On the adiabatic stability of solitons and the matching of conservation laws[a)]

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We derive a series of identities which generalize and simplify the results obtained for adiabatically modulated solitons in the case of perturbed specific integrable equations. It stresses the importance of the variational properties of the solitons, which make an adiabatic theorem plausible. A precise conjecture is made and its validity discussed from different points of view.

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I. INTRODUCTION

In a series of papers Kaup and Newell[1] and Karpman and Maslov[2] independently initiated the study of the perturbation theory for integrable equations. The example of a soliton obeying the KdV equation in a slowly changing medium was afterwards extensively studied (see Refs. 3, 4 and their bibliographies) and the production of a secondary soliton was finally pointed out in Ref. 5. This seems to ruin the validity of the adiabatic ansatz; however, adiabatically modulated solitons do have many features in common with the actual solution, a statement made precise through the study of the modified conservation laws. We show this in a more general and simple way than has yet been done, and next give arguments in favor of a possible adiabatic theorem for solitons, discussing the link with the periodic problem and Whitham's modulation theory.

II. MATCHING THE CONSERVATION LAWS

A. The Korteweg–deVries equation

We start with a typical example, namely

$$u_t + \alpha(t) u u_x + u_{xxx} = 0, \tag{1}$$

where we take $\alpha(t)$ to be a strictly positive C^1 function. When α is a constant, we denote (1) by $(\mathrm{KdV})_\alpha$—they are of course all equivalent by scaling. An equivalent way of writing (1) is found if one sets $q \equiv \alpha u$, $a(t) \equiv \exp(-\int^t \Gamma(s) ds) - \Gamma(s)$ any continuous function, which transforms (1) into

$$q_t + q q_x + q_{xxx} = -\Gamma(t) q. \tag{2}$$

Equation (2) models, for instance, the propagation of a wave governed by the KdV equation along a canal of varying depth.[1] We also mention that (2) includes the so-called cylindrical and spherical KdV equations.

$(\mathrm{KdV})_\alpha$ has an infinite sequence of integrals of the motion

$$H_{n,\alpha}[u] \equiv \int_{-\infty}^{+\infty} T_{n,\alpha}[u] dx, \tag{3}$$

$T_{n,\alpha}$ being a polynomial in $u, \ldots, u^{(n)}$—isobaric of weight n with respect to the grading $\deg u \equiv 1$, $\deg \partial/\partial x \equiv \frac{1}{2}$; one has

$$H_{1,\alpha} \equiv M(u) = \int u, \quad H_{2,\alpha} \equiv E(u) = \int u^2,$$
$$H_{3,\alpha} = \int (\tfrac{1}{3}\alpha u^3 - u_x^2). \tag{4}$$

In order to normalize $T_{n,\alpha}$, we make the convention that it is obtained from $T_n \equiv T_{n,1}$ by changing u into αu and then factoring out a power of α, so that the resulting $T_{n,\alpha}$ is not dividable by α. The first two integrals, $M(u)$ and $E(u)$, are conserved under (1), as is easily checked; in a more general way, the change of these integrals is given by the following:

Lemma: Let u be a solution of (1), then

$$\forall n \geqslant 1, \quad \frac{d}{dt} H_{n,\alpha(t)}[u] = \dot\alpha(t) \cdot \int_{-\infty}^{+\infty} \frac{\partial T_{n,\alpha}}{\partial \alpha}[u] dx. \tag{5}$$

Proof: The variation of $H_{n,\alpha}$ is given by the right-hand side of (5), plus a term coming from the variation of u, α being held constant; this term, however, vanishes, because of the algebraic relations entailing the constancy of $H_{n,\alpha}$ for a constant α. For instance, one has

$$\frac{d}{dt} M(u) = \frac{d}{dt} E(u) = 0; \quad \frac{d}{dt} H_{3,\alpha}[u] = \frac{1}{3}\dot\alpha \int_{-\infty}^{+\infty} u^3. \tag{6}$$

Next we construct the modulated soliton for (1); $(\mathrm{KdV})_\alpha$ has a soliton traveling at speed c,

$$S(x - ct, \alpha, c) \equiv 3c\alpha^{-1} \operatorname{sech}^2[\tfrac{1}{2} c^{1/2}(x - ct)], \tag{7}$$

the mass and energy of which are

$$M(s) = 12 c^{1/2} \alpha^{-1}; \qquad E(s) = 24 c^{3/2} \alpha^{-2}. \tag{8}$$

The modulated soliton is constructed as a traveling wave of constant *energy*. The choice to preserve (in some physical situation this integral may in fact represent the momentum) the energy rather than the mass is essential; it was found by Kaup and Newell in a direct way, as a byproduct of an inverse scattering calculation. We believe that the theorem below gives another justification for this and in the second part, we give yet another, more physical explanation, based on adiabatic invariance, which may have further consequences. We thus define

$$s(x,t,c_0,\delta) \equiv 3c(t)\alpha^{-1}(t) \operatorname{sech}^2\left[\tfrac{1}{2} c^{1/2}(t)\left(x - \int_0^t c(s)\, ds\right) - \delta\right]$$
$$= 3c_0 \alpha^{1/3}(t) \operatorname{sech}^2\left[\frac{c_0^{1/2}}{2} \alpha^{2/3}(t) \times \left(x - c_0 \int_0^t \alpha^{4/3}(s)\, ds\right) - \delta\right], \tag{9}$$

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choosing $c(t) \equiv c_0 \alpha^{4/3}(t)$ $(c_0 > 0)$ to guarantee the constancy of the energy.

Theorem:

$$\forall n \geqslant 2 \qquad \frac{d}{dt} H_{n,\alpha(t)}[s] = \dot{\alpha}(t) \cdot \int_{-\infty}^{+\infty} \frac{\partial T_{n,\alpha}}{\partial \alpha}[s]\, dx, \quad (10)$$

where s is the modulated soliton (9).

Recall that, by the lemma, the same relation holds true for the actual solution of (1), including the case $n = 1$. Only the conservation of mass is broken by s, since by (8), $M(s) = 12 c_0^{1/2} \alpha^{-1/3}(t)$ is not a constant. In fact, the balance of mass will be accounted for by the shelflike tail alluded to in the Introduction. We also point out that α is not assumed to be slowly varying [i.e., $\dot{\alpha}(t)$ small in (1) or $\Gamma(t)$ small in (2)].

Proof of the theorem: As was already shown in the pioneering paper of Lax,[6] one has

$$\forall n \geqslant 2 \qquad \frac{\partial H_{n,\alpha}}{\partial u}(u = s) = c_{n,\alpha} \frac{\partial E}{\partial u}(u = s). \quad (11)$$

This relation holds for any $n \geqslant 2$ $(\partial M / \partial u \equiv 1)$ with a time-dependent—through α—Lagrange multiplier $c_{n,\alpha}$; indeed, at any given time t, the modulated soliton has the shape x-dependence of the usual soliton for the corresponding values $\alpha = \alpha(t)$ and $c = c(t)$ and one needs only notice that (11) does not involve t explicitly. Now

$$\begin{aligned} \frac{dH_{n,\alpha}}{dt} &= \int \frac{\partial H_{n,\alpha}}{\partial u}(u = s) \dot{s}\, dx + \dot{\alpha} \int \frac{\partial T_{n,\alpha}}{\partial \alpha}(u = s) dx \\ &= c_{n,\alpha} \int \frac{\partial E}{\partial u}(u = s) \dot{s}\, dx + \dot{\alpha} \int \frac{\partial T_{n,\alpha}}{\partial \alpha}(u = s) dx \quad (12) \end{aligned}$$

and the first term on the right-hand side vanishes, due to the constancy of $E(s)$. The proof is finished.

Amplification: The situation is not very different with multisoliton states; indeed, start with a N-multisoliton $S_N(x,t,p_1,\ldots,p_{2N})$, containing $2N$ arbitrary parameters $(p_i)_{i=1}^{i=2N}$ which control the amplitudes, speeds, and phases of the soliton components. If the p_i's are modulated so that

$$\frac{dH_{n,\alpha}}{dt}(s_N) = \dot{\alpha} \int \frac{\partial T_{n,\alpha}}{\partial \alpha}(s_N) dx \qquad \text{for} \qquad 1 < n \leqslant N + 1, \quad (13)$$

one can write

$$\forall n > N + 1 \qquad \frac{\partial H_{n,\alpha}}{\partial u}(u = s_N) = \sum_{k=2}^{N+1} C_{k,\alpha} \frac{\partial H_{k,\kappa}}{\partial u}(u = s_N) \quad (14)$$

for some multipliers $c_{k,\alpha}$, and (13) will still hold for $n > N + 1$. Equations (13) yields N equations for the modulation, whereas the multisoliton depends on $2N$ parameters, N of which fix the spectrum of the operator $L = -\partial^2/\partial x^2 + s_N$. Hence, the modulation only determines the isospectral class. For a pure soliton, only a phase is left undetermined, and the modulated soliton should thus be a good approximation of the actual solution, up to translation. For similar problems, we refer to the papers by Benjamin and Mac Kean (Refs. 7 and 8; see also below, Sec. III).

In short, the modulation of the first integrals will be consistent with the whole hierarchy. Again, the mass poses a problem, because its gradient does not vanish at infinity. This suggests that, before generalizing the result of the theorem, we look at the periodic case for (1).

B. The periodic problem for KdV

We refer to Mac Kean and Van Moerbecke[9] for the necessary material and, in order to use standard expressions, we change $u(x,t)$ into $-3u(\frac{1}{2}x,\frac{1}{2}t)$ so that (1) is transformed into

$$u_t = 3\alpha(t) u u_x - \tfrac{1}{2} u_{xxx}. \quad (15)$$

Also, we look for solutions with *primitive* period 1. If we set $q \equiv \alpha u$, (15) transforms into

$$q_t = 3 q q_x - \tfrac{1}{2} q_{xxx} + (\dot{\alpha}/\alpha) q. \quad (16)$$

When $\dot{\alpha} \equiv 0$, solutions corresponding to solitons will have three *simple* eigenvalues, i.e., only one band of instability (λ_1, λ_2) and the main difference with the previous case is that, apart from translation, there are now *two* degrees of freedom, q—which is an elliptic function—being determined, up to translation, by the values of $H_1 \equiv \int q$ and $H_2 \equiv \frac{1}{2} \int q^2$, subject only to $H_2 > \frac{1}{2} H_1^2$. This makes it possible, in the case of a nonconstant α, to determine the modulated $q(x,t)$ such that

(i) for any fixed t, $q(x,t)$ as a function of x has three simple eigenvalues.

(ii) $H_1(q) = \alpha(t) h_1$; $H_2(q) = \alpha^2(t) h_2$ with two constants h_1 and h_2.

For any n, we then have

$$\frac{\partial H_n}{\partial q} = a_n \frac{\partial H_1}{\partial q} + b_n \frac{\partial H_2}{\partial q} = a_n + b_n q. \quad (17)$$

Reverting to the original u, we find

$$H_n(q) = \alpha^{k_n} H_{n,\alpha}(u) \quad (18)$$

for some constants k_n. Hence

$$\frac{\partial H_n}{\partial q} = \alpha^{k_n - 1} \frac{\partial H_{n,\alpha}}{\partial u} \quad (19)$$

and

$$\frac{\partial H_{n,\alpha}}{\partial u} = \alpha^{1 - k_n}(a_n + b_n \alpha u) = \alpha_n + \beta_n u, \quad (20)$$

yielding

$$\begin{aligned} \frac{dH_{n,\alpha}}{dt} &= \int_0^1 \frac{\partial H_{n,\alpha}}{\partial u} u_t\, dx + \dot{\alpha} \int_0^1 \frac{\partial T_{n,\alpha}}{\partial \alpha} dx \\ &= \lambda_{n,\alpha} \frac{d}{dt}\left(\int u\right) + \mu_{n,\alpha} \frac{1}{2} \frac{d}{dt}\left(\int u^2\right) \\ &\quad + \dot{\alpha} \int_0^1 \frac{\partial T_{n,\alpha}}{\partial \alpha} dx = \dot{\alpha} \int_0^1 \frac{\partial T_{n,\alpha}}{\partial \alpha} dx \quad (21) \end{aligned}$$

since $\int u \equiv h_1$ and $\frac{1}{2} \int u^2 \equiv h_2$.

Thus, the modulation is again consistent with the evolution of the $H_{n,\alpha}$ and H_1 is not exceptional any more. It is instructive to check in a direct way that the evolution of the simple spectrum is also correctly matched by the modulation: for a simple eigenvalue λ, if q is a solution of (16), we have

$$\dot{\lambda} = \frac{\dot{\alpha}}{\alpha} \int_0^1 F^2 q\, dx, \quad (22)$$

where F is a normalized eigenfunction for the eigenvalue λ:

$$F'' = (q - \lambda) F; \qquad \int_0^1 F^2 = 1. \quad (22\text{ bis})$$

Suppose q is now the modulated function, according to (i) and (ii), with associated simple spectrum $(\lambda_0(t),\lambda_1(t),\lambda_2(t))$. Then (cf. Ref. 9)

$$F^2(x) = \frac{1}{2(\lambda - \lambda_1')}(-\sigma + 2\lambda + q), \tag{23}$$

where $\sigma(t) = \lambda_0 + \lambda_1 + \lambda_2$ and λ_1' is the zero of the derivative of the Hill discriminant between λ_1 and λ_2 ($\lambda_1' \in (\lambda_1,\lambda_2)$; $\Delta'(\lambda_1') = 0$); hence (22) requires

$$\dot{\lambda} = \frac{\dot{\alpha}}{\alpha}\frac{1}{2(\lambda - \lambda_1')}\int_0^1 (-\sigma + 2\lambda + q)q\,dx$$

$$= \frac{\dot{\alpha}}{\alpha}\frac{1}{2(\lambda - \lambda_1')}[(-\sigma + 2\lambda)H_1 + 2H_2]. \tag{24}$$

But we also have $F^2 = \partial\lambda/\partial q$ and since

$$F^2(x) = \frac{1}{2(\lambda - \lambda_1')}\left[(-\sigma + 2\lambda)\frac{\partial H_1}{\partial q} + \frac{\partial H_2}{\partial q}\right], \tag{25}$$

we obtain

$$\dot{\lambda} = \frac{1}{2(\lambda - \lambda_1')}[(-\sigma + 2\lambda)\dot{H}_1 + \dot{H}_2]. \tag{26}$$

Comparing (22) with (26), we find the conditions

$$H_1 \sim \alpha(t); \qquad H_2 \sim \alpha^2(t) \tag{27}$$

thus confirming the above result. This calculation raises two interesting questions:

(i) How does the discrepancy in the mass flux precisely appear when the period of the motion tends to infinity?

(ii) Does a slow variation of α prevent the opening up of the double spectrum? A positive answer would lead, via trace formula, to a proof that the modulated elliptic function is indeed, for slowly varying α, a good approximation of the solution.

Finally, we briefly mention that N-periodic solutions can be treated in the same way; they depend on $2N+1$ parameters, $N+1$ of which determine the isospectral torus, which has dimension N. Again, the modulation only yields $N+1$ equations, to determine the isospectral class. For a nice geometric formulation of the modulation problem in the KdV (integrable) equation, we refer the reader to H. Flaschka *et al.*[10]

C. Generalization

In this section, we spell out a generalization of the theorem in Sec. A. We keep on assuming that the Cauchy problem is solvable and work with sufficiently fast decreasing functions.

Theorem: Let

$$u_t = K_{\alpha(t)}(u,\ldots,u^{(n)}) \tag{28}$$

be a nonlinear equation such that, when α is fixed, the corresponding equation has a sequence of integrals of the motion

$$H_{n,\alpha} \equiv \int_{-\infty}^{+\infty} T_{n,\alpha}[u]\,dx. \tag{29}$$

We make the further assumption that one of the $H_{n,\alpha}$ (say $n = n_0$) is in fact independent of α [we denote it by $E(u)$] and that there exist solitons $s(x-ct,\alpha,c)$ in the case when α is fixed, which satisfy the condition

— $\forall n \geqslant n_0$, s is a critical point of $H_{n,\alpha}$ when E is held fixed.

The conclusion is the following: start from the soliton profile $u(x,0) \equiv s(x,\alpha(0),c_0)$ and construct the modulated soliton, requiring that E be a constant. Then, the relation

$$\frac{d}{dt}H_{n,\alpha} = \dot{\alpha}\int_{-\infty}^{+\infty}\frac{\partial T_{n,\alpha}}{\partial\alpha}\,dx \tag{30}$$

will hold for $n \geqslant n_0$, both for the actual solution and the modulated soliton.

Proof: It was already given in the general form in Sec. A.

Amplification: Multisolitons can be treated similarly; for details, see again Sec. A.

Remarks: 1. The simplest way of writing equations of type (28) is by scaling, in an integrable equation, on u, x, and t (possibly simultaneously) and then making the scaling time dependent.

2. The condition of the theorem is satisfied at least for the KdV family; in fact, in the paper by Lax, there is indicated an easy way to check it. In particular, it also holds for the modified KdV equation.

Examples: By scaling on x and t, one can study, for instance,

$$\varphi_{tt} - \varphi_{xx} = \alpha(t)\sin\varphi. \tag{31}$$

Here, we look briefly at the equation

$$v_t + \alpha^2(t)v^2v_x + v_{xxx} = 0 \tag{32}$$

because, although it is obviously very similar to (1), there is an important point to be noticed. Equation (32) has again a hierarchy of conserved quantities $\widetilde{H}_{n,\alpha}$ when α is constant, the first of which are

$$M(v) = \int_{-\infty}^{+\infty} v\,dx; \qquad E(v) = \int_{-\infty}^{+\infty} v^2\,dx;$$

$$\widetilde{H}_{3,\alpha} = \int_{-\infty}^{+\infty}(\alpha^2v^4 - 6v_x^2)dx. \tag{33}$$

The soliton can be written

$$s(x,-ct,c,\alpha) = (6c)^{1/2}\alpha^{-1}\,\mathrm{sech}[c^{1/2}(x-ct)] \tag{34}$$

and

$$M(s) = \pi\sqrt{6}/\alpha \quad \text{independent of} \quad c;$$

$$E(s) = 12\sqrt{c}/\alpha^2. \tag{35}$$

Hence, the modulated soliton is given by $c(t) = c_0\alpha^4(t)$;

$$s(x,t,c_0,\delta) = (6c_0)^{1/2}\alpha\,\mathrm{sech}\left[\alpha^2c_0^{1/2}\left(x - c_0\int_0^t \alpha^4(s)\,ds\right) - \delta\right]. \tag{36}$$

It satisfies relation (30) for $n \geqslant 2$. The interesting point is the following: $M(v)$ and $E(v)$ are again both invariant under (32) and one again chooses to preserve $E(s)$ in the modulation; however, (32) and (1) are related via the Miura transform

$$u_t + \alpha uu_x + u_{xxx} = \left(2\alpha v + i\sqrt{6}\frac{\partial}{\partial x}\right)(v_t + v^2v_x + v_{xxx}),$$

$$u \equiv \alpha v^2 + i\sqrt{6}v_x. \tag{37}$$

Under this transform, the energy $\int v^2\,dx$ corresponds to the mass $\int u\,dx$. Why then in both cases does the soliton "choose" to preserve the former? A mathematical answer is given in the above theorem; a different, more physical explanation will be examined at the end of the second section.

III. ADIABATIC STABILITY OF SOLITONS

We first explain, very briefly, why solitons are indeed privileged candidates for being adiabatic invariants. There exist at present two adiabatic theorems, one in classical mechanics, the other one in quantum mechanics. The first states the adiabatic invariance of the canonical momenta of completely integrable, finite-dimensional, Hamiltonian systems (which also includes the invariance of the phase volume); although it is used in physics, it is not rigorously demonstrated in more than one dimension for the configuration space (cf. V. Arnold, Ref. 11). The theorem in quantum mechanics states the invariance of the eigenstates associated with discrete eigenvalues in the spectrum of the Hamiltonian. It appeared in the late twenties (cf. M. Born and V. Fock, Ref. 12) and was neatly demonstrated by T. Kato (cf. Ref. 13).

The first one is thus dealing with finite-dimensional completely integrable systems, and the second with infinite-dimensional linear (hence with trivial action-angle variables) systems; they can be related to each other by considering the Hilbert space of "states" over the classical system, working with the Liouville operator (cf. Ref. 14).

Now, solitons embody the discrete part of the spectrum of infinite-dimensional (nonlinear) completely integrable systems; they are related to canonical momenta and should therefore be adiabatically invariant in some sense (see Ref. 15 for other arguments).

A. The conjecture

Adiabaticity however, is often referred to in a rather loose way, and we feel it may be useful to give a precise formulation of an adiabatic theorem for solitons, modeled on the two existing ones (we again recall that the theorem in classical mechanics is far from being demonstrated, or even properly stated, in a general setting). First, one should work with time-dependent coefficients only; since a soliton is a local object, it experiences spatial gradients as time-dependent fields in the rest frame; in fact, experiments devised to check adiabatic stability in quantum mechanics always involve spatial gradients, rather than time-dependent potentials.

Also, from a purely mathematical point of view, the difficult thing is to find the right function space to work with, i.e., the right notion of distance between the adiabatically modulated soliton and the actual solution (we again refer to Benjamin and Mac Kean for a similar problem in a different kind of stability). The last issue is about the best way of modulating the soliton; we comment more on this later, but first, state our provisional theorem.

Conjecture: Let

$$u_t = K_{\alpha(t)}(u,u',...,u^{(n)}) \tag{38}$$

be a nonlinear equation such that for any fixed α, the corresponding equation is integrable with solitons $s(x-ct,c,\alpha)$ traveling at speed c; (38) preserves an "energy" $E(u)$.

Put $\alpha(t)=\tilde{\alpha}(t/\tau)$, where $\tilde{\alpha}$ is a nice (C^1) function, such that, $\tilde{\alpha}(z)\equiv 0$ for $z\leqslant 0$, $\tilde{\alpha}(z)\equiv 1$ for $z\geqslant 1$, $\tilde{\alpha}$ is monotonous.

Start with the initial data:

$$u(x,0)=s(x,c_0,\alpha(0)) \tag{39}$$

and construct the modulated soliton

$$s\equiv s(x,t,\alpha,c_0,\delta(t)) \tag{40}$$

which preserves energy. Then, there exists $\delta(t)$ such that

$$\lim_{\tau\to\infty}\ \sup_{0<t<\tau^{-1}} \|u(x,t)-s(x,t,c_0,\delta(t))\| = 0, \tag{41}$$

the norm $\|\cdot\|$ referring to some function space.

Thus, up to translation, the modulated soliton and the solution asymptotically ($\tau\to\infty$) coincide. We of course believe that the identities we displayed in part one should perhaps provide a tool for proving such a theorem. We again notice that they hold without any *slowness* in the variation of α being assumed, but one nice thing is that this would connect the stability of solitons with their extremal properties, giving flesh to the appealing picture that when the variation in α is not abrupt, the solution should follow the "bottom of the valley" represented by the modulated soliton. In this respect, the extension to multisoliton states may be misleading, because whereas solitons provide minima of constrained variational problems, multisolitons represent only critical points for such problems.

B. The link with Whitham's modulation theory

We return to the problem of finding the correct constant of the motion to use, in order to display the modulated soliton. To this end, we use Whitham's approach to the modulation theory (see Ref. 16 for a thorough exposition), which, however, deals primarily with *periodic* waves. It does exhibit adiabatic invariance, namely under the form of "conservation of wave action," and since solitons are limits of periodic waves, there should exist a way to relate both theories; we now proceed to describe a first result along this line.

We start with the KdV and MKdV equations

$$u_t + 6uu_x + u_{xxx} = 0, \tag{42}$$

$$v_t + 6v^2v_x + v_{xxx} = 0. \tag{43}$$

These equations are both Lagrangian, if one considers "potentials" φ and ψ such that

$$u=\varphi_x; \qquad v=\psi_x \tag{44}$$

and Lagrangians

$$L[\varphi] = -\tfrac{1}{2}\varphi_t\varphi_x - \varphi_x^3 + \tfrac{1}{2}\varphi_{xx}^2, \tag{45}$$

$$L[\psi] = -\tfrac{1}{2}\psi_t\psi_x - \tfrac{1}{2}\psi_x^4 + \tfrac{1}{2}\psi_{xx}^2. \tag{46}$$

To emphasize the relevant parameters, we write the corresponding solitons in the form

$$s = a\,\mathrm{sech}^2[(a/2)^{1/2}(x-(\omega/k)t)], \tag{47}$$

$$\tilde{s} = a\,\mathrm{sech}[a(x-(\omega/k)t)], \tag{48}$$

a being the amplitude, and ω and k the frequency and wave number.

Following Whitham's prescription, one writes down the "averaged" Lagrangians

$$\mathscr{L}(\omega,k,a) = k\int_{-\infty}^{+\infty} L\,[s]; \qquad \tilde{\mathscr{L}}(\omega,k,a) \equiv \int_{-\infty}^{-\infty} \tilde{L}\,[\tilde{s}]. \tag{49}$$

It is worth noticing that $\mathscr{L}$ and $\tilde{\mathscr{L}}$ represent in fact the averaged Lagrangians for periodic trains of solitons having k waves per unit length. Whitham's equations are expressed as

$$\mathscr{L}_a = 0; \qquad \frac{\partial \mathscr{L}_\omega}{\partial T} - \frac{\partial \mathscr{L}_k}{\partial X} = 0 \tag{50}$$

and similarly for $\tilde{\mathscr{L}}$. The first equation yields the "dispersion relation," namely $\omega = 2ak$ for the KdV equation and $\omega = ka^2$ for the MKdV equation.

The second equation is the conservation of wave action, which expresses a "dual" adiabatic invariance; in Whitham's words, " in the special case of a wave train [...] responding to changes of the medium in time, we have $\mathscr{L}_\omega = \text{const.}$" There is of course a dual statement if one exchanges space and time, which is of interest for the study of equations with space-dependent coefficients. We now compute $\mathscr{L}_\omega$; since $s = s(x - (\omega/k)t) = \varphi_x$, one has

$$-\tfrac{1}{2}\varphi_t\varphi_x = \frac{\omega}{2k}\varphi_x^2 = \frac{\omega}{2k}s^2, \tag{51}$$

and this is the only term in L to give an ω-dependent contribution

$$\begin{aligned} \mathscr{L}_\omega &= \frac{\partial}{\partial\omega}\left(k\int_{-\infty}^{+\infty} - \frac{1}{2}\varphi_t\varphi_x\right) \\ &= \frac{\partial}{\partial\omega}\left(\frac{\omega}{2}\int_{-\infty}^{+\infty} s^2\right) = \frac{1}{2}\int_{-\infty}^{+\infty} s^2 \sim a^{3/2}. \end{aligned} \tag{52}$$

We thus recover the fact that when coefficients are slowly varying, modulation should keep $\mathscr{L}_\omega = \frac{1}{2}\int s^2$ fixed. The computation is the same for the MKdV equation, since we are concerned only with the $-\frac{1}{2}\psi_t\psi_x$ term, in the evaluation of $\tilde{\mathscr{L}}_\omega$. It is in fact obvious that this derivation can be extended to other equations since only the t-dependent part of the Lagrangian contributes. If one looks at Eq. (31) for instance, the above calculation will give the right answer again, namely that in the modulation, one should keep $\mathscr{L}_\omega = \int \varphi_x^2$ fixed.

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