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On the Second Painlevé Equation: The Connection Formula via a Riemann–Hilbert Problem and Other Results

G. LEBEAU AND P. LOCHAK

C.M.A. École normale supérieure, 45, rue d'Ulm, 75230 Paris Cedex 05, France

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We prove the existence of asymptotic expansions to all orders for the solutions of the second Painlevé equation (P II). Using the linearization obtained via a Riemann–Hilbert problem, we then derive the amplitude part of the connection formula for this equation and give some other properties of the expansion.

I. INTRODUCTION

In the past few years, much work has been devoted to the so-called Painlevé equations. Let us recall that these were first studied by Painlevé—and his pupils—around 1900 [1]; he recognized that they were essentially the only second-order equations (besides the elementary ones: linear, Riccati, elliptic equations,...) such that the movable singularities are poles (i.e., movable essential singularities are excluded). It was also shown that Eq. (P VI) is a particular case of the Schlesinger equations [2] which describe the isomonodromic deformations of linear equations with regular singular points. The other equations can be obtained by a limiting procedure. Intere'st in the Painlevé equations revived when Ablowitz, Ramani, and Segur showed [3] that reductions of PDE's solvable by the inverse spectral transform should be of Painlevé type, and consequently should reduce to elementary equations (this almost never occurs) or to one of the six Painlevé transcendants. Shortly after, the Japanese school [4] undertook the generalization of the deformation theory to linear systems with *irregular* singular points and found various results on the Painlevé equations as a by-product.

The present paper deals with some properties of the second Painlevé equation

$$(\mathrm{P\,II})_\alpha \qquad w'' = tw + 2w^3 + \alpha$$

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and we shall in fact consider the special case $\alpha = 0$; it is the only case where it is known that there exist solutions bounded on the real t-axis (we denote $(\mathrm{P\,II})_0$ by $(\mathrm{P\,II})$).

Flachka and Newell [5] wrote $(\mathrm{P\,II})_\alpha$ as the compatibility condition between two linear systems with rational coefficients (see below) and reduced it to a Riemann–Hilbert (R.H.) problem along a certain contour, the solution of which was equivalent to that of a *singular* integral equation. In [6] Fokas and Ablowitz made the solution more explicit by disentangling the R. H. problem into a cascade of three R. H. problems along lines, or equivalently, a sequence of three Fredholm equations, the kernel of each one being given through the solution of the preceding one. It may be amusing to notice that they used a trick already employed by Beals and Coiffman, but which can also be found almost explicitly in Birkhoff's study of the Riemann problem [7].

In the linearization (reduction to linear integral equations) via a R.H. problem, one of the advantages, as pointed out in [5], is that the independent variable t appears only as a parameter and the integration is performed on a spectral parameter z (see below); this is to contrast with the Gelfand–Levitan equation approach, where the integration is over t. Although the difference more or less amounts to a Fourier transform, it was suggested in [5] that the R.H. approach could be put to use in order to solve the so-called connection problem. Let us first recall its formulation; for any $r \in (0, 1)$ there exists a solution of (P II) such that

$$w(t) \sim_{t=+\infty} r.1/(2\sqrt{\pi})\, t^{-1/4} \exp(-2/3\, t^{3/2}) \qquad (\sim r\, Ai(t)). \tag{I.1}$$

The problem consists in describing the asymptotic behaviour of this solution when t approaches $-\infty$; it still has the form of the Airy function, but with other values of the parameters. More precisely (cf. [8]):

$$\begin{aligned} &w(t) \sim_{t=-\infty} d\,|t|^{-1/4} \sin\theta(t)\\ &\theta(t) \sim 2/3\,|t|^{3/2} - 3/4\,d^2 \log|t| + \theta_0. \end{aligned} \tag{I.2}$$

These formulas can be found readily by a *formal* asymptotic expansion of the solution near $-\infty$. The global connection problem consists in finding the two functions $d = d(r)$ (amplitude) and $\theta_0 = \theta_0(r)$ (phase). We show below that the solution does in fact admit an asymptotic expansion of which (1.2) is the first term and we recover the amplitude formula

$$d^2(r) = -1/\pi \log(1 - r^2). \tag{I.3}$$

We are also able to give some details on the terms of the expansion. Formula (1.3) was first obtained in [9] by ingenious roundabout methods and, as the authors themselves pointed out, it was rather a "derivation" than a proof, since many steps rely on formal asymptotic expansions for an associated PDE (the modified KdV equation) which would be extremely difficult to rigorize. Using the Gelfand–Levitan equation, (I.3) was proved recently by MacLeod and Clarkson ([10] and a forthcoming paper) who also say [11] that they have obtained the phase formula

$$\theta_0 = \pi/4 - \arg\{\Gamma(1 - i/2d^2)\} - 3/2 \log 2 \tag{I.4}$$

which was formally derived in [9]. Let us notice that this connection problem is physically important because it provides a rather ubiquitous model for the reflection through a nonlinear caustic (see [8]).

The Japanese school has also abtained [11], using the τ function, results which pertain to the connection problems associated with some of the Painlevé equations, but it may be worth pointing out that, since they consider the theory from the deformation viewpoint, (P VI) appears to be the simplest equation and (P I) and (P II) the most difficult to deal with (even though they are much simpler to write down) because they correspond to the coalescence of many singularities; this explains why, to our knowledge, (I.3) and (I.4) cannot be found explicitly—nor be deduced in a straightforward way—from their work.

The paper is organized as follows. We first prove the existence of an asymptotic expansion (of a certain form) to all orders for the solution of (P II), within a certain range of the initial value parameters; we then give an improved (more explicit) version of the R.H. treatment for (P II) in the particular case we are interested in (i.e., $\alpha = 0$, solutions bounded on the real axis). We then take advantage of this linearization to reduce the connection problem to a—rather intricate—stationary phase analysis; from it we extract the amplitude part of the connection formula (presumably the phase part could also be derived in this way, but we did not manage to dig it out). The form of the expansion and the integral equation also shed some light on the behaviour of the solution, in the large, and it fits especially well with the appearance of poles on the real axis when $|r|$ becomes larger than 1.

Let us add that one of our motivations in this study was to show very concrete information could be extracted from the R.H. formulation, which is often exploited only on a formal and abstract level.

II. Asymptotic Developments

In this part, we will prove the existence of asymptotic developments of a certain form for the solution of (P II). Since we are interested in the oscillating part of the solution, it will be convenient to change t into $-t$, and also to perform a scaling on the unknown function. We thus write the equation

$$f'' + tf = 2c^2 f^3, \qquad (E_c)$$

where f is defined on some interval $[t_0, +\infty[$ and c is a complex number. Notice that for $c = 0$, (E_0) is the Airy equation, and for all c, $cf(-t)$ is the Painlevé transcendant.

II.1. *Formal Asymptotic Solutions*

We will first study formal solutions $g(t)$, of the form

$$\text{(i)} \quad g(t) \sim \sum_{\substack{k \in \mathbb{Z} \\ k \text{ odd}}} g_k(t)\, e^{ik\varphi(t)} \qquad \varphi(t) = 2/3t^{3/2}$$

$$\text{(ii)} \quad g_k(t) \sim \sum_{l=0}^{\infty} g_{k,l}\, t^{\nu_k - 3/2l} \qquad \nu_k = 1/2 - 3/4|k| + i\alpha k, \qquad \text{(II.1)}$$

where the coefficients $g_{k,l}$ are complex numbers, and α is defined by the relation

$$\alpha = -3c^2 g_{1,0} g_{-1,0}. \qquad \text{(II.2)}$$

For simplicity, we will use the following notation: $z^+ = g_{1,0}$, $z^- = g_{-1,0}$, and we denote by Ω the open set in $\mathbb{C}^3$:

$$\Omega = \{(c, z^+, z^-) \in \mathbb{C}^3, \mid \operatorname{Im} \alpha| < \tfrac{3}{4}\}. \qquad \text{(II.3)}$$

Notice that in Ω, Re $\nu_k \to -\infty$ ($|k| \to +\infty$) so that the formal sums in (II.1) are of asymptotic type near $t = +\infty$.

DEFINITION. Let $g(t)$ be a formal series of the form (II.1). $g(t)$ is a formal asymptotic solution of (E_c) if:

(i) $(c, z^+, z^-) \in \Omega$.

(ii) $g'' + tg - 2c^2 g^3$ is zero as a formal series.

Part (ii) is equivalent to the set of relations:

$$(1 - k^2)\, g_{k,n} + 2ik\, g_{k,n-1}(\nu_k - 3/2n + 7/4) + (\nu_k - 3/2n + 3)(\nu_k - 3/2n + 2).g_{k,n-2}$$

$$= 2c^2 \sum g_{j_1,l_1} g_{j_2,l_2} g_{j_3,l_3}\, (j_1 + j_2 + j_3 = k;\ l_1 + l_2 + l_3$$

$$= n - 1/2(|j_1| + |j_2| + |j_3| - |k|) \qquad \text{(II.4)}$$

for all k in $\mathbb{Z}$, k odd and all $n \geqslant 0$.

PROPOSITION II.1. *For every $\beta=(c, z^+, z^-)$ in Ω, there exists a* unique *formal asymptotic solution of (E_c), $g_\beta(t)$ such that $(g_\beta)_{\pm 1, 0}=z^{\pm}$. In addition, for every compact K in Ω, there exist constants A, B, C (depending on K) such that:*

$$\forall\beta \quad \text{in} \quad K, \forall k, n, \; |(g_\beta)_{k,n}| \leqslant A\cdot(B)^{|k|}\cdot C^n\, n! \tag{II.5}$$

and the functions, $\beta\to(g_\beta)_{k,n}$ are polynomials in c, z^+, z^-.

Proof. We shall prove the estimates (II.5) by induction on n, and for every n, by induction on $|k|=1, 2, 3....$ We shall show precisely that there exists $\varepsilon>0$ such that (II.5) holds with $A=\varepsilon$, $B=\varepsilon^{-2}$, $C=\varepsilon^{-7}$. By the same symbol M we denote a finite number of constants which depend only on K. For $n=0$ (II. 4) is the relation $0=0$; for $n=0$ and $|k|\geqslant 3$ it is equivalent to

$$g_{k,0}=1/(1-k^2)\cdot 2c^2 \sum_{\substack{j_1+j_2+j_3=k\\ |j_1|+|j_2|+|j_3|=|k|}} g_{j_1,0}\, g_{j_2,0}\, g_{j_3,0} \tag{II.6}$$

so that $|g_{k,0}|\leqslant M\varepsilon^{3-2|k|}$ and (II.5) holds, provided we ensure $M\varepsilon^2\leqslant 1$.
For $n=1$, $k=\pm 1$ (II.4) is the relation

$$2i(i\alpha)\, z^+=6c^2 z^+ z^-\cdot z^+ \tag{II.7}$$

which is true by definition. We now prove that if the estimates (II.5) are true for $g_{k,n}$, $n<N$, $k=\pm 1$, they are true for $n=N$ and all k, $|k|\geqslant 3$. We start from the identity (II.4) with $n=N$ and use the inductive hypothesis; this reduces the result to the inequality

$$\begin{aligned} &M\,\varepsilon^{1-2|k|-7N+7}\,N!\\ &\quad+M/(1+|k|^2) \sum_{\substack{j_1+j_2+j_3=k\\ l_1+l_2+l_3=N-1/2(|j|-|k|)}} l_1!\, l_2!\, l_3!\, \varepsilon^{3-2|j|-7(l_1+l_2+l_3)}\\ &\leqslant N!\,\varepsilon^{1-2|k|-7N}. \end{aligned} \tag{II.8}$$

We have

$$\sum_{l_1+l_2+l_3=a} l_1!\, l_2!\, l_3! \leqslant 9a! \tag{II.9}$$

and therefore (II.8) is a consequence of

$$\begin{aligned} &M\varepsilon^2/(1+|k|^2) \sum_{j_1+j_2+j_3=k;\; 1/2(|j|-|k|\leqslant N} \varepsilon^{3/2(|j|-|k|)}(N!)^{-1}[N-1/2(|j|-|k|)]!\\ &\quad\leqslant M\varepsilon^2 \end{aligned} \tag{II.10}$$

because we have

$$\#\{j; j_1+j_2+j_3=k,\ |j|-|k|=2m\} \leqslant C^{st}(1+|k|^2)(1+m^2). \qquad \text{(II.11)}$$

It only remains to prove the estimates when $k=\pm 1$; in this case we have to take into account the nonlinear terms on the r.h.s. of (II.4) such that $j_1=1$, $j_2=1$, $j_3=-1$, $l_1=n-1$, $l_2=l_3=0$ and similar terms obtained by permutation of the indices. (II.4) is equivalent to the system $(n-1=N)$:

$$\begin{pmatrix} -\alpha-3/2iN-6c^2z^+z^- & -3c^2z^{+2} \\ -3c^2z^{-2} & -\alpha+3/2iN-6c^2z^+z^- \end{pmatrix}\begin{pmatrix} g_{1,n} \\ g_{-1,n} \end{pmatrix}=\begin{pmatrix} u_+ \\ u_- \end{pmatrix} \qquad \text{(II.12)}$$

with

$$u_\pm=-(v_{\pm 1}-3/2N+3/2)(v_{\pm 1}-3/2N+1/2)\,g_{\pm 1,N-1} + 2c^2 \sum_{\substack{j_1+j_2+j_3=\pm 1 \\ l_1+l_2+l_3=N+1-1/2(|j|-1),\,|j|\geqslant 5}} g_{j_1,l_1}\,g_{j_2,l_2}\,g_{j_3,l_3}. \qquad \text{(II.13)}$$

Using (II.9) and $|j|\geqslant 5$ we see that

$$|u_\pm|\leqslant M\varepsilon^7(N+1)!\,\varepsilon^{-1-7N}+M\varepsilon(N+1)!\,\varepsilon^{-1-7N}. \qquad \text{(II.14)}$$

We conclude from this that the estimate (II.5) holds for all n, k, because the determinant of the matrix on the l.h.s. of (II.12) is $9/4(N+1)^2$ and each entry is less than $M(N+1)$. From (II.4) and (II.12) we see by induction that the $(g_\beta)_{k,n}$ are polynomials in c, z^+, z^-. This concludes the proof of the proposition. ▌

II.2. *Solutions with Asymptotic Developments*

In this section we prove that for every formal asymptotic solution $g(t)$ of (E_c), there exists a unique solution $f(t)$ of (E_c) which is asymptotic to $g(t)$ at $t=+\infty$.

DEFINITION. Let $f(t)$ be a function defined on some interval $[t_0, +\infty)$, and $g(t)$ a formal series of the form (II.1) with $|\operatorname{Im}\alpha|<\frac{3}{4}$. We shall say that f is asymptotic to g ($f\sim g$) if, for any $M>0$, there exist $A, B, C>0$ such that

$$\sup_{t\geqslant t_0} t^M\left| f(t)-\sum_{|k|\leqslant A,\,0\leqslant l\leqslant B} g_{k,l}\,t^{v_k-3/2l}\,e^{ik\varphi(t)}\right|\leqslant C. \qquad \text{(II.15)}$$

Remark. Suppose $f(t)$ is a solution of (E_c), $g(t)$ a formal asymptotic solution of (E_c) and that $f\sim g$; then, the equation itself implies that $f''\sim g''$, and by integration, we conclude that $f' \sim g'$; taking derivatives of the equation, we conclude by induction that for every $j \geqslant 0$, we have $f^{(j)} \sim g^{(j)}$, and so if (II.15) holds, it holds for every derivative. The main result is the following:

THEOREM II.1. *For every compact $K \Subset \Omega$, there exists $t_K > 0$, and for every $\beta = (c, z^+, z^-) \in K$, a unique solution $f_\beta(t)$ of (E_c) defined on $[t_K, +\infty)$ such that $f_\beta \sim g_\beta$. Moreover, there exist $\varepsilon_K > 0$, $D_K > 0$ such that, if we define, for $t \geqslant t_K$, $\beta \in K$, $J_t(\beta) =\text{def } (c, f_\beta(t), f'_\beta(t))$, then $\beta \to J_t(\beta)$ is holomorphic and satisfies the estimate:*

$$|\det[\partial_\beta J_t(\beta)] + 2i| \leqslant D_K\, t^{-\varepsilon_K}$$

for every $t \geqslant t_K$ and $\beta \in K$.

Proof. We fix K and let ω be open in Ω, $K \subset \omega$ with ω of compact closure; for simplicity, we drop the subscripts denoting the domain from A, B, C, etc. We shall first construct for every $\beta \in \omega$ a solution f_β of (E_c), asymptotic to g_β and defined on $[t_1, +\infty)$, t_1 independent of $\beta \in \omega$, such that for every t, $\beta \to f_\beta(t)$ is a holomorphic function.

LEMMA 1. *There exist $A, B, C > 0$ and for every $\beta \in \omega$, every $k \in \mathbb{Z}$ odd, there exists a function $h_{\beta,k}(t)$ defined for $t \geqslant 1$, analytic in $\beta \in \omega$, $t \geqslant 1$, such that, for any $t \geqslant 1$, $N > 0$,*

$$\left| h_{\beta,k}(t)\, t^{-\nu_k} - \sum_{l=0} N_{(g_\beta)_{k,l}}\, t^{-3/2l} \right| \leqslant A B^{|k|} C^{N+1} (N+1)!\, t^{-3/2(N+1)}. \tag{II.16}$$

Proof. We use the Borel summation formula. From Proposition II.1, we know that for every $\beta \in \omega$, we have

$$|(g_\beta)_{k,n}| \leqslant A B^{|k|} C^n\, n!. \tag{II.17}$$

We introduce

$$G_{\beta,k}(x) = \sum_{n=0}^{+\infty} (g_\beta)_{k,n} x^n/n! \tag{II.18}$$

which is analytic for $\beta \in \omega$, $x \in \mathbb{C}$, $|x|\, C < 1$ and satisfies

$$|G_{\beta,k}(x)| \leqslant A B^k (1 - C\,|x|)^{-1} \tag{II.19}$$

Now let $\rho_0 > 0$ such that $\rho_0 C > 1$; we define

$$h_{\beta,k}(t) = t^{\nu_k + 3/2} \int_0^{\rho_0} e^{-x t^{3/2}}\, G_{\beta,k}(x)\, dx. \tag{II.20}$$

Then, $h_{\beta,k}$ is analytic for $t \geqslant 1$, $\beta \in \omega$, and from the formula

$$t^{3/2} \int_0^{+\infty} e^{-xt^{3/2}} x^l \, dx = t^{-3/2l} l! \tag{II.21}$$

we deduce

$$\begin{aligned} h_{\beta,k}(t)\, t^{-\nu_k} - \sum_{l=0}^{N} (g_\beta)_{k,l}\, t^{-3/2l} = {} & t^{3/2} \int_0^{\rho_0} e^{-xt^{3/2}} \left(\sum_{n=N+1}^{+\infty} (g_\beta)_{k,n} x^n/n! \right) dx \\ & + t^{3/2} \int_{\rho_0}^{+\infty} e^{-xt^{3/2}} \left(\sum_{l=0}^{N} (g_\beta)_{k,l}\, x^l/l! \right) dx. \end{aligned} \tag{II.22}$$

From the estimate (II.17) we conclude that the modulus of the l.h.s. of (II.22) is less than

$$\begin{aligned} & A B^{|k|} (1 - \rho_0 C)^{-1} C^{N+1} (N+1)!\, t^{-3/2(N+1)} \\ & + A B^{|k|} e^{-\rho_0 t^{3/2}} \sum_{l=0}^{N} \sum_{l_1 + l_2 = l} C^l\, l!/l_1!\, \rho_0^{l_1} t^{-3/2 l_2}. \end{aligned} \tag{II.23}$$

It is easy to see that there exists a constant D such that

$$e^{-\rho_0 t^{3/2}} \sum_{l=0}^{N} \sum_{l_1 + l_2 = l} C^l\, l!/l_1!\, \rho_0^{l_1}\, t^{-3/2 l_2} \leqslant D^{N+1} (N+1)!\, t^{-3/2(N+1)} \tag{II.24}$$

and the lemma is proved. ∎

Now, because the closure of ω is compact in Ω, there exist $\delta > 0$ such that for every β in the closure of ω, we have $|\operatorname{Im} \alpha| \leqslant 3/4\,(1-\delta)$, and so $\operatorname{Re} \nu_k \leqslant \frac{1}{2} - \frac{3}{4}|k|\,\delta$. If we then choose t_0 such that $t_0^{3/4\delta} > B$, the series

$$H_\beta(t) = \sum_{k \in \mathbb{Z}} h_{\beta,k}(t)\, e^{ik\varphi(t)} \tag{II.25}$$

defines an analytic function of $t \geqslant t_0$, $\beta \in \omega$.

Lemma 2. *Let $\varepsilon_\beta(t)$ be defined as*

$$\varepsilon_\beta(t) = H_\beta(t)'' + t H_\beta(t) - 2c^2 H_\beta^3(t). \tag{II.26}$$

There exists $\varepsilon_0 > 0$ such that, for every $\beta \in \omega$ and $t > t_0$,

$$|\varepsilon_\beta(t)| \leqslant 1/\varepsilon_0\, e^{-\varepsilon_0 t^{3/2}}. \tag{II.27}$$

Proof. One has

$$\varepsilon_\beta(t) = \sum \varepsilon_{\beta,k}(t)\, e^{ik\varphi(t)}$$

$$\varepsilon_{\beta,k}(t) = (1-k^2)\, t h_{\beta,k} + 2ik t^{1/2} h'_{\beta,k} + ik/2t^{-1/2} h_{\beta,k} + h''_{\beta,k}$$
$$- 2c^2 \sum_{j_1+j_2+j_3=k} h_{\beta,j_1} h_{\beta,j_2} h_{\beta,j_3} \qquad \text{(II.28)}$$

but the proof of Lemma 1 shows that the estimate (II.16) is valid not only for $t \geqslant 1$, but also for $t \in \mathbb{C}$, $|\operatorname{Im} t| \leqslant a(\operatorname{Re} t)$, $\operatorname{Re} t \geqslant 1$, provided $a > 0$ is small enough; therefore, the same estimates hold for the derivatives. If in (II.28) we replace $h_{\beta,k}$ by its approximation

$$t^{\nu_k} \sum_{l=0}^{N} (g_\beta)_{k,l}\, t^{-3/2l} \qquad \text{(II.29)}$$

where N is taken to be the integer part of $(C_0 t^{3/2})$ with a small enough C_0, we obtain, by (II.4), (II.5), (II.17) that for some $\varepsilon > 0$,

$$|\varepsilon_{\beta,k}(t)| \leqslant 1/\varepsilon\, B^k\, e^{-\varepsilon t^{3/2}} \qquad \text{(II.30)}$$

and the lemma follows. ∎

LEMMA 3. *There exists $t_1 > 0$, $C_1 > 0$ and for every $\beta \in \omega$ a function $l_\beta(t)$ analytic in t and β such that*

(i) $|l_\beta(t)| \leqslant C_1\, e^{-\varepsilon/2t^{3/2}}$ (II.31)

(ii) $f_\beta(t) =_{\text{def}} H_\beta(t) + l_\beta(t)$ *is a solution of* (E_c) *for* $t \geqslant t_1$. (II.32)

Proof. The function $l_\beta(t)$ should be chosen such that

$$l_\beta(t)'' + t l_\beta(t) = 6c^2 H_\beta^2(t)\, l_\beta(t) + 6c^2 H_\beta(t)\, l_\beta^2(t) + 2c^2 l_\beta^3(t) - \varepsilon_\beta(t). \qquad \text{(II.33)}$$

This equation will now be solved by a fixed point argument.

Let B denote the Banach space of continuous functions l defined on $[t_1, +\infty)$ (t_1 will be chosen later) which satisfy:

$$|l(t)| \leqslant \|l\|\, e^{-\varepsilon/2t^{3/2}} \qquad \text{(II.34)}$$

equipped with the norm $\|l\|$.

Let $Ai(t)$ be the usual Airy function, that is,

$$Ai'' + t\, Ai = 0; \qquad Ai(t) \sim_{t=+\infty} t^{-1/4}\, e^{2/3it^{3/2}}. \qquad \text{(II.35)}$$

The solution of the equation $l'' + tl = a(t)$ which vanishes at $+\infty$ is given by

$$l(t) = L(a)(t) =_{\text{def}} Ai(t) \int_t^\infty 1/(Ai^2(s) \int_s^\infty Ai(\sigma)\, a(\sigma)\, d\sigma \qquad \text{(II.36)}$$

If we define N_β as the nonlinear operator

$$N_\beta(l) = 6c^2\, G_\beta^2 l + 6c^2 G_\beta l^2 + 2cl^3 - \varepsilon_\beta \qquad \text{(II.37)}$$

it is sufficient to prove that if t_1 is large enough, we have

$$\begin{aligned} &\|L \circ N_\beta(l)\| \leqslant \tfrac{1}{2} \qquad \text{if} \quad \|l\| \leqslant 1 \\ &\|L \circ N_\beta(l) - L \circ N_\beta(l')\| \leqslant \tfrac{1}{2} \|l - l'\| \qquad \text{if} \quad \|l\|, \|l'\| \leqslant 1 \end{aligned} \qquad \text{(II.38)}$$

for all β in the closure of ω. But we know that there exist constants C and $\gamma < 1$ such that

$$\forall\, t \geqslant t_0, \qquad |H_\beta(t)| \leqslant C\, t^{\gamma/2} \qquad \text{(II.39)}$$

and if t_1 is large enough, $\varepsilon_\beta \in B$ and is arbitrarily small; also, for $l \in B$, $\|l\| \leqslant 1$, the estimate

$$|N_\beta(l)(t)| \leqslant C^{st}\, t^\gamma\, e^{-\varepsilon/2t^{3/2}}\, \|l\| + \|\varepsilon_\beta\|\, e^{-\varepsilon/2t^{3/2}} \qquad \text{(II.40)}$$

holds true and therefore, we have

$$|L \circ N_\beta(l)(t)| \leqslant C^{st}[t^{\gamma-1}\, e^{-\varepsilon/2t^{3/2}}\, \|l\| + \|\varepsilon_\beta\|/t\, e^{-\varepsilon/2t^{3/2}}] \qquad \text{(II.41)}$$

and also

$$|L \circ N_\beta(l) - L \circ N_\beta(l')| \leqslant C^{st}\, t^{\gamma-1}\, e^{-\varepsilon/2t^{3/2}}\, \|l - l'\|. \qquad \text{(II.42)}$$

Equation (II.38) follows from this inequality.

By construction, $\beta \to N_\beta(l)$ is an analytic function of β ($\varepsilon\omega$), with values in B, and so, the fixed point h_β solution of the equation $l_\beta = L \circ N_\beta(l_\beta)$ is analytic in β. Since the function $f_\beta(t) =_{\text{def}} H_\beta(t) + l_\beta(t)$ is a solution of (E_c), it is an analytic function of t. ∎

For $t \geqslant t_1$, we introduce the function

$$J_t(\beta) = (c, f_\beta(t), f'_\beta(t)). \qquad \text{(II.43)}$$

An elementary computation, using (II.4), (II.17), (II.25), (II.31), (II.32) shows that for t_1 large enough, there exist two strictly positive constants D_K and ε_K such that

$$|\det[\partial_\beta\, J_t(\beta)] + 2i| \leqslant D_K\, t^{-\varepsilon_K}. \qquad \text{(II.44)}$$

Let $\beta_0 = (c_0, z_0^+, z_0^-) \in K$, choose $\rho > 0$, and let B_ρ be the ball:

$$B_\rho = \{(c_0, z^+, z^-) \in \Omega, |z^+ - z_0^+| + |z^- - z_0^-| \leqslant \rho^2\} \subset \Omega. \quad \text{(II.45)}$$

Then, because $\partial_\beta^2 J_t(\beta)$ is bounded by a power of $|t|$, we deduce from (II.44) that there exists t_K large enough, and two positive constants C_0 and C_1 such that, for $t \geqslant t_K$, $J_t(B_\rho)$ contains all the points (c, u, v) such that

$$|u - f_{\beta_0}(t)|^2 + |v - f'_{\beta_0}(t)|^2 \leqslant C_0 |t|^{-C_1}, \quad \text{(II.46)}$$

i.e., all the Cauchy data at time t which are in the ball with a center from the Cauchy data of f_{β_0}, and the radius some inverse power of t. This proves the uniqueness of the solution $f_\beta(t)$ of (E_c) such that $f_\beta \sim g_\beta$ and completes the demonstration of the theorem. ∎

II.3. *Holomorphic Families of Solutions*

Let U be an open subset of $\mathbb{C}$, and $f(t, c)$ defined for $t \in \mathbb{R}, c \in U$, a solution of (E_c) which depends analytically on c. The following result is an immediate consequence of Theorem II.1.

COROLLARY II.1. *Let $V \subset U$ be defined as the set of numbers c such that there exists $\beta = (c, z^+, z^-) \in \Omega$ with $f(\cdot, c) = f_\beta$. Then, V is open, and the function $c \to \beta(c)$ defined by $f(\cdot, c) = f_{\beta(c)}$ from V to Ω is analytic. If $c_0 \in U \cap (\bar{V} - V)$, then $\beta(c)$ goes to infinity in Ω when $c \in V$ tends to c_0.*

If $0 \in U$, then V is nonempty (because the Airy function is a solution of (E_0) of the requested form); if $0 \in U$ and if $f(t, c)$ is a real *solution of (E_c) for $c \in U \cap \mathbb{R}$, then for $c' < 0 < c''$, $[c', c''] \subset V$ if and only if the analytic function $c \to \alpha(c) = -3c^2 z^+(c) z^-(c)$ defined near $c = 0$ extends to this interval (this is because if $f(t, c)$ is a real solution, $z^-(c)$ is the complex conjugate of $z^+(c)$ for c real).*

III. THE RIEMANN-HILBERT PROBLEM FOR P II:

III.1. *The Original Problem.*

We give below a brief account of the R.-H. formulation for (P II) and of its solution; full details can be found in [5] and [6]. We adopt the notations of these authors, so that the reader already familiar with the problem can follow more easily, although, since we are interested in a particular case ($\alpha = 0$, solutions with the asymptotic behaviour (I.1) at infinity) they may look a little too complicated for the situation at hand. As we mentioned in the Introduction, we adopt the stratey of [6], and make the solution more explicit in our case to produce a single Fredholm equation with a known kernel.

Equation (P II) (for $\alpha = 0$) can be written as the compatibility condition between two systems of *linear* equations for the two-component vector $\chi(x, t)$:

$$\partial_t \chi = \begin{pmatrix} -iz & w(t) \\ w(t) & iz \end{pmatrix} \chi \tag{III.1}$$

$$\partial_z \chi = \begin{pmatrix} -i(4z^2 + t + w^2) & 4zw + 2iw' \\ 4zw - 2iw' & i(4z^2 + t + 2w^2) \end{pmatrix} \chi. \tag{III.2}$$

While the IST formulation works mainly with (III.1), the R.H. approach deals primarily with (III.2), which is a *linear* system with *rational* (polynomial) coefficients, and a *single irregular singular point* of rank 2 at infinity. The three Stokes multipliers at infinity will be denoted by a, b, c; they satisfy

$$a + b + c + abc = 0 \tag{III.3}$$

which is equivalent to the absence of monodromy around any finite point. As in [6], we let $\theta(z, t) =_{\text{def}} 4i/3\, z^3 + izt$, $W(z, t)$ a fundamental solution matrix for (III.2) and $\Psi(z, t) = W(z, t)\, Y^{-1}$, $Y = \text{def}\, \text{diag}(e^{-\theta}, e^{+\theta})$ so that $\Psi(z, t)$ tends to the unit matrix as z approaches infinity. Setting $\Psi = \Psi_i$ in each sector S_i (S_i is the angular sector between the rays C_i and C_{i+1}), as shown in Fig. 1.

The R.H. problem is defined by the jump matrices $g_i(z, t)$ such that:

$$\Psi_{j+1} = \Psi_j g_j, \quad 1 \leqslant j \leqslant 5; \qquad \Psi_1(ze^{2i\pi}, t) = \Psi_6(z, t)\, g_6(z, t) \tag{III.4}$$

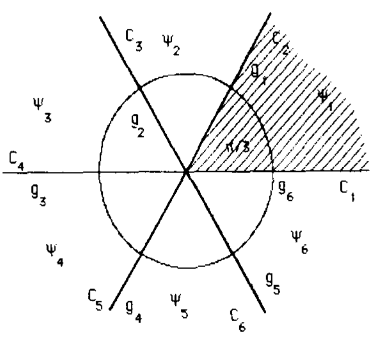

FIGURE 1

and they are given as

$$
\begin{aligned}
&g_1=\begin{pmatrix}1 & 0\\ ae^{2\theta} & 1\end{pmatrix} \qquad g_2=\begin{pmatrix}1 & be^{-2\theta}\\ 0 & 1\end{pmatrix} \qquad g_3=\begin{pmatrix}1 & 0\\ ce^{2\theta} & 1\end{pmatrix}\\
&g_4=\begin{pmatrix}1 & ae^{-2\theta}\\ 0 & 1\end{pmatrix} \qquad g_5=\begin{pmatrix}1 & 0\\ be^{2\theta} & 1\end{pmatrix} \qquad g_6=\begin{pmatrix}1 & ce^{2\theta}\\ 0 & 1\end{pmatrix}.
\end{aligned}
\tag{III.5}
$$

We stress that a, b, and c are *fixed* numbers, i.e., they do not vary as t varies and w satisfies (P II); they are the two free parameters (since a, b, and c are constrained by (III.3)) which label the solution of (P II). Again, we refer to [5 or 6] for a more detailed account of all this. In order to recover the solution $w(t)$ of (P II), one uses the asymptotics

$$w(t)=\lim_{z\in S_1, z\to\infty} -2iz\Psi_{1_{(2,1)}}(z,t), \tag{III.6}$$

where $\Psi_{1_{(2,1)}}(z,t)$ is the lower left element of the matrix Ψ_1. We are interested in a one parameter family of solutions, namely those with the asymptotic behaviour (I.1). As mentioned in [5], this implies $b=0$. However, since we shall use the solution of [6], it will be more convenient to work with $a=0$. In fact, the whole problem is equivariant under the substitution

$$z'=z\,e^{2i\pi/3}, \qquad t'=t\,e^{4i\pi/3} \tag{III.7}$$

which leaves $\theta(z,t)$ invariant and permutes the rays C_i, changing b into a. We therefore set $a=0$, and consequently (because of (III.3)), $b=-c$. The solution of (P II) on the real axis will correspond to arg $t'=\pi/3$ mod π and c will later be identified with r, appearing in the asymptotics (I.1). Henceforth, we drop the primes from the notation.

III.2. *Decompoition of the R.H. problem:*

Since $a=0$, $\Psi_1=\Psi_2$, $\Psi_4=\Psi_5$ and the line C_2-C_5 does not play any role. The R.H. for Ψ has a discontinuity at $z=0$, and we eliminate it, following [6], by setting

$$
\begin{aligned}
&\Psi_1=\Psi_2=\Phi_1\\
&\Psi_3=\Phi_3G_2\\
&\Psi_4=\Psi_5=\Phi_4G_2G_3\\
&\Psi_6=\Phi_6G_6^{-1}.
\end{aligned}
\tag{III.8}
$$

The R.H. problem for the Φ's has its jump matrices g_i' given by (see Fig. 2)

$$g_2' = g_2 G_2^{-1} \qquad g_3' = G_2 g_3 G_3^{-1} G_2^{-1}$$
$$g_5' = (G_2 G_3) g_5 G_5 (G_2 G_3)^{-1} \qquad g_6' = G_6^{-1} g_6, \tag{III.9}$$

where $G_i =_{\text{def}} g_i(B=0)$ and $g_i'(0) = \text{id}$ (any i).

To "decompose" the R.H. problem consists of setting

$$\Phi_1 = \Sigma_1 F_1 a_1 \qquad \Phi_4 = \Sigma_2 F_2 b_1$$
$$\Phi_3 = \Sigma_2 F_1 a_2 \qquad \Phi_6 = \Sigma_1 F_2 b_2, \tag{III.10}$$

where (a_1, a_2, b_1, b_2) are constant matrices, (Σ_1, Σ_2) are analytic matrices in $S_6 \cup S_1 \cup S_2$ and $S_3 \cup S_4 \cup S_5$ respectively and (F_1, F_2) are analytic matrices in the upper and lower half planes, respectively. (Σ_1, Σ_2) and (F_1, F_2) satisfy two R.H. problems along the lines $C_3 - C_6$ and $C_1 - C_4$ (the real axis), respectively; one requires that both R.H. are continuous at the origin and at infinity and all three of the four matrices (say all but Σ_2) tend to the unit matrix at infinity.

We skip the details of the computation of the jumps of these R.H. problems and of the determination of the constants (a_1, a_2, b_1, b_2) which parallel the discussion in [6]. Only one elementary observation helps to make their solution more explicit, even in the general case $(\alpha \neq 0)$. We mention it for the benefit of the interested reader, and to show how the formulae below are obtained. In the notation of [6], it amounts to noting that one can choose $A = B = J(G_1 G_2)^{-1}$, where $J = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$. Then

$$a_1 = 1, \qquad a_2 = J G_2^{-1}, \qquad b_1 = J G_3^{-1} G_2^{-1}, \qquad b_2 = G_6. \tag{III.11}$$

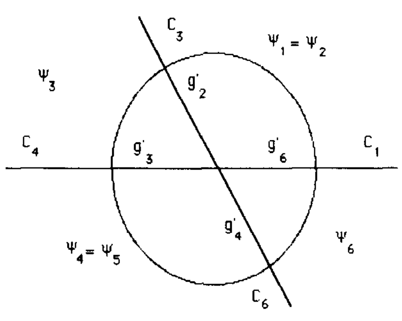

FIGURE 2

(F_1, F_2) satisfy a R.H. problem such that

$$\text{along } C_1,\ F_1 = F_2 g_6; \qquad \text{along } C_4,\ F_1 = F_2 J g_3^{-1} J^{-1}. \tag{III.12}$$

(Σ_1, Σ_2) satisfy a R.H. problem such that

$$\text{along } C_3,\ \Sigma_1 = \Sigma_2 F_1 J g_2^{-1} F_1^{-1}; \qquad \text{along } C_6,\ \Sigma_1 = \Sigma_2 F_2 J g_5 F_2^{-1} \tag{III.13}$$

so that, as anticipated, the jump of this second problem depends on the solution of the first. Let us make one more observation; in fact, we implicitly suppose that the lines C_i have been rotated clockwise by a very small angle, so that the whole problem is set along this new contour, and not the anti-Stokes lines. It is then true that $g_i \to 1$ at infinity because of the decay of $e^{\pm\theta}$ in the various sectors. However, for the sake of clarity of notations, we continue to work formally on the previously defined contours.

In order to have jumps tending to 1 at infinity, and also to translate the problem more easily in terms of integral equations, we make one more modification; setting

$$\Phi_+ = {}^t\Sigma_1, \qquad \Phi_- = -J\,{}^t\Sigma_2 \tag{III.14}$$

and write down the problem for (Φ_+, Φ_-), which we will be dealing with:

$$\Phi_+ = \Gamma_3 \Phi_- \quad \text{along } C_3; \qquad \Phi_+ = \Gamma_6 \Phi_- \quad \text{along } C_6 \text{ (see Fig. 3)}. \tag{III.15}$$

To obtain an explicit expression for Γ_3 and Γ_6, we observe that the jump of the problem for (F_1, F_2) is upper triangular. Consequently, the solution is found in the form

$$F_1 = \begin{bmatrix} 1 & f_1 \\ 0 & 1 \end{bmatrix}, \qquad F_2 = \begin{bmatrix} 1 & f_2 \\ 0 & 1 \end{bmatrix}. \tag{III.16}$$

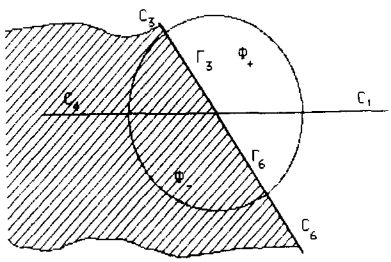

FIGURE 3

An explicit expression for (f_1, f_2) is also available and we mention it, although we shall not need the detailed expression. One has

$$f_1(z, t) = f(z, t), \quad \operatorname{Im} z > 0; \qquad f_2(z, t) = f(z, t), \quad \operatorname{Im} z < 0$$

$$f(z, t) =_{\text{def}} c/2i\pi \left[\int_0^{\infty} e^{-2\theta}(u-z)^{-1}\, du + \int_{-\infty}^{0} e^{-2\theta}(u-z)^{-1}\, du \right]$$

$$\text{for } \operatorname{Im} z \neq 0. \qquad \text{(III.17)}$$

The following symmetry relations are obvious:

$$f_2(z) = -f_1(-z) \quad (\text{all } z); \qquad f_1(0) = c/2 = -f_2(0). \qquad \text{(III.18)}$$

We can now give the explicit expression for the jumps Γ_3 and Γ_6:

$$\Gamma_3 = \begin{pmatrix} 1 & -f_1 \\ -f_1 + c\,e^{-2\theta} & 1 + f_1^2 - cf_1 e^{-2\theta} \end{pmatrix}, \qquad \Gamma_6 = \begin{pmatrix} 1 & -f_2 - C\,e^{2\theta} \\ -f_2 & 1 + f_2^2 - cf_2\, e^{2\theta} \end{pmatrix}. \qquad \text{(III.19)}$$

Note that $\det \Gamma_3 = \det \Gamma_6 = 1$. Tracing back among the various changes of variables, one finds that $\Psi_1 = {}^t\Phi_+ F_1$; upon using (III.16) and (III.6), and taking (III.7) into account, one finds

$$w(t) = \lim_{z \to \infty,\, z \in S_1} -2iz_{+(1,2)}(z, t\, e^{4i\pi/3}) \qquad \text{(III.20)}$$

where $\Phi_{+(1,2)}$ is the *upper right* entry of Φ_+.

III.3. *Reduction to a Fredholm Equation*

We now come to the study of the R.H. problem defined by (III.15) with the expression (III.19) for the jump matrices. As is usual, we define the + and − half z planes, *with respect to the line* $\gamma = C_3 - C_6$ along with the corresponding Hardy spaces $H^{\pm}$ and the Riemann–Hilbert projectors $\Pi_{\pm} : L^2(\gamma) \to H^{\pm}$. In other words, H^+ (resp. H^-) is the subspace of $L^2(\gamma)$ spanned by the functions that are restrictions of functions analytic in the domain of Φ_+ (resp. Φ_-). We shall use the following definition for the Fourier transform:

$$(Ff)(\xi) = \int_{\gamma} e^{-ix\xi} f(x)\, dx. \qquad \text{(III.21)}$$

We denote by $[f]_{\pm} = \Pi_{\pm}(f)$ the projection of a function f on $H^{\pm}$. As is also usual, we add the constants to the span of H^+ and obtain

LEMMA III.1. *One has the decomposition*

$$\Gamma(z)=\begin{pmatrix} 1 & B_- \\ C_+ & D \end{pmatrix}+C\begin{pmatrix} 0 & -e^{2\theta} \\ e^{-2\theta} & 0 \end{pmatrix}, \tag{III.22}$$

where $\Gamma(z)=\Gamma_3(z)$ or $\Gamma_6(z)$ according to whether $z\in C_3$ or $z\in C_6$, $B_-\in H^-$, and $C_+\in H^+$.

Proof. According to (III.19):

(i) $B_- = -f_1+c\,e^{2\theta}$ on C_3; $B_- = -f_2$ on C_4.

(ii) $C_+ = -f_1$ on C_3; $C_+ = -f_1-C\,e^{2\theta}$ on C_6.

The problem for (F_1, F_2) says exactly that the value of B_- (resp. C_+) on C_6 is the continuation of that on C_3 across C_4 (resp. C_1). This completes the proof of the lemma. ∎

For convenience, we write

$$\Phi_\pm=\begin{pmatrix} \alpha_\pm & \beta_\pm \\ \gamma_\pm & \delta_\pm \end{pmatrix}. \tag{III.23}$$

The jump condition is

$$\begin{pmatrix} \alpha_\pm & \beta_\pm \\ \gamma_\pm & \delta_\pm \end{pmatrix}=\begin{pmatrix} 1 & B_- -ce^{2\theta} \\ C_+ +ce^{2\theta} & D \end{pmatrix}\begin{pmatrix} \alpha_- & \beta_- \\ \gamma_- & \delta_- \end{pmatrix} \tag{III.24}$$

and we try to solve for $\beta_+=\Phi_{+(1,2)}$; to this end, we invert the jump matrix, taking advantage of the fact that it has determinant 1, that is,

$$D=1+(C_+ +c\,e^{-2\theta})(B_- -c\,e^{2\theta}). \tag{III.25}$$

In particular we obtain the two equations:

$$\beta_- = D\beta_+ -(B_- -c\,e^{2\theta})\,\delta_+ \tag{III.26}$$

$$\delta_- = -(C_+ +c\,e^{-2\theta})\,\beta_+ +\delta_+ \tag{III.27}$$

Taking the + projection of Eq. (III.27), and using the fact that $\Phi_\pm\to 1$ at infinity,

$$\delta_+ = 1+[\beta_+(C_+ +c\,e^{-2\theta})]_+ . \tag{III.28}$$

Substituting this value of δ_+ in (III.26), and using (III.25) yields

$$\beta_- +B_- -c\,e^{2\theta}=\beta_+ +[e^{-2\theta}\beta_+]_-\cdot(B_- -c\,e^{2\theta}). \tag{III.29}$$

Finally, taking the $+$ part of this equation, one finds

$$-c[e^{2\theta}]_+ = \beta_+ - c^2[e^{2\theta}[e^{-2\theta}\beta_+]_-]_+ . \qquad \text{(III.30)}$$

To clarify the meaning of this equation, we introduce the two operators $J_\pm$ as

$$f \in L^2(\gamma); \qquad J_\pm(f) = \Pi_\pm(e^{\pm 2\theta} f). \qquad \text{(III.31)}$$

The relation $J_-(f) = (J_+(f^*))^*$ is obvious, where $*$ denotes the conjugation (still with respect to the axis γ). If we now set $F = F(z, t, c) =_{\text{def}} -C^{-1}\beta_+$, (III.30) can be translated into the following.

THEOREM III.1. *Let $F(z, t, c) \in H^+$ (t and c enter as parameters) be a solution of the equation*

$$F - c^2(J_+ J_-)(F) = J_+(1), \qquad \text{(III.32)}$$

where $J_\pm$ is defined as in (III.31), *then $w(t)$ is a solution of* (P II), *with*

$$w(t) = 2ic \lim_{z \to \infty, z \in S_1} zF(z, t\, e^{4i\pi/3}, c). \qquad \text{(III.33)}$$

We shall now show that (III.32) is really a Fredholm equation, and that all the quantities mentioned in the theorem are well defined. To this effect, we rotate the t and z planes again, so as to write down everything explicitly with respect to the real axis. Thus, we set $z = |t|^{1/2}\, x\, e^{-i\pi/3}$. Then:

(i) for $t > 0$, $\theta(x, t) = -it^{3/2}(4/3x^3 + x)$,

$$J_\pm(f)(x) = (i^{\pm 1}/2\pi) \int_{-\infty}^{+\infty} e^{\pm 2\theta(y)} f(y)(x \pm i0 - y)^{-1}\, dy \qquad \text{(III.34i)}$$

for $\operatorname{Im} x > 0$ and $\operatorname{Im} x < 0$, respectively;

(ii) for $t < 0$, $\theta(x, t) = -i\,|t|^{3/2}(4/3x^3 - x)$

and the same formula (numbered as (III.34ii)) obtains for $J_\pm$ with the above value of the phase θ.

The major difference between the two cases lies of course in the fact that, in the former the critical points of θ are imaginary ($\pm i/2$) whereas they are real ($\pm 1/2$) in the latter. If now F is defined as a solution of (III.32) with the definition (III.34i) or (III.34ii) for $J_\pm$, (III.33) translates into:

$$w(t) = 2ic\,|t|^{1/2} \lim_{x \to \infty,\, \operatorname{Im} x > 0} x \cdot F(x, t, c). \qquad \text{(III.33bis)}$$

To describe the operator $T = J_+ J_-$ from H^+ into itself, we have the following lemma, which holds for any real t.

LEMMA III.2. *T is self-adjoint, positive, compact, and one-to-one and* $\|T\| < 1$.

Proof. Let $S = \Pi_- e^{-2\theta} \Pi_+$ from H^+ into H^-; S is clearly compact and $T = S^* S$, so that T is self-adjoint, positive, and compact. Using the Fourier transform, we have

$\forall f \in L^2(\mathbb{R})$,

$$F(e^{-2\theta} f_+)(\xi) = 1/2\, |t|^{-1/2} \int_0^\infty Ai(t + 1/2(\eta - \xi)\, |t|^{-1/2})(Ff)(\eta)\, d\eta, \quad \text{(III.35)}$$

where Ai denotes the usual Airy function. Now, if $Tf = 0$, $Sf = 0$, which is the same as $F(e^{-2\theta} f_+)|_{\xi < 0} = 0$. But the Fourier transform of $e^{-2\theta} f_+$ is an analytic function of ξ by (III.35) and the asymptotics of Ai at $+\infty$. Thus $Sf = 0$ implies $\Pi_+ f = 0$ and T is one-to-one. Also, $|e^{\pm 2\theta}| = 1$ so that $\|S\| \leqslant 1$ and $\|T\| \leqslant 1$. Suppose $\|T\| = 1$; then 1 is an eigenvalue (since T is compact positive) and there exists $f \in H^+$, $\|f\| = 1$, $Tf = f$. This implies $\|Sf\| = 1$, or $e^{-2\theta} f \in H^-$, or $F(e^{-2\theta} f)|_{\xi > 0} = 0$, which again is impossible by analyticity. ∎

The following corollary is now easy (and well known).

COROLLARY III.1. *For any $c \notin (-\infty, -1) \cup (1, +\infty)$, Eq.* (III.32) *has a unique solution and the Painlevé transcendent $w(t)$ has no pole on the real axis.*

Poles of $w = w(t, c)$ arise when c^{-2} is an eigenvalue of T; to prove the existence of infinitely many poles when c is real, $|c| > 1$ appears to be difficult, because it amounts to determining whether the particular function $J_+(1)$ is orthogonal to the kernel of $(T - c^{-2} id)$ or not. Equation (III.32) however, places severe restrictions on the possible locations of the poles; in this direction, we have the following:

PROPOSITION III.1. *Let $1 > \mu_1(t) \geqslant \cdots \geqslant \mu_n(t) \geqslant \cdots \geqslant 0$ be the eigenvalues of T, for $t < 0$. Then*

$$\forall n > 0, \qquad \lim_{t \to -\infty} \mu_n(t) = 1. \qquad \text{(III.36)}$$

Proof. It is enough to show that an infinite dimensional subspace V of H^+ exists such that

$$\lim_{t \to -\infty} \sup_{f \in V, \|f\| \leqslant 1} \|\Pi(e^{-2\theta} f)\| = 0 \tag{III.37}$$

because in this case, if $f \in V$, $\|f\| = 1$, we have

$$\|Sf\|^2 = \|\Pi_-(e^{-2\theta} f)\|^2 = 1 - \|\Pi_+(e^{-2\theta} f)\| \to 1 \quad \text{as} \quad t \to -\infty \tag{III.38}$$

and we conclude by the min–max principle.

We use formula (III.35) and perform the scaling

$$(Gf)(\xi) =_{\text{def}} |t|^{-3/4} (Ff)(|t|^{3/2} \xi) \tag{III.39}$$

which is an isometry of L^2; then $A = F \circ \Pi_+ e^{-2\theta} \circ G^{-1}$ writes

$$(Af)(\xi) = 1/2\,|t| \int_0^\infty Ai(1/2\,|t|(\eta - \xi - 2))(Gf)(\eta)\, d\eta. \tag{III.40}$$

We take V to be the space of functions f such that Gf is a combination of characteristic functions of closed subintervals of $(0, 2)$. This implies that Af is a combination of functions $B(1/2\,|t|(\beta + \xi))$, $\beta \in (0, 2)$ with

$$B(z) = 1/2\pi \int_{\mathbb{R}} e^{i(s^3/3 - zs)} s^{-1}\, ds. \tag{III.41}$$

Now, $|B(z)| \leqslant C^{st}\, |z|^{-3/4}$ for $|z| \geqslant 1$, so that

$$B(1/2\,|t|(\beta + \xi)) \leqslant C^{st}\, |t|^{-3/4} (\beta + \xi)^{-3/4} \tag{III.42}$$

and

$$\lim_{|t| \to +\infty} \|B(1/2\,|t|(\beta + \xi))\|_{L^2(\xi \geqslant 0)} = 0. \tag{III.43}$$

This finishes the proof. ∎

Since T has norm less than 1 for $|c| < 1$, (III.32) can be solved by the usual Neumann expansion. That is,

$$F = \sum_0^\infty c^{2n} T^n J_+(1) = \sum_0^\infty c^{2n} (J_+ J_-)^n J_+(1). \tag{III.44}$$

This series has complicated asymptotic behaviour when t approaches $-\infty$ (because the critical points of θ are real), and this will be the subject of the next section; Corollaries II.1 and III.1 show that there *exists* a connection formula $r \to (d(r), \theta_0(r))$ for r small enough. Since we shall prove that $d(r)$ is given by (I.3), this expression can be continued and remains valid for any r, $|r| < 1$. For the time being, we notice that when t is positive, the terms in (III.44) are exponentially decreasing with n (and t) and the asymptotics are given by the first one, which yields

$$w(t, c) \sim_{t=+\infty} -cAi(t) \qquad (\sim -c/(2\pi^{1/2}t^{1/4})\, e^{-2/3t^{3/2}}). \tag{III.45}$$

This enables us to identify the coefficient r in formula (I.1) with $-c$. The minus sign is indifferent since (P II) is invariant under the substitution $w \to -w$; we also recall that c was originally introduced as a Stokes multiplier. We now prove (III.45), taking the first term in (III.44) and using (III.33bis); (III.45) is seen to be equivalent to

$$\lim_{x \to \infty} 2icxJ_+(1)(x) = -ct^{-1/2} Ai(t). \tag{III.46}$$

Using the definition (III.34i) of J_+, we see that

$$\begin{aligned} \lim_{x \to \infty} 2icx\, J_+(1)(x) &= -c/\pi \int_{\mathbb{R}} e^{2it^{3/2}(4/3y^3+y)}\, dy \\ &= -c/(2\pi\sqrt{t}) \int_{\mathbb{R}} e^{i(u^3+tu)}\, du = -ct^{-1/2} Ai(t). \quad \blacksquare \end{aligned} \tag{III.47}$$

Remark. To connect the above with the Gelfand–Levitan equation approach (IST), we can define

$$K(\xi, t, c) =_{\text{def}} 2c\,|t|^{1/2}\, \text{Fourier}(F(|t|^{1/2}\xi, t, c)), \tag{III.48}$$

that is, we take the Fourier transform of F; then (III.32) and (III.33bis) yield

$$\begin{aligned} &K(\xi, t, c) - c^2/4 \int_0^\infty d\theta \int_0^\infty d\eta\, Ai(t+\xi/2+\theta/2) \cdot Ai(t+\eta/2+\theta/2) \cdot K(\eta, t, c) \\ &\quad = cAi(t+\xi/2) \qquad (\xi \geqslant 0) \\ &K(0, t, c) = w(t, c) \end{aligned} \tag{III.49}$$

and these are equivalent to the Gelfand–Levitan equation (see, e.g., [8, p. 243]).

IV. The Connection Formula

This section is devoted to the proof of formula (1.3), using the expression of $w(t)$ given by (III.33 bis). In the sequel, it will be convenient to use $\lambda = t^{3/4}$, which appears as the natural large parameter. We rewrite (I.2) as

$$t^{-1/2}\, w(t) \sim_{t=+\infty} d\lambda^{-1}\, \mathrm{Re}\{\exp i(2/3\,\lambda^2 - \mathrm{id}^2 \ln \lambda + \theta - \pi/2)\}. \quad (\mathrm{I}.2')$$

To find the value of d, we compute the asymptotic expansion of $t^{-1/2}\, w(t)$ as a series in powers of λ and $\ln \lambda$, in the large λ limit. There will be two complex conjugate terms, corresponding to the two critical points in the phase of $J_\pm$ (see (III.34)). Now, the λ^{-1} term can be written $\exp(2i/3\lambda^2)\, \lambda^{-1}[\Sigma_n\, a_n(\log \lambda^{-1})]$ with some coefficients a_n; formula (I.2′) shows that

$$a_n = \mathrm{id}^2/n\; a_{n-1} \quad (\mathrm{IV}.1)$$

and we shall need only the case $n = 1$. Our goal is thus to prove that

$$a_1 = 1/i\pi \log(1 - r^2)\, a_0 \quad (\mathrm{IV}.2)$$

which is equivalent to the connection formula for the amplitude (I.3), because of the results of part III. To obtain the formula for the phase, it would be necessary to compute the value of a_0 (or of any a_n), and not only a ratio. We reduced the computation to—many—elementary integrations, but were not able to find an explicit formula.

We start with formula (III.44) with $J_\pm$ given by (III.34) and we denote the phase by

$$\varphi(y) = 2(4/3y^3 - y). \quad (\mathrm{IV}.3)$$

The general term in the series can be written

$$\begin{aligned}(J_+J_-)^n J_+(1)_{(x)} &= (i/2\pi)^{n+1}(1/2i\pi)^{n+1}(1/2i\pi)^n \\ &\quad\times \int\cdots\int e^{-i\lambda^2(\varphi(x_0)+\cdots+\varphi(x_n))}(x-x_0)^{-1} \\ &\quad\cdot [e^{i\lambda^2\varphi(y_1)}(x_0-y_1)^{-1}(y_1-x_1)^{-1}]\cdots \\ &\quad\times [e^{i\lambda^2\varphi(y_n)}(x_{n-1}-y_n)^{-1}(y_n-x_n)^{-1}]\, dx_0\cdots dy_n \quad (\mathrm{IV}.4)\end{aligned}$$

and we are interested in the coefficient of x^{-1} in the large x expansion, which means we can drop the $(x-x_0)^{-1}$ on the r.h.s. of (IV.4). Here, the integration is taken over the horizontal lines $\mathrm{Im}\, x_k = -\varepsilon_k$, $\mathrm{Im}\, y_k = \varepsilon_k$, ε_k

some small positive numbers. Since we consider the large λ limit, we shall have to apply the saddle point method around the critical points $\pm\frac{1}{2}$. It is easy to see, however, that the terms of order $\lambda^{-1}(\log\lambda)^j$ (any positive j), arise only from the terms where all the x's and all the y's are near the *same* point $\pm\frac{1}{2}$ and the contributions of these two possibilities (i.e., all the x's and y's near $-\frac{1}{2}$ or near $+\frac{1}{2}$) will obviously be complex conjugate numbers. All other terms yield higher order contributions in the asymptotics of $w(t)$.

In order to get the saddle point expansion, we must deform the contours of integration; we first bend the horizontal lines for x_k into the path γ_{x_k} (see Fig. 4), an operation which does not produce any residue-type contribution in the integrals.

We then perform one more deformation to arrive at the following configuration (Fig. 5).

This second deformation, however, does introduce residue terms. The following formula obtains

$$\begin{aligned} 1/2i\pi \int_{\operatorname{Im} y_k=\varepsilon} & [e^{i\lambda^2\varphi(y_k)}(y_k-x_{k-1})^{-1}(x_k-y_k)^{-1}]\,dy_k \\ &= 1/2i\pi \int_{\gamma} [e^{i\lambda^2\varphi(y_k)}(y_k-x_{k-1})^{-1}(x_k-y_k)^{-1}]\,dy_k \\ &\quad - e^{i\lambda^2\varphi(x_k)}(x_k-x_{k-1})^{-1}T_k(x_k)+e^{i\lambda^2\varphi(x_{k-1})}(x_k-x_{k-1})^{-1}T_{k-1}(x_{k-1}), \end{aligned} \tag{IV.5}$$

where T_j denotes the characteristic function of the part of the path γ_{x_j} situated between the real points $\pm\frac{1}{2}$.

In this way, every integral (IV.4) produces a total of 3^n integrals, according to which term is picked up on the r.h.s. of formula (IV.5) and these choices can be specified by a sequence of n symbols σ_i with

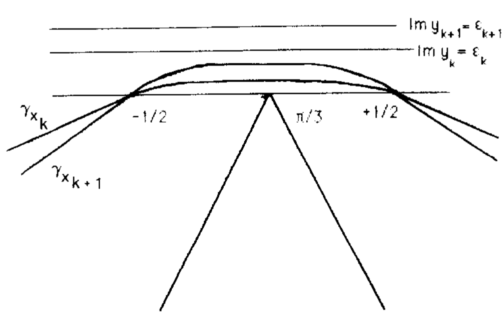

FIGURE 4

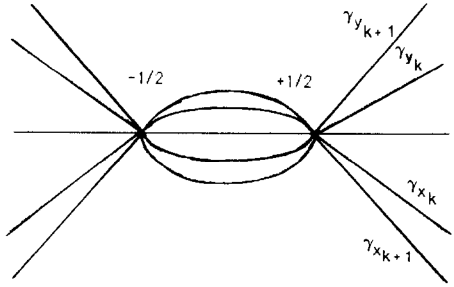

FIGURE 5

$\sigma_i \in \{0, \pm 1\}$. For instance, if $n=5$, the term corresponding to the sequence $(-1, -1, 0, +1, 0)$ reads

$$(i/2\pi)^6(1/2i\pi)^2 \int \cdots \int e^{-i\lambda^2(\varphi(x_2)+\varphi(x_3)+\varphi(x_5))}(x_1-x_0)^{-1}\cdot(x_2-x_1)^{-1}$$
$$[e^{i\lambda^2\varphi(y_1)}(y_1-x_2)^{-1}(x_3-y_1)^{-1}]\cdot(x_4-x_3)^{-1}$$
$$\cdot[e^{i\lambda^2\varphi(y_2)}(y_2-x_4)^{-1}(x_5-y_2)^{-1}], \tag{IV.6}$$

where the variable y_i runs along γ_{y_i} and x_i along γ_{x_i} between $-\frac{1}{2}$ and $+\frac{1}{2}$; these can in fact be taken over the real interval $(-\frac{1}{2}, +\frac{1}{2})$.

We shall not have to compute all these integrals, but will use intermediate ones, arising from sequences of -1's or $+1$'s. A technical proposition is thus needed.

Let $a>0$, $x\in\mathbb{C}$, $\operatorname{Im} x<0$, $y\in\mathbb{C}$, $\operatorname{Im} y>0$; for any integer $k\geqslant 0$, we define the functions $G_k(x, y)$ and $H_k(x)$ by

$$G_0(x, y) = 1/(y-x)$$

$$G_k(x, y) = \int_{-a}^{a} dt_1 \cdots \int_{-a}^{a} dt_k (t_1-x)^{-1}(t_2-t_1)^{-1}\cdots(y-t_k)^{-1}$$
$$k\geqslant 1; \operatorname{Im} x < \operatorname{Im} t_1 < \cdots < \operatorname{Im} t_k < \operatorname{Im} y \tag{IV.7}$$

$$H_k(x) = \int_{-a}^{+a} G_k(x, y)\, dy.$$

Let us also introduce the notation

$$X = \log(a-x)/(a+x); \qquad Y = \log(a-y)/(a+y); \qquad T = X - Y \tag{IV.8}$$

with the determination of the logarithm such that $X|_{x=0} = Y|_{y=0} = 0$. Then:

PROPOSITION IV.1.

$$G_k(x, y) = 1/(y-x) \cdot 1/k! \prod_{j=1}^{k} (T - 2i\pi j) \tag{IV.9.i}$$

$$H_k(x) = 1/(k+1)! \prod_{j=0}^{k} (X - i\pi - 2ij\pi). \tag{IV.9.ii}$$

Proof. We first check by induction on k that

$$(y-x) \cdot G_k(x, y) = \Theta_k(X, Y), \tag{IV.10}$$

where $\Theta_k(X, Y)$ is a polynomial in X and Y of total degree at most k. By definition

$$G_{k+1}(x, y) = \int_{-a}^{+a} 1/(t_1 - x)\, G_k(t_1, y)\, dt_1 \tag{IV.11}$$

and therefore:

$$(y-x) \cdot G_{k+1}(x, y) = \int_0^\infty \Theta_k(\log u, y)[(u - \exp Y)^{-1} - (u - \exp X)^{-1}]\, du. \tag{IV.12}$$

But it is easy to check that the function

$$F_k(z) = \int_0^\infty (\log u)^k [(1+u)^{-1} - (u+z)^{-1}]\, du \tag{IV.13}$$

is a polynomial of degree $k+1$ in $\log z$ (compute $zF_k'(z)$) and (IV.10) follows. Let y now be fixed, $\operatorname{Im} y > 0$. Then, $x \to G_k(x, y)$ is a holomorphic function on the covering of $\mathbb{C} \backslash \{x = a\}$ near $x = a$. We shall investigate the monodromy structure of this function around $x = a$. By (IV.7) and the Cauchy formula, we see that after one loop counterclockwise around this point, $G_k(x, y)$ is changed into $G_k(x, y) + 2i\pi G_{k-1}(x, y)$ and X into $X + 2i\pi$; thus

$$\Theta_k(X + 2i\pi, Y) = \Theta_k(X, Y) + 2i\pi\Theta_{k-1}(X, Y) \tag{IV.14}$$

and by the same argument on the Y variable

$$\Theta_k(X, Y - 2i\pi) = \Theta_k(X, Y) + 2i\pi\Theta_{k-1}(X, Y) \tag{IV.15}$$

from (IV.14) and (IV.15) we conclude by induction that $\Theta_k(X, Y)$ depends on T only, $\Theta_k(X, Y) = Q_k(T)$, and Q_k satisfies

$$Q_k(T + 2i\pi) = Q_k(T) + 2i\pi Q_{k-1}(T). \tag{IV.16}$$

Let us notice that for $k \geqslant 1$, the function $G_k(x, y)$ continuously extends to $x = y > A$, and that $T = 2i\pi$ at this point, so that $Q_k(2i\pi) = 0$. From this and (IV.16) we conclude that

$$Q_k(T) = 1/k! \prod_{j=1}^{k} (T - 2i\pi j) \tag{IV.17}$$

and the proof of (IV.9.i) is complete.

Now, to prove (IV.9.ii), we write

$$H_k(x) = \int_{-a}^{+a} \Theta_k(X, Y)(y - x)^{-1}\, dy \tag{IV.18}$$

and we note that

$$\int_{-a}^{+a} (y - x)^{-1} \log^k((a - y)/(a + y))\, dy$$
$$= \int_{-\infty}^{\infty} (\log u)^k [(1 + u)^{-1} - (u - \exp X)^{-1}]\, du \tag{IV.19}$$

from (IV.13); we conclude that $H_k(x) = R_k(X)$, where R_k is a polynomial of degree at most $k + 1$, and we have $R_0(X) = X - i\pi$. By the monodromy argument at $x = a$, we have

$$R_k(X + 2i\pi) = R_k(X) + 2i\pi R_{k-1}(X) \tag{IV.20}$$

and obviously, $H_k(x) \to 0$, $x = -it$, $t \to +\infty$ $(X \to i\pi)$, and therefore $R_k(i\pi) = 0$; we conclude that

$$R_k(X) = 1/(k + 1)! \prod_{m=0}^{k} (X - i\pi - 2im\pi). \tag{IV.21}$$

The proof is complete. ∎

Our ultimate goal is relation (IV.2) which is equivalent to the amplitude part of the connection formula. Each a_α $(\alpha = 0, 1)$ appears as a sum,

$$a_\alpha = \sum a_{\alpha,n} r^{2n}, \tag{IV.22}$$

and (IV.2) is in turn equivalent to the set of relations:

$$a_{1,n} = -1/(i\pi) \sum 1/k\, a_{0,n-k}. \tag{IV.23}$$

In the neighbourhood of a critical point, say $+\frac{1}{2}$, the stationary phase principal part is obtained by making the change of variable $x=\frac{1}{2}+\lambda^{-1}x'$ and similarly for y. An important remark is now that $\log\lambda$ drops in the expression $(X-Y)$, X and Y being defined as above (see (IV.8)) with $a=\frac{1}{2}$. As a consequence, the $\log\lambda$ contribution in a term of the form

$$\int\cdots\int H_p(x)\,\Gamma(x,y)\,H_q(y)\,dx\,dy \tag{IV.24}$$

will come from the consideration of H_p and H_q only. (IV.24) corresponds to a sequence starting with p times -1 and ending with q times $+1$, and being otherwise arbitrary; in other words, Γ corresponds to an arbitrary sequence of symbols, except it does not begin with -1 nor end with $+1$. In (IV.24), of course, we have dropped a coefficient in front and adopted a compact self-explanatory notation. One has

$$\begin{aligned} H_p(x)=R_p(X) &\sim R_p(\log\lambda^{-1}+\log x)+\text{h.o.t.} \\ &= R_p(\log x)+\log\lambda^{-1}\,R'_p(\log x)+\text{h.o.t.}, \end{aligned} \tag{IV.25}$$

where we applied Taylor formula to a *polynomial.*

The following simple proposition is crucial:

PROPOSITION IV.2. *For any positive n,*

$$R'_n(T)=\sum_{k=1}^{n}(2\pi/i)^{k-1}\,R_{n-k}(T)/k. \tag{IV.26}$$

Proof.

$$R_n(T)=(2i\pi)^n\,P_n(T/(2i\pi)-1/2) \tag{IV.27}$$

with

$$P_n(T)=1/n!\cdot\prod_{m=0}^{n-1}(T-m). \tag{IV.28}$$

The formula

$$P'_n(T)=\sum_{k=1}^{n}(-1)^{k+1}\,P_{n-k}(T)/k \tag{IV.29}$$

can now be checked by induction and is obviously equivalent to (IV.26). ∎

To complete the proof of formula (IV.23) is now a matter of combinatorics, taking the various coefficients into account; for convenience we shall again use compact symbolic notations. The contribution to $a_{0,n}$ of a term like (IV.24) is given by an integral

$$(i/2\pi)^{n+1} \cdot (1/2i\pi)^{n-s} \int \cdots \int R_p(\log x')\, \Gamma(x', y')\, R_q(\log y')\, dx'\, dy' \qquad \text{(IV.30)}$$

here, $n = p + q +$ (length of Γ) is the total order, and s is the number of contractions that have occured, due to the repeated use of formula (IV.5); it is also equal to the number of ± 1's in the sequence defining the integral.

By the proposition above, the contribution of (IV.24) to $a_{1,n}$ will be given by a sum of contributions to $a_{0,k}$ ($k = 0$ to $n-1$). To be more precise, it will be easier to work "backwards": consider a diagram of the form $H_1 \Gamma H_m$ of length $n-k$. It contributes to $a_{0,n-k}$ by an integral of the form (IV.30) with $n-k$ in place of n. What are the corresponding terms in $a_{1,n}$? It is easy to check that they arise from the two terms $\int R'_{1+k} \Gamma R_m$ and $\int R_1 \Gamma R'_{m+k}$ and from them *only*, because all other terms give higher order contributions. So, they both give a contribution of the form $\int R_1 \Gamma R_m$ in $a_{1,n}$ with a coefficient

$$\begin{aligned} &1/k \cdot (i/2\pi)^{n+1} \cdot (1/2i\pi)^{n-s} \cdot 1/k \cdot (2\pi/i)^{k-1} \\ &\quad = (i/2\pi)^{n-k+1} \cdot (1/2i\pi)^{n-s} \cdot 1/k \cdot (i/2\pi) \end{aligned} \qquad \text{(IV.31)}$$

which is to be compared to the coefficient $(i/2\pi)^{n-k+1}(1/2i\pi)^{n-s}$ in front of the contribution to $a_{0,n-k}$. The l.h.s. of the above formula comes from the following facts: $(i/2\pi)^{n+1}$ appears because we are dealing with diagrams of length n; $n - s = (n-k) + (s+k)$ arises because the number of contractions in this term is greater by k than the corresponding number in the term of length $n-k$; $1/k \cdot (2\pi/i)^{k-1}$ comes from formula (IV.26). All these match nicely together to produce the coefficient $(i/2\pi)^{n-k+1} \cdot (1/2i\pi)^{n-s}$ multiplied by $2 \cdot (1/k) \cdot i/2\pi = -1/i\pi \cdot (1/k)$ (the factor 2 comes in because there are two terms). This finishes the proof of (IV.23) and consequently of (I.3).

We only notice that the above algebraic properties give the impression that there should be other ways to exploit the integral equation (III.32), or even the expansion (III.44) in order to get more global informations on $w(t)$.

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