Physica 17D (1985) 313–322
North-Holland, Amsterdam

SOLITON GENERATION IN THE FORCED NON-LINEAR SCHRÖDINGER EQUATION

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Received 17 June 1985

We derive a finite-dimensional model which describes the generation of solitons in the forced non-linear Schrödinger equation. Its qualitative behaviour is in good agreement with previous numerical computations.

1. Introduction

The transformation of an electromagnetic wave (the laser-beam) into an electrostatic wave which is subsequently absorbed by the plasma constitutes an important absorption mechanism in experiments on laser–matter interaction. The absorption in the inhomogeneous plasma occurs at the critical density, where the local plasma frequency coincides with the E.M. wave frequency [1]. In this region the E.M. wave is transformed into an electrostatic plasma wave which propagates down the density gradient. This backscattering phenomenon reduces the absorption, and therefore may impede fusion reactions. This problem may be described by the non-linear Schrödinger equation

$$\left(\mathrm{i}\,\partial_t + \nabla^2 + gx\right)q + p|q|^2 q = s_0, \tag{1}$$

which is derived as an approximation of the Zakharov equations. The constant s_0 accounts for the incident electromagnetic wave; the point $x = 0$ corresponds to the perfect resonance between the e.m. wave and the plasma-wave, and the term gx describes the detuning due to the plasma inhomogeneity; the term $p|q|^2q$ accounts for the plasma non-linearity in the subsonic approximation for the acoustic waves [2]. This nonlinear Schrödinger equation with a forcing term is also a good model equation for pulse propagation in optical fibres [3]. Due to the source term, that equation cannot be solved by the Inverse Scattering Transform. Numerical solutions have been obtained by J.C. Adam et al. [4]. They exhibit the following dependence on p. For $p = 0$, the stationary solution is stable and close to the Airy function (see fig. 1); for small values of p, say $0 \le p < p_0$, the non-linearity modifies only slightly its profile, by mainly steepening its shape: the larger p is, the more the solution looks like a soliton in the vicinity of $x = 0$. In this range of the parameter p, $0 \le p < p_0$, the solution remains stable in time, so that the envelope of the plasma waves does not convect away from the resonance region $x \approx 0$. There is a marked transition at $p = p_0$ such that the solution changes dramatically as compared with the previous case: indeed, for $p \ge p_0$, the stationary solution becomes unstable and periodic emission of pseudo-solitons occurs (fig. 2); a soliton-like envelope grows in time near $x = 0$ until its amplitude reaches a critical value which makes it drift away and move towards the low density region; a new soliton-like wave packet grows then again around $x = 0$ and the process repeats itself. When the non-linearity parameter p is further increased, new

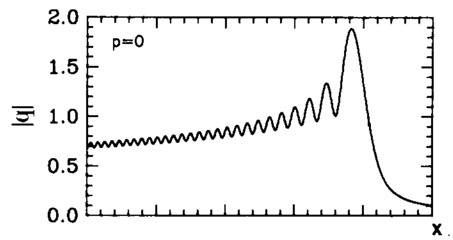

Fig. 1. Stationary solution of eq. (2) for $p = 0$.

bifurcations take place (maybe subharmonic ones) and for $p > p_1$, pseudo-solitons are emitted randomly (fig. 3). In this regime, eq. (1) exhibits a physical example of turbulent behaviour, where spatially coherent structures are chaotically emitted in time.

In this paper, we attempt to describe the birth of solitons in the forced NLS equation in the presence of a gradient and a source term. Eq. (1) cannot be treated, using classical perturbation techniques [5, 6]; these would at most account, in the present example for the slow modulation of the solitons, as they convect down the gradient, certainly not for their creation. In fact, these perturbation schemes always assume a constant number of solitons (in practice none or one) and this is one of their weaknesses. Moreover, the r.h.s. of eq. (1) is not "small", and there is little hope that it could be treated as a perturbation term at all.

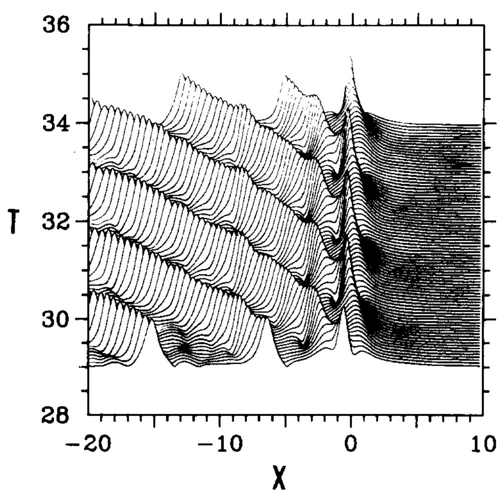

Fig. 2. Periodic emission of solitons in eq. (2) for p slightly above the threshold p_0 in a spatio-temporal representation. Each curve pictures the x dependence of q at a given time.

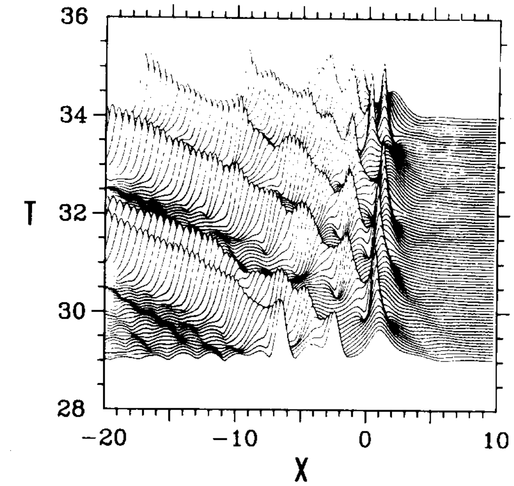

Fig. 3. Random emission of solitons in eq. (2) for p above the threshold p_1 in the same spatio-temporal representation as in fig. 2.

We therefore introduce a finite dimensional model, which makes it possible to describe – albeit in a crude way – the energy transfer between the soliton and the radiation part of the solution. That phenomenon was not taken into account in previous studies [7]. In order to build that model, we consider the physical picture of soliton emission, which numerical computations reveal. Indeed, they show two stages in soliton growth (fig. 4a and 4b). In the first one, the solution grows up near the resonance where it interacts with the source. It then splits and the pseudo-soliton moves away down the gradient leaving behind a tail which in turn interacts with the source via the non-linear equation (see fig. 4a and b). We shall modelise the first stage as follows. We describe the solution by a five parameters trial function localized near $x = 0$, similar to the soliton solution of the usual unforced NLS equation in an inhomogeneous medium. Of course, due to the forcing term, the internal parameters depend on time. Starting from that model, we derive – in section 2 – a finite sys-

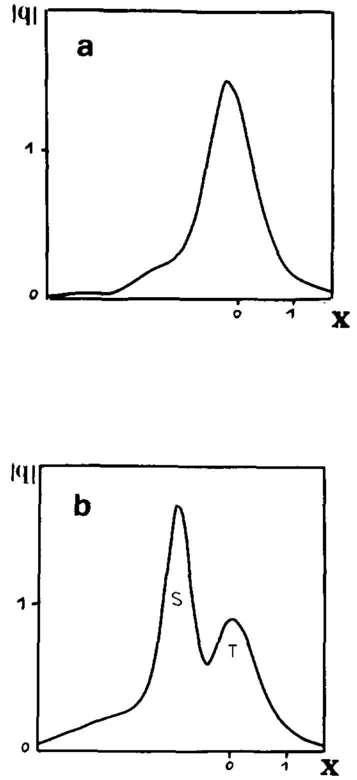

Fig. 4. a) Growth of an initial condition of the forced N.L.S. equation (2). b) Splitting of the solution of (2) at the end of resonant growth. S is the soliton being emitted. T denotes the tail.

tem of differential equations, which governs the evolution of the internal parameters. It enables us to determine the saturation parameters of the pseudo-soliton.

In the second stage, a pseudo-soliton is emitted after the solution has reached its maximum amplitude. We compute the characteristics of the tail left behind, by "subtracting" from the saturation solution, its solitonic part, as explained in section 3.

Thus, we derive a model map, which associates to the initial conditions for the birth of a pseudo-soliton, the initial conditions for the growth of the next one. That map exhibits properties similar to the dynamics of the force NLS equation, as shown in section 4.

2. Finite-dimensional model for the resonant growth of the solution in the NLS equation

Setting

$$q = g^{2/3} S_0 q', \quad p = (g S_0)^{-2} p',$$
$$x = g^{1/3} x', \quad t = g^{2/3} t',$$

in eq. (1), we obtain (dropping the primes) the reduced equation

$$\mathrm{i} q_t + q_{xx} - xq + p|q|^2 q = 1, \tag{2}$$

where only one relevant parameter p is left.

The same equation without forcing term is integrable, since the simple change of variables

$$q'(x', t') = q(x, t) \exp\left[\mathrm{i}(xt + t^3/3)\right],$$
$$x' = \sqrt{\frac{p}{2}}\,(x + t^2), \quad t' = pt,$$

yields (dropping the primes again) the usual NLS equation

$$\mathrm{i} q_t + \tfrac{1}{2} q_{xx} + |q|^2 q = 0. \tag{3}$$

When the source term is taken into account, the same change of variables yields

$$\mathrm{i} q_t + \tfrac{1}{2} q_{xx} + |q|^2 q = S(x, t), \tag{4}$$

with the forcing term given by

$$S(x, t) = \frac{1}{p} \exp\left[\mathrm{i}\left(\frac{\sqrt{2}\, xt}{p^{3/2}} - \frac{2}{3}\frac{t^3}{p^3}\right)\right].$$

We now derive a finite-dimensional model which enables us to study the growth of the solution in the non-linear regime. We suppose that the solution can be represented, at least locally – near $x = 0$ – by a function of the form

$$q(x, t) = A \operatorname{sech}\left[B(x - x_0)\right] \times \exp\left[\mathrm{i}k(x - x_0) + \varphi\right], \tag{5}$$

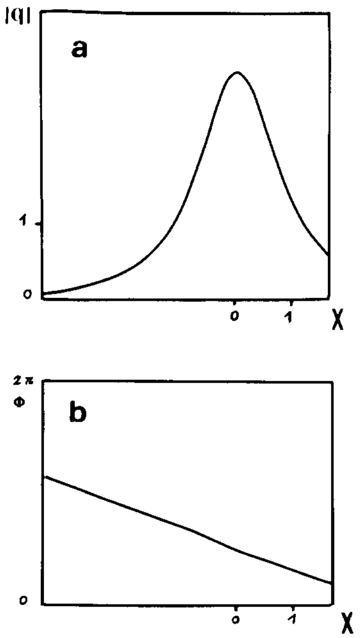

Fig. 5. a) Spatial variation of the modulus $|q|$ of the solution of eq. (2) at a fixed time during resonant growth. b) Spatial variation of the phase of the solution.

where the five real parameters (A, B, k, x_0, φ) are all time dependent; we thus assume that the solution is well described by an amplitude A, a width B, a position x_0, a wavelength of the carrier wave, $2\pi/k$, and a local phase φ. That this hypothesis is not unrealistic can be checked on fig. 5a and b. Since A and B are independent quantities, we are not restricted to solutions close to solitary waves of the unforced equation.

One could think of deriving equations for the internal parameters by using a Rayleigh–Ritz method in the space of trial functions for the Lagrangian density

$$\mathscr{L}=\frac{\mathrm{i}}{2}(qq_t^* - q^*q_t) + \tfrac{1}{2}|q_x|^2 - \tfrac{1}{2}|q|^4 + S^*q + Sq^*.$$

However, one is thus led to a relation which links the parameters and forbids us to consider solutions which differ too much from the soliton of the unforced equation. Rather than introducing a larger trial function space (in fact, the solution given under the form (5) can be fitted quite well – near $x = 0$ – with the numerically computed solution of the equation) we found more advisable to rely on a different method, based on the invariants of the NLS equation. We consider the first three of the infinite sequence of invariants H_n $(n \geq 1)$ for the NLS equation:

$$\begin{aligned} H_1(q) &= \int_{-\infty}^{+\infty} |q|^2\,\mathrm{d}x, \\ H_2(q) &= \mathrm{i}\int_{-\infty}^{+\infty} (q_x q^* - q_x^* q)\,\mathrm{d}x, \\ H_3(q) &= \int_{-\infty}^{+\infty} (|q|^4 - |q_x|^2)\,\mathrm{d}x. \end{aligned} \tag{6}$$

These are the most "physical" ones, representing respectively the total "mass" M, the total momentum P and the energy E. When a source term is introduced, these quantities vary according to the equations

$$\begin{aligned} \frac{\mathrm{d}M}{\mathrm{d}t} &= -2\int_{-\infty}^{+\infty} \mathrm{Im}(S^*q)\,\mathrm{d}x, \\ \frac{\mathrm{d}P}{\mathrm{d}t} &= 4\int_{-\infty}^{+\infty} \mathrm{Re}(S_x q^*)\,\mathrm{d}x, \\ \frac{\mathrm{d}E}{\mathrm{d}t} &= 4\int_{-\infty}^{+\infty} \mathrm{Im}(Sq^*)|q|^2\,\mathrm{d}x \\ &\quad -2\int_{-\infty}^{+\infty} \mathrm{Im}(S_{xx}^* q)\,\mathrm{d}x. \end{aligned} \tag{7}$$

Introducing the trial functions internal parameters we obtain

$$\begin{aligned} \frac{\mathrm{d}M}{\mathrm{d}t} &= -\frac{2\pi}{p}\frac{A}{B}\,\mathrm{sech}\left[\frac{\pi}{2B}\left(k - \sqrt{2}\,p^{-3/2}t\right)\right] \\ &\quad \times \sin\delta(t), \\ \frac{\mathrm{d}P}{\mathrm{d}t} &= -\left(\frac{2}{p}\right)^{3/2} t\,\frac{\mathrm{d}M}{\mathrm{d}t}, \\ \frac{\mathrm{d}E}{\mathrm{d}t} &= \left(A^2 + \frac{A^2}{B}\left(k - \sqrt{2}\,p^{-3/2}t\right)^{-2}\right. \\ &\quad \left. -2p^{-3}t^2\right)\frac{\mathrm{d}M}{\mathrm{d}t}, \end{aligned} \tag{8}$$

where we have set

$$\delta(t)=\frac{2}{3}\frac{t^3}{p^3}-\sqrt{2}\,p^{-3/2}tx_0+\varphi(t).$$

We now introduce the variable $N=A/B$ which indicates – as we show below – the number of solitons contained in the solution with parameters A, B; we also scale the time by setting: $s\equiv\sqrt{2}\,p^{-3/2}t$. M, P, E have simple expressions in terms of A, B, k (or N, B, k):

$$M=\frac{2A^2}{B}=2N^2B,$$

$$P=-4k\frac{A^2}{B}=-4kN^2B,$$

$$E=\frac{3}{4}\frac{A^4}{B}-\frac{2k^2A^2}{B}-\frac{2}{3}A^2B$$
$$=\frac{4}{3}N^4B^3-2k^2N^2B-\frac{2}{3}N^2B^3.$$

It is now easy to derive the system which governs the evolution of the internal parameters N, B, k,

$$\frac{\mathrm{d}N}{\mathrm{d}s}=-\frac{3\pi}{4}\sqrt{\frac{p}{2}}\,\frac{1}{B}\left[1-\left(\frac{s-k}{B}\right)^2\right]\times\operatorname{sech}\left[\frac{\pi}{2B}(k-s)\right]\sin\delta,$$

$$\frac{\mathrm{d}B}{\mathrm{d}s}=-\frac{3\pi}{2}\sqrt{\frac{p}{2}}\,\frac{1}{N}\left[\left(\frac{s-k}{B}\right)^2-\frac{1}{3}\right]\times\operatorname{sech}\left[\frac{\pi}{2B}(k-s)\right]\sin\delta, \tag{9}$$

$$\frac{\mathrm{d}k}{\mathrm{d}s}=-\pi\sqrt{\frac{p}{2}}\,\frac{1}{N}\left(\frac{s-k}{B}\right)\operatorname{sech}\left[\frac{\pi}{2B}(k-s)\right]\sin\delta.$$

This is not a closed set of equations, and we now derive evolution equations for the position x_0 and the phase φ.

To this end, we come back to eq. (4), the solution of which we write as $q(x,t)=C(x,t)\exp[\mathrm{i}\psi(x,t)]$; we arrive at the real system

$$C\psi_t-\frac{1}{2}C_{xx}+\frac{1}{2}C\psi_x^2-C^3=-\frac{1}{p}\cos(\sigma-\psi), \tag{10}$$

$$C_t+\psi_xC_x=\frac{1}{p}\sin(\sigma-\psi), \tag{11}$$

in which $\sigma(x,t)$ denotes the phase of the source term $S(x,t)$. We assume that the amplitude $C(x,t)$ is an even function centered at x_0 and that the phase $\psi(x,t)$ varies slowly – as a function of x – in the region where the solution $q(x,t)$ is localized. Both assumptions are in good agreement with numerical results on the resonant growth of the solution (see fig. 5a and 5b). Therefore, we set $\psi(x,t)=\varphi(t)+k(t)(x-x_0(t))$ (a linear approximation near $x=0$); we do not yet impose however a sech law for the amplitude $C(x,t)$. Eq. (11) then writes

$$C_t+kC_x=-\frac{1}{p}\sin\left[\left(k-\sqrt{2p}^{\,-3/2}t\right)(x-x_0)+\delta(t)\right],$$

which is a transport equation for the amplitude C, with speed $k(t)$. Projecting that equation on the even function C_x yields

$$\int_{-\infty}^{+\infty}C_tC_x\,\mathrm{d}x+k\int_{-\infty}^{+\infty}C_x^2\,\mathrm{d}x=-\frac{1}{p}\cos\delta\times\int_{-\infty}^{+\infty}\sin\left[\left(k-\sqrt{2}\,p^{-3/2}t\right)(x-x_0)\right]C_x\,\mathrm{d}x.$$

In order to find an equation for the position x_0, we now insert the trial function $C(x,t)=A(t)\operatorname{sech}[B(t)(x-x_0(t))]$; in this way, we obtain

$$\frac{\mathrm{d}x_0}{\mathrm{d}t}=k+\frac{3}{pAB}\cos\delta\left[\frac{\pi}{2B}\left(k-\sqrt{2}\,p^{-3/2}t\right)\right]\times\operatorname{sech}\left[\frac{\pi}{2B}\left(k-\sqrt{2}\,p^{-3/2}t\right)\right]. \tag{12}$$

From this equation and eq. (10) at $x=x_0$ we find

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t}=A^2+\frac{1}{2}(k^2-B^2)+\frac{1}{pAB}\cos\delta\left\{\left[\frac{3\pi}{2B^2}\left(k-\sqrt{2}\,p^{-3/2}t\right]\times\operatorname{sech}\left[\frac{\pi}{2B}\left(k-\sqrt{2}\,p^{-3/2}t\right)\right]-1\right\}. \tag{13}$$

In terms of the variable N and the scaled time s, eq. (12) and (13) can be rewritten as

$$\frac{\mathrm{d}x_0}{\mathrm{d}s}=\frac{p^{3/2}}{\sqrt{2}}\left\{k+\frac{3}{pNB^2}\cos\delta\left[\frac{\pi}{2B}(k-s)\right]\times\operatorname{sech}\left[\frac{\pi}{2B}(k-s)\right]\right\}, \tag{12'}$$

$$\frac{\mathrm{d}\varphi}{\mathrm{d}s}=\frac{p^{3/2}}{\sqrt{2}}\left\{N^2B^2+\frac{1}{2}(k^2-B^2)+\frac{1}{pNB}\cos\delta\left(\frac{3}{B}\left[\frac{\pi}{2B}(k-s)\right]\times\operatorname{sech}\left[\frac{\pi}{2B}(k-s)\right]-1\right)\right\}. \tag{13'}$$

Numerical integration of the complete set of equation on the variables (N, B, k, x_0, φ) – namely eqs. (9), (12′) and (13′) – yields typical solutions like those on fig. 6, for an initial phase φ_0 close to $3\pi/2$. The most striking fact is the existence of a long time interval during which the phase δ remains approximately locked (see fig. 6a and the solution keeps growing (see fig. 6b). The choice of the initial phase can be justified by the following argument; for the first invariant of the unforced equation, we have

$$\frac{\mathrm{d}M}{\mathrm{d}t}=-\frac{2}{p}\int_{-\infty}^{+\infty}C(x,t)\times\sin(\psi(x,t)-\sigma(x,t))\,\mathrm{d}x,$$

with the same notations as above and the same assumptions on the amplitude and the phase. Therefore, during the initial growth of the solution

$$\frac{\mathrm{d}M}{\mathrm{d}t}\simeq-\frac{2}{p}\sin\delta(t)\int_{-\infty}^{+\infty}C(y,t)\times\cos\left[\left(k(t)-\sqrt{2}p^{-3/2}t\right)y\right]\mathrm{d}y,$$

which shows that initially, δ must be close to $3\pi/2$ to account for the initial growth of the solution.

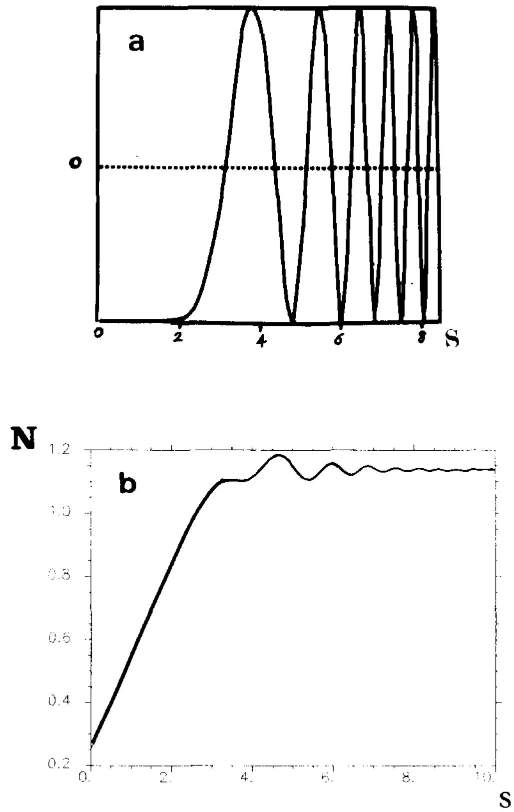

Fig. 6. Typical solution of the differential system (for $p = 0.5$). a) Evolution of $\cos\delta$. Notice the phase locking period; b) growth of the number of solitons N.

Numerical results on eq. (2) also reveal a phase locking during the resonant growth of the solution (see fig. 7) which shows a good qualitative agreement between the original partial differential equation and our finite dimensional model.

We now define an unlocking or saturation time as the time S needed for the phase to unlock and $\sin\delta$ to become positive for the first time (see fig. 6a). This time represents the duration of resonant growth of the solution or also the elapsed time between successive emissions of solitons. As a phase locking underlies the growth of the solution it seems reasonable to define this time as we do. In what follows we shall call saturation parameters the values of the internal parameters of time S. Notice that they differ only slightly from the asymptotic values (see fig. 6b).

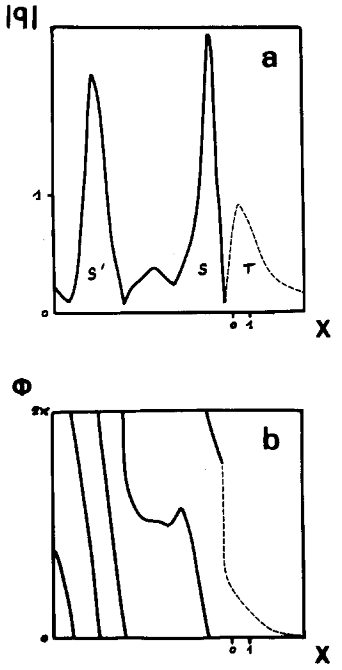

Fig. 7. a) Modulus of the solution of eq. (2) in periodic regime. T denotes the tail, S the soliton being emitted, S' the previous one. Notice that s' is a true soliton since $A = B$ within 10% of accuracy. b) Phase of the solution. Notice the phase locking phenomenon.

3. Splitting of the solution and emission of one soliton

At the end S of the growth period, a soliton is emitted and convects down the density gradient, leaving behind a tail which in turn receives energy from the source near the point $x = 0$. The emission phenomenon itself is so short that the energy transfer between the source and the solution plays a negligible role in this process; it can therefore be described, using the usual, unforced, NLS equation.

Let us briefly review the exactly solvable case of the inverse scattering problem associated with the NLS equation, for a sech potential. Eq. (3) has envelope soliton solutions which write

$$q(x,t) = 2\eta\,\mathrm{sech}\,[2\eta(x-2\xi t)] \times \exp\left[2\mathrm{i}\left(\xi x - (\xi^2-\eta^2)t + \delta\right)\right]. \tag{14}$$

The amplitude 2η and speed 2ξ are associated with a pair of eigenvalues (ζ, ζ^*), $\zeta = \xi + \mathrm{i}\eta$ for the Zakharov–Shabat spectral problem

$$\begin{aligned} V_{1x} + \mathrm{i}\zeta V_1 &= \mathrm{i}q^* V_2, \\ V_{2x} - \mathrm{i}\zeta V_2 &= \mathrm{i}q V_1. \end{aligned} \tag{15}$$

Following [8], we give the solution of this problem for the potential

$$q(x) = \mathrm{i}A\,\mathrm{sech}\,(Bx)\exp(\mathrm{i}kx) \tag{16}$$

(the factor i is introduced for convenience).

The left Jost function (ϕ_1, ϕ_2) solution of (15) is defined by the asymptotic behaviour

$$\lim_{x\to-\infty} \phi_1(x)\,\mathrm{e}^{\mathrm{i}\zeta x} = 1; \qquad \lim_{x\to-\infty} \phi_2(x) = 0.$$

ϕ_1 and ϕ_2 are expressible in terms of the hypergeometric function F and one has in particular

$$\phi_1(x,\zeta) = \mathrm{e}^{-\mathrm{i}\zeta x} F\left(N, -N; \gamma, \tfrac{1}{2}[1 + \mathrm{th}(Bx)]\right),$$

with $N = A/B$ and $\gamma \equiv \frac{1}{2} + (\mathrm{i}/2B)(k - 2\zeta)$. The coefficients $a(\zeta)$ and $b(\zeta)$, which are related to the transmission (T) and reflection (R) coefficients via the formulae

$$T(\zeta) = a^{-1}(\zeta); \quad R(\zeta) = b(\zeta)a^{-1}(\zeta)$$

are given by

$$a(\zeta) = \lim_{x\to+\infty} \phi_1(x,\zeta)\,\mathrm{e}^{\mathrm{i}\zeta x} = \frac{\Gamma^2(\gamma)}{\Gamma(\gamma - N)\Gamma(\gamma + N)} \tag{17}$$

and

$$b(\zeta) = \lim_{x\to+\infty} \phi_2(x,\zeta)\,\mathrm{e}^{-\mathrm{i}\zeta x} = \frac{\mathrm{i}\gamma}{N}\,\frac{\Gamma(\gamma)\Gamma(-\gamma)}{\Gamma(N)\Gamma(-N)}, \tag{18}$$

where Γ denotes the Euler gamma function.

The poles of $a(\zeta)$ in the upper half-plane ($\eta > 0$) determine the discrete eigenvalues of the spectral problem; they are given by the condition

$$\gamma - N = n; \quad n \in \mathbb{N}. \tag{19}$$

There are thus $[N+\frac{1}{2}]$ ($[x]$ denotes the integer part of x) of them, located on the vertical line $\mathrm{Re}\,\zeta \equiv \xi = k/2$.

From the above, one finds that any solution of the NLS equation with an initial condition of the form (16) such that $B/2 < A < 3B/2$ contains one – and one only – soliton of amplitude $(2A - B)$,

$$q_s(x,t) = \mathrm{i}(2A-B)\,\mathrm{sech}\left[(2A-B)(x-kt-x_c)\right] \times \exp\left[\mathrm{i}kx - \mathrm{i}\left(\frac{k^2-B^2}{2} - 2A^2 + 2AB\right)t + \mathrm{i}\delta\right] \tag{20}$$

towards which it converges (for suitable x_c and δ) asymptotically – in L^∞ norm for instance.

We shall assume that when the solution of the full equation splits at the end of the period of resonant growth the emitted part q_s is in fact the soliton contained in the saturation solution (considered as an initial condition of the unforced NLS equation). It will be characterized by an amplitude $2A - B$ and a speed k where A, B and k are the parameters of the solution at the end of growth. If we modelise the tail q_r as a trial function $(A'/\mathrm{ch}\,B'x)\mathrm{e}^{\mathrm{i}k'x}$ with $A' = A - B$, $B' = B$, $k' = k$, in the transformation from q to q_r, N is decreased by one. As $a(\gamma, N-1) = a(\gamma, N)\cdot(\gamma + N - 1)/(\gamma - N)$, a is changed by a rational factor and the subtraction of the soliton appears as a simple Bäcklund transformation. This decomposition of the solution into a soliton and a tail is the only one that conserves the invariants H_n of the N.L.S. equation

$$H_n(q) = H_n(q_s) + H_n(q_r), \quad \text{for } n \geq 1.$$

However, numerical results show that while the soliton drifts away the tail remains pinned near the resonance point (see fig. 4b). This splitting of the solution in two parts with different speeds is linked with the phase relationship between the source and the solution. The soliton which is emitted does not exhibit any phase relationship with the source and its evolution may be described by the unforced equation. The tail is reconstructed after the emission to phase lock again with the source. Therefore in what follows we modelize the tail associated to the solution

$$\frac{A}{\mathrm{ch}\,B(x - x_0(S))} \exp\left[\mathrm{i}k(x - x_0(S)) + \mathrm{i}\varphi(S)\right]$$

by the trial function

$$\frac{A-B}{\mathrm{ch}\,B(x - x_0(S))} \exp\left[3\mathrm{i}\pi/2\right].$$

That modelization is supported on the following "physical" and mathematical grounds:

- it assumes a pinning of the tail at the resonance point in agreement with numerical results;
- it conserves the first invariant of the NLS equation;
- in the transformation from q to q_r, N is decreased by one which corresponds to the emission of one soliton;
- we may also notice that changing the speed of the solution, that is replacing q by $q\exp[\mathrm{i}\,kx]$ in the unforced N.L.S. equation does not alter the spectrum of the Zakharov–Shabat problem (15) except for a trivial translation along the real axis of amplitude $-k/2$.

4. The recurrence map and its properties

Our next goal is the study of the periodic generation of solitons. To this end we construct a recurrence map in the parameter space (N, B) as follows. We start from an initial condition with internal parameter values N_0, B_0, $k = 0$, $x_0 = 0$, $\varphi = 3\pi/2$. We let the internal parameters of the trial function evolve according to the differential system we derived in section 2. When the phase δ unlocks at time S we subtract from the saturation solution thus obtained with internal parameters N_s, B_s, k_s, x_{0_s}, φ_s, its soliton part by the procedure explained in section 3. Thus we obtain the

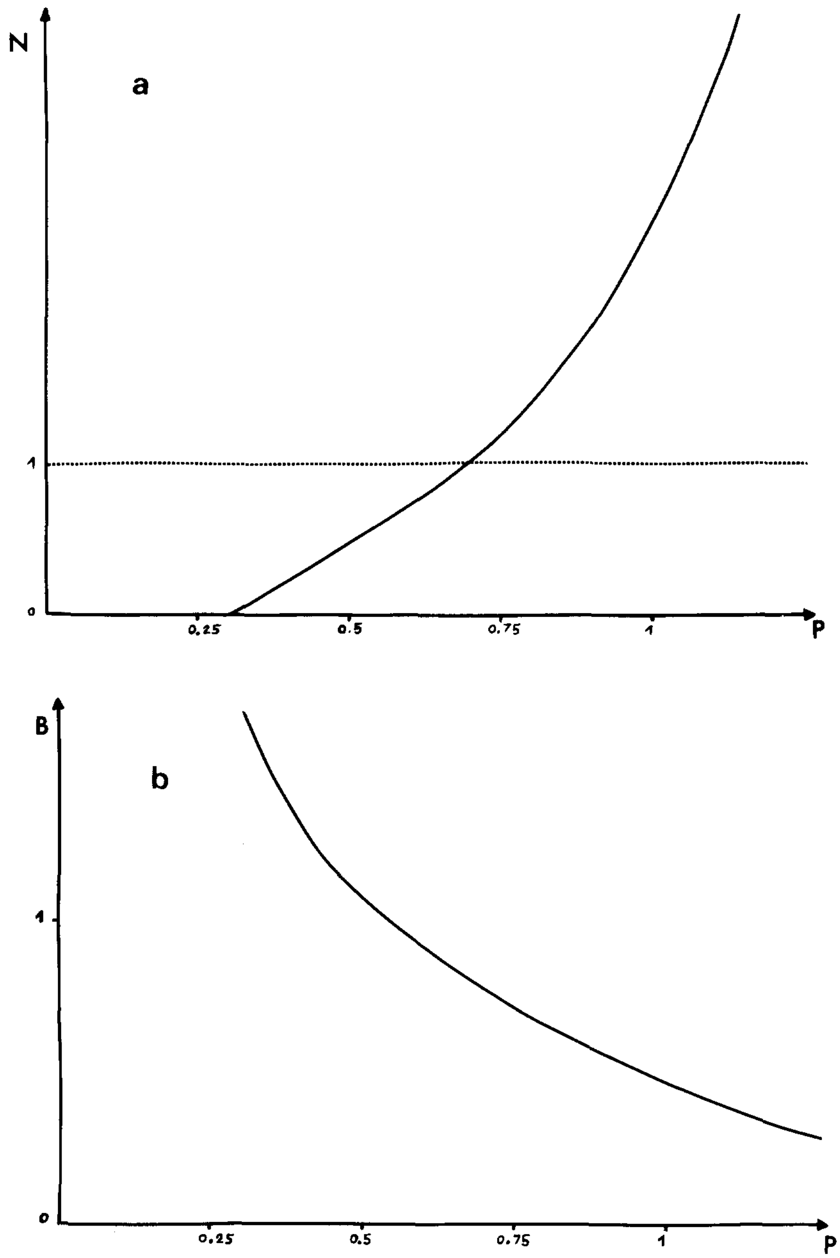

Fig. 8. Location of the fixed point in parameter space as a function of the parameter p. a) Number of solitons in the initial condition; b) inverse of the width.

parameters $N_1 = N_s - 1$, $B_1 = B_s$, $k = 0$, $x_0 = 0$, $\varphi = 3\pi/2$ of the tail. Such a scheme defines a recurrence map $\mathcal{T}$ in the parameter space (N, B) as the time S map of the flow (9), (12), (13) (where S is the unlocking time) followed by the subtraction map which accounts for the emission of one soliton:

$$\mathcal{T}(N_0, B_0) = (N_1 = N_s - 1, B_1 = B_s).$$

We found a stable strongly attracting fixed point for the map $\mathcal{T}$ when p exceeds 0.3. It accounts for the periodic emission of solitons. The corre-

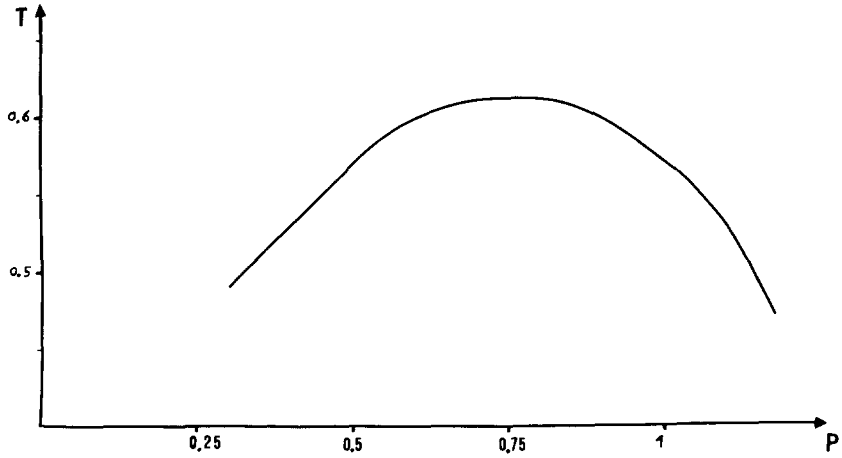

Fig. 9. Duration of the phase locking expressed in physical time t as a function of p.

sponding values of N, B and the duration of growth T in physical time $t = p^{3/2}s/\sqrt{2}$ are displayed on fig. 8a and b, and fig. 9. The value of N does strongly increase with p; we think that values greater than 1 (obtained for $p > 0.7$) must be rejected as unrealistic and due to the limitations of our model.

5. Conclusion

What precedes accounts for the following qualitative facts:

- the forced N.L.S. equation supports the emission of solitons;
- for a certain range of values of p this emission is periodic;
- there is no emission below a certain value of p and periodicity breaks down beyond an other value.

The model we derived does certainly not display all of the dynamics of the forced N.L.S. equation. However, it suggests that important features of the dynamics are of finite dimensional nature. Moreover it is one step beyond preceding models and therefore more realistic. Since the forced N.L.S. equation cannot be treated by perturbative methods we think our work brings some insight into the dynamics of this physically relevant equation.

Acknowledgement

We are deeply indebted to D. Pesme for fruitful discussions related to this paper.

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