

# Automorphism groups of profinite complexes of curves and the Grothendieck-Teichmüller group

P.Loachak

## Abstract

This article is primarily devoted to the study of the automorphism group  $Aut(\hat{C}(S))$  of the profinite completion of the complex of curves  $C(S)$  attached to a topological surface  $S$ . We obtain a complete description in terms of the Grothendieck-Teichmüller group. We also show that  $Aut(\hat{C}(S))$  coincides (apart from two well-known low dimensional exceptions) with the automorphism group of  $\hat{\Gamma}(S)$  the corresponding profinite Teichmüller modular group (or mapping class group). We hope that this circle of ideas can provide a natural geometric setting for further developments in Grothendieck-Teichmüller theory and related topics.

## 0. Introduction

This article can be considered a sequel to [BL], especially §4 of that paper. It uses the same objects with very similar motivations. Because of the variety of notions which come into play, it seemed expedient to gather the main definitions in an Appendix to which we refer below. We have also recalled there some known results which should help put the present text in perspective, along with this introduction and general remarks scattered in [B] in [BL]. That Appendix is of course meant to be consulted when necessary rather than read through.

Given a connected hyperbolic surface of finite type  $S$  (see §A.1, i.e. Section 1 of the Appendix), one classically associates to it a (discrete) complex of curves  $C(S)$  (§A.6) equipped with a natural action of the modular group  $\Gamma(S)$ . The profinite completion  $\hat{C}(S)$  of  $C(S)$  with respect to that action was introduced and first studied in [B] (see also §A.11). We will be mainly interested in the automorphism group  $Aut(\hat{C}(S))$  of that complex; we restrict attention as usual to *continuous* automorphisms with respect to the natural profinite topology on the group  $Aut(\hat{C}(S))$ , which is recalled below. A main point, anticipated in [BL], is to show that  $Aut(\hat{C}(S))$  is given by a version of the Grothendieck-Teichmüller group (see Theorem 4.5 and 5.12 below). More generally it seems that profinite complexes of curves provide a rich geometric setting which is very well suited to the full *profinite* Grothendieck-Teichmüller theory, the version of the theory from which all others can in principle be derived by various linearization processes (see [Lo]). In particular, just as in the discrete case (see §A.13), but in a more subtle way and a richer setting, the automorphism group of the *complex*  $\hat{C}(S)$  carries information about the automorphism groups of *every open subgroup* of  $\hat{\Gamma}(S)$ , indeed also about morphisms between these subgroups.

The present paper revolves in some sense around the following short exact sequence, which describes the automorphism group  $Aut(\hat{C}(S))$  of the profinite curves complex for  $S$  connected

hyperbolic of finite type with  $d(S) > 1$  (§A.2) and  $S$  not of type (1,2):

$$1 \rightarrow \text{Inn}(\hat{\Gamma}(S)) \rightarrow \text{Aut}(\hat{C}(S)) \rightarrow \mathcal{G}(S) \rightarrow 1. \quad (0.1)$$

First it says that  $\text{Inn}(\hat{\Gamma}(S))$  is a normal subgroup, an assertion which is far from obvious and is proved in §3 below. Granted this fact we write  $\mathcal{G}(S) = \text{Out}(\hat{C}(S)) = \text{Aut}(\hat{C}(S))/\text{Inn}(\hat{\Gamma}(S))$  and remark that the exceptional nature of type (1,2) is well understood and elementary (see §A.4 for a start). The next very nontrivial piece of information is that the group  $\mathcal{G}(S)$  is essentially independent of  $S$ , for  $S$  as above; this is completely in line with the main intuition of Grothendieck in his famous *Esquisse d'un programme* ([G]). In fact we prove in §4 that for  $g(S) = 0$ , that is for  $S$  of type  $(0, n)$ ,  $n \geq 5$ , one has  $\mathcal{G}(S) = \widehat{GT}$ , where  $\widehat{GT}$  is the original version of the Grothendieck-Teichmüller group introduced by V.Drinfeld in [D]. For  $g(S) > 0$ , more precisely when  $d(S) > 1$ ,  $g(S) > 0$  and  $\Gamma(S)$  is centerfree (i.e.  $S$  not of type (1,2) or (2,0)), we show in §5 that  $\mathcal{G}(S) = \mathbb{I}$ , where  $\mathbb{I} \subset \widehat{GT}$  is defined there. To be complete,  $\text{Out}(\hat{C}(S_{2,0})) = \widehat{GT}$  and the two exceptional types (1,2) and (2,0) can be dealt with in detail (see Proposition 5.1). The group  $\mathbb{I}$  is close to but *a priori* not identical with the groups which have been introduced in [HLS] and [NS] in order to deal with the case of strictly positive genus. The Galois group  $G_{\mathbb{Q}} = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  injects into  $\mathbb{I}$  and the morphism is canonical up to conjugacy (see also [BL] see Propositions 4.12 and 4.13).

**Remark 0.1** (on terminology): The notation  $\mathbb{I}$  comes from Grothendieck's manuscripts, especially *Longue marche à travers la théorie de Galois* and *Esquisse d'un programme*; he uses it to denote – in a rather non standard fashion – the Galois group  $G_{\mathbb{Q}}$ . We take it up here because the group defined in the present paper seems fairly universal, as will emerge below. It contains the group denoted by the same letter in [NS]. Whether or not these groups actually coincide is part of a network of questions on which we comment briefly at the end of §5.

Sequence (0.1) itself should be compared with sequence (A2) in §A.13, which is essentially due to N.Ivanov (see [Iv1]) and summarizes the situation in the *discrete* case. There instead of  $\mathcal{G}(S)$  one gets as a quotient  $\text{Out}(C(S)) = \mathbb{Z}/2 \simeq \text{Gal}(\mathbb{C}/\mathbb{R})$ . We will refer to this fundamental property as the *rigidity* of the discrete curves complex and remark that on the way we will sketch a proof of this result from a slightly different perspective.

We now have to recall or summarize the information which is already available and which we will use concerning automorphisms of profinite curves complexes. We will treat again the case where  $S$  is a connected hyperbolic surface of finite type and modular dimension  $d(S) > 1$ . The generalization to non connected surfaces is easy and the cases  $d(S) = 0, 1$  are well understood. We first recall from [BL], Lemma 4.2, that the exceptional isomorphisms between complexes of surfaces of different types are the same as in the discrete case (see e.g. [FL1] for the latter). Namely, if  $S$  and  $S'$  are two connected hyperbolic surfaces of different finite types,  $\hat{C}(S)$  and  $\hat{C}(S')$  (§§A.5,11) are *not* isomorphic, save for the following cases:  $\hat{C}(S_{1,2}) \simeq \hat{C}(S_{0,5})$  and  $\hat{C}(S_{2,0}) \simeq \hat{C}(S_{0,6})$  (recall we assume  $d(S) > 1$ ). We give a name to the first complex, which is actually a (pro)graph, writing  $\mathcal{C} = C(S_{0,5}) \simeq C(S_{1,2})$  and  $\hat{\mathcal{C}}$  for the completion. Let us now move to automorphisms of complexes. The set  $\text{Aut}(\hat{C}(S))$  of continuous automorphisms of the complex  $\hat{C}(S)$  has a natural structure of profinite group which we recall from the beginning of §4 in [BL]. The complex  $\hat{C}(S)$  is defined as the inverse limit of the finite complexes  $C^\lambda(S) = C(S)/\Gamma^\lambda$  for the system of levels  $\lambda \in \Lambda$ . A continuous

automorphism, which is also open, defines for any  $\lambda \in \Lambda$  a map  $C^\lambda(S) \rightarrow C^\mu(S)$  for some  $\mu \in \Lambda$  ( $\lambda \geq \mu$ ). When varying  $\lambda \in \Lambda$ , a basis of neighborhoods of the identity in  $Aut(\hat{C}(S))$  is given by those automorphisms which induce the natural projection; note that these open neighborhoods are not subgroups. The same description holds with  $\hat{C}(S)$  replaced by  $\hat{C}_P(S)$  and  $\hat{C}_*(S)$  (§§A.8,9,10,11) and shows that the automorphism groups of these graphs are also naturally profinite. One can define continuous morphisms between more general profinite complexes in the same fashion.

The first – easy – piece of information is the inclusion  $Aut(\hat{C}_P(S)) \subset Aut(\hat{C}_*(S))$  ([BL], Lemma 4.6) which parallels the discrete case ([BL], Lemma 2.8). Next and much less easy is the fact that the natural injective map  $Aut(\hat{C}(S)) \hookrightarrow Aut(\hat{C}_*(S))$  is an isomorphism ([BL], Theorem 4.4). Here we recall that  $C_*(S)$  is actually the 1-skeleton of the dual of  $C(S)$ . For the 1-skeleton  $C^{(1)}(S)$  itself, it is easy to prove, like in the discrete case, that there is a natural isomorphism  $Aut(\hat{C}^{(1)}(S)) \simeq Aut(\hat{C}(S))$ . The case of  $C_*(S)$  is more difficult and more interesting. In the sequel we will often identify  $Aut(\hat{C}_*(S))$  and  $Aut(\hat{C}(S))$ , sometimes at the expense of a slight abuse of notation. In particular, combining this isomorphism with the above mentioned inclusion, we get a natural injective map  $Aut(\hat{C}_P(S)) \hookrightarrow Aut(\hat{C}(S))$ . The corresponding assertion in the discrete case, namely  $Aut(C_P(S)) \hookrightarrow Aut(C(S))$  constitutes the main result of [M] (and is reproved in [BL], Theorem 2.13).

Surely the main novelty from the point of view of automorphisms, when dealing with profinite complexes, is the fact that the natural inclusion map  $Aut(\hat{C}_P(S)) \hookrightarrow Aut(\hat{C}(S))$  is *not* an isomorphism – and far from it. It is indeed an isomorphism in the discrete setting as an immediate consequence of the inclusion  $Aut(C_P(S)) \hookrightarrow Aut(C(S))$  and the rigidity of  $C(S)$ , leading immediately to  $Aut(C_P(S)) = Aut(C(S)) = Mod(S)$  (§A.3). In other words the rigidity of the discrete curves complex  $C(S)$  and the above inclusion imply the rigidity of the discrete pants complex (or graph):  $Out(C_P(S)) = Out(C(S)) = \mathbb{Z}/2$ .

Now one the main results of [BL] is that the analog essentially holds true for the profinite pants graph  $\hat{C}_P(S)$ . Namely, its automorphism group is described by the following exact sequence ([BL], Theorem 4.15):

$$1 \rightarrow Inn(\hat{\Gamma}(S)) \rightarrow Aut(\hat{C}_P(S)) \rightarrow \mathbb{Z}/2 \rightarrow 1. \quad (0.2)$$

Recall again that we have assumed  $S$  connected hyperbolic with  $d(S) > 1$  and  $S$  not of type (1,2). In fact, as explained in [BL], the result also holds true for  $d(S) = 1$  with the appropriate definitions. It breaks for  $S$  of type (1,2) if taken literally but its failure to hold true is well-understood. Modulo the appropriate and usual qualifications in the low dimensional cases, this result says that the profinite pants graph (*a fortiori* complex; see §A.8) is rigid:  $Out(\hat{C}_P(S)) = \mathbb{Z}/2$ . The importance and usefulness of that rigidity property will be amply illustrated below.

The connection between automorphisms of curves complexes and of the modular groups and their open subgroups goes roughly as in the discrete case. Namely let  $Aut^*(\hat{\Gamma}(S)) \subset Aut(\hat{\Gamma}(S))$  denote as usual the group of continuous *inertia preserving* automorphisms of the profinite modular group  $\hat{\Gamma}(S)$ . An element of that group maps a (profinite) power of a (profinite) twist to an element of the same kind (for more detail, see e.g. the introduction of [BL]). In the discrete case,  $Aut^*(\Gamma(S)) = Aut(\Gamma(S))$  and this might also be the case in the profinite setting but that is an independent and difficult question (see the end of §5 for a short discussion). Then if  $\Gamma(S)$  is

centerfree, there is a natural injection  $Aut^*(\hat{\Gamma}(S)) \hookrightarrow Aut(\hat{C}(S))$ . More generally, if  $\Gamma^\lambda \subset \Gamma(S)$  is a finite index subgroup, the same conclusion applies, with a natural injection  $Aut^*(\hat{\Gamma}^\lambda) \hookrightarrow Aut(\hat{C}(S))$  ([BL], Proposition 4.11). For  $d(S) > 1$ , this leaves aside the cases of types (1, 2) and (2, 0) with a center of order 2, which can easily be treated in a specific way (see §5 for detail). The upshot is that a description of  $Aut(\hat{C}(S))$  also essentially provides a description of the groups  $Aut^*(\hat{\Gamma}^\lambda)$  for all  $\lambda \in \Lambda$ . In fact, this shows that for  $d(S) > 1$  the Grothendieck-Teichmüller group uniformly describes the automorphism groups of all the open subgroups of  $\hat{\Gamma}(S)$  (see §§4,5).

As a last item in this introduction, we will develop a geometric picture which is essentially vindicated below and may be helpful in order to get the overall meaning of the present paper. It demonstrates among other things how close we are, at least in spirit, to the original intuitions developed in [D] (in the case of genus 0 and for pronilpotent completions) and which lead to the introduction of the Grothendieck-Teichmüller group. That picture stems from profinite geometry and thus from the use of profinite complexes of curves. These carry a kind of homotopical information at infinity for the inverse system of the finite covers of a given moduli stack  $\mathcal{M}(S)$ . This in itself can make it plausible that although they have no group structure *a priori*, their automorphism groups do enjoy amazing rigidity properties. In a way, these complexes enable one to work directly with covers rather than with fundamental groups, as far as moduli spaces of curves are concerned. Note that different or partial completions of the complexes of curves and related objects are used in particular in [HLS] and [NS] and that the so-called universal setting for Teichmüller theory has also been considered from that angle, especially in [Pe], [LS2] and [FK]. It can perhaps be said that these other partial completions are best suited for studying the moduli spaces themselves (and thus the full modular groups), rather than take the whole tower of the finite covers into account, as we do here, which in a way amounts to developing a kind of algebraic substitute for Teichmüller theory. Perhaps the universal viewpoint could also be reconsidered from the present perspective.

So let  $S$  be connected hyperbolic of finite type and introduce:

**Definition 0.2:** An *oriented embedding* is a  $\hat{\Gamma}(S)$ -orbit for the natural left action of  $\hat{\Gamma}(S)$  on the set of injective maps  $j : \hat{C}_P(S) \hookrightarrow \hat{C}_*(S)$ . The set of oriented embeddings is denoted  $J(S)$ .

Quite explicitly, we thus identify two embeddings  $j$  and  $j' = g \circ j$ ,  $g \in \hat{\Gamma}$ . Let us now recall that, as explained in [BL],  $\hat{C}_P(S)$  is equipped (contrary to  $\hat{C}_*(S)$  or  $\hat{C}(S)$ ) with a natural orientation, once the surface  $S$  has itself been given an orientation, which we assume once and for all. It is thus fairly natural to define the set of oriented embeddings by identifying two injective maps  $j_1$  and  $j_2$  as above if the composite map  $j_1^{-1} \circ j_2$  lies in  $Aut^+(\hat{C}_P(S))$ , the group of oriented automorphisms of  $\hat{C}_P(S)$ . Now assume that  $d(S) > 1$ , and that  $S$  is not of type (1, 2) in order to avoid some niceties. Then the rigidity statement from [BL] quoted above states that any such automorphism is induced by an element of the modular group:  $Aut^+(\hat{C}_P(S)) \simeq Inn(\hat{\Gamma}(S))$ . Recalling that  $\hat{\Gamma}(S)$  acts on  $\hat{C}(S)$  via  $Inn(\hat{\Gamma}(S))$ , we find that the above two definitions coincide. As usual the one dimensional cases and the peculiarities of type (1, 2) are easily understood.

Next, there is a natural action of  $Aut(\hat{C}(S))$  on  $J(S)$  obtained by first identifying (sometimes implicitly in the sequel)  $Aut(\hat{C}(S))$  with  $Aut(\hat{C}_*(S))$  and then postcomposing:  $\phi \cdot j = \phi \circ j$  for  $j \in J$ ,  $\phi \in Aut(\hat{C}_*(S))$ . The action factors through the quotient  $Aut(\hat{C}(S))/\hat{\Gamma}(S) = Out(\hat{\Gamma}(S))$  of  $Aut(\hat{C}(S))$  by  $\hat{\Gamma}(S)$ , where  $\hat{\Gamma}(S)$  acts effectively via  $Inn(\hat{\Gamma}(S))$ , provided  $\hat{\Gamma}(S)$  is normal in

$Aut(\hat{C}(S))$ , which is shown in §3 below. The upshot is that the natural action of  $\mathcal{G}(S) = Out(\hat{C}(S))$  on  $J(S)$  should be free and transitive, and indeed we have:

**Proposition 0.3:** *If  $d(S) > 1$  and  $S$  is not of type  $(1, 2)$ ,  $J(S)$  is a  $\mathcal{G}(S)$ -torsor.*

*Proof:* The group  $\mathcal{G}(S)$  acts on  $J(S)$  as explained above and the action is faithful by the definition of  $J(S)$ . There remains to show that it is also transitive, which is equivalent to showing that any embedding  $j : \hat{C}_P(S) \hookrightarrow \hat{C}_*(S)$  defines a unique automorphism of  $\hat{C}_*(S)$ . Since  $\hat{C}_P(S)$  and  $\hat{C}_*(S)$  have the same set  $\hat{V}$  of vertices (§A.10), namely the completion of the set  $V$  of maximal multicurves (pants decompositions),  $j$  defines an automorphisms of the vertices of  $\hat{C}_*(S)$  which has to extend to the edges of that graph. The extension, if it exists, is unique, since  $\hat{C}_*(S)$  is a flag complex (there is at most one edge between two vertices). But then the problem is local, that is it can be reduced to modular dimension 1, in which case it boils down to the easy assertion that an embedding  $\hat{F} \hookrightarrow \hat{G}$  (see §§A.9,11) can be extended to an automorphism of the profinite complete graph on  $\hat{G}$ ; see also [BL] §4 for detailed proofs of similar facts. As usual type  $(1, 2)$  is only a mild exception.  $\square$

Now in fact there is a privileged point in  $J(S)$ , say  $j_0$ , which is just the completion of the topological embedding  $C_P(S) \hookrightarrow C_*(S)$ . So we have a torsor which is “naturally” trivialized and which for most purposes can be identified with the group  $\mathcal{G}(S)$  itself, which in turn we will identify with (versions of) the Grothendieck-Teichmüller group.

How does the above compare with the situation in [D], which gave rise to the original definition of the Grothendieck-Teichmüller group? In [D], §4, the pronilpotent genus 0 version of the Grothendieck-Teichmüller group  $GT(k)$  ( $k$  a field of characteristic 0) appears via universal deformations of quasi-Hopf quasitriangular (or braided) universal enveloping algebras. The profinite version  $\widehat{GT}$  is then introduced by analogy (top of p.846). Note that the pronilpotent (or pronilpotent) version is not deduced from the profinite version via the natural (functorial) procedure described e.g. in [De], §9. In the first place  $\widehat{GT}$ , like  $G_{\mathbb{Q}}$ , is of course not the profinite completion of a discrete group; in fact its discrete part reduces to  $\mathbb{Z}/2$  ([D], Proposition 4.1) and this is indeed what one finds here in the discrete setting (see e.g. (A2)). The point we would like to make here is that we are in fact exploring a deformation theory in the full *profinite* geometric setting. How can one interpret  $J$  as a set of deformations? Classically, if  $\alpha$  and  $\beta$  are simple (isotopy classes of) curves on a surface we say that they have *minimal intersection* if either they are supported on a subsurface of type  $(1, 1)$  and intersect at 1 point only, or they are supported on a subsurface of type  $(0, 4)$  and intersect at 2 points. This topological definition is the essential ingredient in the definition of the graph  $C_P(S)$  (see §A.8). That graph *and* its profinite completion are essentially rigid by the results of [M] (or [BL], §2) in the discrete setting and by the result recalled above ([BL], Theorem 4.15) in the profinite case. However  $\hat{C}(S)$  turns out to have a lot of (non inner) automorphisms and these will be parametrized by the *profinite deformations of the minimal intersection rule*. Note again that the graphs  $\hat{C}_P(S)$  and  $\hat{C}_*(S)$  share the same set  $\hat{V}$  of vertices (see §§A.8,9) which is nothing but the completion of the set  $V$  of pants decompositions of the surface  $S$ . We will see in §3 below that a kind of *transversality* property obtains, that is, for two embeddings  $j, j' \in J$ , either their images coincide or they have *no* edge in common. An embedding is thus entirely specified by giving one of its edges, and such an edge deserves to be called

a rule for minimal intersection. The topological rule recalled above corresponds to the topological embedding  $j_0$ . In some sense much of the mystery is (still) hidden in the profinite Farey graph  $\hat{F}$ .

So what profinite complexes of curves are doing for us here is to enable one to develop a *profinite geometric deformation theory* in all genera. The set  $J$  closely corresponds to the set of associators (see [D], §§5, 6 and e.g. [R]). However we are in the full profinite setting, which is far from the motivic, that is here the prounipotent – or pronilpotent – setting. Motivic parlance is thus not very useful here as there is so to speak only a Betti side, thus no analog of the famous  $KZ$ -associator (see [D] and [R]) involving iterated integrals and the polyzeta values, and which also embodies the Betti-de Rham comparison isomorphism. Instead, the topological (Betti) embedding  $j_0$  plays the role of the the “natural” associator represented simply by the open interval  $(0, 1)$ . This may look a little disappointing but probably should not be taken negatively. After all, the profinite “nonlinear” world is just different (see [Lo] for a partial review of some of these differences). It would however surely be interesting to develop part of what we do here in a pro- $\ell$  setting. Lots of constructions can be transposed literally but hopefully there may arise some new and surprising features.

*Acknowledgements:* It is a great pleasure to thank Leila Schneps and Marco Boggi for numerous if not innumerable conversations on various topics related to the themes of the present paper. I warmly thank Wolfgang Herfort for telling me of his joint result with Luis Ribes, which is used in the proof of Lemma 2.6, as well as Ivan Marin for enlightening discussions about generalized Artin braid groups. Finally, I am very grateful to the Mathematics Department of Tel Aviv University and especially Prof. M.Jarden for a very nice and interesting stay, during which part of the present article was completed.

## 1. Injectivity, induction and the two levels principle

In this section we will as much as possible reduce the problem to the case when  $S$  has modular dimension 2. That this is at all plausible stems from a remarkable intuition of Grothendieck in [G] which is known as the two levels principle (“principe des deux premiers étages”). It already has several different incarnations. In particular, the very existence and definition (in [D]) of the Grothendieck-Teichmüller group finds its origin there. We postpone to the end of the section some further observations about the results described below.

Given a surface  $S$ , we define a *subsurface*  $T$  as  $T = S \setminus \sigma$  where  $\sigma \in C(S)$ ; we denote it  $S_\sigma$  and it is nothing but  $S$  cut or slit along the multicurve representing  $\sigma$ . In this definition, the curves are defined as usual up to isotopy. By a *piece* of a surface we will intend a connected component of a subsurface. We will always assume that such a piece  $S' \subset S$  is incompressible, that is each boundary circle of  $S'$  is an essential loop (an element of  $\mathcal{S}(S)$ ). Associated to a subsurface  $S_\sigma$  is a natural inclusion  $C_*(S_\sigma) \subset C_*(S)$ :  $C_*(S_\sigma)$  is the full subgraph of  $C_*(S)$  whose vertices correspond to those pants decompositions of  $S$  which include  $\sigma$ . Combined with the results of [BL], this inclusion is one of the features which makes the use of  $C_*(S)$  (and  $\hat{C}_*(S)$ ) convenient and flexible. A similar description clearly holds for the pants graph of a subsurface. It will also be useful to recall that any cell of  $\hat{C}(S)$  lies in the  $\hat{\Gamma}(S)$ -orbit of a cell of the discrete complex  $C(S)$  (see [BL], especially the beginning of §4); the finitely many orbits are in one-to-one correspondence with the

$\Gamma(S)$ -orbits in the discrete case ([BL], Lemma 3.6) and determine the *topological type* of the cells.

For  $\sigma \in C(S)$ , we let  $|\sigma|$  denote the number of curves which constitute  $\sigma$ . So  $|\sigma| = \dim(\sigma) + 1$  if  $\dim(\sigma)$  denotes the dimension of  $\sigma \in C(S)$ . The quantity  $|\sigma|$  simply turns out to be more convenient in our context; in particular we have  $d(S_\sigma) = d(S) - |\sigma|$ . We include throughout the case of an empty cell (of dimension  $-1$ ):  $S_\emptyset = S$ . We refer to §A.10 (or [BL]) for a few remarks on non connected surfaces. For example if  $\sigma$  is a maximal multicurve (pants decomposition),  $S_\sigma$  is a disjoint union of *trinions* (pants) and  $C_*(S_\sigma)$  is reduced to a point. We say that two simplices  $\rho, \sigma \in C(S)$  are *compatible* if the curves which compose  $\rho$  and  $\sigma$  do not intersect properly, that is they are either disjoint or coincide. Complex theoretically it means that  $\rho$  and  $\sigma$  lie in the closure of a common higher dimensional simplex of  $C(S)$ . If  $\rho$  and  $\sigma$  are compatible, we define their union and intersection  $\rho \cup \sigma, \rho \cap \sigma \in C(S)$  in the obvious way.

We can now rephrase our goal in this section as a comparison between the groups  $Aut(\hat{C}(S))$  and  $Aut(\hat{C}(S_\sigma))$ , for  $\sigma \in C(S)$ . We have that  $d(S_\sigma) = d(S) - |\sigma|$  is the sum of the dimensions of the components and we will have to assume that there is at least one piece (connected component of  $S_\sigma$ ) with dimension at least 2. This may sound a little awkward but is in fact completely in line with the overall philosophy: the complexes associated to surfaces of dimension 1 simply do not carry enough structure. More technically, this is also optimal: if for instance a surface of type  $(0, 6)$  is cut into two pieces of type  $(0, 4)$ , the automorphism groups attached to the surface and the subsurface are wildly different. By induction we can restrict attention to the case when  $\sigma$  consists of just one curve, in which case we use the notation  $\alpha \in \mathcal{S}$  rather than  $\sigma \in C(S)$ . Now if  $d(S) > 2$  and  $S_\alpha$  is connected, that is  $\alpha$  is non separating, we get  $d(S_\alpha) > 1$  and we expect an injectivity statement of the sort stated in Theorem 1.2 below. But if for instance  $S$  has genus 0, all curves are separating. Complexes suggest what seems to be the right notion:

**Definition 1.1:** Given a connected surface  $S$ , a curve  $\alpha \in \mathcal{S}$  is called *complex theoretically non separating* (CTNS) if either it is non separating in the usual sense ( $S_\alpha$  is connected) or one of the two components of  $S_\alpha$  is a trinion (i.e. of type  $(0, 3)$ ).

The rationale for the above definition and the denomination is as follows. If  $\alpha$  is CTNS, then either  $S_\alpha$  is connected or  $S_\alpha = S' \amalg S''$  with – say –  $S''$  a trinion. Then it is not quite true that  $C_*(S_\alpha) \simeq C_*(S')$  but it *is* true that these complexes and their profinite completions have the same automorphisms if  $d(S) > 1$ . In fact  $S''$  just adds one point which is connected to all vertices (see [BL] or §A.10) and is left fixed by any automorphism, provided  $d(S) > 1$ . So we find that in both cases  $Aut(\hat{C}(S_\alpha)) = Aut(\hat{C}(S'))$  is the automorphism group associated to a connected surface with one less dimension. Of course this holds true for the discrete groups well. This can be extended to non connected surfaces in a fairly obvious way but we will not require such an extension.

So the notion of CTNS curve seems to be the right one when dealing with the general case. It is also interesting to check that it is in accordance with the exceptional isomorphisms in dimensions 1, 2 and 3, namely:  $C(S_{0,4}) \simeq C(S_{1,1})$ ,  $C(S_{0,5}) \simeq C(S_{1,2})$  and  $C(S_{0,6}) \simeq C(S_{2,0})$ . In the one dimensional case all curves are CTNS, and so it is in dimension 2, including the *separating* curves for type  $(1, 2)$ . Then in dimension 3, the separating curves for type  $(2, 0)$ , which are the only non CTNS curves, break the surface into two pieces of type  $(1, 1)$  and correspond to the only type in  $(0, 6)$  which is *not* CTNS, namely those curves which cut the surface into two pieces of type  $(0, 4)$ .

We are ready to state the first main result:

**Theorem 1.2:** *Let  $S$  be a connected hyperbolic surface of finite type with  $d(S) > 1$  and let  $\alpha \in \mathcal{S}(S)$  be a complex theoretically non separating curve. Let then  $F \in \text{Aut}(\hat{C}(S))$  be an automorphism of the profinite curves complex fixing the curve  $\alpha$ . If  $F$  restricts to the identity on  $\hat{C}(S_\alpha)$ , then  $F \in \langle \tau_\alpha \rangle$ , the procyclic group generated by the twist along  $\alpha$ .*

Here we view  $\langle \tau_\alpha \rangle$  as a subgroup of  $\text{Aut}(\hat{\Gamma}(S))$ , hence also as a subgroup of  $\text{Aut}(\hat{C}(S))$ ; note that even if  $\hat{\Gamma}(S)$  has non trivial center,  $\langle \tau_\alpha \rangle$  injects into  $\text{Aut}(\hat{\Gamma}(S))$ . In the statement itself we have used quite a few non trivial results from [B] and [BL]. We have implicitly identified  $\text{Aut}(\hat{C}(S))$  with  $\text{Aut}(\hat{C}_*(S))$ . Then we use the natural inclusion  $C_*(S_\sigma) \hookrightarrow C_*(S)$ , the resulting completed map between the completed graphs and especially the deep fact that the closure of  $C_*(S_\sigma)$  in  $C_*(S)$  is isomorphic to the full completion of  $C_*(S_\sigma)$  (see [B]). One should also recall that it is shown in [BL], Proposition 4.1, that automorphisms of  $\hat{C}(S)$  preserve the topological type of curves; this is true but empty in dimension 1 and it breaks only for  $S$  of type (1,2), in the usual, mild and well-understood way. We may however leave this case aside anyway because of the isomorphism  $C(S_{1,2}) \simeq C(S_{0,5})$ . Then starting from an arbitrary  $F$ ,  $F(\alpha) = g \cdot \alpha$  for some  $g \in \hat{\Gamma}(S)$  and replacing  $F$  by  $F' = g^{-1} \circ F$  the statement can be tested on  $F'$ .

This is a typical injectivity result (whose proof will be complete by the end §2) and yet another embodiment of the two levels principle, perhaps more geometric or conceptually more satisfying than the existing results. For instance, consider a group automorphism  $F \in \text{Aut}^*(\hat{\Gamma}(S))$ , let  $\alpha$  be a CTNS curve and assume that  $F$  restricts to the identity on the centralizer of the corresponding twist  $\tau_\alpha$ . Then  $F$  induces an automorphism of the complex  $\hat{C}(S)$  and applying the statement above we find that  $F$  is just conjugacy by a power of  $\tau_\alpha$ . This would work with an inertia preserving automorphism of any open subgroup of  $\hat{\Gamma}(S)$ , using a finite power of  $\tau_\alpha$  lying in that subgroup (which of course always exists). Here we are again implicitly using the results of [B], §7 about centralizers of twists.

Let  $(C_d)$  denote the assertion of the Theorem for  $d(S) = d$ . Note that  $(C_0)$  is empty, whereas  $(C_1)$  is false. It turns out that, contrary to what happens with most inductive proofs, settling the initial case  $(C_2)$  is just as difficult as completing the inductive step. In this section we prove the inductive step. The initial case  $(C_2)$  will be dealt with in the next section. In other words, for the time being we are interested in showing:

**Proposition 1.3:** *Assertion  $(C_{d-1})$  implies  $(C_d)$  for  $d > 2$ .*

The proof uses a topological property, which we first proceed to state. For a connected surface  $S$ , let  $C_{ctns}(S) \subset C^{(1)}(S)$  be the full subgraph of the 1-skeleton of  $C(S)$  whose vertices are given by the CTNS curves. In other words, the vertice are all CTNS curves and two vertices are joined by an edge if the corresponding curves do not intersect. We will also consider a subgraph of  $C_{ctns}(S)$ , say  $C_{ctns}^\vee(S)$ , which is defined as follows. Recall that a pair of curves  $\alpha, \beta \in \mathcal{S}$  is called a *cut pair* if neither of the two curves is separating but their union is. We let  $C_{ctns}^\vee(S)$  be the subgraph of  $C_{ctns}(S)$  obtained by removing all edges corresponding to cut pairs. We need the following:

**Proposition 1.4:** *The graph  $C_{ctns}(S)$  is connected if  $d(S) > 1$ ; its subgraph  $C_{ctns}^\vee(S)$  is connected if  $d(S) > 1$  and  $S$  is not of type (1,2).*

*Proof of Proposition 1.3, granted Proposition 1.4:* We start from  $F \in \text{Aut}(\hat{C}(S))$  with  $S$  connected hyperbolic of finite type and of dimension  $d = d(S) > 2$ . We assume that  $F$  fixes  $\alpha$  and restricts to the identity on  $\hat{C}(S_\alpha)$  for a CTNS curve  $\alpha$ . We write  $S'_\alpha$  for the connected component of  $S_\alpha$  which is of dimension of  $d - 1$ ; in particular, if  $\alpha$  is non separating in the ordinary sense of the word,  $S'_\alpha = S_\alpha$ . Consider now  $\beta$ , a curve such that  $\alpha$  and  $\beta$  are joined by an edge in  $C_{ctns}^\vee(S)$ , that is  $\beta$  is also CTNS with respect to  $S$ , it is disjoint from  $\alpha$  and  $(\alpha, \beta)$  is not a cut pair. Then we can consider the subsurface  $S_{\alpha, \beta}$  obtained by cutting  $S$  along  $\alpha$  and  $\beta$ , which is the same as cutting  $S'_\alpha$  along  $\beta$  because  $\beta$  cannot live on a trinion if there is any. Recalling that  $d(S) > 2$ , let  $S'_{\alpha, \beta}$  be again the only connected component of  $S_{\alpha, \beta}$  which is not a trinion. Now reverse the order of the operations, denoting  $S'_\beta$  the component of  $S_\beta$  not a trinion. The surface  $S'_\beta$  has dimension  $d - 1$  and  $S'_{\alpha, \beta}$  is nothing but  $S'_\beta$  slit along  $\alpha$ , possibly discarding as usual a chopped off trinion.

We wish to apply the induction hypothesis to the surface  $S'_\beta$ , of dimension  $d - 1$ , and curve  $\alpha$ . First  $\alpha$  is indeed CTNS with respect to  $S'_\beta$ . In order to check this, two cases are to be excluded, bearing in mind of course that  $\alpha$  and  $\beta$  are disjoint and that  $\alpha$  is CTNS with respect to the original surface  $S$ . First  $\alpha$  could cut off a trinion from  $S$  but not from  $S'_\beta$ ; however this is clearly impossible. Second  $\alpha$  could be non separating for  $S$  but separating for  $S'_\beta$ ; but then  $\beta$  has to be non separating for  $S$  and  $(\alpha, \beta)$  would be a cut pair for  $S$ , a case which is excluded by assumption. So we can indeed apply the inductive hypothesis to the pair  $(S'_\beta, \alpha)$  and we get that, after possibly composing by a (profinite) power of the twist along  $\alpha$ , the automorphism  $F$ , which fixes  $\beta$  because  $\beta \in C(S_\alpha)$ , actually fixes  $\hat{C}(S_\beta)$ ; that is it fixes all (pro)curves that are disjoint from  $\beta$ .

Next we may replace  $\alpha$  by  $\beta$  and do as above, except that now we clearly do not have to correct  $F$  by a twist along  $\beta$ . Proceeding in this way and applying Proposition 1.4 we conclude that  $F$  restricts to the identity on  $\hat{C}(S_\gamma)$  for *any* CTNS curve  $\gamma \in \mathcal{S}$ . Finally let  $\delta \in \mathcal{S}$  be a curve which is *not* CTNS; by considering the subsurface  $S_\delta$ , one finds a CTNS curve  $\gamma$  disjoint from  $\delta$ . As a result  $\delta \in C(S_\gamma)$  and so is fixed by  $F$ . So  $F$  is the identity on  $C(S)$  which is dense in  $\hat{C}(S)$ , hence  $F = 1$ , finishing the proof of the proposition, modulo Proposition 1.4.  $\square$

We now return to the:

*Proof of Proposition 1.4:* The statement is classical in the sense that quite a few complexes derived from  $C(S)$  have been introduced and shown to be connected. Let us start with the low dimensional case. If  $d(S) = 1$ ,  $C_{ctns}^\vee(S) = C_{ctns}(S)$  is the full complex  $C(S)$ , which is not connected. If  $S$  is of type  $(0, 5)$ , again  $C_{ctns}^\vee(S_{0,5}) = C_{ctns}(S_{0,5}) = C(S_{0,5}) = \mathcal{C}$  which is connected. When  $S$  has type  $(1, 2)$ , one still has the equality  $C_{ctns}(S_{1,2}) = C(S_{1,2}) = \mathcal{C}$  so that  $C_{ctns}(S_{1,2})$  is connected. But  $C_{ctns}^\vee(S_{1,2})$  is not, because one cannot join the two elements of a cut pair inside it.

Next, for  $d(S) > 2$  we first reduce to the first assertion of the proposition. This reduction amounts to showing that with this dimensionality assumption two elements of a cut pair *can* be joined inside  $C_{ctns}^\vee(S)$ . So let  $(\alpha, \beta)$  be a cut pair. The subsurface  $S_{\alpha, \beta}$  has two components  $S'_{\alpha, \beta}$  and  $S''_{\alpha, \beta}$ , whose dimensions add up to  $d(S) - 2 > 0$ . So at least one of them, say  $S'_{\alpha, \beta}$ , has strictly positive dimension. If  $S'_{\alpha, \beta}$  has strictly positive genus, take  $\gamma$  a nonseparating curve of  $S'_{\alpha, \beta}$  and the path  $(\alpha, \gamma, \beta)$  connects  $\alpha$  to  $\beta$  inside  $C_{ctns}^\vee(S)$ . If on the other hand  $S'_{\alpha, \beta}$  has genus 0, take  $\gamma$  on  $S'_{\alpha, \beta}$  bounding a trinion not containing  $\alpha$  and  $\beta$ . Again the path  $(\alpha, \gamma, \beta)$  then connects  $\alpha$  to  $\beta$  inside  $C_{ctns}^\vee(S)$ .

So we are now reduced to proving that the graph  $C_{ctns}(S)$  is connected if  $d(S) > 1$  (the case  $d(S) = 2$  was actually dealt with above). We will use a technique which in essence is fairly classical (see in particular [MS]). One could however use other and in fact more powerful approaches; in particular, “Putnam’s trick”, which provides a general way of showing the connectivity of complexes with group actions, does apply to the present situation (see [Pu]).

Curves cutting off trinions will be called *trinion curves* for short. We write as usual  $i(\alpha, \beta)$  for the intersection number of the two isotopy classes of curves  $\alpha$  and  $\beta$  and assume (often implicitly) that *tight* representatives have been chosen, that is actual curves whose geometric intersection number is exactly  $i(\alpha, \beta)$ . We use induction on  $i(\alpha, \beta)$ , the cases  $i(\alpha, \beta) = 0, 1$  being easy. So, given  $\alpha, \beta \in C_{ctns}(S)$  with  $i(\alpha, \beta) > 1$  we need only find  $\gamma \in C_{ctns}(S)$  with  $i(\alpha, \gamma) < i(\alpha, \beta)$  and  $i(\beta, \gamma) < i(\alpha, \beta)$ ; we will actually achieve more. In order to construct  $\gamma$  we proceed as follows. First we define a *wave*  $\beta'$  of  $\beta$  (w.r.t.  $\alpha$ ) as a segment of  $\beta$  between two intersection points, that is  $\beta' \cap \alpha = \partial\beta'$  (we borrow the terminology from [MS] but give it a looser meaning: we do not require any additional property on  $\beta'$ ). Now start from  $\alpha$  and a wave  $\beta'$  of  $\beta$ . Let  $\alpha'$  and  $\alpha''$  be the two segments of  $\alpha$  cut out by  $\beta'$  ( $\alpha = \alpha' \cup \alpha''$ ). We may now find close translates of  $\alpha'$  and  $\beta'$ , say  $\alpha'_\varepsilon$  and  $\beta'_\varepsilon$  such that one can tie them together into (the isotopy class of a simple closed curve)  $\gamma = \alpha'_\varepsilon \cup \beta'_\varepsilon$  and one has either  $i(\alpha, \gamma) = 0$  and  $i(\beta, \gamma) \leq i(\alpha, \beta) - 2$  or  $i(\alpha, \gamma) = 1$  and  $i(\beta, \gamma) \leq i(\alpha, \beta) - 1$ .

Next, this  $\gamma$  can be essential or not and it can be CTNS or not. If it is inessential, then we can replace  $\alpha'$  by  $\alpha''$  in the above and get  $\delta = \alpha''_\varepsilon \cup \beta'_\varepsilon$ . If both  $\gamma$  and  $\delta$  are inessential, then since  $\alpha$  itself is essential, it has to be a trinion curve. In particular it is separating and we now choose another wave of  $\beta$ , not situated on the side of the trinion cut out by  $\alpha$ . Because  $d(S) > 1$ , so that  $S$  is not of type  $(0, 4)$ , we can proceed as above and find an essential curve. We thus may and will assume that the original  $\gamma = \alpha'_\varepsilon \cup \beta'_\varepsilon$  is essential. Now if  $\gamma$  is non separating or is a trinion curve we are done. So we assume that  $\gamma$  is separating and not a trinion curve; let  $S'$  and  $S''$  the two surfaces cut out by  $\gamma$ , labeled in such a way that the segment  $\alpha''$  is contained in  $S''$ . We can now select a wave  $\beta_1$  of  $\beta$  located in  $S'$ . Indeed if no such wave exists, then  $i(\alpha, \beta) = 2$  and since  $\gamma$  is not a trinion curve we can find a CTNS curve on  $S'$ , which is thus disjoint of  $\alpha$  and  $\beta$  and connects these two curves in  $C_{ctns}(S)$ . We now construct  $\gamma_1$  essentially as above, taking into account the simplifying fact that  $\gamma$  is separating. The wave  $\beta_1$  determines two complementary segments  $\gamma'$  and  $\gamma''$  of  $\gamma$ . Consider one of them, say  $\gamma'$ , and the union  $\gamma' \cup \beta_1$ . Construct a sufficiently small regular tubular neighborhood of the latter curve and let  $\gamma_1$  be the connected component of the boundary contained in  $S'$ . If  $\gamma_1$  is not essential, use  $\gamma''$  instead of  $\gamma'$ ; one of the two curves has to be essential because  $\gamma$  is not a trinion curve. Then we find that  $\gamma_1$  is essential, disjoint from  $\alpha$  ( $i(\alpha, \gamma_1) = 0$ ) and that  $i(\beta, \gamma_1) \leq i(\alpha, \beta) - 3$ . This completes the proof of Proposition 1.4, hence also of Proposition 1.3. Clearly the above proof is algorithmic. In particular one gets an upper bound on the distance from  $\alpha$  to  $\beta$  in  $C_{ctns}(S)$  which is linear as a function of their intersection number  $i(\alpha, \beta)$ .  $\square$

We have thus completed the inductive step of Theorem 1.2 but are yet to prove the two dimensional case, which is the main object of the next section. Assuming that statement for the time being, we will state and prove a consequence of Theorem 1.2 which geometrically embodies

the uniqueness part of the two levels principle. Although this is actually a corollary, we state it as a theorem, given its independent interest:

**Theorem 1.5:** *Let  $S$  be connected of finite type and let  $F, F' \in \text{Aut}(\hat{C}(S))$  be two automorphisms. If  $F$  and  $F'$  coincide on a piece  $T \subset S$  with  $d(T) > 0$  they differ by an element of  $\hat{\Gamma}(S)$ .*

*Proof* (granted Theorem 1.2): Considering  $F'^{-1} \circ F$  we can assume that  $F'$  is the identity, that  $F$  restrict to the identity on  $T$  and we wish to show that  $F \in \hat{\Gamma}(S)$ . If  $d(S) = 0, 1$ , there is nothing to prove. For  $d(S) > 1$ , we may for instance proceed by descending induction on the dimension of  $T$ , the statement being certainly true if  $d(T) = d(S)$ , that is  $T = S$ . The injection  $i : T \hookrightarrow S$  induces a map  $i_* : C(T) \rightarrow C(S)$  which is not necessarily a monomorphism (although it is on the vertices). By the results of [B], the closure of the image  $i_*(C(T))$  in  $\hat{C}(S)$  coincides with  $i_*(\hat{C}(T))$ , the image of the completion of  $C(T)$  via the completed map (still denoted  $i_*$  for simplicity). Let us now complete the relative boundary  $\partial T \setminus \partial S$ , which is a cell of  $C(S)$ , into a pants decomposition  $\sigma$  of  $S$ , that is a top dimensional cell. After twisting by an element of  $\hat{\Gamma}(S)$ , we may assume that  $F$  fixes  $\sigma$  because by the results of [BL],  $F$  preserves the topological type of cells. Consider now a trinion  $T'$  cut off by  $\sigma$  which is adjacent to  $T$ , that is has at least a boundary curve  $\alpha$  in common. Then apply Theorem 1.2 to the connected surface  $T \cup T'$ ; after possibly twisting along  $\alpha$ , we get a new automorphism which restricts to the identity on  $T \cup T'$  and differs from the original  $F$  by an element of  $\hat{\Gamma}(S)$ . This completes the induction and thus the proof (modulo again the completion of the proof of Theorem 1.2).  $\square$

## 2. The two dimensional case: a pentagonal story

This section is devoted to the two dimensional case, which governs the local structure of the curves complexes and their automorphisms. Note that it is clear from the definitions (see §§A.8,9,11) that for  $d(S) = 1$  (types  $(0, 4)$  and  $(1, 1)$ ) the curves complex  $C_*(S)$  not carry enough structure to control the corresponding automorphism group. However, again in dimension 1, both the pants graph, *alias* the Farey tessellation  $F$  and its completion  $\hat{F}$  are rigid:  $\text{Out}(F) = \text{Out}(\hat{F}) = \mathbb{Z}/2$ . A second observation is that  $C_*(S_{0,4})$  and  $C_*(S_{1,1})$  are isomorphic via an isomorphism which induces an isomorphism on the corresponding pants graphs (and ditto for the completions). That this is *not* the case for  $d(S) = 2$  will play a role and be discussed in §5 below.

We consider  $S$  with  $d(S) = 2$ ; because of the isomorphism  $C(S_{0,5}) \simeq C(S_{1,2}) (= \mathcal{C})$  and the corresponding isomorphism between the completed complexes, in this section we may and do assume that  $S$  has type  $(0, 5)$ . We write either  $C(S)$ ,  $C_*(S)$ ,  $C_P(S)$  or  $\mathcal{C}$ ,  $\mathcal{C}_*$ ,  $\mathcal{C}_P$ . They are all graphs,  $\mathcal{C}_P \subset \mathcal{C}_*$  and  $\mathcal{C}_*$  is dual to  $\mathcal{C}$ . The same relations apply to the respective completions  $\hat{\mathcal{C}}$ ,  $\hat{\mathcal{C}}_*$  and  $\hat{\mathcal{C}}_P$ . Also,  $\Gamma = \Gamma(S) \simeq \Gamma_{0,[5]}$ .

We also recall that, generally speaking (i.e. for any  $S$ ),  $\Gamma(S)$  acts on  $C(S)$  via its natural action on simple closed curves  $\mathcal{S}(S)$ . If  $\alpha \in \mathcal{S}(S)$ ,  $\tau_\alpha \in \Gamma(S)$  denotes the twist along  $\alpha$  (after fixing an orientation of  $S$ ). For  $g \in \Gamma(S)$ , these are related by  $t_{g \cdot \alpha} = g \tau_\alpha g^{-1}$ . It is remarkable (and non trivial) that these notions make sense in the completed setting, with the same relation, that is for  $\alpha \in \hat{\mathcal{S}}(S)$  and  $g \in \hat{\Gamma}(S)$  (see [B] and [BL] for details).

Now in the discrete setting, we have the following *standard pentagon* which we denote  $\varpi_0$  (see Figure 1). It consists of the curves  $\alpha_i$ ,  $i \in \mathbb{Z}/5$  and we view it as a pentagon  $\varpi_0 \subset \mathcal{C}_P \subset \mathcal{C}_*$ , each

vertex being a pants decomposition. Explicitly:

$$(\alpha_1, \alpha_4) \rightarrow (\alpha_2, \alpha_4) \rightarrow (\alpha_2, \alpha_5) \rightarrow (\alpha_3, \alpha_5) \rightarrow (\alpha_3, \alpha_5) \rightarrow (\alpha_4, \alpha_1).$$

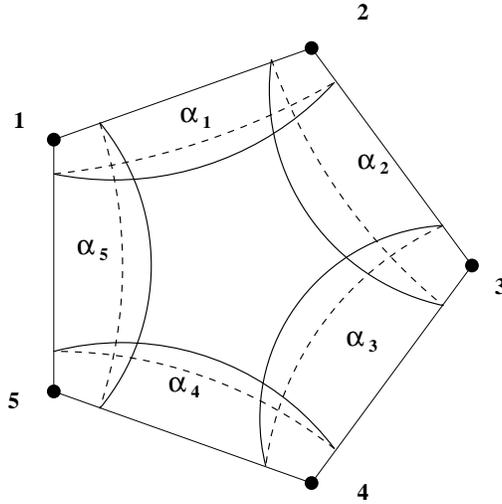


Figure 1

In the discrete case, recall that we say that two (isotopy classes of) curves  $\alpha, \beta \in \mathcal{S}$  *intersect minimally* if they intersect in two points, that is their geometric intersection number (as defined by Thurston) is 2; since they are separating, their algebraic intersection is zero. It is not clear *a priori* how to *define* a pentagon in the profinite setting because one usually defines a discrete pentagon using minimal intersection; this is precisely what has to be avoided in the profinite case because one *cannot* extend to that setting the notion of geometric intersection (the notion of algebraic intersection number readily extends, as given by the cup product on  $H^1(S, \hat{\mathbb{Z}})$ ). As explained in the introduction, we are in fact interested in *deforming* the minimal intersection rule. However, still in the discrete setting, we have the following elementary topological statement:

**Lemma 2.1:** *Let  $S$  be of type  $(0,5)$  and let  $\alpha_i \in \mathcal{S}$  ( $i \in \mathbb{Z}/5$ ) be 5 curves on  $S$ ; then the following conditions are equivalent:*

- i)  $\alpha_i$  and  $\alpha_{i+2}$  are disjoint for all  $i \in \mathbb{Z}/5$ ;
- ii)  $\alpha_i$  and  $\alpha_{i+1}$  have minimal intersection for all  $i \in \mathbb{Z}/5$ . □

If the  $\alpha_i$  satisfy i) and ii) they are said to form a *pentagon*. Note that i) simply says that  $P_i = (\alpha_i, \alpha_{i+2})$  is a pants decomposition for all  $i \in \mathbb{Z}/5$ , that is  $P_i$  is a vertex of both complexes  $C_*$  and  $C_P$ , which indeed share the same set  $V$  of vertices. Now ii) says that  $P_i$  and  $P_{i+2}$  are joined by an edge in  $C_P$ :  $(P_i, P_{i+2}) \in E_P$ .

Since condition ii) is not even defined in the profinite case, we define profinite pentagons via condition i). Formally:

**Definition 2.2:** On a surface  $S$  of type  $(0,5)$ , five (pro)curves  $\alpha_i \in \hat{\mathcal{S}}$  ( $i \in \mathbb{Z}/5$ ) form a *profinite pentagon* if  $\alpha_i$  and  $\alpha_{i+2}$  are disjoint for all  $i \in \mathbb{Z}/5$ , that is  $P_i = (\alpha_i, \alpha_{i+2}) \in \hat{V}$ .

The definition is clearly graph theoretic; it also makes sense for a surface  $S$  of bigger modular

dimension and curves which sit on a subsurface (which makes sense in the profinite setting as well) of type  $(0, 5)$ .

There is another elementary topological statement which turns out to be quite useful:

**Lemma 2.3:** *Let  $S$  be of type  $(0, 5)$  and let  $\alpha, \beta \in \mathcal{S}$  be two curves on  $S$ . Then there exists at most one curve  $\gamma \in \mathcal{S}$  which is disjoint from both  $\alpha$  and  $\beta$ .*  $\square$

It turns out that Lemma 2.3 does hold true in the profinite case, which we record as:

**Lemma 2.4:** *The profinite version of Lemma 2.3 holds true: If  $\alpha, \beta \in \hat{\mathcal{S}}$  are (pro)curves on  $S$  of type  $(0, 5)$ , there is at most one curve  $\gamma \in \hat{\mathcal{S}}$  which is disjoint from both  $\alpha$  and  $\beta$ .*

*Proof:* The group theoretic proof is easy. It is enough to consider the intersection  $Z(\tau_\alpha) \cap Z(\tau_\beta) \subset \hat{\Gamma}$  of the centralizers of the twists on  $\alpha$  and  $\beta$ . The description of the centralizers in the discrete case is elementary and it extends to the profinite setting. In this genus 0 case, this can be proved rather easily, without recourse to the general results of [B], §7 (although these can of course be applied). It could be interesting to find a nice graph theoretic proof of the lemma.  $\square$

It is interesting to observe that Lemma 2.4 implies that in a pentagon  $(\gamma_i)$ ,  $i \in \mathbb{Z}/5$ ,  $\gamma_i \in \hat{\mathcal{S}}$ , the (pro)curves  $\gamma_i$  and  $\gamma_{i+1}$  determine  $\gamma_{i+3}$  uniquely. Note also that in the discrete case,  $(\gamma_i, \gamma_{i+3})$  and  $(\gamma_{i+1}, \gamma_{i+3})$  define an edge of  $\mathcal{C}_P \subset \mathcal{C}_*$ ; in the profinite case, they define an edge of  $\hat{\mathcal{C}}_*$ .

At this point we unfortunately have to break the exposition in order to recall some standard notation and elementary facts on ordinary braid groups and genus 0 mapping class groups. We will use very little of this material, but prepare the ground for sections 4 and 5 below, where it will be (again moderately) put to use as well. Everything we need (and beyond) is contained e.g. in [LS1], in a way which is geared toward Grothendieck-Teichmüller theory.

So,  $B_n$  denotes the plane Artin braid group on  $n$  strands ( $(n \geq 1)$ ), generated by the  $\tau_i$ ,  $i = 1, \dots, n-1$ , representing the simple crossing of the  $i$ -th and  $(i+1)$ -th strands. They satisfy the usual braid relations. It is sometimes useful to use a redundant but symmetric system of generators:  $\tau_{ij} = \tau_{j-1} \dots \tau_{i+1} \tau_i \tau_{i+1}^{-1} \dots \tau_{j-1}^{-1}$ . Here  $1 \leq i < j \leq n$  and one sets  $\tau_{ji} = \tau_{ij}$ ,  $\tau_{ii} = 1$ ;  $\tau_{ij}$  corresponds to the simple crossings of the  $i$ -th and  $j$ -th strands; see [LS1], Proposition 1 for more.

There is a natural surjection  $B_n \rightarrow \mathcal{S}_n$  which maps  $\tau_{ij}$  to the transposition  $(i j)$ . The kernel  $K_n$  is the pure plane braid group, generated by the  $x_{ij}$ 's with  $x_{ij} = \tau_{ij}^2$ . The following mutually commuting pure elements, namely  $y_i = \tau_{i-1} \dots \tau_2 \tau_1 \tau_2 \dots \tau_{i-1}$  ( $1 < i \leq n$ ;  $y_1 = 1$ ) play an important role. In particular the center of  $B_n$  is free cyclic, generated by the product  $\omega_n = y_1 y_2 \dots y_n = (\tau_1 \dots \tau_{n-1})^n$ . Note that the  $y_i$ 's, hence also  $\omega_n$ , are pure and that  $\omega_n$  also generates the center of  $K_n$ .

The sphere braid group  $B_n(S^2)$  is naturally isomorphic to  $B_n / \langle y_n \rangle$  where  $\langle y_n \rangle$  denotes the normal closure of  $y_n$  in  $B_n$ . The image of the center  $\omega_n$  generates the center of  $B_n(S^2)$  and has order 2. Finally the modular group  $\Gamma_{0,[n]}$  is determined as:  $\Gamma_{0,[n]} \simeq B_n(S^2) / \langle \omega_n \rangle \simeq B_n / \langle y_n, \omega_n \rangle$ . The elements  $\tau_i$  are of course interpreted as twists exchanging points  $i$  and  $i+1$  on a marked sphere and this generalizes to the  $\tau_{ij}$ .

As for the first nontrivial case, the group  $B_3$  is generated by two elements  $\tau_1$  and  $\tau_2$ . Its center  $Z$  is free cyclic generated by  $\omega_3 = (\tau_1 \tau_2 \tau_1)^2 = (\tau_1 \tau_2^2)^2 = (\tau_1 \tau_2)^3$ . Then  $B_3 / Z \simeq PSL_2(\mathbb{Z})$ . The pure braid group  $K_3 \subset B_3$  is the direct product of the free group  $F_2 = \langle x_{12}, x_{23} \rangle$  ( $x_{12} = \tau_1^2$ ,

$x_{23} = \tau_2^2$ ) and the center:  $K_3 = \langle \tau_1^2, \tau_2^2 \rangle \times \langle (\tau_1 \tau_2 \tau_1)^2 \rangle$ .

We will add some pieces of information to this terse list when needed but it is now time to return to considering an automorphism  $F \in \text{Aut}(\hat{\mathcal{C}})$ . The starting point is that  $F$  clearly has to take pentagons (in the sense of Definition 2.2) to pentagons; this alone will turn out to be a very stringent requirement indeed. By simply requiring  $F(\varpi_0)$  to be a pentagon, we *a priori* we get 5 conditions on  $F$ , but we are interested in *outer* automorphisms, so we may and do normalize  $F$  by twisting by the action of  $\hat{\Gamma}$ . First, after labeling the points, which are encoded in  $\hat{\mathcal{C}}(S)$  or equivalently  $\hat{\mathcal{C}}_*(S)$ , we find that  $F$  induces a permutation of the five points, that is an element of  $\text{Aut}(\mathcal{S}_5)$ . Using the fact that  $\text{Out}(\mathcal{S}_5) = \{1\}$  we can twist by an element of  $\hat{\Gamma}$  in order to eliminate this automorphism and get a *pure* element of  $\text{Aut}(\hat{\mathcal{C}})$ , that is one which does not permute the points. Note that any curve separates  $S$  into a trinion and a piece of type  $(0, 4)$  and thus determines a partition of the set of points into two subsets of 2 and 3 points respectively. An automorphism  $F$  is pure if and only if  $\alpha$  and  $F(\alpha)$  determine the same partition for any  $\alpha \in \mathcal{S}$ . As for a second normalization, we know that  $\hat{\Gamma}$  acts transitively on  $\hat{V}$ , the set of pants decompositions (this is of course specific of type  $(0, 5)$ ). So twisting again by an element of  $\hat{\Gamma}$ , we can assume that  $F$  fixes a pants decomposition, which for definiteness and to conform with almost standard conventions we choose to be  $(\alpha_1, \alpha_4)$ . So we may and do assume that  $F$  is pure,  $F(\alpha_1) = \alpha_1$  and  $F(\alpha_4) = \alpha_4$ .

From there, one derives simple *a priori* formulas for the action of  $F$  on the  $\alpha_i$ 's. In order to write them down “explicitly”, we first note that the twist on the curve  $\alpha_i$  of the standard pentagon  $\varpi_0$  is nothing but  $x_{i,i+1} \in \Gamma_{0,5}$ . It is also interesting to recall that these five elements  $x_{12}, x_{23}, x_{34}, x_{45}, x_{51}$  generate  $\Gamma_{0,5}$ . Next piece of well-known information: the explicit structure of the centralizers of twists in  $\Gamma_{0,5}$ . It is enough of course to recall that the centralizer  $Z(x_{12})$  is given by  $Z(x_{12}) = \langle x_{12}, x_{34}, x_{45} \rangle$ . The centralizer of  $x_{12}$  in the completion  $\hat{\Gamma}_{0,5}$  is nothing but the completion of  $Z(x_{12})$ . Moreover it coincides with the (*a priori* larger) normalizer of any open subgroup of the procyclic group  $\langle x_{12} \rangle$ . As mentioned above, these results are relatively easy in genus 0; much less so in the general case, for which we refer especially to [BL], §3. Finally we explain how elementary “change of variables” works in our case. Let  $f \in \hat{F}_2$  and specify two topological generators. For instance write  $\hat{F}_2$  as the completion of  $F_2 = \langle x, y \rangle$ , that is, choose a “discretification” and two generators in it (this is too particular but will meet our needs). Then write  $f = f(x, y)$ ; we want to make sense of  $f(a, b)$  for any elements  $a, b$  of a profinite group  $G$ . To do this, simply observe there is a unique morphism  $\hat{F}_2 \rightarrow G$  mapping the ordered pair  $(x, y)$  to  $(a, b)$ ; write  $f(a, b)$  for the image of  $f = f(x, y)$ . The desirable naturality properties are plain from this definition.

This leads to the following formulas for the action of  $F$  on  $\varpi_0$ :

$$F(\alpha_1) = \alpha_1, \quad F(\alpha_2) = f \cdot \alpha_2, \quad F(\alpha_3) = g \cdot \alpha_3, \quad F(\alpha_4) = \alpha_4, \quad F(\alpha_5) = fh \cdot \alpha_5. \quad (2.1)$$

In these formulas,  $f, g$  and  $h$  are elements of  $\hat{\Gamma}_{0,5}$  and more precisely, we can assume that:

$$f = f(x_{12}, x_{23}) \in \langle x_{12}, x_{23} \rangle, \quad g = g(x_{34}, x_{45}) \in \langle x_{34}, x_{45} \rangle, \quad h = h(x_{45}, x_{51}) \in \langle x_{45}, x_{51} \rangle. \quad (2.2)$$

Here we have used nothing but the structure of the centralizers of twists as recalled above, and the fact that  $F(\varpi_0)$  is a pentagon, i.e.  $F(\alpha_i)$  and  $F(\alpha_{i+2})$  are disjoint. The elements  $f, g, h$  are by

no means unique. Here we performed a first normalization: for instance  $f$  might depend on  $x_{45}$  but that does not change the action at all so we may as well write  $f$ ,  $g$ , and  $h$  as above.

In terms of twists, this means for example that  $F(x_{23}) = fx_{23}^\lambda f^{-1}$  for some  $\lambda \in \hat{\mathbb{Z}}^*$  and these preliminary formulas can be compared to the ones describing the standard action of the Grothendieck-Teichmüller group (see e.g. [LS1] or [N1] and §4 below). We insist that formulas (2.1) and (2.2) are forced upon us by just requiring that  $F(\varpi_0)$  be a pentagon, as it should. Now for *every* triplet  $(f, g, h)$  of elements of  $\hat{\Gamma}$  as in formulas (2.2), 4 out of the 5 conditions for  $F(\varpi_0)$  to be a pentagon are satisfied, that is all of them save for the disjointness of  $F(\alpha_3)$  and  $F(\alpha_5)$ . This last condition is equivalent to the equation:

$$(gx_{34}g^{-1}, fhx_{51}h^{-1}f^{-1}) = 1, \quad (\text{II})$$

where  $(a, b) = aba^{-1}b^{-1}$  denotes the commutator of  $a$  and  $b$  in  $\hat{\Gamma}$ . Equation (II) can thus be seen as the “fundamental equation”, which should rigidify everything. It is nonlinear and profinite, so *a priori* not easy to handle.

We may normalize  $f$ ,  $g$  and  $h$  a little further, which eventually will bring us to exploring the notion of adjacency for pentagons. First, we may assume that  $f$  (resp.  $g$ ,  $h$ ) has weight 0 with respect to  $x_{23}$  (resp.  $x_{34}$ ,  $x_{51}$ ) by multiplying on the right by suitable (profinite) powers of these twists. Then we may twist  $F$  by powers of  $x_{12}$  and  $x_{45}$  in such a way that  $f$  and  $g$  will become *commutators*, that is have zero weight in both variables. This rigidifies the situation completely. Namely if we require that  $F$  fix the pants decomposition  $(\alpha_1, \alpha_4)$  and that  $f$  and  $g$  be commutators, there are no free parameters left. This is actually intrinsic and general (in any dimension): an automorphism which fixes a pants decomposition is determined up to a multitwist along that decomposition. It is precisely the role of tangential base points to rigidify by killing that possible twist: requiring that  $f$  and  $g$  be commutators amounts to fixing a kind of *topological tangential base point*; this is used and goes under various names in the literature, e.g. in an analytic setting in order to define Fenchel-Nielsen twists as real numbers (not only on the circle); see [NS] for a use of such a rigidification in the framework of Grothendieck-Teichmüller theory. Here we adopt a rather practical viewpoint on the matter, without spelling out the (known) underlying geometric picture.

We come back to our standard pentagon  $\varpi_0$  and automorphism  $F$ . It is interesting to note first of all that  $h$ , if it exists, is determined by the pair  $(f, g)$  – assuming  $h$  is normalized to have weight 0 with respect to  $x_{51}$ . Indeed, this is a direct application of Lemma 2.4:  $F(\alpha_5)$  is nothing but the only curve which is disjoint from both  $F(\alpha_2)$  and  $F(\alpha_3)$ . Next in the discrete case, *all* solutions of Eq. (II) are obtained by twisting along the multicurve  $(\alpha_1, \alpha_4)$  we singled out. After the first normalization mentioned above, one gets:  $f = x_{12}^m$ ,  $g = x_{45}^n$  and then  $h = x_{45}^n$ . Here one can cross check that in this discrete toy situation,  $F(\alpha_5)$  is indeed determined by  $f$  and  $g$ : the exponent  $n$  in  $h$  is the only one which ensures that  $F(\alpha_5)$  commutes with  $F(\alpha_3)$ . This also gives the complete classification of pentagons in the discrete case and show that they all belong to a single  $\Gamma$ -orbit. Simple as this may sound, it is key in showing the rigidity of the discrete curves complex  $\mathcal{C}$ .

In the profinite setting, using a topological tangential base point, i.e. normalizing as described above, we precisely get rid of the multitwists along  $(\alpha_1, \alpha_4)$ , that is of the elements  $m, n$ , now in

$\hat{\mathbb{Z}}$ . We will write  $\hat{F}'_2 = \langle x_{12}, x_{23} \rangle'$  for the commutator subgroup of the free group generated by  $x_{12}$  and  $x_{23}$  and analogously for other such pairs. Ideally, we would like to show that  $f$  determines  $g$ , that is, given  $f = f(x_{12}, x_{23}) \in \hat{F}'_2 = \langle x_{12}, x_{23} \rangle'$ , there should exist at most one pair  $(g, h)$ , where  $g \in \langle x_{34}, x_{45} \rangle'$  and  $h \in \langle x_{45}, x_{51} \rangle$  has zero weight with respect  $x_{51}$ , such that the triplet  $(f, g, h)$  yields a solution of Eq. (II).

This rigidity statement would provide an essentially complete classifications of pentagons in  $\hat{\mathcal{C}}$ . However we are not able to prove it at the moment; we will however add some comments on its geometric and group theoretic meanings in the next section. We also note that it could be interesting to study Eq. (II) in the pro- $\ell$  setting, where in particular it can be translated into Lie theoretic terms and is amenable to a step by step solving procedure.

We will now state and prove the case  $f = 1$  of the above statement, which will suffice for our needs, in particular to complete the proof of Theorem 1.2; it also has a clear geometric meaning, as detailed below. We are thus interested in showing:

**Proposition 2.5:** *Consider the equation:*

$$(gx_{34}g^{-1}, hx_{51}h^{-1}) = 1,$$

with  $g \in \langle x_{34}, x_{45} \rangle'$  and  $h \in \langle x_{45}, x_{51} \rangle$ . Then  $g = 1$  and  $h \in \langle x_{51} \rangle$ .

Since  $g \in \langle x_{34}, x_{45} \rangle'$  and  $h \in \langle x_{45}, x_{51} \rangle$ , this equation takes place in the group (topologically) generated by  $x_{34}, x_{45}$  and  $x_{51}$  inside  $\hat{\Gamma}_{0,5}$ , which is nothing but the completion of the corresponding subgroup of  $\Gamma_{0,5}$ . The structure of the latter group is relatively simple: it is isomorphic to the group  $G$  with 3 generators  $x, y$  and  $z$  and only relation  $(x, z) = 1$ , stating that  $x$  and  $z$  commute. So  $G \simeq \mathbb{Z}^2 * \mathbb{Z}$  and to get the isomorphism, just map  $(x_{34}, x_{45}, x_{51})$  to  $(x, y, z)$ . For the sake of clarity, we thus restate Proposition 2.5 under the following equivalent form:

**Lemma 2.6:** *Let  $G = \langle x, y, z \rangle \simeq \mathbb{Z}^2 * \mathbb{Z}$  be as above and let  $g \in \langle x, y \rangle' \simeq \hat{F}'_2$ ,  $h \in \langle y, z \rangle \simeq \hat{F}_2$  be two elements of  $\hat{G}$  satisfying the commutation relation:  $(gxg^{-1}, hzh^{-1}) = 1$ . Then  $g = 1$  and  $h \in \langle z \rangle$ .*

*Proof:* This is a direct application of a result by W.Herfort and L.Ribes; see Theorem 9.1.12 of [RZ]. By assumption,  $(g^{-1}h)z(g^{-1}h)^{-1}$  centralizes  $x$  and the quoted result directly implies that  $g^{-1}h \in \langle x, z \rangle$ . So we can write:  $h(y, z) = g(x, y)w(x, z)$  for some  $w \in \langle x, z \rangle$ . Moding out by the normal closure of  $y$  and using that  $g$  is a commutator, we find that in fact  $w = w(z) \in \langle z \rangle$ . We then obtain that  $g$  depends on  $y$  only and so that  $g = 1$  since it is a commutator. This finishes the proof of the lemma, hence also of Proposition 2.5.  $\square$

We remark that we heavily used the structure of  $G$  as a free product; this suggests that the study of Eq. (II) in the general case, that is for a non trivial  $f$ , may well be substantially more difficult. Let us now exploit the above in order to complete the proof of Theorem 1.2. First we introduce an easy notion:

**Definition 2.7:** Two pentagons (in the sense of Definition 2.2) are called *adjacent* if they have an edge in common in  $\hat{\mathcal{C}}_*(S)$ .

This of course makes sense and can be used on a surface  $S$  of any modular dimension. If  $\varpi$  and  $\varpi'$  lie on a surface of type  $(0, 5)$  and are adjacent, they share 3 curves in common, corresponding

to two pants decompositions. By Lemma 2.4, it is equivalent to require that  $\varpi$  and  $\varpi'$  have 2 consecutive curves in common, or equivalently again two curves which intersect (that is which are not non intersecting...). Proposition 2.5 can now be viewed as a description of the pentagons which are adjacent to the standard pentagon  $\varpi_0$  (on a surface of type  $(0, 5)$ ). Indeed, if  $\varpi$  is adjacent to  $\varpi_0$  one can assume after relabeling that  $\varpi$  contains the curves  $\alpha_1, \alpha_2$  and  $\alpha_4$ . Then Proposition 2.5 indeed describes all pentagons which contain these curves: they are obtained by twisting  $\varpi_0$  by a (profinite) power of  $x_{45}$ .

This notion of adjacency may look a little uninteresting at first sight, but it will turn out to be surprisingly useful, as will become even clearer in the next section. Here is a first consequence of Proposition 2.5 which will be used a lot in the next section. Namely, on a surface of type  $(0, 5)$ , assume that  $\varpi$  and  $\varpi'$  are adjacent pentagons and that  $\varpi \subset \hat{\mathcal{C}}_P$ ; then  $\varpi' \subset \hat{\mathcal{C}}_P$ . In fact we can assume that  $\varpi = \varpi_0$  is the standard pentagon and then this is an immediate consequence of Proposition 2.5; see Proposition 3.1 and Lemma 3.2 below for details and a slight generalization. Now we prove:

**Lemma 2.8:** *Let  $F \in \text{Aut}(\hat{\mathcal{C}})$  fixing the standard pentagon  $\varpi_0$ , that is  $F(\alpha_i) = \alpha_i$  for all  $i \in \mathbb{Z}/5$ . Then either  $F = 1$  or it is a reflection, that is an orientation reversing involution.*

*Proof:* Indeed, consider any discrete pentagon  $\varpi \subset \mathcal{C}_P$  which is adjacent to  $\varpi_0$ . Then  $F$  fixes the side which is common to  $\varpi_0$  and  $\varpi$ , so  $F(\varpi)$  is adjacent to  $\varpi_0$  and by the remark above  $F(\varpi) \subset \hat{\mathcal{C}}_P$ . Continuing in the same fashion, that is considering a pentagon adjacent to  $\varpi$ , we find that  $F(\mathcal{C}_P) \subset \hat{\mathcal{C}}_P$ , because any two pentagons of the discrete complex  $\mathcal{C}_P$  can be connected by a finite sequence of mutually adjacent pentagons. So by density  $F(\hat{\mathcal{C}}_P) \subset \hat{\mathcal{C}}_P$  and in fact  $F(\hat{\mathcal{C}}_P) = \hat{\mathcal{C}}_P$  by changing  $F$  into  $F^{-1}$ . So  $F \in \text{Aut}(\hat{\mathcal{C}}_P)$  and using the rigidity of  $\hat{\mathcal{C}}_P$  (i.e. [BL], Theorem 4.15) and the fact that  $F$  fixes  $\varpi_0$ , it is easy to see that  $F$  is indeed the identity or a reflection.  $\square$

We are now finally in a position to prove assertion  $(C_2)$  of section 1, that is:

**Proposition 2.9:** *The statement of Theorem 1.2 holds true for  $d = 2$ .*

*Proof:* Let  $F \in \text{Aut}(\hat{\mathcal{C}}(S))$ , with  $d(S) = 2$ ; we may assume that  $S$  of type  $(0, 5)$ , that  $F(\alpha) = \alpha$  for a curve  $\alpha \in \mathcal{S}$  and that  $F$  restricts to the identity on  $\hat{\mathcal{C}}(S_\alpha)$ . Then we can further assume, again without loss of generality, that we are in the standard situation with, say,  $\alpha = \alpha_4$ . So  $F$  fixes  $\alpha_1$  and  $\alpha_2$  as well. Applying Proposition 2.5 we conclude that, possibly after twisting along  $\alpha_4$ ,  $F$  fixes  $\varpi_0$  and then, by Lemma 2.8, that  $F$  is the identity, a reflection being excluded by the assumed triviality of  $F$  on  $\hat{\mathcal{C}}(S_\alpha)$ .  $\square$

This completes the proof of Theorem 1.2. In the next section, we will see some further applications of adjacency, combined with other simple geometric notions. We will now add a few observations which we hope may help put the results in perspective or suggest lines for further research. None of them will be used below so that they can be skipped without impairing further technical understanding.

**Remarks 2.10:**

1. Technically speaking, it should be noted that injectivity becomes much easier if one is *a priori* given *two* intersecting curves  $\alpha$  and  $\beta$  such that  $F \in \text{Aut}(\mathcal{C})$  restricts to the identity both on  $\hat{\mathcal{C}}(S_\alpha)$

and  $\hat{C}(S_\beta)$  (compare [N1], Lemma 3.2.2, for the group theoretic counterpart). The whole point here is to start from just one such curve.

2. Theorems 1.2 and 1.5 should hold essentially *verbatim* both in the discrete (i.e. non completed) and pro- $\ell$  (equivalently pronilpotent) settings. In the discrete case, the only results from [B] or [BL] that are required are (part of) those in [BL], which compare the automorphism groups of  $C(S)$  and  $C_*(S)$ . The proof of the two-dimensional case is much easier than its profinite counterpart to which we turn in the next section. In the pro- $\ell$  setting, one literally follows the proofs given in the present paper, using the pro- $\ell$  analogs of the results from [B] or [BL], which are for the most part at least mentioned in those papers, or are easy variants.

3. Starting from the discrete version of Theorem 1.5 (injectivity) and in order to prove the rigidity of the discrete complex, that is  $Out(C(S)) = Mod(S)$  (except in a handful of well-understood low dimensional cases), it is enough to show that this is the case for  $S$  of type  $(0,5)$ . But in the discrete case we have already observed that it is easy to classify all pentagons, which form a unique  $\Gamma(S)$ -orbit. So in order to reprove the classical (Ivanov-Korkmaz-Luo) rigidity result, there only remains to prove the discrete analog of Lemma 2.8. This in turn amounts to devising a direct elementary proof of the rigidity of the discrete pants graph  $\mathcal{C}_P$ , which looks like a feasible and perhaps interesting task.

4. Let us clarify the overall strategy a little more. We now have Theorem 1.2 and thus Theorem 1.5 at our disposal, that is an *injectivity* statement: two outer automorphisms coincide if they coincide on a piece of modular dimension 1. Ideally speaking, we should then move to an *extension* result stating in substance that if  $F \in Aut(\hat{C}(T))$  is an automorphism of  $\hat{C}(T)$ , where  $T$  is a piece of a connected surface  $S$  with  $d(T) > 1$ , then it can be extended to an automorphism of the full complex  $\hat{C}(S)$ . The real situation we will study in §§4,5 is not quite as neat but almost. In the meantime we need also worry, among other things, about the *normality* of  $\hat{\Gamma}(S)$  in  $Aut(\hat{C}(S))$ , which we do below in §3.

5. To put things slightly differently, we are trying to localize the problem on pieces of dimension 1 (injectivity) or 2 (extension). For a piece  $T \subset S$  one can define a *restriction map*  $r_T : Out(\hat{C}(S)) \rightarrow Out(\hat{C}(T))$  which is well-defined and injective for  $d(T) > 0$  by the injectivity result. Combining this with an extension property ensures that it is also surjective and independent of  $T$  for  $d(T) > 1$ . This will work literally in genus 0 (§4) but reality will turn out to be a little more complicated for  $S$  of strictly positive genus (§5).

6. In [I2,3], Y.Ihara implemented a similar strategy (which in some sense comes from [G]) in the simplest possible non trivial case, that is in a group theoretic (actually Lie theoretic) context, for pronilpotent (i.e. not full profinite) completions of braid groups (i.e. genus 0). Curiously enough his injectivity result in [I2] is the only one we know of in the literature which resembles Theorem 1.5. He already notices that his result is valid in the discrete case as well and provides a new proof of the Dyer-Grossman result on the rigidity of Artin's braid groups:  $Out(B_n) = \mathbb{Z}/2$ ,  $n > 2$ .

7. In recent years several complexes related to the curves complex  $C(S)$  have been introduced (e.g. the Torelli complex). In certain cases, they were then shown to be rigid i.e. their automorphisms reduce to the "obvious" ones. It could be interesting to see whether the strategy of the present

paper can be transposed, including to the completions of these complexes. One should note however that one of the main insights of the *Esquisse* is again about the *locality* of certain phenomena. Yet, thinking for instance of the Torelli group, the property for a twist (or the underlying curve) to be separating is *not* a local property in that sense.

8. We insist that injectivity is proved here as a first step in the exploration of the automorphism group. For instance, in genus 0, it is already known that  $Out^*(\hat{\Gamma}_{0,[n]}) = \widehat{GT}$  for  $n \geq 5$ . This is Proposition 4.14 of [BL], which draws heavily on [HS], itself using techniques of [LS1] and [N1]. Using this result it is of course easy to recover the group theoretic version of Theorem 1.5 in genus 0. But this is a “bad” proof. In the next section, working in the opposite direction we will in fact reprove (and beyond) the result on  $Out^*(\hat{\Gamma}_{0,[n]})$  using injectivity, which laid buried in the previous rather circuitous proof.

9. We finish with the rather obvious remark that all the work to-date has been done in a group theoretic rather than complex theoretic setting. The latter is in principle more powerful as it encompasses the full system of all the open subgroups of the modular groups. It is also often more flexible, as perhaps exemplified above, although of course group theoretic techniques can at times be not only helpful but also hardly dispensed with.

### 3. Transversality and normality

Such are the keywords of the present section, as will soon become clearer. We first summarize some conclusions from the last section for the sake of clarity:

**Proposition 3.1:** *Let  $\varpi \subset \hat{C}_*(S)$  be a pentagon, with  $S$  of type  $(0,5)$ . The following conditions are equivalent:*

- i)  $\varpi$  is in the  $\hat{\Gamma}(S)$ -orbit of the standard pentagon  $\varpi_0$ ;
- ii)  $\varpi$  is an inverse limit of pentagons in  $C_P(S)$ ;
- iii)  $\varpi$  is contained in  $\hat{C}_P(S)$ ;
- iv) One side of  $\varpi$  is contained in  $\hat{C}_P(S)$ .

Let us make ii) and iii) completely explicit. Condition ii) means that there exists a sequence  $(\varpi_\lambda)_{\lambda \in \Lambda}$  (see §A.5 for the notation) of pentagons in  $C_P(S)$  such that  $\varpi$  is the inverse limit of the  $\varpi^\lambda$ 's, where  $\varpi^\lambda = \varpi_\lambda \mod \Gamma^\lambda$ . Condition iii) means that the sides of  $\varpi$  lie in  $\hat{E}_P \subset \hat{E}$ , the (pro)set of edges of  $\hat{C}_P(S)$  (see §A.10 for the notation). Recall that  $\hat{C}_*(S)$  and  $\hat{C}_P(S)$  have the same set  $\hat{V}$  of vertices, namely the completion of the set of pants decompositions, and that since  $S$  is of type  $(0,5)$ ,  $\hat{V}$  consists of just one  $\hat{\Gamma}$ -orbit.

*Proof of Proposition 3.1:* Conditions are stated in order of apparent decreasing strength. So i) implies ii) which implies iii), which in turn implies iv) and we need only show that iv) actually implies i). So let  $\varpi \subset \hat{C}_*(S)$  with one side in  $\hat{C}_P(S)$ . That side consists of 2 pants decompositions  $P$  and  $P'$ . We can assume that  $P = (\alpha_1, \alpha_4)$ , with  $\alpha_1, \alpha_4$  as in the standard pentagon  $\varpi_0$ , and that  $P' = (\alpha_1, \beta)$  for some  $\beta \in \hat{S}$ . Because  $(P, P') \in \hat{E}_P$  there exists a sequence  $(\beta_\lambda) \subset \mathcal{S}$  such that  $\beta$  is the inverse limit of the  $\beta^\lambda$ , with  $\beta^\lambda \in \mathcal{S}^\lambda = \mathcal{S}/\Gamma^\lambda$ ,  $\beta^\lambda = \beta_\lambda \mod \Gamma^\lambda$ . Now every pair  $(\alpha_1, \beta_\lambda)$  lies, up to reflection, in the  $\Gamma$ -orbit of the pair  $(\alpha_1, \alpha_2)$ . Actually, since  $\beta_\lambda$  is disjoint from  $\alpha_4$ , this takes place on the surface of type  $(0,4)$  obtained by cutting  $S$  along  $\alpha_4$  and we are simply

asserting that  $SL_2(\mathbb{Z})$  acts transitively on the edges of the Farey tessellation. Since the sequence  $\beta^\lambda$  converges, we find that  $(\alpha_1, \beta)$  lies in the  $\hat{\Gamma}$ -orbit of  $(\alpha_1, \alpha_2)$ ; so twisting by an element of  $\hat{\Gamma}$  and possibly reflecting, we can assume that  $\varpi$  contains  $P = (\alpha_1, \alpha_4)$  and  $P' = (\alpha_2, \alpha_4)$ . Proposition 2.5 then completes the proof.  $\square$

We already know that a pentagon which is adjacent to a pentagon contained in  $\hat{C}_P(S)$  is itself contained in  $\hat{C}_P(S)$ . Now consider an automorphism of  $F \in \text{Aut}(\hat{C}(S))$ . Clearly it preserves adjacency. If we take a pentagon  $\varpi \subset F(\hat{C}_P(S))$  and consider  $\varpi'$  adjacent to it, then  $F^{-1}(\varpi)$  and  $F^{-1}(\varpi')$  are adjacent as well, the first one being contained in  $\hat{C}_P(S)$ . So  $F^{-1}(\varpi')$  is contained in  $\hat{C}_P(S)$  and  $\varpi'$  is contained in  $F(\hat{C}_P)$ . In other words we have shown:

**Lemma 3.2:** *Let  $S$  be of type  $(0, 5)$ ,  $F \in \text{Aut}(\hat{C}(S))$ ,  $\varpi \subset F(\hat{C}_P(S))$  a pentagon and  $\varpi'$  adjacent to  $\varpi$ ; then  $\varpi' \subset F(\hat{C}_P(S))$ .*  $\square$

Let now  $F, F' \in \text{Aut}(\hat{C}(S))$  be two automorphisms and consider the images  $F(\hat{C}_P(S))$  and  $F'(\hat{C}_P(S))$ . As usual only the edges are interesting here since the sets of vertices of these images both coincide with  $\hat{V}$ . Under these circumstances, we have the following *transversality property*:

**Proposition 3.3:** *Let  $S$  be hyperbolic of finite type with  $d(S) > 1$  and  $S$  not of type  $(1, 2)$ , let  $F, F' \in \text{Aut}(\hat{C}(S))$  and assume that  $F(\hat{C}_P(S))$  and  $F'(\hat{C}_P(S))$  have at least one edge in common. Then these graphs coincide and, up to a possible reflection,  $F' = F \circ g$  with  $g \in \hat{\Gamma}(S)$ .*

*Proof:* To the first assertion, precomposing by  $F'^{-1}$ , we can assume that  $F' = 1$  and we have to show that if one of the edges of  $F(\hat{C}_P)$  is an edge of  $\hat{C}_P$ , then  $F(\hat{C}_P) = \hat{C}_P$ . Assume  $e \in \hat{E}_P$  is an edge in  $F(\hat{C}_P)$ ; then  $e = (P, P')$  with  $P$  and  $P'$  two pants decompositions. Because  $F$  is type preserving ([BL], Proposition 4.1),  $P$  and  $F(P)$  are in the same  $\hat{\Gamma}$ -orbit. So twisting  $F$  by an element of  $\hat{\Gamma}$  we can assume that  $F$  fixes  $P$ . Then  $P$  and  $P'$  have  $d - 1$  curves in common ( $d = d(S)$ ) and these  $d - 1$  curves will cut off a surface  $T$  of dimension 2 in  $S$ , so  $T$  is either of type  $(0, 5)$  or of type  $(1, 2)$ . We may and do assume as usual that that  $T$  is of type  $(0, 5)$ , because the assertions are complex theoretic. Restricting attention to  $T$ , we are reduced to the case where  $S$  is itself of type  $(0, 5)$ ; then we can conclude that actually, up to reflection,  $F \in \Gamma$ . Indeed, any side of  $\hat{C}_P$  can be completed into a pentagon (see the proof of Proposition 3.1). So by Proposition 3.1,  $F$  maps a pentagon of  $\hat{C}_P$  into a pentagon of  $\hat{C}_P$ . By Proposition 3.1 again, after twisting by an element of  $\hat{\Gamma}$ , we obtain an automorphism which fixes the standard pentagon  $\varpi_0$  and Lemma 2.8 finishes the proof.

As for the last part of the statement, if  $F'(\hat{C}_P) = F(\hat{C}_P)$ , then  $F^{-1} \circ F'(\hat{C}_P) = \hat{C}_P$ , so up to a possible reflection  $F^{-1} \circ F'$  coincides with the action of some element of  $\hat{\Gamma}(S)$ . Since we have not yet proved the normality of  $\hat{\Gamma}$  in  $\text{Aut}(\hat{C})$ , using  $F' \circ F^{-1}$  we could write just as well  $F' = h \circ F$  for some  $h \in \hat{\Gamma}$ .  $\square$

This statement can be seen as a strengthening of Theorem 1.5, modulo the easy exception of type  $(1, 2)$  and so this transversality statement also features a kind of strong *injectivity* property.

As usual, the situation for type  $(1, 2)$  is easy to unravel. The proposition then breaks for the following reason (and no other). We have  $\hat{C}(S_{1,2}) \simeq \hat{C}(S_{0,5})$ ,  $\hat{\Gamma}_{1,[2]}$  acts on  $\hat{C}(S_{1,2})$  via  $\hat{\Gamma}_{1,2}$  and  $\hat{\Gamma}_{1,2} \subset \hat{\Gamma}_{0,[5]}$  (see §A.4). So  $\hat{\Gamma}_{0,[5]}$  acts on  $\hat{C}(S_{1,2})$  and an element which is not in  $\hat{\Gamma}_{1,2}$  (i.e. whose permutation does not fix 5, with the convention of §A.4) clearly does not act via  $\hat{\Gamma}_{1,[2]}$ .

We now come to the normality of  $\hat{\Gamma}(S)$  inside  $Aut(\hat{C}(S))$ , that is:

**Theorem 3.4:** *Let  $S$  be hyperbolic of finite type with  $d(S) > 1$  and  $S$  not of type  $(1, 2)$ . Then  $\hat{\Gamma}(S)$  is a normal subgroup of  $Aut(\hat{C}(S))$ .*

*Proof:* Again the exceptional case  $(1, 2)$  is easy to understand. Here the failure of the statement to hold true boils down to the fact that  $\Gamma_{1,2}$  is not normal in  $\Gamma_{0,[5]}$ .

The proof is now fairly easy. First we observe that the statement is equivalent to the fact that for any  $F \in Aut(\hat{C}(S))$ ,  $\hat{\Gamma}$  acts on  $F(\hat{C}_P(S))$ , that is the natural  $\hat{\Gamma}$ -action on  $\hat{C}(S)$  (or  $\hat{C}_*(S)$ ) leaves  $F(\hat{C}_P(S))$  globally invariant. Indeed if  $\hat{\Gamma}$  is normal, then for  $g \in \hat{\Gamma}$ ,  $F^{-1}gF = h \in \hat{\Gamma}$ , so that  $F^{-1}gF(\hat{C}_P) = h\hat{C}_P = \hat{C}_P$  and  $gF(\hat{C}_P) = F(\hat{C}_P)$ . Conversely, if the latter is true, then  $F^{-1}gF$  is an orientation preserving automorphism of  $\hat{C}_P$ , so it is an element of  $\hat{\Gamma}$  and we are done.

So we need to show that  $gF(\hat{C}_P) = \hat{C}_P$  for every  $g \in \hat{\Gamma}$  and we can restrict attention to twists since they topologically generate  $\hat{\Gamma}$ . Let  $\tau = \tau_\alpha \in \hat{\Gamma}$  be a twist along the curve  $\alpha$ . It actually does not matter here whether  $\alpha$  is a curve or a procurve. Consider the inverse image  $\beta = F^{-1}(\alpha)$ . We can include  $\beta$  in a pentagon  $\varpi$  which is contained in  $\hat{C}_P(S)$  and in fact in  $\hat{C}_P(T)$  for  $T$  a 2 dimensional connected subsurface of  $S$ . We may assume (as in the proof of Proposition 3.3), that  $T$  is of type  $(0, 5)$  and then use, say, Proposition 3.1, which describes *all* pentagons in  $\hat{C}_P = \hat{C}_P(T)$ . So  $\varpi \subset \hat{C}_P$  is a pentagon, and thus  $F(\varpi) \subset F(\hat{C}_P)$  is again a pentagon, now in  $F(\hat{C}_P)$ ; moreover  $\alpha$  is one of the five curves it contains. We have simply shown that any curve  $\alpha$  can be included in a pentagon inside  $F(\hat{C}_P)$ .

Now  $F$  and  $F' = \tau \circ F$  are two automorphisms of  $\hat{C}_*(S)$ . Moreover,  $F(\varpi)$  and  $F'(\varpi)$  are adjacent pentagons by construction, the second being obtained from the first by a twist along one of the curves it contains. Since  $\varpi \subset \hat{C}_P$ , there remains only to apply Proposition 3.3.  $\square$

We end this section by briefly coming back to the classification of pentagons in  $\hat{C}_*(S)$ ; we consider a surface of type  $(0, 5)$  although results could probably be extended to the general case. We could not complete that task of classifying pentagons which, as explained above amounts to a better understanding of Eq. (II). This can be rephrased in several equivalent ways. For instance, is it true that any pentagon in  $\hat{C}_*(S)$  is the image of a pentagon in  $\hat{C}_P(S)$  by an element of  $Aut(\hat{C}(S)) = Aut(\hat{C}_*(S))$ ? Recall from Proposition 0.3 that  $Out(\hat{C}(S))$  can be identified with the oriented injections of  $\hat{C}_P(S)$  into  $\hat{C}_*(S)$ . So the question can also be rephrased as: Is it true that any pentagon in  $\hat{C}_*(S)$  can be (uniquely) completed into a copy of  $\hat{C}_P(S)$  inside  $\hat{C}_*(S)$ ? That would give a one-to-one correspondence between pentagons and elements of  $Out(\hat{C}(S))$  up to reflection. Finally, in group theoretic terms, a pentagon corresponds to 5 twists  $\tau_i$  such that  $\tau_i$  commutes with  $\tau_{i+2}$  for all  $i \in \mathbb{Z}/5$ . For the standard pentagon, the  $\tau_i = x_{i,i+1}$  generate  $\Gamma_{0,5}$  and satisfy one ‘‘pentagonal’’ relation (see e.g. [LS1]). This remains valid for any pentagon in  $\hat{C}_P(S)$  by, say, Proposition 3.1, except of course that now the  $\tau_i$ ’s topologically generate  $\hat{\Gamma}_{0,5}$ . Is it valid for *every* pentagon in  $\hat{C}_*(S)$ ? Using Grothendieck’s terminology in his *Longue Marche à travers la théorie de Galois*, that would indicate that pentagons are in one-to-one correspondence with the *discretifications* of  $\hat{\Gamma}_{0,[5]}$  (up to reflection).

## 4. The Grothendieck-Teichmüller group, genus zero

We start with a very minimal reminder about the full profinite version of the Grothendieck-

Teichmüller group in genus 0, as originally introduced in [D]; we aim at no more than fixing notation. We refer the reader to the seminal papers [G], [D], [I1] for the early history of the subject and to the following papers (and a few others) for background material from various viewpoints and with various degrees of generality: [I1], [S1], [LS1], [HS], [HLS], [NS], [Lo] etc.

There is a nested sequence of inclusions;

$$G_{\mathbb{Q}} \subset \widehat{GT} \subset \text{Aut}^*(\hat{F}_2),$$

where  $G_{\mathbb{Q}}$  is the Galois group of  $\mathbb{Q}$ ,  $\widehat{GT}$  will be defined presently and  $\text{Aut}^*(\hat{F}_2)$  is the group of continuous inertia preserving automorphism of  $\hat{F}_2$ , the full profinite completion of  $F_2 = \langle x, y \rangle$ , itself the free group on 2 generators  $x$  and  $y$ . Inertia preserving means that the procyclic groups  $\langle x \rangle$  and  $\langle y \rangle$  are respectively mapped to conjugate groups by an element  $F \in \text{Aut}^*(\hat{F}_2)$  and so is  $\langle z \rangle$ , with  $xyz = 1$ . To be honest, we add that the above inclusions are not completely “natural” as they depend on the choice of a rational (tangential) basepoint.

Twisting by inner automorphisms of  $\hat{F}_2$ , one can normalize the elements of  $\text{Aut}^*(\hat{F}_2)$  by requiring that the group  $\langle x \rangle$  be globally fixed. Behind that normalization and in greater generality are again such notions as tangential basepoints, splitting of certain sequences etc. but the long and the short is that, concretely speaking, the elements of  $\text{Aut}^*(\hat{F}_2)$  we are interested in, including of course the elements of  $\widehat{GT}$ , are given as pairs  $F = (\lambda, f)$  with  $\lambda \in \hat{\mathbb{Z}}^*$  (the invertible elements of  $\hat{\mathbb{Z}}$ ) and  $f \in \hat{F}'_2$  (the topological derived subgroup of  $\hat{F}_2$ ). The action on  $\hat{F}_2$  is defined by:

$$F(x) = x^\lambda, \quad F(y) = f^{-1}y^\lambda f. \quad (4.1)$$

One requires that these formulas define an automorphism, that is an invertible morphism, and there is no effective way to ensure invertibility, in contrast with the pronilpotent case where such formulas *always* define invertible morphisms (because  $\lambda$  is invertible). Multiplication is given by composition in the automorphism group  $\text{Aut}(\hat{F}_2)$ . This leads to the following formula for the product of  $F = (\lambda, f)$  and  $F' = (\lambda', f')$ :

$$F' \circ F = (\lambda\lambda', f'F'(f)). \quad (4.2)$$

Then for an automorphism  $F$  to define an element of  $\widehat{GT}$ , the associated pair  $(\lambda, f)$  has to satisfy the following 3 relations:

- (I) (2-cycle)  $f(x, y)f(y, x) = 1$ ;
- (II) (3-cycle)  $f(x, y)x^\mu f(z, x)z^\mu f(y, z)y^\mu = 1$  where  $xyz = 1$  and  $\mu = (\lambda - 1)/2$ ;
- (III) (5-cycle)  $f(x_{12}, x_{23})f(x_{34}, x_{45})f(x_{51}, x_{12})f(x_{23}, x_{34})f(x_{45}, x_{51}) = 1$ ;

Thus  $\widehat{GT} \subset \text{Aut}^*(\hat{F}_2)$  is the subgroup whose elements are defined by pairs  $F = (\lambda, f) \in \hat{\mathbb{Z}}^* \times \hat{F}'_2$  acting as in (4.1) and satisfying (I), (II) and (III). Note that these are often referred to as “relations” although “equations” would be more correct:  $\widehat{GT}$  is a subgroup, not a quotient of  $\text{Aut}^*(\hat{F}_2)$ .

**Remark 4.1:** It was noted that by H.Furusho that (I) is in fact an easy consequence of (III). We nevertheless retain (I) in the definition because of its geometric meaning. Recently, the same author proved (in [F]) the surprising and beautiful result that (II) is a consequence of (III) in the

pro-unipotent setting, that is for  $GT(k)$ , where  $k$  is a field of characteristic zero (you may need a quadratic extension to define  $\mu$ ). This of course raises the question as to whether this is also true in the present full profinite setting. Furusho's result immediately implies that this is the case in the pro- $\ell$  setting ( $\ell$  a prime). The group  $\widehat{GT}$  would then be characterized by the 5-cycle only. We observe that what we have done hitherto, and in fact what we do below, essentially involves pentagons only. So it is in line with this idea that they tell the whole story. One needs "only" find all solutions of Eq. (II) of section 2.

If  $F = \sigma \in G_{\mathbb{Q}}$ , we denote the parameters by  $(\lambda_{\sigma}, f_{\sigma})$  and in fact  $\lambda_{\sigma} = \chi(\sigma)$  coincides with the value of the cyclotomic character  $\chi : G_{\mathbb{Q}} \rightarrow \hat{\mathbb{Z}}^*$ . In particular the first projection map  $\widehat{GT} \rightarrow \hat{\mathbb{Z}}^*$ , defined by  $F = (\lambda, f) \rightarrow \lambda$ , is surjective since it is already surjective when restricted to  $G_{\mathbb{Q}}$ . Its kernel is denoted  $\widehat{GT}^1$  and is an important subgroup of  $\widehat{GT}$ , containing the Galois group of  $\mathbb{Q}^{ab}$ , the maximal abelian extension of  $\mathbb{Q}$ .

The only "discrete" elements of  $\widehat{GT}$ , that is those given by pairs  $(\lambda, f) \in \mathbb{Z}^* \times F'_2$  are  $(\pm 1, 1)$  ([D], Proposition 4.1) and the only nontrivial element among these, given by the pair  $c = (-1, 1)$ , corresponds to complex conjugacy. About the second projection  $F = (\lambda, f) \rightarrow f$ , it is interesting to note here that it is exactly two-to-one. Namely if  $F = (\lambda, f)$ , then  $F' = F \circ c = (-\lambda, f)$  is the only other element with the same  $f$ , as can be readily inferred from (4.2). In the language we have been using in the previous sections,  $F$  and  $F'$  coincide up to reflection. So  $f$  determines  $\lambda$  up to a sign, corresponding to a possible reflection. Finally there is no global uniform determination of  $\lambda$ , that is the projection  $(\lambda, f) \rightarrow f$  does not have a section.

**Remark 4.2:** At this point we unfortunately have to notice a minor notational clash. It is customary to use formulas (4.1) for the action of an element  $F = (\lambda, f)$ . This is a convention stemming in particular from [D] (just like  $\lambda$  which might as well be called  $\chi$ ). But it clashes with the equivariance of the action of  $\hat{\Gamma}$  on (pro)curves and (pro)twists, namely  $f\tau_{\alpha}f^{-1} = \tau_{f \cdot \alpha}$ . Using profinite complexes, one can literally view the action of  $\widehat{GT}$  as permuting the procyclic groups  $\langle \tau_{\alpha} \rangle$  for  $\alpha \in \hat{\mathcal{S}}$  ranging over the (pro)set of (pro)curves. The real and nontrivial justification for this lies in the isomorphism between the (profinite) group theoretic complex and the profinite curves complex (see §A.12). So it seems that the right convention, which is more than a convention, is to write  $F(y) = fy^{\lambda}f^{-1}$ , that is change  $f$  into its inverse. Because of the 2-cycle relation, this also amounts to switching arguments ( $f^{-1}(x, y) = f(x, y)$ ). In the sequel, partly out of habit and partly for purposes of comparison with previous papers, we will however retain the hitherto usual convention for the action of  $\widehat{GT}$ , using formulas (4.1) above.

It is interesting to keep in mind the action of  $\widehat{GT}$  on  $\hat{\Gamma}_{0, [5]}$  which extends (4.1), considered as an action on  $\hat{\Gamma}_{0,4} \simeq \hat{F}_2$  (see [LS1], [N1]). We will use the notation of the reminder below Lemma 2.4. Given  $F = (\lambda, f) \in \widehat{GT}$ , we write down its action on the twists  $\tau_1 = \tau_{12}$ ,  $\tau_2 = \tau_{23}$ ,  $\tau_3 = \tau_{34}$ ,

$\tau_4 = \tau_{45}$  and  $\tau_{15}$  which generate  $\hat{\Gamma}_{0,[5]}$ , namely:

$$\begin{aligned}
F(\tau_1) &= \tau_1^\lambda, \\
F(\tau_2) &= f(x_{23}, x_{12})\tau_2^\lambda f(x_{12}, x_{23}), \\
F(\tau_3) &= f(x_{34}, x_{45})\tau_3^\lambda f(x_{45}, x_{34}), \\
F(\tau_4) &= \tau_4^\lambda, \\
F(\tau_{15}) &= f(x_{23}, x_{12})f(x_{51}, x_{45})\tau_{15}^\lambda f(x_{45}, x_{51})f(x_{23}, x_{12}).
\end{aligned} \tag{4.3}$$

These formulas are to be compared with formulas (2.1), modulo the notational problem explained in the remark above. They say in particular that  $\widehat{GT}$  provides, as it should, solutions of Eq. (II), with  $h = g = f$ . Are these the only solutions? This question is a rephrasing of those raised at the end of Section 3.

We now concentrate on the group  $\mathcal{G} = \text{Out}(\hat{\mathcal{C}})$ . Here and below we always consider that  $\mathcal{C} = C(S)$  with  $S$  of type  $(0, 5)$ , not  $(1, 2)$ . So  $\mathcal{G} = \text{Aut}(\hat{\mathcal{C}})/\hat{\Gamma}$  with  $\Gamma = \Gamma(S) \simeq \Gamma_{0,[5]}$  and this is well-defined by Theorem 3.4. In particular  $\hat{\Gamma}$  acts transitively on  $\hat{V}$ , the set of pants decomposition i.e. the vertices of  $\hat{\mathcal{C}}$  and  $\hat{\mathcal{C}}_P$ . By contrast the action of  $\hat{\Gamma}_{1,[2]}$  is not transitive because it preserves the set of separating curves on the surface of type  $(1, 2)$ .

As a particular case of Proposition 3.3, we first get the following statement which we record explicitly because of its suggestive character:

**Proposition 4.3:** *Let  $F, F' \in \text{Aut}(\hat{\mathcal{C}})$  and  $e \in \hat{E}_P$  be an edge of  $\hat{\mathcal{C}}_P$ . If  $F(e) = F'(e)$ , then up to a possible reflection  $F$  and  $F'$  coincide as elements of  $\mathcal{G} = \text{Out}(\hat{\mathcal{C}})$ .  $\square$*

Let us make this completely explicit, using the standard pentagon  $\varpi_0 \subset \hat{\mathcal{C}}$ . We denote by  $a$  the ‘‘first side’’ of  $\varpi_0$ , given by the curves  $(\alpha_1, \alpha_2, \alpha_4)$ , or equivalently the pants decomposition  $P = (\alpha_1, \alpha_4)$  and  $P' = (\alpha_2, \alpha_4)$ , which are vertices of  $\mathcal{C}_P$  and define the edge  $a$ ; here  $a$  stands for ‘associativity’. By Lemma 2.4,  $a$  is determined by  $(\alpha_1, \alpha_2)$  alone. Let  $F \in \text{Out}(\hat{\mathcal{C}})$ ; reversing the normalization process detailed in Section 2, we find a representative, still denoted  $F$ , which fixes the curves  $\alpha_1$  and  $\alpha_4$ , and such that  $F(\alpha_2) = f \cdot \alpha_2$  with  $f \in \langle x_{12}, x_{23} \rangle'$ . By Proposition 4.3,  $F$  viewed again as an element of  $\text{Out}(\hat{\mathcal{C}})$  is determined by  $f$  up to reflection. Note that if  $f = 1$ , that is  $F(a) = a$ , then, as an automorphism,  $F \in \langle x_{45} \rangle \subset \hat{\Gamma}$ , up to reflection as usual.

So to any  $F \in \mathcal{G} = \text{Out}(\hat{\mathcal{C}})$  we have attached  $f \in \langle x, y \rangle'$  ( $x = x_{12}$ ,  $y = x_{23}$ ) which determines  $F$  up to reflection, just as what happens for the elements of  $\widehat{GT}$ , as recalled above. In fact our next goal is to prove:

**Theorem 4.4:**

$$\text{Out}(\hat{\mathcal{C}}) = \widehat{GT}.$$

*Proof:* First there is a canonical injection  $\widehat{GT} \hookrightarrow \mathcal{G} = \text{Out}(\hat{\mathcal{C}})$ , as already established in [BL]. It comes as the composition of two other natural injections. First we have  $\widehat{GT} \hookrightarrow \text{Out}^*(\hat{\Gamma})$  essentially by definition. From [BL], Proposition 4.14, we already know that this injection is in fact an isomorphism but we stress that we do *not* use this here; rather Theorem 4.2 will reprove this isomorphism. Second there is a natural injection  $\text{Out}^*(\hat{\Gamma}) \hookrightarrow \mathcal{G}$  (see [BL], Proposition 4.11) which comes from the fact that an element of  $\text{Out}^*(\hat{\Gamma})$  permutes the conjugacy classes of the procyclic

groups generated by twists, thus inducing an automorphism of the complex  $\hat{\mathcal{C}}$ . More precisely it actually induces an automorphism of the profinite group theoretic complex (see §A.12) and one then uses the fact (from [B]) that the latter complex coincides with the profinite curves complex. Finally the map  $Out^*(\hat{\Gamma}) \rightarrow \mathcal{G}$  is injective because  $\hat{\Gamma}$  is centerfree.

We now prove the opposite inclusion, namely  $\mathcal{G} \subset \widehat{GT}$ . Given  $F \in \mathcal{G}$  we singled out above the edge  $a \in \hat{E}_P$ , that is the first side of the standard pentagon, in order to attach the element  $f \in \langle x, y \rangle'$  to  $F$ . We could use any other edge  $e \in \hat{E}_P$  of the pants graph  $\hat{\mathcal{C}}_P$ , and we get a map  $f : \hat{E}_P \rightarrow \hat{F}'_2$  which to  $e \in \hat{E}_P$  assigns an element  $f_e \in \hat{F}'_2$ . We wish to show that this map is constant, that is  $f_e$  is in fact independent of  $e$ . Now in our case in mind that in our case  $\hat{\Gamma}$  acts transitively on  $\hat{E}_P$ , as was mentioned already in the last section. Indeed, any edge of  $\hat{\mathcal{C}}_P$  can be completed into a pentagon and there remains to apply Proposition 3.1. So in order to show that  $f$  is the constant map, it is enough to show that it is  $\hat{\Gamma}$ -invariant. Let  $F \in Aut(\hat{\mathcal{C}})$ ,  $e, e' \in \hat{E}_P$ . There exists  $g \in \hat{\Gamma}$  with  $g \cdot e = e'$ ; considering  $g^{-1}Fg$ , we are reduced to showing that  $F$  and  $g^{-1}Fg$  define the same value of  $f = f_e$ . But this is clear since  $\hat{\Gamma}$  is normal in  $Aut(\hat{\mathcal{C}})$ , so that  $g^{-1}Fg = g^{-1}FgF^{-1} = F = hF$  for some  $h \in \hat{\Gamma}$ .

More explicitly, this equivariance property goes as follows. Let again  $e = (\alpha, \beta)$  and  $e' = (\alpha', \beta')$  be two edges of  $\hat{\mathcal{C}}_P$ , defined by the corresponding pairs of curves. Write  $f_e = f$ ,  $f_{e'} = f'$  and suppose  $e' = g \cdot e$  with  $g \in \hat{\Gamma}$ , so that  $g\tau_\alpha g^{-1} = \tau_{g\alpha} = \tau_{\alpha'}$  and  $g\tau_\beta g^{-1} = \tau_{\beta'}$ . Then:

$$f'(\tau_{\alpha'}, \tau_{\beta'}) = gf(\tau_\alpha, \tau_\beta)g^{-1} = f(g\tau_\alpha g^{-1}, g\tau_\beta g^{-1}) = f(\tau_{\alpha'}, \tau_{\beta'}), \quad (4.4)$$

where the second equality is a formal algebraic fact.

There remains to show that  $f \in \hat{F}'_2$  coming from an element  $F \in \mathcal{G}$  satisfies relations (I), (II), (III) recalled above. The parameter  $\lambda \in \hat{\mathbb{Z}}^*$  appears explicitly in (II) and will be discussed in due time. We already mentioned in Remark 4.1 that (I) is an easy consequence of (III) and so does not really need to be proved independently. We will nevertheless first give a proof of (I) which emphasizes the roles of reflections, but because this proof is not logically necessary we will remain a bit sketchy. Given the above, especially thanks to the equivariance property embodied in (4.4), we are in any event essentially back to the usual group theoretic  $\widehat{GT}$  setting and the proofs very much follow the usual pattern (see e.g. [LS1]).

Everything in relations (I) and (II) is local on a piece  $S' \subset S \simeq S_{0,5}$  of type (0,4). We write as usual  $x = x_{1,2}$ ,  $y = x_{2,3}$ ,  $f = f(x, y) \in \langle x, y \rangle'$ . We can find a reflection  $r$  of the surface  $S$  ( $r^2 = 1$ ) such that  $r(x) = y$  and  $r(y) = x$ ; of course  $r \in Mod(S)$  but  $r$  does not belong to  $\Gamma(S) = Mod^+(S)$ . Then we compute that:

$$rFr(x) = fF(y) = f^{-1}(y, x)yf(y, x), \quad rFr(y) = rF(x) = r(x) = y.$$

Here we use a group theoretic notation which should be taken *cum grano salis*. We should for instance use procyclic groups instead of generators, that is replace  $x$  and  $y$  by  $\langle x \rangle$  and  $\langle y \rangle$ . Next we show that  $rFr$  and  $F$  define the same  $f$ . In fact we can find another reflection  $r'$  such that  $r'(x) = x^{-1}$  and  $r'(y) = y^{-1}$  and  $r'Fr'$  certainly defines the same element  $f$  as  $F$ . But  $rr' \in \Gamma(S)$  and so  $rFr = r'Fr'$  as elements of  $Out(\hat{\mathcal{C}})$ . We can now twist  $rFr$  by  $f^{-1}(y, x)$  (that is consider  $Inn(f^{-1}(y, x))rFr$ ) which does not alter it as an outer automorphism, and from the formulas

above we readily obtain that it is also defined by  $f'(x, y) = f^{-1}(y, x)$ . This implies relation (I). We remark that in terms of finite groups, we have been playing around with the dihedral group of order 10, preserving the cyclic order on 5 points. Elements of  $\Gamma(S)$  induce the cyclic subgroup of index 2 and the action of the reflections leads to relation (I).

We now come to relations (II) and (III), actually starting with (III) which is a little easier to deal with at this point and also much more important (see again Remark 4.1 above). Considering again the standard pentagon  $\varpi_0 \subset \mathcal{C}_P$  whose vertices are the pants decompositions  $P_i$ ,  $i \in \mathbb{Z}/5$ , let  $F \in \text{Aut}(\hat{\mathcal{C}})$  and normalize it so that  $F(P_1) = P_1$  ( $P_1 = (\alpha_1, \alpha_4)$ ). It acts on  $\varpi_0$  via formulas (2.1) (see however Remark 4.2) and we normalize it further so that  $f$  and  $g$  are commutators. We will freely use these topological tangential basepoints in what follows, using twists (that is profinite multitwists) on pants decompositions. The group  $\Gamma = \Gamma_{0,[5]}$  contains an element  $\rho$  of order 5, which is nothing but the ordinary rotation of the same order. Explicitly  $\rho = \tau_4\tau_3\tau_2\tau_1$  in terms of the standard generators of Artin braid group  $B_5$ , of which  $\Gamma$  is a quotient. Then  $\rho$  acts on the  $x_{i,i+1}$  ( $i \in \mathbb{Z}/5$ ) by shifting indices:  $\rho^{-1}x_{i,i+1}\rho = x_{i+1,i+2}$ , that is, it rotates the pentagon  $\varpi_0$ , as it should. Now let  $F' = \text{Inn}(f(x_{12}, x_{23})x_{12}^a x_{45}^b) \circ F$  with  $a, b \in \hat{\mathbb{Z}}$ . Using the equivariance property (4.4), Proposition 2.5 and injectivity (Theorem 1.2) we obtain that *there exist*  $a$  and  $b$  such that  $F' = \rho^{-1}(F) = \rho F \rho^{-1}$ . We can continue in this way and, going around the pentagon, we find that  $F = \text{Inn}(\pi)F$  for a certain element  $\pi \in \hat{\Gamma}$  of the form:

$$\pi = f(x_{12}, x_{23})x_{12}^{a_1}f(x_{34}, x_{45})x_{34}^{a_3}f(x_{51}, x_{12})x_{51}^{a_5}f(x_{23}, x_{34})x_{23}^{a_2}f(x_{45;51})x_{45}^{a_4}.$$

Here we used the fact that one element of the multitwist commutes with  $f$  (e.g.  $x_{45}$  commutes with  $f(x_{12}, x_{23})$ ) so that only one twist gets sandwiched between two  $f$ 's. So we find that  $\pi = 1$  and this is just the usual pentagon (see relation (III)) except for these possible twists with exponents  $a_i \in \hat{\mathbb{Z}}$ . But now the abelianization of  $\hat{\Gamma}$  is  $\hat{\Gamma}^{ab} \simeq \hat{\mathbb{Z}}^5$  generated by the  $x_{i,i+1}$ 's and looking at the image of  $\pi$  in  $\hat{\Gamma}^{ab}$  we obtain that  $a_i = 0$  for all  $i$ , because  $f$  is a commutator, which completes the proof that  $f$  satisfies relation (III).

Relation (II) is proved in much the same way. Just as relation (I), it takes place on  $S' \subset S \simeq S_{0,5}$  of type (0, 4), cut out by  $\alpha_4$ . We have an inclusion of the braid group on 3 strands  $B_3$  into  $\Gamma$ ;  $B_3$  is generated by  $\tau_1$  and  $\tau_2$  with  $\tau_1^2 = x_{12}$ ,  $\tau_2^2 = x_{23}$ . We recall from §2 that  $\tau_{13} = \sigma_2\sigma_1\sigma_2^{-1}$  and  $x_{13} = \tau_{13}^2$ . With this notation the lantern relation on  $S'$  reads:  $x_{12}x_{13}x_{23} = x_{45}$ , which commutes of course with  $x_{12}$ ,  $x_{13}$  and  $x_{23}$ .

There is now a triangle in  $\mathcal{C}_P$ , given by the pants decompositions  $P_1 = (\alpha_1, \alpha_4)$ ,  $P_2 = (\alpha_2, \alpha_4)$  and  $P_3 = (\alpha_{13}, \alpha_4)$ ,  $x_{13}$  being the twist along the curve  $\alpha_{13}$  which we refrain from drawing (it is the “third curve” in the classical lantern). Instead of  $\rho$ , one then uses  $\theta = \tau_1\tau_2$  which, regarded as an element of  $\Gamma_{0,[5]}$  satisfies  $\theta^3 = x_{12}x_{13}x_{23} = x_{45}$  and permutes  $x_{12}$ ,  $x_{23}$  and  $x_{13}$  cyclically. As noticed in [HLS], relation (II) actually takes place in any group generated by elements  $x$ ,  $y$  and  $z$  such that  $xyz$  is central. So we use of course the three twists  $x = x_{12}$ ,  $y = x_{23}$  and  $z = x_{13}$  and can just as well set  $x_{45} = 1$ , collapsing the curve  $\alpha_4$  to a point. Proceeding as above one gets a triangular identity of the form:

$$f(x, y)x^a f(z, x)z^b f(y, z)y^c = 1,$$

for some  $a, b, c \in \hat{\mathbb{Z}}$ . This time however, we are in the colored braid group on 3 strands, whose abelianization is generated by  $x, y$  and  $z$  with only relation  $x + y + z = 0$ . So looking at the abelianization of that relation we obtain that  $a = b = c$  and call that common value  $\mu$ , recovering relation (II). The value of  $\mu$  is actually not well-defined; it can be changed to  $-(1 + \mu)$  or in other words, setting  $\lambda = 2\mu + 1$  the sign of  $\lambda$  is still to be determined as can be proved algebraically (see in particular [N3], §4). One gets two elements of  $\widehat{GT}$  differing from each other by a reflection, namely  $(\pm\lambda, f)$ , and one has simply to compare with the original  $F$  to determine which of them coincides with it.  $\square$

As has been noticed several times already, the projection  $Aut(\hat{\mathcal{C}}) \rightarrow \mathcal{G} \simeq \widehat{GT}$  is split; a splitting is obtained in particular by using a tangential basepoint. One can consult e.g. [LS1] or the Appendix of [N1] for details. As is plain from the above, relations (I), (II) and (III) are closely connected with the torsion of  $\Gamma_{0,[5]} = \Gamma(S_{0,5})$  or in fact of the full mapping class group  $Mod(S_{0,5})$  ( $\Gamma_{0,[5]}$  has no 2-torsion). For further consequences of this remark, we refer to [LS3]. Returning to our setting, we finally notice that  $\lambda$  should be detected complex theoretically and it would be interesting to find more geometric characterizations than the one used above.

Theorem 4.4 pretty much settles the matter in genus 0 and it seems worthwhile to draw the consequences at this point, before moving to the case of arbitrary genus. Although it is an easy corollary of the above results we call again the next statement theorem, because of its intrinsic interest:

**Theorem 4.5:** *Let  $S \simeq S_{0,n}$ ,  $n > 4$ , a sphere with  $n$  marked point, so  $d(S) = n - 3 > 1$  and  $\hat{\Gamma}(S) \simeq \hat{\Gamma}_{0,[n]}$ . Then:*

i)  $Out(\hat{\mathcal{C}}(S)) = Out^*(\hat{\Gamma}(S)) = \widehat{GT}$ ;

ii) Any (inertia preserving) automorphism of an open subgroup of  $\hat{\Gamma}(S)$  is induced by an automorphism of  $\hat{\Gamma}(S)$ .

*Proof:* Let  $F \in Aut(\hat{\mathcal{C}}(S))$ ; after composing by an element of  $\hat{\Gamma}(S)$  we can assume that  $F$  fixes a pants decomposition. So we may consider the restriction of  $F$  to a piece  $T \subset S$  of type  $(0, 5)$ . By Theorem 4.4 that restriction is an element of  $\widehat{GT}$ , viewed as a subgroup  $Aut(\hat{\mathcal{C}}(T))$ , using a splitting as recalled above. But then it defines an automorphism of the group  $\hat{\Gamma}(S)$ , which in turn induces an automorphism of the full complex  $\hat{\mathcal{C}}(S)$ , whose image in  $Out(\hat{\mathcal{C}}(S))$  coincides with the image of  $F$ . By injectivity, i.e. Theorem 1.2, they coincide. This shows both equalities in i), reproving in particular the second one (see Proposition 4.14 in [BL]).

Let now  $\Gamma^\lambda \subset \Gamma(S)$  be a finite index subgroup (see §A.5). By Proposition 4.11 of [BL] and noting that all the intervening groups are centerfree, we get a natural injective map:  $Aut^*(\hat{\Gamma}^\lambda(S)) \hookrightarrow Aut(\hat{\mathcal{C}}(S))$ . When combined with i), this proves ii).  $\square$

To be more explicit, if for instance  $\Gamma^\lambda$  is normal in  $\Gamma(S)$  we get an exact sequence:

$$1 \rightarrow \hat{\Gamma}(S) \rightarrow Aut^*(\hat{\Gamma}^\lambda) \rightarrow \mathcal{G}^\lambda \rightarrow 1, \quad (4.5)$$

since  $\hat{\Gamma}(S) = Inn(\hat{\Gamma}(S))$ . We can also write:

$$1 \rightarrow \Gamma(S)/\Gamma^\lambda \rightarrow Out^*(\hat{\Gamma}^\lambda) \rightarrow \mathcal{G}^\lambda \rightarrow 1. \quad (4.6)$$

Both sequences are split and  $\mathcal{G}^\lambda \subset \mathcal{G} = \widehat{GT}$  is a subgroup whose meaning is clear at the level of Galois groups. In fact  $\Gamma^\lambda$  corresponds to a level structure  $\mathcal{M}^\lambda$ , that is a finite Galois (stack unramified) cover of  $\mathcal{M} = \mathcal{M}(S) \simeq \mathcal{M}_{0,[n]}$ . To the covering  $\mathcal{M}^\lambda/\mathcal{M}$  is attached a field of moduli  $K = K^\lambda$ , which is also a field of definition because the covering is Galois. Then  $K$  is a finite extension of  $\mathbb{Q}$  with Galois group  $G_K$  and there is an inclusion  $G_K \subset \mathcal{G}^\lambda$ , which is natural up to inner automorphism.

We remark that assertion ii) seems hardly accessible to the group theoretic methods which have been used hitherto in the profinite setting. Even the case of the pure group  $\hat{\Gamma}_{0,n} \subset \hat{\Gamma}_{0,[n]}$  would apparently be hard to settle (see [HS]). Assertion ii) could also be refined, much as in the discrete case (see [Iv1] and later articles on the subject), starting in particular with the consideration of morphisms between open subgroups of  $\hat{\Gamma}(S)$  rather than just automorphisms.

Before we slowly leave the world of genus 0, it is worth noting the following result which parallels Dyer-Grossman rigidity statement in the discrete case, namely  $Out^*(B_n) = \mathbb{Z}/2$  for  $n > 2$ , and improves the results of [LS1]:

**Theorem 4.6:**  $Out^*(\hat{B}_n) = \widehat{GT}$  for  $n > 3$ .

*Proof:* This is an easy consequence of Theorem 4.5 and the following isomorphism and inclusions (see [LS1], Proposition A.4):

$$\Gamma_{0,n+1} \simeq K_n/Z \simeq B_n/Z \subset \Gamma_{0,[n+1]}.$$

This is valid for all  $n > 1$  and  $Z = \langle \omega_n \rangle \simeq \mathbb{Z}$ . It readily extends to the respective completions. For  $n > 3$ , one has  $Out^*(\hat{\Gamma}_{0,[n+1]}) = \widehat{GT}$  by i) of Theorem 4.5 and  $Out^*(\hat{\Gamma}_{0,n+1})$  is an extension of  $\mathcal{S}_{n+1}$  by  $\widehat{GT}$ , according to assertion ii) of the same theorem. Now in fact  $B_n/Z$  can be identified with the subgroup of  $\Gamma_{0,[n+1]}$  which maps to the stabilizer of the – say – last point via the natural projection  $\Gamma_{0,[n+1]} \rightarrow \mathcal{S}_{n+1}$ . This stabilizer is isomorphic to  $\mathcal{S}_n$  and self-normalizing in  $\mathcal{S}_{n+1}$ . We thus obtain  $Out^*(\hat{B}_n/Z) = \widehat{GT}$  and one then shows that  $Aut(\hat{B}_n/Z) = Aut(\hat{B}_n)$ .  $\square$

## 5. The Grothendieck-Teichmüller group, strictly positive genus

We now move to the case of arbitrary genus. In this section we will have to rely in part on the material and results developed in [HLS] and [NS] to which we refer for more detail and geometric background. In that sense, what follows should not be considered as self-contained because some notions and constructions are simply too heavy to recall in complete detail, but we will try to emphasize the main points. The proofs of the statements are of course complete, using the results in the two above references (and some others).

In modular dimension 2, one finds types (0, 5) and (1, 2), which are of course closely related. In particular the corresponding curves graphs  $C(S_{0,5})$  and  $C(S_{1,2})$  are isomorphic but we will have to spell out this isomorphism and the differences between these two cases are in fact responsible for the introduction of the subgroup  $\mathbb{F} \subset \widehat{GT}$  governing the situation in strictly positive genus. In modular dimension 3, there are 3 types of surfaces, namely (0, 6), (1, 3) and (2, 0). The first and last one are again closely related (e.g.  $C(S_{2,0}) \simeq C(S_{0,6})$ ) but the middle one will turn out to represent the generic case; in particular:  $\mathbb{F} = Out^*(\hat{\Gamma}_{1,[3]})$ . This universality is explored in a different way in the Annex.

We will start by disposing of the two exceptional cases with nontrivial center, that is types  $(1, 2)$  and  $(2, 0)$ , from the point of view of group automorphisms. Of course there is nothing to do here concerning curves complexes because of the above mentioned isomorphisms. We thus state (writing  $\Gamma_2 = \Gamma_{2,0}$ ):

**Proposition 5.1:**

- i)  $Out^*(\hat{\Gamma}_{1,[2]}/Z) = Out^*(\hat{\Gamma}_2/Z) = \widehat{GT}$ ;
- ii)  $Out^*(\hat{\Gamma}_{1,[2]}) = Out^*(\hat{\Gamma}_2) = \widehat{GT} \times \mathbb{Z}/2$ .

*Proof:* Here  $Z = \langle \iota \rangle \simeq \mathbb{Z}/2$  denotes the center of the group at hand, which in both cases is of order 2, generated by the (hyper)elliptic involution; for simplicity we denote the latter by the common letter  $\iota$ . As to the first assertion,  $\Gamma_{1,[2]} = \Gamma_{1,2} \times Z$  is the direct product of the pure group by its center and the same assertion is valid for the completed groups. So  $\hat{\Gamma}_{1,[2]}/Z = \hat{\Gamma}_{1,2} \subset \hat{\Gamma}_{0,[5]}$ . Moreover  $\hat{\Gamma}_{1,2}$  is of index 5 and self-normalizing in  $\hat{\Gamma}_{0,[5]}$  (cf. §A.4). Combining this with ii) of Theorem 4.5 (for  $n = 5$ ) we get the first assertion. The case of  $\hat{\Gamma}_2$  is immediate, since  $\hat{\Gamma}_2/Z = \hat{\Gamma}_{0,[6]}$  and there remains to apply Theorem 4.5 again.

Granted i), the proof of assertion ii) follows that of the analogous statement in the discrete case. In particular, the case of  $\Gamma_2$  is treated in detail in [McC] (with  $Out(\Gamma_2/Z) = \mathbb{Z}/2$ ). The crux of the matter is to compute the abelianizations of the intervening groups. One has  $\hat{\Gamma}_2^{ab} = \Gamma_2^{ab} \simeq \mathbb{Z}/10$  and  $\hat{\Gamma}_{1,[2]}^{ab} = \Gamma_{1,[2]}^{ab} = \mathbb{Z}/2 \times \Gamma_{1,2}^{ab} \simeq \mathbb{Z}/2 \times \mathbb{Z}/12$ . Here we used that  $\Gamma_{1,2}^{ab}$  is cyclic of order 12, which can be seen in several ways. For instance one can notice that  $\Gamma_{1,2} \simeq B_4/Z$  and that  $B_4^{ab}$  is free cyclic, whereas the image of its center in the abelianization has order 12; this is because it is generated by the element  $\omega_4$  which is a product of 12 twists.

Using this information and proceeding as in [McC], one gets ii). In both cases the exceptional outer automorphism, indeed a *bona fide* automorphism, is defined by mapping any twist  $\tau$  to the product  $\tau\iota$ . This fixes the involution  $\iota$  (so does the action of  $\widehat{GT}$ ) and defines an automorphism which commutes with the action of  $\widehat{GT}$  and has order 2.  $\square$

Type  $(1, 2)$  plays an important part in the geometric understanding of the situation and we thus have to examine it in more detail, first summarizing known and for a large part already mentioned facts. We denote as in [BL] by  $Aut^\#(\hat{C}(S)) \subset Aut(\hat{C}(S))$  the subgroup of *type preserving* automorphisms, that is those automorphisms such that a curve and its image have the same topological type. The following facts are proved in [BL] (adapting [FL1] to the profinite setting). First  $F \in Aut(\hat{C}(S))$  is type preserving if (and of course only if) it preserves the set of separating curves. Second, if  $S$  is not of type  $(1, 2)$ , we actually have an equality  $Aut^\#(\hat{C}(S)) = Aut(\hat{C}(S))$ , this being true but empty for  $d(S) = 0, 1$ . This property has already been used above a few times. The proposition below summarizes some information, part of which either follows readily from the above and the rest comes from [BL], §4:

**Proposition 5.2:** *Let  $S \simeq S_{1,2}$  be a surface of type  $(1, 2)$ , then:*

- i)  $Aut^\#(\hat{C}(S)) \subset Aut(\hat{C}(S))$  has index 5 and  $F \in Aut(\hat{C}(S))$  belongs to  $Aut^\#(\hat{C}(S))$  if and only if there exists a separating curve whose image by  $F$  is separating;
- ii)  $Aut^\#(\hat{C}(S))/\hat{\Gamma}(S) = \widehat{GT}$ ;
- iii) Let  $S$  be connected with  $d(S) > 2$ ,  $\alpha \in \hat{S}(S)$  a separating curve such that  $S_\alpha = S' \amalg S''$  with

$S'$  of type  $(1,2)$ . Let  $F \in \text{Aut}(\hat{C}(S))$  fixing  $\alpha$ , so that it induces  $F' \in \text{Aut}(\hat{C}(S'))$ . Then  $F'$  is type preserving i.e.  $F' \in \text{Aut}^\#(\hat{C}(S'))$ .  $\square$

Recall now the elementary topological construction of the isomorphism  $C(S_{1,2}) \simeq C(S_{0,5})$ . Namely, associated with the (topological) elliptic involution  $\iota$  there is a unramified covering of degree 2, say  $p : S_{1,2} \rightarrow S_{0,5} \simeq S_{1,2}/\langle \iota \rangle$  (see Figure 2 below) where the points on  $S_{0,5}$  are marked rather than deleted, so that  $p$  ramifies exactly at the (topological) Weierstrass points. We assume, as in §A.4, that the two marked points on  $S_{1,2}$  project to the 5-th point on  $S_{0,5}$ . The projection  $p$  then looks as on the following figure:

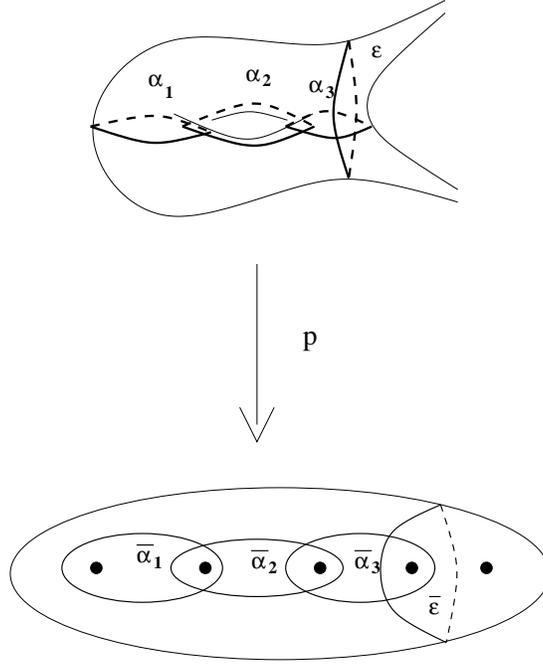


Figure 2

Because  $\iota$  is central in  $\Gamma_{1,[2]}$ , any loop  $\alpha \in \mathcal{S}(S_{1,2})$  coincides with its image  $\iota(\alpha)$  (proof:  $\tau_{\iota(\alpha)} = \iota\tau_\alpha\iota = \tau_\alpha$ ). So to any loop  $\alpha \in \mathcal{S}(S_{0,5})$  one can associate *one* (any) component of its preimage  $p^{-1}(\alpha)$  and this yields a well-defined map  $\mathcal{S}(S_{1,2}) \rightarrow \mathcal{S}(S_{0,5})$  which is actually an isomorphism. It will be a little more convenient to use the inverse map, which is readily extended to the full complexes in a simplicial way. We thus get an explicit isomorphism  $\phi : C(S_{1,2}) \rightarrow C(S_{0,5})$ .

It is quite interesting to remark that  $\phi$  does *not* induce an isomorphism between the respective pants graphs  $C_P(S_{1,2})$  and  $C_P(S_{0,5})$  and these are in fact not isomorphic. This elementary topological fact, which will be further illustrated below, is in some sense responsible for the difference between types  $(0,5)$  and  $(1,2)$  and much more generally between genus 0 and strictly positive genus. An important point is to distinguish between separating and non separating curves on  $S_{1,2}$ . This is in contrast with the case of  $S_{1,1}$  and  $S_{0,4}$  where this phenomenon does not arise, since there are no separating curves on  $S_{1,1}$ . In order to make these observations a little more precise, we list the following easy facts, which however will not be used directly below: 1) if  $\alpha, \beta \in \mathcal{S}(S_{1,2})$  intersect in exactly one point, they are both non separating; 2) if they intersect in 2 points, they

are either both non separating or of opposite types (separating and non separating); in other words 2 separating curves cannot intersect in 2 points and this is a specific feature of type (1, 2); 3) if  $\alpha \in \mathcal{S}(S_{0,5})$ , it determines a partition of 1, 2, 3, 4, 5 into a pair and a triplet; then  $p^{-1}(\alpha)$  is separating if and only if 5 belongs to the pair; 4) by the above, if  $\alpha, \beta \in \mathcal{S}(S_{0,5})$  are joined by an edge in  $C_P(S_{0,5})$  but their preimages  $\alpha' = p^{-1}(\alpha)$  and  $\beta' = p^{-1}(\beta)$  are separating (which is determined using 3), then  $\alpha'$  and  $\beta'$  are *not* joined by an edge in  $C_P(S_{1,2})$ .

In this section we will try to conform to the common convention according to which curves are denoted by Greek letters and the corresponding twists by the “corresponding” Roman letters ( $a = \tau_\alpha$ ,  $b = \tau_\beta$ , etc.). Of course the equally standard notation for the twists along the curves of the standard pentagon make a very bad start. So we compromise. In the figure above, the twists  $a_i$  ( $i = 1, 2, 3$ ) along the curves  $\alpha_i$  generate a copy of  $B_4$  and  $\phi$  induces the familiar injective map (with the same name)  $\phi : \Gamma_{1,2} \simeq B_4/Z \hookrightarrow \Gamma_{0,[5]}$  defined by  $\phi(a_i) = \tau_i$ . We use bars to denote the image of a curve by the projection  $p$ . One should beware of the fact that  $\bar{\alpha}_i$  does not go through the marked points; in fact  $p^{-1}(\bar{\alpha}_i)$  consists of two copies of  $\alpha_i$ , that is  $p^{-1}(\bar{\alpha}_i) = 2\alpha_i$ , counting with multiplicity, which by the definition of  $\phi$  means that  $\phi(\alpha_i) = \bar{\alpha}_i$ . Note that the  $\bar{\alpha}_i$ 's are no other than the  $\alpha_i$ 's of Figure 1 in Section 2. Clearly,  $\bar{\varepsilon}$  on the figure is nothing but  $\bar{\alpha}_4$  but we avoid that piece of notation, as  $\alpha_4$  does not really exist (see below). Here we do not discuss the – equivalently up to relabeling – loops involving points 1 and 5, but they will play a role below (see Figures 4 and 5).

Consider again  $F \in \text{Aut}^\sharp(\hat{C}(S_{1,2}))$ , a type preserving automorphism. We can translate assertion ii) of Proposition 5.2 as follows; twisting by an element of  $\hat{\Gamma}_{1,[2]}$ , and using that  $F$  is type preserving, we may assume that it fixes the pants decomposition  $(\alpha_1, e)$ . Then  $F_1 = \phi \circ F \circ \phi^{-1}$  is an element of  $\widehat{GT}$ , say  $F_1 = (\lambda, f)$  which acts on  $\hat{C}(S_{0,[5]})$  and also on  $\hat{\Gamma}_{0,[5]}$  in the “standard way”, that is according to formulas (4.3). Viewed from upstairs,  $F$  acts on the  $\tau_i$ 's ( $i = 1, 2, 3$ ) as in those formulas, where  $x_{i,i+1} = \tau_i^2 = a_i^2 = \bar{\alpha}_i$  (again for  $i = 1, 2, 3$ ). Please pay attention to the exponents. Then  $\tau_4$  does not belong to the image of  $\Gamma_{1,2}$  (it would permute marked points and Weierstrass points) but  $\tau_4^2 = x_{45}$  does belong to that image. Explicitly, by a form of the lantern relation:  $x_{45} = (\tau_1\tau_2\tau_1)^2 = (\tau_1\tau_2)^3 = x_{12}x_{13}x_{23}$  ( $x_{13} = \tau_1^{-1}x_{23}\tau_1$ ). Finally, and this will turn out to be quite important, one can compute the image of  $e \in \Gamma_{12}$  (the twist along  $\varepsilon$ ) under  $\phi$  and find that  $\phi(e) = x_{45}^2$ . This can be traced to the fact that the preimage  $p^{-1}(\bar{\varepsilon})$  of the loop encircling points 4 and 5 has only one component, namely  $e$ , counted with a multiplicity 2 which the map  $\phi$  does not directly take into account. This can also be seen from a well-known identity for a surface of genus 1 with one boundary component, namely the boundary twist  $e$  can be expressed as:  $e = (\tau_1\tau_2\tau_1)^4 = (\tau_1\tau_2)^6$ . In particular, we find that of course  $F(e) = e^\lambda$ .

We are finally in a position to address the cases of strictly positive genus. We will say that a surface  $S$  (hyperbolic of finite type) is *generic* or of *generic type* if  $d(S) > 1$ ,  $g(S) > 0$  and  $Z(\Gamma(S)) = \{1\}$ . Generic surfaces in that sense exist only for  $d(S) \geq 3$  and the only generic type in modular dimension 3 is type (1, 3), the other two types of the same dimension being (0, 6) and (2, 0). We will see in this section and in the Annex that type (1, 3) indeed deserves to be called generic. In that direction, we first note the following straightforward but suggestive topological lemma:

**Lemma 5.3:** *A surface  $S$  (hyperbolic of finite type) is of generic type (i.e.  $d(S) > 1$ ,  $g(S) > 0$  and  $Z(\Gamma(S)) = \{1\}$ ) if and only if it contains a piece of type  $(1, 3)$ , i.e. there exists  $\sigma \in C(S)$  such that the subsurface  $S_\sigma$  has a connected component of type  $(1, 3)$ .  $\square$*

Let  $S$  be of generic type and  $F \in \text{Out}(\hat{C}(S))$  an outer automorphism of its profinite curves complex of which we pick a representative in  $\text{Aut}(\hat{C}(S))$  (which we call again  $F$ ). We may find in  $S$  a piece  $T$  of type  $(1, 3)$  and select  $\sigma \in C(S)$  so that  $T$  is a component of  $S_\sigma$ . We then complete  $\sigma$  into a pants decomposition, i.e. select a top dimensional cell of  $C(S)$  whose closure contains  $\sigma$ . Since  $F$  is type preserving (Cf. [BL]), we may then twist  $F$  by an element of  $\hat{\Gamma}(S)$  so that the resulting automorphism preserves that decomposition. So we may concentrate on  $T$  and simply assume that  $S = T$ . Here we are actually using two nontrivial facts from [B] and [BL]. Explicitly we first identify  $F$  with an element of  $\text{Aut}(\hat{C}_*(S))$  using that the automorphism groups of  $\hat{C}(S)$  and  $\hat{C}_*(S)$  are isomorphic. Then we use that there is a natural injection  $\hat{C}_*(T) \hookrightarrow \hat{C}_*(S)$  which is just the completion of the injection in the discrete case. Equivalently, the closure of  $C_*(T) \subset C_*(S)$  inside  $\hat{C}_*(S)$  is indeed the full completion  $\hat{C}_*(T)$  and not just a quotient thereof. Finally we remark that by the usual injectivity property, the restriction of  $F$  to  $\text{Aut}(\hat{C}(T))$  completely determines  $F \in \text{Aut}(\hat{C}(S))$ . We will soon discover that conversely, any outer automorphism of  $\hat{C}(T)$  extends an automorphism of  $\hat{C}(S)$ . So, for the time being we take a look at the local situation and set  $S = T$ .

Pictorially, the situation looks like on the following figure:

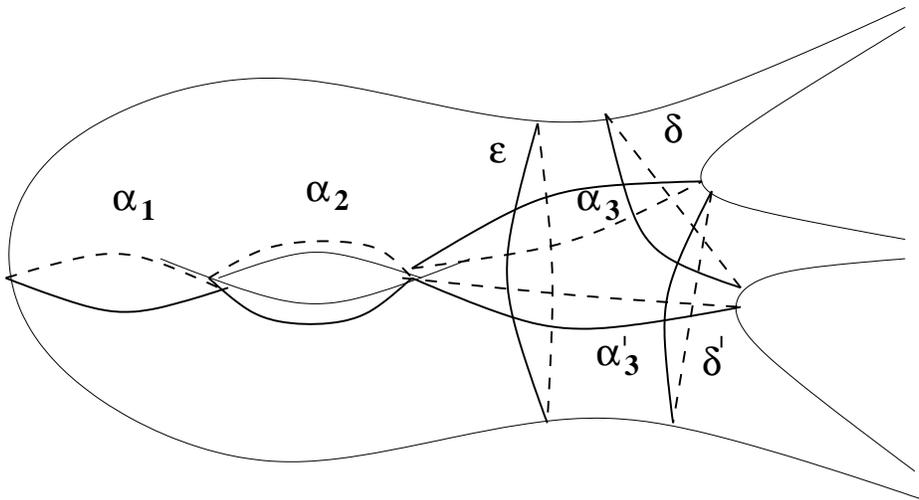


Figure 3

The surface  $S$  is covered by a piece  $S_0$  of type  $(0, 5)$  obtained by cutting along the loop  $\alpha_1$  and by a piece  $S_1$  of type  $(1, 2)$  obtained by cutting along  $\delta'$ . The intersection  $S_0 \cap S_1$  is of type  $(0, 4)$ . One should be a little cautious at this point and below, concerning the differences between boundary curves and marked points and also about allowed permutations of points or boundary components. For the sake of simplicity and clarity, we will usually leave the routine justifications to the reader; one should remember for instance that boundary twists are central in the appropriate modular groups and that the elements of  $\widehat{GT}$ , viewed as automorphisms, do not permute points.

Starting again from  $F \in \text{Aut}(\hat{C}(S))$  we may and do assume that it fixes the decomposition

$(\alpha_1, \varepsilon, \delta')$ . It thus determines two restrictions  $r_0(F)$  and  $r_1(F)$  to  $S_0$  and  $S_1$  respectively, using the same facts about localization that were recalled above. By assertion iii) of Proposition 5.2,  $r_1(F)$  is type preserving. So by Theorem 4.4 and ii) of Proposition 5.2, we derive two elements  $F_0, F_1 \in \widehat{GT}$ , where  $F_0$  is just the image of  $r_0(F)$  in the outer automorphism group of  $S_0$ , and  $F_1$  is constructed as detailed in the discussion below Figure 2 that is, roughly speaking:  $F_1 = \phi \circ r_1(F) \circ \phi^{-1}$ .

We now have to express that the actions of  $F_0$  and  $F_1$  match on the intersection  $S_0 \cap S_1$ . To this end, we introduce:

**Definition 5.4:** Let  $F_0 = (\lambda_0, f_0)$  and  $F_1 = (\lambda_1, f_1)$  be two elements of  $\widehat{GT}$ . They will be called a *compatible pair* if  $\lambda_0 = \lambda_1$  and there exist  $a, b, c \in \hat{\mathbb{Z}}$  such that:

$$f_1(\tau_1^2, \tau_2^2) = \tau_2^{2a} f_0(\tau_1, \tau_2^4) \tau_1^{2b} (\tau_1 \tau_2^2)^{2c}. \quad (5.1)$$

Before returning to the geometric motivation behind this definition, we show that things are tighter than they may look at first sight. In particular the values of  $a, b$  and  $c$  are in fact determined from that identity. Computing them requires a slight adaptation of computations appearing in [NS]. As we have seen,  $\lambda$  extends the cyclotomic character on the Galois group (and should therefore be denoted  $\chi$ ;  $\lambda$  is Drinfeld's original piece of notation and has become customary). Similarly one can extend to  $\widehat{GT}$  the additive Kummer characters  $G_{\mathbb{Q}} \rightarrow \hat{\mathbb{Z}}$ . We will need only the case of  $\rho_2 : G_{\mathbb{Q}} \rightarrow \hat{\mathbb{Z}}$  the Kummer character at 2, which can be extended to a character of  $\widehat{GT}$  in several equivalent ways: see [NS], §5, including the closing remark of that paragraph. If  $F = (\lambda, f) \in \widehat{GT}$ , we write  $\rho_2(f)$ , as the value of  $\rho_2$  depends on  $f$  only. Then we have:

**Proposition 5.5:**

i)  $\rho_2(f_0) = \rho_2(f_1)$ ;

ii) Writing  $\rho$  for this common value, one has:  $a = -4\rho, b = -2\rho, c = 2\rho$ .

*Proof:* We start with ii), for which it suffices to read [NS] §§2,5 carefully; the arguments presented there literally apply in our case and produce the values:  $a = -4\rho_2(f_0), b = -2\rho_2(f_0), c = 2\rho_2(f_0)$ . There remains to prove i), to which end we use a result from [N3] which we first proceed to recall. There is a natural morphism  $B_3 = \langle \tau_1, \tau_2 \rangle \rightarrow GL_2(\hat{\mathbb{Z}})$  defined as usual by:

$$\tau_1 \rightarrow \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \tau_2 \rightarrow \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

The same formulas define a natural map:  $\hat{B}_3 \rightarrow \widehat{GL}_2(\hat{\mathbb{Z}}) \rightarrow GL_2(\hat{\mathbb{Z}})$ , where the second map is natural surjection to the modular completion. We write  $\bar{\tau}_i$   $i = 1, 2$  for the image of  $\tau_i$  under the composite map. Given  $F = (\lambda, f)$ , we may consider the specialization  $f(a, b) \in GL_2(\hat{\mathbb{Z}})$  for any  $a, b \in GL_2(\hat{\mathbb{Z}})$ . Then (4.2) in [N3] asserts that:

$$f\left(\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} 1 & -4\rho_2(f) \\ 0 & 1 \end{pmatrix}.$$

Note that the first argument of  $f$  is just  $\bar{\tau}_1^2$ . We insist that this is valid for any  $F \in \widehat{GT}$ , not necessarily in  $G_{\mathbb{Q}}$  (many more such formulas are available in that case). Now let  $A_3 = \langle \tau_1, \tau_2^2 \rangle \subset B_3$  be the subgroup of the braids such that their third strand returns to its place, i.e. the preimage of

the stabilizer of 3 for the natural surjection  $B_3 \rightarrow \mathcal{S}_3$ . This subgroup has only one defining relation, namely that  $(\tau_1 \tau_2^2)^2$  is central. Moreover the inclusion can be completed into a natural injection  $\hat{A}_3 \hookrightarrow \hat{B}_3$ , i.e. the closure of  $A_3$  in  $\hat{B}_3$  is the full completion  $\hat{A}_3$ . We may now specialize Eq.(5.1) using the composite map  $\hat{A}_3 \hookrightarrow \hat{B}_3 \rightarrow GL_2(\hat{\mathbb{Z}})$  defined via:

$$\tau_1 \rightarrow \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \tau_2^2 \rightarrow \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}.$$

This is admissible because the image of  $(\tau_1 \tau_2^2)^2$  is central, and in fact that element maps to the identity matrix. We now compare the two sides of this specialization of (5.1). By the result above, on the left hand side, we get an upper triangular matrix with 1's on the diagonal and  $-4\rho_2(f_1)$  as the upper right entry. As for the right hand side, since  $\tau_2^4$  maps to the identity and  $f_0(\tau_1^2, \tau_2^2)$  is a commutator, that is has weight 0 in both arguments, the factor  $f_0(\tau_1, \tau_2^4) \tau_1^{2b}$  maps to 1. So do the factors  $\tau_2^{2a} = \tau_2^{-8\rho_2(f_0)} = (\tau_2^4)^{-2\rho_2(f_0)}$  and  $(\tau_1 \tau_2^2)^{2c}$ , since  $\tau_2^4$  and  $(\tau_1 \tau_2^2)^2$  both map to 1. There remains only  $\tau_1^{-4\rho_2(f_0)}$  which maps to the upper triangular matrix with 1's on the diagonal and  $-4\rho_2(f_0)$  in the upper right corner. This proves the equality  $\rho_2(f_0) = \rho_2(f_1)$ .  $\square$

We return to automorphisms, that is to  $F \in \text{Aut}(\hat{C}(S))$ , giving rise to  $F_0$  and  $F_1$  as above. Although we started from an automorphism of  $\hat{C}(S)$ ,  $F_0$  and  $F_1$  can be viewed as group automorphisms. We write  $F_0 = (\lambda_0, f_0)$ ,  $F_1 = (\lambda_1, f_1)$  and examine what happens on  $S_0 \cap S_1$ . Both  $F_0$  and  $F_1$  induce an automorphism of the modular group of that surface, which is in fact pure, since  $F$  fixes  $\alpha_1$  and  $\delta'$ . Collapsing these loops to points, we are left with two (inertia preserving) automorphisms of  $\hat{F}_2$  which must coincide as outer automorphisms, that is in  $\text{Out}^*(\hat{F}_2)$ , since they both induce the original  $F$ . First it is plain that  $\lambda_0 = \lambda_1 (= \lambda)$  and  $F_0(e) = F_1(e) = e^\lambda$ , recalling that  $F$  fixes  $\varepsilon$ .

There only remains to recall the standard  $\widehat{GT}$ -action on  $\hat{C}(S_{0,5})$  (namely (4.3)) and the one we detailed on  $\hat{C}(S_{1,2})$ , using the isomorphism  $\phi$ . The crux of the matter is to compare the actions of  $F_0$  and  $F_1$  on  $\alpha_3$ . On the one hand:

$$F_0(\alpha_3) = f_0(a_3, e) a_3^\lambda f_0(e, a_3),$$

because of the action of  $\widehat{GT}$  on  $\hat{\Gamma}(S_0) \simeq \hat{\Gamma}_{0,[5]}$  (boundary curves do not play any role here). This formula can for instance be deduced immediately looking at the pentagon  $(\varepsilon, \alpha_3, \delta, \delta', \alpha'_3)$  in  $S_0$  (or of course using the more general yoga associated with the notion of ‘‘lego’’; see below). On the other hand, recalling the discussion below Figure 2,  $a_3$  can be identified with  $\tau_3$ , so that  $F_1(\alpha_3) = F_1(\tau_3)$  is given by:

$$F_1(\tau_3) = f_1(x_{34}, x_{45}) \tau_3^\lambda f_1(x_{45}, x_{34}),$$

where one has to carefully translate notations as in that discussion. As a result  $F_0$  and  $F_1$  coincide in  $\text{Out}^*(\hat{F}_2)$  if and only if  $\lambda_0 = \lambda_1$  and:

$$f_1(x_{34}, x_{45}) = x_{45}^a f_0(a_3, e) x_{34}^b \omega_3^c,$$

that is, using  $a_3 = \tau_3$ ,  $e = x_{45}^2$ , if and only if they form a compatible pair of elements of  $\widehat{GT}$ .

We remark that the computation above is of course closely related to the one leading to the so-called relation (IV) (see [NS] and references therein) which was also a motivating factor underlying the development of the “lego” in [HLS] and [NS]. But it is not identical. Here we are trying to “glue” two *a priori* distinct actions and thus do not require that  $F_0 = F_1$ , that is  $f_0 = f_1$ . This remark will be expanded below.

**Proposition-Definition 5.6:** *The set of compatible pairs forms a group  $\mathbb{I}$ , which can be viewed as a subgroup of  $\widehat{GT}$  by mapping  $(F_0, F_1) \in \mathbb{I}$  to  $F_0 \in \widehat{GT}$ .*

*Proof:* Use a fixed decomposition of  $S_{1,3}$  as above, then  $(F_0, F_1) \in \mathbb{I}$  form a compatible pair if and only if they induce the same outer automorphism on the intersection, a condition which is clearly preserved under inversion and composition. So we get a group  $\mathbb{I}$  which is indeed a subgroup of  $\widehat{GT}$  as indicated in the statement. Note that  $F_1$  is obviously determined by  $F_0$  according to (5.1), which we can view as the definition of  $f_1$  (see Proposition 5.5 ii)). In other other words  $F_0 = (\lambda, f_0) \in \widehat{GT}$  lies in  $\mathbb{I}$  if and only if  $F_1 = (\lambda, f_1)$  is an element of  $\widehat{GT}$ , where  $f_1$  is defined by the right hand side of (5.1). Conversely,  $F_0$  is also determined by  $F_1$  because the occurrence of two different  $F_0$ 's for a given  $F_1$  would contradict the injectivity result.  $\square$

The above definition and description may sound a little artificial at this point. Things will appear much more intrinsic below. The Annex gives a rather different view of  $\mathbb{I}$  by means of an additional (somewhat implicit) relation in  $\widehat{GT}$ .

Starting from a compatible pair  $F_0, F_1$  of elements of  $\widehat{GT}$ , we will now describe a procedure which constructs from it an automorphism  $F \in Out^*(\hat{\Gamma}(S))$  and thus also  $F \in Out(\hat{C}(S))$ . To this end we basically follow the strategy of [HLS] and [NS] (see in particular [HLS], Theorem 4 and [NS], Proposition 8.1) which as mentioned above we will not however recall in full detail. The novelty here is that we start from *two* possibly distinct elements of  $\widehat{GT}$  and the point is that the constructions presented in these papers generalize very smoothly in that direction. The (outer) automorphisms (or actions) we get as an output will be said to be *of lego type* and it will evolve that they describe in fact *all* the automorphisms of  $\hat{C}(S)$  and  $\hat{\Gamma}(S)$ .

So we start again from a generic surface  $S$  and a compatible pair  $F_0 = (\lambda, f_0), F_1 = (\lambda, f_1)$  of elements of  $\widehat{GT}$ . We also fix a pants decomposition  $\mathcal{P} = (\alpha_1, \dots, \alpha_d)$ , with  $d = d(S)$ , or actually from a topological tangential basepoint associated to it (that last point will remain implicit in the actual recipe). In the parlance of [NS], pick a quilt on  $\mathcal{P}$ , that is add seams to the pants; in terms of moduli spaces,  $\mathcal{P}$  defines a maximally degenerate point on  $\overline{\mathcal{M}}(S)$  and then pick a tangential basepoint there. We now wish to construct  $F \in Aut^*(\hat{\Gamma}(S))$  from these data, and  $F$  will coincide with  $F_0$  and  $F_1$  on the appropriate pieces. First  $F$  is normalized so that  $F(\alpha_i) = \alpha_i$  or equivalently  $F(a_i) = a_i^\lambda$ . Let now  $\beta \in \mathcal{S}(S)$  be an arbitrary curve. There exists a finite sequence of pants decompositions, say  $\mathcal{P}_1 = \mathcal{P}, \mathcal{P}_2, \dots, \mathcal{P}_n = \mathcal{P}'$  such that  $\beta$  appears in the end decomposition  $\mathcal{P}'$  and two consecutive decompositions  $\mathcal{P}_k$  and  $\mathcal{P}_{k+1}$  are joined by an edge in the pants complex  $C_{\mathcal{P}}(S)$ . In other words, two consecutive elements of the chain differ by an elementary move, of type  $A$  or  $S$  (see A.8). We phrase everything in the language of [HLS] and [NS]; clearly the language of complexes could be used as well and the translation is obvious.

Now we define  $F(b)$  as  $F(b) = t^{-1}b^\lambda t$  where  $t = t(\mathcal{P}, \mathcal{P}')$  is a *transfer factor*. In turn  $t$  decomposes as  $t = t_n \dots t_2 t_1$  where  $t_k = t(\mathcal{P}_k, \mathcal{P}_{k+1})$  is an *elementary transfer factor*, describing the

passage between two decompositions differing by an elementary move only. In terms of complexes the edges of the pants graph  $C_P(S)$  are of two types,  $A$  and  $S$  and we will define an elementary transfer factor associated to each of the two types.

There remains to define the elementary transfer factors, which we do as follows. Let  $\mathcal{P}$  and  $\mathcal{P}'$  now be two decompositions differing by an elementary move. Up to relabeling we may assume that:  $\mathcal{P} = (\alpha_1, \dots, \alpha_d)$ ,  $\mathcal{P}' = (\alpha'_1, \dots, \alpha'_d)$ ,  $d = d(S)$ ,  $\alpha_i = \alpha'_i$  for  $i > 1$  and  $(\alpha_1, \alpha'_1)$  sit either on a surface of genus 0 (type  $A$ ) or 1 (type  $S$ ). In the former case  $\mathcal{P}, \mathcal{P}'$  define an  $A$ -move, in the second an  $S$ -move. We set:

$$\begin{aligned} t(\mathcal{P}, \mathcal{P}') &= f_0(a_1, a'_1), & \text{if } (\mathcal{P}, \mathcal{P}') \text{ is of type } A, \\ t(\mathcal{P}, \mathcal{P}') &= a_1'^{-8\rho} f_1(a_1^2, a_1'^2) a_1^{8\rho} (a_1 a_1' a_1)^{\lambda-1}, & \text{if } (\mathcal{P}, \mathcal{P}') \text{ is of type } S. \end{aligned}$$

Here  $\rho = \rho_2(f_0) = \rho_2(f_1)$  as above. Since  $\lambda - 1 = 2\mu$  is even (in  $\hat{\mathbb{Z}}$ ), we can also write  $(a_1 a_1' a_1)^{2\mu}$  in the second line and  $(a_1 a_1' a_1)^2$  generates the center of the group (isomorphic to  $\hat{B}_3$ ) generated by  $a_1$  and  $a_1'$ .

**Remark 5.7:** In [HLS] we treated the case  $l = 1$ ,  $\rho_2 = 0$ , so the factors in the elementary transfer  $S$ -factor above disappear. One thus obtains a group (denoted  $\Lambda$  there) which in particular contains the closed subgroup of  $G_{\mathbb{Q}}$  which is the intersection of the kernels of the cyclotomic character and the Kummer character at 2. That is, it corresponds to the extension of  $\mathbb{Q}$  obtained by adjoining all roots of 1 and 2. The main purpose of [NS] is to remove these conditions. In geometric terms, it was enough in [HLS] to work with maximally degenerate basepoints whereas in [NS] one had to take a closer look at and keep track of the tangential basepoints. This is done by using a rigidifying structure, consisting of pants with seams (there dubbed “quilts”), as is done for instance when defining a Nielsen twist as a real number (not just modulo  $2\pi$ ). Although this technically complicates the analysis the guidelines remains very much the same so that the newcomer might want to first take a look at the simpler (but more restricted) version of the construction appearing in the first paper.

In order for the above recipe to make sense at all, one needs to prove that the value of  $F(b)$  does not depend on either the choice of  $\mathcal{P}'$  containing  $\beta$  nor on the sequence connecting  $\mathcal{P}$  and  $\mathcal{P}'$ . This done (see below) one then needs to show that this assignment of the value  $F(b)$  for any twist defines an automorphism of  $\hat{\Gamma}(S)$ . The first and crucial step, that is the fact that  $F(b)$  is well-defined, relies essentially on a result from [HLS] (Theorem 2) asserting that the (full, two dimensional) pants complex  $C_P(S)$  is simply connected. What one needs to show is thus that the definition of  $F(b)$  is invariant under the elementary homotopies, that is when going around a face of the pants complex  $C_P(S)$  (for the remainder of this section and at variance with the rest of the paper,  $C_P(S)$  will by default refer to the full two-dimensional pants complex). There are 4 types of such faces, labeled respectively  $(3A)$ ,  $(3S)$ ,  $(5A)$  and  $(6AS)$  in [HLS] (see also [NS] and [M]). As the names indicate,  $(3A)$  (resp.  $(3S)$ ) defines a triangle of  $A$ -moves (resp.  $S$ -moves) on a surface of type  $(0, 4)$  (resp.  $(1, 1)$ ); in turn  $(5A)$  corresponds to the familiar pentagon on a surface of type  $(0, 5)$ , giving rise to a pentagon of  $A$ -moves. Finally  $(6AS)$  defines an hexagon on a surface of type  $(1, 2)$  and mixes  $A$ - and  $S$ -moves. This is a keypoint here.

Before coming back to it in detail, we briefly turn to the second step, namely how to show that the assignment of the image to every twist defines a morphism (that it is invertible is no problem).

In other words, one has to check that these images satisfy the defining relations of  $\Gamma(S)$  or rather  $\widehat{\Gamma}(S)$ , but it amounts to the same because completion is right exact as a functor. At this point one uses the infinite presentation of  $\Gamma(S)$  with all twists as generators, given in [Ge1] (and refined in [FL2]) and recalled in [HLS] (Theorem 1; see also [NS], Theorem 9.2). As far as we are concerned here, the point is that the relations are all localized on surfaces of types  $(0, 4)$  and  $(1, 1)$ , so involve only sequences of  $A$ -moves or  $S$ -moves but do not mix the two types.

The conclusion of the above analysis is that, when starting from a pair  $(F_0, F_1)$  of elements of  $\widehat{GT}$  (rather from just one element as in [HLS], [NS]), the only point of difference lies in the analysis of the faces of type  $(6AS)$ . All the rest can be copied quite literally, namely the verification of the good behavior of the definition with respect to faces of types  $(3A)$ ,  $(3S)$  and  $(5AS)$  ([HLS], Lemma 6; [NS], Proposition 8.1) and the fact that the procedure described above, which we refer to as *lego* (a word used by Grothendieck in his *Esquisse*), defines an endomorphism, and actually an automorphism, of  $\widehat{\Gamma}(S)$  ([HLS], §3, Step 2; [NS], Proposition 9.1). Note that in these proofs one effectively uses the  $\widehat{GT}$ -relations for both  $F_0$  and  $F_1$  but not the fact that they form a compatible pair.

We come to the analysis of the cycles of type  $(6AS)$ , which is easy but conceptually intriguing and hopefully revealing. Indeed this is in some sense the *only* place where the role of the *genus* is really visible. One way to put it is to recall that  $\phi$ , as defined at the beginning of this section provides an isomorphism between  $C(S_{1,2})$  and  $C(S_{0,5})$  and between the respective completions as well, but *not* between the pants complexes (or graphs)  $C_P(S_{1,2})$  and  $C_P(S_{0,5})$ . Here is what  $(6AS)$  looks like:

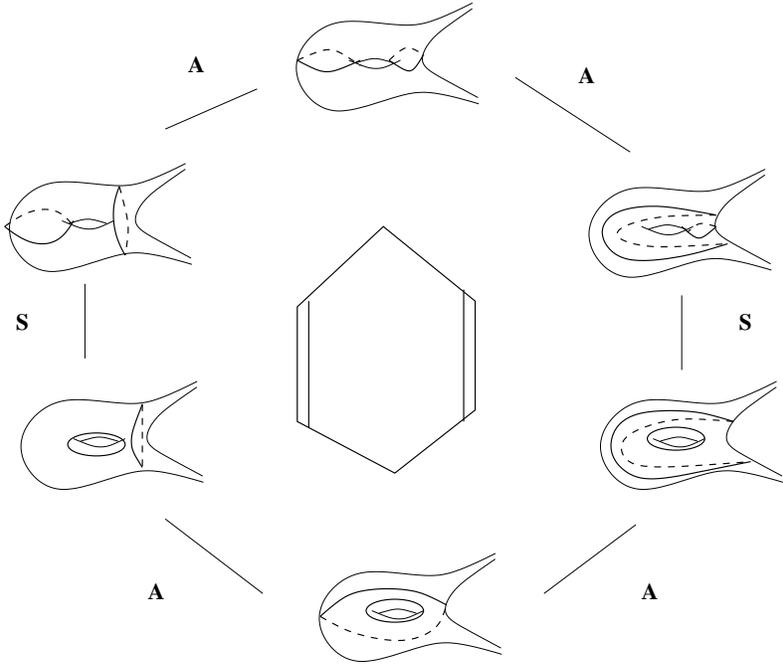


Figure 4

Now the projection onto  $S_{0,5}$ , obtained by quotienting via the elliptic involution as in Figure 2 looks as follows:

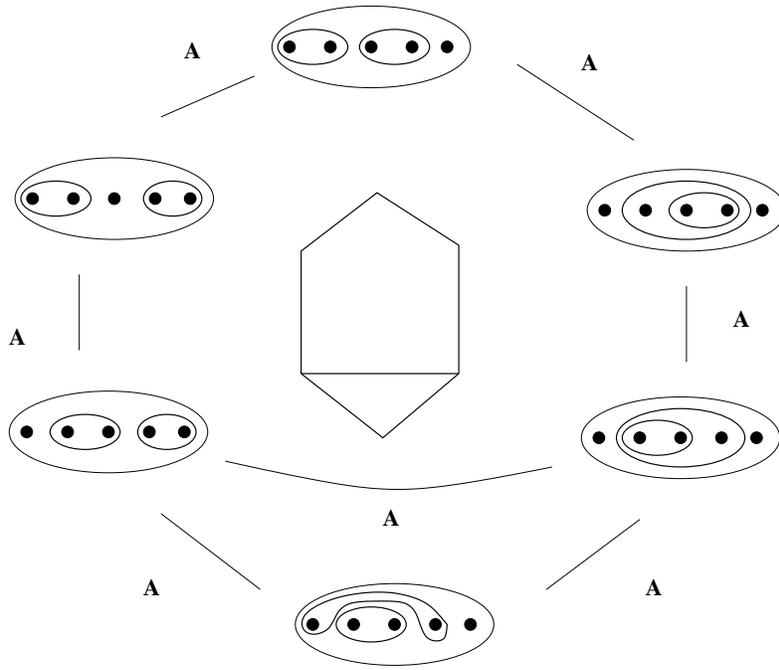


Figure 5

The point is that the projection of the hexagon  $6AS$  in  $C_P(S_{1,2})$  decomposes into a pentagon (of type  $(5A)$ ) and a triangle (of type  $(3A)$ ) in  $C_P(S_{0,5})$  as is schematically indicated by the two hexagons inside the figures. We used single lines for edges of type  $A$  and double lines for edges of type  $S$  in the pants graphs. This is an illustration of the phenomenon we already came across (see the discussion after Proposition 5.2). More precisely, Let  $e = e_1$  (resp.  $e' = e_3$ ) be the separating curve appearing on the right hand (resp. left hand) vertical side in Figure 4. These curves  $e$  and  $e'$  intersect in 4 points in  $\mathcal{S}(S_{1,2})$  and so are not joined by an edge in  $C_P(S_{1,2})$ , whereas their images in  $\mathcal{S}(S_{0,5})$  do give rise to an edge of the pants graph  $C_P(S_{0,5})$ . In order to get a closed circuit “upstairs”, that is in  $C_P(S_{1,2})$  one thus has to insert, in a somewhat arbitrary way, an additional non separating curve  $e_2$ , as in the bottom picture of Figure 4. It is then easy to see that the images of these three curves  $e_1, e_2, e_3$  form a triangle in  $C_P(S_{0,5})$ . We insist on these elementary topological considerations because in the end they seem to form the core of the difference of the analysis in genus 0 and that in strictly positive genus.

Return to a compatible pair  $(F_0, F_1)$ , that is an element of  $\mathbb{F}$ . Assume for the moment that  $\lambda = 1, \rho = 0$ ; this is only for the sake of clarity and the appropriate factors will then be restored, following [NS] (see also Remarks 5.10 below). Then one finds that  $F(b)$  has defined above does not change when going along a circuit of type  $(6AS)$  if and only if the following relation holds:

$$f_0(e_3, a_1) f_1(a_2^2, a_3^2) f_0(e_2, e_3) f_0(e_1, e_2) f_1(a_1^2, a_2^2) f_0(a_3, e_1) = 1. \quad (R)$$

This is indeed a generalization of the relation with the same name introduced in [HLS], except that there, one has  $f_0 = f_1 = f$  (we will return to that point below). It contains 4 factors of type  $f_0$  (with no squares) corresponding to the 4 edges of type  $A$  (genus 0), and 2 factors of type  $f_1$  (with squares) corresponding to the 2 edges of type  $S$  (genus 1). Now we want to show that if  $(F_0, F_1)$  is

a compatible pair (with the above restrictions on  $\lambda$  and  $\rho$ ) relation  $(R)$  above is satisfied. Indeed, one transforms all  $A$ -terms in the relation using the compatibility condition (5.1) in Definition 5.4. For instance,  $f_0(a_3, e_1) = f_1(x_{34}, x_{45})$ . This done, one gets a 6-terms relations in  $\widehat{\Gamma}_{0,5}$ , which decomposes into a triangle and a pentagon; it holds true because  $F_1 \in \widehat{GT}$ , so satisfies relations  $(II)$  (triangle) and  $(III)$  (pentagon) and this completes that sketch of proof.

We do not detail this because we can use [NS] at this point, also removing the restrictions on  $\lambda$  and  $\rho$ . Indeed one finds there a version  $(R')$  of  $(R)$  which is jazzed up to all elements of  $\widehat{GT}$  (see [NS], §1); we simply need to distinguish between the  $f_0$  and  $f_1$  factors (which is immediate) in order to get the analog of  $(R)$  above, as refined to *all* compatible pairs. The proof that compatible pairs satisfies  $(R')$  then literally follows the computation in [NS], p. 543, whose geometric significance is as above: the projection of the hexagon of mixed type  $(6AS)$  decomposes into a triangle  $(3A)$  and a pentagon  $(5A)$ .

At this point we have actually shown the following:

**Proposition 5.8:** *Given a generic surface  $S$ , any element  $F \in \mathbb{F}$ , that is any compatible pair  $(F_0, F_1)$  of elements of  $\widehat{GT}$  ( $F = F_0$ ) gives rise to a well-defined element of  $Out^*(\widehat{\Gamma}(S))$ .*

*Proof:* This has been already proved, except for the following. We started from a richer set of data, namely not only from  $S$  but also a piece of  $S$  of type  $(1,3)$  and a pants decomposition of it (or of the whole of  $S$ ). But, following again [HLS] and [NS], the procedure described above produces an element of  $Out^*(\widehat{\Gamma}(S))$  which is independent of these additional data.  $\square$

We can now essentially reinterpret this and see that the situation is much more “natural” that it may still appear at this point. It should also clarify the meaning of the above. Let us first formally introduce the “lego”:

**Definition 5.9:** Let  $S$  be a surface of generic type and let  $F \in Aut^*(\widehat{\Gamma}(S))$  be an automorphism. We say that it is *of lego type* (or gives rise to an action of lego type) if there exists a – necessarily compatible – pair  $(F_0, F_1)$  of elements of  $\widehat{GT}$  such that  $F$  induces the outer automorphism defined by  $F_0$  (resp.  $F_1$ ) on a piece of type  $(0,4)$  (resp.  $(1,1)$ ). The action is of *strict lego type* if  $F_0 = F_1$ .

Here we did implicitly collapse the boundary curves of the pieces of types  $(0,4)$  or  $(1,1)$  and the action is just as in (4.1) where in the case of genus 1,  $x = \tau_1^2$ ,  $y = \tau_2^2$ . Note that if  $g(S) = 0$ , any  $F \in \widehat{GT}$  induces an action of strict lego type. If  $S$  is of generic type and an automorphism is of lego type, it is of strict lego type if and only if  $F = F_0 = F_1$  and by Definition 5.4 and Proposition 5.5, that element  $= F = (\lambda, f) \in \widehat{GT}$  satisfies the following equation ( $\rho = \rho_2(f)$ ):

$$f(\tau_1^2, \tau_2^2) = \tau_2^{-8\rho} f(\tau_1, \tau_2^4) \tau_1^{-4\rho} (\tau_1 \tau_2^2)^{4\rho}. \quad (IV)$$

This translates the equivalent fact that an action is strictly of lego type if it is of lego type and commutes with the elliptic involution. We will see below, and in fact have essentially already proved that any automorphism is of lego type, but first we pause briefly in order to insert two observations:

**Remarks 5.10:**

1. Relation (or equation)  $(IV)$  appears in [NS] and was discovered a little earlier in particular by computing the action of the Galois group on the modular groups (see the contribution of

H.Nakamura in [GGA]). Any element  $\sigma \in G_{\mathbb{Q}}$  gives rise to  $F_{\sigma} \in \widehat{GT}$  inducing an action of strict lego type. Relation (IV) for Galois elements can be proved in a very simple way, as is done in [NS], Theorem 2. It is remarkable that the proof involves only  $\mathbb{P}^1 \setminus \{0, 1, \infty\}$  and order two coverings of that space which are also of genus 0. Abstractly speaking, granted that the Galois action is of lego type, it is clear that it should commute with the elliptic involution and thus be of strict lego type. It amounts roughly to the observation that the generic elliptic involution, that is the order 2 automorphism of the generic point of the stack  $\mathcal{M}_{1,1}$ , if defined over  $\mathbb{Q}$ .

2. Using these facts on the action of  $G_{\mathbb{Q}}$ , we could in the proofs of the above statements (and below as well) restrict ourselves to considering compatible pairs elements  $(F_0, F_1) \in \widehat{GT}$  with  $\lambda = 1, \rho = 0$ . This does simplify the matter (and the writing) quite a bit and can be achieved as follows. Any  $\sigma \in G_{\mathbb{Q}}$  gives rise to an action of strict lego type, that is the corresponding  $F_{\sigma} = (\lambda_{\sigma} = \chi(\sigma), f_{\sigma})$  satisfies relation (IV) above. On the other hand adding (all) roots of unity and (all) roots of 2 gives rise to disjoint extensions of  $\mathbb{Q}$ , so that given a pair  $(\alpha, \beta) \in \hat{\mathbb{Z}}^* \times \hat{\mathbb{Z}}$ , there exists  $\sigma \in G_{\mathbb{Q}}$  with  $\chi(\sigma) = \alpha, \rho_2(\sigma) = \beta$ . We also recall that  $\rho_2$  extends the character with the same name on  $G_{\mathbb{Q}}$  ( $\rho_2(\sigma) = \rho_2(f_{\sigma})$ ) and that it is a cocycle on  $\widehat{GT}$ , that is  $\rho_2(F \circ G) = \rho_2(F) + \lambda(F)\rho_2(G)$  for  $F, G \in \widehat{GT}$  (see e.g. [NS], Corollary 5.2). Consider now  $(F_0, F_1) \in \mathbb{I}$  as above, and let  $\sigma \in G_{\mathbb{Q}}$  such that  $\chi(\sigma) = \lambda^{-1}, \rho_2(\sigma) = -\lambda^{-1}\rho$ . Then  $F' = (F'_0 = F_{\sigma} \circ F_0, F'_1 = F_{\sigma} \circ F_1) \in \mathbb{I}$  satisfies  $\lambda(F') = 1, \rho_2(F') = 0$  and it would be enough to prove e.g. Proposition 5.8 for such elements. One can then set  $\lambda = 1, \rho = 0$  both in the compatibility condition (5.1) of Definition 5.4 and in definition of the transfer factors.

In [NS], the computation on p.543 actually shows that (IV) implies (R'). Recall that (R') is like (R) above, only with  $f_0 = f_1 = f$  and some correcting  $\lambda$ - and  $\rho$ -factors added (see [NS], Introduction). In fact, (R') is, almost by its very definition, a necessary and sufficient condition for an element of  $\widehat{GT}$  to extend to strictly positive genus and give rise to an action of strict lego type. Now we can show the following statement, which arose in a discussion with Leila Schneps:

**Proposition 5.11:** *Relations (R') and (IV) are equivalent.*

*Proof:* That (IV) implies (R') is well-known and was mentioned above. Conversely, if  $F = (\lambda, f) \in \widehat{GT}$  satisfies (R'), one gets an action of strict lego type on  $\hat{\Gamma}_{1,[2]}$  from  $F$  by applying the above construction. It coincides with the original action of  $F$  on a piece of type (1, 1) (cutting along  $\varepsilon$  in Figure 2) and so by injectivity the two actions coincide everywhere (i.e.  $F_0 = F_1$ ), which in fact means that they coincide on the twist  $a_3$  (see again Figure 2). This translates into the fact that  $F$  satisfies (IV). The new point here is of course the use of injectivity as a rigidifying factor.  $\square$

We will denote by  $\mathbb{I}_s \subset \mathbb{I}$  the subgroup of  $\mathbb{I}$  given by the elements which induce an action of strict lego type. In other words  $F \in \widehat{GT}$  belongs to  $\mathbb{I}_s$  if and only if it satisfies (IV), or equivalently (R'). Note that  $\mathbb{I}_s$  is indeed a subgroup of  $\widehat{GT}$ , as can be seen by its geometric definition and is proved directly in [NS], §6 (which does not use (III')).

We now gather the information in:

**Theorem 5.12:** *Let  $S$  be of generic type, that is  $d(S) > 1, g(S) > 0, Z(\Gamma(S)) = \{1\}$ . Equivalently  $S$  is of type  $(g, n)$  with  $3g - 3 + n > 2, g > 0, (g, n) \neq (2, 0)$ ; then:*

*i)  $Aut^*(\hat{\Gamma}(S)) = Aut(\hat{C}(S))$ ;*

- ii)  $Out^*(\hat{\Gamma}(S)) = Out(\hat{C}(S)) = \mathbb{I}$ ; in particular, every automorphism of  $\hat{\Gamma}(S)$  is of lego type;
- iii) Every inertia preserving automorphism of every open subgroup of  $\hat{\Gamma}(S)$  is induced by an inertia preserving automorphism of the full group  $\hat{\Gamma}(S)$ ;
- iv) In particular if a subgroup of  $\hat{\Gamma}(S)$  is open and normal, then it is of the form  $\hat{\Gamma}^\lambda$  with  $\Gamma^\lambda$  a finite index subgroup of  $\Gamma(S)$  and there is a short exact sequence:

$$1 \rightarrow \Gamma(S)/\Gamma^\lambda \rightarrow Out^*(\hat{\Gamma}^\lambda) \rightarrow \mathbb{I}^\lambda \rightarrow 1, \quad (5.2)$$

with  $\mathbb{I}^\lambda$  an open subgroup of  $\mathbb{I}$ .

*Proof:* There remains only to gather information. First i) is a consequence of ii). It is nonetheless stated explicitly because of its importance. Since it obtains in genus 0 as well (Cf. Theorem 4.5), the equality  $Aut^*(\hat{\Gamma}(S)) = Aut(\mathbb{C}(S))$  is actually valid for  $d(S) > 1$ ,  $Z(\Gamma(S)) = \{1\}$  i.e.  $3g - 3 + n > 1$ ,  $(g, n)$  not equal  $(1, 2)$  or  $(2, 0)$ . These exceptions are completely understood (Cf. Proposition 5.1).

Assertion ii) has essentially been proved already. Let us quickly review the argument. Start from an element of  $Out(\hat{C}(S))$  and select a representative  $F \in Aut(\hat{C}(S))$ . Choose a piece  $T \subset S$  of type  $(1, 3)$ ; this is possible since  $S$  is of generic type. Then pick a pants decomposition  $\mathcal{P}$  adapted to  $T$ , that is containing the boundary curves of  $T$ . Next twist  $F$  by an element of  $\hat{\Gamma}(S)$  so that it preserves  $\mathcal{P}$  and call the new element  $F$  again. Get two subsurfaces of  $T$ , say  $T_0$  and  $T_1$  of types  $(0, 5)$  and  $(1, 2)$  respectively. We can assume that they are determined by curves of  $\mathcal{P}$ . Then  $F$  induces automorphisms of the graphs  $\hat{C}(T_0)$  and  $\hat{C}(T_1)$ , which are both isomorphic to  $\hat{C}$ . Applying Theorem 4.4 produces two elements  $F_0$  and  $F_1$  of  $\widehat{GT}$ , which form a compatible pair because  $F$  acts on the complex  $\hat{C}(T)$ . So we get an element of  $\mathbb{I}$  which we can identify with  $F_0$  (or with  $F$  itself). By Proposition 5.8, which uses the constructions of [HLS] and [NS],  $F$  induces an element of  $Out^*(\hat{\Gamma}(S))$ . That element in turn uniquely determines an automorphism of  $\hat{C}(S)$  (recall that  $\hat{\Gamma}(S)$  is centerfree). Finally injectivity (Theorem 1.5) guarantees that we have gone full circle and recovered our original outer automorphism of  $\hat{C}(S)$ , which completes the proof of ii). Note that *a posteriori* everything is of course intrinsic, that is independent of the choices that were made in the course of the proof.

Assertions iii) and iv) are easy consequences and are proved exactly as in genus 0; see Theorem 4.5 and the explicitation below the proof of that result. The interpretation of  $\mathbb{I}^\lambda$  in the Galois case, that is of the intersection  $\mathbb{I}^\lambda \cap G_{\mathbb{Q}}$  is also as discussed there. It can perhaps be added (and this is valid of course in genus 0 as well) that by [NiSe] the open subgroups of  $\hat{\Gamma}(S)$  coincide with the subgroups of finite index (that is, any subgroup of finite index is open). So the sequences (4.5) and (4.6), as well as (5.2) above (we omitted the analog of (4.5) for the sake of brevity only) describe the situation for a general subgroup of finite index in  $\hat{\Gamma}(S)$ . Again, one could address and classify morphisms between open subgroups, not just automorphisms.  $\square$

So for a generic surface  $S$ ,  $Out^*(\hat{\Gamma}(S)) = Out^*(\hat{\Gamma}(S_{1,3}))$ . We reprove part of this statement in the Annex. Type  $(1, 3)$  is the only non exceptional (!) type in modular dimension 3, the other two being  $(0, 6)$  and  $(2, 0)$ . It can be seen as copies of  $S_{0,5}$  and  $S_{1,2}$  (the two pieces in dimension 2) intersecting along a piece of dimension 1 (necessarily of type  $(0, 4)$ ). The Annex gives a more symmetric view of the situation. At any rate, the above does illustrate again the two levels

principle, but *cum grano salis* through the important role played by type (1, 3). We also note that one has explicit formulas for the action of  $\mathbb{I}$  on generators of  $\hat{\Gamma}_{g,[n]}$ . They are obtained by an easy modification of the formulas displayed in [NS], §11, changing elementary transfer factors from  $f$  to  $f_0$  or  $f_1$  in the appropriate way, as we did above for relation (R).

In closing we add again a few remarks and questions. We have seen that any automorphism is of lego type (see Definition 5.9). Let  $\mathbb{I}_s \subset \mathbb{I}$  be the subgroup of automorphism of *strict* lego type. An element of  $\mathbb{I}$  belongs to  $\mathbb{I}_s$  if and only if it satisfies the equivalent relations (R') and (IV) (Cf. Proposition 5.11), or equivalently again if it commutes with the elliptic involution. Note that  $\mathbb{I}_s$  is contained in the group denoted  $\mathbb{I}$  in [NS]. That  $\mathbb{I}_s$  is a subgroup of  $\mathbb{I}$  results from its geometric characterization; this is also proved directly in [NS], §6 (in which relation (III') is not used).

So we have a nested sequence:

$$G_{\mathbb{Q}} \subset \mathbb{I}_s \subset \mathbb{I} \subset \widehat{GT} \subset \text{Out}^*(\hat{F}_2)$$

and it may be interesting to speculate on the possible equalities occurring in that chain. Clearly of course  $\widehat{GT}$  is strictly smaller than  $\text{Out}^*(\hat{F}_2)$ , which is a very big group indeed. At the other end, comparing  $G_{\mathbb{Q}}$  with  $\mathbb{I}_s$  or any reasonably explicit group “of geometric origin” represents a major challenge.

Let us very briefly comment on the possible coincidences of the *GT*-like groups by simply stressing again their intrinsic meaning. First  $\mathbb{I}_s \stackrel{?}{=} \mathbb{I}$  if and only if all automorphisms are *strictly* of lego type, or else commute with the elliptic involution. Put a little differently, if  $(F_0, F_1)$  is a compatible pair of elements of  $\widehat{GT}$ , then in fact  $F_0 = F_1$ .

The possible equality  $\mathbb{I} \stackrel{?}{=} \widehat{GT}$  would be most interesting, as it would mean that “everything already happens in genus 0”. Recall that in terms of motives, that is in the pro- $\ell$  (or pronilpotent, or prounipotent) setting, the situation (including polyzeta values etc.) can be understood (mostly conjecturally though!) at present only in genus 0, by means of the unconditional (not conjectural) mixed Tate motives, whereas the situation in strictly positive genus is quite murky (and not of Tate type). Also one may venture to imagine that the equality  $\mathbb{I} \stackrel{?}{=} \widehat{GT}$  holds only if in fact  $\mathbb{I}_s \stackrel{?}{=} \widehat{GT}$ , that is all elements of  $\widehat{GT}$  would satisfy relation (IV). Recall (Cf. Remark 4.1 above) that by H.Furusho’s recent work the pro- $\ell$  (or prounipotent) version of  $\widehat{GT}$  is defined by the 5-cycle relation only, and this may well be true of the full profinite group  $\widehat{GT}$  itself. The equality  $\mathbb{I}_s \stackrel{?}{=} \widehat{GT}$ , that is showing that relation (IV) is actually redundant, would thus reduce everything to pentagons, including in higher genus.

Another very interesting question is to try and see whether one can “remove the star”, that is determine whether for  $d(S) > 1$  (not for  $d(S) = 1$ ), any automorphism is inertia preserving. In other words, do we have  $\text{Aut}^*(\hat{\Gamma}(S)) \stackrel{?}{=} \text{Aut}(\hat{\Gamma}(S))$  for  $d(S) > 1$ ? This is certainly not just a matter of technicality and actually leads to a hoard of questions and analogies. We will content ourselves with the barest remarks. On the one hand this is true in the discrete setting, that is for  $\Gamma(S)$ , and comes for instance from the characterization of powers of twists inside  $\Gamma(S)$  in pure group theoretic terms (which can be traced to the work of N.Ivanov in the early 1980’s). Of course, this is ultimately clear from the equality  $\text{Out}(\Gamma(S)) = \mathbb{Z}/2$  but it seems useful to isolate that statement

(as is done in [McC]). On the other hand this can be seen as the analog at the level of fundamental groups and for the moduli stacks of curves, of the local correspondence in birational anabelian geometry, developed by J.Neukirch, F.Bogomolov, F.Pop and J.Koenigsmann in particular (see [Sz] for a beautiful review). Here the close analogy comes between the decomposition group of a (rank 1) valuation inside the Galois group of finitely generated field and the centralizer of a twist in the Teichmüller group. Now in order to characterize (powers of) twists inside Teichmüller groups, Thurston’s theory seems hard to dispense with. In particular one can isolate the fact that pseudo-Anosov mapping classes are self-centralizing. This is easy using Thurston’s completion of Teichmüller space (i.e. measured foliations); see e.g. [Iv2], Lemma 8.13 (this was written up in full detail by J.D.McCarthy in the early 1980’s). So let us finish with a test question, still a far cry from what is really desirable: Let  $g \in \Gamma(S)$  be a pseudo-Anosov diffeomorphism; is it true that the centralizer of  $g$  in the completion  $\hat{\Gamma}(S)$  is procyclic?

**Annex: On surfaces of type (1, 3)**

Here we add two observations with the goal of clarifying the “universal” character of type (1, 3). The first one has to do with the finite presentation of  $\Gamma_{g,n}$  derived in [Ge2] and the second one is the connection with generalized Artin braid groups. In the main body of the text we considered  $S_{1,3}$  as covered by two subsurfaces of type (0, 5) and (1, 2) respectively, intersecting along a piece of type (0, 4). Here is a more symmetric view of the situation:

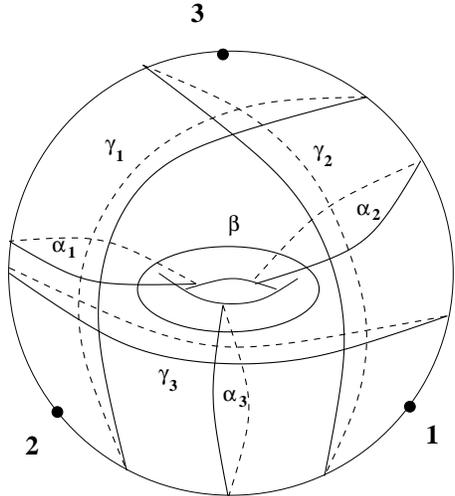


Figure 6

We use again the standard convention of Greek letters ( $\alpha, \beta, \gamma, \delta, \dots$ ) for loops and Latin letters ( $a, b, c, d, \dots$ ) for the corresponding twists. We already know that it will make little difference to consider surfaces with boundary twists or points and to allow or not for permutations of points. So let us for a moment consider that we are working with boundary components (which cannot be permuted) and let  $\delta_i$  ( $i = 1, 2, 3$ ) denote a small loop encircling point  $i$  (the  $d_i$ ’s do not appear on Figure 6).

Let  $\Gamma_1^3$  denote the modular group of genus 1 with 3 boundary components (this is not the topologists standard notation). Then following [Ge2], the group  $\Gamma_1^3$  is generated by the  $a_i, d_i$

( $i = 1, 2, 3$ ) and  $b$  with three kinds of relations: 1) the obvious commutation relations for non intersecting curves (in particular all  $a_i$ 's and  $d_i$ 's commute; the  $d_i$ 's are central), 2) the classical braid relations for each  $a_i$  with  $b$  ( $a_i b a_i = b a_i b$ ) and 3) the *star relation*:

$$(a_1 a_2 a_3 b)^3 = d_1 d_2 d_3. \quad (\star)$$

The main content of [Ge2] (see also [Ge1]) is that this is a completely general phenomenon. For any  $(g, n)$  one finds a similar collection of generating twists and the relations are of the three types above. The star relation enables one to recover all classical relations supported on surfaces of modular dimensions 1 and 2 (see [Ge2], §2 for details).

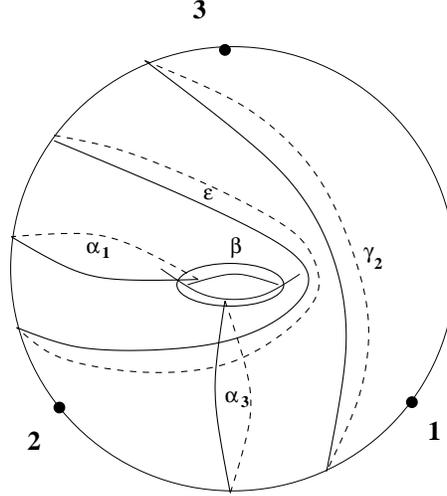


Figure 7

Consider for instance the surface of genus 1 with one marked point (labeled 2) and one boundary curve ( $\gamma_2$ ; see Figure 7). Add a fictitious marked point (say between  $\alpha_1$  and the marked point 2) and apply  $(\star)$  to get:  $(a_1^2 a_3 b)^3 = c_2$ . This does not look very symmetric but can in fact readily be transformed into  $(a_1 b a_3)^4 = (a_1 a_3 b)^4 = c_2$ , using only braid relations. Similarly one can consider the surface of genus 1 with one boundary component  $\varepsilon$  and recover in this way the relation  $(a_1 b)^6 = e$ , which was used in the text in a crucial way, when pasting a subsurface of type  $(0, 5)$  obtained by cutting along  $\alpha_1$  with a subsurface of type  $(1, 2)$  obtained by cutting along  $\gamma_2$ .

Let now  $S_i$  (resp.  $S'_i$ ) be the subsurface of type  $(0, 5)$  (resp.  $(1, 2)$ ) obtained by cutting along  $\alpha_i$  (resp.  $\gamma_i$ ). Forgetting again about boundary components, let  $F \in \text{Aut}^*(\widehat{\Gamma}_{1,3})$ , fixing the pants decomposition  $(\alpha_1, \alpha_2, \alpha_3)$ . Restricting  $F$  to  $S_i$  we get an element of  $\widehat{GT}$  and these three elements coincide; this way we get  $F_0$ . Similarly, after twisting, we can restrict the action to  $S'_i$ , retrieve again three elements of  $\widehat{GT}$ , which again coincide with an element  $F_1$  and  $(F_0, F_1)$  form a compatible pair, so that  $F_0 = F \in \mathbb{F}$ .

But we can also forget about  $F_1$  and ask: When will  $F$  (identified with  $F_0 \in \widehat{GT}$ ) extend to an automorphism of  $\widehat{\Gamma}_{1,3}$  (equivalently of  $\widehat{C}(S_{1,3})$ )? Clearly the answer is: If and only if it preserves relation  $(\star)$ , namely  $(a_1 a_2 a_3 b)^3 = 1$ . Here we already have  $F(\alpha_i) = \alpha_i$ , that is  $F(a_i) = a_i^\lambda$  for  $i = 1, 2, 3$  and the only missing generator is  $b$  (if one adds boundary components,  $F(d_i) = d_i^\lambda$ ). So we need only find – say –  $h$  such that  $F(\beta) = h^{-1} \cdot \beta$ , i.e.  $F(b) = h^{-1} b^\lambda h$ . Putting this into  $(\star)$  we

get an “equation” for  $h$  (which can perhaps be simplified; see [Ge2], §2). Of course, since  $F$  must be of lego type, we can actually compute  $h$  using the lego; we leave it as an exercise to compute the transfer factor (that is  $h$ ) in this way. It takes 3 moves to extricate  $\beta$  from the  $\alpha_i$ ’s and of course  $f_1$  (with  $F_1 = (\lambda, f_1)$ ) appears in the answer.

We conclude from the above that [Ge2] displays the universality of type  $(1, 3)$  in the form of the star relation  $(\star)$ . In the text, we pasted pieces of types  $(0, 5)$  and  $(1, 2)$  in order to define compatible pairs and the group  $\Gamma$ . That same group can be described using  $(\star)$ , which occurs as a kind of monodromy relation for the covering of  $S_{1,3}$  by the three subsurfaces  $S_i$  of type  $(0, 5)$ .

We will be extremely brief about the connection with generalized Artin braid groups, which hopefully can lead to other developments. In [Ma1], the Galois action on the braid group  $A(E_7)$  associated with the root system (Dynkin diagram)  $E_7$  is explicitly computed and related to the Galois action on  $\hat{\Gamma}_3$ . That occurred at the same time when the Galois action on the  $\hat{\Gamma}_{g,n}$  was also being computed (see H.Nakamura’s contribution in [GGA]). Then the connection between generalized braid groups and mapping class groups was made more general and explicit; we refer in particular to [Ma2] and its bibliography. We will content ourselves here to pointing the suggestive isomorphism:  $\Gamma_{1,[3]} \simeq A(D_4)/Z$  where  $A(D_4)$  is the generalized braid group associated with the root system  $D_4$  and  $Z$  denotes its center. The isomorphism is “natural” and suggests a notation for  $A(D_4)$  which closely parallels our notation above for  $\Gamma_{1,3}$ . The center  $Z$  is free cyclic and in that notation, it is generated precisely by the element  $(a_1 a_2 a_3 b)^3$  which occurs in the star relation. Moreover, this is an instance of a quite general phenomenon (see [Ma2]). Finally we mention that the  $Y$ -shaped diagram corresponding to  $D_4$  can be considered a central piece in the lego of Coxeter graphs giving rise to the generalized braid groups.

## Appendix: Definitions and some known results

We have gathered here a number of definitions, most of which but not all are classical, and a number of results, most of which but not all are used in the text. We have left aside the results about automorphisms of profinite complexes of curves which were proved in [BL] and the definitions and results related to the Grothendieck-Teichmüller group. The material related to the first theme is dealt with in the main body of the text. We essentially rely on the existing references (see the beginning of §4) as far as Grothendieck-Teichmüller theory is concerned.

**A.1** A finite *type* is a pair  $(g, n)$  of non negative integers; it is *hyperbolic* if  $2g - 2 + n > 0$ . Given a type, we let  $S_{g,n}$  denote the – unique up to diffeomorphism – differentiable surface of genus  $g$  with  $n$  deleted points. We occasionally write  $g(S)$  for the genus of  $S$ . The points are considered setwise, i.e. are not labelled; they can also be seen as “holes” provided isotopies do not fix the boundary. Conversely a surface is of type  $(g, n)$  if it diffeomorphic to  $S_{g,n}$  where the points are considered setwise, i.e. are not labelled. The Euler characteristic of  $S_{g,n}$  is  $\chi(S) = 2 - 2g - n$  and it is hyperbolic if its type is.

**A.2** Attached to a surface  $S$  of type  $(g, n)$  are the *Teichmüller space*  $\mathcal{T}(S)$  and *moduli space*  $\mathcal{M}(S)$ . We restrict henceforth to hyperbolic surfaces. The Teichmüller space  $\mathcal{T}(S)$  is noncanonically identified with the standard Teichmüller space  $\mathcal{T}_{g,n}$  associated with the given type. It has dimension  $d(S) = d_{g,n} = 3g - 3 + n$ , which we call the *modular dimension* of  $S$  or of the given type – we

will often drop the adjective “modular”. In turn  $\mathcal{M}(S)$  is – again noncanonically – identified with  $\mathcal{M}_{g,[n]}$ , the moduli space of curves of the given type, with unlabelled marked points. We use brackets  $([n])$  when the points are unlabelled, that is are considered setwise. Note that to be consistent we should write  $S_{g,[n]}$  rather than  $S_{g,n}$  but we nevertheless retain the latter piece of notation for simplicity; also,  $\mathcal{T}_{g,[n]} = \mathcal{T}_{g,n}$ .

**A.3** We let  $Mod(S) = \pi_0(Diff(S))$  denote the (extended) *mapping class group* of  $S$ , i.e. the group of isotopy classes of diffeomorphisms of  $S$ . The index 2 subgroup of orientation preserving isotopy classe is denoted  $Mod^+(S)$ . More generally an upper + will mean *orientation preserving*. We usually write  $\Gamma(S) = Mod^+(S)$  and call it the (*Teichmüller*) *modular group*. It can be seen as the orbifold fundamental group of  $\mathcal{M}(S)$  and as the Galois group of the orbifold unramified cover  $\mathcal{T}(S)/\mathcal{M}(S)$ . So we have the tautological exact sequence:

$$1 \rightarrow \Gamma(S) \rightarrow Mod(S) \rightarrow \mathbb{Z}/2 \rightarrow 1. \quad (A1)$$

The group  $\Gamma(S)$  is (noncanonically) isomorphic to  $\Gamma_{g,[n]}$ , defined as the fundamental group of the complex orbifold  $\mathcal{M}_{g,[n]}$ . The group  $\Gamma_{g,[n]}$  is centerfree, with 4 low-dimensional exceptions, i.e. types  $(0, 4)$ ,  $(1, 1)$ ,  $(1, 2)$  and  $(2, 0)$ . In the first case the center is Klein’s Vierergruppe ( $\simeq \mathbb{Z}/2 \times \mathbb{Z}/2$ ); in the other three cases the center is isomorphic to  $\mathbb{Z}/2$ , generated by the (hyper)elliptic involution. We refer to any elementary text on the subject for more detail and to [BL] for some less elementary remarks about the stacky side of these non centerfree cases (see especially the Remark at the end of the Introduction there).

**A.4** *Permutations* of points play a certain role in the text. The moduli space of curves of genus  $g$  with  $n$  *ordered points* is denoted  $\mathcal{M}_{g,n}$ . The cover  $\mathcal{M}_{g,n}/\mathcal{M}_{g,[n]}$  is finite, orbifold unramified (stack étale) and Galois with group  $\mathcal{S}_n$ , the permutation group on  $n$  symbols.

Let us detail one low dimensional example (or exception) which is used in the text. The group  $\Gamma_{1,[2]}$  has center  $Z$  isomorphic to  $\mathbb{Z}/2$  as mentioned above. It is the *direct* product of that center and the corresponding ordered group:  $\Gamma_{1,[2]} = \Gamma_{1,2} \times Z$ . Moreover  $\Gamma_{1,2} \subset \Gamma_{0,[5]}$  is the subgroup which corresponds to the permutations stabilizing the – say – fifth point. Geometrically speaking, to a genus 1 curve with 2 marked points one can associate 5 points, namely the 4 Weierstrass points plus the orbit of the two points under the elliptic involution; the two points can be indeed made to form an orbit, after a suitable translation. The 4 points can be permuted but the fifth one should be kept labeled under the action of the modular group, hence the above description. Finally, it is useful to note that  $\Gamma_{1,2}$  is self-normalizing in  $\Gamma_{0,[5]}$ , so in particular not normal.

**A.5** *Profinite completion* is denoted with a hat as usual. So  $\hat{\Gamma}(S)$  is the completion of  $\Gamma(S)$  and can be seen as the “geometric” fundamental group of  $\mathcal{M}(S)$ , i.e. the fundamental group of that space as a  $\mathbb{C}$ - or equivalently  $\overline{\mathbb{Q}}$ -stack. Ditto for  $\hat{\Gamma}_{g,[n]}$  and  $\mathcal{M}_{g,[n]}$ . In the present paper we use only the full profinite completion but many results and proofs hold for other (profinite) completions especially the pro- $\ell$  quotients, and the adaptation would only require notational changes, except when remarked otherwise.

We index the inverse system of the finite index subgroups of  $\Gamma = \Gamma(S) \simeq \Gamma_{g,[n]}$  by the set  $\Lambda$ ,

so that for any  $\lambda \in \Lambda$  we have a subgroup  $\Gamma^\lambda$  and by definition:

$$\hat{\Gamma} = \varprojlim_{\lambda \in \Lambda} \Gamma/\Gamma^\lambda.$$

To any  $\Gamma^\lambda$  there corresponds a finite orbifold unramified cover  $\mathcal{M}^\lambda/\mathcal{M}$ . We call the subgroups  $\Gamma^\lambda$  “levels” and the corresponding covers  $\mathcal{M}^\lambda$  “level structures”, a traditional terminology in this context ( $\lambda$  stands for “level” in barbaric parlance). In particular, for  $m \geq 2$  a positive integer, the abelian level  $\mathcal{M}(m)$  is defined by the subgroup  $\Gamma(m)$  which is the kernel of the natural map  $\Gamma \rightarrow Sp_{2g}(\mathbb{Z}/m)$ , that is  $\Gamma(m)$  is the group of diffeomorphisms of  $S$  which fix the homology of the associated unmarked or compact surface modulo  $m$ . For  $\lambda, \mu \in \Lambda$  we write  $\mu \geq \lambda$  if  $\Gamma^\mu \subseteq \Gamma^\lambda$  i.e. if  $\mathcal{M}^\mu$  is a covering of  $\mathcal{M}^\lambda$ , and we say that  $\mathcal{M}^\mu$  (resp.  $\Gamma^\mu$ ) dominates  $\mathcal{M}^\lambda$  (resp.  $\Gamma^\lambda$ ).

Let us gather some results about these profinite modular groups, which for the most part are effectively used in the main body of the text. First  $\Gamma(S)$  is residually finite, that is the natural map  $\Gamma(S) \rightarrow \hat{\Gamma}(S)$  is an injection. We thus usually identify  $\Gamma(S)$  with its image in  $\hat{\Gamma}(S)$ . Then a number of important results were obtained in [B] about the center of  $\hat{\Gamma}(S)$  and its open subgroups, as well as about the centralizers of Dehn twists in that same group. For that second topic we refer to [B], §7. The main point about the center  $Z(\hat{\Gamma}(S))$  is that it is simply equal to  $Z(\Gamma(S))$ , the center of  $\Gamma(S)$ . Let now  $U \subset \hat{\Gamma}(S)$  be an open subgroup and let  $H = U \cap \Gamma(S)$  so that  $U \simeq \hat{H}$ . Then it is known that  $Z(H) = H \cap \Gamma(S)$  and it is shown in [B] that the same holds for the profinite completion, namely  $Z(U) = U \cap Z(\hat{\Gamma}(S)) = U \cap Z(\Gamma(S))$ . We stress that these results are technically quite hard; they are also useful, reducing the problems with centers in profinite modular groups to the discrete case, which is well-understood. The same holds for centralizers of twists, in contrast with – say – centralizers of pseudo-Anosov elements, which are unknown in the profinite case.

**A.6** We now briefly summarize the definitions pertaining to various *complexes of curves*, referring to [B] and [BL] for more detail or of course to any of the many references (e.g. [Iv1,2], [FL1] etc.). It is remarkable that we will actually need only consider *graphs* (and *prographs*), that is complexes of dimension 1.

Given a surface  $S$ , hyperbolic and of finite type (see §A1), we let  $\mathcal{S}(S)$  denote the set of isotopy classes of simple closed curves on  $S$  not isotopic to boundary curves. A *multicurve* is a set of non intersecting elements of  $\mathcal{S}$ . Here, non intersecting means that there exist representatives which do not intersect (see [FLP] or any standard reference for details – or a summary in [BL]).

The first complex  $C(S)$  is the one originally defined by W.J.Harvey in the late sixties. A  $k$ -simplex of  $C(S)$  is defined by a multicurve  $\underline{s} = (s_0, \dots, s_k)$ , so the vertices of  $C(S)$  correspond to elements of  $\mathcal{S}(S)$ . Boundary and face operators are defined by deletion and inclusion respectively. That makes  $C(S)$  into a (non locally finite) simplicial complex of dimension  $d(S) - 1$  where  $d(S)$  is the modular dimension of  $S$  (see §A.2). We will write  $C^{(k)}(S)$  for the  $k$ -dimensional skeleton of  $C(S)$  and use a similar notation for the other complexes. Note that  $\mathcal{S}(S)$  is just the 0-skeleton (vertex set) of  $C(S)$  but it is nonetheless useful to retain a specific piece of notation.

There is a natural action of  $\Gamma(S)$  on  $C(S)$  determined by saying that to  $g \in \Gamma$  and a curve  $\alpha \in \mathcal{S}$  one associates  $g \cdot \alpha$ , the image of the curve by  $g$ , everything of course up to isotopy.

**A.7** Next we define  $C_G(S)$ , the *group theoretic complex*. It is useful only in the profinite case, so is included in the present discrete setting essentially to fix notation. Here all objects pertain to the discrete topology, so we add a superscript “disc”. Let  $\Gamma = \Gamma(S)$  and  $\mathcal{G}^{disc}(\Gamma)$  denote the set of all subgroups of  $\Gamma$ . To every simplex (i.e. multicurve)  $\sigma \in C(S)$  we assign the (discrete) free abelian group  $G_\sigma^{disc} \in \mathcal{G}(\Gamma)$  spanned by the (Dehn) twists associated to  $\sigma$ . We then make this set of subgroups into a complex by using the boundary and face operators as for  $C(S)$  which makes  $\mathcal{G}^{disc}(\Gamma)$  into a complex, in fact a Boolean lattice.

The long and the short is that  $C_G(S)$  is trivially isomorphic to  $C(S)$ . Let us define a  $\Gamma$ -action on  $C_G(S)$  so as to make the isomorphism equivariant. To  $\alpha \in \mathcal{S}$ , that is a vertex of  $C(S)$ , one thus assigns the cyclic group generated by  $\tau_\alpha$ , the twist along  $\alpha$ . Note that at that point, we should and do fix an orientation for  $S$ . Then for  $g \in \Gamma$  one has the well-known formula:  $\tau_{g \cdot \alpha} = g\tau_\alpha g^{-1} \in \Gamma$ . The right-hand side of this equality defines an action of  $\Gamma$  on  $C_G(S)$  which makes the natural isomorphism between  $C(S)$  and  $C_G(S)$  equivariant.

**A.8** We then come to the *pants complex*  $C_P(S)$ . It was briefly mentioned the appendix of the classical 1980 paper by A.Hatcher and W.Thurston (see [HLS] or [M]) and first studied in [HLS] where it is shown to be connected and simply connected. It is a two dimensional, not locally finite complex whose vertices are given by the pants decomposition (i.e. maximal multicurves) of  $S$ ; these correspond to the simplices of highest dimension ( $= d(S) - 1$ ) of  $C(S)$ . Given two vertices  $\underline{s}, \underline{s}' \in C_P(S)$ , they are connected by an edge if and only if  $\underline{s}$  and  $\underline{s}'$  have  $d(S) - 1$  curves in common, so that up to relabeling (and of course isotopy)  $s_i = s'_i, i = 1, \dots, d(S) - 1$ , whereas  $s_0$  and  $s'_0$  differ by an *elementary move*, which means the following. Cutting  $S$  along the  $s_i$ 's,  $i > 0$ , there remains a surface  $\Sigma$  of modular dimension 1, so  $\Sigma$  is of type  $(1, 1)$  or  $(0, 4)$ . Then  $s_0$  and  $s'_0$ , which are supported on  $\Sigma$ , should intersect in a minimal way, that is they should have geometric intersection number 1 in the first case, and 2 in the second case (in the latter case their algebraic intersection number is 0). In the first case (genus 1), the edge (and move) is said to be of type  $S$  (for “simple”, see [HLS]); in the second case (genus 0) of type  $A$  (for “associativity”, see [HLS]). For  $d(S) = 1$ , the 1-skeleton of  $C_P(S)$  is the Farey graph  $F$ .

We have thus defined the 1-skeleton  $C_P^{(1)}(S)$  of  $C_P(S)$  which, following [M], we call the *pants graph* of  $S$ . We will not give here the definition of the 2-cells of  $C_P(S)$  (see [HLS] or [M]), as we will actually not use it. They describe certain relations between elementary moves, that is they can be considered as elementary homotopies; as mentioned above, pasting them in makes  $C_P(S)$  simply connected (cf. [HLS]). It is shown in [M] how to recover the full 2-dimensional pants complex from the pants graph. The definition of the 2-cells is briefly recalled and used in §5 of the present paper. For  $d(S) = 1$  the pants complex is the Farey tessellation, which we again denote  $F$ . Apart from there, in the present paper we only use the 1-skeleton of  $C_P(S)$ , so that in order to simplify notation  $C_P(S)$  will refer by default to the pants graph.

**A.9** We finally define the *graph*  $C_*(S)$  which is studied in [BL] and plays an important role in the profinite case, while actually clarifying a number of issues even in the discrete case (see [BL], §2). The graph  $C_*(S)$  has the same set of vertices as  $C_P(S)$ , namely the pants decomposition of  $S$ . The edges are defined simply by relaxing the minimal intersection condition in the definition of the edges of  $C_P(S)$ . In other words two vertices represented by multicurves  $\underline{s} = (s_i)_i$  and  $\underline{s}' = (s'_i)_i$

( $i = 0, \dots, d(S) - 1$ ) are joined by an edge if up to relabeling  $s_i = s'_i$  for  $i > 0$ ; then  $s_0$  and  $s'_0$  lie on a surface of type  $(0, 4)$  or  $(1, 1)$ . So  $C_P(S) \subset C_*(S)$  is a subgraph with the same set of vertices.

If  $S$  is connected (see §A.10 below) of dimension 0, it is of type  $(0, 3)$  (a “trinion” or “pair of pants”); by convention,  $C_P(S) = C_*(S)$  is reduced to a point with no edge attached; note that usually one defines  $C(S_{0,3}) = \emptyset$ . If  $S$  is connected of dimension 1, it is of type  $(0, 4)$  or  $(1, 1)$ . In both cases  $C_P(S) = F$  coincides with the Farey graph. On the other hand, it is easily checked that  $C_*(S)$  is the complete graph with the same vertices as  $F$ , which we denote by  $G$ . This is simply because two curves on a surface of (modular) dimension 1 always intersect. Finally if  $d(S) > 1$ ,  $C_*(S)$  is nothing but the 1-skeleton of  $C(S)^*$ , the complex dual to  $C(S)$ . For that reason, when  $d(S) = 1$ , it becomes natural to define  $C(S)$  as the dual of  $G$ , which is not the usual convention but seems to be the right one for our purposes.

**A.10** It is useful to extend the definitions of the graphs  $C_P(S)$  and  $C_*(S)$  to *non connected surfaces*. The extension is rather trivial but it shows that these two graphs are particularly well-behaved. The definitions are simply unchanged. We will write  $V(S)$  for the set of vertices common to  $C_*(S)$  and  $C_P(S)$  (i.e. pants decompositions),  $E(S)$  (resp.  $E_P(S)$ ) for the edges of  $C_*(S)$  (resp.  $C_P(S)$ ); so  $E_P(S) \subset E(S)$ .

Let  $S = S' \amalg S''$  be given as the disjoint sum of  $S'$  and  $S''$ , which themselves need not be connected. First note that modular dimension is additive:  $d(S) = d(S') + d(S'')$ . Then it is easy to describe  $C_*(S)$  and  $C_P(S)$  in terms of the graphs associated to  $S'$  and  $S''$ . For the vertices we get:  $V(S) = V(S') \times V(S'')$ ; and for the edges of  $C_*(S)$ :  $E(S) = E(S') \times V(S'') \amalg V(S') \times E(S'')$ . Simply change  $E$  into  $E_P$  for the case of  $C_P$ . These prescriptions immediately generalize to an arbitrary number  $r$  of not necessarily connected pieces. If  $S = \amalg_i S_i$ ,  $d(S) = \sum_i d(S_i)$ ,  $V(S) = \prod_i V(S_i)$  and  $E(S) = \prod_i V(S_1) \times \dots \times E(S_i) \times \dots \times \dots \times V(S_r)$ ; replace again  $E$  with  $E_P$  when dealing with  $C_P$ .

**A.11** We now come to *profinite complexes* of curves. More generally, let  $X_\bullet$  be a simplicial complex endowed with an action of  $\Gamma = \Gamma(S)$ . Then we can define its profinite completion as the inverse limit:

$$\hat{X}_\bullet = \varprojlim_{\lambda \in \Lambda} X_\bullet / \Gamma^\lambda,$$

which we regard as a simplicial object in the category of profinite sets. The above definition would of course be valid for other groups than  $\Gamma$  and spaces  $X$  which are not necessarily simplicial complexes. However the action of  $\Gamma$  on  $X$  has to satisfy certain geometric conditions which in our cases are easily met (see [B], §5).

We apply the above to  $\mathcal{S}(S)$ ,  $C(S)$ ,  $C_P(S)$  and  $C_*(S)$ , obtaining the respective completions  $\hat{\mathcal{S}}(S)$ ,  $\hat{C}(S)$ ,  $\hat{C}_P(S)$  and  $\hat{C}_*(S)$ . As in the above section, we have dropped the “bullet” from the notation but stress that these are indeed simplicial objects. The proset  $\hat{\mathcal{S}}(S)$  is thus the set of procurves and it is again the proset of vertices of  $\hat{C}(S)$ . The complexes  $\hat{C}_P(S)$  and  $\hat{C}_*(S)$  are in fact *prographs*. We will usually drop the prefix “pro” for simplicity but it should definitely be emphasized that these profinite spaces are complicated objects, just like profinite groups and even more so; note that the group completion  $\hat{\Gamma}$  is obtained via the above procedure by letting  $\Gamma$  act on itself by translation. We refer to [B] and [BL] for the basic properties of the profinite complexes of curves.

We add one “obvious” remark which is also of primary importance. It is clear from the above general definition that replacing  $\Gamma$  with any finite index subgroup  $\Gamma^\mu$  does not change  $\hat{X}$ . So the latter carries a kind of asymptotic information and this explains in part why profinite complexes of curves can give information about all open subgroup of  $\hat{\Gamma}(S)$  (e.g. about their automorphisms). As a trivial application, if  $d(S) = 1$ , one has  $\hat{C}_P(S) = \hat{F}$  and  $\hat{C}_*(S) = \hat{G}$ , where  $F$  and  $G$  are as in §A.9 and completion is with respect to the action of  $SL_2(\mathbb{Z})$  or that of its subgroup  $F_2$  or any of its finite index subgroups..

**A.12** Closure of the *group theoretic complex*. Let  $\mathcal{G} = \mathcal{G}(\hat{\Gamma})$  denote the set of closed subgroups of  $\hat{\Gamma}$ . This is again a profinite set since it can be written as:

$$\mathcal{G} = \varprojlim_{\lambda \in \Lambda} \mathcal{G}(\Gamma/\Gamma^\lambda),$$

where  $\Gamma^\lambda$  runs over the normal subgroups of finite index in  $\Gamma$  and  $\mathcal{G}(\Gamma/\Gamma^\lambda)$  denotes the finite set of the subgroups of  $\Gamma/\Gamma^\lambda$ . We also have an action of  $\hat{\Gamma}$  on  $\mathcal{G}$  by conjugation. There is a natural map  $C_G(S) \rightarrow \mathcal{G}$ , of the group theoretic complex (see §A.7) to  $\mathcal{G}$  which maps every simplex  $\sigma$  to the closure  $G_\sigma$  of the group  $G_\sigma^{disc}$  in  $\mathcal{G}$ . Let  $\sigma \in C(S)$  be a  $k$ -simplex, determined by a multicurve  $\underline{s} = (s_0, \dots, s_k)$ , that is by the twists  $\tau_i$  along the  $s_i$ . Then it is true, although far from obvious, that  $G_\sigma \subset \hat{\Gamma}$  is the profree abelian group generated by the  $\tau_i$ .

Next one considers the closure  $\overline{C}_G(S)$  of the image of  $C_G(S)$  in  $\mathcal{G}$ ; it is not hard to extend the face and boundary operators so as to make it into a prosimplicial complex with  $\hat{\Gamma}$  action. By naturality and density, there is a natural  $\hat{\Gamma}$ -equivariant *surjective* map  $\hat{C}(S) \rightarrow \overline{C}_G(S)$ . An important result is that this map is injective, providing an equivariant isomorphism between these complexes which extends the natural (“trivial”) isomorphism in the discrete case (see [B], Theorem 7.8). We will often identify the two complexes  $\overline{C}_G(S)$  and  $\hat{C}(S)$  in practice but the reader should be aware that this isomorphism result is not banal in the profinite case. Finally there is a refinement which is crucial when dealing with open subgroups of  $\hat{C}(S)$  (see [BL] Theorem 3.7).

**A.13** Our last item here will deal briefly with *automorphisms of discrete modular groups and of complexes of curves*. We refer to [Iv1], [FL1], [M] and [BL] for more detailed statements and proofs. As in [BL] our statements are geared towards the profinite case and we have extracted what seems to be the significant minimum in that direction. We let  $S$  be connected hyperbolic and of finite type; we assume that  $d(S) > 1$  and  $S$  is not of type  $(1, 2)$ , that is  $S$  is of type  $(0, 5)$  or  $d(S) > 2$ . This last assumption we make simply in order to avoid discussing well-known low-dimensional peculiarities (see e.g. [FL1] or [BL]).

Then the automorphisms of the curves complex are described by the exact sequence:

$$1 \rightarrow Inn(\Gamma(S)) \rightarrow Aut(C(S)) \rightarrow \mathbb{Z}/2 \rightarrow 1, \quad (A2)$$

where, in view of the profinite case, the group  $\mathbb{Z}/2$  should be considered as generated by complex conjugacy, so isomorphic to the Galois group  $Gal(\mathbb{C}/\mathbb{R})$ .

With our assumptions, the only case when  $\Gamma(S)$  is not centerfree, hence  $Inn(\Gamma(S))$  is different from  $\Gamma(S)$  is when  $S$  is of type  $(2, 0)$  (and the center has order 2). Yet it is best to think of the left-hand group as  $Inn(\Gamma(S)) \subset Aut(\Gamma(S))$ .

We add that, denoting by  $C^{(1)}(S)$  the 1-skeleton of  $C(S)$ , there is a natural injective map  $Aut(C(S)) \rightarrow Aut(C^{(1)}(S))$  and this map is actually an isomorphism. This is an easy result, coming from a graph-theoretic characterization of the simplices of  $C(S)$  inside the graph  $C^{(1)}(S)$ : they are in one-to-one correspondence with the finite complete subgraphs, so have to be preserved by any automorphism of the graph.

Using the sequence (A2) it is fairly easy to derive a description of the *group* automorphisms under the form of the following exact sequence:

$$1 \rightarrow Inn(\Gamma(S)) \rightarrow Aut(\Gamma(S)) \rightarrow \mathbb{Z}/2 \rightarrow 1. \quad (A3)$$

In other words  $Aut(\Gamma(S)) = Mod(S)$ ,  $Out(\Gamma(S)) \simeq \mathbb{Z}/2$  and the only non inner automorphism is generated by a reflection of the surface, that is an orientation reversing involution of the surface  $S$ , alias complex conjugacy, the generator of  $Gal(\mathbb{C}/\mathbb{R})$ . Note that the existence of such a reflection shows that the three sequences (A1), (A2) and (A3) are split; see the third section of [Mo] for more elaborate considerations on the subject. Using (A2) again, it is fairly easy to extend the above to any finite index subgroup of  $\Gamma(S)$  (cf. [Iv1]) which is quite a substantial improvement. In fact (A3) remains valid if one replaces the middle group  $\Gamma(S)$  by a normal finite index subgroup  $\Gamma^\lambda$  without changing the left and right hand groups.

Put somewhat differently, there is an *a priori* injective map  $Aut(\Gamma(S)) \rightarrow Aut(C(S))$  and (A3) asserts it is an isomorphism. This is a close analog of a famous result of Tits which states that under suitable assumptions, the automorphisms of the building of an algebraic group come from automorphisms of the group itself.

Now concerning the *graphs*  $C_P(S)$  and  $C_*(S)$ . It is clear from the definitions that we have an inclusion  $Aut(C_P(S)) \subset Aut(C_*(S))$ . It is then shown in [BL], §2, that  $Aut(C_*(S)) = Aut(C(S))$ . This is achieved by showing how  $C(S)$  is actually graph theoretically encoded in  $C_*(S)$  – recall that the latter is the 1-skeleton of the *dual* of  $C(S)$ . Putting this together with the above inclusion, this yields an inclusion  $Aut(C_P(S)) \subset Aut(C(S))$ , reproving the main result of [M]. Finally, (A3) shows that any automorphism of  $C(S)$  can be viewed as an automorphism of the pants graph  $C_P(S)$ , so that  $Aut(C_P(S)) = Aut(C(S))$ ; one can remark that this is also *ipso facto* the automorphism group of the full, 2-dimensional pants complex. One of the main differences between the discrete and profinite cases is that the above equality breaks completely in the profinite setting.

## References

- [B] M.Boggi, Profinite Teichmüller theory, *Math. Nachr.* **279** (2006), 953-987.
- [BL] M.Boggi, P.Lochak, On profinite complexes of curves, their automorphisms and the weak anabelianity of the moduli stacks of curves, preprint, 2006.
- [D] V.G. Drinfel'd, On quasitriangular quasi-Hopf algebras and a group closely connected with  $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ , *Leningrad Math. J.* **2** (1991), 829-860.
- [De] P.Deligne, Le groupe fondamental de la droite projective moins trois points, in *Galois Groups over  $\mathbb{Q}$* , Y.Ihara et al. eds. MSRI Publication **16**, Springer, 1989, 79-297.
- [DE] *The Grothendieck theory of Dessins d'Enfants*, L.Schneps ed., London Math. Soc. Lect. Note Ser. **200**, Cambridge University Press, 1994.

- [F] H.Furusho, Pentagon and hexagon equations, preprint, 2007.
- [FK] L.Funar, C.Kapoudjian, On a universal mapping class group of genus zero, *Geom. Funct. Anal.* **14** (2004), 965–1012.
- [FL1] F.Luo, Automorphisms of the complex of curves, *Topology* **39** (2000), 283–298.
- [FL2] F.Luo, A presentation of the mapping class groups, *Math. Research Letters* **4** (1997), 735–739.
- [FLP] *Travaux de Thurston sur les surfaces* (seconde édition), A.Fathi, F.Laudenbach, V.Poénaru eds., Astérisque **66-67**, SMF Publ., 1991.
- [G] A.Grothendieck, *Esquisse d'un Programme*, 1984, in [GGA], 5-47.
- [Ge1] S.Gervais, Presentation and central extensions of mapping class groups, *Transactions AMS* **348** (1996), 3097-3132.
- [Ge2] S.Gervais, A finite presentation of the mapping class group of a punctured surface, *Topology* **40** (2001), 703-725.
- [GGA] *Geometric Galois Actions*, L.Schneps and P.Lochak eds., London Math. Soc. Lect. Note Ser. **242**, Cambridge University Press, 1997.
- [GGA2] *Geometric Galois Actions 2*, L.Schneps and P.Lochak eds., London Math. Soc. Lect. Note Ser. **243**, Cambridge University Press, 1997.
- [HLS] A.Hatcher, P.Lochak, L.Schneps, On the Teichmüller tower of mapping class groups, *J. reine und angew. Math.* **521** (2000), 1–24.
- [HS] D.Harbater, L.Schneps, Fundamental groups of moduli and the Grothendieck-Teichmüller group, *Transactions AMS* **352** (2000), 3117-3148.
- [I1] Y. Ihara, Profinite braid groups, Galois representations and complex multiplications, *Ann. of Math.* **123** (1986), 43-106.
- [I2] Y. Ihara, Automorphisms of pure sphere braid groups and Galois representations, in *The Grothendieck Festschrift*, Vol. II, 353-373, Birkhäuser, 1990.
- [I3] Y.Ihara, On the stable derivation algebra associated with some braid groups, *Israel J. Math.* **80** (1992), 135-153.
- [Iv1] N.V.Ivanov, Automorphisms of complexes of curves and of Teichmüller spaces, *Internat. Math. Res. Notices* **14** (1997), 651–666.
- [Iv2] N.V.Ivanov, *Subgroups of Teichmüller modular groups*, Translations of Mathematical Monographs **115**, AMS Publ., 1992.
- [Lo] P.Lochak, Fragments of nonlinear Grothendieck-Teichmüller theory, *Woods Hole Mathematics*, 225-262, Series on Knots and Everything **34**, World Scientific, 2004.
- [LS1] P.Lochak et L.Schneps, The Grothendieck-Teichmüller group and automorphisms of braid groups, in [DE], 323-358.
- [LS2] P.Lochak et L.Schneps, The universal Ptolemy-Teichmüller groupoid, in [GGA2], 325-347.
- [LS3] P.Lochak et L.Schneps, A cohomological interpretation of the Grothendieck-Teichmüller group, *Inventiones math.* **127** (1997), 571-600.

- [M] D.Margalit, Automorphisms of the pants complex, *Duke Math. J.* **121** (2004), 457–479.
- [Ma1] M.Matsumoto, Galois group  $G_{\mathbb{Q}}$ , Singularity  $E_7$ , and Moduli  $\mathcal{M}_3$ , in [GGA2], 179-218.
- [Ma2] M.Matsumoto, A presentation of mapping class groups in terms of Artin groups and geometric monodromy of singularities, *Math. Ann.* **316** (2000), 401-418.
- [Mo] S.Mochizuki, Topics surrounding the anabelian geometry of hyperbolic curves, in *Galois groups and fundamental groups*, 119–165, MSRI Publ. **41**, Cambridge Univ. Press, 2003.
- [McC] J.D.McCarthy, Automorphisms of surface mapping class groups: A recent theorem of N.Ivanov, *Inventiones math.* **84** (1986), 49-71.
- [MS] H.A.Masur, S.Schleimer, The pants complex has only one end, in *Spaces of Kleinian groups*, London Math. Soc. Lec. Notes **329** (2006), 209-218.
- [MNT] S.Mochizuki, H.Nakamura, A.Tamagawa, Grothendieck’s conjectures concerning fundamental groups of algebraic curves, *Sugaku* **50** (1998), 113-129.
- [N1] H.Nakamura, Galois rigidity of pure sphere braid groups and profinite calculus, *J. Math. Sci. Univ. Tokyo* **1** (1994), 72-136.
- [N2] H.Nakamura, Galois rigidity of profinite fundamental groups, *Sugaku* **47** (1995), 1-17; translated in *Sugaku Expositions* **10** (1997), 195-215.
- [N3] H.Nakamura, Some classical views on the parameters of the Grothendieck-Teichmüller group, in *Progress in Galois theory*, 123-133, *Dev. Math.* **12**, Springer Verlag, 2005.
- [NS] H.Nakamura, L.Schneps, On a subgroup of the Grothendieck-Teichmüller group acting on the tower of profinite Teichmüller modular groups, *Inventiones math.* **141** (2000), 503-560.
- [NiSe] N.Nikolov, D.Segal, Finite index subgroups in profinite groups, *C.R. Acad. Sci. Paris*, **337** (2003), 303-308.
- [Pe] R.C.Penner, Universal constructions in Teichmüller theory, *Advances in Mathematics* **98** (1993), 143-215.
- [Pu] A.Putnam, A note on the connectivity of certain complexes associated to surfaces, preprint, 2006.
- [R] G.Racinet, Doubles mélanges des polylogarithmes multiples aux racines de l’unité, *Publ. Math. IHES* **95** (2002), 185–231.
- [RZ] L.Ribes, P.Zalesskii, *Profinite Groups*, Springer Verlag, 2000.
- [S1] L.Schneps, The Grothendieck–Teichmüller group  $\widehat{GT}$ : a survey, in [GGA], 183–203.
- [S2] L.Schneps, Automorphisms of curves and their role in the Grothendieck-Teichmüller theory, *Math. Nachr.* **279** (2006), 656-671.
- [Sz] T.Szamuely, Groupes de Galois de corps de type fini (d’après Pop), *Astrisque* **294** (2004), SMF Publ., 403–431.

Institut de Mathématiques de Jussieu, 175 rue du Chevaleret, F-75013 Paris  
 lochak@math.jussieu.fr.