ELLIPTIC MULTIZETAS AND THE ELLIPTIC DOUBLE SHUFFLE RELATIONS

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Abstract. We define an *elliptic generating series* whose coefficients, the *elliptic multizetas*, are related to the elliptic analogs of multiple zeta values introduced by Enriquez as the coefficients of his elliptic associator; both sets of coefficients lie in $\mathcal{O}(\mathfrak{H})$, the ring of functions on the Poincaré upper half-plane $\mathfrak{H}$. The elliptic multizetas generate a $\mathbb{Q}$-algebra $\mathcal{E}$ which is an elliptic analog of the algebra of multiple zeta values. Working modulo $2\pi i$, we show that the algebra $\mathcal{E}$ decomposes into a geometric and an arithmetic part, and study the precise relationship between the elliptic generating series and the elliptic associator defined by Enriquez. We show that the elliptic multizetas satisfy a double shuffle type family of algebraic relations similar to the double shuffle relations satisfied by multiple zeta values. We prove that these elliptic double shuffle relations give all algebraic relations among elliptic multizetas if (a) the classical double shuffle relations give all algebraic relations among multiple zeta values and (b) the elliptic double shuffle Lie algebra has a certain natural semi-direct product structure analogous to that established by Enriquez for the elliptic Grothendieck-Teichmüller Lie algebra.

1. Introduction

1.1. **Elliptic multizetas.** An elliptic analog of the multiple zeta values first made an explicit appearance in Enriquez' article [15] under the name "analogues elliptiques des nombres multizetas". They arise as coefficients of his elliptic associator constructed in [14], which is closely related to the elliptic Knizhnik–Zamolodchikov–Bernard (KZB) equation [8, 24] and to multiple elliptic polylogarithms [7, 24]; more recently, they have even found applications to computations in high energy physics [1]. Taking the regularized limit $\tau \to i\infty$ of elliptic multizetas, one retrieves the classical multiple zeta values [15, 26], which gives the explicit connection between the genus zero and genus one multizetas. The idea of considering the graded $\mathbb{Q}$-algebra generated by these coefficients, was introduced in [2, 26, 27], which provide some explicit dimension results in depth 2.

Recall that the Drinfel'd associator Φ_{KZ}, first introduced in [11], is a power series in two non-commutative variables[1], which is the generating series for the usual multiple zeta values, by the work of Le and Murakami [**?**]. In analogy with this, Enriquez's elliptic associator, which is defined as a pair of monodromies (cf. §5.2 of [14]), takes the form of a pair of group-like power series in two non-commutative

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[1]Throughout this paper we use the same definitions and conventions as [14]; in particular Φ_{KZ} is defined in §5.1.

variables a and b:

$$A(\tau), B(\tau) \in \mathcal{O}(\mathfrak{H})\langle\langle a, b\rangle\rangle,$$

where $\mathcal{O}(\mathfrak{H})$, denotes the ring of holomorphic functions of one variable τ running through the Poincaré upper half-plane. We call the coefficients of $A(\tau)$ and $B(\tau)$ *A-elliptic multizetas* and *B-elliptic multizetas*, or A-EMZs and B-EMZs. The acronym EMZ stands for *elliptic multizetas*; since they are functions of τ and not real numbers like the coefficients of Φ_{KZ}, we drop the word "values" in the elliptic situation[2].

The main new object introduced in this article is a third power series

$$C(\tau) \in \mathcal{O}(\mathfrak{H})\langle\langle a, b\rangle\rangle$$

and its logarithm $E(\tau) = \log C(\tau)$ (§3.1). The series $E(\tau)$ is called the *elliptic generating series*, and its coefficients, in $\mathcal{O}(\mathfrak{H})$, are called *$E$-elliptic multizetas* or E-EMZs. We write $\mathcal{E}$, $\mathcal{A}$ and $\mathcal{B}$ for the vector spaces generated by the coefficients of $C(\tau)$, the A-EMZs and the B-EMZs respectively, which are by definition subspaces of $\mathcal{O}(\mathfrak{H})$. We show in Lemma 3.2 that like $A(\tau)$ and $B(\tau)$, $C(\tau)$ is a group-like power series, which implies that each of the three subspaces $\mathcal{E}$, $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{O}(\mathfrak{H})$ actually forms a $\mathbb{Q}$-algebra. In Lemma 3.3, we show that that the coefficients of $E(\tau)$, the E-MZVs, form a system of algebra generators for $\mathcal{E}$. This is the generating system we will study in the rest of the article.

One of the main results in [14] concerning the power series $A(\tau)$, $B(\tau)$ is that they can be written in the form[3] $g(\tau)\cdot A$ and $g(\tau)\cdot B$, where $g(\tau)$ is an automorphism of $\mathcal{O}(\mathfrak{H})\langle\langle a, b\rangle\rangle$ (introduced in §5.1 of [14] but recalled in §2 below), and A and B are power series in $\mathcal{Z}[2\pi i]\langle\langle a, b\rangle\rangle$ (defined in §3.5 of [14] with the notation A_+, A_-, but recalled in §3.1 below). This property of the elliptic associator is the motivation for our definition of the power series $C(\tau)$ directly in the form $g(\tau)\cdot C$, where C is a group-like power series in $\mathcal{Z}[2\pi i]\langle\langle a, b\rangle\rangle$ closely related to A and B (§3.1). Since $g(\tau)$ is an automorphism, the power series $E(\tau) = \log C(\tau)$ then naturally takes the form $g(\tau)\cdot E$ with $E = \log C \in \mathcal{Z}[2\pi i]\langle\langle a, b\rangle\rangle$.

We let $\mathcal{E}^{\mathrm{geom}}$ denote the $\mathbb{Q}$-algebra generated by the coefficients of $g(\tau)$ (in a precise sense explained in §2). These coefficients lie in $\mathcal{O}(\mathfrak{H})$, and are realized as particular linear combinations of iterated integrals of Eisenstein series for $\mathrm{SL}_2(\mathbb{Z})$ (see [4, 25]). We note that for any ring R, $g(\tau)$ induces an automorphism of $\big(\mathcal{E}^{\mathrm{geom}}\otimes R\big)\langle\langle a, b\rangle\rangle$. We use this fact frequently below.

The structure of the $\mathbb{Q}$-algebra $\mathcal{E}^{\mathrm{geom}}$ is the main topic of §2. It is related to the bigraded Lie algebra $\mathfrak{u}^{\mathrm{geom}}$ of the prounipotent radical of $\pi_1^{\mathrm{geom}}(MEM)$, where MEM denotes the Tannakian category of universal mixed elliptic motives [19]. More precisely, $\mathcal{E}^{\mathrm{geom}}$ is related to the bigraded Lie algebra $\mathfrak{u}$ which is the image of $\mathfrak{u}^{\mathrm{geom}}$ under the monodromy representation from $\mathfrak{u}^{\mathrm{geom}}$ to the Lie algebra of derivations of a free Lie algebra on two generators whose existence is shown in [19], §22. The explicit generators of this Lie algebra are well-known, cf. Definition 2.1.

[2]These values, which are Enriquez' "elliptic analogs of MZV's", and the E-MZVs introduced below, are very different from Brown's "multiple modular values" [4], which are complex numbers.

[3]Throughout this article we use the dot notation $\cdot$ to indicate the action of automorphisms or derivations on elements.

The first results of this article are summarized in the following theorem, whose proof rests in large part on the $\mathbb{C}$-linear independence of iterated integrals of Eisenstein series proved in Theorem 2.8 and its corollary 2.9[4].

Theorem. *(i) [Thm. 2.6] There is a natural isomorphism*

$$\mathcal{E}^{\rm geom} \cong \mathcal{U}(\mathfrak{u})^\vee,$$

where $\mathcal{U}(\mathfrak{u})^\vee$ *is the graded dual of the universal enveloping algebra of the Lie algebra* $\mathfrak{u}$*. In particular,* $\mathcal{E}^{\rm geom}$ *is a commutative, graded, Hopf* $\mathbb{Q}$*-algebra.*

(ii) [Cor. 2.10] The subalgebra of $\mathcal{O}(\mathfrak{H})$ *generated by* $\mathcal{E}^{\rm geom}$ *and* $\mathcal{Z}[2\pi i]$ *is isomorphic to the tensor product*

$$\mathcal{E}^{\rm geom} \otimes_{\mathbb{Q}} \mathcal{Z}[2\pi i]. \tag{1.1}$$

Corollary. *The* $\mathbb{Q}$*-algebras* $\mathcal{A}$*,* $\mathcal{B}$ *and* $\mathcal{E}$ *are subalgebras of the tensor product* (1.1).

Proof. Since $A(\tau)$ has the form $g(\tau)\cdot A$, the coefficients of $A(\tau)$ are algebraic expressions in elements of $\mathcal{E}^{\rm geom}$ and $\mathcal{Z}[2\pi i]$. The same holds for $B(\tau)$ and $C(\tau)$. The result then follows from (ii) of the Theorem. □

1.2. **Structure of the elliptic multizeta algebras mod $2\pi i$.** For technical reasons linked to our use of Grothendieck-Teichmüller theory and Écalle's mould theory, we restrict our study of the three different types of elliptic multizetas to objects to their reductions modulo $2\pi i$ in the following sense.

Let $\overline{\mathcal{Z}}$ denote the quotient of $\mathcal{Z}[2\pi i]$ by the ideal generated by $2\pi i$, which is isomorphic to the quotient of the algebra of multiple zeta values $\mathcal{Z}$ by the ideal generated by $\zeta(2) = -\frac{(2\pi i)^2}{24}$. The quotient of $\mathcal{E}^{\rm geom}\otimes_{\mathbb{Q}} \mathcal{Z}[2\pi i]$ by the ideal $1\otimes 2\pi i$ is isomorphic to

$$\mathcal{E}^{\rm geom} \otimes_{\mathbb{Q}} \overline{\mathcal{Z}}.$$

Definition 1.1. *Let* $\overline{\mathcal{E}}$ *(resp.* $\overline{\mathcal{B}}$*) denote the image of the subalgebra* $\mathcal{E}$ *(resp.* $\mathcal{B}$*) of* $\mathcal{E}^{\rm geom}\otimes\mathcal{Z}[2\pi i]$ *in the quotient* $\mathcal{E}^{\rm geom}\otimes\overline{\mathcal{Z}}$*. Let the reduced power series* $\overline{E}(\tau)$ *(resp.* $\overline{B}(\tau)$*) to be obtained from* $E(\tau)$ *(resp.* $B(\tau)$*) by reducing the coefficients from* $\mathcal{E}$ *to* $\overline{\mathcal{E}}$ *(resp. from* $\mathcal{B}$ *to* $\overline{\mathcal{B}}$*).*

The case of $\mathcal{A}$ is slightly different, because it follows from the definition of $A(\tau)$ given in (3.2) and (3.3) that the ring $\mathcal{A}$ lies in the ideal $\mathbb{Q}\cdot 1 + \mathcal{E}^{\rm geom}\otimes 2\pi i\mathcal{Z}[2\pi i]$, and therefore the image of this ring in the quotient $\mathcal{E}^{\rm geom}\otimes\overline{\mathcal{Z}}$ is just $\mathbb{Q}\cdot 1$. We get around this as follows. We set $A' = A\frac{1}{2\pi i}$; this power series lies in $\mathcal{Z}\langle\langle a,b\rangle\rangle$ by the definition of A (cf. (3.2)). We then set

$$A'(\tau) = g(\tau)\cdot A' \in \big(\mathcal{E}^{\rm geom}\otimes\mathcal{Z}[2\pi i]\big)\langle\langle a,b\rangle\rangle.$$

Definition 1.2. *Let* $\mathcal{A}'\subset\mathcal{E}^{\rm geom}\otimes\mathcal{Z}$ *denote the* $\mathbb{Q}$*-algebra generated by the coefficients of* $A'(\tau)$*. Let* $\overline{\mathcal{A}}$ *denote the image of* $\mathcal{A}'$ *in the reduced ring* $\mathcal{E}^{\rm geom}\otimes\overline{\mathcal{Z}}$*. Let* $\overline{A}'$ *be the power series obtained from* A' *by reducing the coefficients from* $\mathcal{Z}$ *to* $\overline{\mathcal{Z}}$*, and* $\overline{A}'(\tau)$ *to be the power series obtained from* $A'(\tau)$ *by reducing the coefficients from* $\mathcal{E}^{\rm geom}\otimes\mathcal{Z}$ *to* $\mathcal{E}^{\rm geom}\otimes\overline{\mathcal{Z}}$*. We have* $\overline{A}'(\tau) = g(\tau)\cdot\overline{A}'$*.*

[4] The second author subsequently generalized this result to the case of arbitrary quasimodular forms for $\mathrm{SL}_2(\mathbb{Z})$.

The coefficients of the three reduced power series $\overline{E}(\tau)$, $\overline{B}(\tau)$ and $\overline{A}'(\tau)$, which generate the three subalgebras $\overline{\mathcal{E}}$, $\overline{\mathcal{B}}$ and $\overline{\mathcal{A}}$ of $\mathcal{E}^{\rm geom}\otimes\overline{\mathcal{Z}}$, are called $\overline{E}$-EMZs, $\overline{B}$-EMZs and $\overline{A}$-EMZs.

Our first goal is to compare the four algebras $\overline{\mathcal{E}}$, $\overline{\mathcal{A}}$, $\overline{\mathcal{B}}$ and $\mathcal{E}^{\rm geom}\otimes_{\mathbb{Q}}\overline{\mathcal{Z}}$. The element $2\pi i\tau\in\mathcal{O}(\mathfrak{H})$ plays a special role in this comparison. It lies in $\mathcal{E}^{\rm geom}$ since it is the coefficient of ε_0 in $g(\tau)$ (see (2.6) below), so $2\pi i\tau\otimes 1$ lies in the tensor product $\mathcal{E}^{\rm geom}\otimes_{\mathbb{Q}}\overline{\mathcal{Z}}$, but it does not lie in $\mathcal{E}$ or $\mathcal{A}$, although it does lie in $\mathcal{B}$. Working mod $2\pi i$ allows us to make a much more precise statement than the simple inclusion, which also has the advantage of showing that the three reduced algebras are highly non-trivial.

Theorem (Cor. **??**). *We have the equalities*[5]

$$\overline{\mathcal{E}}[2\pi i\tau\otimes 1]=\overline{\mathcal{A}}[2\pi i\tau\otimes 1]=\overline{\mathcal{B}}=\mathcal{E}^{\rm geom}\otimes_{\mathbb{Q}}\overline{\mathcal{Z}}.$$

As noted above, the elliptic generating series $E(\tau)$ has the form $g(\tau)\cdot E$; thus its reduction $\overline{E}(\tau)$ has the form $g(\tau)\cdot\overline{E}$ where the reduction $\overline{E}$ of E has coefficients in $\overline{\mathcal{Z}}$. Throughout this article, we consider the power series $\overline{\Phi}_{\rm KZ}$ obtained by reducing the coefficients of $\Phi_{\rm KZ}$ from $\mathcal{Z}$ to $\overline{\mathcal{Z}}$. This reduced power series, a priori an associator belonging to the torsor of associators $M(\overline{\mathcal{Z}})$, can be considered as lying in the group $GRT_1(\overline{\mathcal{Z}})$ (defined in [11], §5), since the associator relations become equivalent to the group relations for any associator whose degree 2 part is zero (cf. Prop. 5.9 of [11]). The key point (Theorem 3.4) of the proof of the equality $\overline{\mathcal{E}}[2\pi i\tau\otimes 1]=\mathcal{E}\otimes\overline{\mathcal{Z}}$ is the fact that

$$\overline{E}=\widetilde{\Gamma}(\overline{\Phi}_{\rm KZ}),\tag{1.2}$$

where $\widetilde{\Gamma}$ is the composition

$$GRT_1(\overline{\mathcal{Z}})\xrightarrow{\Gamma}GRT_{ell}(\overline{\mathcal{Z}})\xrightarrow{\pi}\overline{\mathcal{Z}}\langle\langle a,b\rangle\rangle,$$

where Γ is Enriquez' section ([14], §4), and the projection π maps an element $(\lambda,f,g_+,g_-)\in GRT_{ell}(\overline{\mathcal{Z}})$ to the component $g_+\in\overline{\mathcal{Z}}\langle\langle a,b\rangle\rangle$. In particular, we show that the coefficients of $\overline{E}$ generate all of $\overline{\mathcal{Z}}$ (Corollary 3.5). The ring $\overline{\mathcal{E}}$ is the $\mathbb{Q}$-algebra generated by the coefficients of $\overline{E}(\tau)$, which, as for $E(\tau)$, are all algebraic expressions in the coefficients of $g(\tau)$ and those of $\overline{E}$. The delicate part of the proof consists in showing that the coefficients of $\overline{E}(\tau)$ can be "untangled" to separately recover a set of generators of $\overline{\mathcal{Z}}$ and, with the addition of $2\pi i\tau$, a set of generators of $\mathcal{E}^{\rm geom}$. The same arguments hold for $\overline{\mathcal{A}}$ and $\overline{\mathcal{B}}$, except that instead of (1.2), we use [26], Theorem 5.4.2, to show that the coefficients of the arithmetic parts $\overline{A}'$ and $\overline{B}$ generate all of $\overline{\mathcal{Z}}$ and the same untangling argument. In fact, the result can be framed for more general power series, as is done in Theorem 3.6. Thanks to the above theorem, we call the $\mathbb{Q}$-algebra $\mathcal{E}^{\rm geom}\otimes_{\mathbb{Q}}\overline{\mathcal{Z}}$ the *$\mathbb{Q}$-algebra of elliptic multizetas (modulo $2\pi i$).*

1.3. The elliptic double shuffle relations for $\overline{E}$-EMZs.

The theorem in the previous section shows that the $\overline{E}$-EMZs together with $2\pi i\tau$ form a set of generators of $\mathcal{E}^{\rm geom}\otimes\overline{\mathcal{Z}}$, as do the $\overline{A}$-EMZs and the $\overline{B}$-EMZs taken together. These three sets of generators are quite different from each other. A natural question, given a set of generators of a ring, is to try to establish the relations they satisfy, and if possible a

[5]For two $\mathbb{Q}$-algebras $A\subset B$ and an element $b\in B$, the notation $A[b]$ here denotes the subring of B generated by A and b.

complete set thereof. The fact that the power series $\overline{A}(\tau)$ and $\overline{B}(\tau)$ are group-like provides some relations; another family, the "Fay relations", is partially known for $\overline{A}$-EMZs (cf. [26]). Enriquez gives a complete set of *associator relations* satisfied by the elliptic associator, derived from the fact that the power series $A(\tau)$ and $B(\tau)$ induce an automorphism of the prounipotent 2-strand torus braid group. However, these relations mingle the A-EMZs and the B-EMZs along with genus zero multiple zeta values, and do not provide separate relations for each generating set. For the $\overline{E}$-EMZs, however, we can say more, both about explicit relations satisfied by $\overline{E}(\tau)$, and about the question of whether these relations may be a complete set.

The third main result of this article (Theorem 4.4) shows that the $\overline{E}$-EMZs satisfy an explicit set of algebraic relations called the *elliptic double shuffle relations*, given in the form of two *elliptic double shuffle equations* satisfied by the power series $\overline{E}(\tau)$.

These elliptic double shuffle relations arise as follows. Since we have $\overline{E}(\tau) = \Gamma(\overline{\Phi}_{\mathrm{KZ}})$, the power series $\overline{E}(\tau)$ will necessarily satisfy equations that are transports by Γ of the usual double shuffle equations satisfied by $\overline{\Phi}_{\mathrm{KZ}}$ (shuffle and stuffle relations, also known as double mélange in Racinet's terminology [32], and symmetrality/symmetrility in Écalle's [13]). These transported relations can be determined explicitly thanks to a major theorem of Écalle together with the results of [34]. We call them the *elliptic double shuffle equations*, and show that they are similar in nature to the well-known (regularized) double shuffle relations for multiple zeta values, but in fact, surprisingly, closer to their depth-graded version.

The proof of the theorem relies on several difficult known results, in particular a crucial theorem of Écalle ([13] but see [33], Theorem 4.6.1 for a complete proof).

On the question of whether the elliptic double shuffle relations generate all algebraic relations between $\overline{E}$-EMZs, we show in §4.2 that the elliptic double shuffle relations are a complete set in depth 2 (Prop. 4.6), thanks to the fact that depth 2 is too small for the real multiple zeta values to occur. In higher depth, however, we naturally encounter problems related to the unknown transcendence properties of the real multiple zeta values, exactly as we do when conjecturing that the usual double shuffle relations generate all algebraic relations between multizeta values. In Prop. 4.5, we show that the elliptic double shuffle relations do form a complete set of algebraic relations between $\overline{E}$-EMZs under the following familiar conjectures from multizeta theory:

(a) The double shuffle relations generate all algebraic relations among the multiple zeta values modulo $2\pi i$.

(b) The elliptic double shuffle Lie algebra $\mathfrak{ds}_{ell}$ [34] is isomorphic to a semi-direct product $\mathfrak{ds}_{ell} \cong \mathfrak{u} \rtimes \gamma(\mathfrak{ds})$, where $\mathfrak{ds}$ is the usual double shuffle Lie algebra and γ is the extension of Enriquez' section to $\mathfrak{ds}$ obtained using mould theory (cf. [**?**], end of §1).

Conjecture (a) is a standard conjecture in multizeta theory (cf. [21], Conjecture 1). It would imply strong transcendence results for multiple zeta values, and therefore seems out of reach at the moment. Conjecture (b), however, is purely algebraic, and may therefore be more tractable, although it still seems difficult. It would follow for example from Enriquez' generation conjecture ([14], §10) together with the conjecture that $\mathfrak{grt}_{ell} \subset \mathfrak{ds}_{ell}$ (an elliptic version of Furusho's theorem [17]).

The last question addressed in the paper, in §4.3, concerns a family of algebraic relations satisfied by the $\overline{A}$-EMZs (Theorem 4.8), which are the *Fay relations* on A-EMZs studied in [2, 27], but here considered mod $2\pi i$; we compare them to the closely related *push-neutrality* relations. These families are identical in depth 2 (although not in higher depths). In [2, 27], the depth[6] 2 Fay relations are given explicitly and it is shown that the Fay and shuffle relations give a complete set of $\mathbb{Q}$-linear relations between A-EMZs in depth 2 The possible completeness of the relations in all depths (depending on conjectures such as those cited above), the precise comparison between the algebras $\overline{\mathcal{E}}$ and $\overline{\mathcal{A}}$, and above all the lifting of the questions considered here to the situation not modulo $2\pi i$ are all topics for further research.

1.4. **Outline of the article.** The contents of this paper are organized as follows. In §2, we introduce the algebra $\mathcal{E}^{\mathrm{geom}}$ of *geometric elliptic multizetas*, describe their relation to iterated integrals of Eisenstein series, and prove the crucial linear independence of iterated Eisenstein integrals, as well as the relation between $\mathcal{E}^{\mathrm{geom}}$ and the Lie algebra $\mathfrak{u}$. In §3 we construct the elliptic generating series $E(\tau)$ and define the E-EMZs to be its coefficients, and $\mathcal{E}$ to be the $\mathbb{Q}$-algebra they generate. Passing modulo $2\pi i$, we prove the main structural result $\overline{\mathcal{E}}[2\pi i\tau] \simeq \mathcal{E}^{\mathrm{geom}} \otimes \overline{\mathcal{Z}}$ and its analogs for $\overline{\mathcal{A}}$ and $\overline{\mathcal{B}}$. In §4, we study the elliptic double shuffle equations satisfied by the mod $2\pi i$ elliptic generating series $\overline{E}(\tau)$ (or more precisely, the linearized version satisfied by its Lie version), and give evidence for the completeness of the resulting system of algebraic relations between the $\overline{E}$-EMZs. Finally, we study a family of relations satisfied by $\overline{A'}(\tau)$. The necessary background concerning moulds is briefly summarized in §4.1.

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2. Geometric Elliptic Multizetas

In the first two sections, we respectively recall the definition of a certain Lie algebra $\mathfrak{u}$ of derivations [30, 36] and of iterated integrals of Eisenstein series [4, 25].

In §2.3, we introduce the algebra of geometric elliptic multizetas, and prove that it is isomorphic to the graded dual of the universal enveloping algebra of $\mathfrak{u}$. The crucial step is a linear independence result for iterated integrals of Eisenstein series, which we prove (in slightly greater generality than needed) in §2.4.

2.1. **A family of special derivations.** We begin by fixing our notation. For a $\mathbb{Q}$-algebra A, let $\mathfrak{f}_2(A) = \mathrm{Lie}_A[\![x_1, y_1]\!]$ be the *completed* (with respect to the descending central series) free Lie algebra over A on two generators x_1, y_1 with Lie bracket $[\cdot,\cdot]$. Its (topological) universal enveloping algebra will be denoted by $\mathcal{U}(\mathfrak{f}_2)_A$, and $F_2(A) := \exp(\mathfrak{f}_2(A)) \subset \mathcal{U}(\mathfrak{f}_2)_A$ is the set of exponentials of Lie series. Note that $\mathcal{U}(\mathfrak{f}_2)_A$ is canonically isomorphic to $A\langle\langle x_1, y_1\rangle\rangle$, the A-algebra of formal power series in non-commuting variables x_1, y_1. Moreover, $\mathcal{U}(\mathfrak{f}_2)_A$ is a complete Hopf A-algebra, whose (completed) coproduct Δ is uniquely determined by $\Delta(w) = w \otimes 1 + 1 \otimes w$, for $w \in \{x_1, y_1\}$. The group $F_2(A)$ can also be characterized as the set of group-like elements of $\mathcal{U}(\mathfrak{f}_2)_A$. Likewise, the Lie algebra $\mathfrak{f}_2(A) \subset \mathcal{U}(\mathfrak{f}_2)_A$ is precisely the subset

[6]The depth is called "length" in [2, 27].

of Lie-like (or primitive) elements. If $A = \mathbb{Q}$, we will write $\mathfrak{f}_2$ instead of $\mathfrak{f}_2(\mathbb{Q})$ and likewise $\mathcal{U}(\mathfrak{f}_2)$ and F_2 instead of $\mathcal{U}(\mathfrak{f}_2)_A$ and $F_2(A)$. Now let $\mathrm{Der}(\mathfrak{f}_2)$ denote the Lie algebra of continuous derivations of the completed Lie algebra $\mathfrak{f}_2$, and define $\mathrm{Der}_0(\mathfrak{f}_2)$ as the subalgebra of those $D \in \mathrm{Der}(\mathfrak{f}_2)$ which (i) annihilate the bracket $[x_1, y_1]$:

$$D([x_1, y_1]) = 0$$

and (ii) are such that $D(y_1)$ contains no linear term in x_1. Since $\mathfrak{f}_2$ is the completion of a free Lie algebra, the commutator of y_1 is $\mathbb{Q}\cdot y_1$, from which it follows easily that every derivation $D \in \mathrm{Der}_0(\mathfrak{f}_2)$ is uniquely determined by its value on x_1. Similarly, the only non-zero derivation $D \in \mathrm{Der}_0(\mathfrak{f}_2)$ which annihilates y_1 is the derivation ε_0 defined by $x_1 \mapsto y_1$, $y_1 \mapsto 0$.

We next recall the definition of a family of derivations, which was first considered in [36], also played an important role in [8], and was studied in detail in [30].

Definition 2.1. For $k \geq 0$, define a derivation $\varepsilon_{2k} \in \mathrm{Der}_0(\mathfrak{f}_2)$ by

$$\varepsilon_{2k}(x_1) = \mathrm{ad}(x_1)^{2k}(y_1),$$

and denote

$$\mathfrak{u} = \widehat{\mathrm{Lie}}(\varepsilon_{2k}; k \geq 0) \subset \mathrm{Der}_0(\mathfrak{f}_2)$$

the graded completion of the Lie subalgebra generated by the ε_{2k}. We say that a derivation D of $\mathfrak{f}_2$ is of homogeneous degree $m \geq 0$ if for every element $f \in \mathfrak{f}_2$ of homogeneous degree n, $D(f)$ is of homogeneous degree $n + m$. Let $\mathrm{Der}_0'(\mathfrak{f}_2)$ be the subspace of $\mathrm{Der}_0(\mathfrak{f}_2)$ spanned by derivations of homogeneous degree ≥ 1, and let $\mathfrak{u}' = \mathrm{Der}_0'(\mathfrak{f}_2) \cap \mathfrak{u}$. We have isomorphisms

$$\mathrm{Der}_0(\mathfrak{f}_2) \simeq \mathbb{Q}\varepsilon_0 \oplus \mathrm{Der}_0'(\mathfrak{f}_2)$$

and

$$\mathfrak{u} \simeq \mathbb{Q}\varepsilon_0 \oplus \mathfrak{u}'. \tag{2.1}$$

Observe that $\varepsilon_2 = -\,\mathrm{ad}([x_1, y_1])$, and thus ε_2 is central in $\mathfrak{u}$. Thus we have a generating set for $\mathfrak{u}'$ given by

$$\mathfrak{u}' = \mathrm{Lie}[\mathrm{ad}^n(\varepsilon_0)(\varepsilon_{2k}); n \geq 0,\ k \geq 1]. \tag{2.2}$$

As seen above, every ε_{2k} is uniquely determined by its value on x_1, while ε_0 is the only non-zero derivation $D \in \mathfrak{u}$, which annihilates y_1. From this, we get

Proposition 2.2. *The $\mathbb{Q}$-linear evaluation maps*

$$v_{x_1} : \mathrm{Der}_0(\mathfrak{f}_2) \to \mathfrak{f}_2, \quad D \mapsto D(x_1),$$
$$v_{y_1} : \mathrm{Der}_0'(\mathfrak{f}_2) \to \mathfrak{f}_2, \quad D \mapsto D(y_1),$$

are injective.

For the applications to elliptic multizetas, it will be more natural to scale the derivations ε_{2k} as follows:

$$\widetilde{\varepsilon}_{2k} := \begin{cases} \frac{2}{(2k-2)!}\varepsilon_{2k} & k > 0 \\ -\varepsilon_0 & k = 0. \end{cases}$$

In this way, $\widetilde{\varepsilon}_{2k}$ is the image of the Eisenstein generator $\mathbf{e}_{2k}$ under the monodromy representation $\mathfrak{u}^{\mathrm{geom}} \to \mathrm{Der}_0(\mathfrak{f}_2)$ (cf. [19], Theorem 22.3).

2.2. **Iterated Eisenstein Integrals.** In a sense to be made precise below, the derivation ε_{2k} naturally corresponds to integrals of Hecke-normalized Eisenstein series of weight $2k$ (for $\mathrm{SL}_2(\mathbb{Z})$), whereas commutators of ε_{2k} correspond to *iterated integrals of Eisenstein series*. These are special cases of *iterated Shimura integrals* (or *iterated Eichler integrals*) of modular forms introduced by Manin [25], and later generalized by Brown [4].[7]

For $k \geq 0$, let $G_{2k}(q)$ be the Hecke-normalized Eisenstein series, defined by $G_0(q) := -1$ and for $k \geq 1$

$$G_{2k}(q) = -\frac{B_{2k}}{4k} + \sum_{n\geq 1} \sigma_{2k-1}(n) q^n, \quad q = e^{2\pi i\tau}$$

Here, $\sigma_\ell(n) = \sum_{d|n} d^\ell$ denotes the ℓ-th divisor function, and the B_{2k} are the Bernoulli numbers defined by

$$\frac{z}{e^z - 1} = 1 - \frac{z}{2} + \sum_{n\geq 1} B_{2n} \frac{z^{2n}}{(2n)!}.$$

Via the exponential map $\exp : \mathfrak{H} \to D^*$, $\tau \mapsto q = \exp(2\pi i\tau)$, from the upper half-plane to the punctured unit disc

$$D^* = \{q \in \mathbb{C}, 0 < |q| < 1\},$$

we may consider G_{2k} as a function of either variable q or τ, and we shall do so according to context.

Next, we define iterated integrals of Eisenstein series. More generally, if $f(q) = \sum_{n=0}^\infty a_n q^n$ is such that $a_0 = 0$, (e.g. f is a cusp form), then the definition of the indefinite integral $\int_\tau^{i\infty} f(\tau_1)\mathrm{d}\tau_1$ poses no problem, as by definition f vanishes at $i\infty$. This is not the case for the Eisenstein series G_{2k}, and consequently $\int_\tau^{i\infty} G_{2k}(\tau_1)\mathrm{d}\tau_1$ diverges. It can be regularized by setting, for $k \geq 1$,

$$\int_\tau^{i\infty} G_{2k}(\tau_1)\mathrm{d}\tau_1 := \int_\tau^{i\infty} \Big[G_{2k}(\tau_1) - G_{2k}^\infty\Big]\mathrm{d}\tau_1 - \int_0^\tau G_{2k}^\infty \mathrm{d}\tau_1,$$

where $G_{2k}^\infty = -\frac{B_{2k}}{4k}$ is the constant term in the Fourier expansion of G_{2k} (if $k = 0$, a similar method works). Note that the integral of G_{2k} so defined satisfies the differential equation $\mathrm{d}f(\tau) = -G_{2k}(\tau)\mathrm{d}\tau$. The definition of regularized iterated integrals of Eisenstein series in [4], which is a special case of Deligne's tangential base point regularization ([9], §15) generalizes this construction, and runs as follows.

Let $W = \mathbb{C}[\![q]\!]^{<1}$ be the $\mathbb{C}$-algebra of formal power series, which converge on $D = \{q \in \mathbb{C} \,|\, |q| < 1\}$. We may decompose $W = W^0 \oplus W^\infty$ with $W^0 = q\mathbb{C}[\![q]\!]$ and $W^\infty = \mathbb{C}$. For a power series $f \in W$, define f^0 to be its image in W^0 under the natural projection, and define $f^\infty \in W^\infty$ likewise. For example, in the case of the Eisenstein series $G_{2k}(q)$ with $k > 0$, we have

$$G_{2k}^\infty = -\frac{B_{2k}}{4k}, \quad G_{2k}^0(q) = \sum_{n\geq 1} \sigma_{2k-1}(n) q^n.$$

We denote by $T^c(W)$ the *shuffle algebra* on the $\mathbb{C}$-vector space W. As a $\mathbb{C}$-vector space, $T^c(W)$ is simply the graded (for the length of tensors) dual of the tensor

[7]To be precise, Manin defined iterated Shimura integrals of cusp forms between base points on the upper half-plane (possibly cusps), and the extension to Eisenstein series (which requires a regularization procedure) is due to Brown.

algebra $T(W) = \bigoplus_{n\geq 0} W^{\otimes n}$. It is customary to write down elements of the dual space $(W^{\otimes n})^\vee$ using bar notation $[f_1|, \ldots, |f_n]$. Moreover, $T^c(W)$ is naturally a commutative $\mathbb{C}$-algebra, whose product is the shuffle product ⧢, defined by

$$[f_1|\ldots|f_r] ⧢ [f_{r+1}|\ldots|f_{r+s}] = \sum_{\sigma\in\Sigma_{r,s}} f_{\sigma^{-1}(1)}\cdots f_{\sigma^{-1}(r+s)},$$

where $\Sigma_{r,s}$ denotes the set of permutations σ on $\{1,\ldots,r+s\}$, such that σ is strictly increasing on both $\{1,\ldots r\}$ and on $\{r+1,\ldots,r+s\}$.

Now define a map $R : T^c(W) \to T^c(W)$ by the formula

$$R[f_1|\ldots|f_n] = \sum_{i=0}^{n}(-1)^{n-i}[f_1|\ldots|f_i] ⧢ [f_n^\infty|\ldots|f_{i+1}^\infty].$$

Following [4], eq. (4.11), we can now make the

Definition 2.3. Given $f_1,\ldots,f_n \in W$ as above, their *regularized iterated integral* is defined as

$$I(f_1,\ldots,f_n;\tau) := (2\pi i)^n \sum_{i=0}^{n}\int_\tau^{i\infty} R[f_1|\ldots|f_i]_{\mathrm{d}\tau}\int_\tau^0 [f_{i+1}^\infty|\ldots|f_n^\infty]_{\mathrm{d}\tau},$$

where

$$\int_a^b [f_1|\ldots|f_n]_{\mathrm{d}\tau} := \int\cdots\int_{a\leq\tau_1\leq\ldots\leq\tau_n\leq b} f_1(\tau_1)\ldots f_n(\tau_n)\mathrm{d}\tau_1\ldots\mathrm{d}\tau_n.$$

Remark 2.4. The reason for the $(2\pi i)^n$-prefactor is to preserve the rationality of the Fourier coefficients. More precisely, if $f_1,\ldots,f_n$ have rational coefficients (i.e. $f_i \in W_{\mathbb{Q}} := \mathbb{Q}[\![q]\!]^{<1}$), then $I(f_1,\ldots,f_n;\tau) \in W_{\mathbb{Q}}[\log(q)]$, where $\log(q) := 2\pi i\tau$.

As is the case for usual iterated integrals ([20], Sect. 2), regularized iterated integrals satisfy the differential equation

$$\frac{\partial}{\partial\tau}\Big|_{\tau=\tau_0} I(f_1,\ldots,f_n;\tau) = -f_1(\tau_0)I(f_2,\ldots,f_n;\tau_0), \tag{2.3}$$

as well as the shuffle product formula

$$I(f_1,\ldots,f_r;\tau)I(f_{r+1},\ldots,f_{r+s};\tau) = \sum_{\sigma\in\Sigma_{r,s}} I(f_{\sigma(1)},\ldots,f_{\sigma(r+s)};\tau). \tag{2.4}$$

The only case of interest for us will be when $f_1,\ldots,f_n$ are given by Eisenstein series $G_{2k_1},\ldots,G_{2k_n}$. In this case, we set

$$\mathcal{G}_{\underline{k}}(\tau) := I(G_{2k_1},\ldots,G_{2k_n};\tau), \tag{2.5}$$

where $\underline{k} = (k_1,\ldots,k_n)$ and likewise denote by

$$\mathcal{I}^{\mathrm{Eis}} := \mathrm{Span}_{\mathbb{Q}}\{\mathcal{G}_{\underline{k}}(\tau)\} \subset \mathcal{O}(\mathfrak{H})$$

the $\mathbb{Q}$-span of all iterated Eisenstein integrals $\mathcal{G}_{\underline{k}}(\tau)$ for all multi-indices $\underline{k}$ (including $\mathcal{G}_\emptyset := 1$ for the empty index). Note that $\mathcal{I}^{\mathrm{Eis}}$ is a $\mathbb{Q}$-subalgebra of $\mathcal{O}(\mathfrak{H})$ by (2.4), and that it contains $\mathbb{Q}[2\pi i\tau]$ as a subalgebra, since by (2.5) we have

$$\mathcal{G}_0(\tau) = 2\pi i\tau. \tag{2.6}$$

2.3. **The τ-evolution equation and the algebra of geometric elliptic multizetas.** We now put together the special derivations $\widetilde{\varepsilon}_{2k}$ and the iterated Eisenstein integrals into a single, formal series

$$g(\tau) := \mathrm{id} + \sum_{\underline{k}} \mathcal{G}_{\underline{k}}(\tau)\widetilde{\varepsilon}_{\underline{k}}, \tag{2.7}$$

where the sum is over all multi-indices $\underline{k} \in \mathbb{Z}^n_{\geq 0}$ for $n > 0$, and for each tuple $\underline{k} = (k_1, \ldots, k_n)$, we set $\widetilde{\varepsilon}_{\underline{k}} := \widetilde{\varepsilon}_{2k_1} \circ \ldots \circ \widetilde{\varepsilon}_{2k_n} \in \mathcal{U}(\mathfrak{u})$, the universal enveloping algebra of $\mathfrak{u}$. From (2.3), it is clear that $g(\tau)$ satisfies the differential equation

$$\frac{1}{2\pi i}\frac{\partial}{\partial\tau} g(\tau) = -\Big(\sum_{k\geq 0} G_{2k}(\tau)\widetilde{\varepsilon}_{2k}\Big) g(\tau),$$

and it follows that $g(\tau)$ is group-like, i.e. it is the exponential $g(\tau) = \exp(r(\tau))$ of a Lie series

$$r(\tau) \in \mathfrak{u} \otimes_{\mathbb{Q}} \mathcal{I}^{\mathrm{Eis}}. \tag{2.8}$$

Definition 2.5. Define the $\mathbb{Q}$-algebra $\mathcal{E}^{\mathrm{geom}}$ of geometric elliptic multizetas to be the $\mathbb{Q}$-algebra generated by the coefficients of $r(\tau) \cdot x_1$.

Equivalently, $\mathcal{E}^{\mathrm{geom}}$ is equal to the $\mathbb{Q}$-vector space linearly spanned by the coefficients of the series $g(\tau) \cdot e^{x_1}$, because the coefficients of each of the power series $r(\tau) \cdot x_1$ and $g(\tau) \cdot e^{x_1}$ can be written as algebraic expressions in the coefficients of the other. Also, note that since every derivation in $\mathfrak{u}$ is uniquely determined by its value on x_1, the $\mathbb{Q}$-algebra $\mathcal{E}^{\mathrm{geom}}$ is also the same as the $\mathbb{Q}$-algebra spanned by the coefficients of $g(\tau)$ when written in a basis of the $\mathbb{Q}$-algebra generated by the ε_{2k}. Thus in particular we have the inclusion of commutative algebras

$$\mathcal{E}^{\mathrm{geom}} \subset \mathbb{Q}[\mathcal{G}_{\underline{k}}(\tau), \underline{k} \in \mathbb{N}^n, n \geq 0] \subset \mathcal{O}(\mathfrak{H}).$$

We can now state the main result of §2.

Theorem 2.6. *For every $\mathbb{Q}$-subalgebra $A \subset \mathbb{C}$, there is an isomorphism*

$$\mathcal{U}(\mathfrak{u})^{\vee} \otimes_{\mathbb{Q}} A \cong \mathcal{E}^{\mathrm{geom}} \otimes_{\mathbb{Q}} A$$

of A-algebras. In particular, $\mathcal{E}^{\mathrm{geom}}$ is a commutative, graded Hopf algebra in a natural way.

Proof. The main ingredient in the proof is that the iterated Eisenstein integrals $\mathcal{G}_{\underline{k}}(\tau)$ are linearly independent over $\mathbb{C}$, as functions in τ. More precisely, by Corollary 2.9, proved in the next section, there is a natural isomorphism

$$\mathcal{I}^{\mathrm{Eis}} \otimes_{\mathbb{Q}} A \cong T^c(V_{\mathrm{Eis}}) \otimes_{\mathbb{Q}} A,$$

where $T^c(V_{\mathrm{Eis}})$ is the shuffle algebra on the $\mathbb{Q}$-vector space V_{Eis} spanned by all Eisenstein series G_{2k}, $k \geq 0$.

Assuming Corollary 2.9 for the moment, the proof of Theorem 2.6 proceeds as follows. Since the tensor algebra $T(V_{\mathrm{Eis}})$ is freely generated by one element in every even degree $2k \geq 0$, we get a canonical surjection $T(V_{\mathrm{Eis}}) \to \mathcal{U}(\mathfrak{u})$ of $\mathbb{Q}$-algebras, which induces by duality an injection

$$\iota : \mathcal{U}(\mathfrak{u})^{\vee} \hookrightarrow T^c(V_{\mathrm{Eis}}) \cong \mathcal{I}^{\mathrm{Eis}}.$$

On the other hand, choosing a (homogeneous) linear basis $\mathcal{B}$ of $\mathcal{U}(\mathfrak{u})$, the element $g(\tau)$ naturally defines a map

$$\begin{aligned}\widetilde{\iota}: \mathcal{U}(\mathfrak{u})^\vee &\hookrightarrow \mathcal{I}^{\mathrm{Eis}}\\ b^\vee &\mapsto b^\vee(g(\tau)),\end{aligned}$$

where $b^\vee \in \mathcal{B}^\vee$ are the dual basis elements. Clearly, the image of $\widetilde{\iota}$ does not depend on the choice of basis, and equals $\mathcal{E}^{\mathrm{geom}}$ by definition. On the other hand, it is easy to see that the maps $\iota, \widetilde{\iota}: \mathcal{U}(\mathfrak{u})^\vee \to \mathcal{I}^{\mathrm{Eis}}$ are equal, whence the result for $A = \mathbb{Q}$, and the general case follows simply by extension of scalars. Finally, it is well-known that the universal enveloping algebra of any graded Lie algebra has a natural structure of a (cocommutative) graded Hopf algebra, thus $\mathcal{U}(\mathfrak{u})^\vee$ is naturally a (commutative) graded Hopf algebra. □

2.4. **Linear independence.** In this subsection, we complete the proof of Theorem 2.6 by showing that the family of iterated Eisenstein integrals is linearly independent over $\mathbb{C}$, and that as a consequence $\mathcal{I}^{\mathrm{Eis}} \otimes_{\mathbb{Q}} \mathbb{C} \cong T^c(V_{\mathrm{Eis}}) \otimes_{\mathbb{Q}} \mathbb{C}$ as $\mathbb{C}$-algebras. Although these results can meanwhile also be deduced from [28], which proves linear independence of iterated integrals of quasimodular forms for $\mathrm{SL}_2(\mathbb{Z})$ over the (fraction field of the) ring of quasimodular forms for $\mathrm{SL}_2(\mathbb{Z})$, we give a slightly different proof here in the special case of Eisenstein series which has the advantage that it works over a larger field of coefficients.

The main idea is to use the following general linear independence result.

Theorem 2.7 ([10, Theorem 2.1])**.** *Let $(\mathcal{A}, \mathrm{d})$ be a differential algebra over a field k of characteristic zero, whose ring of constants* $\ker(\mathrm{d})$ *is precisely equal to k. Let $\mathcal{C}$ be a differential subfield of $\mathcal{A}$ (i.e. a subfield such that $\mathrm{d}\mathcal{C} \subset \mathcal{C}$), X any set with associated free monoid X^*. Suppose that $S \in \mathcal{A}\langle\langle X \rangle\rangle$ is a solution to the differential equation*

$$\mathrm{d}S = M \cdot S,$$

where $M = \sum_{x \in X} u_x x \in \mathcal{C}\langle\langle X \rangle\rangle$ is a homogeneous series of degree 1, with initial condition $S_1 = 1$, where S_1 denotes the coefficient of the empty word in the series S. The following are equivalent:

(i) *The family of coefficients $(S_w)_{w \in X^*}$ of S is linearly independent over $\mathcal{C}$.*
(ii) *The family $\{u_x\}_{x \in X}$ is linearly independent over k, and we have*

$$\mathrm{d}\mathcal{C} \cap \mathrm{Span}_k(\{u_x\}_{x \in X}) = \{0\}. \tag{2.9}$$

Using this theorem, we can now prove linear independence of iterated Eisenstein integrals.

Theorem 2.8. *The family $\{\mathcal{G}_{\underline{k}}(\tau)\}$ is linearly independent over* $\mathrm{Frac}(\mathbb{Z}[\![q]\!])$.

Proof. We will apply Theorem 2.7 with the following parameters:

- $k = \mathbb{Q}$, $\mathcal{A} = \mathbb{Q}[\log(q)](\!(q)\!)$ with differential $\mathrm{d} = q\frac{\partial}{\partial q}$, and $\mathcal{C} = \mathrm{Frac}(\mathbb{Z}[\![q]\!])$ (the latter is a differential field by the quotient rule for derivatives)
- $X = \{a_{2k}\}_{k \geq 0}$, $u_{a_{2k}} = -G_{2k}(q)$, hence

$$M(q) = -\sum_{k \geq 0} G_{2k}(q) a_{2k}.$$

With these conventions, it follows from (2.3) that the formal series

$$1+\int_q^0 [M]_{\mathrm{d}\log q}+\int_q^0 [M|M]_{\mathrm{d}\log q}+\ldots \in \mathcal{O}(\mathfrak{H})\langle\langle X\rangle\rangle,$$

with the iterated integrals regularized as in Section 2.2, is a solution of the differential equation $\mathrm{d}S = M\cdot S$, with $S_1 = 1$. Consequently, the coefficient of the word $w = a_{2k_1}\ldots a_{2k_n}$ in S is equal to $\mathcal{G}(2k_1,\ldots,2k_n;\tau)$. Moreover, since the $\mathbb{Q}$-linear independence of the Eisenstein series is well-known (cf. e.g. [35], VII.3.2), it remains to verify (2.9) in our situation.

To this end, assume that there exist $\alpha_{2k}\in\mathbb{Q}$, all but finitely many of which are equal to zero, such that

$$\sum_{k\geq 0}\alpha_{2k}G_{2k}(q)\in \mathrm{d}\mathcal{C}. \tag{2.10}$$

Clearing denominators, we may assume that $\alpha_{2k}\in\mathbb{Z}$. Furthermore, from the definition of $\mathrm{d} = q\frac{\partial}{\partial q}$, one sees that the image $\mathrm{d}\mathcal{C}$ of the differential operator d does not contain any constant except for zero. Therefore, the coefficient of the trivial word 1 in (2.10) vanishes; in other words

$$\sum_{k\geq 0}\alpha_{2k}G_{2k}(q)=\sum_{k\geq 1}\alpha_{2k}E^0_{2k}(q)\in q\mathbb{Q}[\![q]\!].$$

Now the differential d is invertible on $q\mathbb{Q}[\![q]\!]$, and inverting d is the same as integrating. Hence (2.10) is equivalent to

$$\sum_{k\geq 1}\alpha_{2k}\mathcal{G}^0_{2k}(\tau)\in\mathcal{C},\quad \mathcal{G}^0_{2k}(\tau):=\int_q^0 E^0_{2k}(q_1)\frac{\mathrm{d}q_1}{q_1}. \tag{2.11}$$

But this is absurd, unless all the α_{2k} vanish, as we shall see now. Indeed, if $f\in\mathcal{C}=\mathrm{Frac}(\mathbb{Z}[\![q]\!])$, then there exists $m\in\mathbb{Z}\setminus\{0\}$ such that $f\in\mathbb{Z}[m^{-1}](\!(q)\!)$. This follows from the well-known inversion formula for power series. On the other hand, the coefficient of q^p in $\mathcal{G}^0_{2k}(\tau)$, for p a prime number, is given by

$$\frac{\sigma_{2k-1}(p)}{p}=\frac{p^{2k-1}+1}{p}\equiv\frac{1}{p}\mod\mathbb{Z}.$$

Thus, we must have $\frac{1}{p}\sum_{k\geq 1}\alpha_{2k}\in\mathbb{Z}[m^{-1}]$, for every prime number p, in particular $\sum_{k\geq 1}\alpha_{2k}$ is divisible by infinitely many primes (namely, at least all the primes which don't divide m), which implies $\sum_{k\geq 1}\alpha_{2k}=0$.

Now assume that k_1 is the smallest positive, even integer with the property that $\alpha_{k_1}\neq 0$. Consider the coefficient of $q^{p^{k_1}}$ in $\mathcal{G}^0_{2k}(\tau)$, which is equal to

$$\frac{\sigma_{2k-1}(p^{k_1})}{p^{k_1}}=\frac{1}{p^{k_1}}\sum_{j=0}^{k_1}p^{j(2k-1)}\equiv\begin{cases}\frac{1}{p^{k_1}}\mod\mathbb{Z} & \text{if } 2k>k_1\\ \frac{1}{p^{k_1}}+\frac{1}{p}\mod\mathbb{Z} & \text{if } 2k=k_1.\end{cases}$$

By (2.11), we have $\frac{\alpha_{k_1}}{p}+\frac{1}{p^{k_1}}\sum_{k\geq 1}\alpha_{2k}\in\mathbb{Z}[m^{-1}]$, and by what we have seen before, $\sum_{k\geq 1}\alpha_{2k}=0$. Hence $\frac{\alpha_{k_1}}{p}\in\mathbb{Z}[m^{-1}]$, for every prime number p, which again implies $\alpha_{k_1}=0$, in contradiction to our assumption $\alpha_{k_1}\neq 0$. Therefore, in (2.11), we must have $\alpha_{2k}=0$ for all $k\geq 1$ and (2.9) is verified. □

Corollary 2.9. *The iterated Eisenstein integrals $\mathcal{G}_{\underline{k}}(\tau)$ are $\mathbb{C}$-linearly independent, and for every $\mathbb{Q}$-subalgebra $A \subset \mathbb{C}$, we have a natural isomorphism of A-algebras*

$$\psi_A : T^c(V_{\mathrm{Eis}}) \otimes_{\mathbb{Q}} A \to \mathcal{I}^{\mathrm{Eis}} \otimes_{\mathbb{Q}} A$$
$$[G_{2k_1}|\dots|G_{2k_n}] \mapsto \mathcal{G}_{\underline{k}}(\tau),$$

where $\underline{k} = (k_1, \dots, k_n)$ and $V_{\mathrm{Eis}} = \mathrm{Span}_{\mathbb{Q}}\{G_{2k}(\tau) \,|\, k \geq 0\} \subset \mathcal{O}(\mathfrak{H})$.

Proof. Since $\mathbb{Q} \subset \mathrm{Frac}(\mathbb{Z}[\![q]\!])$, Theorem 2.8 shows in particular that the $\mathcal{G}_{\underline{k}}$ are linearly independent over $\mathbb{Q}$. Since the Eisenstein series G_{2k} have coefficients in $\mathbb{Q}$, it follows from the definition that $\mathcal{G}_{\underline{k}} \in \mathbb{Q}(\!(q)\!)[\log(q)]$, and elements of $W_{\mathbb{Q}}[\log(q)] = \mathbb{Q}(\!(q)\!)[\log(q)]$ are linearly independent over $\mathbb{Q}$, if and only they are so over $\mathbb{C}$.

For the second statement, it is clear that ψ_A is a homomorphism of $\mathbb{Q}$-algebras (since both sides are endowed with the shuffle product) and that it is surjective. The injectivity of ψ_A is just the A-linear independence of iterated Eisenstein integrals. □

Corollary 2.10. *For any $\mathbb{Q}$-subalgebra $A \subset \mathbb{C}$ (viewed inside $\mathcal{O}(\mathfrak{H})$ as constant functions), the $\mathbb{Q}$-subalgebra of $\mathcal{O}(\mathfrak{H})$ generated by $\mathcal{I}^{\mathrm{Eis}}$ and A is canonically isomorphic to $\mathcal{I}^{\mathrm{Eis}} \otimes_{\mathbb{Q}} A$, and the $\mathbb{Q}$-subalgebra generated by $\mathcal{E}^{\mathrm{geom}}$ and A is canonically isomorphic to $\mathcal{E}^{\mathrm{geom}} \otimes_{\mathbb{Q}} A$.*

Proof. By the previous corollary, the elements of $\mathcal{I}^{\mathrm{Eis}}$ are linearly independent over $\mathbb{C}$, so over A. Since both A and $\mathcal{I}^{\mathrm{Eis}}$ are $\mathbb{Q}$-algebras, any element of the algebra generated by A and $\mathcal{I}^{\mathrm{Eis}}$ can be expressed as a linear combination of elements of $\mathcal{I}^{\mathrm{Eis}}$ with coefficients in A which is unique up to scalar multiplication by rationals. This implies the isomorphism with the tensor product. The result carries over to $\mathcal{E}^{\mathrm{geom}}$ trivially since $\mathcal{E}^{\mathrm{geom}}$ lies inside $\mathcal{I}^{\mathrm{Eis}}$. □

3. The generating series of elliptic multizetas

In the first paragraph of this section, we recall an important fact about the elliptic associator defined by Enriquez in [14], or more precisely, the structure of the group-like power series $A(\tau)$, $B(\tau)$: namely, there exist power series A and B (whose definitions are recalled in (3.2) below) such that

$$A(\tau) = g(\tau) \cdot A, \qquad B(\tau) = g(\tau) \cdot B,$$

where $g(\tau)$ denotes the automorphism of (2.7). In analogy with this structure, we will define a new power series $E(\tau)$, which will also take the form $g(\tau) \cdot E$ for a power series $E \in F_2(\mathcal{Z})$. We call $E(\tau)$ the *elliptic generating series*, and its coefficients the *E-elliptic multizetas* or E-EMZs; similarly we call the coefficients of $A(\tau)$ the A-EMZs and those of $B(\tau)$ the B-EMZs. We define $\mathcal{E}$ (resp. $\mathcal{A}$, resp. $\mathcal{B}$) to be the $\mathbb{Q}$-algebra generated by the E-EMZs (resp. the A-EMZs, resp. the B-EMZs; these coefficients are Enriquez's "elliptic analogs of multiple zeta values"). The algebras $\mathcal{E}$, $\mathcal{A}$ and $\mathcal{B}$ are closely related but not equal. More importantly, each family of EMZs satisfies different algebraic relations.

In the remainder of the section, for technical reasons related to the use of mould theory, we work modulo $2\pi i$ in the sense explained in §1.2. In particular, we consider the power series $\overline{\Phi}_{\mathrm{KZ}}$ and $\overline{E}$, which are the power series obtained from Φ_{KZ} and E by reducing the coefficients from $\mathcal{Z}$ to $\overline{\mathcal{Z}} = \mathcal{Z}/\langle(2\pi i)^2\rangle$. In §3.2, we give an expression for $\overline{E}$ which relates it explicitly to the Drinfel'd associator $\overline{\Phi}_{\mathrm{KZ}}$, and which will be essential in determining the algebraic relations satisfied by the reduced E-EMZs (denoted $\overline{E}$-EMZs) in §4.

In §3.3, we give a result on the structure of the $\mathbb{Q}$-algebra generated by the coefficients of a power series of the form $F(\tau) = g(\tau) \cdot F$ where $F \in F_2(\overline{\mathcal{Z}})$ and $g(\tau) \cdot F \in F_2(\mathcal{E}^{\text{geom}} \otimes \overline{\mathcal{Z}})$. We define the reduced power series $\overline{A}(\tau)$ and $\overline{B}(\tau)$; together with $\overline{E}(\tau)$, they are all power series of the form $g(\tau) \cdot F$, so we apply the theorem to obtain our main result of the section, an explicit structural result relating the $\mathbb{Q}$-algebras $\overline{\mathcal{E}}$, $\overline{\mathcal{A}}$ and $\overline{\mathcal{B}}$ generated by the coefficients of $\overline{E}(\tau)$, $\overline{A}(\tau)$ and $\overline{B}(\tau)$ respectively.

3.1. **Definition of the elliptic generating series $E(\tau)$.** Throughout this section, we use the following change of variables: $a = y_1$ and $b = x_1$. This change of variables will be applied to all the expressions in x_1, y_1 encountered in the previous section, such as $g(\tau) \cdot y_1$, and we will also express other quantities studied by B. Enriquez in terms of a and b, in particular the elliptic associator. The purpose of this change of variables is for the application of mould theory in §4.

Let Ass_μ denote the set of genus zero associators $\Phi \in F_2(\mathbb{C})$ such that the coefficient of ab in Φ is equal to $\mu^2/24$ [11]. We will use the same elements t_{01}, t_{02}, t_{12} as in [14], §5.3, but rewritten in the variables a, b:

$$t_{01} = Ber_b(-a), \quad t_{02} = Ber_{-b}(a), \quad t_{12} = [a, b], \tag{3.1}$$

where

$$Ber_x(y) = \frac{\mathrm{ad}(x)}{e^{\mathrm{ad}(x)} - 1}(y),$$

so that $t_{01} + t_{02} + t_{12} = 0$. Recall that Enriquez showed that a section from Ass_μ to the set of elliptic associators is given by mapping $\Phi \in Ass_\mu$ to the elliptic associator (μ, Φ, A, B) defined by

$$\begin{aligned} A &= \Phi(t_{01}, t_{12}) e^{\mu\, t_{01}} \Phi(t_{01}, t_{12})^{-1} \\ B &= e^{\mu\, t_{12}/2} \Phi(t_{02}, t_{12}) e^{b} \Phi(t_{01}, t_{12})^{-1} \end{aligned} \tag{3.2}$$

(A and B are denoted A_+ and A_- in [14], Proposition 4.8). In this article we fix the values $\mu = 2\pi i$ and $\Phi = \Phi_{\mathrm{KZ}}$, the Drinfel'd associator, whose coefficients generate $\mathcal{Z}$ ([16]). The elliptic associator $\big(A(\tau), B(\tau)\big)$ defined in §6.2 of [14] satisfies the relations

$$A(\tau) = g(\tau) \cdot A, \quad B(\tau) = g(\tau) \cdot B \tag{3.3}$$

(see §5.2 of [15]).

The (completed) Lie algebra $\mathfrak{f}_2 = \mathrm{Lie}[\![a, b]\!]$ is topologically generated by a and b, but since the operator Ber_b is invertible, we have

$$a = -Ber_b^{-1}(t_{01}) = \Big(\frac{e^{\mathrm{ad}(b)} - 1}{\mathrm{ad}(b)}\Big)(-t_{01}), \tag{3.4}$$

so that we can just as well take t_{01} and b as generators. Similarly, we can take $e^{t_{01}}$ and e^b as topological generators of the prounipotent group $F_2 = F_2(\mathbb{Q}) = \exp(\mathfrak{f}_2)$, which is a priori topologically generated by e^a and e^b.

Let us define an automorphism σ of $F_2(\mathcal{Z}[2\pi i])$ by

$$\begin{aligned} \sigma(e^{t_{01}}) &= \Phi_{\mathrm{KZ}}(t_{01}, t_{12}) e^{t_{01}} \Phi_{\mathrm{KZ}}(t_{01}, t_{12})^{-1} \\ \sigma(e^{b}) &= e^{\pi i t_{12}} \Phi_{\mathrm{KZ}}(t_{02}, t_{12}) e^{b} \Phi_{\mathrm{KZ}}(t_{01}, t_{12})^{-1}. \end{aligned}$$

We set

$$E=\sigma(a),\quad C=\exp(E)=\sigma(e^a). \tag{3.5}$$

The automorphism σ extends to an automorphism of the completed enveloping algebra $\mathcal{U}(\mathfrak{f}_2)$, and restricts to an automorphism of $\mathfrak{f}_2$. Thus the power series $E=\sigma(a)$ is Lie-like. All Lie-like and group-like power series discussed in this section are contained in the free non-commutative power series ring $R\langle\langle a,b\rangle\rangle$ where R is either $\mathcal{E}^{\rm geom}\otimes\mathcal{Z}[2\pi i]$ or $\mathcal{E}^{\rm geom}\otimes\overline{\mathcal{Z}}$. When we speak of the ring generated by the coefficients of such a power series, we mean that we take coefficients of the power series written in any linear basis of $R\langle\langle a,b\rangle\rangle$ (for example the basis of monomials in a and b), all of which lie in R, and consider the subring of R generated by these coefficients. The "degree" is the degree in the variables a, b.

Up to degree 5, the explicit expansion of E is given by

$$E=a-\frac{\pi i}{2}c_3+\frac{\pi i}{12}[c_1,c_3]+\zeta(3)c_5+\frac{\pi^2}{36}[c_1,c_4]+\frac{\pi^2}{9}[c_3,c_2], \tag{3.6}$$

where $c_i=ad(a)^{i-1}(b)$.

In analogy with (3.3), we now set

$$E(\tau)=g(\tau)\cdot E\ \text{ and }\ C(\tau)=g(\tau)\cdot C=g(\tau)\cdot\sigma(e^a). \tag{3.7}$$

These power series lie in $\big(\mathcal{E}^{\rm geom}\otimes\mathcal{Z}[2\pi i]\big)\langle\langle a,b\rangle\rangle$.

Definition 3.1. The Lie-like power series $E(\tau)$ is called the *elliptic generating series*, and its coefficients are called the *E-EMZs* or *E-elliptic multizetas*. For $\underline{k}=(k_1,\dots,k_r)$ we write $E(\underline{k})$ for the coefficient in $E(\tau)$ of the monomial $c_{k_1}\cdots c_{k_r}$, which lies in the tensor product $\mathcal{E}^{\rm geom}\otimes\mathcal{Z}[2\pi i]$. The $\mathbb{Q}$-algebra generated inside $\mathcal{E}^{\rm geom}\otimes\mathcal{Z}[2\pi i]$ by the E-elliptic multizetas $E(\underline{k})$ is denoted by $\mathcal{E}$.

Lemma 3.2. *The power series $E(\tau)$ is Lie-like and $C(\tau)$ is group-like. The element $2\pi i$ appears as a coefficient in each of the three power series $A(\tau)$, $B(\tau)$ and $E(\tau)$ expanded in the variables c_i. The element $2\pi i$ does not lie in the $\mathbb{Q}$-algebra $\mathcal{A}'$ generated by the coefficients of $A'(\tau)$, but π^2 does.*

Proof. Since $g(\tau)$ can be considered as an automorphism of the universal enveloping algebra of $\mathfrak{f}_2$, it preserves the Lie algebra $\mathfrak{f}_2\otimes_{\mathbb{Q}}(\mathcal{E}^{\rm geom}\otimes_{\mathbb{Q}}\mathcal{Z}[2\pi i])$; thus $E(\tau)$ is Lie-like, and $C(\tau)$ is group-like. To check that a rational multiple of $2\pi i$ occurs as a coefficient in each of the three power series in the statement, it suffices to give their expansions in the c_i in low weights using the explicit formulas (3.2), (3.3), (3.6) and (3.7), together with formula (2.7) defining $g(\tau)$. For the first three, we obtain

$$\begin{aligned}E(\tau)&=a+\big(G_2(\tau)-\frac{\pi i}{2}\big)c_3+\frac{\pi i}{12}G_0(\tau)c_4+\frac{\pi i}{12}[c_1,c_3]+\cdots\\ A(\tau)&=1-2\pi ic_1-2\pi^2c_1^2+\pi ic_2+\cdots\\ B(\tau)&=1+c_1+\frac{1}{2}c_1^2+\pi ic_2+\frac{\pi^2}{3}c_3+\cdots\end{aligned}$$

which shows that the coefficient $2\pi i$ appears as a coefficient in low weight.

For the final statement, it is easy to see that $2\pi i$ does not appear in the ring of coefficients of $A'(\tau)$ since this power series is given by applying $g(\tau)$ to the product of three terms

$$A'=\Phi_{\rm KZ}(t_{01},t_{12})\,e^{t_{01}}\,\Phi_{\rm KZ}(t_{01},t_{12})^{-1},$$

none of which have $2\pi i$ in their coefficient rings, since these lie in $\mathbb{R}$ and $2\pi i$ does not. Since $\Phi_{\rm KZ}$ has $\zeta(2)$ as a coefficient, we may ask whether π^2 lies in the coefficient ring of $A'(\tau)$. The expansion of the power series $A'(\tau)$ is quite complicated and necessitates the help of a computer. It is necessary to go up to weight 5 in order to find enough coefficients to isolate π^2. In weight 5, however, we find that the sum of the coefficient of $c_1^3c_2$ and $c_2c_1^3$ in the expansion of $A'(\tau)$ is equal to $\frac{4\pi^2-1}{24}$, which shows that π^2 does lie in the coefficient ring $\mathcal{A}'$. $\square$

Lemma 3.3. *The underlying vector space of $\mathcal{E}$ is spanned by the coefficients of $C(\tau)$.*

Proof. Let $\mathcal{E}'$ denote the $\mathbb{Q}$-vector space spanned by the coefficients of $C(\tau)$. We first note that $\mathcal{E}'$ is in fact a $\mathbb{Q}$-algebra, because $C(\tau)$ is a group-like power series, which means that the product of two of its coefficients can be written as a linear combination of such by using the (multiplicative) shuffle relations. Next, we note that since $C(\tau) = \exp E(\tau)$, the coefficients of $C(\tau)$ can be expressed as algebraic combinations of the coefficients of $E(\tau)$, so they lie inside the subring $\mathcal{E} \subset \mathcal{E}^{\mathrm{geom}} \otimes \mathcal{Z}[2\pi i]$. Thus $\mathcal{E}' \subset \mathcal{E}$. But conversely, since $E(\tau) = \log C(\tau)$, the coefficients of $E(\tau)$ are all algebraic and thus linear combinations of those of $C(\tau)$, so they lie in $\mathcal{E}'$, so $\mathcal{E} \subset \mathcal{E}'$. This completes the proof. $\square$

3.2. **An expression for E modulo $2\pi i$.** From now until the end of this article, we work modulo $2\pi i$, in the sense that if a series has coefficients in $\mathcal{Z}[2\pi i]$, we reduce these modulo the ideal generated by $2\pi i$. The quotient ring $\overline{\mathcal{Z}}$ is equal to the quotient of $\mathcal{Z}$ by $(2\pi i)^2$, or equivalently, by $\zeta(2)$. We use overlining to denote the reduced objects. The goal of the section is to obtain an expression for $\overline{E}$ that relates it directly to the reduced Drinfeld associator $\overline{\Phi}_{\mathrm{KZ}}$.

In order to approach this result, we will move from the Lie algebra of derivations over to power series in a and b by using the map given by evaluation at a. This is important because it allows us to compare derivations with power series in a and b such as $\overline{\Phi}_{\mathrm{KZ}}$.

Let v_a denote the linear map given by evaluation at a. In Prop. 2.2 we considered this map restricted to $D \in \mathrm{Der}_0'(\mathfrak{f}_2)$; we have $v_a(D) = D(a)$ for $D \in \mathrm{Der}_0'(\mathfrak{f}_2)$. Let the push-operator be the linear automorphism of $\mathbb{Q}\langle\langle a, b\rangle\rangle$ defined by cyclically permuting the powers of a between the letters b in a monomial:

$$push(a^{k_0}b\cdots ba^{k_r}) = a^{k_r}ba^{k_0}b\cdots a^{k_{r-1}}, \tag{3.8}$$

extended to polynomials and power series by linearity. A power series is said to be *push-invariant* if $push(p) = p$. It is shown in [34], Lemma 2.1.1, that the restriction of v_a to the Lie subalgebra $\mathrm{Der}_0'(\mathfrak{f}_2)$, which is injective by Prop. 2.2, has image equal to the space of push-invariant Lie series $\mathfrak{f}_2^{\mathrm{push}} \subset \mathfrak{f}_2$. The map v_a transports the Lie bracket on $\mathrm{Der}_0'(\mathfrak{f}_2)$ to a Lie bracket $\langle\cdot,\cdot\rangle$ on $\mathfrak{f}_2^{\mathrm{push}}$ as follows:

$$\langle D(a), D'(a)\rangle = [D, D'](a). \tag{3.9}$$

We also use v_a to transport the exponential map $\exp : \mathrm{Der}_0'(\mathfrak{f}_2) \to \mathrm{Aut}(\mathfrak{f}_2)$ to an exponential map $\exp_a$ which makes the following diagram commute:

$$\begin{array}{ccc} \mathrm{Der}_0'(\mathfrak{f}_2) & \xrightarrow{\exp} & \mathrm{Aut}_0'(\mathfrak{f}_2) \\ \Big\downarrow{\scriptstyle v_a} & & \Big\downarrow{\scriptstyle v_a} \\ \mathfrak{f}_2^{\mathrm{push}} & \xrightarrow{\exp_a} & \mathfrak{f}_2, \end{array} \tag{3.10}$$

where $\mathrm{Aut}_0'(\mathfrak{f}_2) = \exp\big(\mathrm{Der}_0'(\mathfrak{f}_2)\big)$. We observe that the right-hand vertical map v_a is injective on $\mathrm{Aut}_0'(\mathfrak{f}_2)$. Indeed, if $\exp(D)\cdot a = \exp(D')\cdot a$ for two derivations $D \neq D' \in \mathrm{Der}_0'(\mathfrak{f}_2)$, then letting $\tilde{D} = ch_{\mathrm{Der}_0'(\mathfrak{f}_2)}(-D', D)$, where $ch_{\mathrm{Der}_0'(\mathfrak{f}_2)}$ denotes the Campbell-Hausdorff law on $\mathrm{Der}_0'(\mathfrak{f}_2)$, we would have

$$\exp(\tilde{D})\cdot a = a = a + \tilde{D}(a) + \frac{1}{2}\,\tilde{D}^2(a) + \cdots \tag{3.11}$$

By the assumption that $D \neq D'$, we have $\tilde{D}(a) \neq 0$. Let Let d denotes the lowest degree term occurring in $\tilde{D}(a)$. Then $d > 1$, and the degree d part of $\exp(\tilde{D})\cdot a$ can only come from the term $\tilde{D}(a)$ in (3.11), contradicting the desired identity $\exp(\tilde{D})\cdot a = a$.

This shows that v_a is injective on $\mathrm{Aut}_0'(\mathfrak{f}_2)$, which by the diagram (3.10) then shows that $\exp_a$ is also injective. Let $\mathcal{G}_a$ denote the image of $\mathfrak{f}_2^{\mathrm{push}}$ under $\exp_a$, or equivalently, the image of $\mathrm{Aut}_0'(\mathfrak{f}_2)$ under v_a. Then $\mathcal{G}_a$ is a set of elements in $\mathfrak{f}_2$, which forms a group when equipped with the group law transported from $\mathrm{Aut}_0'(\mathfrak{f}_2)$ by v_a. This group law, which we denote by $\star_a$, is compatible with the Campbell-Hausdorff law on the Lie algebra $\mathfrak{f}_2^{\mathrm{push}}$, since for two derivations $D, D' \in \mathrm{Der}_0'(\mathfrak{f}_2)$, we have

$$\begin{aligned}
\exp_a\big(D(a)\big) \star_a \exp_a\big(D'(a)\big) &= \big(\exp(D)\cdot a\big) \star_a \big(\exp(D')\cdot a\big)\\
&= \big(\exp(D)\circ\exp(D')\big)\cdot a\\
&= \exp\big(ch_{\mathrm{Der}_0'(\mathfrak{f}_2)}(D, D')\big)\cdot a\\
&= \exp_a\big(ch_{\mathfrak{f}_2^{\mathrm{push}}}\big(D(a), D'(a)\big)\big). \qquad (3.12)
\end{aligned}$$

We also have $\exp_a(0) = v_a\big(\exp(0)\big) = v_a(id) = a$, so in fact the element a is the unit element of the group $\mathcal{G}_a$ equipped with the multiplication $\star_a$. Explicitly, for $D \in \mathrm{Der}_0'(\mathfrak{f}_2)$, we have

$$\exp_a\big(D(a)\big) = v_a\big(\exp(D)\big) = \exp(D)\cdot a = a + D(a) + \frac{1}{2}D^2(a) + \cdots \tag{3.13}$$

Let $\mathfrak{grt}_{ell}$ be the elliptic Grothendieck-Teichmüller Lie algebra defined by B. Enriquez in §5.6 of [14]. Not surprisingly, this Lie algebra will be an essential tool in proving our results. Let us recall some of the basic facts concerning it. Firstly, Enriquez showed that there is a natural Lie morphism $\mathfrak{grt}_{ell} \to \mathrm{Der}_0(\mathfrak{f}_2)$. It was further shown in [34], equation (1.2.4), that this map is injective.[8] We will identify $\mathfrak{grt}_{ell}$ with its image in $\mathrm{Der}_0(\mathfrak{f}_2)$.

Enriquez also proved the following results. There is a canonical surjection $\mathfrak{grt}_{ell} \to \mathfrak{grt}$. Let $\mathfrak{r}_{ell}$ denote the kernel; then it is easy to see that $\mathfrak{u} \subset \mathfrak{r}_{ell}$. Finally, Enriquez gave a section $\gamma : \mathfrak{grt} \to \mathfrak{grt}_{ell}$ of the canonical surjection, and showed that $\mathfrak{grt}_{ell}$ has the form of a semi-direct product

$$\mathfrak{grt}_{ell} \cong \mathfrak{r}_{ell} \rtimes \gamma(\mathfrak{grt}).$$

We write γ_a for the composition map $v_a \circ \gamma$, so that

$$\gamma_a : \mathfrak{grt} \to \mathfrak{f}_2^{\mathrm{push}}. \tag{3.14}$$

Let $\exp^{\odot}$ denote the ("twisted Magnus") exponential map $\exp^{\odot} : \mathfrak{grt} \to GRT$ [[31], (2.14), where it is denoted $\circledast$]. Recall that $\mathrm{Der}^*\big(\mathrm{Lie}[[x,y]]\big)$ is the space of

[8]Note that what is denoted $\mathrm{Der}_0(\mathfrak{f}_2)$ in [34] is denoted here by $\mathrm{Der}_0'(\mathfrak{f}_2)$.

derivations that annihilate x and take y to a bracket $[y,f]$ and $z=-x-y$ to a bracket $[z,g]$ for some $f,g\in \mathrm{Lie}[x,y]$. Writing $\mathrm{Aut}^*\big(\mathrm{Lie}[[x,y]]\big)$ for the group of automorphisms $\exp(D)$ with $D\in \mathrm{Der}^*\big(\mathrm{Lie}[[x,y]]\big)$, we have the commutative diagram

$$\begin{array}{ccccccc} \mathrm{Der}^*(\mathrm{Lie}[[x,y]]) & \longleftarrow & \mathfrak{grt} & \xrightarrow{\ \gamma\ } & \mathfrak{grt}_{ell} & \xrightarrow{\ v_a\ } & \mathfrak{f}_2^{\mathrm{push}} \\ \downarrow{\scriptstyle \exp} & & \downarrow{\scriptstyle \exp^{\odot}} & & \downarrow{\scriptstyle \exp} & & \downarrow{\scriptstyle \exp_a} \\ \mathrm{Aut}^*(\mathrm{Lie}[[x,y]]) & \longleftarrow & GRT & \xrightarrow{\ \Gamma\ } & GRT_{ell} & \xrightarrow{\ v_a\ } & \mathcal{G}_a. \end{array} \tag{3.15}$$

where Γ is the group homomorphism that makes the middle square commute, and the upper map $\mathfrak{grt}\to \mathrm{Der}^*(\mathrm{Lie}[[x,y]])$ in the left-hand square is the map that takes a Lie element $\psi\in\mathfrak{f}_2$ to the associated *Ihara derivation* D_ψ defined by

$$D_\psi(x)=0,\qquad D_\psi(y)=[\psi(x,y),y]. \tag{3.16}$$

Ihara [22, 23] studied these derivations in detail, and in particular, he showed that if $\Psi=\exp^{\odot}(\psi)$ and A_Ψ denotes the automorphism $\exp(D_\psi)$ of $\mathcal{U}(\mathrm{Lie}[[x,y]])$, then

$$A_\Psi(x)=x,\qquad A_\Psi(y)=\Psi\ y\ \Psi^{-1}. \tag{3.17}$$

We can now state the main result of this subsection.

Theorem 3.4. *Let $\overline{E}$ be obtained from E by reducing the coefficients from $\overline{\mathcal{Z}}$ to $\mathcal{Z}/\langle(2\pi i)^2\rangle$. Then*

$$\overline{E}=\Gamma(\overline{\Phi}_{\mathrm{KZ}})\cdot a.$$

Proof. Let $\psi\in\mathfrak{grt}$, and let $\Psi=\exp^{\odot}(\psi)\in GRT$. Then $\gamma(\psi)\in\mathfrak{grt}_{ell}\subset \mathrm{Der}_0(\mathfrak{f}_2)$ and $\Gamma(\Psi)=\exp\big(\gamma(\psi)\big)\in GRT_{ell}\subset \mathrm{Aut}_0(\mathfrak{f}_2)$, the group of automorphisms preserving $[a,b]$. The proof is based on a result from [14], Lemma-Definition 4.6, which states that the automorphism $\Gamma(\Psi)$ acts as follows:

$$\begin{aligned} \Gamma(\Psi)(t_{01}) &= \Psi(t_{01},t_{12})t_{01}\Psi(t_{01},t_{12})^{-1} \\ \Gamma(\Psi)(b) &= \log\big(\Psi(t_{02},t_{12})e^b\Psi(t_{01},t_{12})^{-1}\big), \end{aligned} \tag{3.18}$$

where t_{01} is as in (3.1). Recall from (3.4) that we can take t_{01} and b as generators of $\mathfrak{f}_2$.

Recall that $\overline{\Phi}_{\mathrm{KZ}}\in GRT(\overline{\mathcal{Z}})$. (This is the reason for which we work mod $2\pi i$, since the term $-\zeta(2)[x,y]$ in Φ_{KZ} means that it does not lie in GRT, preventing us from taking advantage of the results on $\mathfrak{grt}_{ell}$.) Let $\overline{\sigma}$ be the automorphism of $F_2(\overline{\mathcal{Z}})$ obtained from σ by reducing modulo $2\pi i$, i.e.

$$\begin{aligned} \overline{\sigma}(e^{t_{01}}) &= \overline{\Phi}_{\mathrm{KZ}}(t_{01},t_{12})e^{t_{01}}\overline{\Phi}_{\mathrm{KZ}}(t_{01},t_{12})^{-1}=\overline{A'} \\ \overline{\sigma}(e^{b}) &= \overline{\Phi}_{\mathrm{KZ}}(t_{02},t_{12})e^{b}\overline{\Phi}_{\mathrm{KZ}}(t_{01},t_{12})^{-1}=\overline{B}, \end{aligned} \tag{3.19}$$

where we set $A'=A^{1/2\pi i}$ and $\overline{A'}$ and $\overline{B}$ denote the reductions of A' and B mod $2\pi i$.

Comparing with the values of $\Gamma(\overline{\Phi}_{\mathrm{KZ}})$ from (3.18) on the generators t_{01}, b of $\mathfrak{f}_2$, we find that $\overline{\sigma}=\Gamma(\overline{\Phi}_{\mathrm{KZ}})$. Since $E=\sigma(a)$ by (3.5), we find that

$$\overline{E}=\overline{\sigma}(a)=\Gamma(\overline{\Phi}_{\mathrm{KZ}})\cdot a,$$

which concludes the proof. □

Corollary 3.5. *The $\mathbb{Q}$-algebra generated by the coefficients of $\overline{E}$ is all of $\overline{\mathcal{Z}}$.*

Proof. Set $\phi_{\rm KZ}=\log^{\odot}(\overline{\Phi}_{\rm KZ})$, so that $\phi_{\rm KZ}\in\mathfrak{grt}\otimes_{\mathbb{Q}}\overline{\mathcal{Z}}$. We first show that the coefficients of $\phi_{\rm KZ}$ (written in a basis of $\mathfrak{grt}$) multiplicatively generate the same ring as the coefficients of $\overline{\Phi}_{\rm KZ}$, namely all of $\overline{\mathcal{Z}}$. To do this, we use an argument analogous to the one in the proof of Lemma 3.3. Let $\overline{\mathcal{Z}}'$ denote the $\mathbb{Q}$-algebra generated multiplicatively by the coefficients of $\phi_{\rm KZ}$. We of course know that the $\mathbb{Q}$-vector space $\overline{\mathcal{Z}}$ spanned by the (reduced) multizeta values which are the coefficients of $\overline{\Phi}_{\rm KZ}$ is actually a $\mathbb{Q}$-algebra, since $\overline{\Phi}_{\rm KZ}$ is group-like. The definition of the twisted Magnus exponential (cf. [31], (2.14), or [33], (3.5.2)) shows that the coefficients of $\overline{\Phi}_{\rm KZ}$ are all algebraic expressions in the coefficients of $\phi_{\rm KZ}$; thus $\overline{\mathcal{Z}}\subset\overline{\mathcal{Z}}'$. But similarly, since $\phi_{\rm KZ}=\log^{\odot}\overline{\Phi}_{\rm KZ}$, the coefficients of $\phi_{\rm KZ}$ are also all algebraic expressions in elements of $\overline{\mathcal{Z}}$; thus $\overline{\mathcal{Z}}'=\overline{\mathcal{Z}}$; in other words, the coefficients of $\phi_{\rm KZ}$ multiplicatively generate $\overline{\mathcal{Z}}$.

Since γ is an injective map from $\mathfrak{grt}$ to $\mathfrak{grt}_{ell}$, the coefficients of $\gamma(\phi_{\rm KZ})$ in a basis of $\mathfrak{grt}_{ell}$ also generate all of $\overline{\mathcal{Z}}$. Recall that Enriquez showed that $\mathfrak{grt}_{ell}$ is isomorphic to a semi-direct product of two of its subspaces, $\gamma(\mathfrak{grt})\rtimes\mathfrak{r}_{ell}$, and that $\varepsilon_{2k}\in\mathfrak{r}_{ell}$ for $k\geq 0$. Therefore we see that $\varepsilon_0\notin\gamma(\mathfrak{grt})\subset\mathfrak{grt}_{ell}\subset{\rm Der}_0(\mathfrak{f}_2)$. Since the natural bigrading on ${\rm Der}_0(\mathfrak{f}_2)$ restricts to a bigrading on $\gamma(\mathfrak{grt})$ (cf. [14]), we find that in fact $\gamma(\mathfrak{grt})\subset{\rm Der}_0'(\mathfrak{f}_2)$. Therefore, by Prop. 2.2, the evaluation map v_a is injective on $\gamma(\mathfrak{grt})$, so the coefficients of $\gamma(\phi_{\rm KZ})\cdot a$ in a basis of $v_a(\mathfrak{grt}_{ell})$ also generate $\overline{\mathcal{Z}}$. By the same argument as above, thanks to the definition of $\exp_a$ in (3.13), the coefficients of $\exp_a\bigl(\gamma(\phi_{\rm KZ})\cdot a\bigr)$ then span $\overline{\mathcal{Z}}$. But by (3.13) and the diagram (3.15), we have

$$\exp_a\bigl(\gamma(\phi_{\rm KZ}\cdot a\bigr)=\exp\bigl(\gamma(\phi_{\rm KZ})\bigr)\cdot a=\Gamma(\overline{\Phi}_{\rm KZ})\cdot a=\overline{E},\tag{3.20}$$

which completes the proof. □

3.3. **Structure of the $\mathbb{Q}$-algebras $\overline{\mathcal{E}}$, $\overline{\mathcal{A}}$ and $\overline{\mathcal{B}}$.** Since the rings generated by the coefficients of E, A and B all lie inside $\mathcal{Z}[2\pi i]$ and the ring generated by the coefficients of $g(\tau)$ is $\mathcal{E}^{\rm geom}$, and since $E(\tau)=g(\tau)\cdot E$, $A(\tau)=g(\tau)\cdot A$ and $B(\tau)=g(\tau)\cdot B$, the rings $\mathcal{E}$, $\mathcal{A}$ and $\mathcal{B}$ generated by the coefficients of $E(\tau)$, $A(\tau)$ and $B(\tau)$ respectively are all contained inside the ring generated inside $\mathcal{O}(\mathfrak{H})$ by the subrings $\mathcal{E}^{\rm geom}$ and $\mathcal{Z}[2\pi i]$, which as we saw is isomorphic to the tensor product of these two rings:

$$\mathcal{E},\mathcal{A},\mathcal{B}\subset\mathcal{E}^{\rm geom}\otimes_{\mathbb{Q}}\mathcal{Z}[2\pi i].$$

We have

$$\mathcal{E}^{\rm geom}\otimes_{\mathbb{Q}}\mathcal{Z}[2\pi i]\rightarrow\bigl(\mathcal{E}^{\rm geom}\otimes_{\mathbb{Q}}\mathcal{Z}[2\pi i]\bigr)/\langle 1\otimes 2\pi i\rangle\simeq\mathcal{E}^{\rm geom}\otimes_{\mathbb{Q}}\overline{\mathcal{Z}}.$$

We saw in Lemma 3.3 that $2\pi i\in\mathcal{E}$. The $\mathbb{Q}$-algebra $\overline{\mathcal{E}}$ generated by the coefficients of $\overline{E}(\tau)$ is equal to the quotient of $\mathcal{E}\subset\mathcal{E}^{\rm geom}\otimes\mathcal{Z}[2\pi i]$ by the intersection of $\mathcal{E}$ with the ideal $\langle 1\otimes 2\pi i\rangle$, so we have an inclusion

$$\overline{\mathcal{E}}\subset\mathcal{E}^{\rm geom}\otimes_{\mathbb{Q}}\overline{\mathcal{Z}}.$$

Recall from Definitions 1.1 and 1.2 that we set $\overline{A'}(\tau)=g(\tau)\cdot\overline{A'}$ and $\overline{B}(\tau)=g(\tau)\cdot\overline{B}$, where $\overline{A'}$ and $\overline{B}$ are as in (3.19) above. We write $\overline{\mathcal{A}}$ for the $\mathbb{Q}$-algebra generated by the coefficients of $\overline{A'}(\tau)$ and $\overline{\mathcal{B}}$ for that generated by the coefficients of $\overline{B}(\tau)$. Then like $\overline{\mathcal{E}}$, we have inclusions

$$\overline{\mathcal{A}},\overline{\mathcal{B}}\subset\mathcal{E}^{\rm geom}\otimes_{\mathbb{Q}}\overline{\mathcal{Z}}$$

(see Definition 1.2). The goal of this paragraph is to compare the subrings $\overline{\mathcal{E}}$, $\overline{\mathcal{A}}$ and $\overline{\mathcal{B}}$ of $\mathcal{E}^{\mathrm{geom}} \otimes_{\mathbb{Q}} \overline{\mathcal{Z}}$.

Theorem 3.6. *We have the following equalities:*

$$\overline{\mathcal{E}}[2\pi i\tau] = \overline{\mathcal{A}}[2\pi i\tau] = \overline{\mathcal{B}} = \mathcal{E}^{\mathrm{geom}} \otimes_{\mathbb{Q}} \overline{\mathcal{Z}}.$$

Proof. Let $r(\tau) = \log g(\tau) \in \mathrm{Der}_0(\mathfrak{f}_2)$. Recall from (3.19) that $\overline{A'} = \overline{\sigma}(e^{t_{01}})$, $\overline{B} = \overline{\sigma}(e^b)$, and $\overline{E} = \overline{\sigma}(a)$, where $\overline{\sigma} = \Gamma(\overline{\Phi}_{\mathrm{KZ}}) = \exp\bigl(\gamma(\phi_{\mathrm{KZ}})\bigr)$. Let us write $ch_{[,]} = ch_{\mathrm{Der}_0(\mathfrak{f}_2)}$ for the Campbell-Hausdorff law in the derivation algebra $\mathrm{Der}_0(\mathfrak{f}_2)$. We set

$$\delta(\tau) = ch_{[\,,\,]}\bigl(r(\tau), \gamma(\phi_{\mathrm{KZ}})\bigr) \in \mathrm{Der}_0(\mathfrak{f}_2).$$

Then we have

$$\begin{cases} \log \overline{A'}(\tau) = g(\tau)\cdot \log \overline{A'} = \exp\bigl(r(\tau)\bigr)\circ \exp\bigl(\gamma(\phi_{\mathrm{KZ}})\bigr)\cdot t_{01} = \exp\bigl(\delta(\tau)\bigr)\cdot t_{01} \\ \log \overline{B}(\tau) = g(\tau)\cdot \log \overline{B} = \exp\bigl(r(\tau)\bigr)\circ \exp\bigl(\gamma(\phi_{\mathrm{KZ}})\bigr)\cdot b = \exp\bigl(\delta(\tau)\bigr)\cdot b \\ \overline{E}(\tau) = g(\tau)\cdot \overline{E} = \exp\bigl(r(\tau)\bigr)\circ \exp\bigl(\gamma(\phi_{\mathrm{KZ}})\bigr)\cdot a = \exp\bigl(\delta(\tau)\bigr)\cdot a. \end{cases} \tag{3.21}$$

We also set

$$\mathfrak{a}(\tau) = \delta(\tau)\cdot t_{01}, \quad \mathfrak{b}(\tau) = \delta(\tau)\cdot b, \quad \mathfrak{e}(\tau) = \delta(\tau)\cdot a.$$

Step 1: The case of $\overline{\mathcal{B}}$. This case turns out to be the easiest one of the three, for the reason that by Prop. 2.2, the map v_b evaluating derivations on $b \in \mathfrak{f}_2$ is injective on all of $\mathfrak{u}$, which is not the case for v_a or $v_{t_{01}}$. Let $V_b \subset \mathfrak{f}_2$ denote the vector space $v_b\bigl(\mathrm{Der}_0(\mathfrak{f}_2)\bigr)$. Recall the whole situation with the exponential map $\exp_a$ and the group $\mathcal{G}_a$ that we introduced in (3.9), (3.10), (3.12), (3.13). Thanks to the injectivity of v_b, we can set up the analogous situation for b instead of a, but now using all $\mathrm{Der}_0(\mathfrak{f}_2)$.

We first transport the Lie bracket from $\mathrm{Der}_0(\mathfrak{f}_2)$ onto V_b via v_b, setting

$$\langle D(b), D'(b)\rangle_b = [D, D'](b),$$

which makes V_b into a Lie algebra. We then define $\exp_b$ to be the map that makes the diagram

$$\begin{array}{ccc} \mathrm{Der}_0(\mathfrak{f}_2) & \xrightarrow{\exp} & \mathrm{Aut}_0(\mathfrak{f}_2) \\ \downarrow{\scriptstyle v_b} & & \downarrow{\scriptstyle v_b} \\ V_b & \xrightarrow{\exp_b} & \mathfrak{f}_2 \end{array} \tag{3.22}$$

commute.

Exactly as in the case of a, we can show that the right-hand vertical map v_b induced on $\mathrm{Aut}_0(\mathfrak{f}_2)$ is still injective. Thus $\exp_b$ is also injective on V_b. We write

$$\mathcal{G}_b = \exp_b(V_b) = v_b\bigl(\mathrm{Aut}_0(\mathfrak{f}_2)\bigr) \subset \mathfrak{f}_2,$$

and equip this set with a group law $\star_b$ in analogy with (3.12), transported from $\mathrm{Aut}_0(\mathfrak{f}_2)$ by v_b:

$$\begin{aligned} \exp_b\bigl(D(f)\bigr) \star_b \exp_b\bigl(D'(b)\bigr) &= v_b\bigl(\exp(D)\bigr) \star_b v_b\bigl(\exp(D')\bigr) \\ &= v_b\bigl(\exp(D)\circ \exp(D')\bigr) \\ &= \bigl(\exp(D)\circ \exp(D')\bigr)\cdot b. \end{aligned} \tag{3.23}$$

We write $\log_b : \mathcal{G}_b \to V_b$ for the inverse of $\exp_b$, so that

$$\log_b\big(\exp(D)\cdot b\big) = D(b). \tag{3.24}$$

We will use all this information in the following calculation. By (3.21) and the diagram (3.22), we have

$$\log \overline{B}(\tau) = \exp\big(\delta(\tau)\big)\cdot b = \exp_b\big(\delta(\tau)\cdot b\big) \in \mathcal{G}_b, \tag{3.25}$$

so

$$\mathfrak{b}(\tau) := \delta(\tau)\cdot b = \log_b \log \overline{B}(\tau) \in V_b. \tag{3.26}$$

Thus by (3.24) with $D = \delta(\tau) = ch_{[\,,\,]}\big(r(\tau), \gamma(\phi_{KZ})\big)$, we have

$$\begin{aligned}
\mathfrak{b}(\tau) &= \delta(\tau)\cdot b \\
&= ch_{[\,,\,]}\big(r(\tau), \gamma(\phi_{\mathrm{KZ}})\big)\cdot b \\
&= r(\tau)\cdot b + \gamma(\phi_{\mathrm{KZ}})\cdot b + \frac{1}{2}[r(\tau), \gamma(\phi_{\mathrm{KZ}})]\cdot b + \cdots \\
&= r(\tau)\cdot b + \gamma(\phi_{\mathrm{KZ}})\cdot b + \frac{1}{2}\langle r(\tau)\cdot b, \gamma(\phi_{\mathrm{KZ}})\cdot b\rangle_b + \cdots ,
\end{aligned}$$

which we rewrite as

$$\mathfrak{b}(\tau) = r(\tau)\cdot b + \gamma(\phi_{\mathrm{KZ}})\cdot b + s(\tau)\cdot b, \tag{3.27}$$

where $s(\tau)$ is the sum of all the bracketed terms in $ch_{[\,,\,]}\big(r(\tau), \gamma(\phi_{\mathrm{KZ}})\big)$.

By the argument of Lemma 3.3, the $\mathbb{Q}$-algebra generated by the coefficients of $\mathfrak{b}(\tau) = \log_b \log \overline{B}(\tau)$ is equal to the one generated by the coefficients of $\log \overline{B}(\tau)$, which in turn is equal to the one generated by the coefficients of $\overline{B}(\tau)$, namely $\overline{\mathcal{B}} \subset \mathcal{E}^{\mathrm{geom}} \otimes \overline{\mathcal{Z}}$. In order to show that these two algebras are equal, we will consider $\overline{\mathcal{B}}$ as the $\mathbb{Q}$-algebra generated by the coefficients of $\mathfrak{b}(\tau)$, and use properties of the right-hand side of (3.27) to show separately that it contains $\overline{\mathcal{Z}}$ and $\mathcal{E}^{\mathrm{geom}}$.

Let us write $\overline{\mathcal{Z}}_{>0}$ for the $\mathbb{Q}$-vector space spanned by the images in $\overline{\mathcal{Z}}$ of the multizeta values in $\mathcal{Z}$ under the surjection $\mathcal{Z} \twoheadrightarrow \overline{\mathcal{Z}}$. Then $\overline{\mathcal{Z}}$ is generated by $\mathbb{Q} = \overline{\mathcal{Z}}_0$ and $\mathcal{Z}_{>0}$. Let us write $\mathfrak{nz}$ for the vector space quotient $\overline{\mathcal{Z}}_{>0}/\overline{\mathcal{Z}}^2_{>0}$, where $\overline{\mathcal{Z}}^2$ denotes the vector subspace of $\overline{\mathcal{Z}}$ generated by products of elements of $\overline{\mathcal{Z}}_{>0}$, which can be viewed as linear combinations thanks to the shuffle multiplication of multizetas. The vector space $\mathfrak{nz}$ is called the space of *new multiple zeta values*.

Let $\mathcal{MZ}$ denote the $\mathbb{Q}$-algebra of *motivic multiple zeta values* defined by Goncharov (see [18]); let $\mathcal{MZ}_{>0}$ denote the $\mathbb{Q}$-vector subspace generated by the motivic multizeta values, and let $\mathfrak{nmz} = \mathcal{MZ}_{>0}/\mathcal{MZ}^2_{>0}$ be the space of *new motivic multizeta values*. Goncharov showed that $\mathcal{MZ}$ is a Hopf algebra, so that $\mathfrak{nmz}$ is a Lie co-algebra. He further showed that the motivic $\zeta^{\mathfrak{m}}(2) = 0$ in $\mathcal{MZ}$, that there is a surjection $\mathcal{MZ} \twoheadrightarrow \overline{\mathcal{Z}}$, and that the motivic multizeta values satisfy the associator relations of $\overline{\Phi}_{\mathrm{KZ}}$. It follows that there are injective maps in the dual situation

$$\mathfrak{nz}^\vee \subset \mathfrak{nmz}^\vee \subset \mathfrak{grt}. \tag{3.28}$$

All three spaces are vector subspaces of $\mathfrak{f}_2$, so that these injections can be considered as inclusions. The Lie series ϕ_{KZ} lies in the vector space $\mathfrak{nz}^\vee \otimes_{\mathbb{Q}} \overline{\mathcal{Z}}$, so by (3.28), it can also be considered as lying in the Lie algebras $\mathfrak{nmz}^\vee \otimes_{\mathbb{Q}} \overline{\mathcal{Z}} \subset \mathfrak{grt} \otimes_{\mathbb{Q}} \overline{\mathcal{Z}}$. Thus $\gamma(\phi_{\mathrm{KZ}}) \in \gamma(\mathfrak{nmz}^\vee) \otimes_{\mathbb{Q}} \overline{\mathcal{Z}}$ and

$$\gamma(\phi_{\mathrm{KZ}})\cdot b \in v_b\big(\gamma(\mathfrak{nmz}^\vee)\big) \otimes_{\mathbb{Q}} \overline{\mathcal{Z}}.$$

An important theorem by Brown ([5]) identified the Lie algebra $\mathfrak{nmz}^\vee$ with the fundamental Lie algebra of the category of mixed Tate motives over $\mathbb{Z}$, which is free on one generator in each odd weight ≥ 3 (the weight used here corresponds to the degree of the Lie polynomials, and is the negative of the usual motivic weight). In [19], Hain and Matsumoto defined a category of universal mixed elliptic motives, and they showed that the fundamental Lie algebra of that category has a monodromy representation in $\mathrm{Der}_0(\mathfrak{f}_2)$ whose image Π is isomorphic to a semi-direct product $\Pi \cong \mathfrak{u}' \rtimes \gamma(\mathfrak{nmz}^\vee)$. Thus $D_1 \in \mathfrak{u}'$ and $D_2 \in \gamma(\mathfrak{nmz}^\vee)$, then any bracketed word in these two derivations lies in $\mathfrak{u}'$. R and $D_2 \in \gamma(\mathfrak{nmz}^\vee) \otimes_{\mathbb{Q}} R$,

In fact, we can actually say more, and show that even if $D_1 \in \mathfrak{u}$ and not just $\mathfrak{u}'$, all bracketed words of D_1 and D_2 still lie in $\mathfrak{u}'$; in other words, the Lie algebra generated inside $\mathrm{Der}_0(\mathfrak{f}_2)$ by $\mathfrak{u}$ and $\gamma(\mathfrak{nmz}^\vee)$ has the structure of a semi-direct product $\mathfrak{u} \rtimes \gamma(\mathfrak{nmz}^\vee)$. Indeed, we have $\varepsilon_0 \in \mathfrak{sl}_2$, and Hain and Matsumoto show that there is an action of $\mathfrak{sl}_2$ on the semi-direct product $\mathfrak{u}' \rtimes \gamma(\mathfrak{nmz}^\vee)$. But $\mathfrak{sl}_2$ is included in the Lie algebra $\mathfrak{r}_{ell}$ defined by Enriquez, who shows that $\mathfrak{r}_{ell}$ is a normal Lie subalgebra of $\mathfrak{grt}_{ell}$, and in fact that $\mathfrak{grt}_{ell}$ has the structure of a semi-direct product $\mathfrak{grt}_{ell} = \mathfrak{r}_{ell} \rtimes \gamma(\mathfrak{grt})$. Enriquez's result shows that a bracketed word combining ε_0 with any derivations in $\mathfrak{grt}_{ell}$ lies in $\mathfrak{r}_{ell}$. In particular, a bracketed word combining ε_0 with any derivations in the subspace $\gamma(\mathfrak{nmz}^\vee) \subset \gamma(\mathfrak{grt})$ lies in $\mathfrak{r}_{ell}$. But Hain-Matsumoto's result that $\mathfrak{u}' \rtimes \gamma(\mathfrak{nmz}^\vee)$ is an $\mathfrak{sl}_2$-module shows that such a bracket lies inside $\mathfrak{u}' \rtimes \gamma(\mathfrak{nmz}^\vee)$. Thus it lies inside the intersection of $\mathfrak{u}' \rtimes \gamma(\mathfrak{nmz}^\vee)$ with $\mathfrak{r}_{ell}$, which is just $\mathfrak{u}'$. The above statements all hold when the Lie algebra $\mathfrak{u}$, $\mathfrak{u}'$ and $\gamma(\mathfrak{nmz}^\vee)$ are tensored with any $\mathbb{Q}$-algebra R.

In our situation, we set

$$D_1 = r(\tau) \in \mathfrak{u} \otimes_{\mathbb{Q}} \mathcal{E}^{\mathrm{geom}} \quad \text{and} \quad D_2 = \gamma(\phi_{\mathrm{KZ}}) \in \gamma(\mathfrak{nmz}^\vee) \otimes_{\mathbb{Q}} \overline{\mathcal{Z}}.$$

Then from the above, any bracketed word combining these two derivations lies in the space $\mathfrak{u}' \otimes_{\mathbb{Q}} \big(\mathcal{E}^{\mathrm{geom}} \otimes_{\mathbb{Q}} \overline{\mathcal{Z}}\big)$. Thus in particular, we have $s(\tau) \in \mathfrak{u}' \otimes_{\mathbb{Q}} \big(\mathcal{E}^{\mathrm{geom}} \otimes_{\mathbb{Q}} \overline{\mathcal{Z}}\big)$ and $s(\tau) \cdot b \in v_b(\mathfrak{u}') \otimes_{\mathbb{Q}} \big(\mathcal{E}^{\mathrm{geom}} \otimes_{\mathbb{Q}} \overline{\mathcal{Z}}\big)$, where $s(\tau)$ is the sum of bracketed terms in $ch_{[\,,\,]}(D_1, D_2)$ as in (3.27). Altogether, we thus have

$$\begin{cases} \gamma(\phi_{\mathrm{KZ}}) \cdot b \in v_b\big(\gamma(\mathfrak{nmz}^\vee)\big) \otimes_{\mathbb{Q}} \overline{\mathcal{Z}} \\ r(\tau) \cdot b \in v_b(\mathfrak{u}) \otimes_{\mathbb{Q}} \mathcal{E}^{\mathrm{geom}} \\ s(\tau) \in v_b(\mathfrak{u}') \otimes_{\mathbb{Q}} \big(\mathcal{E}^{\mathrm{geom}} \otimes_{\mathbb{Q}} \overline{\mathcal{Z}}\big) \end{cases} \tag{3.29}$$

for the three terms in the right-hand side of (3.27).

We are now ready to show that $\overline{\mathcal{B}} \supset \overline{\mathcal{Z}}$. Since the evaluation map v_b is injective on $\mathrm{Der}_0(\mathfrak{f}_2)$, it is in particular injective on the subspace $\mathfrak{u} \rtimes \gamma(\mathfrak{nmz}^\vee)$. Let V denote the underlying vector space of $v_b(\mathfrak{u})$, and W that of $v_b\big(\gamma(\mathfrak{nmz}^\vee)\big)$. Then the underlying vector space of the semi-direct product $v_b\big(\mathfrak{u} \rtimes \gamma(\mathfrak{nmz}^\vee)\big)$ is the direct sum $V \oplus W$. Writing $R = \mathcal{E}^{\mathrm{geom}} \otimes_{\mathbb{Q}} \overline{\mathcal{Z}}$, we deduce from (3.29) that

$$\gamma(\phi_{\mathrm{KZ}}) \cdot b \in W \otimes_{\mathbb{Q}} R \quad \text{and} \quad r(\tau) \cdot b,\ s(\tau) \cdot b \in V \otimes_{\mathbb{Q}} R. \tag{3.30}$$

Let us take a linear basis of $V \oplus W$ adapted to the direct product, i.e. in which every element belongs either to V or to W. Write $\mathfrak{b}(\tau)$ in this basis, and consider the coefficient of a basis element $w \in W$. By (3.30), $\mathfrak{b}(\tau)$ decomposes as a sum of two terms, $\gamma(\phi_{\mathrm{KZ}}) \cdot b \in W \otimes_{\mathbb{Q}} R$ and $r(\tau) \cdot b + s(\tau) \cdot b \in V \otimes_{\mathbb{Q}} R$. Therefore, the coefficient in $\mathfrak{b}(\tau)$ of any basis element $w \in W$ is equal to the coefficient of w in $\gamma(\phi_{\mathrm{KZ}}) \cdot b$ written in the same basis of W. But since we know that the coefficients of $\phi_{\rm KZ}$ written in a basis of $\mathfrak{nmz}^\vee$ multiplicatively generate all of $\overline{\mathcal{Z}}$, and since γ is injective and defined over $\mathbb{Q}$, the same holds for the coefficients of $\gamma(\phi_{\rm KZ})$ written in a basis of $\gamma(\mathfrak{nmz}^\vee)$, and then since v_b is injective on this space and defined over $\mathbb{Q}$, the same again holds for the coefficients of $v_b\big(\gamma(\phi_{\rm KZ})\big)$ written in a basis of W. Thus the coefficients in $\mathfrak{b}(\tau)$ of the elements of the basis of W span all of $\overline{\mathcal{Z}}$, so $\overline{\mathcal{B}}\supset\overline{\mathcal{Z}}$.

Now we will show that $\overline{\mathcal{B}}\supset\mathcal{E}^{\rm geom}$. Here we need to deal with the $s(\tau)$ term. For this, we will proceed by induction on the weight. We take a basis for V which is the image under v_b of a weight-graded basis of $\mathfrak{u}$. The Lie series $\mathfrak{b}(\tau)$ starts in weight 1 with the term $2\pi i\tau a$, which comes from the $2\pi i\tau\varepsilon_0$ term of $r(\tau)$ acting on b. Since $r'(\tau):=r(\tau)-2\pi i\tau\varepsilon_0$ and $s(\tau)$ lie in $\mathfrak{u}'$, these derivations are strictly weight-increasing, so there are no other weight 1 terms in $\mathfrak{b}(\tau)$. Thus $2\pi i\tau\in\overline{\mathcal{B}}$. We use this result as the base case, fix $n>1$, and make the induction hypothesis that for all $m<n$, the coefficients in $\mathfrak{b}(\tau)$ of the weight m basis elements of V span the weight m graded part of $\mathcal{E}^{\rm geom}$, so that $\overline{\mathcal{B}}$ contains the weight graded parts of $\mathcal{E}^{\rm geom}$ for all weight $m<n$. Consider the coefficient in $\mathfrak{b}(\tau)$ of a weight n basis element $v\in V$. Each coefficient is of the form r_v+s_v where r_v is the coefficient of v in $r(\tau)\cdot b$ and s_v is the coefficient of v in $s(\tau)\cdot b$. But the part of the derivation $s(\tau)$ that takes b to a weight n polynomial is made up of brackets of parts of $r(\tau)$ and of $\gamma(\phi_{\rm KZ})$ of strictly smaller weight, and whose coefficients are thus algebraic combinations of coefficients of $r(\tau)$ of lower weight, which already appear in $\overline{\mathcal{B}}$ by the induction hypothesis, and of coefficients of $\gamma(\phi_{\rm KZ})$, i.e. elements of $\overline{\mathcal{Z}}$, which already lie in $\overline{\mathcal{B}}$ by the result above that $\overline{\mathcal{B}}\supset\overline{\mathcal{Z}}$. Thus not just the coefficient r_v+s_v, but also the term s_v lies in $\overline{\mathcal{B}}$, which proves that $r_v\in\overline{\mathcal{B}}$. Thus all the coefficients of the weight n part of $r(\tau)\cdot b$ lie in $\overline{\mathcal{B}}$, so by induction, all the coefficients of $r(\tau)\cdot b$ lie in $\overline{\mathcal{B}}$; since v_b is injective, these coefficients generate the same $\mathbb{Q}$-algebra as the coefficients of $r(\tau)$, namely $\mathcal{E}^{\rm geom}$. This proves that $\overline{\mathcal{B}}\supset\mathcal{E}^{\rm geom}$, and completes the proof of the desired result $\overline{\mathcal{B}}\simeq\mathcal{E}^{\rm geom}\otimes_{\mathbb{Q}}\overline{\mathcal{Z}}$.

Step 2. The case of $\overline{\mathcal{E}}$. The argument is similar to the one for $\overline{\mathcal{B}}$, but there is an added subtlety coming from the fact that $r(\tau)$ lies in ${\rm Der}_0(\mathfrak{f}_2)$, but v_a is not injective on ${\rm Der}_0(\mathfrak{f}_2)$ since $\varepsilon_0(a)=0$. To get around this, we will use the fact that

$$\mathfrak{u}\simeq\mathfrak{u}'\rtimes\mathbb{Q}\varepsilon_0$$

(see (2.1)). The universal enveloping algebra of a semi-direct product of Lie algebras is isomorphic to the tensor product of the two Lie algebras, and its graded dual is isomorphic to the tensor product of the two duals. Therefore we have

$$(\mathcal{U}\mathfrak{u})^\vee\simeq(\mathcal{U}\mathfrak{u}')^\vee\otimes_{\mathbb{Q}}(\mathcal{U}\mathbb{Q}\varepsilon_0)^\vee.$$

Under the identification $(\mathcal{U}\mathfrak{u})^\vee\simeq\mathcal{E}^{\rm geom}$ of Theorem 2.6, this translates to

$$\mathcal{E}^{\rm geom}\simeq\mathcal{E}_0^{\rm geom}\otimes_{\mathbb{Q}}\mathbb{Q}[2\pi i\tau],$$

where $\mathcal{E}_0^{\rm geom}$ is multiplicatively generated by the coefficients of $r'(\tau)=r(\tau)-2\pi i\tau\varepsilon_0$. In particular, the subspace inclusion $\mathfrak{u}'\subset\mathfrak{u}$ corresponds in the dual to the surjection

$$\mathcal{E}^{\rm geom}\twoheadrightarrow\mathcal{E}^{\rm geom}/\langle 2\pi i\tau\rangle\simeq\mathcal{E}_0^{\rm geom}.$$

The derivation $\delta(\tau)=ch_{[\,,\,]}\big(r(\tau),\gamma(\phi_{\rm KZ})\big)$ lies in $\mathfrak{u}\otimes_{\mathbb{Q}}\mathcal{E}^{\rm geom}\otimes\overline{\mathcal{Z}}$, but if we consider the derivation $\hat\delta(\tau)$ obtained by reducing its coefficients mod $2\pi i\tau$, we see that

$$\hat\delta(\tau)=ch_{[\,,\,]}\big(r'(\tau),\gamma(\phi_{\rm KZ})\big)\in\mathfrak{u}'\otimes_{\mathbb{Q}}\mathcal{E}_0^{\rm geom}\otimes_{\mathbb{Q}}\overline{\mathcal{Z}},$$

where $r'(\tau) = r(\tau) - 2\pi i\tau\varepsilon_0$.

Let $\hat{\overline{E}}(\tau)$ and $\hat{\mathfrak{e}}(\tau)$ be the power series obtained from $\overline{E}(\tau)$ and $\mathfrak{e}(\tau) \in \mathfrak{f}_2 \otimes_{\mathbb{Q}} \big(\mathcal{E}^{\mathrm{geom}} \otimes_{\mathbb{Q}} \overline{\mathcal{Z}}\big)$ by reducing the coefficients mod $2\pi i\tau$. Then we have

$$\hat{\overline{E}}(\tau),\ \hat{\mathfrak{e}}(\tau) \in \mathfrak{f}_2 \otimes \big(\mathcal{E}_0^{\mathrm{geom}} \otimes_{\mathbb{Q}} \overline{\mathcal{Z}}\big)$$

and

$$\hat{\overline{E}}(\tau) = \exp\big(\hat{\delta}(\tau)\big) \cdot a, \quad \hat{\mathfrak{e}}(\tau) = \hat{\delta}(\tau) \cdot a,$$

so

$$\hat{\mathfrak{e}}(\tau) = log_a \,\hat{\overline{E}}(\tau).$$

Thus the $\mathbb{Q}$-algebras generated by the coefficients of $\mathfrak{e}(\tau)$ and by $\hat{\overline{E}}(\tau)$ are equal. Denote this $\mathbb{Q}$-algebra by $\mathfrak{E} \subset \mathcal{E}_0^{\mathrm{geom}} \otimes_{\mathbb{Q}} \overline{\mathcal{Z}}$. Since

$$\hat{\mathfrak{e}}(\tau) = ch_{[\,,\,]}\big(r'(\tau), \gamma(\phi_{\mathrm{KZ}})\big) \cdot a = r'(\tau) \cdot a + \gamma(\phi_{\mathrm{KZ}}) \cdot a + \hat{s}(\tau) \cdot a$$

where $\hat{s}(\tau)$ denotes the bracketed terms in the Campbell-Hausdorff product, we can use the identical arguments to the case of $\mathfrak{b}(\tau)$ above to prove that $\mathfrak{E} \supset \overline{\mathcal{Z}}$. We also again use induction on the weight to prove that $\mathfrak{E}$ contains all the coefficients of $r'(\tau)$. The only difference with the case of $\mathfrak{b}(\tau)$ is the base case, which is no longer in weight 1. The lowest weight term of $\hat{\mathfrak{e}}(\tau)$ is of weight 3, and it comes from the term $\mathcal{G}_2(\tau)\varepsilon_2$ of $r'(\tau)$ acting on a (note that $\hat{s}(\tau) \cdot a$ has no terms of weight lower than 7). Thus the same induction as above works to prove that every coefficient of $r'(\tau)$ lies in $\mathfrak{E}$, so we find that $\mathfrak{E} \simeq \mathcal{E}_0^{\mathrm{geom}} \otimes_{\mathbb{Q}} \overline{\mathcal{Z}}$.

Since $\mathfrak{E}$ is the $\mathbb{Q}$-algebra generated by the reduction of $\overline{E}(\tau)$ mod $2\pi i\tau$ and it is equal to $\mathcal{E}_0^{\mathrm{geom}} \otimes_{\mathbb{Q}} \overline{\mathcal{Z}}$, we see that $\mathfrak{E}[2\pi i\tau] \simeq \mathcal{E}^{\mathrm{geom}} \otimes_{\mathbb{Q}} \overline{\mathcal{Z}}$. This means that the composition map

$$\overline{\mathcal{E}} \hookrightarrow \mathcal{E}^{\mathrm{geom}} \otimes_{\mathbb{Q}} \overline{\mathcal{Z}} \twoheadrightarrow \big(\mathcal{E}^{\mathrm{geom}} \otimes_{\mathbb{Q}} \overline{\mathcal{Z}}\big)/\langle 2\pi i\tau \otimes 1\rangle \simeq \mathcal{E}_0^{\mathrm{geom}} \otimes_{\mathbb{Q}} \overline{\mathcal{Z}}$$

is surjective, which gives us the desired equality $\overline{\mathcal{E}}[2\pi i\tau] \simeq \mathcal{E}^{\mathrm{geom}} \otimes_{\mathbb{Q}} \overline{\mathcal{Z}}$.

Step 3. The case of $\overline{\mathcal{A}}$. The argument here is identical to the one for $\overline{\mathcal{E}}$. We again have the problem that, as discovered independently by Enriquez and by Hain and Matsumoto, there exists a unique derivation $\eta \in \mathfrak{u}$ that annihilates t_{01}; η is defined over $\mathbb{Q}$ and is a linear combination of the ε_{2k}, $k \geq 0$, with rational coefficients

$$\eta = \varepsilon_0 - \frac{1}{12}\varepsilon_2 + \frac{1}{240}\varepsilon_4 - \frac{1}{6048}\varepsilon_6 + \cdots$$

The exact nature of η is not important, only the fact that it has a term in ε_0 and is defined over $\mathbb{Q}$; this shows that $v_{t_{01}}$ is injective on $\mathfrak{u}'$. We let $W = v_{t_{01}}(\mathfrak{nmz}^\vee)$ and $V = v_{t_{01}}(\mathfrak{u}')$, and choose a basis of V which is the image under $v_{t_{01}}$ of a weight-graded basis of $\mathfrak{u}'$ as above. We then proceed exactly as in the case of $\overline{\mathcal{E}}$ to show that the $\mathbb{Q}$-algebra generated by the coefficients of $\exp\big(\hat{\delta}(\tau)\big) \cdot t_{01}$, (which is the reduction of $\log \overline{A}(\tau)$ mod $2\pi i\tau$) contains $\overline{\mathcal{Z}}$. For the induction argument, even if V is not itself graded by the weight, we can simply transport the weight-grading of $\mathfrak{u}'$ to V and use induction on that (or equivalently, do the induction on the lowest weight parts of the basis elements). Since t_{01} starts with $-a$, the argument is identical to the one for a above, and shows that the $\mathbb{Q}$-algebra generated by the coefficients of $\hat{\delta}(\tau) \cdot t_{01}$, and thus also by those of $\exp\big(\hat{\delta}(\tau)\big) \cdot t_{01}$, is isomorphic to $\mathcal{E}_0^{\mathrm{geom}} \otimes_{\mathbb{Q}} \overline{\mathcal{Z}}$, so that again we have $\overline{\mathcal{A}}[2\pi i\tau] \simeq \mathcal{E}^{\mathrm{geom}} \otimes_{\mathbb{Q}} \overline{\mathcal{Z}}$ as desired. This concludes the proof. $\square$

4. The elliptic double shuffle and push-neutrality relations

In this section we use mould theory to explore and compare algebraic relations between the $\overline{E}$-EMZs with algebraic relations between the $\overline{A}$-EMZs. The first paragraph, §4.1, gives a brief exposition of the necessary definitions and results from mould theory.

Our main result on $\overline{E}$-elliptic multizetas, in §4.2, arises as a corollary of the preceding theorem and the main result of [34]. We show that $\overline{E}(\tau)$ satisfies a certain double family of algebraic relations called the *elliptic double shuffle relations*, related to the familiar double shuffle properties of Φ_{KZ}, but more similar to the graded double shuffle relations studied for example in [3]. Further, we show that if one assumes certain reasonable conjectures from multizeta and Grothendieck-Teichmüller theory, the elliptic double shuffle relations can be expected to form a *complete* set of algebraic relations for the $\overline{E}$-EMZs. We compute these relations and the associated dimensions in detail in depth 2.

Finally, in §4.3 we consider a double family of relations satisfied by $\overline{A'}(\tau)$ (or more precisely by the log of this series). The first family is just the usual shuffle, but the second is very different from the second shuffle relation satisfied by $\overline{E}(\tau)$. We call it the family of *push-neutrality relations*, and show that it is related to the *Fay relations* studied in [27]. We compute the relations and the associated dimensions in depth 2 and show that they are different from those of $\overline{E}(\tau)$, which means that while we know by Theorem 3.6 that $\overline{\mathcal{E}}[2\pi i\tau] = \overline{\mathcal{A}}[2\pi i\tau]$, the algebras $\overline{\mathcal{E}}$ and $\overline{\mathcal{A}}$ themselves are not equal nor even isomorphic as filtered algebras (i.e. the dimensions of the associated gradeds are not equal).

4.1. **A very brief introduction to moulds.**

We recall some notions from Ecalle's theory of moulds [12, 13] that we will need in order to study algebraic relations between elliptic multizetas. Besides the original references, a more detailed introduction to moulds can be found in [33].

4.1.1. *Moulds and bialternality.*

In this article, we use the term 'mould' to refer only to rational-function valued moulds with coefficients in $\mathbb{Q}$. Thus, a mould is a family of functions

$$\{P(u_1,\ldots,u_r) \mid r \geq 0\}$$

with $P(u_1,\ldots,u_r) \in \mathbb{Q}(u_1,\ldots,u_r)$. In particular $P(\emptyset)$ is a constant. The *depth r* part of a mould is the function $P(u_1,\ldots,u_r)$ in r variables. By defining addition and scalar multiplication of moulds in the obvious way, i.e. depth by depth, moulds form a $\mathbb{Q}$-vector space that we call $Moulds$. We write $Moulds_{pol}$ for the subspace of polynomial-valued moulds. The vector space ARI is the subspace of $Moulds$ consisting of moulds P with constant term $P(\emptyset) = 0$, and ARI_{pol} is again the subspace of polynomial-valued moulds in ARI.

The standard mould multiplication mu is given by

$$mu(P,Q)(u_1,\ldots,u_r) = \sum_{i=0}^{r} P(u_1,\ldots,u_i)Q(u_{i+1},\ldots,u_r). \tag{4.1}$$

For simplicity, we write $P\,Q = mu(P,Q)$. This multiplication defines a Lie algebra structure on ARI with Lie bracket lu defined by $lu(P,Q) = mu(P,Q) - mu(Q,P)$.

We now introduce four operators on moulds. The Δ-operator on moulds is defined as follows: if $P \in ARI$, then

$$\Delta(P)(u_1,\ldots,u_r) = u_1\cdots u_r(u_1+\cdots+u_r)P(u_1,\ldots,u_r). \tag{4.2}$$

The *dar*-operator is defined by

$$dar(P)(u_1,\ldots,u_r) = u_1\cdots u_r\,P(u_1,\ldots,u_r). \tag{4.3}$$

The *push*-operator is defined by

$$push(B)(u_1,\ldots,u_r) = B(u_2,\ldots,u_r,-u_1-\cdots-u_r). \tag{4.4}$$

Finally, the *swap* operator is defined by

$$swap(A)(v_1,\ldots,v_r) = A(v_r,v_{r-1}-v_r,\ldots,v_1-v_2). \tag{4.5}$$

Here the use of the alphabet $v_1, v_2, \ldots$ instead of $u_1,\ldots,u_r$ is purely a convenient way to distinguish a mould from its swap.

The main property on moulds that we will need to consider is *alternality*. A mould P is said to be *alternal* if for all $r>1$ and for $1 \le i \le [r/2]$, we have

$$\sum_{\mathbf{u}\in sh((u_1,\ldots,u_i),(u_{i+1},\ldots,u_r))} P(\mathbf{u}) = 0, \tag{4.6}$$

where the set of r-tuples $sh\big((u_1,\ldots,u_i),(u_{i+1},\ldots,u_r)\big)$ is the set

$$\big\{(u_{\sigma^{-1}(1)},\ldots,u_{\sigma^{-1}(r)})\,\big|\,\sigma\in S_r \text{ such that } \sigma(1)<\cdots<\sigma(i),\ \ \sigma(i{+}1)<\cdots<\sigma(r)\big\}.$$

The mould $swap(A)$ is alternal if it satisfies the same property (4.6) in the variables v_i.

We write ARI^{al} for the space of alternal moulds in ARI, and $ARI^{al/al}$ for the space of moulds which are alternal and whose swap is also alternal. We also consider moulds which are alternal and whose swap is alternal up to addition of a constant-valued mould. The space of these moulds is denoted ARI^{al*al} and we call them *bialternal*.

We say that a mould P is Δ-bialternal if $\Delta^{-1}(P)$ is bialternal, and we write $ARI^{\Delta\text{-}al*al}$ for the space of such moulds.

4.1.2. *From power series to moulds.* Let $c_i = \mathrm{ad}(a)^{i-1}(b)$ for $i \ge 1$ as in §3.1. Let the depth of a monomial $c_{i_1}\cdots c_{i_r}$ be the number r of c_i in the monomial; the weight (degree in a and b) and the depth form a topological bigrading on the formal power series ring $\mathbb{Q}\langle\langle C\rangle\rangle = \mathbb{Q}\langle\langle c_1,c_2,\ldots\rangle\rangle$ on the free variables c_i. Here, by "topological bigrading" we mean that $\mathbb{Q}\langle\langle C\rangle\rangle$ is the direct product (not the direct sum) $\prod_{n,d\ge 0} V_{n,d}$ of its components of weight n and depth d. Similarly, we write $L[\![C]\!] = \mathrm{Lie}[\![c_1,c_2,\ldots]\!]$ for the corresponding Lie algebra. By Lazard elimination, we have an isomorphism

$$\mathbb{Q}a \oplus L[\![C]\!] \cong \mathfrak{f}_2 = \mathrm{Lie}[\![a,b]\!].$$

Following Écalle, let ma denote the standard vector space isomorphism from $\mathbb{Q}\langle\langle C\rangle\rangle$ to the space $(Moulds)^{pol}$ defined by

$$\begin{aligned} ma : \mathbb{Q}\langle\langle C\rangle\rangle &\xrightarrow{\sim} (Moulds)^{pol} \\ c_{k_1}\cdots c_{k_r} &\mapsto (-1)^{k_1+\cdots+k_r-r}u_1^{k_1-1}\cdots u_r^{k_r-1} \end{aligned} \tag{4.7}$$

on monomials, extended by linearity to all power series.

It is well-known that $p \in \mathbb{Q}\langle\langle C\rangle\rangle$ satisfies the shuffle relations if and only if p is a Lie series, i.e. $p \in \mathrm{Lie}[\![C]\!]$. The alternality property on moulds is analogous to these shuffle relations, that is a series $p \in \mathbb{Q}\langle\langle C\rangle\rangle$ satisfies the shuffle relations if and only if $ma(p)$ is alternal (see e.g. [33], §2.3 and Lemma 3.4.1]). Writing ARI^{al} for the subspace of alternal moulds and ARI^{al}_{pol} for the subspace of alternal polynomial-valued moulds, this shows that the map ma restricts to a Lie algebra isomorphism

$$ma : \mathrm{Lie}[\![C]\!] \xrightarrow{ma} ARI^{al}_{lu,pol}.$$

Finally, we recall that for any mould $P \in ARI$, Écalle defines a derivation $arit(P)$ of the Lie algebra ARI_{lu}. We do not need to recall the definition of $arit$ here (but it is given in §4.4 below where we prove a technical lemma). For now it is enough to know that when restricted to polynomial-valued moulds, it is related to the Ihara derivations (3.16) via the morphism ma:

$$ma\big(D_f(g)\big) = -arit\big(ma(f)\big) \cdot ma(f).$$

For each $P \in ARI$, we also define the derivation

$$arat(P) = -arit(P) + \mathrm{ad}(P), \tag{4.8}$$

where $\mathrm{ad}(P) \cdot Q = lu(P,Q)$.

4.1.3. *Reminders on the elliptic double shuffle Lie algebra $\mathfrak{ds}_{ell}$.* We end this subsection by recalling the definition and a few facts about the elliptic double shuffle Lie algebra $\mathfrak{ds}_{ell}$ from [34].

Definition 4.1. The *elliptic double shuffle Lie algebra* $\mathfrak{ds}_{ell}$ is the subspace of $\mathfrak{f}_2$ such that

$$ma\big(\mathfrak{ds}_{ell}\big) = ARI^{\Delta\text{-}al*al}_{pol}, \tag{4.9}$$

i.e. $\mathfrak{ds}_{ell}$ consists of the Lie power series $f \in \mathfrak{f}_2$ such that $ma(f)$ is Δ-bialternal.

The following results are essentially contained in [?] and [34]. We give some details of the proofs for the convenience of the reader.

Proposition 4.2. *The space $\mathfrak{ds}_{ell}$ satisfies the following properties.*

(i) $\mathfrak{ds}_{ell} \subset \mathfrak{f}_2^{\mathrm{push}}$*, where* $\mathfrak{f}_2^{\mathrm{push}}$ *has been defined in Section 3.2;*
(ii) $\mathfrak{ds}_{ell}$ *is a Lie algebra under the bracket* $\langle\, , \rangle$ *on* $\mathfrak{f}_2^{\mathrm{push}}$ *defined in* (3.9)*.*
(iii) *There is a Lie algebra inclusion*

$$\widetilde{\mathfrak{grt}}_{ell} \subset \mathfrak{ds}_{ell},$$

where $\widetilde{\mathfrak{grt}}_{ell}$ *is the Lie subalgebra of* $\mathfrak{grt}_{ell}$ *generated by* $\gamma(\mathfrak{grt})$ *and* $\mathfrak{u}$*.*

Proof. For (i), by definition, elements of $ma(\mathfrak{ds}_{\mathrm{ell}})$ are alternal moulds whose swap is also alternal up to the addition of a constant-valued mould. It is shown in Lemma 2.5.5 of [33] that such moulds are push-invariant and Δ trivially respects push-invariance which shows that $\mathfrak{ds}_{\mathrm{ell}} \subset \mathfrak{f}_2^{\mathrm{push}}$.

For (ii), first note that $ARI^{\Delta\text{-}al*al}$ is a Lie algebra under the ari-bracket ([?], Theorem 3.3; note that Δ is denoted "sing" there). Therefore by definition, $\mathfrak{ds}_{ell} = \Delta(ARI^{\Delta\text{-}al*al})$ is a Lie algebra under the Dari-bracket, which is the transport of the ari-bracket via the map Δ, by Proposition 3.2.1 of [34]. But when restricted to polynomials, the Dari-bracket is nothing but the Lie-bracket of derivations, i.e. $\mathrm{Darit}(\mathrm{Dari}(P,Q)) = [\mathrm{Darit}(P), \mathrm{Darit}(Q)]$, so it is the same as the bracket of derivations in $\mathrm{Der}_0(\mathfrak{f}_2)$.

Finally, for (iii), we first show that both $\gamma(\mathfrak{grt})$ and $\mathfrak{u}$ lie in $\mathfrak{ds}_{\mathrm{ell}}$. The inclusion $\gamma(\mathfrak{grt}) \subset \mathfrak{ds}_{\mathrm{ell}}$ is Theorem 1.3.1 of [34]. For the other inclusion $\mathfrak{u} \subset \mathfrak{ds}_{\mathrm{ell}}$, it is shown in Corollary 3.6 of [?] that $\Delta^{-1} \circ ma$ gives an injective map from $\mathfrak{u}$ (which is called $\mathcal{E}$ in [?]) into $ARI^{\Delta\text{-}al/al}$ which is clearly equivalent to ma giving an injection from $\mathfrak{u}$ to $\mathfrak{ds}_{\mathrm{ell}} = \Delta(ARI^{\Delta\text{-}al/al})$. Finally, the semi-direct action is nothing other than the Dari-bracket, and we saw in (ii) that this is the same as the bracket of derivations in Der_0. □

Remark 4.3. In [6], a Lie algebra called $\mathfrak{pls}$ (for "polar linearized shuffle") is introduced, which is essentially equivalent to $\mathfrak{ds}_{ell}$. It is also shown that $\mathfrak{u}$ embeds into $\mathfrak{pls}$ ([6], Proposition 4.6) and, moreover, it is asked whether the equality $\mathfrak{u} = \mathfrak{pls}$ holds. Proposition 4.2.(iii) implies that $\mathfrak{ds}_{ell}$ is, in fact, much larger than $\mathfrak{u}$. More precisely, Enriquez ([14], §7) has shown that $\mathfrak{u}$ lies in the kernel of the surjection $\mathfrak{grt}_{ell} \to \mathfrak{grt}$ from which it follows that the image $\gamma(\mathfrak{grt}) \subset \widetilde{\mathfrak{grt}}_{ell}$ of $\mathfrak{grt}$ under the splitting γ is disjoint from $\mathfrak{u}$. In particular, the Lie algebra $\mathfrak{u}$ cannot equal $\mathfrak{ds}_{ell}$.

4.2. **The elliptic double shuffle relations.** We can now give the elliptic double shuffle property satisfied by the reduced elliptic generating series $\overline{E}(\tau)$. It is in fact phrased more directly as a property of the log power series

$$\mathfrak{e}(\tau) = \log_a\big(\overline{E}(\tau) - a + 1\big),$$

where $\log_a$ is the inverse of the exponential $\exp_a$ defined in (3.9), or rather, on the mould version of this power series

$$\mathfrak{e}_m(\tau) = ma\big(\mathfrak{e}(\tau)\big).$$

Theorem 4.4. *The mould $\mathfrak{e}_m(\tau)$ is Δ-bialternal, i.e. $\Delta^{-1}\big(\mathfrak{e}_m(\tau)\big)$ is a bialternal mould.*

Proof. Consider equation (3.27) from the proof of Theorem 3.6 in the case where $f = a$, $F = \overline{E}$, $F(\tau) = g(\tau) \cdot \overline{E}$. The left-hand side of (3.27) is equal to $\mathfrak{e}(\tau)$, so we find that $\mathfrak{e}(\tau) = r(\tau) \cdot a + \gamma(\phi_{\mathrm{KZ}}) \cdot a + s(\tau)$. Let $\mathfrak{e} = \gamma(\phi_{\mathrm{KZ}}) \cdot a$, so that $\mathfrak{e} \in v_a\big(\gamma(\mathfrak{grt})\big) \otimes_{\mathbb{Q}} \overline{\mathcal{E}}$. By the proof of Theorem 3.6, we have $r(\tau) \cdot a + s(\tau) \in V \otimes_{\mathbb{Q}} \overline{\mathcal{E}}$ where $V = v_a(\mathfrak{u}')$. Therefore, $\mathfrak{e}(\tau) \in \widetilde{\mathfrak{grt}}_{ell}$ by the definition of $\widetilde{\mathfrak{grt}}_{ell}$, and since $\widetilde{\mathfrak{grt}}_{ell} \subset \mathfrak{ds}_{ell}$ by Proposition 4.2 (iii), we also have $\mathfrak{e}(\tau) \in \mathfrak{ds}_{ell} \otimes_{\mathbb{Q}} \overline{\mathcal{E}}$, which proves the theorem thanks to (4.9). □

We conjecture that the elliptic double shuffle relations form a complete set of algebraic relations between the $\overline{E}$-elliptic multizetas. This statement really breaks down into two statements, one concerning the arithmetic part $\overline{\mathcal{Z}}$ of $\overline{\mathcal{E}}$ and the other the geometric part $\mathcal{E}^{\mathrm{geom}} = \mathcal{U}(\mathfrak{u})^\vee$. We show in Proposition 4.5 that indeed, the completeness follows from two conjectures: the first one a standard conjecture from multizeta theory, and the second a similar conjecture from elliptic multizeta theory. Due to the fact that it is much easier to work in the geometric situation than the arithmetic situation (as there are no problems of transcendence), we are actually able to prove that the elliptic double shuffle relations are complete in depth 2, without any recourse to conjectures (see Proposition 4.6).

The first conjecture amounts to the inclusions in (3.28) being all isomorphisms as well as the standard conjecture that the inclusion $\mathfrak{grt} \subset \mathfrak{ds}$ (proved by Furusho in [17]) is actually also an isomorphism. All these are implied by the simplified statement:

Conjecture 1: $$\mathfrak{nz}^\vee \cong \mathfrak{ds}.$$

This is the standard conjecture that the double shuffle relations suffice to generate all the algebraic relations satisfied by multiple zeta values [21].

The second conjecture amounts to the existence of a canonical semi-direct product structure on the elliptic double shuffle Lie algebra $\mathfrak{ds}_{ell}$. This is inspired by Enriquez result that the elliptic Grothendieck–Teichmüller Lie algebra $\mathfrak{grt}_{ell}$ is isomorphic to a semi-direct product $\mathfrak{r}_{ell} \rtimes \gamma(\mathfrak{grt})$ where $\mathfrak{r}_{ell}$ is a certain Lie ideal of $\mathfrak{grt}_{ell}$ containing $\mathfrak{u}$. Analogously, we have

Conjecture 2: $$\mathfrak{u} \rtimes \gamma(\mathfrak{ds}) \cong \mathfrak{ds}_{ell}.$$

This conjecture is closely related to Enriquez' "generation conjecture" for $\mathfrak{grt}_{ell}$ [14], §10, namely that $\mathfrak{u} \cong \mathfrak{r}_{ell}$. If Enriquez' conjecture were true, then the left hand side of our Conjecture 2 would be isomorphic to $\mathfrak{grt}_{ell}$, and Conjecture 2 would reduce to showing that $\mathfrak{grt}_{ell} \cong \mathfrak{ds}_{ell}$ which is the elliptic analog of the well-known conjecture $\mathfrak{grt} \cong \mathfrak{ds}$.

One can also merge Conjectures 1 and 2 into a single conjecture, thereby extending (3.28) to the elliptic setting. The elliptic analog of $\mathfrak{nmz}^\vee$ is the elliptic motivic fundamental Lie algebra, which is conjecturally isomorphic to its image $\Pi = V \rtimes \mathfrak{nmz}^\vee$ in the derivation algebra $\mathrm{Der}_0(\mathfrak{f}_2)$ (cf. the proof of Theorem 3.6). Then we get inclusions

$$V \rtimes \mathfrak{nz}^\vee \subset V \rtimes \mathfrak{nmz}^\vee \cong \Pi \subset \widetilde{\mathfrak{grt}}_{ell}, \tag{4.10}$$

which conjecturally are all equalities. Note that the first equality would also follow from Conjecture 1 above.

Proposition 4.5. *If Conjectures 1 and 2 are true, then the elliptic double shuffle relations generate all algebraic relations between elliptic multizetas.*

Proof. By Conjecture 1, we would have $\overline{\mathcal{Z}} \cong \mathcal{U}(\mathfrak{ds})^\vee$, so since $\mathcal{E}^{\mathrm{geom}} \cong \mathcal{U}(\mathfrak{u})^\vee \cong \mathcal{U}(V)^\vee$ by Theorem 2.6, we would have

$$\overline{\mathcal{E}}[2\pi i\tau] \cong \mathcal{U}(V)^\vee \otimes_{\mathbb{Q}} \mathcal{U}(\mathfrak{ds})^\vee.$$

It is known that the underlying vector space of the universal enveloping algebra $\mathcal{U}(R \rtimes L)$ of a semi-direct product of Lie algebras $R \rtimes L$ is the space $\mathcal{U}(R) \otimes_{\mathbb{Q}} \mathcal{U}(L)$; in fact $\mathcal{U}(R \rtimes L)$ is a Hopf algebra equipped with the smash product ([29]) and with the standard coproduct for which elements of $R \rtimes L$ are primitive. The dual $\mathcal{U}(R \rtimes L)^\vee$ has underlying $\mathbb{Q}$-algebra $\mathcal{U}(R)^\vee \otimes_{\mathbb{Q}} \mathcal{U}(L)^\vee$ (and is equipped with the smash coproduct).

Thus by Conjecture 2, we would have the isomorphism of $\mathbb{Q}$-algebras

$$\overline{\mathcal{E}}[2\pi i\tau] \cong \mathcal{U}(\mathfrak{u})^\vee \otimes_{\mathbb{Q}} \mathcal{U}(\mathfrak{ds})^\vee \cong \mathcal{U}(\mathfrak{ds}_{ell})^\vee.$$

Now, for any Lie algebra $\mathfrak{g}$ defined over $\mathbb{Q}$ and any $\mathbb{Q}$-algebra R, if f is an element of $\mathfrak{g} \otimes_{\mathbb{Q}} R$, then the subring of R generated by the coefficients of f (in a linear basis of $\mathfrak{g}$) generate a subring of R which is necessarily isomorphic to a quotient of $\mathcal{U}(\mathfrak{g})^\vee$; in other words, the coefficients of f satisfy relations that are imposed by the fact that f lies in the Lie algebra $\mathfrak{g}$, and possibly others. If this quotient is actually isomorphic to $\mathcal{U}(\mathfrak{g})^\vee$, this signifies that the coefficients do not satisfy any further algebraic relations than those imposed on them by the fact that f lies in $\mathfrak{g}$.

In our case, we have $\mathfrak{e}(\tau) \in \mathfrak{ds}_{ell} \otimes_{\mathbb{Q}} \overline{\mathcal{E}}$, and the coefficients of $\mathfrak{e}(\tau)$, together with $2\pi i\tau$, generate $\overline{\mathcal{E}}[2\pi i\tau]$, which by the conjectures is isomorphic to $\mathcal{U}(\mathfrak{ds}_{ell})^\vee$, implying that the coefficients of $\mathfrak{e}(\tau)$ do not satisfy any other algebraic relations than those imposed by the fact that $\mathfrak{e}(\tau)$ lies in $\mathfrak{ds}_{ell}$, i.e. is Δ-bialternal. $\square$

Explicit elliptic double shuffle relations. Let us take a closer look at what the Δ-bialternality properties are. The first property is that $\mathfrak{e}_m(\tau)$ is Δ-alternal, i.e. that $\Delta^{-1}(\mathfrak{e}_m(\tau))$ is alternal. But Δ trivially preserves alternality, so this is equivalent to saying that $\mathfrak{e}_m(\tau)$ is alternal, i.e. that for each $r > 1$,

$$(EDS.1) \qquad \sum_{u \in sh\big((u_1,\ldots,u_k),(u_{k+1},\ldots,u_r)\big)} \mathfrak{e}_m(\tau)(u) = 0$$

for $1 \le k \le [r/2]$. This condition is equivalent to the statement that the power series $\mathfrak{e}(\tau)$ is a Lie series.

The new relations on $\mathfrak{e}_m(\tau)$ are the second set, which say that up to adding on a constant-valued mould, the swap of the mould $\Delta^{-1}\big(\mathfrak{e}_m(\tau)\big)$ is also alternal, where the swap-operator is defined in (4.5). This alternality is given by the equalities for $r > 1$

$$(EDS.2) \qquad \sum_{v \in sh\big((v_1,\ldots,v_k),(v_{k+1},\ldots,v_r)\big)} swap\big(\Delta^{-1}\mathfrak{e}_m(\tau)\big)(v) = 0$$

for $1 \le k \le [r/2]$.

The swapped mould is given explicitly by

$$swap\big(\Delta^{-1}\mathfrak{e}_m(\tau)\big) = \frac{1}{v_1(v_1-v_2)\cdots(v_{r-1}-v_r)v_r}\,\mathfrak{e}_m(\tau)(v_r, v_{r-1}-v_r,\ldots,v_1-v_2).$$

Thus the alternality conditions in (EDS.2) are all sums of rational functions with denominators that are products of terms of the form v_i and $(v_i - v_j)$, which sum to zero. Therefore, by multiplying through by the common denominator

$$v_1\cdots v_r \prod_{i>j}(v_i - v_j),$$

the second elliptic shuffle equation can be expressed as a family of polynomial conditions on the mould $swap(\mathfrak{e}_m(\tau))$.

Elliptic double shuffle relations in depth 2. Let us work this out explicitly in depth 2. The usual alternality condition reduces to

$$(EDS.1\text{-depth } 2) \qquad \mathfrak{e}_m(\tau)(u_1,u_2) + \mathfrak{e}_m(\tau)(u_2,u_1) = 0.$$

The swap alternality condition reads

$$\frac{1}{v_1(v_1-v_2)v_2} swap(\mathfrak{e}_m(\tau))(v_1,v_2) + \frac{1}{v_1(v_2-v_1)v_2} swap(\mathfrak{e}_m(\tau))(v_2,v_1) = 0,$$

which, clearing denominators, reduces simply to

$$swap(\mathfrak{e}_m(\tau))(v_1,v_2) - swap(\mathfrak{e}_m(\tau))(v_2,v_1) = 0.$$

Since $swap(\mathfrak{e}_m(\tau))(v_1,v_2) = e_m(v_2, v_1 - v_2)$, this is given by the relation

$$\mathfrak{e}_m(\tau)(v_2, v_1 - v_2) = \mathfrak{e}_m(\tau)(v_1, v_2 - v_1)$$

directly on $\mathfrak{e}_m(\tau)$. Applying the depth 2 swap operator from $\overline{ARI}$ to ARI (given by $v_1 \mapsto u_1+u_2$, $v_2 \mapsto u_1$), we transform this relation into

$$\mathfrak{e}_m(\tau)(u_1,u_2) = \mathfrak{e}_m(\tau)(u_1+u_2,-u_2).$$

Finally, $\mathfrak{e}_m(\tau)$ is of odd degree, so by the depth 2 version of (EDS.1), we have $\mathfrak{e}_m(\tau)(-u_2,-u_1) = \mathfrak{e}_m(\tau)(u_1,u_2)$, which gives

$$\mathfrak{e}_m(\tau)(u_1,u_2) = \mathfrak{e}_m(\tau)(u_2,-u_1-u_2). \qquad (EDS.2\text{-depth } 2)$$

Note that this is nothing other than $\mathfrak{e}_m(\tau)(u_1,u_2) = push\big(\mathfrak{e}_m(\tau)\big)(u_1,u_2)$ where the push-operator is defined in (4.4). Thus in depth 2, the Δ-bialternality conditions correspond to alternality and push-invariance of $\mathfrak{e}_m(\tau)$ (which in turn correspond to the fact that $\mathfrak{e}(\tau)$ is a Lie series that is push-invariant in depth 2 in the sense of power series, as in (3.8)). This simple reformulation is special to depth 2; the Δ-bialternal property does not lend itself so easily to a direct expression as a property of $\mathfrak{e}(\tau)$ in higher depths.

We end this subsection by showing that the conjecture that the Δ-bialternal relations are sufficient holds in depth 2.

Proposition 4.6. *The relations (EDS.1) and (EDS.2) in odd degrees are the only relations satisfied by $\mathfrak{e}_m(\tau)$ in depth 2.*

Proof. We can prove this result without recourse to any conjectures, essentially because depth 2 is too small to contain any of the arithmetic part of $\mathfrak{e}_m(\tau)$ (we qualify this statement below), and the geometric part $V = v_a(\mathfrak{u})$ is well-understood in depth two. We know that $\mathfrak{e}(\tau) \in \mathfrak{ds}_{ell} \subset \mathfrak{f}_2^{push}$. The graded dimensions of $\mathfrak{f}_2$ in depth 2 are given by

$$dim(\mathfrak{f}_2^{\mathrm{push}})_n^2 = \left\lfloor \frac{n-5}{6} \right\rfloor + 1. \tag{4.11}$$

Now the depth two part of $\mathfrak{ds}_{ell} \supset V \rtimes \gamma(\mathfrak{nz}^\vee)$ is contained in the depth two part of V, since $\gamma(\mathfrak{nz}^\vee)$ is of depth ≥ 3. Thus

$$\dim\big(\mathfrak{ds}_{ell}\big)_n^2 = \dim V_n^2 = \begin{cases} \left\lfloor \frac{n-5}{6} \right\rfloor + 1 & \text{if } n \text{ is odd} \geq 5 \\ 0 & \text{otherwise.} \end{cases} \tag{4.12}$$

Indeed, the last equality follows from the fact that in depth 2, V is spanned by the $[\varepsilon_{2j},\varepsilon_{2k}](a)$ with $j<k$, $j,k \neq 1$, which are all of odd weight, and the fact that, as shown in [30], the only relations between these $\left\lfloor \frac{n-3}{4} \right\rfloor$ brackets come from period polynomials, whose number is given by $\left\lfloor \frac{n-7}{4} \right\rfloor - \left\lfloor \frac{n-5}{6} \right\rfloor$. Thus $V^2 = \mathfrak{ds}_{ell}^2 = (\mathfrak{f}_2^{push})^2$, so the Lie relation (EDS.1) and the push-invariance relation (EDS.2) suffice to characterize elements of $\mathfrak{ds}_{ell}$ in depth 2. □

Depth 2 elements of $\mathfrak{ds}_{ell}$ in low weights:

- in weight 5,

$$ma\big([\varepsilon_0,\varepsilon_4](a)\big) = 2u_1^3 + 3u_1^2u_2 - 3u_1u_2^2 - 2u_2^3.$$

- in weight 7,

$$ma\big([\varepsilon_0,\varepsilon_6](a)\big) = 2u_1^5 + 5u_1^4u_2 + 2u_1^3u_2^2 - 2u_1^2u_2^3 - 5u_1u_2^4 - 2u_2^5.$$

- in weight 9,

$$ma\big([\varepsilon_0,\varepsilon_8](a)\big) = 2u_1^7 + 7u_1^6u_2 + 9u_1^5u_2^2 + 5u_1^4u_2^3 - 5u_1^3u_2^4 - 9u_1^2u_2^5 - 7u_1u_2^6 2u_2^7.$$

- in weight 11,

$$ma\big([\varepsilon_0,\varepsilon_{10}](a)\big) = 8u_1^9 + 36u_1^8u_2 + 74u_1^7u_2^2 + 91u_1^6u_2^3 + 41u_1^5u_2^4 - 41u_1^4u_2^5 - 91u_1^3u_2^6 - 74u_1^2u_2^7 - 36u_1u_2^8 - 8u_2^9$$

$$ma\big([\varepsilon_4,\varepsilon_6](a)\big) = -2u_1^7u_2^2 - 7u_1^6u_2^3 - 5u_1^5u_2^4 + 5u_1^4u_2^5 + 7u_1^3u_2^6 + 2u_1^2u_2^7.$$

4.3. **The elliptic associator and the push-neutrality relations mod $2\pi i$.**

Definition 4.7. Let $\mathfrak{a}$ be the power series with coefficients in $\overline{\mathcal{Z}}$ given by

$$\mathfrak{a} = \frac{1}{2\pi i}\log(A) \bmod 2\pi i = \log(\overline{A'}) = \overline{\Phi}_{\mathrm{KZ}}(t_{01},t_{12})t_{01}\overline{\Phi}_{\mathrm{KZ}}(t_{01},t_{12})^{-1},$$

and let $\mathfrak{a}(\tau) = g(\tau)\cdot\mathfrak{a}$.

The coefficients of $\mathfrak{a}(\tau)$ generate the $\mathbb{Q}$-algebra $\overline{\mathcal{A}}$ of $\overline{A}$-EMZs. In this paragraph we will consider certain relations satisfied by the coefficients of $\mathfrak{a}(\tau)$, different from the linearized elliptic double shuffle relations satisfied by $\mathfrak{e}(\tau)$. The first family of relations on the coefficients of $\mathfrak{a}(\tau)$ is the usual family of *alternality* relations, but the second is the family of *push-neutrality* relations. These relations are related (mod $2\pi i$) to the *Fay-shuffle relations* introduced in [27], and studied explicitly in depth 2. We show that in depth 2, the push-neutrality relations are identical to the Fay-shuffle relations. We also show that even in depth 2 and mod $2\pi i$, the alternality and push-neutrality relations are strictly weaker than the linearized elliptic double shuffle relations.

We will give our relations in terms of mould theory (but see Corollary 4.11 for a translation into power series terms at the end). For this we recall the *push* and *dar*-operators defined in (4.4) and (4.3). We will say that a mould B is *push-neutral* if

$$B(u_1,\ldots,u_r) + push(B)(u_1,\ldots,u_r) + \cdots + push^r(B)(u_1,\ldots,u_r) = 0 \qquad (4.13)$$

for all $r \geq 1$, where *push* denotes the push-operator on moulds defined in (4.4).

Theorem 4.8. *Let $\mathfrak{a}_m(\tau) = ma\big(\mathfrak{a}(\tau)\big)$. Then $\mathfrak{a}_m(\tau)$ is alternal and $dar^{-1}\big(\mathfrak{a}_m(\tau)\big)$ is push-neutral in depth $r > 1$.*

Proof. Recall the derivation *arat* defined in (4.8). For any $P \in ARI$, set

$$\mathrm{Darit}(P) = dar \circ arat\big(\Delta^{-1}(P)\big) \circ dar^{-1}. \qquad (4.14)$$

It is shown in [34], Lemma 3.1.2, that the map

$$\mathrm{Der}_0(\mathfrak{f}_2) \hookrightarrow \mathrm{Der}(ARI_{lu})$$
$$D \mapsto \mathrm{Darit}\big(ma(v_a(D))\big) \qquad (4.15)$$

is an injective Lie morphism, so that we have

$$ma\big(D(f)\big) = \mathrm{Darit}\big(ma(v_a(D))\big)\cdot ma(f). \qquad (4.16)$$

Let $\mathfrak{a}_m = ma(\mathfrak{a})$, $\mathfrak{a}_m(\tau) = ma\big(\mathfrak{a}(\tau)\big)$, and $r_m(\tau) = ma\big(r_a(\tau)\big)$. Under the map (4.15), we have $r(\tau) \mapsto \mathrm{Darit}\big(r_m(\tau)\big)$, so

$$ma\big(r(\tau)\cdot\mathfrak{a}\big) = \mathrm{Darit}\big(r_m(\tau)\big)\cdot\mathfrak{a}_m.$$

Since

$$\mathfrak{a}(\tau) = g(\tau)\cdot\mathfrak{a} = \sum_{n\geq 0}\frac{1}{n!}r(\tau)^n\cdot\mathfrak{a}, \qquad (4.17)$$

we have

$$\mathfrak{a}_m(\tau) = \sum_{n\geq 0} \frac{1}{n!}\mathrm{Darit}\big(r_m(\tau)\big)^n \cdot \mathfrak{a}_m. \tag{4.18}$$

Let $\overline{\sigma}$ denote the automorphism of $\mathfrak{f}_2$ defined in §3.2. We have

$$\mathfrak{a} = \overline{\sigma}(t_{01}).$$

Recall from §3.2 that $\overline{\sigma} = \gamma(\phi_{\mathrm{KZ}})$, where $\phi_{\mathrm{KZ}} = \log_a\big(\overline{\Phi}_{\mathrm{KZ}}\big)$.

The derivation $\gamma(\phi_{\mathrm{KZ}})$ lies in $\mathrm{Der}_0(\mathfrak{f}_2)$, so $\gamma(\phi_{\mathrm{KZ}}) \cdot t_{01} \in \mathfrak{f}_2$; thus $\mathfrak{a}$ is a Lie series. Since $r(\tau) \in \mathrm{Der}_0(\mathfrak{f}_2)$, we have $r(\tau)^n \cdot \mathfrak{a} \in \mathfrak{f}_2$ for all $n \geq 1$, so by (4.17), $\mathfrak{a}(\tau) = g(\tau) \cdot \mathfrak{a} \in \mathfrak{f}_2$, which means that $\mathfrak{a}_m(\tau)$ is alternal. This settles the first property of $\mathfrak{a}_m(\tau)$ stated in the theorem.

Let us consider the second property. Since $\gamma(\phi_{\mathrm{KZ}}) \in \mathrm{Der}_0(\mathfrak{f}_2)$, it annihilates t_{12}. Therefore, setting $t'_{01} = t_{01} + \frac{1}{2}t_{12}$, we have

$$\mathfrak{a} = \gamma(\phi_{\mathrm{KZ}}) \cdot t_{01} = \gamma(\phi_{\mathrm{KZ}}) \cdot t'_{01}. \tag{4.19}$$

Set $T'_{01} = ma(t'_{01})$, and set

$$\mathfrak{z} = ma\Big(v_a\big(\gamma(\phi_{\mathrm{KZ}})\big)\Big) = ma\big(\gamma_a(\phi_{\mathrm{KZ}})\big).$$

Then by (4.16), the equality (4.19) translates into moulds as

$$\mathfrak{a}_m = \mathrm{Darit}(\mathfrak{z}) \cdot T'_{01}.$$

To complete the proof of the second property, we will use the following lemma, whose proof is deferred to the final subsection of this paper.

Lemma 4.9. *Let $A \in ARI$. If A is push-neutral, then $arat(P) \cdot A$ is push-neutral for all $P \in ARI$. If $dar^{-1}A$ is push-neutral, then $dar^{-1} \cdot \mathrm{Darit}(P) \cdot A$ is push-neutral for all $P \in ARI$.*

It is easy to see that if A is a push-invariant mould, then $dar^{-1}A$ is push-neutral, since

$$dar^{-1}(A)(u_1,\ldots,u_r) + push\big(dar^{-1}(A)\big)(u_1,\ldots,u_r) + \cdots + push^r\big(dar^{-1}(A)\big)(u_1,\ldots,u_r)$$

$$= \left(\frac{1}{u_1\cdots u_r} + \frac{1}{u_2\cdots u_0} + \cdots + \frac{1}{u_0u_1\cdots u_{r-1}}\right)A(u_1,\ldots,u_r)$$

$$= \left(\frac{u_0 + u_1 + \cdots + u_r}{u_0u_1\cdots u_r}\right)A(u_1,\ldots,u_r) = 0,$$

where $u_0 = -u_1 - \cdots - u_r$. By Proposition 4.10 below, $dar^{-1}T'_{01}$ is push-neutral and by Lemma 4.9, so is

$$dar^{-1}\mathfrak{a}_m = dar^{-1} \cdot \mathrm{Darit}(\mathfrak{z}) \cdot T'_{01}.$$

To show that $dar^{-1}\mathfrak{a}_m(\tau)$ is push-neutral we use the same lemma again. Since $dar^{-1}\mathfrak{a}_m$ is push-neutral, so is $dar^{-1} \cdot \mathrm{Darit}\big(r_m(\tau)\big) \cdot \mathfrak{a}_m$, and then successively, so is $dar^{-1} \cdot \mathrm{Darit}\big(r_m(\tau)\big)^n \cdot \mathfrak{a}_m$ for all $n \geq 1$. Thus $dar^{-1}\mathfrak{a}_m(\tau)$ is push-neutral by (4.18). This proves the theorem. □

The following proposition was used in the proof of Theorem 4.8.

Proposition 4.10. *The mould*

$$ma([t'_{01},a]) = -\sum_{n=2}^{\infty} \frac{B_n}{n!} ma([\mathrm{ad}^n(b)(a),a]) \tag{4.20}$$

is push-neutral.

Proof. It is enough to show the push-neutrality of $f_n := ma([\mathrm{ad}^n(b)(a),a])$ for all $n \geq 2$ separately. Using the definition of ma (cf. Section 4.1), we see that

$$ma(\mathrm{ad}^n(b)(a)) = -\sum_{k=1}^{n} (-1)^{n-k} \binom{n-1}{k-1} u_k \in \mathbb{Q}[u_1,\ldots,u_n]. \tag{4.21}$$

Now in depth n, the operator $\mathrm{ad}(a)$ on $\mathbb{Q}\langle\langle C\rangle\rangle$ corresponds to multiplication by $-(u_1+\ldots+u_n)$. Consequently,

$$\begin{aligned} ma([\mathrm{ad}^n(b)(a),a]) &= -ma([a,\mathrm{ad}^n(b)(a)]) \\ &= -(u_1+\ldots+u_n)\sum_{k=1}^{n}(-1)^{n-k}\binom{n-1}{k-1}u_k \\ &= -\sum_{j,k=1}^{n}(-1)^{n-k}\binom{n-1}{k-1}u_j u_k. \end{aligned} \tag{4.22}$$

On the other hand, by the definition of the push-operator (4.4), we have $push(f_n) = -\sum_{j,k=1}^{n}(-1)^{n-k}\binom{n-1}{k-1}u_{j+1}u_{k+1}$, where the indices are to be taken mod n (so that $u_{k+n} = u_k$). Using the elementary fact that $\sum_{k=1}^{n}(-1)^{n-k}\binom{n-1}{k-1} = 0$ for $n \geq 2$, it is now clear that

$$\sum_{i=0}^{n-1} push^i(f_n) = 0, \tag{4.23}$$

i.e. f_n is push-neutral for all $n \geq 2$, as was to be shown. □

We end this subsection by studying these relations more explicitly in depth 2 and comparing them with the elliptic double shuffle relations on $\mathfrak{e}_m(\tau)$. The alternality relation is of course the same:

$$(FS.1) \qquad \mathfrak{a}_m(\tau)(u_1,u_2) + \mathfrak{a}_m(\tau)(u_2,u_1) = 0.$$

The push-neutrality relation in depth 2 is given by

$$(FS.2) \qquad \frac{1}{u_1u_2}\mathfrak{a}_m(\tau)(u_1,u_2) + \frac{1}{u_2u_0}\mathfrak{a}_m(\tau)(u_2,u_0) + \frac{1}{u_0u_1}\mathfrak{a}_m(\tau)(u_0,u_1) = 0$$

where $u_0 = -u_1 - u_2$. Multiplying by the common denominator $u_0u_1u_2$ yields the polynomial relation

$$u_0\mathfrak{a}_m(\tau)(u_1,u_2) + u_1\mathfrak{a}_m(\tau)(u_2,u_0) + u_2\mathfrak{a}_m(\tau)(u_0,u_1) = 0.$$

It was shown in [27] Theorem 3.11, that the dimension of the space of polynomials in u_1, u_2 of odd degree d satisfying (FS.1) and (FS.2) is given by $\lfloor \frac{d}{3} \rfloor + 1$. In terms of the weight $n = d+2$ of the corresponding polynomials in $\mathfrak{f}_2$, this is

$$\left\lfloor \frac{n-2}{3} \right\rfloor + 1.$$

In weight 5, for example, there are two independent such polynomials:

$$u_1^2u_2 - u_1u_2^2 \quad \text{and} \quad u_1^3 - u_2^3.$$

In weight 7, there are again two independent polynomials, given by

$$u_1^4u_2 - u_1u_2^4 \quad \text{and} \quad u_1^5 + u_1^3u_2^2 - u_1^2u_2^3 - u_2^5.$$

In weight 9, the space is three-dimensional, given by

$$u_1^7 - 2u_1^4u_2^3 + 2u_1^3u_2^4 - u_2^7$$
$$u_1^6u_2 - u_1u_2^6$$
$$u_1^5u_2^2 + u_1^4u_2^3 - u_1^3u_2^4 - u_1^2u_2^5.$$

Finally, we work out the case of weight 11, where the dimension is four:

$$u_1^9 + 3u_1^5u_2^4 - u_1^4u_2^5 - u_2^9$$
$$u_1^8u_2 - u_1u_2^8$$
$$u_1^7u_2^2 - u_1^5u_2^4 + u_1^4u_2^5 - u_1^2u_2^7$$
$$u_1^6u_2^3 + u_1^5u_2^4 - u_1^4u_2^5 - u_1^3u_2^6$$

Observe that these dimensions are significantly bigger than those given by the elliptic double shuffle equations (EDS.1) and (EDS.2) in depth 2. This is explained by the fact that the vector space generated by the coefficients of $\mathfrak{a}_m(\tau)$ in a given weight and depth is not equal to the one generated by the analogous coefficients of $\mathfrak{e}_m(\tau)$, or in terms of the algebras, that while $\overline{A}[2\pi i\tau] = \overline{\mathcal{E}}[2\pi i\tau]$ by virtue of Corollary **??**, the $\mathbb{Q}$-algebras $\overline{\mathcal{E}}$ and $\overline{\mathcal{A}}$ are quite different and do not even have the same graded dimensions.

Under the conjecture $\overline{\mathcal{Z}} \cong \mathcal{U}(\mathfrak{grt})^\vee$, the $\mathbb{Q}$-algebra $\overline{\mathcal{E}}$ is isomorphic to $\mathcal{U}(\mathfrak{grt}_{ell})^\vee$, and thus inherits a natural bigrading dual to that of $\mathfrak{grt}_{ell}$. Together with products of elements of $\overline{\mathcal{E}}$ of smaller depth and weight (including $\mathcal{G}_0$), the coefficients of $\mathfrak{e}_m(\tau)$ in a given weight n and depth d span the bigraded part $\overline{\mathcal{E}}_n^d$, whereas those of $\mathfrak{a}_m(\tau)$ do not.

For example, in weight 5 and depth 2, the coefficients of $\mathfrak{e}_m(\tau)$ generate the 1-dimensional space $\langle 2\mathcal{G}_{0,4} + \mathcal{G}_0\mathcal{G}_4\rangle$. The bigraded subspace $\overline{\mathcal{E}}_5^2$ is spanned by $\mathcal{G}_2^2$, $\mathcal{G}_0\mathcal{G}_4$ and $\mathcal{G}_{0,4}$, but it is also spanned by the two products $\mathcal{G}_2^2$ and $\mathcal{G}_0\mathcal{G}_4$ and the single coefficient $2\mathcal{G}_{0,4} + \mathcal{G}_0\mathcal{G}_4$ of $\mathfrak{e}_m(\tau)$ in weight 5 and depth 2.

The weight 5, depth 2 coefficients of $\mathfrak{a}_m(\tau)$, however, do not lie in $\overline{\mathcal{E}}_5^2$. They span the 2-dimensional subspace $\langle -\frac{1}{12}\mathcal{G}_0\mathcal{G}_2 + \frac{3}{2}\mathcal{G}_0\mathcal{G}_4 + 3\mathcal{G}_{0,4} - \frac{1}{360}\mathcal{G}_0^2 + \frac{1}{2}\mathcal{G}_2^2, \frac{1}{240}\mathcal{G}_0^2 - 2\mathcal{G}_{0,4} - \mathcal{G}_0\mathcal{G}_4\rangle$ of $\overline{\mathcal{E}}$.

We end this subsection with a power series statement of the alternality and push-neutrality relations on $\mathfrak{a}_m(\tau)$.

Corollary 4.11. *The power series $\mathfrak{A} = [a, \mathfrak{a}(\tau)]$ is push-neutral in the sense that, if $\mathfrak{A}^r$ denotes the depth r part of $\mathfrak{A}$ for $r > 1$, then*

$$A^r + push(A^r) + \cdots + push^r(A^r) = 0$$

where push denotes the push-operator on power series defined in (3.8).

Proof. By Theorem 4.8, the mould $dar^{-1}\mathfrak{a}_m(\tau)$ is push-neutral. Consider the operator

$$-\Delta(A)(u_1, \ldots, u_r) = u_1 \cdots u_r(-u_1 - \ldots - u_r)A(u_1, \ldots, u_r).$$

Since the factor $u_1 \ldots u_r(-u_1 - \ldots - u_r)$ is push-invariant, the mould $-\Delta(A)$ is push-neutral if A is. Therefore in particular $-\Delta\big(dar^{-1}\mathfrak{a}_m(\tau)\big)$ is push-neutral. But this mould is given by

$$\begin{aligned}-\Delta\big(dar^{-1}\mathfrak{a}_m(\tau)\big)(u_1,\ldots,u_r) &= -(u_1+\cdots+u_r)\,\mathfrak{a}_m(\tau)(u_1,\ldots,u_r)\\ &= ma\big([a,\mathfrak{a}(\tau)]\big)(u_1,\ldots,u_r),\end{aligned}$$

where the last equality is a standard identity (see Appendix A of [31] or (3.3.1) of [33]). Therefore the mould $ma([a,\mathfrak{a}(\tau)])$ is a push-neutral mould, i.e. $[a,\mathfrak{a}(\tau)]$ is push-neutral as a power series. □

4.4. **Proof of Lemma 4.9.** In order to prove this lemma, we need to have recourse to the complete formula for the action of $arat$. We first recall Écalle's formula for $arit$ (cf. [13] or [33]), which is given as

$$\big(arit(P)\cdot A\big)(w) = \sum_{\substack{w=abc\\ c\neq\emptyset}} A(a\lceil c)P(b) - \sum_{\substack{w=abc\\ a\neq\emptyset}} A(a\rceil c)P(b),$$

where if the word $u=(u_1,\ldots,u_r)$ is decomposed into three chunks as $u=abc$, $a=(u_1,\ldots,u_i)$, $b=(u_{i+1},\ldots,u_{i+j})$, $c=(u_{i+j+1},\ldots,u_r)$, then we use Écalle's notation

$$\begin{aligned}a\rceil &= (u_1,\ldots,u_{i-1},u_i+u_{i+1}+\cdots+u_{i+j})\\ \lceil c &= (u_{i+1}+\cdots+u_{i+j+1},u_{i+j+2},\ldots,u_r).\end{aligned}$$

Moreover

$$\mathrm{ad}(P)\cdot A = mu(P,A) - mu(A,P)$$

where mu is the mould multiplication defined in (4.1); these correspond precisely to the 'missing' terms $a=\emptyset$ and $c=\emptyset$, so that $arat(P)\cdot A$ actually has the simpler expression

$$\big(arat(P)\cdot A\big)(w) = \sum_{w=abc} \big(A(a\lceil c)P(b) - A(a\rceil c)P(b)\big). \tag{4.24}$$

Now let A be push-neutral, and let $P\in ARI$. We need to show that (4.24) is push-neutral. In fact we will show that the two terms

$$\sum_{w=abc} A(a\lceil c)P(b) \quad\text{and}\quad \sum_{w=abc} A(a\rceil c)P(b) \tag{4.25}$$

of (4.24) are separately push-neutral.

Because the push-neutrality relations take place in fixed depth, we may assume that A is concentrated in depth s and P in depth t, with $s+t=r$. We will prove the push-neurality of the first term in (4.25); the proof for the second term is completely analogous.

Therefore the decompositions $w=abc$ we need to consider are those of the form

$$w = abc = (u_1,\ldots,u_i)(u_{i+1},\ldots,u_{i+t})(u_{i+t+1},\ldots,u_r),$$

and we can rewrite the first term of (4.25) as

$$\sum_{i=0}^{r-t} A(u_1,\ldots,u_i,u_{i+1}+\cdots+u_{i+t+1},u_{i+t+2},\ldots,u_r)P(u_{i+1},\ldots,u_{i+t}).$$

The k-th power of the push-operator acts by $u_i\mapsto u_{i-k}$, with indices considered modulo $(r+1)$. The push-neutrality condition thus reads

$$\sum_{k=0}^{r}\sum_{i=0}^{r-t} A(u_{1-k},\ldots,u_{i-1-k},u_{i-k},u_{i+1-k}+\cdots+u_{i+t+1-k},u_{i+t+2-k},\ldots,u_{r-k})$$

$$\cdot P(u_{i+1-k},\dots,u_{i+t-k})=0.$$

We will show that the coefficients of each term $P(u_{m+1},\dots,u_{m+t})$ sums to zero due to the push-neutrality of A. In fact it is enough to show that the coefficient of $P(u_1,\dots,u_t)$ sums to zero, as all the other terms are obtained from this one by applying powers of the push-operator.

The terms containing $P(u_1,\dots,u_t)$ are those for which the index $k=i$, so that $k\in\{0,\dots,r-t=s\}$, and we must show that the sum

$$\sum_{k=0}^{s} A(u_{r-k+2},\dots,u_r,u_0,u_1+\cdots+u_{t+1},u_{t+2},\dots,u_{r-k})$$

vanishes, where $u_0=-u_1-\cdots-u_r$ and we have shifted some of the indices modulo $(r+1)$ in order to make them positive. Note now that

$$u_1+\cdots+u_{t+1}=-u_0-u_{t+2}-\cdots+u_r.$$

As a result the last sum runs over the $(s+1)$ cyclic permutations of $u_{t+2},\dots,u_r,u_0$ and $-u_{t+2}-\cdots-u_r-u_0$, so it is equal to the sum over the push_s-orbit of just one term, say the one with $k=s$, i.e. to

$$\sum_{k=0}^{s} A(u_{t+2},\dots,u_r,u_0),$$

which indeed vanishes since A is push-neutral. This concludes the proof of Lemma 4.9. □

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