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Appendix

The action of the absolute Galois group on the moduli space of spheres with four marked points

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A large part of this volume is devoted to the theory of "dessins d'enfants" which is concerned with the action of (finite quotients of) the absolute Galois group on (finite quotients of) $\hat{F}_2$, the profinite completion of the free group on two generators. This last group appears as the algebraic fundamental group of $\mathbb{P}^1\overline{\mathbb{Q}}\setminus\{0,1,\infty\}$, which is the moduli space of spheres with 4 ordered marked points. As explained in paragraph 2 of Grothendieck's *Esquisse d'un programme*, this is but the first nontrivial instance of the action of the absolute Galois group on the fundamental group of a moduli space $\mathcal{M}_{g,n}$ (genus g, n marked points), namely here $\mathcal{M}_{0,4}$. We refer to Grothendieck's grandiose sketch for a – very – broad perspective. We only note that in genus 0, these fundamental groups are quotients of the Artin braid groups, and that the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ lifts to an action on the profinite completions of the braid groups. This has been interpreted in a strikingly different way by Drinfeld in [Dr]; we refer to the article by Lochak and Schneps in this volume, and also to [Ih-Ma], for more details.

Our aim in this appendix is to provide the necessary setting for a concrete study of the action of the Galois group on $\pi_1(\mathbb{P}^1\overline{\mathbb{Q}}\setminus\{0,1,\infty\})$. The main properties are summarized in Theorem 1 below. Although Theorem 1 is no more than Ihara's Proposition 1.6 and I and II of Theorem 1.7 from the main body of this article, we have put a particular emphasis on proving the results from scratch, recalling the definitions and constructions of the basic objects (such as tangential base points for example), and in particular, seeking to lay the groundwork for generalizations to the study of the higher dimensional moduli spaces, such as is done by Ihara in the main body of this paper for the next simplest case, namely the two-dimensional moduli space of spheres with 5 marked points. In studying the one dimensional situation, we follow Ihara's sketch in [Ih] (sections 2,3). The concrete realisation and use of tangential base points by means of convergent Puiseux series was introduced by Anderson and Ihara in [A-I] and we note that it allows for implementation on a computer and actual computations of dessins (see the paper by J.-M.Couveignes and L.Granboulan in this volume).

The Galois action on $\pi_1(\mathbb{P}^1\overline{\mathbb{Q}}\setminus\{0,1,\infty\})$. For a field k, we write $X_4(k) := \mathbb{P}^1k\setminus\{0,1,\infty\}$; similarly $X_n(k)$ will denote the moduli space over k of spheres with n ordered marked points. Let $\mathbb{\Gamma} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, as in Grothendieck's *Esquisse.*

Recall a basic property of the algebraic fundamental group (cf. [SGA 1]): given a smooth absolutely irreducible variety X over a field k of characteristic 0, we have the exact sequence of algebraic fundamental groups:

$$(1) \qquad 1 \to \pi_1(X \times_k \overline{k}) \to \pi_1(X) \to \mathrm{Gal}(\overline{k}/k) \to 1,$$

where $\overline{k}$ denotes an algebraic closure of k. When $k = \mathbb{Q}$, $\pi_1(X \times_k \overline{k})$ coincides with the profinite completion of $\pi_1^{\mathrm{top}}(X(\mathbb{C}))$, the topological fundamental group of the analytic variety $X(\mathbb{C})$, where we choose an embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ (we write π_1 for the algebraic fundamental group, π_1^{top} for the topological fundamental group – when it exists). So we get in particular:

$$(2) \qquad 1 \to \hat{F}_2 \to \pi_1(\mathbb{P}^1\mathbb{Q} \setminus \{0, 1, \infty\}) \to \mathbb{I}\Gamma \to 1,$$

because $\pi_1^{\mathrm{top}}(X_4(\mathbb{C})) \simeq F_2$, the free group on two generators. More explicitly, $\pi_1^{\mathrm{top}}(X_4(\mathbb{C}))$ is generated by x and y where x (resp. y) is a closed loop around 0 (resp. 1); this is made more precise below. So here $F_2 = \langle x, y\rangle$ and we write the elements of $\hat{F}_2$, the profinite completion of F_2, as "prowords" $f(x, y)$. This notation allows in particular to make sense of substitutions $(x, y) \mapsto (\xi, \eta)$, where ξ and η are elements of any profinite group. As does any exact sequence, (2) defines an outer action of $\mathbb{I}\Gamma$ on $\hat{F}_2$, that is a map $\mathbb{I}\Gamma \to \mathrm{Out}(\hat{F}_2)$. But (2) is actually split, and by choosing a splitting (see below), we can define a map $\mathbb{I}\Gamma \to \mathrm{Aut}(\hat{F}_2)$; if $\sigma \in \mathbb{I}\Gamma$, we write $\phi_\sigma \in \mathrm{Aut}(\hat{F}_2)$ for the corresponding automorphism or simply $\sigma \cdot \gamma := \phi_\sigma(\gamma)$ for the action of σ on $\gamma \in \hat{F}_2$. We may now state

Theorem 1: *There exists a faithful action of* $\mathbb{I}\Gamma = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ *on* $\hat{F}_2$*, that is, an injective map*

$$\sigma \in \mathbb{I}\Gamma \to \phi_\sigma \in \mathrm{Aut}(\hat{F}_2).$$

Writing $F_2 = \langle x, y\rangle$*, we have:*

$$(3) \qquad \phi_\sigma(x) = x^{\chi(\sigma)},\ \phi_\sigma(y) = f_\sigma^{-1}(x, y)\, y^{\chi(\sigma)} f_\sigma(x, y);$$

here $\chi : \mathbb{I}\Gamma \to \hat{\mathbb{Z}}^\times$ *denotes the cyclotomic character and* $f_\sigma \in \hat{F}_2$*. Actually* $f_\sigma \in \hat{F}_2'$*, the derived group of* $\hat{F}_2$*, so that we have an injective map:*

$$(4) \qquad \sigma \in \mathbb{I}\Gamma \to (\chi(\sigma), f_\sigma) \in \hat{\mathbb{Z}}^\times \times \hat{F}_2'.$$

Moreover, f_σ *satisfies the following two equations:*

$$\text{(I)}\ f_\sigma(x, y) f_\sigma(y, x) = 1$$

$$\text{(II)}\ f_\sigma(z, x) z^m f_\sigma(y, z) y^m f_\sigma(x, y) x^m = 1$$

where $z = (xy)^{-1}$ *and* $m = m(\sigma) = \frac{1}{2}(\chi(\sigma) - 1)$*.*

The rest of this appendix is essentially devoted to introducing the basic notions behind Ihara's proof of this theorem. Let $\tilde{X}_4$ be the (unique up to isomorphism) universal covering of X_4; to the family of finite coverings of X_4, which appear as quotients of $\tilde{X}_4$, there corresponds a sequence of extensions of function fields

$$\mathbb{Q}(T) \hookrightarrow \overline{\mathbb{Q}}(T) \hookrightarrow \Omega \tag{5}$$

where Ω is the maximal extension of $\overline{\mathbb{Q}}(T)$ unramified outside $\{0, 1, \infty\}$; actually (2) is nothing but the sequence of Galois groups corresponding to this sequence of extensions. We let $Ext(\overline{\mathbb{Q}}(T))$ denote the set – actually the category – of the finite extensions of $\overline{\mathbb{Q}}(T)$ unramified outside $0, 1, \infty$; they correspond to the finite connected unramified algebraic coverings of X_4.

Tangential base points (following Anderson-Ihara [A-I]). We shall first interpret the *base points* of the fundamental groups (which have been purposefully, if hastily, omitted from (1)), in terms of function fields. Let $t_0 \in \mathbb{Q} \setminus \{0, 1\} = X_4(\mathbb{Q})$ be a rational point, to be taken as base point (in the general case, fix a rational point of $X_n(\mathbb{Q})$). Although we shall ultimately need that $t_0 \in \mathbb{Q}$, we first work geometrically, i.e. over $\mathbb{C}$. Let $Y(\mathbb{C})$ be a finite connected unramified covering of X_4, F its function field, p the projection $Y \to X_4$. Identifying $\mathbb{Q}(X_4)$ with $\mathbb{Q}(t)$ (and thus $\mathbb{C}(X_4)$ with $\mathbb{C}(t)$), to every point $y_0 \in p^{-1}(t_0)$, the fibre over t_0, we now associate an embedding $\phi : F = \mathbb{C}(Y) \hookrightarrow \mathbb{C}((t - t_0))$, the formal series in $t - t_0$. Indeed, any $f \in F$ can be viewed locally (in the analytic sense) as a function of $(t - t_0)$, through the local isomorphism p (between a neighbourhood of y_0 and one of t_0); it is uniquely determined by its Laurent expansion at any such point $y_0 \in Y$ and fixing y_0 determines such an embedding, where the image of F is actually contained in the ring of *convergent* Laurent series, of which $\mathbb{C}((t - t_0))$ is the completion (with respect to the obvious valuation). Also, if $f \in \overline{\mathbb{Q}}(Y)$, $\phi(f) \in \overline{\mathbb{Q}}((t - t_0))$.

The – discrete – group $\pi_1^{\mathrm{top}}(X_4; t_0) \simeq F_2$ defines an action on this set of embeddings $\{\phi_{y_0}\}, y_0 \in p^{-1}(t_0)$, via the *monodromy* at t_0, as follows. Let $\gamma \in \pi_1^{\mathrm{top}}(X_4; t_0)$ be a loop based at t_0, $f \in F$ a rational function, $\phi : F \hookrightarrow \mathbb{C}((t - t_0))$ an embedding (equivalently a choice of $y_0 \in p^{-1}(t_0)$). Then define $\gamma \cdot \phi$ by

$$\gamma.\phi(f) = \phi(\gamma^{-1}.f) \in \mathbb{C}((t - t_0)),$$

where $\gamma^{-1} f$ is the analytic continuation of f (viewed locally as a function on $\mathbb{P}^1\mathbb{C}$ as explained above) along γ. Since we denote $\gamma' \circ \gamma$ the compostition of γ' with γ (γ' *after* γ) we need to have γ^{-1} in the above definition so that the action be covariant.

Notice now that the above contructions are algebraic (defined over $\overline{\mathbb{Q}}$) and yield a coherent system when varying the covering Y. We may thus pass to the projective limit on the coverings and their fundamental groups, and to the inductive limit on the algebraic function fields. As a result we get:

Proposition 2: *(i) To a coherent family of places of Ω "over" t_0 is associated an embedding $\phi : \Omega \hookrightarrow \overline{\mathbb{Q}}((t - t_0))$.*

(ii) The image of a finite extension $E \in Ext(\overline{\mathbb{Q}}(T)) \subset \Omega$ is contained in the field of convergent Laurent series.

(iii) Any $\gamma \in \pi_1(X_4 \times_{\mathbb{Q}} \mathbb{C}; t_0)$, the profinite algebraic fundamental group, determines an action $\phi \to \gamma \cdot \phi$ on the set of these embeddings.

In the above, t_0 was an arbitrary rational point of X_4. To remove this arbitrariness, it is tempting to take $\{0, 1, \infty\}$ as base points. Although they are at infinity (not in X_4), Deligne described in [De] a method of using them at least as starting points: let us recall the description of his *tangential base points.* Fix $t_0 = 0$ for definiteness; the trouble is that if we look at a neighbourhood of 0 in $X_4^* = \mathbb{P}^1$ and intersect it with X_4, we find a punctured disk, which is not simply connected, so that we can't use such a region as a "base point" for a fundamental group (a similar phenomenon occurs in any dimension). So consider instead the point 0 *together with* the direction $0 \to 1$; denote this as $\overrightarrow{01}$, and similarly $\overrightarrow{0\infty}$ for the opposite direction. Altogether, we find a set $\mathcal{B}$ of six such tangential base points, namely

$$\mathcal{B} = \{\overrightarrow{01}, \overrightarrow{0\infty}, \overrightarrow{10}, \overrightarrow{1\infty}, \overrightarrow{\infty 0}, \overrightarrow{\infty 1}\}.$$

Any $\vec{u} = \overrightarrow{AB} \in \mathcal{B}$ defines a sector $S_{\vec{u}}$, obtained from a small disk around A by removing the semiaxis *opposite* the direction $\overrightarrow{AB}$. Picking $\overrightarrow{01}$ again as an example, it defines $S_{\overrightarrow{01}}$, obtained from a small disk around 0 by removing $(0, \infty)$.

Puiseux series. Now let $p : Y \to X_4$ again be a finite connected covering, $F = \overline{\mathbb{Q}}(Y) \in Ext(\overline{\mathbb{Q}}(T))$ the corresponding function field. There is a unique compactification Y^* of Y as a covering of $\mathbb{P}^1$ ramified (at most) at $\{0, 1, \infty\}$. The geometric situation is as follows; take $\vec{u} = \overrightarrow{01} \in \mathcal{B}$ for ease of notation, and also to make the connection with the "dessins d'enfants" clearer. Pick $y_0 \in p^{-1}(0)$; if the ramification index of Y^* at y_0 is e, there are e edges of type $\overrightarrow{01}$ originating from y_0. We define the fibre $\mathcal{F}_{\overrightarrow{01}}(Y^*)$ of Y (or Y^*) above $\overrightarrow{01}$ as the set of pairs formed by a point $y_0 \in p^{-1}(0)$ and an edge of type $\overrightarrow{01}$ originating at y_0. The set $\mathcal{F}_{\vec{u}}(Y^*)$ is defined analogously for any $\vec{u} \in \mathcal{B}$. Note that the cardinal of $\mathcal{E}_{\vec{u}}(Y^*)$ is equal to the degree of the covering Y. Let us now explain (again assuming $\vec{u} = \overrightarrow{01}$ for convenience) how the choice of an element in $\mathcal{F}_{\vec{u}}(Y^*)$ determines an embedding of $F = \overline{\mathbb{Q}}(Y)$ in the field of Puiseux series, namely here $\overline{\mathbb{Q}}\{\{T\}\}$, the field of series with fractional exponents.

First we can find a local (in the analytic sense) coordinate z on Y^* near y_0 such that the covering $Y^*(\mathbb{C}) \to \mathbb{P}^1\mathbb{C}$ is locally isomorphic to the cyclic covering $z \to t = z^e$ of the unit disk. The choice of an edge of type $\overrightarrow{01}$ at y_0 is equivalent to the choice of a uniformizing parameter z such that its restriction to that given edge is real positive. Any $f \in F = \overline{\mathbb{Q}}(Y)$ can be expanded near y_0 as a convergent Laurent series in z,

$$f(z) = \sum_{n \geq -N} a_n z^n, \text{ for some } N \geq 0.$$

Define $\tilde{f} = \sum_n a_n T^{n/e}$ to be the corresponding formal Puiseux series. The map $f \to \tilde{f}$ defines an embedding $F \hookrightarrow \overline{\mathbb{Q}}\{\{T\}\}$, extending the natural embedding $\overline{\mathbb{Q}}(T) = \overline{\mathbb{Q}}(\mathbb{P}^1) \hookrightarrow \overline{\mathbb{Q}}\{\{T\}\}$. More precisely, any $\tilde{f} = \sum_n a_n T^{n/e}$ associated to $f \in \overline{\mathbb{Q}}(Y)$ is convergent and thus defines a germ of analytic function in the sector $S_{\overrightarrow{01}}$ (we use the word "analytic" to refer to functions into $\mathbb{P}^1$, i.e. meromorphic functions if one prefers). We denote $\mathcal{P}_{\overrightarrow{01}}(\overline{\mathbb{Q}})$ the field of such germs of analytic functions on $S_{\overrightarrow{01}}$ or, equivalently, convergent Puiseux series with coefficients in $\overline{\mathbb{Q}}$.

To summarize the above, we find that any edge of type $\overrightarrow{01}$, that is any element of $\mathcal{F}_{\overrightarrow{01}}(Y)$, defines an embedding:

$$F = \overline{\mathbb{Q}}(Y) \stackrel{\phi}{\hookrightarrow} \mathcal{P}_{\overrightarrow{01}}(\overline{\mathbb{Q}}) \subset \overline{\mathbb{Q}}\{\{T\}\}$$

which extends the natural embedding $\overline{\mathbb{Q}}(T) = \overline{\mathbb{Q}}(\mathbb{P}^1) \subset \overline{\mathbb{Q}}\{\{T\}\}$.

Conversely, any such embedding first defines a place in F above T, i.e. a point $y_0 \in \overline{Y}$ in the fibre over 0, together with a uniformizing parameter z, namely the pullback of $T^{1/e}$, in the completion of F at this place. The uniformizing parameter in turn determines an edge of the dessin originating at y_0, that is an element of $\mathcal{F}_{\overrightarrow{01}}(Y)$, by requiring that it be real positive on it.

This can be done for any of the six tangential base points $\vec{u} \in \mathcal{B}$; $\mathcal{P}_{\vec{u}}(\overline{\mathbb{Q}}) = \mathcal{P}_{\vec{u}}$ will denote the field of convergent Puiseux series (with coefficients in $\overline{\mathbb{Q}}$) in the sector $S_{\vec{u}}$, considered as an extension of $\overline{\mathbb{Q}}(\mathbb{P}^1)$. One should however beware of the fact that this refers to six *different* embeddings of $\overline{\mathbb{Q}}(\mathbb{P}^1)$ into fields of Puiseux series. Let us illustrate this important caveat in the case of $\vec{u} = \overrightarrow{0\infty}$. In that case we expand functions in the sector $S_{\overrightarrow{0\infty}}$ around the *negative* real axis. We now look for z such that $z^e = -t$, with z real positive for t real negative. The choice of an element of $\mathcal{F}_{\overrightarrow{0\infty}}(Y)$, the fibre over $\overrightarrow{0\infty}$, determines an embedding

$$F \hookrightarrow \mathcal{P}_{\overrightarrow{0\infty}}(\overline{\mathbb{Q}}) \subset \overline{\mathbb{Q}}\{\{T\}\}$$

extending the injective morphism $\overline{\mathbb{Q}}(T) \to \overline{\mathbb{Q}}\{\{T\}\}$ such that T is mapped to $-T$. The same phenomenon occurs at $\overrightarrow{1\infty}$, $\overrightarrow{10}$, $\overrightarrow{\infty 0}$ and $\overrightarrow{\infty 1}$, corresponding to series in fractional powers of $(T-1)$, $(1-T)$, $-T^{-1}$ and T^{-1} respectively.

We have now proved the first assertion in the following proposition.

Proposition 3: *(i) For any $\vec{u} = \overrightarrow{AB} \in \mathcal{B}$ and any finite connected unramified covering Y of X_4 (i.e. a covering of $\mathbb{P}^1$ unramified outside $0, 1, \infty$), there is a one to one correspondence between $\mathcal{F}_{\vec{u}}(\overline{Y})$, the fibre of the compactified covering $\overline{Y}$ over $\vec{u}$, and the set $\mathcal{E}_{\vec{u}}(\overline{\mathbb{Q}}(Y))$ of embeddings of the function field $\overline{\mathbb{Q}}(Y)$ in $\mathcal{P}_{\vec{u}}$, above $\overline{\mathbb{Q}}(T)$.*

(ii) $\mathcal{F}_{\vec{u}}$ defines a covariant functor from the category of finite connected unramified coverings of X_4 to the category Ens of sets.

(iii) $\mathcal{E}_{\vec{u}}$ defines a contravariant functor from the category $Ext(\overline{\mathbb{Q}}(T))$ of the finite extensions of $\overline{\mathbb{Q}}(T)$ unramified outside $0, 1, \infty$ to the category Ens of sets.

The abstract formulation of (ii) and (iii) turns out to be useful in order to phrase the analogue of (iii) in Proposition 2 (see Proposition 4 below). Proving these two assertions essentially amounts to proving that we can compose morphisms. We sketch it for (iii). For two extensions F, F', a morphism $\alpha : F \to F'$ is a morphism of fields fixing $\overline{\mathbb{Q}}(T)$, corresponding to a map between the associated coverings. Given α and $\phi \in \mathcal{E}_{\vec{u}}(F')$, we define $\mathcal{E}_{\vec{u}}(\alpha)(\phi) = \phi \circ \alpha \in \mathcal{E}_{\vec{u}}(F)$. With this in mind, the proof is reduced to some rather trivial verifications.

We note that the corresponding constructions behave nicely with respect to the inverse (resp. direct) system of coverings (resp. field extensions). This is actually included in (ii) and (iii). In particular, in much the same way as in Proposition 2 (i), we obtain a set of embeddings:

$$\mathcal{E}_{\vec{u}} : \Omega \hookrightarrow Puis_{\vec{u}}.$$

Moreover, one has the following property of surjectivity at any finite level: given $F_0 \in Ext(\overline{\mathbb{Q}}(T))$, and $\phi_0 \in \mathcal{E}_{\vec{u}}(F_0)$, there exists a coherent family $(F, \phi \in \mathcal{E}_{\vec{u}}(F))$ of function fields and embeddings, representing an element of $\mathcal{E}_{\vec{u}}(\Omega)$, and containing (F_0, ϕ_0).

The fundamental groupoid. Now, instead of the fundamental group, we want to consider the *fundamental groupoid* $\pi_1(X_4; \mathcal{B})$ based at the set $\mathcal{B}$ of the six tangential base point. A path $\gamma \in \pi_1^{\mathrm{top}}(X_4; \vec{u}, \vec{v})$ from $\vec{u} = \overrightarrow{AB}$ to $\vec{v} = \overrightarrow{CD}$ is a path in X_4 from A to C (the endpoints being excluded), starting tangentially to $\overrightarrow{AB}$ and arriving tangentially to $\overrightarrow{CD}$ (in the "direction" $\overrightarrow{DC}$). The path γ is allowed to go through "points" of $\mathcal{B}$ but it has to "arrive" and "leave" along the same direction. We leave it to the reader to describe the corresponding notion of homotopy, which allows to define the topological groupoid $\pi_1^{\mathrm{top}}(X_4; \mathcal{B})$ as the groupoid of these paths modulo homotopy. As an example, the path γ_0 obtained from going from 0 to 1 and back along the interval (0,1) is well-defined and null homotopic (equivalent to the unit of the *group* $\pi_1^{\mathrm{top}}(X_4; \overrightarrow{01}, \overrightarrow{01})$). Similarly, the path γ_1 obtained by going from 0 to 1, then from 1 to ∞ and back to 0 along straight lines with curls at the corners to switch direction is also well defined and null homotopic. γ_0 and γ_1 will play an important role in what follows, ultimately giving equations (I) and (II) in Theorem 1.

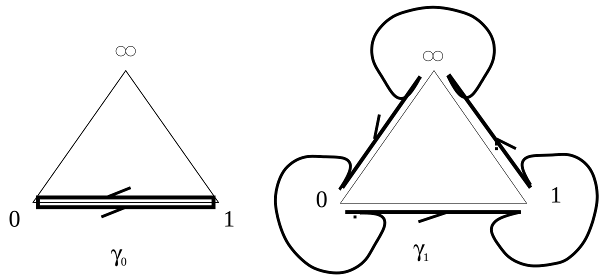

Figure 1

As for the ordinary fundamental group, the above can be expressed in terms of coverings and, passing to the projective limit yields the algebraic – profinite – groupoid $\pi_1(X_4; \mathcal{B})$. For $\vec{u} \in \mathcal{B}$, we write $\pi_1(X_4; \vec{u}) = \pi_1(X_4; \vec{u}, \vec{u})$ for the *group* of (homotopy classes of) paths from $\vec{u}$ to itself.

The next step (and the last one before describing the action of $\mathbb{\Gamma}$!) consists in viewing any $\gamma \in \pi_1(X_4; \vec{u}, \vec{v})$ as a map from $\mathcal{E}_{\vec{u}}$ to $\mathcal{E}_{\vec{v}}$, actually as a natural transformation between these two functors. Again we describe the geometric finite situation and sketch what is needed to pass to the algebraic and profinite case. So let Y be a finite covering again, $f \in F = \overline{\mathbb{Q}}(Y)$, $\phi \in \mathcal{E}_{\vec{u}}(F)$ an embedding $F \hookrightarrow \mathcal{P}_{\vec{u}}$, and $\gamma \in \pi_1^{\mathrm{top}}(X_4; \vec{u}, \vec{v})$ a geometric path. As in the non ramified case, $\gamma \cdot \phi \in \mathcal{E}_{\vec{v}}$ is defined by analytic continuation. Namely, for $f \in F$, $\phi(f)$ defines a germ of analytic function on $S_{\vec{u}}$, which can be continued along γ; as a result, we get a germ on $S_{\vec{v}}$, whose expansion is by definition $\gamma \cdot \phi(f) \in \mathcal{P}_{\vec{v}}$. Before looking briefly at morphisms, we state this formally:

Proposition 4: *For any $\gamma \in \pi_1(X_4; \vec{u}, \vec{v})$, the assignment $\phi \in \mathcal{E}_{\vec{u}} \to \gamma \cdot \phi \in \mathcal{E}_{\vec{v}}$ defines a natural transformation between $\mathcal{E}_{\vec{u}}$ and $\mathcal{E}_{\vec{v}}$.*

We have defined the action at the discrete level above. As usual one should check that it is compatible with the directed system of coverings, which is a consequence of the naturality of the transformation, that is, of the compatibility with the morphisms. This in turn amounts to the following. Let Y and Y' be two finite coverings, with compactifications $\overline{Y}$ and $\overline{Y}'$; let $a : Y \to Y'$ be a covering map, $\alpha : F' \to F$ the corresponding morphism of the function fields over $\overline{\mathbb{Q}}(T)$. Given ϕ and γ as above, we have defined $\mathcal{E}_{\vec{u}}(\alpha)(\phi)$ by $\mathcal{E}_{\vec{u}}(\alpha)(\phi) = \phi \circ \alpha \in \mathcal{E}_{\vec{u}}(F')$, and we want to check that for any ϕ, α, γ,

$$\mathcal{E}_{\vec{u}}(\alpha)(\gamma \cdot \phi) = \gamma \cdot \big(\mathcal{E}_{\vec{u}}(\alpha)(\phi)\big). \tag{6}$$

This boils down in effect to a simple path lifting property, granted that we only need to prove this at a finite discrete level. Pick an oriented edge $\vec{U}$ over $\vec{u}$ in Y (think in terms of drawings and assume that $\vec{u} = \overrightarrow{01}$). $\vec{U}$ uniquely determines a lift Γ of γ in $\overline{Y}$; (6) asserts that $a(\Gamma)$ is a lift of γ to $\overline{Y}'$, which is obvious since a is a covering map that is, preserves the natural projections of the coverings onto $\mathbb{P}^1$.

There is a rich dictionary between the various ways of describing coverings, of which we have already used quite a few. In the simplest case of unramified finite connected coverings $p : Y \to X$ and ordinary, discrete, fundamental group, the monodromy action on the fibre over the base point x_0 of the fundamental group $\pi_1^{\mathrm{top}}(X; x_0)$ sets up a category equivalence between the connected coverings of X and the finite sets endowed with a transitive action of the fundamental group. To a covering is associated the finite fibre over the base point with the monodromy action and, conversely, any finite set with a transitive π_1 action gives rise to such a connected covering. Starting from a covering, yet another way to describe this situation is to consider the subgroup $H \subset \pi_1$ of finite index having trivial action on the fibre at x_0 and to view this fibre as the homogeneous space π_1/H.

Using the above constructions, essentially the same description can be given, using the fibre at a *ramified* point. Specifically, the category of the connected unramified coverings of X_4 is equivalent to that of sets together with a transitive action of the group $\pi_1^{\mathrm{top}}(X_4; \overrightarrow{01}) \simeq F_2$. Given a covering Y, the action of $\pi_1(X_4; \overrightarrow{01})$ is of course that on $\mathcal{E}_{\overrightarrow{01}}(\overline{\mathbb{Q}}(Y))$ which we have described above. Here we used $\overrightarrow{01}$ only to underline the connection with dessins d'enfants; we may naturally use any $\vec{u} \in \mathcal{B}$ and also, slightly less trivially, replace the monodromy at $\vec{u}$ by the "transport" from $\vec{u}$ to $\vec{v}$ via elements of $\pi_1(X_4; \vec{u}, \vec{v})$ which induce a map $\mathcal{E}_{\vec{u}} \to \mathcal{E}_{\vec{v}}$, as shown above.

The point of all this resides in the fact that, just as in the case of the fundamental group (cf. [SGA 1]), we have the following statement, which appears as a converse to Proposition 4 and could actually serve as a definition for the algebraic fundamental groupoid:

Proposition 5: *Any natural transformation from $\mathcal{E}_{\vec{u}}$ to $\mathcal{E}_{\vec{v}}$ is induced by an element of* $\pi_1(X_4; \vec{u}, \vec{v})$.

Explicit action of the Galois group on the fundamental groupoid. We may now turn to the action of the Galois group $\mathbb{\Gamma}$ and to the proof of Theorem 1. First we make the identification $\pi_1(X_4; \overrightarrow{01}) \simeq \hat{F}_2$ explicit by choosing a generating set $\{x, y\}$ (those which occur in the statement of Theorem 1) as in Figure 2; x (resp. y) winds around 0 (resp. 1).

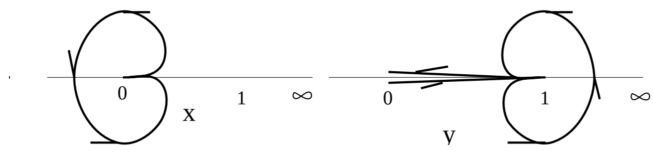

Figure 2

We then turn back to the exact sequence (2) which may now be rewritten more accurately as

$$1 \to \pi_1(X_4(\overline{\mathbb{Q}}); \overrightarrow{01}) \to \pi_1(X_4(\mathbb{Q}); \overrightarrow{01}) \to \mathbb{\Gamma} \to 1. \tag{7}$$

This is equivalent, as we already noticed, to the sequence of Galois groups:

$$1 \to \mathrm{Gal}(\Omega/\overline{\mathbb{Q}}(T)) \to \mathrm{Gal}(\Omega/\mathbb{Q}(T)) \to \mathbb{\Gamma} \to 1. \tag{8}$$

Again, (7) (or (8)) determines an outer action of $\mathbb{\Gamma}$ on $\hat{F}_2$, i.e. a map $\mathbb{\Gamma} \to \mathrm{Out}(\hat{F}_2)$; that this action is faithful, i.e. that the map is injective, is a direct consequence of Belyi's theorem for which we refer to [B] (see also the article by L. Schneps in this volume). Here we want to take advantage of the fact that (7) and (8) are actually split to define an explicit map $\mathbb{\Gamma} \to \mathrm{Aut}(\hat{F}_2)$, as in the statement of Theorem 1. Of course it will then still be injective.

If $F \in Ext(\overline{\mathbb{Q}}(T))$ is a finite extension of $\overline{\mathbb{Q}}(T)$ unramified outside $0, 1, \infty$, if $\vec{u} \in \mathcal{B}$, $\phi \in \mathcal{E}_{\vec{u}}(F)$ and $\sigma \in \mathbb{\Gamma}$, we define $\sigma \cdot \phi : F \to \mathcal{P}_{\vec{u}}$ by "action on the coefficients". Namely, fixing $\vec{u} = \overrightarrow{01}$ for definiteness again, for $f \in F$, $\phi(f) = \sum_n a_n t^{n/e}$ with $a_n \in \overline{\mathbb{Q}}$ for all n and $\sigma \cdot \phi(f) := \sum_n \sigma(a_n) t^{n/e}$.

Note that $\sigma \cdot \phi$ is *not* an element of $\mathcal{E}_{\vec{u}}(F)$ because it is an embedding $F \hookrightarrow \mathcal{P}_{\vec{u}}$ over $\mathbb{Q}(T)$, *not* over $\overline{\mathbb{Q}}(T)$. For $\gamma \in \pi_1^{\mathrm{top}}(X_4; \vec{u}, \vec{v})$, we now define $\sigma \cdot \gamma := \sigma \circ \gamma \circ \sigma^{-1}$, that is,

$$\sigma \cdot \gamma(\phi) := \sigma \cdot \big(\gamma \cdot (\sigma^{-1} \cdot \phi)\big). \tag{9}$$

The explicit computational recipe, which will be applied several times below, goes as follows. Pick $\phi \in \mathcal{E}_{\vec{u}}(F)$, $f \in F$, so that $\phi(f) \in \mathcal{P}_{\vec{u}}(\overline{\mathbb{Q}})$; apply σ^{-1} on the coefficients to get $\sigma^{-1} \cdot \phi(f) \in \mathcal{P}_{\vec{u}}(\overline{\mathbb{Q}})$; analytically continue $\sigma^{-1} \cdot \phi(f)$, viewed as a germ of analytic function on $S_{\vec{u}}$, along γ and expand the result at $\vec{v}$ to find $\gamma \cdot (\sigma^{-1} \cdot \phi)(f) \in \mathcal{P}_{\vec{v}}$. Lastly, apply σ to the coefficients to obtain $(\sigma \cdot \gamma)\,(\phi)(f)$.

Now, notice that if $\phi \in \mathcal{E}_{\vec{u}}$, $(\sigma \cdot \gamma)\,(\phi)$ *is* indeed an element of $\mathcal{E}_{\vec{v}}$ because the action of γ and σ commute on the elements of $\overline{\mathbb{Q}}(T)$ (with the embeddings $\overline{\mathbb{Q}}(T) \hookrightarrow \mathcal{P}_{\vec{u}}$ for $\vec{u} \in \mathcal{B}$), so that $\sigma \cdot \gamma(\phi) : F \hookrightarrow \mathcal{P}_{\vec{v}}(\overline{\mathbb{Q}})$ is an embedding over $\overline{\mathbb{Q}}(T)$. The usual argument using the coherence of the various actions allows to extend the above to the profinite setting. Any $\sigma \cdot \gamma \in \pi_1(X_4; \vec{u}, \vec{v})$ thus defines a map $\mathcal{E}_{\vec{u}} \to \mathcal{E}_{\vec{v}}$ and it is easy to check that it is in fact a natural transformation between $\mathcal{E}_{\vec{u}}$ and $\mathcal{E}_{\vec{v}}$. Applying Proposition 5 we have proved the following:

Proposition 6: $\sigma \cdot \gamma \in \pi_1(X_4; \vec{u}, \vec{v}) \quad \text{for all} \quad \gamma \in \pi_1(X_4; \vec{u}, \vec{v}),\ \sigma \in \mathbb{\Gamma}.$

It is clear that the following associativity and commutativity relations between the arithmetic and geometric actions hold true:

$$(\sigma\,\tau) \cdot \gamma = \sigma \cdot (\tau \cdot \gamma), \quad \sigma \cdot (\gamma' \circ \gamma) = (\sigma \cdot \gamma') \circ (\sigma \cdot \gamma), \tag{10}$$

for all $\sigma, \tau \in \mathbb{\Gamma}$, $\gamma, \gamma' \in \pi_1(X_4; \mathcal{B})$, whenever the composition of paths is possible. In particular, we have thus defined an action of the Galois group on the – profinite – fundamental groupoid, that is, a morphism

$$\mathbb{\Gamma} \to \mathrm{Aut}\big(\pi_1(X_4; \mathcal{B})\big).$$

We briefly indicate how this is related to the a priori outer action determined by the exact sequences (7) and (8). Actually, if we fix a coherent family $\big(\mathcal{E}_{\vec{u}}(F)\big), F \subset \Omega$, of embeddings $F \hookrightarrow \mathcal{P}_{\vec{u}}(\overline{\mathbb{Q}})$ over $\overline{\mathbb{Q}}(T)$, we get $\phi : \Omega \hookrightarrow Puis_{\vec{u}}(\overline{\mathbb{Q}})$. Any such ϕ determines a section of (8), $s_\phi : \mathbb{\Gamma} \to \mathrm{Gal}(\Omega/\mathbb{Q}(T))$ via the defining relation

$$\phi\big(s_\phi(\sigma)(f)\big) := \sigma \cdot \phi(f),$$

for any $f \in F \subset \Omega$, which means, put in a more down-to-earth fashion, that given a "prescription" to expand a function as a Puiseux series on $S_{\vec{u}}$ or, equivalently, given a coherent system of "oriented edges" over $\vec{u}$, there is a natural action of $\mathbb{\Gamma}$ defined by the action on the coefficients of the Puiseux series. The action of $\mathbb{\Gamma}$ on $\pi_1(X_4;\mathcal{B})$ we have constructed above is obtained from the outer action defined by (8) when using this family of sections.

We now turn to the explicit computations which will complete the proof of Theorem 1. Besides x and y (see Figure 2), we shall make use of the elements $p, q, r \in \pi_1(X_4;\mathcal{B})$, defined as follows (see Figure 3):

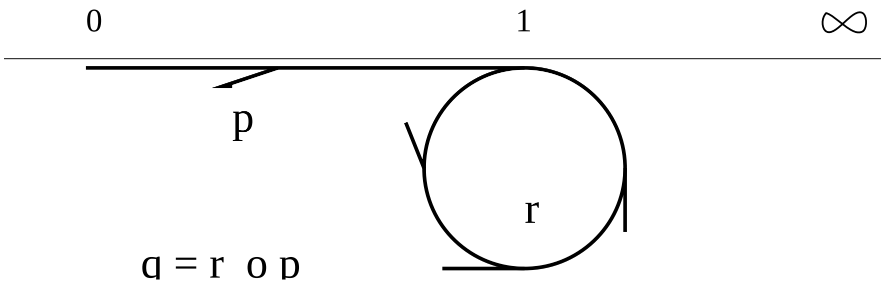

Figure 3

– $p \in \pi_1(X_4;\overrightarrow{01},\overrightarrow{10})$ is the path form 0 to 1 along the interval (0,1);
– $r \in \pi_1(X_4;\overrightarrow{10},\overrightarrow{1\infty})$ is obtained by rotating a half turn counterclockwise around 1;
– $q = r \circ p \in \pi_1(X_4;\overrightarrow{01},\overrightarrow{1\infty})$.

Let us first compute $\sigma \cdot x$ and $\sigma \cdot y$ for $\sigma \in \mathbb{\Gamma}$. Let $F \in Ext(\overline{\mathbb{Q}}(T)), \phi \in \mathcal{E}_{\overrightarrow{01}}(F), f \in F$, so that $\phi(f)(t) = \sum_n a_n t^{n/e} \in \mathcal{P}_{\overrightarrow{01}}$; $x \cdot \phi(f)$ is obtained by continuing this series around 0:

$$x \cdot \phi(f)(t) = \sum_n a_n \zeta^n t^{n/e} \quad \text{where} \quad \zeta = \exp(\frac{2i\pi}{e}). \tag{11}$$

We now compute $\sigma \cdot x$, following the recipe detailed after formula (9):

$$\phi(f) = \sum_n a_n t^{n/e} \xrightarrow{\sigma^{-1}} \sum_n \sigma^{-1}(a_n) t^{n/e} \xrightarrow{x} \sum_n \sigma^{-1}(a_n)\zeta^n t^{n/e} \xrightarrow{\sigma}$$

$$\xrightarrow{\sigma} \sum_n \sigma\big(\sigma^{-1}(a_n)\zeta^n\big) t^{n/e} = \sum_n a_n \zeta^{n\chi(\sigma)} t^{n/e} = (\sigma \cdot x) \cdot \phi(f) \tag{12}$$

Comparing (11) and (12), we find that $\sigma \cdot x = x^{\chi(\sigma)}$, as stated in Theorem 1.

We shall now compute $\sigma \cdot y$ somewhat formally, and then justify the computation. Let $\theta \in \mathrm{Aut}(X_4)$ defined by $\theta(t) = 1 - t$, and notice that $y = p^{-1}\,\theta(x)\,p \in \pi_1(X_4;\overrightarrow{01})$ (we drop the $\circ$ from the notation). Define

$$f_\sigma(x,y) := p^{-1}\ \sigma \cdot p$$

and compute (we write $\sigma \cdot \gamma$ or $\sigma(\gamma)$ indifferently, hopefully making the formulae as readable as possible):

$$\begin{aligned}\sigma \cdot y &= \sigma \cdot (p^{-1}\theta(x)p) = \sigma(p^{-1})p\, p^{-1}\theta(\sigma \cdot x)p\, p^{-1}\sigma(p)\\ &= f_\sigma^{-1}\, p^{-1}\theta(x^{\chi(\sigma)})p\, f_\sigma = f_\sigma^{-1}\, p^{-1}(\theta(x))^{\chi(\sigma)}p\, f_\sigma = f_\sigma^{-1}\, y^{\chi(\sigma)} f_\sigma,\end{aligned}$$

as in the statement of Theorem 1.

To complete the proof of this theorem, it only remains to justify the above computation, to show that $f_\sigma \in \hat{F}_2'$, and to derive equations (I) and (II). Justifying the above computation actually means studying the effect of the automorphisms of X_4 on the whole situation, and in particular justifying the intervertion $\sigma \circ \theta = \theta \circ \sigma$ which was used in the first line of the above computation. We shall be somewhat sketchy at this point, which presents no special difficulty. The key point is *that the automorphisms of X_4 are $\mathbb{Q}$-rational.* Indeed, $\mathrm{Aut}(X_4) \simeq S_3$ is generated by $\theta : t \to 1-t$ and $\omega : t \to (1-t)^{-1}$, corresponding to the transposition (01) and the three cycle $(0\,1\,\infty)$ respectively. Recall that more generally, for any $n \geq 5$, $\mathrm{Aut}(X_n) \simeq S_n$ and these automorphisms are $\mathbb{Q}$-rational (that $S_n \subset \mathrm{Aut}(X_n)$ is obvious; that there are no other automorphisms is not so obvious); one can also work over other fields. For $n = 4$, the action of S_4 factors through S_3.

We shall briefly describe the action of $\mathrm{Aut}(X_4)$ on $\pi_1(X_4; \mathcal{B})$ and the commutation with the Galois action, due to the $\mathbb{Q}$-rationality of the elements of $\mathrm{Aut}(X_4)$. In view of what was recalled above, the extension to the higher dimensional case is essentially obvious.

First any $\lambda \in \mathrm{Aut}(X_4)$ acts on $\mathcal{B}$ by looking at the derivatives of λ at the points $(0, 1, \infty)$; for example we have that $\omega(\overrightarrow{01}) = \overrightarrow{1\infty}$. Actually, $\mathrm{Aut}(X_4)$ acts simply transitively on $\mathcal{B}$ and any λ is completely determined by – say – $\lambda(\overrightarrow{01})$. Any automorphism λ also determines a map (still denoted λ):

$$Puis_{\vec{u}} \xrightarrow{\lambda} Puis_{\lambda(\vec{u})}$$

for any $\vec{u}$ in $\mathcal{B}$, simply by changing variable in the Puiseux series. Hence a map:

$$\mathcal{E}_{\vec{u}} \xrightarrow{\lambda} \mathcal{E}_{\lambda(\vec{u})}.$$

As for the action of the automorphisms on the fundamental groupoid, one first let $\lambda \in \mathrm{Aut}(X_4)$ act on $\pi_1^{\mathrm{top}}(X_4; \mathcal{B})$ by simply taking the image of a geometric path by λ; this defines a morphism:

$$\gamma \in \pi_1^{\mathrm{top}}(X_4; \vec{u}, \vec{v}) \to \lambda(\gamma) \in \pi_1^{\mathrm{top}}(X_4; \lambda(\vec{u}), \lambda(\vec{v})).$$

Now if $E \in Ext(\overline{\mathbb{Q}}(T))$and $\phi \in \mathcal{E}_{\vec{u}}(E)$, $\gamma \cdot \phi$ stabilizes $\mathcal{E}_{\vec{v}}(E) \subset \mathcal{P}_{\vec{v}}$ and $\lambda(\gamma)$ maps $\mathcal{E}_{\lambda(\vec{u})}(\lambda(E))$ to $\mathcal{E}_{\lambda(\vec{v})}(\lambda(E))$. Since $\lambda(E) \in Ext(\overline{\mathbb{Q}}(T))$ because λ is $\mathbb{Q}$-rational, we may pass to the direct limit and define

$$\pi_1(X_4; \vec{u}, \vec{v}) \xrightarrow{\lambda} \pi_1(X_4; \lambda(\vec{u}), \lambda(\vec{v})),$$

on the algebraic, profinite groupoid. Note that in the above, we more than once took $\vec{u} = \overrightarrow{01}$ for "definiteness" in the proof of some statement. Now, with the above in mind, we could also appeal to the fact that for any $\vec{u} \in \mathcal{B}$ there is a unique $\lambda \in \mathrm{Aut}(X_4)$ such that $\lambda(\overrightarrow{01}) = \vec{u}$, in order to restrict ourselves to the case $\vec{u} = \overrightarrow{01}$.

We have the following easy proposition:

Proposition 7: *For any $\lambda \in \mathrm{Aut}(X_4)$, $\gamma \in \pi_1(X_4; \vec{u}, \vec{v})$, $\phi \in \mathcal{E}_{\vec{u}}$, we have:*

$$\lambda(\gamma \cdot \phi) = \lambda(\gamma) \cdot \lambda(\phi). \tag{13}$$

The two sides actually define two ways of analytically continuing the *same* function, when $\gamma \in \pi_1^{\mathrm{top}}(X_4; \vec{u}, \vec{v})$. We leave the details to the reader.

We now add in the action of the Galois group. We work with the following objects: $\sigma \in \mathbb{\Gamma}$, $\lambda \in \mathrm{Aut}(X_4)$, $E \in Ext(\overline{\mathbb{Q}}(T))$ and $\phi \in \mathcal{E}_{\vec{u}}(E)$ an embedding $E \hookrightarrow \mathcal{P}_{\vec{u}}$ continuing the embedding $\overline{\mathbb{Q}}(T) \hookrightarrow \mathcal{P}_{\vec{u}}$ defined by $\vec{u} \in \mathcal{B}$; the first – crucial – relation is :

$$\lambda(\sigma \cdot \phi) = \sigma \cdot \lambda(\phi). \tag{14}$$

Recall that $\sigma \cdot \phi$ is *not* an element of $\mathcal{E}_{\vec{u}}(E)$, as it does not preserve the embedding $\overline{\mathbb{Q}}(T) \hookrightarrow \mathcal{P}_{\vec{u}}$. It is defined by action on the coefficients; assuming $\vec{u} = \overrightarrow{01}$, then for $f \in E$:

$$\phi(f) = \sum_n a_n t^{n/e} \longrightarrow \sigma \cdot \phi(f) = \sum_n \sigma(a_n) t^{n/e};$$

so $\sigma \cdot \phi$ is actually an element of $\mathcal{E}_{\vec{u}}({}^{\sigma}E)$ where ${}^{\sigma}E$ is the conjugate of the field E under σ.

In (14), $\lambda(\phi)$ is obtained by making the change of variable: $t = \lambda^{-1}(s)$, and expanding at $\lambda(\overrightarrow{01})$. As λ is defined over $\mathbb{Q}$, the commutation relation (14) is clear.

We now come to the last relation we need to justify the above computation of $\sigma \cdot y$ and undertake the proof of relations (I) and (II) in Theorem 1:

Proposition 8: *For all $\sigma \in \mathbb{\Gamma}$, $\gamma \in \pi_1(X_4; \mathcal{B})$, $\lambda \in \mathrm{Aut}(X_4)$,*

$$\lambda(\sigma \cdot \gamma) = \sigma \cdot \lambda(\gamma). \tag{15}$$

This is a direct consequence of the defining relation (9), and relations (13) and (14); we need only work at a finite level and test both sides on $\phi \in \mathcal{E}_{\vec{u}}$:

$$\begin{aligned}\lambda(\sigma \cdot \gamma) \cdot \lambda(\phi) &= \lambda\big((\sigma \cdot \gamma) \cdot \phi\big) = \lambda(\sigma\,\gamma\,\sigma^{-1}\,(\phi)) = \sigma \cdot \lambda(\gamma\,\sigma^{-1}\,(\phi)) \\ &= \sigma \cdot \lambda(\gamma)\,\lambda(\sigma^{-1}\,\phi) = \sigma \cdot \lambda(\gamma)\,\sigma^{-1} \cdot \lambda(\phi) = \big(\sigma \cdot \lambda(\gamma)\big) \cdot \lambda(\phi),\end{aligned}$$

which proves relation (15).

Before deriving equations (I) and (II) in Theorem 1, we first prove that $f_\sigma(x,y) \in \hat{F}_2'$, the derived group of $\hat{F}_2$. In order to see this, it is enough to show that $f_\sigma \in \pi_1(X_4; \overrightarrow{01}) \simeq \mathrm{Gal}(\Omega/\overline{\mathbb{Q}}(T))$ fixes the maximal *abelian* extension of $\overline{\mathbb{Q}}(T)$ contained in Ω; this is generated by cyclic coverings over 0 and 1, that is, by $T^{1/e}$ and $(1-T)^{1/e}$ for all integers e. Now, applying f_σ to $t^{1/e}$ for a given embedding in $\mathcal{E}_{\overrightarrow{01}}$, we have the chain

$$t^{1/e} \xrightarrow{\sigma^{-1}} t^{1/e} \xrightarrow{p} \big(1-(1-t)\big)^{1/e} = \textstyle\sum_n a_n (1-t)^{n/e} \quad (a_n \in \mathbb{Q})$$
$$\xrightarrow{\sigma} \big(1-(1-t)\big)^{1/e} \xrightarrow{p^{-1}} t^{1/e} = f_\sigma(t^{1/e}).$$

The proof that f_σ fixes cyclic coverings ramified at 1 is analogous.

Equations (I) and (II) come from the following two geometric relations, which take place in $\pi_1^{\mathrm{top}}(X_4; \overrightarrow{01})$:

$$\theta(p)\,p = 1; \quad \omega^2(q)\,\omega(q)\,q = 1. \tag{16}$$

Recall that p and q are as on Figure 2, and that θ and ω are elements of $\mathrm{Aut}(X_4)$ of order 2 and 3 respectively ($\theta(t) = 1-t$, $\omega(t) = (1-t)^{-1}$). Relations (16) simply express that the paths γ_0 and γ_1 of Figure 1 are trivial in $\pi_1^{\mathrm{top}}(X_4; \overrightarrow{01})$.

The proof of equation (I) is reduced to the following computation (again we use $\sigma(\gamma)$ or $\sigma \cdot \gamma$ according to which is more readable):

$$1 = \sigma \cdot \big(\theta(p)\,p\big) = \theta(\sigma \cdot p)\ \sigma \cdot p = \theta(pf_\sigma(x,y))\ pf_\sigma(x,y) =$$
$$\theta(p)\,\theta(f_\sigma(x,y))\,pf_\sigma(x,y) = p^{-1}\,\theta(f_\sigma(x,y))\,p\ f_\sigma(x,y) =$$
$$f_\sigma\big(p^{-1}\theta(x)p, p^{-1}\theta(y)p\big)\ f_\sigma(x,y),$$

which proves equation (I) because of the geometric relations:

$$p^{-1}\,\theta(x)\,p = y, \qquad p^{-1}\,\theta(y)\,p = x. \tag{17}$$

The proof of (II) is completely analogous, only more involved; since $q = r\,p$, we have $\sigma(q) = \sigma(r)\,\sigma(p) = \sigma(r)\,pf_\sigma$. We first prove:

$$\sigma \cdot r = r\,\theta(x)^{1/2(\chi(\sigma)-1)}. \tag{18}$$

This is obtained by an elementary computation, just as for $\sigma \cdot x$; the only difficulty is that one should not get mixed up with the determinations of the logarithm... We only sketch it; start from $f(t) \in \mathcal{P}_{\overrightarrow{10}}$,

$$f(t) = \textstyle\sum_n a_n (1-t)^{n/e} = \sum_n a_n \exp\big(\frac{n}{e}\log(1-t)\big)$$

with the principal determination of "log" ($t \notin (1\infty)$). Continuing this along r, we find that $1-t$ goes from $\overrightarrow{01}$ to $\overrightarrow{0\infty}$, going around 0 from *above* (t goes around 1 from below). So we get, slightly abusing notation:

$$r(f) = \sum_n a_n \exp\big(\frac{n}{e}\log(t-1) + i\pi\big) = \sum_n a_n\ \exp(\frac{n}{2}\frac{2i\pi}{e})\exp\big(\frac{n}{e}\log(t-1)\big),$$

with again the principal determination of "log". Now, following around $\sigma \cdot r = \sigma\, r\, \sigma^{-1}$, we find:

$$\sigma \cdot r(f) = \sum_n a_n \big(\exp \frac{i\pi}{e}\big)^{n\chi(\sigma)} \exp\big(\frac{n}{e}\log(t-1)\big).$$

Applying r^{-1} and setting $\zeta = \exp \frac{2i\pi}{e}$ we arrive at:

$$r^{-1}\,\sigma \cdot r(f) = \sum_n a_n \zeta^{\frac{n}{2}(\chi(\sigma)-1)}(1-t)^{n/e},$$

and since $\theta(t) = 1-t$, we find that (18) actually holds true. It immediately implies:

$$\sigma \cdot q = q\, y^m\, f_\sigma, \ \text{where } m = m(\sigma) = \frac{1}{2}(\chi(\sigma)-1). \tag{19}$$

We now have a situation with order 3 symmetry. Set $s := \omega(q)\, q$; then the analogue of relations (17) reads:

$$q^{-1}\omega(x)q = y,\ q^{-1}\omega(y)q = z,\ q^{-1}\omega(z)q = x, \ \text{with } xyz = 1;$$

$$s^{-1}\omega^2(x)s = z,\ s^{-1}\omega^2(y)s = x,\ s^{-1}\omega^2(z)s = y. \tag{20}$$

Starting from the second of relations (16) we find that:

$$1 = \omega^2(\sigma \cdot q)\,\omega(\sigma \cdot q)\,\sigma \cdot q = \omega^2(q\, y^m\, f_\sigma)\,\omega(q\, y^m\, f_\sigma)\, q\, y^m\, f_\sigma. \tag{21}$$

Moreover, using the first set of relations (20), we find that:
$\omega(q\, y^m\, f_\sigma) = \omega(q) q\, q^{-1}\omega(y^m) q\, q^{-1} f_\sigma\big(\omega(x), \omega(y)\big) q = \omega(q) q\, z^m\, f_\sigma(y,z).$
Similarly the second set of relations (20) implies:
$\omega^2(q)\,\omega^2(y^m)\, f_\sigma\big(\omega^2(x), \omega^2(y)\big)\,\omega(q) q = x^m\, f_\sigma(z,x).$
Substituting into (21) gives equation (II) in Theorem 1, thus completing its proof.