STABILITY OF NEARLY INTEGRABLE CONVEX HAMILTONIAN SYSTEMS OVER EXPONENTIALLY LONG TIMES

P. LOCHAK[1], A. I. NEISTADT[2] AND L. NIEDERMAN[3]

I INTRODUCTION

In the present paper, we shall study the stability of a near integrable Hamiltonian system over finite but very long intervals of times. So we look at the system governed by the Hamiltonian:

$$H(p,q) = h(p) + f(p,q) \text{ with } (p,q) \in R^n \times T^n,\ T = R/Z,$$

where (p,q) are action-angle variables of the integrable Hamiltonian h. We assume that H is **analytic** over some domain $G \times T^n$ ($G \subset R^n$ a "nice" domain say convex open) and that h is a **convex** function ($\nabla^2 h(p)$ is a sign definite symmetric matrix). The perturbation f is of size ε (see below).

We prove that for initial conditions $(p(0), q(0))$ with $p(0) \in G$ not too close from the boundary, one has:

$$||p(t) - p(0)|| \leq R(\epsilon) \text{ for } |t| \leq \mathcal{T}(\epsilon)\ , \tag{1}$$

provided $\varepsilon \leq \varepsilon_0$, with $R(\varepsilon)$ of order ε^b, $\mathcal{T}(\varepsilon)$ of order $\exp(c/\varepsilon^a)$.

For the *stability exponents* (a,b) one may take: $a = b = 1/(2n)$. Such estimates as (1) were first obtained by N. N. Nekhorochev ([2]) in the case where h is assumed to be *steep*, a weaker requirement than convexity, but with different values of the exponents a and b. Recently P. Lochak ([1]) introduced a very different proof method which in particular allowed him to reach the value $a = 1/(2n+2)$ in the convex case for the exponent which governs the time of stability $\mathcal{T}(\varepsilon)$. Our purpose here is to present a short self-contained proof leading to the value $a = 1/(2n)$; we follow [1] closely, except for an improvement of an analytic lemma (see below II.a) which originates in Neistadt ([3]) .

The value of this improvement lies mainly in that $1/(2n)$ is likely to be **optimal** in the sense that on longer timescales, Arnold's diffusion should switch on, leading to a drift of the action variables of order 1; this is predicted by heuristic reasonnings and computations to be found in papers of B. V. Chirikov ([5]) and, under a different guise, in [1].

We refer once and for all to [1] for background informations, corollaries and further applications of the method we use.

We notice that this value of the exponent has been recently obtained by Pöschel (Preprint [7]) where he uses the original proof method of N. N. Nekhorochev. But we stress that the method developped here (from [1]) suggests a rather different picture than the usual one and draws a sharp qualitative distinction between convex systems which are encountered frequently in nature and other strictly non linear (steep) systems. The most striking qualitative new feature that emerges is probably the role of the resonance phenomenon over finite but exponentially long times.

The scheme of the proof is as follows. Let $\omega(p) = \nabla h(p) \in R^n$ be the frequency vector; if $\omega_0 = \omega(p_0)$ is rational (i.e. multiple of an integer one) then the torus $p = p_0$ is filled with closed orbits of the unperturbed system with common period T such that $T\omega_0 \in Z^n$. We shall first study stability properties of the system in the neighbourhood of such a periodic torus (Part II) where we will reach the exponents $a = b = 1/2$. Then we use Dirichlet's theorem which prescribes a minimal rate of approximation of the vectors of R^n by rational ones; it allows here to approach an arbitrary point in the space of the action variables by points corresponding to periodic tori and apply to the latter the stability results of Part II in order to obtain (1) for an arbitrary initial condition.

II Stability in the neighbourhood of a periodic torus

The Hamiltonian H is supposed to be defined and *analytic* in a neighbourhood of the origin, more precisely on a complex domain $\mathcal{D} = D(R, \rho, \sigma)$, $(\rho > 0, \sigma > 0)$ defined as follows: let B_R be the ball of radius R around the origin, then:

$$\mathcal{D} = D(R, \rho, \sigma) = \left\{(p, q) \in C^{2n},\ \mathrm{dist}\,(p, B_R) \leq \rho\ ;\ \Re(q) \in T^n\ ;\ |\Im(q)| \leq \sigma\right\}\ , \tag{2}$$

with $|\Im(q)| = \mathrm{Sup}_i(|\Im(q_i)|)$; H is supposed to be continuous on the boundary of $\mathcal{D}$. Note that the real part of $\mathcal{D}$ is $B_{R+\rho} \times T^n$. When $0 \leq \delta \leq \rho$ and $0 \leq \xi \leq \sigma$ we denote by $\mathcal{D} - (\delta, \xi)$ the domain $D(R, \rho - \delta, \sigma - \xi)$.

The norm $||.||_{\mathcal{D}}$ is the sup norm (L^∞) over $\mathcal{D}$ and we define E and ε by:

$$||H||_{\mathcal{D}} = E\ ,\ ||f||_{\mathcal{D}} = \varepsilon E\ ,\ \varepsilon \geq 0\ . \tag{3}$$

The euclidian norm is denoted $||.||$, and $||.||_{\mathcal{D}}$ for a vector valued function on $\mathcal{D}$ is defined as the supremum over $\mathcal{D}$ of the euclidian norm of its value.

Let $\omega(p) = \nabla h(p) \in R^n$ and $A(p) = \nabla^2 h(p) \in \mathcal{M}_n(R)$ be the frequency vector and the hessian matrix. In this section $\omega_0 = \omega(0)$ is supposed to be

rational of (minimal) period T, i.e. $T\omega_0 \in Z^n$; so the unperturbed flow is periodic of period T on the torus $p = 0$. We denote $\Omega = ||\omega_0||$ (euclidian norm). Since h is supposed here to be convex so that $A(p)$ is sign definite – say – positive. More precisely if m is a lower bound of the spectrum of A over the real part of the domain, then:

$$\forall p \in B_{R+\rho} \subset R^n,\ \forall v \in R^n\ :\ A(p)v.v \geq m||v||^2\ , \tag{4}$$

where $m > 0$ and $u.v$ denotes the scalar product of two vectors u, v. We will denote M the operator norm of A on $\mathcal{D}$ (the complex domain): $||A(p)v|| \leq M||v||$.

All the constants will be explicitely computed as simple functions of the parameters Ω, m, M, ρ, σ, E, T.

a) Analytic part of the proof:

For $g(q)$ a function defined on the torus, we denote $< g >$ its *time average* along the orbits of the flow directed along ω_0, i.e.:

$$< g > (q) = \frac{1}{T}\int_0^T g(q + \omega_0 t)\, dt.$$

We shall say that g is *resonant* (with respect to ω_0) if $g =< g >$, which means that g is constant along the orbits of the flow directed along ω_0. We say that g is *non resonant* if $< g >= 0$.

Viewing p as a parameter, the Hamiltonian can be decomposed as:

$$H(p,q) = h(p) + Z(p,q) + N(p,q) \tag{5}$$

with:
- $Z(p,q)$ resonant with respect to ω_0,
- $N(p,q)$ non resonant with respect to ω_0.

Using an averaging procedure we will construct a near-identity canonical transformation which permits to reduce the size of the non resonant part of H. This analytic lemma will first be used in a preliminary transformation to reduce significantly this non resonant part. It will then be used iteratively to define a sequence of transformations which permit at each step to reduce geometrically the size of the non resonant part of the perturbation. In the end the composition of all the previous transformations gives a Hamiltonian with an exponentially small non resonant part (in the new variables).

We have $||Z+N||_{\mathcal{D}} = ||f||_{\mathcal{D}} = \varepsilon E$ and define η by $||N||_{\mathcal{D}} = \eta E$, with $\eta \leq 2\varepsilon$ because $N = f - < f >$ and $|| < f > ||_{\mathcal{D}} \leq ||f||_{\mathcal{D}}$ ($< . >$ is an orthogonal projection operator).

Let $\mu > 0$ and $\nu > 0$ be such that:

$$\left|\left|\frac{\partial Z}{\partial p}\right|\right|_{\mathcal{D}-(\delta/4,\xi/4)} \leq \mu \;,\quad \left|\left|\frac{\partial Z}{\partial q}\right|\right|_{\mathcal{D}-(\delta/4,\xi/4)} \leq \nu. \tag{6}$$

In this setting, the following obtains:

Analytic lemma:

Let δ and ξ be two real numbers satisfying:

$$0 < \delta < \rho \;;\; 0 < \xi < \sigma \text{ and } 2\eta TE < \xi\delta. \tag{7}$$

Then there exists a canonical transformation $\mathcal{C} : \mathcal{D}' \to \mathcal{D}$ with $\mathcal{D}' = \mathcal{D}-(\delta,\xi)$ such that $\mathcal{C}$ is one-to-one and its image $\mathcal{C}(\mathcal{D}')$ satisfies:

$$\mathcal{D} - \left(\frac{3\delta}{2};\frac{3\xi}{2}\right) \subseteq \mathcal{C}(\mathcal{D}') \subseteq \mathcal{D} - \left(\frac{\delta}{2};\frac{\xi}{2}\right). \tag{8}$$

If we denote $(p,q) = \mathcal{C}(p',q')$ and $H' = H \circ \mathcal{C}$, then H' can be written as (5) (using primed letters) with:

$$||Z' + N'||_{\mathcal{D}'} = \varepsilon' E \;,\; ||N'||_{\mathcal{D}'} = \eta' E,$$

where: $\varepsilon' \leq \varepsilon + \dfrac{\eta\alpha}{2}$, $\eta' \leq \eta\alpha$ and $\alpha = 9M(R+\rho)\dfrac{T}{\xi} + 70\dfrac{\eta TE}{\xi\delta} + 5\dfrac{\mu T}{\xi} + 5\dfrac{\nu T}{\delta}$. (9)

Finally, let δ' and ξ' such that $0 < \delta' < \rho - \delta$ and $0 < \xi' < \sigma - \xi$ then:

$$\left|\left|\frac{\partial Z'}{\partial p'}\right|\right|_{\mathcal{D}'-(\delta'/4,\xi'/4)} \leq \mu' = \mu + \frac{2\eta\alpha E}{\delta'} \;,\quad \left|\left|\frac{\partial Z'}{\partial q'}\right|\right|_{\mathcal{D}'-(\delta'/4,\xi'/4)} \leq \nu' = \nu + \frac{2\eta\alpha E}{\xi'}. \tag{10}$$

Proof:

We use the formalism of Lie series, so that the transformation $\mathcal{C}$ will be defined as the time 1 map of an auxiliary Hamiltonian $\chi(p',q')$ on $\mathcal{D}'$ (see e.g.[6] for some information about Lie series). Using the Poisson bracket:

$$L_\chi(f) = \{\chi, f\} = \frac{\partial \chi}{\partial p}.\frac{\partial f}{\partial q} - \frac{\partial \chi}{\partial q}.\frac{\partial f}{\partial p},$$

we have $\mathcal{C} = \exp(L_\chi)$ and $H' = \exp(L_\chi)(H)$.

Let f be analytic on $\mathcal{D}$ (continuous on the boundary), in order to estimate the size of the derivatives of f on $\mathcal{D} - (\delta, \xi)$, one writes that at a given point (p, q)

$$\left\| \frac{\partial f}{\partial q}(p,q) \right\| = \mathrm{Sup}_{||e||=1} \left\| \frac{d}{dt}_{|t=0} f(p, q+te) \right\|.$$

One then applies Cauchy formula to the function $t \mapsto f(p, q+te)$ of the complex variable t, defined for $|t| \leq \xi$ and continuous on the boundary when $(p,q) \in \mathcal{D} - (\delta, \xi)$, and obtains

$$\left\| \frac{\partial f}{\partial q}(p,q) \right\|_{\mathcal{D}-(\delta,\xi)} \leq \frac{1}{\xi} ||f||_{\mathcal{D}},$$

the equivalent inequality for $\partial f / \partial p$ is proved in the same way. These inequalities for the Poisson bracket provides:

$$\begin{aligned} ||\{f,g\}||_{\mathcal{D}-(\delta,\xi)} &\leq \frac{||f||_{\mathcal{D}-(\delta',\xi')}}{\delta - \delta'} \frac{||g||_{\mathcal{D}-(\delta',\xi')}}{\xi - \xi'} + \frac{||f||_{\mathcal{D}-(\delta',\xi')}}{\xi - \xi'} \frac{||g||_{\mathcal{D}-(\delta',\xi')}}{\delta - \delta'} \\ &\leq \frac{2}{(\delta-\delta')(\xi-\xi')} ||f||_{\mathcal{D}-(\delta',\xi')} ||g||_{\mathcal{D}-(\delta',\xi')} \end{aligned}$$

for two functions defined on $\mathcal{D}$ (this estimate could be slightly improved).

Then we can write:

$$H' = h + Z + N + \{\chi, h\} + \{\chi, Z+N\} + H' - H - \{\chi, H\} = h + Z + \mathcal{R},$$

with $\mathcal{R} = H' - h - Z$.

The terms of order 1 in $\mathcal{R}$ (w.r.t. ε) are:

$$N(p', q') + \{\chi, h\}(p', q') = N(p', q') + \omega(p', q') . \frac{\partial \chi}{\partial q}(p', q').$$

We will choose as χ a solution, on $\mathcal{D}$, of the equation:

$$\omega_0 . \frac{\partial \chi}{\partial q}(p, q) = N(p, q). \tag{11}$$

In fact, this equation is satisfied by: $\chi(p,q) = \frac{1}{T} \int_0^T N(p, q + \omega_0 t) t \, dt$. Then:

$$\begin{aligned} \mathcal{R}(p', q') = (\omega_0 - \omega(p')) \frac{\partial \chi}{\partial q}(p', q') + \{\chi, Z+N\}(p', q') \\ + H'(p', q') - H(p', q') - \{\chi, H\}(p', q'). \end{aligned}$$

We write:

$$H' = h + Z' + N' \text{ where } Z' = Z + < \mathcal{R} > \ , \ N' = \mathcal{R} - < \mathcal{R} >;$$

one has: $\eta' \leq \dfrac{2||\mathcal{R}||_{\mathcal{D}'}}{E}$, so there only remains to estimate $||\mathcal{R}||_{\mathcal{D}'}$.

From the explicit expression of χ, one immediately obtains: $||\chi||_{\mathcal{D}} < \dfrac{T}{2}||N||_{\mathcal{D}} = \dfrac{\eta T E}{2}$, which implies, taking (7) into account,

$$\left|\left|\frac{\partial \chi}{\partial p}\right|\right|_{\mathcal{D}-(\delta/2;0)} < \frac{\eta T E}{\delta} < \frac{\xi}{2} \ ; \ \left|\left|\frac{\partial \chi}{\partial q}\right|\right|_{\mathcal{D}-(0;\xi/2)} < \frac{\eta T E}{\xi} < \frac{\delta}{2};$$

this in turn ensures the validity of (8).

Using Taylor formula at order two and the fact that $\mathcal{C}(\mathcal{D}') \subset \mathcal{D} - (\delta/2, \xi/2)$, one finds the estimate:

$$||\mathcal{R}||_{\mathcal{D}'} \leq ||\omega_0 - \omega||_{\mathcal{D}'} \left|\left|\frac{\partial \chi}{\partial q}\right|\right|_{\mathcal{D}'} + ||\{\chi, Z + N\}||_{\mathcal{D}'} + \frac{1}{2}||\{\chi, \{\chi, H\}\}||_{\mathcal{D}-(\delta/2,\xi/2)}. \tag{12}$$

Cauchy inequalities and the properties of the hessian matrix imply the following estimates:

$$\bullet ||\omega_0 - \omega||_{\mathcal{D}'} \left|\left|\frac{\partial \chi}{\partial q}\right|\right|_{\mathcal{D}'} \leq M(R + \rho)\frac{\eta T E}{2\xi},$$

$$\bullet ||\{\chi, N\}||_{\mathcal{D}'} \leq 2\frac{||\chi||_{\mathcal{D}}||N||_{\mathcal{D}}}{\xi\delta} = \frac{\eta^2 T E^2}{\xi\delta};$$

on the other hand, since $\mathcal{D}' \subset \mathcal{D} - (\delta/4, \xi/4)$, we can write:

$$||\{\chi, Z\}||_{\mathcal{D}'} \leq \left|\left|\frac{\partial \chi}{\partial p}\right|\right|_{\mathcal{D}'} \left|\left|\frac{\partial Z}{\partial q}\right|\right|_{\mathcal{D}'} + \left|\left|\frac{\partial \chi}{\partial q}\right|\right|_{\mathcal{D}'} \left|\left|\frac{\partial Z}{\partial p}\right|\right|_{\mathcal{D}'} \leq \frac{\eta\nu T E}{2\delta} + \frac{\eta\mu T E}{2\xi}.$$

To estimate the third term we insert again the definition of χ into the Poisson bracket to get:

$$\{\chi, H\} = -N + (\omega_0 - \omega)\frac{\partial \chi}{\partial q} + \{\chi, N + Z\},$$

so that:

$$\begin{aligned} \|\{\chi, \{\chi, H\}\}\|_{\mathcal{D}-(\delta/2,\xi/2)} \leq\ & \|\{\chi, N\}\|_{\mathcal{D}-(\delta/2,\xi/2)} \\ & + \left\|\left\{\chi, (\omega_0 - \omega)\frac{\partial \chi}{\partial q}\right\}\right\|_{\mathcal{D}-(\delta/2,\xi/2)} \\ & + \|\{\chi, \{\chi, N\}\}\|_{\mathcal{D}-(\delta/2,\xi/2)} \\ & + \|\{\chi, \{\chi, Z\}\}\|_{\mathcal{D}-(\delta/2,\xi/2)}. \end{aligned}$$

Using again Cauchy inequalities and (7) we can estimate the previous terms. The sum of these inequalities yields (9) with the given value of α.

To control the derivative of Z', we use $Z'(p',q') - Z(p',q') =< \mathcal{R} >$ and Cauchy inequality gives (10).

At this point we will make a few remarks:

– The first term on (12) represents a frequency shift and comes in because we solve (11) instead of adapting the frequency, that is solve the same equation with $\omega(p)$ substituted for ω_0. Because of this frequency shift which grows with R, we do not work directly on the whole of $\mathcal{D}$, but restrict attention to the smaller domain: $\mathcal{D}' = \mathcal{D}(R(\varepsilon), \rho(\varepsilon), \sigma) \subset \mathcal{D}(R, \rho, \sigma)$, where $R(\varepsilon)$ will be the confinement radius in our theorem. In fact convexity implies that the unperturbed energy increases at least as the square of the distance to the origin; adding a perturbation of order ε, both terms have to be on the same size to ensure confinement. This implies that $R(\varepsilon)$ is at least on the order of $\sqrt{\varepsilon}$, which will be the value chosen here ($b = 1/2$). For a fixed value of the radius of confinement, we cannot improve the frequency shift term, hence we will try to reduce (9) essentially to the first term.

– The analytic lemma will be used $s(\varepsilon)$ times to have a non resonant part in the transformed Hamiltonian of size $\eta' = 2^{-s}\eta$, here $s(\varepsilon)$ will be on the order of $\varepsilon^{-a} = \varepsilon^{-1/2}$. To stay on the domain where H is defined, δ and ξ will be $\mathcal{O}(R(\varepsilon)/s)$ and $\mathcal{O}(\sigma/s)$. If one directly uses Cauchy inequalities to estimate μ and ν (as in [1]), the last two terms in (9) are $\mathcal{O}(\varepsilon^{-1/2}T/\sigma)$ while the frequency shift term is $\mathcal{O}(T/\sigma)$. To obtain terms of the same size in (9), one must first perform a preliminary transformation to have a small enough non resonant part (i.e. η) before using (9) and (10) in the iterative scheme.

– In order to simplify the expression of α we will choose $R(\varepsilon) = \rho(\varepsilon)$; this is an arbitrary but not essential specialisation. We shall often write R and ρ, without making the dependence on ε explicit and rewrite the conditions $R(\varepsilon) \leq R$ and $\rho(\varepsilon) \leq \rho$ at the very end.

b) Preliminary transformation:

As previously said, the iterative scheme must begin with a nonresonant part of size η_0 such that the second term in (9) is on the same order as the frequency shift term (i.e. srT with our value of ξ in the iterative part). Hence, we will first build a preliminary transformation which will be denoted $\mathcal{C}^{(0)}$.

The analytic lemma will be used with: $\delta = R/3$; $\xi = \sigma/3$. Taking into account inequality $\eta < 2\varepsilon$, (7) becomes:

$$\frac{\varepsilon T}{R} \leq \frac{\sigma}{36E}.$$

Also, by Cauchy inequalities:

$$\left\|\frac{\partial Z}{\partial p}\right\|_{\mathcal{D}-(\delta/4,\xi/4)} \leq \frac{12\varepsilon E}{R} \text{ and } \left\|\frac{\partial Z}{\partial q}\right\|_{\mathcal{D}-(\delta/4,\xi/4)} \leq \frac{12\varepsilon E}{\sigma},$$

so (8) becomes:

$$\mathcal{D}-\left(\frac{R}{2},\frac{\sigma}{2}\right) \subset \mathcal{C}^{(0)}\left(\mathcal{D}-\left(\frac{R}{3},\frac{\sigma}{3}\right)\right) \subset \mathcal{D}-\left(\frac{R}{6},\frac{\sigma}{6}\right).$$

The new size of the non resonant part satisfies: $\eta' \leq 54\eta\left(\frac{M}{\sigma}TR + 30\frac{\varepsilon TE}{\sigma R}\right)$.

We suppose that:

$$R = R(\varepsilon) \geq \left(\frac{6E}{M}\right)^{1/2}\varepsilon^{1/2};$$

then (9) yields:

$$\varepsilon' \leq \varepsilon\left(1 + 162\frac{M}{\sigma}TR\right) = \varepsilon_0 \text{ and } \eta' \leq 324\frac{M}{\sigma}\varepsilon TR = \eta_0.$$

c) Main transformation:

We will now define the intermediate transformations: $\mathcal{C}^{(j)} : \mathcal{D}^{(j)} \longrightarrow \mathcal{D}^{(j-1)}$ for $j \in \{1,\dots,s\}$. The initial Hamiltonian, $H = H^{(0)}$ is the one obtained after the preliminary transformation; it is defined on:

$$\mathcal{D}^{(0)} = \mathcal{D}-\left(\frac{R}{3},\frac{\sigma}{3}\right);$$

the intermediate quantities are denoted $H^{(j)}$, ε_j, η_j, μ_j, ν_j. We shall impose that the sequence (η_j) decrease at least geometrically.

Here we take: $\mathcal{D}^{(j)} = \mathcal{D}^{(j-1)} - (\delta_j,\xi_j)$ with $\delta_j = \delta = \frac{R}{3s}$, $\xi_j = \xi = \frac{\sigma}{3s}$, in order that the image $\mathcal{C}^{(0)} \circ \dots \circ \mathcal{C}^{(n)}\left(\mathcal{D}^{(n)}\right)$ contains the real ball B_R in action space.

Since

$$\left\|\frac{\partial Z}{\partial p}\right\|_{\mathcal{D}-(R/12,\sigma/12)} \leq \frac{12\varepsilon E}{R}, \quad \left\|\frac{\partial Z}{\partial q}\right\|_{\mathcal{D}-(R/12,\sigma/12)} \leq \frac{12\varepsilon E}{\sigma},$$

after the preliminary transformation we get (see (10)):

$$\mu_0 \leq \frac{12\varepsilon E}{R} + \frac{6\eta_0 Es}{R} = \mu_0', \quad \nu_0 \leq \frac{12\varepsilon E}{\sigma} + \frac{6\eta_0 Es}{\sigma} = \frac{R}{\sigma}\mu_0'.$$

With the previous inequalities, our values of δ and ξ at each step of the iterative scheme, and formula (10), we can connect μ_j and ν_j for $j \in \{1, \ldots, s\}$ by the relations:

$$\nu_j = \mu_j \frac{\delta}{\xi} = \mu_j \frac{R}{\sigma}.$$

Assume that for $j \in \{0, \ldots, n\}$, one has $\eta_j \leq 2^{-j}\eta_0$, then:

$$\mu_n - \mu_0 \leq \left(2^{-1} + \ldots + 2^{-n}\right) \frac{2\eta_0 E}{\delta} \leq \frac{6E\eta_0 s}{R}.$$

We can apply the lemma if (see (7)): $18s^2TE\eta_n \leq \sigma R$; since $\eta_n < \eta_0$, the previous inequality is satisfied if one imposes the threshold condition:

$$\frac{s^2T\eta_0}{R} \leq \frac{\sigma}{18E}.$$

Under these assumptions, (9) becomes:

$$\eta_{n+1} \leq 54\eta_n \left[\frac{M}{\sigma}sRT + 7\frac{E}{\sigma}\frac{\varepsilon sT}{R} + 20\frac{\eta_0 TEs^2}{R\sigma}\right].$$

We will now require that the second and third terms in the bracket be smaller than the first, which gives two more threshold conditions to satisfy. Under this condition,

$$\eta_{n+1} \leq 162\ \eta_n \frac{M}{\sigma}sRT;$$

in order to get a geometric decrease for the sequence (η_j), we assume that:

$$sRT \leq 3 \cdot 10^{-3}\frac{\sigma}{M}. \tag{13}$$

This relation represents the natural link between the three parameters s, R, T, namely that $s(\varepsilon)$ should be on the order of $(RT)^{-1}$. This implies, $\eta_{n+1} \leq \eta_n/2 \leq 2^{-(n+1)}\eta_0$.

We denote: $(p,q) = \mathcal{C}(p',q') = \left(\mathcal{C}^{(0)} \circ \ldots \circ \mathcal{C}^{(n)}\right)(p',q')$ by composing the transformations built in the two previous parts, $\mathcal{C}$ is defined on $\mathcal{D}-(2R/3, 2\sigma/3)$ and satisfies:

$$c(\varepsilon) = ||p-p'||_{\mathcal{D}-(2R/3,2\sigma/3)} \leq \frac{3sTE}{\sigma}(\eta_0+\ldots+\eta_{s-1}) + \frac{6\varepsilon TE}{\sigma} \leq 6(\varepsilon+\eta_0 s)\frac{TE}{\sigma},$$

with the same estimate for $||q-q'||$.

If we assume:

$$36(\varepsilon+\eta_0 s)TE \leq R\sigma,$$

then, one can state:

$$\mathcal{D}-(R,\sigma) \subseteq \mathcal{C}\left(\mathcal{D}-\left(\frac{5R}{6},\frac{5\sigma}{6}\right)\right),$$

so that we can use $\mathcal{C}$ to change variables on the real part of $\mathcal{D}$. The threshold conditions can be simplified if we use (13) and the definition of η_0, which gives $\eta_0 \leq \varepsilon/s$.

Gathering everything together we arrive at the following:

Lemma:

There exists a canonical transformation $\mathcal{C} : \mathcal{D}' \longrightarrow \mathcal{D}$ with $\mathcal{D}' = \mathcal{D} - (5R/6, 5\sigma/6)$, which is one-to-one and satisfies:

$$||p-p'||_{\mathcal{D}'} \leq \frac{R}{6} \; ; \;\; ||q-q'||_{\mathcal{D}'} \leq \frac{\sigma}{6}.$$

The transformed Hamiltonian can be decomposed as in (5) (using primed letters) with:

$$\varepsilon' \leq \varepsilon\left(1+324\frac{M}{\sigma}TR\right) \quad \text{and} \quad \eta' \leq 2^{-s}\left(324\frac{M}{\sigma}\varepsilon TR\right),$$

provided the following thresholds are satisfied ($s = s(\varepsilon)$, $R = R(\varepsilon)$):

$$1)\ sRT \leq 3\cdot 10^{-3}\frac{\sigma}{M}\ ; \qquad 2)\ \frac{\varepsilon T}{R} \leq \frac{\sigma}{72E}\ ; \qquad 3)\ \varepsilon^{1/2} \leq \left(\frac{M}{20E}\right)^{1/2} R\ .$$

d) Intermediate statement with free parameters:

In order to obtain a stability result one must complement the above lemma with a simple geometric argument which uses convexity and energy conservation. We are aiming at a statement of the following type: every trajectory with initial condition (in action space) lying in the (real) ball of radius $r(\varepsilon)$ around the origin will stay for $|t| \leq \mathcal{T}(\varepsilon)$ in the ball of radius $R(\varepsilon) \leq R$. One can expand the unperturbed Hamiltonian around the initial condition:

$$h(p'(t)) = h(p'(0)) + \omega(p'(0)).(p'(t)-p'(0)) + \frac{1}{2}A(p^*)(p'(t)-p'(0)).(p'(t)-p'(0)),$$

where p^* is located between $p'(t)$ and $p'(0)$ (the domain is convex). The *convexity* of h then allows to estimate the last term from below; more precisely, (4) implies:

$$\frac{1}{2}m||p'(t)-p'(0)||^2 \leq |h(p'(t)) - h(p'(0))| + |\omega(p'(0)).(p'(t)-p'(0))|.$$

To estimate the first term of the right hand side one uses energy conservation for the (perturbed) system, $H'(p'(t), q'(t)) = H'(p'(0), q'(0))$ which implies:

$$\begin{aligned}|h(p'(t)) - h(p'(0))| &\leq |Z'(p'(t))| + |Z'(p'(0))| + |R'(p'(t))| + |R'(p'(0))| \\ &\leq 2\left[\varepsilon E + \frac{\eta_0 E}{2} + (2^{-1} + \ldots + 2^{-s})\frac{\eta_0 E}{2}\right] \\ &\quad + 2^{-s}\eta_0 E + 2^{-s}\eta_0 E \leq 2(\varepsilon + 2\eta_0)E\ .\end{aligned}$$

To estimate the term $|\omega(p'(0)).(p'(t)-p'(0))|$, one considers the projections of the vectors $\omega(p'(0))$ and $p'(t)-p'(0)$ on ω_0 and the orthogonal complement (we denote Π and $\Pi^\perp$ the corresponding orthogonal projection operators).

First, $||\Pi^\perp(\omega(p'(0))|| \leq ||\omega(p'(0)) - \omega_0|| \leq M||p'(0)|| \leq 2rM$ provided that:

$$c(\varepsilon) \leq 6(\varepsilon + \eta_0 s)\frac{TE}{\sigma} \leq 12\frac{\varepsilon TE}{\sigma} \leq r(\varepsilon)\ . \tag{14}$$

On the other hand, since Z' is resonant with respect to ω_0 we have the crucial fact that:

$$\frac{\partial Z'}{\partial q}\ .\ \omega_0 = 0,$$

and because of the Hamiltonian character of the equation: $\Pi(\dot{p}') = -\Pi\left(\frac{\partial R'}{\partial q}\right)$, which implies:

$$\begin{aligned}||\Pi(p'(t)-p'(0))|| &\leq |t|\left|\left|\frac{\partial R'}{\partial q}\right|\right|_{\mathcal{D}-(5r/6,5\sigma/6)} \\ &\leq \frac{6}{\sigma}|t|\cdot||R'||_{\mathcal{D}-(2R/3,2\sigma/3)} \leq \frac{6}{\sigma}\mathcal{T}(\varepsilon)\eta_s E.\end{aligned}$$

Here $|t| \leq \mathcal{T}(\varepsilon)$ where the later is still to be specified.

Gathering the previous inequalities together we obtain:

$$|\omega(p'(0)).(p'(t) - p'(0))| \leq 2rM||p'(t) - p'(0)|| + \frac{6}{\sigma}\mathcal{T}(\varepsilon)\eta_s E||\omega(p'(0))||.$$

With the threshold (14) we have $p'(0) \in \mathcal{B}_{2r}$ and if we add the hypothesis: $2rM \leq \Omega$, we get: $|\omega(p'(0))| \leq \Omega + 2rM \leq 2\Omega$.

Finally, provided all the previous conditions are satisfied and denoting $a = ||p'(t) - p'(0)||$ we find that:

$$\frac{1}{2}ma^2 \leq 2(\varepsilon + 2\eta_0)E + 12\frac{\Omega}{\sigma}\mathcal{T}(\varepsilon)\eta_s E + 2rMa. \tag{15}$$

This inequality can be simplified if we first impose $s \geq 2$ which ensures $\eta_0 \leq \varepsilon/s \leq \varepsilon/2$.

On the other hand, we also want that: $12\frac{\Omega}{\sigma}\mathcal{T}(\varepsilon)\eta_s E \leq \varepsilon E$, that is satisfied if:

$$\mathcal{T}(\varepsilon) \leq \frac{\sigma}{12\Omega}s2^s,$$

which is the value we finally adopt. Now, solving (15), we find that:

$$a \leq 2r\frac{M}{m} + \left(4r^2\frac{M^2}{m^2} + 10\frac{\varepsilon E}{m}\right)^{1/2}.$$

Let g be an arbitrary positive constant such that:

$$10m\varepsilon E \leq 4gr^2M^2 \quad \text{or} \quad r^2 \geq \frac{5mE}{2M^2 g}\varepsilon. \tag{16}$$

Then one gets: $a \leq (4+g)r(\varepsilon)\frac{M}{m}$. Moreover, using (14) we can state:

$$||p(t)|| \leq (7+g)\frac{M}{m}r(\varepsilon) \quad \text{for} \quad t \leq \mathcal{T}(\varepsilon);$$

we will now connect r and R by the relation: $r(\varepsilon) = \frac{m}{(7+g)M}R(\varepsilon)$.

With $s = s(\varepsilon)$ big enough and $R(\varepsilon) = \rho(\varepsilon)$ small enough to stay in the domain where H is defined, we obtain our first statement about stability over exponentially long times.

Model statement:

If the following inequalities are satisfied:

$$1)\ \varepsilon \leq \frac{2M^2 g}{5mE} r^2(\varepsilon)\ ;\ \ 2)\ \frac{\varepsilon T}{r(\varepsilon)} \leq \frac{\sigma}{12E}\ ;\ \ 3)\ s(\varepsilon) r(\varepsilon) T \leq 3 \cdot 10^{-3} \frac{m\sigma}{(7+g)M^2}\ ;$$

$$4)\ r(\varepsilon) \leq \frac{\Omega}{2M}\ ;\ \ 5)\ s(\varepsilon) \geq 2\ ;\ \ 6)\ (7+g)\frac{M}{m} r(\varepsilon) \leq \inf(R,\rho)\ ;$$

and the initial conditions are such that $||p(0)|| \leq r(\varepsilon)$, then one has:

$$||p(t)|| \leq R(\varepsilon) = (7+g)\frac{M}{m} r(\varepsilon) \text{ for } |t| \leq \mathcal{T}(\varepsilon) \text{ with } \mathcal{T}(\varepsilon) = \frac{\sigma}{6\Omega} 2^{s(\varepsilon)}\ .$$

e) Stability results in the neighbourhood of the periodic solutions:

In this section, we prove three theorems which follow directly from the above model statement. The first one is not the most natural statement at this point, but it yields a time of validity which is independant of the periodic orbit one considers and this will be crucial in the next part, when we approach an arbitrary initial condition by rational points corresponding to periodic tori. The second theorem is nothing but a particuliar case of the first; it costs a stronger condition on the period T so that in Part III, it will not allow to obtain the best time exponent (i.e. a), but with a better threshold for the size of the perturbation. The last statement in this section gives the best stability theorem around a given, fixed, periodic torus.

Theorem 1.1 :

Let α, $0 < \alpha \leq 1/2$, λ an arbitrary positive constant and assume that:

$$||p(0)|| \leq r(\varepsilon) = \lambda \frac{\varepsilon^\alpha}{T}\ ,\ \text{with } T \text{ satisfing } 1 \leq T < \varepsilon^{\alpha - 1/2}.$$

Then:

$$||p(t)|| \leq R(\varepsilon) = (7+g(\lambda))\frac{M}{m} r(\varepsilon)\ ,\ \text{when } |t| \leq \mathcal{T}(\varepsilon) = \frac{\sigma}{6\Omega} \exp\left(K(\lambda)\varepsilon^{-\alpha}\right),$$

where the constants are:

$$g(\lambda) = \frac{5mE}{2M^2\lambda^2}\ ,\quad \text{and}\quad K(\lambda) = 10^{-3}\frac{m\sigma}{(7+g(\lambda))M^2\lambda}\ ,$$

provided that ε satisfies the following threshold conditions:

$$i)\ \varepsilon^\alpha < 10^{-3}\frac{m\sigma}{(7+g(\lambda))M^2\lambda}\ ;\quad ii)\ \varepsilon^\alpha \leq \frac{\sigma\lambda}{12E}\ ;$$
$$iii)\ \varepsilon^\alpha \leq \frac{\Omega}{2M\lambda}\ ;\quad iv)\ \varepsilon^\alpha \leq \frac{m\inf(R,\rho)}{(7+g(\lambda))M\lambda}.$$

Proof:

To prove the theorem, we only need to apply the model statement with $r(\varepsilon)$ as in the statement above and

$$s(\varepsilon) = E\left[\frac{K(\lambda)}{\log(2)}\varepsilon^{-\alpha}\right] + 1,$$

where $E[x]$ is the integer part of a real x. In this case inequalities 1) and 3) of the model statement are always satisfied with our values of g and K (we use inequality 5) to check condition 3)) and the other inequalities give the four thresholds of the theorem.

In the following two theorems we set:

$$\lambda_1 = 10^{-4}\frac{m\sigma}{M^2}\ ,\quad \text{and} \quad T_1 = 5\cdot 10^{-5}\left(\frac{m}{E}\right)^{1/2}\frac{\sigma}{M}\ ,$$

then the following statement holds:

Theorem 1.2 :

Let $\alpha > 0$, $\alpha < 1/2$ (notice the <u>strict</u> inequality), if $||p(0)|| \leq r(\varepsilon) = \lambda_1\varepsilon^\alpha/T$ with T such that $1 \leq T < T_1\varepsilon^{\alpha 9-1/2}$, then:

$$||p(t)|| \leq R(\varepsilon) = 8\frac{M}{m}r(\varepsilon) \text{ when } |t| \leq \tau(\varepsilon) = \frac{\sigma}{6\Omega}\exp(\varepsilon^{-\alpha}),$$

provided ε satisfies:

$$i)\ \varepsilon^{1/2-\alpha} < T_1\ ;\quad ii)\ \varepsilon^\alpha \leq 10^3\frac{M}{\sigma}\inf(R,\rho)\ ;\quad iii)\ \varepsilon^\alpha \leq 5\cdot 10^3\frac{M\Omega}{m\sigma}.$$

Proof:

We apply the model statement with:

$$g = 1 \ , \ R(\varepsilon) = \ 8\frac{M}{m}r(\varepsilon) \text{ and } \ s(\varepsilon) = E\left[\frac{\varepsilon^{-\alpha}}{\log(2)}\right] + 1,$$

then conditions 1) and 3) in the model statement are always satisfied with our values of λ_1 and T_1. The other inequalities give the last two threshold conditions of the theorem, and condition $1 < T_1\varepsilon^{\alpha-1/2}$ has to be added separately.

Theorem 1.3 :

If $||p(0)|| \le r(\varepsilon) = r_0\sqrt{\varepsilon}$, then:

$$||p(t)|| \le 8\frac{M}{m}r_0\sqrt{\varepsilon} \ , \ \text{ if } |t| \le \frac{\sigma}{6\Omega}\exp\left(\frac{T_1}{T\sqrt{\varepsilon}}\right),$$

with:

$$r_0 = \left(\frac{5mE}{2M^2}\right)^{1/2}.$$

Provided ε satisfies:

$$i) \ \ \varepsilon \le \frac{\Omega^2}{10mE} \ ; \qquad ii) \ \ \varepsilon \le 6\cdot 10^{-3}\left(\frac{m}{E}\right)(\inf(R,\rho))^2 \ ;$$

$$iii) \ \ \sqrt{\varepsilon} \le 5 \ 10^{-5}\left(\frac{m}{E}\right)^{1/2}\frac{\sigma}{MT} \ .$$

Proof:

We apply the model statement with:

$$s(\varepsilon) = E\left[\frac{T_1}{\log(2)T\sqrt{\varepsilon}}\right] + 1 \ , \qquad g = 1 \ , \text{ and } \qquad R(\varepsilon) = 8\frac{M}{m}r(\varepsilon) \ .$$

In this case, it follows from the values of r_0 and T_1 that conditions 1) and 3) of the model statement are satisfied; the threshold conditions in the theorem ensure the validity of the remaining ones.

III Stability for arbitrary initial conditions

We now combine Theorems 1.1 and 1.2 with a simple approximation argument to complete the proof of the stability estimates for arbitrary initial conditions. Let us first recall the following elementary approximation theorem (due to Dirichlet):

Let $\alpha \in R^n$ and $Q > 1$; there exists $\zeta \in Z^n$ and an *integer* q with $1 \leq q < Q$ such that $||q\alpha - \zeta|| \leq \sqrt{n}Q^{-1/n}$, $||\ ||$ denotes the usual euclidian norm.

This immediately implies the following

Proposition:

Let $Q > 1$, $q > 1$, $q < Q$ and $\omega^* \in R^n$ with $||\omega^*||_\infty = w$ then there exists a rational vector ω of period $T = q/w$ such that

$$||\omega - \omega^*|| \leq \frac{\sqrt{n-1}}{TQ^{1/(n-1)}} .$$

Proof:

Indeed, relabeling the components if necessary, one can write $\omega^* = w\ \omega_1^*$ with $\omega_1^* = (\pm 1, \omega^{*\prime})$. Then, apply Dirichlet theorem with $\alpha = \omega^{*\prime} \in R^{n-1}$ and pick $\omega = w(1, q^{-1}\zeta)$, which is a rational vector of period $T = q/w$ which satisfies the statement (one uses the fact that q is an integer).

Let now $(p(0), q(0))$ be an arbitrary initial condition; we write $p^* = p(0)$, $\omega^* = \omega(p^*)$ and apply the above proposition which yields some rational ω.

Here, we can always be reduced to the case when $w = 1$ (i.e. $T = q$) if one introduces the scalings:

$$t' = wt\ ,\ H' = \frac{H}{w}\ ,\ \omega^{*\prime} = \frac{\omega^*}{w}\ ,\ m' = \frac{m}{w}\ ,\ M' = \frac{M}{w}\ ,\ E' = \frac{E}{w}\ ,\ \varepsilon' = \varepsilon\ . \quad (17)$$

Below, for the sake of clarity, we write everything using the original quantities and shall remember at the very end that one should *first* perform the transformations (1) and change the results accordingly.

We assume that ω is closed enough to ω^* so that the frequency map can be inverted, by an easy application of the implicit function theorem, this is the case over the ball $B\left(\omega^*, m^2/(4|h|_3)\right)$, where $|h|_3$ is an estimate from above of the third derivative of h. So assume:

$$\frac{\sqrt{n-1}}{Q^{1/(n-1)}} < \frac{m^2}{4|h|_3}.$$

Under this condition, there exists an action $p \in R^n$ such that

$$\nabla h(p) = \omega \text{ and } ||p - p^*|| \leq \frac{\sqrt{n-1}}{m} \frac{1}{TQ^{1/(n-1)}} , \tag{18}$$

where the factor $1/m$ estimates from above the norm of the inverse of the frequency map.

In order to apply Theorem 1.1 *around the point* p, which is a rational point of period T thereby obtaining a stability estimate for p^*, the latter point must lie in the "*influence zone*" of p, i.e.one should have (see Theorem 1.1):

$$r(\varepsilon) = \lambda \frac{\varepsilon^\alpha}{T} \geq ||p - p^*||.$$

In turn, from (18), this is ensured if:

$$\lambda = \frac{\sqrt{n-1}}{m} \qquad \text{and} \qquad Q = \varepsilon^{-(n-1)\alpha}.$$

Another important requirement to keep in mind is that the period should not be too long, more precisely, one must require that $T < \varepsilon^{\alpha - 1/2}$; since $T < Q$, we want:

$$Q = \varepsilon^{-(n-1)\alpha} \leq \varepsilon^{\alpha - 1/2}, \text{ that is } \varepsilon^{1/2 - n\alpha} \leq 1,$$

which is satisfied for $\alpha \leq \dfrac{1}{2n}$ (and $\varepsilon \leq 1$). We set $\alpha = \dfrac{1}{2n}$.

We can then apply Theorem 1.1 and find that: $||p(t) - p(0)|| \leq (6 + g(\lambda))\dfrac{M}{m} r(\varepsilon)$.

When making the threshold conditions, one should beware of the fact that, in the third one (iii) Theorem 1.1), $\Omega = ||\omega(p)||$ refers to p and not p^*. Everything should then be expressed with parameters centered at p^*, this is easy because, denoting $\Omega^* = ||\omega^*||$, one has:

$$|\Omega - \Omega^*| \leq Mr(\varepsilon) \leq \frac{\Omega^*}{2} ,$$

so we need only substitute Ω with Ω^* in the third condition and with $3\Omega^*/2$ in the definition of $\mathcal{T}(\varepsilon)$. We can also simplify the expressions for the parameters $g(\lambda)$ and K. This leads to the following

Theorem 2.1:

Any trajectory starting from the initial point $(p(0), q(0)) \in \mathcal{D} \cap R^{2n}$ satisfies:

$$||p(t) - p(0)|| \leq 3M \left(E + \frac{2}{m} \right) \varepsilon^{1/(2n)} \quad \text{for} \quad |t| \leq \frac{2\sigma}{9\Omega^*} \exp\left(K \varepsilon^{-1/(2n)}\right),$$

with the parameter: $K = 10^{-3}\dfrac{m^2\sigma}{(7\sqrt{n-1}+3mE)M^2}$, provided ε satisfies the following inequalities:

$$i)\quad \varepsilon^{1/(2n)} < 10^{-3}\frac{m^2\sigma}{(7\sqrt{n-1}+3mE)M^2}\ ;$$

$$ii)\quad \varepsilon^{1/(2n)} \leq \frac{m^2}{(7\sqrt{n-1}+3mE)M^2}\ \inf(R,\rho)\ ;$$

$$iii)\quad \varepsilon^{1/(2n)} < \frac{m^2}{4|h|_3\sqrt{n-1}}\ ;\ iv)\quad \varepsilon^{1/(2n)} \leq \frac{m\Omega^*}{2M\sqrt{n-1}}$$

where $|h|_3$ is an estimate (from above) of the third derivative of h on the domain $\mathcal{D}$.

In this statement, all the parameters connected with the Hamiltonian, along with time t, are those which are obtained *after* the rescalings (17) have been performed, i.e. one should use the primed quantities in (17).

This theorem gives the desired "optimal" exponents but the time of validity and the threshold conditions deteriorate strongly when the number of degrees of freedom increases. In fact, for "reasonably" small values of ε, the next theorem may very well provide better estimates in spite of a worse exponent.

In the same way, Theorem 1.2 around p can be applied if $r(\varepsilon) = \lambda_1\dfrac{\varepsilon^\alpha}{T} \geq ||p-p^*||$ — here we use the constants λ_1 and T_1 defined in the previous part, the inequality is ensured with (18) if

$$Q^{1/(n-1)} = 10^4\sqrt{n-1}\frac{M^2}{m^2\sigma}\varepsilon^{-\alpha}.$$

To apply Theorem 1.2, the period must not be too long, so one wants the inequality:

$$Q = \left(\frac{\sqrt{n-1}}{\lambda_1 m}\right)^{n-1}\varepsilon^{-(n-1)\alpha} \leq T_1\varepsilon^{-(1-2\alpha)/2} \text{ or } \varepsilon^{1/2-n\alpha} \leq T_1\left(\frac{\lambda_1 m}{\sqrt{n-1}}\right)^{n-1}. \tag{19}$$

This defines a threshold condition provided that $1/2-n\alpha > 0$. Consequently the value $\alpha = 1/(2n)$ is not accessible, but any smaller value is. Here we shall take, for example:

$$\alpha = \frac{1}{2(n+1)}\ .$$

Under this condition, Theorem 1.2 may be applied and one gets:

$$||p(t) - p(0)|| \le 7\frac{M}{m}\lambda_1\frac{\varepsilon^{1/[2(n+1)]}}{T}.$$

With the same substitution as in 2.1 for the parameter Ω and a simplification of the inequality (19), we arrive at:

Theorem 2.2:

For any initial point $(p(0), q(0))$ a trajectory starting at $(p(0), q(0))$ satisfies:

$$||p(t) - p(0)|| \le \; 10^{-3}\frac{\sigma}{M}\varepsilon^{1/[2(n+1)]} \text{ when } |t| \le \frac{2\sigma}{9\Omega^*}\exp(\varepsilon^{-1/[2(n+1)]}) \; ,$$

provided that ε satisfies:

$$i) \; \varepsilon^{1/[2(n+1)]} \le \; 10^3\frac{M}{\sigma}\inf(R,\rho) \; ; \quad ii) \; \varepsilon^{1/[2(n+1)]} \le \; 5 \; 10^3\frac{M\Omega}{m\sigma}$$

$$iii) \; \varepsilon^{1/[2(n+1)]} < \; 5 \; 10^{-5}\left(\frac{m}{E}\right)^{1/2}\frac{\sigma}{M} \; ; \quad iv) \; \varepsilon^{1/[2(n+1)]} \le \; \frac{m}{4\lambda_1|h|_3} \; ;$$

$$v) \; \varepsilon^{1/[2(n+1)]} \le \; \left(\sqrt{mE}\right)^{-1}\left(10^{-4}\;\frac{m^2\sigma}{\sqrt{n-1}M^2}\right)^n \; .$$

This statement also refers to the primed quantities in (17).

So far, we have only used basic approximation results and all the previous theorems may be improved, exactly as in [1], if one restricts attention to certain classes of initial conditions with further arithmetical properties. Among other things, one can prove that *almost all* points in phase space admit, for any $\eta > 0$, the stability exponents:

$$(a, b) = \left(\frac{1}{2n} - \eta, \frac{1}{2} - \eta\right) \; ;$$

we recall that the pair $(1/(2n), 1/2)$ corresponds to B.V. Chirikov's prediction for the maximal speed of Arnold's diffusion.

REFERENCES

[1] P. Lochak. Canonical Perturbation Theory via Simultaneous Approximation, Uspekhi. Math. Nauk., **6**, 1992, p.59 (English translation in Russian Math. Surveys).

[2] N. N. Nekhorochev, Russian Math. Surveys, **32**, p.1, 1977.

[3] A. I. Neistadt, J. Appl. Math. Mech., **45**, p. 80, 1981.

[4] A. I. Neistadt, J. Appl. Math. Mech., **48**, p. 197, 1981.
[5] B. V. Chirikov, Physics Reports, **52**, p.263, 1979.
[6] G. Benettin, L. Galgani, A. Giorgilli, Celestial Mechanics, **37**, p.1, 1985.
[7] J. Pöschel, Preprint E.T.H., 1992, to appear in Math. Zeitschrift (See also this volume).

[1] D.M.I., Ecole Normale Supérieure 45 rue d'Ulm, F-75230 Paris Cedex 05

[2] Space Research Institute Profsoyuznaya ul. 84/32, 117810, Moscow, Russia

[3] Université P. et M. Curie U.F.R. de Mathématiques, 4 place Jussieu, F-75252 Paris Cedex 05