AB}) = \eta^{-1}\gamma^{-1}\big(x^{T_r}_{AB}\big)^\lambda \gamma\eta = \eta^{-1}\big(x^T_{AB}\big)^\lambda \eta. \tag{4}$$

Step 2: This action of $\widehat{GT}$ on the generators of $\hat{\mathcal{P}}_\infty$ extends to a groupoid automorphism of $\hat{\mathcal{P}}_\infty$. We must check that all the relations of the groupoid are respected.

Lemma 6. *The action defined above of $\widehat{GT}$ on the generators $x^{T^*}_{AB}$ of $\hat{K}^*_{T^*}$ for all pairs of adjacent or neighboring clumps A and B of T^* determines an automorphism of $\hat{K}^*_{T^*}$.*

Proof. Let $F \in \widehat{GT}$. Then the action of F on the generators of each of the groups $\hat{K}^*(0, S_n)$ for $n \geq 2$ extends to an automorphism. Indeed, it is known (cf. [PS], chapter II) that $\widehat{GT}$ is an automorphism group of the pure mapping class group $\hat{K}(0, 2^n)$ in many ways, corresponding to the trivalent trees with 2^n edges; the action we consider here corresponds to the trivalent tree dual to the polygon S_n. Now, we have the exact sequence

$$1 \to \mathbb{Z}^{2^n} \to \hat{K}^*(0, S_n) \to \hat{K}(0, 2^n) \to 1,$$

and it is easily seen that the $\widehat{GT}$-action on $\hat{K}(0, 2^n)$ extends to an automorphism of $\hat{K}^*(0, S_n)$ simply by letting $F = (\lambda, f) \in \widehat{GT}$ act on each twist t^T_A by sending it to $(t^T_A)^\lambda$.

Now let us show that the automorphisms of each $\hat{K}^*(0, S_n)$ given by $F \in \widehat{GT}$ respect the natural inclusions $\hat{K}^*(0, S_n) \hookrightarrow \hat{K}^*(0, S_{n+1})$. Recall that S_{n+1} is obtained from S_n by subdividing each interval of S_n into two. If A and B are neighboring wide ribbons of T, then x^T_{AB} lies in $\hat{K}^*(0, S_n)$ if and only if A and B are actually supported on S_n, i.e. correspond to intervals delimited by a finite number of neighboring edges of S_n. Supposing this is the case, then of course x^T_{AB} is also supported on S_{n+1}, so x^T_{AB} also lies in $\hat{K}^*(0, S_{n+1})$, as it should since $K^*(0, S_n)$ injects into $\hat{K}^*(0, S_{n+1})$. Furthermore, in order to compute $F(x^T_{AB})$, we need to use a finite series of elementary morphisms as explained in the definition of the $\widehat{GT}$-action on braids, and they consist of moves on edges lying in S_n, and are therefore the same whether x^T_{AB} is considered as lying in $\hat{K}^*(0, S_n)$ or $\hat{K}^*(0, S_{n+1})$, so that the expression of $F(x^T_{AB})$ is not dependent on n, i.e. F respects the injection $\hat{K}^*(0, S_n) \hookrightarrow \hat{K}^*(0, S_{n+1})$. ♢

Lemma 7. *For A and B disjoint intervals of the circle and T and T' different marked tesselations, the commutativity relations $x^{T'}_{AB} = \gamma^{-1} x^T_{AB} \gamma$ of equation (1) are respected by the action of $\widehat{GT}$.*

Proof. Recall that γ is a finite chain of morphisms in the universal Ptolemy groupoid taking T' to T. In the case where A and B are actually adjacent ribbons for T, the lemma follows immediately from the definition of the action of F on the braids x^T_{AB}. If A and B are not adjacent ribbons for T, it suffices to take a third tesselation T'' such that they are adjacent ribbons for T'' and then again use the definition of F on the braids x^T_{AB} and $x^{T'}_{AB}$, by commuting them to T'' via an element of the universal Ptolemy groupoid. ♢

Lemmas 5, 6 and 7 show that the action of $\widehat{GT}$ respects all defining relations of the universal Ptolemy-Teichmüller groupoid, and this concludes the proof of theorem 4. ♢

§5. Relations with the ordered Teichmüller groupoids

Let us very briefly sketch the relationship between the universal Ptolemy-Teichmüller groupoid and the fundamental Teichmüller groupoids of genus zero moduli space. Let $\mathcal{M}_{0,n}$ denote the moduli space of Riemann spheres with n ordered marked points. The Teichmüller groupoids are the fundamental groupoids $\pi_1(\mathcal{M}_{0,n}; \mathcal{B}_n)$ of the moduli spaces $\mathcal{M}_{0,n}$ for $n \geq 4$ of genus zero Riemann surfaces with n ordered marked points, on the set $\mathcal{B}_n$ of base points *near infinity* of maximal degeneration. The use of these groupoids was suggested in [D], page 847, and their structure was investigated in detail in [PS], chapters I.2 and II. The set $\mathcal{B}_n$ is essentially described by isotopy classes of numbered trivalent trees with n leaves. Let us define *ordered* Teichmüller groupoids to be the fundamental groupoids of the moduli spaces $\mathcal{M}_{0,n}$ on a certain subset $\mathcal{C}_n$ of the base point set $\mathcal{B}_n$, so that the ordered Teichmüller groupoids are subgroupoids of the full Teichmüller groupoids. We define the base point sets $\mathcal{C}_n$ of the ordered Teichmüller groupoids to be the set of base points near infinity in $\mathcal{M}_{0,n}$ corresponding to trivalent trees whose n leaves are numbered in cyclic order $1,\ldots,n$. The set of associativity moves acts transitively on such trees, so that the paths of the ordered Teichmüller groupoids are of two types: the braids (local groups), i.e. the fundamental groups of $\mathcal{M}_{0,n}$ based at each base point, which are all isomorphic to $K(0,n)$, and associativity moves going from one base point to another.

The universal Ptolemy-Teichmüller groupoid covers all the ordered Teichmüller groupoids for $n\geq 4$ in the sense that these groupoids naturally occur as quotients of subgroupoids in many ways. Indeed, for $n\geq 4$, choose a n-sided polygon S in any given tesselation T, and consider the set of elementary paths on T which act only on edges inside the polygon. In other words, consider the finite set of tesselations T' differing from T only inside the polygon S. Now consider the set of marked tesselations obtained from these by marking any chosen edge of T not in the interior of S, and the same edge on the other tesselations differing from T only inside S. This gives a subgroupoid of $\mathcal{P}_\infty$ on a finite number of base points. Now we quotient the local group at each tesselation by suppressing all the ribbons except those attached to intervals delimited by edges of the polygon S. The quotient of K_T^* obtained in this way is exactly the pure mapping class group $K^T(S)$ on these ribbons, isomorphic to $K^*(0,n)$; thus we obtain the ordered Teichmüller groupoid as a quotient of $\mathcal{P}_\infty$. It is shown in [S1] that there is a $\widehat{GT}$-action on the profinite completion of the fundamental groupoid $\pi_1(\mathcal{M}_{0,n};\mathcal{B}_n)$; this action fixes the objects of the groupoid, i.e. the elements of $\mathcal{B}_n$, so it restricts to an action of $\hat\pi_1(\mathcal{M}_{0,n};\mathcal{C}_n)$. The relation with the main theorem of this article is that our $\widehat{GT}$-action on $\hat{\mathcal{P}}_\infty$ passes to the quotient described here, and gives exactly the usual one on $\hat\pi_1(\mathcal{M}_{0,n};\mathcal{C}_n)$.

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