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Higher elliptic Gamma functions, cohomology of linear groups and higher elliptic units

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Abstract

Multiple elliptic Gamma functions are functions originated from mathematical physics which form a hierarchy of multivariate meromorphic functions satisfying modular transformation properties for special linear groups $\mathrm{SL}_n(\mathbb{Z})$ involving an attached hierarchy of Bernoulli rational functions. In this work we study the connections between these functions and the arithmetic of number fields and especially the possibility of relating invariants in class field theory to special values of multiple elliptic Gamma functions. To achieve this, we define geometric families of the multiple elliptic Gamma functions attached to families of linear forms on integral lattices and recast the transformation properties satisfied by these functions in terms of cocycle relations for $\mathrm{SL}_n(\mathbb{Z})$. This construction of higher elliptic Gamma functions upgrades the construction carried out by Felder, Henriques, Rossi and Zhu for the elliptic Gamma function to the whole hierarchy of multiple elliptic Gamma functions. We also define geometric families of Bernoulli rational functions in a similar way and show that these functions form a collection of $(n-1)$-cocycles for groups of totally positive units in number fields and which may be used to evaluated partial zeta functions at $s = 0$ in totally real number fields. Moreover, we investigate the cocycle properties of both collections of functions for congruence subgroups in $\mathrm{SL}_n(\mathbb{Z})$ by performing a standard smoothing operation, and show that they simplify considerably.

The main goal of this work is the construction of conjectural higher elliptic units above number fields with exactly one complex place as special values of higher elliptic Gamma functions, upgrading the construction carried out by Bergeron, Charollois and García for complex cubic fields. These higher elliptic units are obtained by evaluating the multiplicative $(n-2)$-cocycle for a congruence subgroup in $\mathrm{SL}_n(\mathbb{Z})$ built from higher elliptic Gamma functions against a $(n-2)$-cycle on the group of totally positive units of a given number field of degree n with exactly one complex place. We conjecture that these higher elliptic units are indeed algebraic units which belong to prescribed abelian extensions of the base field where they are evaluated and that they satisfy a Kronecker limit formula which relates the logarithm of their modulus to values of derivatives of partial zeta functions at $s = 0$ in the base field. We present algorithms we used to produce examples of higher elliptic units and examples of such computations for number fields of degree 3, 4, 5 and 6.

Résumé

Les fonctions Gamma elliptiques multiples sont des fonctions provenant de la physique mathématique. Elles forment une hiérarchie de fonctions méromorphes à plusieurs variables satisfaisant des propriétés de transformations modulaires sous l'action des groupes spéciaux linéaires $\mathrm{SL}_n(\mathbb{Z})$ qui font intervenir une hiérarchie de fonctions rationnelles de Bernoulli. Dans ce travail, nous étudions les connexions entre ces fonctions et l'arithmétique des corps de nombres et en particulier la possibilité de relier des invariants de la théorie du corps de classes à des valeurs speciales de fonctions Gamma elliptiques multiples. Pour ce faire, nous définissons des familles géométriques de fonctions Gamma elliptiques multiples associées à des familles de formes linéaires sur des réseaux entiers et exprimons leurs propriétés de transformations en termes de relations de cocycle pour $\mathrm{SL}_n(\mathbb{Z})$. Cette construction de fonctions Gamma elliptiques supérieures généralise la construction proposée par Felder, Henriques, Rossi et Zhu pour la fonction Gamma elliptique à l'ensemble de la hiérarchie des fonctions Gamma elliptiques multiples. Nous définissons de manière similaire des familles géométriques de fonctions rationnelles de Bernoulli et montrons que ces fonctions forment une collection de $(n-1)$-cocycles pour des groupes d'unités totalement positives dans les corps de nombres qui peuvent être utilisés pour évaluer les fonctions zêta partielles de corps de nombres totalement réels en $s = 0$. De plus, nous étudions les propriétés de cocycle plus simples qui sont satisfaites par ces deux collections de fonctions pour les sous-groupes de congruence de $\mathrm{SL}_n(\mathbb{Z})$ en appliquant un processus de régularisation standard.

Le but principal de ce travail est la construction d'unités elliptiques supérieures au-dessus de corps de nombres avec une seule place complexe comme valeurs spéciales des fonctions Gamma elliptiques supérieures, ce qui généralise la construction proposée par Bergeron, Charollois et García pour les corps cubiques complexes. Ces unités elliptiques supérieures sont obtenues par évaluation d'un $(n-2)$-cocycle multiplicatif pour un sous-groupe de congruence de $\mathrm{SL}_n(\mathbb{Z})$ fabriqué à partir des fonctions Gamma elliptiques supérieures contre un $(n-2)$-cycle sur le groupe des unités totalement positives d'un corps de nombres de degré n avec une seule place complexe. Nous conjecturons que ces unités elliptiques supérieures sont en effet des unités algébriques qui appartiennent à des extensions abéliennes prescrites du corps de base où elles sont évaluées, et qu'elles satisfont une formule limite de Kronecker, reliant le logarithme de leur module à des valeurs de fonctions zêta partielles associées au corps de base, dérivées en $s = 0$. Nous présentons les algorithmes que nous avons utilisés pour produire des exemples d'unités elliptiques supérieures ainsi que des exemples de calculs de ces unités pour des corps de nombres de degré 3, 4, 5 et 6.

Foreword

This thesis consists of four chapters. The first two chapters correspond to the first two articles in the series *Geometric families of multiple elliptic Gamma functions and arithmetic applications*. They are already submitted for publication and are available at https://arxiv.org/abs/2510.16515 and https://arxiv.org/abs/2602.06561 respectively. These articles are included in this thesis with minor reformatting to avoid unnecessary redundancy. The third and fourth chapters correspond to what will be the third article in this series which is essentially an upgraded version of the working paper *Elliptic units above number fields with exactly one complex place* which is available at https://arxiv.org/abs/2406.06094. The third and fourth chapters also contain the discussion of optimal cases in our algebraicity conjecture presented in the short paper *Computations of higher elliptic units* which has been submitted for publication and is available at https://arxiv.org/abs/2601.11961.

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General introduction

In this work we are interested in a hierarchy of functions satisfying modular transformation properties for special linear groups and whose special values are conjectured to yield algebraic units. The first function in this hierarchy is the elliptic θ function defined on $\mathbb{C} \times \mathbb{H}$ by the following absolutely convergent infinite product:

$$\theta(z, \tau) = \prod_{n \geq 0} \left(1 - e^{-2i\pi z} e^{2i\pi(n+1)\tau}\right) \left(1 - e^{2i\pi z} e^{2i\pi n\tau}\right).$$

It is well-known that this function has modular symmetries for the group $\mathrm{SL}_2(\mathbb{Z})$ and it is used in the theory of *Complex Multiplication* to build elliptic units above imaginary quadratic fields. These elliptic units are given by evaluations of the form

$$\frac{\theta\left(\frac{1}{q}, \tau\right)^N}{\theta\left(\frac{N}{q}, N\tau\right)} \tag{0.1}$$

for well-chosen $q, N \in \mathbb{Z}_{\geq 2}$ and τ an irrational algebraic number in an imaginary quadratic field $\mathbb{K}$. These evaluations yield q-units inside ray class fields of $\mathbb{K}$ which are smoothed versions of Ramachandra's elliptic units (see [Ram64]). As a consequence of this construction, Hilbert's 12th problem is solved for imaginary quadratic fields as evaluations of the θ function describe the abelian extensions of imaginary quadratic fields. We may illustrate this with the following example where the two complex numbers

$$\begin{aligned} u_1 &= \theta\left(\frac{7}{91}, \frac{10 + e^{2i\pi/3}}{91}\right)^7 \theta\left(\frac{7}{13}, \frac{10 + e^{2i\pi/3}}{13}\right)^{-1} = \frac{-3 - 7\sqrt{-3}}{4} + \sqrt{\frac{-41 + 17\sqrt{-3}}{8}}, \\ u_2 &= \theta\left(\frac{14}{91}, \frac{10 + e^{2i\pi/3}}{91}\right)^7 \theta\left(\frac{14}{13}, \frac{10 + e^{2i\pi/3}}{13}\right)^{-1} = \frac{-3 - 7\sqrt{-3}}{4} - \sqrt{\frac{-41 + 17\sqrt{-3}}{8}} \end{aligned} \tag{0.2}$$

are algebraic numbers. More precisely, they are conjugated 13-units in an abelian extension of $\mathbb{K} = \mathbb{Q}(e^{2i\pi/3})$ ramified only above a prime ideal of norm 13 which is the splitting field of their minimal polynomial $x^4 + 3x^3 + 32x^2 + 13$. We review the main results on the θ function and on the construction of elliptic units above imaginary quadratic fields in chapter 0.

The second function in the hierarchy of functions which we study is the so-called elliptic Gamma function of Ruijsenaars [Rui97]. It is a meromorphic function of three variables defined on $\mathbb{C} \times \mathbb{H}^2$ by the infinite product:

$$\Gamma(z, \tau, \sigma) = \prod_{m,n \geq 0} \left(\frac{1 - \exp(2i\pi((m+1)\tau + (n+1)\sigma - z))}{1 - \exp(2i\pi(m\tau + n\sigma + z))}\right).$$

As a function of z it is well-defined outside of the discrete set of poles $\mathbb{Z} + \mathbb{Z}_{\geq 0}\tau + \mathbb{Z}_{\geq 0}\sigma$. This function satisfies modular transformation properties for the special linear group $\mathrm{SL}_3(\mathbb{Z})$ similar to those of the θ function for $\mathrm{SL}_2(\mathbb{Z})$, as shown by Felder and Varchenko in [FV00]. In their recent article [BCG23] Bergeron, Charollois and García used the elliptic Gamma function to construct conjectural algebraic units above complex cubic fields, by generalising the type of evaluation (0.1) which yields elliptic units above imaginary quadratic fields. This is showcased by the following example. Let us set $z = e^{2i\pi/3}10^{1/3}$ and $\mathbb{K} = \mathbb{Q}(z)$ the splitting field of the polynomial $x^3 - 10$. Denote by $\mathcal{O}_{\mathbb{K}}$ the ring of integers of $\mathbb{K}$. Let us set $\tau = 5z^2 + 11z - 5230$ and $\sigma = 2z^2 - z - 2335$. We may compute the four evaluations:

$$u_k = \frac{\Gamma\left(\frac{k}{5}, \frac{\tau}{1485}, \frac{\sigma}{1485}\right)^{11}}{\Gamma\left(\frac{11k}{5}, \frac{11\tau}{1485}, \frac{11\sigma}{1485}\right)} \approx \begin{cases} -27.5333588... - i \cdot 32.7146180... & \text{for } k = 1 \\ -2.2349933... - i \cdot 4.9384566... & \text{for } k = 2 \\ -0.0760627... + i \cdot 0.1680687... & \text{for } k = 3 \\ -0.0150592... + i \cdot 0.0178931... & \text{for } k = 4 \end{cases} \tag{0.3}$$

to high precision, say up to 1000 digits. The four values we obtain coincide up to this precision with the four roots of the palindromic relative polynomial $P_{\text{rel}} \in \mathcal{O}_{\mathbb{K}}[X]$:

$$P_{\text{rel}} = X^4 + (-7z^2 + 5z + 19)X^3 + (-19z^2 + 70z - 59)X^2 + (-7z^2 + 5z + 19)X + 1$$

which defines a $\mathbb{Z}/4\mathbb{Z}$ extension $\mathbb{L}$ of $\mathbb{K}$ ramified only above the prime ideal of norm 5 in $\mathcal{O}_{\mathbb{K}}$. Alternatively, we may check that these four values coincide up to 1000 digits of precision with four of the roots of a palindromic integral polynomial P_{abs} which defines an absolute equation of $\mathbb{L}$ over $\mathbb{Q}$:

$$P_{\text{abs}} = x^{12} + 57x^{11} + 1956x^{10} + 4640x^9 + 35415x^8 - 109818x^7 + 150139x^6 + \dots$$

Thus, u_1, u_2, u_3 and u_4 are expected to be algebraic units which are Galois conjugates over $\mathbb{K}$, and they are given by evaluations of the elliptic Gamma function at points in $\mathbb{K}$. From another point of view, we may see the evaluations given in (0.3) as an analytic parametrisation of the roots of the polynomial P_{rel}.

The elliptic units given by (0.2) as well as the conjectural higher elliptic units presented in (0.3) were computed using our general method for the construction of higher elliptic units which we develop in chapter III. These examples are presented with more detail in sections 0.3 and IV.2.1.3 respectively. The elliptic units of the form (0.1) above imaginary quadratic fields possess many other properties including a version of Kronecker's second limit formula (see Theorem 0.7), and the higher elliptic units we construct are expected to share similar properties. A key difference however when considering higher degree number fields is that the evaluations need to be very precise to obtain algebraic numbers. Indeed, it is not true that all evaluations of the form

$$\frac{\Gamma(z, \tau, \sigma)^N}{\Gamma(Nz, N\tau, N\sigma)}$$

yield algebraic numbers for any $z \in \mathbb{Q} - \mathbb{Z}$ and any τ, σ in a fixed complex cubic field $\mathbb{K}$: the parameters τ and σ must be very carefully chosen.

Our main goal throughout this work is to generalise the construction of elliptic units to general number fields with exactly one complex place (also called Almost Totally Real number fields, or ATR fields for short) using the multiple elliptic Gamma functions. These functions were introduced by Nishizawa [Nis01] to generalise both the theta function and elliptic Gamma function. They are meromorphic functions of several variables defined on $\mathbb{C} \times \mathbb{H}^{r+1}$ defined by the infinite product:

$$G_r(z, \tau_0, \dots, \tau_r) = \prod_{m_0,\dots,m_r \geq 0} \left(1 - e^{2i\pi(-z + \sum_{j=0}^r (m_j+1)\tau_j)}\right)\left(1 - e^{2i\pi(z + \sum_{j=0}^r m_j\tau_j)}\right)^{(-1)^r}$$

for all $r \in \mathbb{Z}_{\geq 0}$. As a function of z they are holomorphic over $\mathbb{C}$ when r is even and meromorphic when r is odd, in which case the set of poles is given by $\mathbb{Z} + \mathbb{Z}_{\geq 0}\tau_0 + \cdots + \mathbb{Z}_{\geq 0}\tau_r$. From this writing we may identify $\theta = G_0$ and $\Gamma = G_1$. These multiple elliptic Gamma functions satisfy modular transformation properties for the corresponding special linear group $\mathrm{SL}_{r+2}(\mathbb{Z})$ similar to those satisfied by the θ and Γ functions for $\mathrm{SL}_2(\mathbb{Z})$ and $\mathrm{SL}_3(\mathbb{Z})$ respectively.

In general, by analogy with the form (0.1) of elliptic units above imaginary quadratic fields, the higher elliptic units we aim to construct above a degree n number field $\mathbb{K}$ with exactly one complex place should be given by products of the form:

$$\prod_{j=1}^{\kappa} \frac{G_{n-2}(z_j, \tau_{1,j}, \ldots, \tau_{n-1,j})^N}{G_{n-2}(Nz_j, N\tau_{1,j}, \ldots, N\tau_{n-1,j})}$$

where N is a choice of smoothing index and the z_j's and the $\tau_{l,j}$'s should be carefully chosen elements in $\mathbb{K}$. Most of this work is dedicated to the formulation of a precise conjecture on the algebraic nature of these values, which is supported by numerical evidence. Our main conjecture concerns evaluations of this form which can be obtained as the evaluation of a multiplicative $(n-2)$-cocycle built from the G_{n-2} functions against a $(n-2)$-cycle in $\mathrm{SL}_n(\mathbb{Z})$. Indeed, if we introduce the following generalised rational Bernoulli functions:

$$\frac{1}{n!} B_{n,n}(z, \omega_1, \ldots, \omega_n) = \mathrm{coeff}[t^0] \left(\frac{e^{zt}}{\prod_{j=1}^{n} (e^{\omega_j t} - 1)} \right)$$

by a coefficient extraction of the series in the right-hand side, then the so-called modular property of the G_{n-2} functions is given by:

$$\prod_{j=1}^{n} G_{n-2}\left(\frac{z}{\omega_j}, \left(\frac{\omega_k}{\omega_j} \right)_{j \neq k} \right) = \exp\left(-\frac{2i\pi}{n!} B_{n,n}(z, \omega_1, \ldots, \omega_n) \right)$$

for any $\omega_1, \ldots, \omega_n \in \mathbb{C} - \{0\}$ satisfying $\omega_k/\omega_j \notin \mathbb{R}$ for $1 \leq k \neq j \leq n$ (see [Nar04]). This modular property for $\mathrm{SL}_n(\mathbb{Z})$ allows us to construct collections of partial modular symbols for $\mathrm{SL}_n(\mathbb{Z})$ from the G_{n-2} and $B_{n,n}$ functions, which yield multiplicative $(n-2)$-cocycles and additive $(n-1)$-cocycles respectively when restricted to specific subgroups of $\mathrm{SL}_n(\mathbb{Z})$. The first two chapters in this work are dedicated to the study of these cocycle relations, while the third and fourth chapter are dedicated to the construction and computation of higher elliptic units above number fields with exactly one complex place.

To make precise statements on the cocycle properties satisfied by these functions we introduce geometric families of these functions, upgrading the construction carried out by Felder, Henriques, Rossi and Zhu for the elliptic Gamma function in [FHRZ08] to the whole hierarchy of G_r functions. If $L \simeq \mathbb{Z}^n$ is a lattice of rank $n = r + 2$ in a $\mathbb{Q}$-vector space $V \simeq \mathbb{Q}^n$, $\Lambda = \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$ and $a_0, \ldots, a_r \in \Lambda$ we define a function

$$G_{r,a_0,\ldots,a_r} := \begin{cases} V/L \times \mathbb{C} \times \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{C}) & \to \mathbb{C} \\ (v, w, x) & \to G_{r,a_0,\ldots,a_r}(v)(w, x) \end{cases}$$

which generalises the ordinary G_r function which can be seen as a geometric G_r function for the abstract lattice $L = \mathbb{Z} + \mathbb{Z}\tau_0 + \cdots + \mathbb{Z}\tau_r$. A similar construction gives a geometric version of the Bernoulli rational functions $B_{n,n}$ attached to n linear forms $a_1, \dots, a_n$ in Λ:

$$B_{n,a_1,\dots,a_n} := \begin{cases} V/L \times \mathbb{C} \times \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{C}) & \to \mathbb{C} \\ (v, w, x) & \to B_{n,a_1,\dots,a_n}(v)(w, x). \end{cases}$$

For simplicity, we shall often use the general denomination of *higher elliptic Gamma functions* to refer to the geometric families of $G_{r,a_0,\dots,a_r}$ functions and the denomination of *higher Bernoulli rational functions* to refer to the geometric families of $B_{n,a_1,\dots,a_n}$ functions.

We show that these two collections of functions are equivariant under the action of $\mathrm{SL}_n(\mathbb{Z})$ on L and Λ and that they satisfy a modular property which extends the modular property satisfied by the ordinary G_r and $B_{n,n}$ functions:

Theorem 0.1: *For most configurations of the linear forms $a_1, \dots, a_n \in \Lambda$ and for w, x in a dense open subset of $\mathbb{C} \times \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{C}) \simeq \mathbb{C} \times \mathbb{C}^n$:*

$$\prod_{j=1}^{n} G_{n-2,a_1,\dots,\widehat{a_j},\dots,a_n}(v)(w, x)^{(-1)^{j+1}} = \exp(2i\pi B_{n,a_1,\dots,a_n}(v)(w, x)).$$

This statement generalises the classic modular property for the θ function given in (0.10) and it is made more precise in Theorems I.1 and II.2. This modular property may be rephrased as the splitting of a $(n-1)$-cocycle constructed from the higher Bernoulli rational functions by the higher elliptic Gamma functions. The higher Bernoulli rational functions indeed satisfy cocycle relations of their own as:

Theorem 0.2: *For most configurations of $n+1$ linear forms $a_0, \dots, a_n \in \Lambda$:*

$$\sum_{j=0}^{n} (-1)^j B_{n,a_0,\dots,\widehat{a_j},\dots,a_n} = 0.$$

This generalises the known cocycle properties of the Dedekind-Rademacher function (see section 0.1) and it is proven in Corollary I.1 as a consequence of a more general theorem on cocycle relations satisfied by certain indicator functions of closed rational polyhedral cones (see Theorem I.2). The collections of geometric G_r and B_n functions thus produce partial modular symbols for $\mathrm{SL}_n(\mathbb{Z})$ and the cocycle properties they satisfy become simpler when performing a smoothing operation which produces smoothed versions of these collections of functions, satisfying transformation properties for congruence subgroups in $\mathrm{SL}_n(\mathbb{Z})$. This is done by introducing a sublattice L' of L such that L/L' is cyclic of order N and by considering the smoothed objects:

$$G_{n-2,a_1,\dots,a_{n-1}}(v)(w, x, L, L') = \frac{G_{n-2,a_1,\dots,a_{n-1}}(v)(w, x, L')^N}{G_{n-2,a_1,\dots,a_{n-1}}(v)(w, x, L)}$$

and

$$B_{n,a_1,\dots,a_n}(v)(w, x, L, L') = N B_{n,a_1,\dots,a_n}(v)(w, x, L') - B_{n,a_1,\dots,a_n}(v)(w, x, L)$$

which are defined for $a_1, \dots, a_n \in \Lambda \cap \Lambda'$ where $\Lambda' = \mathrm{Hom}_{\mathbb{Z}}(L', \mathbb{Z})$. The most striking fact about these smoothed objects is that under mild assumptions on the lattice L', the function $B_{n,a_1,\dots,a_n}(v)(w, x, L, L')$ is independent of $w, x \in \mathbb{C} \times \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{C})$ and is a rational number related to higher Dedekind sums whose denominator is bounded independently of $a_1, \ldots, a_n$. This is expressed by the following theorem which shows that the modularity property simplifies considerably when performing this smoothing operation:

Theorem 0.3: *Let $a_1, \ldots, a_n \in \Lambda \cap \Lambda'$. Then, under mild assumptions on the smoothing lattice L' satisfying $L/L' \simeq \mathbb{Z}/N\mathbb{Z}$, the smoothed value $B_{n,a_1,\ldots,a_n}(v)(w,x,L,L')$ is independent of $w, x \in \mathbb{C} \times \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{C})$ and:*

$$B_{n,a_1,\ldots,a_n}(v)(w,x,L,L') \in \mathcal{D}(N,n)^{-1}\mathbb{Z}$$

where $\mathcal{D}(N,n) = \prod_{p|N} p^{\lfloor \frac{n}{p-1} \rfloor}$. Therefore, for most configurations of the linear forms $a_1, \ldots, a_n \in \Lambda \cap \Lambda'$, the smoothed version of the modular property gives the cocycle relation:

$$\left(\prod_{j=1}^{n} G_{n-2,a_1,\ldots,\widehat{a_j},\ldots,a_n}(v)(w,x,L,L')^{(-1)^{j+1}} \right)^{\mathcal{D}(N,n)} = 1.$$

These statements which can be seen as higher analogues of (0.16) and (0.15) are made more precise in Theorems II.3 and II.1. In arithmetic applications, we restrict the collections of geometric G_{n-2} and B_n functions to algebraic tori corresponding to groups of totally positive units in number fields, for which the linear forms are always in good configurations. A first arithmetic application of these functions is the computation of values of partial zeta functions in totally real number fields at $s = 0$ using the B_n functions, building on prior work on the computations of such values using Bernoulli rational functions (see [Shi76], [Col88], [DyDF14] and [CDG15]).

Theorem 0.4: *Let $\mathbb{F}$ be a totally real number field of degree n. Let $\mathfrak{f}$ be an integral ideal in $\mathcal{O}_{\mathbb{F}}$ and fix an integral ideal $\mathfrak{b}$ representing a class in the narrow ray class group $\mathrm{Cl}^+(\mathfrak{f})$. Then there are explicitly computable linear forms $a_{j,\rho}$ on the $\mathbb{Q}$-vector space $\mathbb{F} \simeq \mathbb{Q}^n$ for $1 \leq j \leq n$ and $\rho \in \mathfrak{S}_{n-1}$ and explicitly computable signs $\nu_\rho \in \{-1, 0, +1\}$ such that:*

$$\zeta_{\mathfrak{f}}(\mathfrak{b}, 0) = \frac{1}{n} \sum_{k=1}^{n} \sum_{\rho \in \mathfrak{S}_{n-1}} \nu_\rho B_{n,a_{1,\rho},\ldots,a_{n,\rho}}(1_{\mathbb{F}})(0, -\sigma_k)$$

where $\sigma_1, \ldots, \sigma_n$ are the real embeddings of $\mathbb{F}$.

The fact that these values may be expressed in terms of Bernoulli rational functions was already known for a long time, but the novelty here lies in the link between the arithmetic of totally real number fields and the higher elliptic Gamma functions through the associated collections of higher Bernoulli rational functions. The second arithmetic applications concerns the construction of conjectural higher elliptic units above ATR fields. The general result we have in mind is the following vague form of our conjecture, for which we give a more precise statement in chapter III.

Conjecture 0.5: *Let $\mathbb{K}$ be an ATR field of degree n which we view as a subset of $\mathbb{C}$ via one of its two complex embeddings. Fix an integral ideal $\mathfrak{f} \neq \mathcal{O}_{\mathbb{K}}$ and set $\mathfrak{f} \cap \mathbb{Z} = q\mathbb{Z}$ for some $q \in \mathbb{Z}_{>0}$. Fix an integral ideal $\mathfrak{b}$ of $\mathcal{O}_{\mathbb{K}}$ representing a class in $\mathrm{Cl}^+(\mathfrak{f})$ and an integral ideal $\mathfrak{a}$ of $\mathcal{O}_{\mathbb{K}}$ of norm N such that $\mathcal{O}_{\mathbb{K}}/\mathfrak{a}$ is cyclic. Lastly, fix a system of fundamental units $\varepsilon_1, \ldots, \varepsilon_r$ for the group $\mathcal{O}_{\mathbb{K}}^{+,\times}$ of totally positive units in $\mathcal{O}_{\mathbb{K}}$. Then there is an explicit* *integer κ and explicit parameters z_j, $\tau_{1,j},\dots,\tau_{n-1,j}$ in $\mathbb{K}$, all depending on $\mathfrak{f},\mathfrak{b},\mathfrak{a},\varepsilon_1,\dots,\varepsilon_r$ such that the evaluation*

$$u = \left(\prod_{j=1}^{\kappa} \frac{G_{n-2}(z_j,\tau_{1,j},\dots,\tau_{n-1,j})^N}{G_{n-2}(Nz_j,N\tau_{1,j},\dots,N\tau_{n-1,j})}\right)^{\mathcal{D}(N,n)} \tag{0.4}$$

is an algebraic unit in the narrow ray class field $\mathbb{K}^+(\mathfrak{f})$ at $\mathfrak{f}$ (or q-unit if $n=2$). This algebraic unit is expressed as the evaluation of a multiplicative $(n-2)$-cocycle built from N-smoothed G_{n-2} functions, which depends on the choice of some set of base points $\underline{h}=(h_\rho)_{\rho\in\mathfrak{S}_r}$, against an $(n-2)$-cycle $\Upsilon=\Upsilon(\varepsilon_1,\dots,\varepsilon_r)\in H_{n-2}(\mathcal{O}_{\mathbb{K}}^{+,\times},\mathbb{Z})$. In addition, the algebraic unit u satisfies a Kronecker limit formula of the form:

$$N\zeta_{\mathfrak{f}}'(\mathfrak{b},0)-\zeta_{\mathfrak{f}}'(\mathfrak{ab},0)=\frac{1}{\mathcal{D}(N,n)}\log|u|^2.$$

This conjecture is an attempt at generalising the known results for the elliptic units built from the θ function (see Theorems 0.7 and 0.8). Higher analogues of these results are notoriously harder to prove, since there is no clear generalisation of the theory of *Complex multiplication* for higher degree number fields with exactly one complex place.

Most of chapter III is dedicated to the formulation of a precise geometric setup which gives explicit formulas for the parameters $z_j,\tau_{1,j},\dots,\tau_{n-1,j}$ where the N-smoothed G_{n-2} functions are evaluated to produce the elliptic units. This is done by generalising the geometric setup in [BCG23] for $n=3$ to higher degree ATR fields. The specific evaluations of a $(n-2)$-cocycle depending on a choice of base points $\underline{h}$ against the $(n-2)$-cycle Υ is presented in section III.3.1. The remainder of section III.3 concerns the precise analysis of the underlying geometric setup and the determination of the size κ of the product (0.4) as well as the parameters z_j and $\tau_{l,j}$ involved in the evaluation. A notable technical difficulty that arises for $n\geq 4$, which is absent from the cases $n=2$ and $n=3$, is that the base points $h_\rho,\rho\in\mathfrak{S}_r$ upon which the evaluation greatly depends cannot be chosen independently. It is the subject of section III.3.4 where we define a compatibility condition that these base points must satisfy. Our main conjecture is supported by numerical evidence which we present in chapter IV together with the algorithms we used to produce such examples.

Our main conjecture is related to Hilbert's 12th problem and to the rank one abelian Stark conjectures for number fields with exactly one complex place. Indeed, if proven, a precise conjecture in the flavour of Conjecture 0.5 would give a partial answer to Hilbert's 12th problem for ATR fields, by building abelian extensions of these number fields using the general multivariate meromorphic functions that are the multiple elliptic Gamma functions (see section III.4.2.4). This would also prove some cases of the rank one abelian Stark conjectures for number fields with exactly one complex place (see section III.4.2.5 for a discussion of those) by giving an explicit formula for the Stark unit.

Let us now give an outline of the present work:

0. In chapter 0 we briefly review the basic results of *Complex Multiplication* regarding elliptic units above imaginary quadratic fields. We present the properties satisfied by the θ function and by the elliptic units obtained by evaluation of this function at points of an imaginary quadratic field and explain how the results of this work generalise each of these properties for the higher elliptic Gamma functions and the higher elliptic units.

I. In chapter I we recall the properties of the collection of multiple elliptic Gamma functions and the associated collection of generalised Bernoulli rational functions and we focus on the construction of their associated cocycles for $\mathrm{SL}_n(\mathbb{Z})$. To this end, we introduce geometric families of multiple elliptic Gamma functions, upgrading the construction carried out by Felder, Henriques, Rossi and Zhu for the elliptic Gamma function in [FHRZ08] to the whole hierarchy of G_r functions. We also introduce geometric families of Bernoulli rational functions associated to the B_n and show how the modularity property satisfied by the G_r functions may be rephrased in terms of cocycle properties for these collections of functions. A first arithmetic application is then given, as we use the collection of geometric B_n functions to express values of partial zeta functions at $s = 0$ in totally real number fields.

II. In chapter II we expand on the cocycle properties of the collections of geometric G_{n-2} and B_n functions by introducing a smoothing operation inspired by previous work on smoothed partial zeta functions (see for instance [CN79], [Das08], [CD14]). The smoothed versions of the cocycles built from these functions are then shown to define partial modular symbols on congruence subgroups in $\mathrm{SL}_n(\mathbb{Z})$.

III. In chapter III we carry out the construction of conjectural higher elliptic units above number fields of degree $n \geq 3$ with exactly one complex place given by evaluations of G_{n-2} functions. These evaluations must be done in a very precise manner and we give a very detailed description of our construction to highlight the main theoretical and computational difficulties.

IV. In chapter IV we present the algorithms we use to compute our higher elliptic units and we provide numerical evidence to support our general conjecture for fields of degree $3, 4, 5$ and 6.

We recommend that the reader who is mostly interested in the construction of higher elliptic units skips the contents of sections I.3 to II.5 and focuses on chapter III after they have become familiar with the construction of higher elliptic Gamma functions in section I.2. The reader might then revisit these sections where we prove many statements on the cocycle properties of the higher elliptic Gamma functions which constitute important steps towards a proof of the Main Conjecture III.37.

Chapter 0

Elliptic units for imaginary quadratic fields

0.1 The Dedekind η-function

The Dedekind η-function is a modular function for $\mathrm{SL}_2(\mathbb{Z})$ defined for $\tau \in \mathbb{H}$ by the infinite product:

$$\eta(\tau) = q^{1/24} \prod_{n \geq 1} (1 - q^n)$$

where as usual $q = \exp(2i\pi\tau)$. It satisfies a modularity property of the form

$$\eta\left(\frac{a\tau + b}{c\tau + d}\right) = \epsilon(a, b, c, d)\sqrt{c\tau + d}.\eta(\tau) \tag{0.5}$$

where for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, $\epsilon(a, b, c, d) = \epsilon(\gamma)$ is a 24-th root of unity. More precisely, this root of unity is expressed in terms of Dedekind sums. These sums are defined for $c > 0$ and $(c, d) = 1$ by:

$$s(c, d) = \sum_{k=1}^{c-1} b_1\left(\frac{k}{c}\right) b_1\left(\frac{kd}{c}\right) \tag{0.6}$$

where $b_1 : t \to t - \lfloor t \rfloor - 1/2$ is the classic periodic version of the Bernoulli polynomial $B_1(t) = t - 1/2$. The Dedekind-Rademacher function ϕ_{DR} is defined on $\mathrm{SL}_2(\mathbb{Z})$ by:

$$\phi_{DR}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} \frac{b}{d} & \text{if } c = 0 \\ \frac{a+d}{c} - 12.\mathrm{sign}(c)s(|c|, d) & \text{if } c \neq 0 \end{cases}$$

where $\mathrm{sign}(c) = c/|c|$ when $c \neq 0$ and $\mathrm{sign}(c) = 0$ if $c = 0$. The function ϕ_{DR} takes values in $\mathbb{Z}$ and satisfies the modularity relation:

$$\eta\left(\frac{a\tau + b}{c\tau + d}\right) = \sqrt{c\tau + d}\,.e^{\frac{i\pi}{12}(\phi_{DR}(\gamma) - 3.\mathrm{sign}(c))}\eta(\tau).$$

(see [Rad32]). Thus the unit $\epsilon(a, b, c, d)$ is explicitly given by

$$\epsilon(a, b, c, d) = e^{\frac{i\pi}{12}(\phi_{DR}(\gamma) - 3.\mathrm{sign}(c))}. \tag{0.7}$$

The function ϕ_{DR} is almost a group morphism as for any $\gamma, \gamma' \in \mathrm{SL}_2(\mathbb{Z})$:

$$\phi_{DR}(\gamma'') = \phi_{DR}(\gamma) + \phi_{DR}(\gamma') - 3.\mathrm{sign}(cc'c'')$$

where $\gamma'' = \left(\begin{smallmatrix} a'' & b'' \\ c'' & d'' \end{smallmatrix}\right) = \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \gamma'\gamma$. The level N function Ψ_N defined on the congruence subgroup $\Gamma_0(N)$ by

$$\Psi_N \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \phi_{DR} \begin{pmatrix} a & bN \\ \frac{c}{N} & d \end{pmatrix} - \phi_{DR} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

(see [Rad32], [Maz79]) then defines a 1-cocycle in $H^1(\Gamma_0(N), \mathbb{Z})$.

The Dedekind η-function appears as the key ingredient for Kronecker's first limit formula which is essentially the statement that when τ is an imaginary quadratic number:

$$\lim_{s\to 1} \left(\sum_{m,n \in \mathbb{Z}^2 - (0,0)} Q(m,n)^{-s} - \frac{\pi}{s-1} \right) = 2\pi \left(\gamma - \log 2 - \log \sqrt{y} - \log |\eta(\tau)|^2 \right) \tag{0.8}$$

where $y = \Im(\tau)$, $Q(m,n) = y^{-1}|m + n\tau|^2$ is the quadratic form attached to τ and $\gamma = \lim_{n\to\infty}(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n)$ is Euler's gamma constant (see [[Sie80], Theorem 1]). This limit formula may be used to compute values of L-functions for imaginary quadratic fields at $s = 1$ (see for instance [[Sie80], Chapter II, §1]). It may also be used to compute values of partial zeta functions at $s = 1$ (or equivalently at $s = 0$ thanks to the functional equation) in real quadratic fields (see [[Sie80], Theorem 12]). We may express this result as:

Theorem 0.6 [Meyer's Theorem] : *Let $\mathbb{K}$ be a real quadratic field. Fix an integral ideal $\mathfrak{b}$ of $\mathbb{K}$ representing a class in the narrow Hilbert class group $\mathrm{Cl}^+(\mathbb{K})$ of $\mathbb{K}$. Fix $[\beta_1, \beta_2]$ a $\mathbb{Z}$-basis of $\mathfrak{b}$ satisfying $\beta_1/\beta_2 > \beta_1'/\beta_2'$ where $\beta \to \beta'$ represents algebraic conjugation in $\mathbb{K}$. Let $\mathcal{O}_{\mathbb{K}}^{+,\times}$ be the group of totally positive units of $\mathcal{O}_{\mathbb{K}}$ and ε be the unique generator of $\mathcal{O}_{\mathbb{K}}^{+,\times}$ satisfying $\varepsilon > \varepsilon'$. Denote by a, b, c, d the unique integers satisfying $\varepsilon\beta_1 = a\beta_1 + b\beta_2$ and $\varepsilon\beta_2 = c\beta_1 + d\beta_2$. Then the matrix $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ belongs to $\mathrm{SL}_2(\mathbb{Z})$ and the value at $s = 0$ of the partial zeta function attached to the class of $\mathfrak{b}$ in $\mathrm{Cl}^+(\mathbb{K})$ is given by:*

$$\zeta_{\mathcal{O}_{\mathbb{K}}}(\mathfrak{b}, 0) = \frac{\mathcal{N}(\beta_2)}{12.|\mathcal{N}(\beta_2)|}\phi_{DR}(\gamma). \tag{0.9}$$

A similar statement holds for partial zeta functions attached to ray class fields using the modularity defect of θ functions (see [[Sie80], Theorems 12 and 13]). This statement implies in particular that the value $\zeta_{\mathcal{O}_{\mathbb{K}}}(\mathfrak{b}, 0)$ is rational, and, more generally, it is well-known from the theorem of Klingen and Siegel that partial zeta functions in totally real number fields take rational values at non-positive integers. In this work we generalise (0.9) to higher degree totally real number fields using our higher Bernoulli rational functions (see Theorem 0.4).

The Dedekind-Rademacher function and its level N avatars have also played a key role in the construction of conjectural elliptic units above real quadratic fields by Darmon and Dasgupta [DD06] and they are also connected to Gross-Stark units above real quadratic fields (see [DPV24]). In the next section we recall the modular transformation properties of the θ function using those of the η function.

0.2 Transformation properties of the θ function

Recall the definition of the θ function for $(z,\tau) \in \mathbb{C} \times \mathbb{H}$ by:

$$\theta(z,\tau) = \prod_{n\geq 0} \left(1 - e^{-2i\pi z} e^{2i\pi(n+1)\tau}\right)\left(1 - e^{2i\pi z} e^{2i\pi n\tau}\right).$$

This function enjoys modular transformation properties under an action of $\mathrm{SL}_2(\mathbb{Z})$. Indeed, there is a rational function $P_2 : \mathrm{SL}_2(\mathbb{Z}) \to \mathbb{Q}(z,\tau)$ such that for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$:

$$\theta\left(\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right) = \theta(z,\tau).e^{2i\pi P_{2,\gamma}(z,\tau)}. \tag{0.10}$$

The rational function $P_{2,\gamma}$ may be expressed in terms of the classic Dedekind sums as:

$$P_{2,\gamma}(z,\tau) = \begin{cases} 0 & \text{if } c = 0 \text{ and } d = 1 \\ z - \frac{1}{2} & \text{if } c = 0 \text{ and } d = -1 \\ \frac{z^2c^2+zc+1/6}{2c(c\tau+d)} - \frac{z}{2} + \frac{c\tau+d}{12c} - \mathrm{sign}(c)(s(|c|,d) + \frac{1}{4}) & \text{if } c \neq 0 \end{cases}. \tag{0.11}$$

This can be derived from [[Sie80], Proposition 4] in conjunction with [[Rad32], Formula (3.26)] as follows. Let us introduce another theta function ϑ_1 defined on $\mathbb{C} \times \mathbb{H}$ by:

$$\vartheta_1(z,\tau) = e^{2i\pi(\tau/8-1/4)}(e^{i\pi z} - e^{-i\pi z}) \prod_{m\geq 1}(1 - e^{2i\pi z}e^{2i\pi m\tau})(1 - e^{2i\pi m\tau}e^{-2i\pi z})(1 - e^{2i\pi m\tau}).$$

It follows from a straightforward computation that:

$$\vartheta_1(z,\tau) = e^{2i\pi(\tau/12 - z/2 + 1/4)}.\theta(z,\tau).\eta(\tau)$$

for any $z \in \mathbb{C}$ and any $\tau \in \mathbb{H}$. Let us now prove (0.11) for $c > 0$ using the η and ϑ_1 functions as well as results from [Sie80].

Proof of formula (0.11):

Let us write as before $\epsilon = \epsilon(a,b,c,d)$ for the 24-th root of unity satisfying

$$\eta\left(\frac{a\tau+b}{c\tau+d}\right) = \epsilon.\sqrt{c\tau+d}.\eta(\tau).$$

The modular transformation property of ϑ_1 given in [[Sie80], Proposition 4] may be written as:

$$\vartheta_1\left(\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right) = \epsilon^3\sqrt{c\tau+d}.e^{i\pi cz^2/(c\tau+d)}.\vartheta_1(z,\tau). \tag{0.12}$$

To simplify notations we adopt notations in the style of Siegel : $\tau^* = (a\tau+b)/(c\tau+d)$ and $z^* = z/(c\tau+d)$. Then (0.12) reads:

$$e^{2i\pi\left(\frac{\tau^*}{12} - \frac{z^*}{2} + \frac{1}{4}\right)}\theta(z^*,\tau^*)\eta(\tau^*) = \epsilon^3\sqrt{c\tau+d}.e^{i\pi cz^2/(c\tau+d)}.e^{2i\pi\left(\frac{\tau}{12} - \frac{z}{2} + \frac{1}{4}\right)}\theta(z,\tau)\eta(\tau).$$

Dividing by the modularity relation (0.5) for η gives:

$$e^{2i\pi\left(\frac{\tau^*}{12} - \frac{z^*}{2} + \frac{1}{4}\right)}\theta(z^*,\tau^*) = \epsilon^2 e^{i\pi cz^2/(c\tau+d)}.e^{2i\pi\left(\frac{\tau}{12} - \frac{z}{2} + \frac{1}{4}\right)}\theta(z,\tau).$$

Therefore:

$$\theta(z^*,\tau^*) = \epsilon^2 e^{2i\pi\left(\frac{z^2c^2}{2c(c\tau+d)}\right)}.e^{2i\pi\left(\frac{\tau}{12} - \frac{z}{2}\right)}e^{-2i\pi\left(\frac{\tau^*}{12} - \frac{z^*}{2}\right)}\theta(z,\tau).$$

$$\theta(z^*,\tau^*) = \epsilon^2 e^{2i\pi\left(\frac{z^2c^2+zc}{2c(c\tau+d)}-\frac{z}{2}+\frac{\tau}{12}-\frac{a\tau+b}{12(c\tau+d)}\right)}\theta(z,\tau).$$

We may now rewrite $\tau-\tau^*$ as:

$$\tau-\tau^* = \frac{c(c\tau+d)\tau - c(a\tau+b)}{c(c\tau+d)}$$
$$\tau-\tau^* = \frac{(c\tau+d)^2 - d(c\tau+d) - ca\tau - (ad-1))}{c(c\tau+d)}$$
$$\tau-\tau^* = \frac{c\tau+d}{c} + \frac{1}{c(c\tau+d)} - \frac{a+d}{c}.$$

Thus:

$$\theta(z^*,\tau^*) = \epsilon^2 e^{2i\pi\left(\frac{z^2c^2+zc+1/6}{2c(c\tau+d)}-\frac{z}{2}+\frac{c\tau+d}{12c}-\frac{a+d}{12c}\right)}\theta(z,\tau).$$

It follows from (0.7) that ϵ is expressed in terms of the Dedekind-Rademacher function ϕ_{DR} as:

$$\epsilon(a,b,c,d) = e^{\frac{i\pi}{12}(\phi_{DR}(\gamma)-3\mathrm{sign}(c))} = e^{\frac{i\pi}{12}(\frac{a+d}{c}-12.\mathrm{sign}(c)(s(|c|,d)+1/4))}.$$

This gives:

$$\theta(z^*,\tau^*) = e^{2i\pi\left(\frac{z^2c^2+zc+1/6}{2c(c\tau+d)}-\frac{z}{2}+\frac{c\tau+d}{12c}-\mathrm{sign}(c)(s(|c|,d)+1/4)\right)}\theta(z,\tau)$$

as claimed. □

The modular transformation property for the θ function involving the rational function P_2 may be generalised to the whole hierarchies of higher elliptic Gamma functions and higher Bernoulli rational functions, as proven in Theorem 0.1.

It is classical to perform a smoothing operation on the θ function to obtain modular units of one of the two following shapes:

$$\frac{\theta(z,\tau)^{N^2}}{\theta(Nz,\tau)} \quad or \quad \frac{\theta(z,\tau)^N}{\theta(Nz,N\tau)}.$$

In the first case, by adding a small exponential prefactor we obtain the basic ingredient for Siegel units:

$${}_N\theta(z,\tau) = e^{2i\pi\left(\frac{N^2-1}{12}\tau+\frac{N-N^2}{2}\left(z-\frac{1}{2}\right)\right)}\frac{\theta(z,\tau)^{N^2}}{\theta(Nz,\tau)} \tag{0.13}$$

when $z \in \mathbb{Q}$, τ is an imaginary quadratic number and N is a rational prime which is inert in the field $\mathbb{Q}(\tau)$ (see [[Kat04], Proposition 1.3]). The second expression gives smoothed versions of Ramachandra's elliptic units (see [Ram64]) in the setting where N is a rational prime which splits in $\mathbb{Q}(\tau)$. In this work we focus on the second type of smoothing and if we set:

$$\theta^{(N)}(z,\tau) = \frac{\theta(z,\tau)^N}{\theta(Nz,N\tau)} \tag{0.14}$$

then (0.10) gives for any $N \mid c$:

$$\theta^{(N)}\left(\frac{z}{c\tau+d},\frac{a\tau+b}{c\tau+d}\right) = \theta^{(N)}(z,\tau).e^{2i\pi P_{2,\gamma}^{(N)}(z,\tau)} \tag{0.15}$$

where

$$P_{2,\gamma}^{(N)}(z,\tau)=\begin{cases}0 & \text{if } c=0 \text{ and } d=1\\ \frac{1-N}{2} & \text{if } c=0 \text{ and } d=-1\\ \operatorname{sign}(c)\left(s\left(\frac{|c|}{N},d\right)-Ns(|c|,d)+\frac{1-N}{4}\right) & \text{if } c\neq 0.\end{cases}$$

Crucially, the smoothed modularity defect $P_{2,\gamma}^{(N)}$ depends only on the matrix $\gamma\in\Gamma_0(N)=\{\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\mid\; N\mid c\}$ and not on z and τ. Moreover, it satisfies an integrality property as the values of $P_{2,\gamma}^{(N)}$ lie in $\frac{1}{12}\mathbb{Z}$. The application $(\gamma\to P_{2,\gamma}^{(N)})$ is essentially a smoothed version of the classic Dedekind-Rademacher function ϕ_{DR}, which differs slightly from the 1-cocycle Ψ_N on the congruence subgroup $\Gamma_0(N)$. Explicitly:

$$P_{2,\gamma}^{(N)}=\frac{1}{12}\left(N\phi_{DR}\begin{pmatrix}a&b\\c&d\end{pmatrix}-\phi_{DR}\begin{pmatrix}a&bN\\\frac{c}{N}&d\end{pmatrix}\right)+\begin{cases}0 & \text{if } c=0 \text{ and } d=1\\ \frac{1-N}{2} & \text{if } c=0 \text{ and } d=-1\\ \operatorname{sign}(c)\frac{1-N}{4} & \text{if } c\neq 0\end{cases} \tag{0.16}$$

and its reduction mod $\mathbb{Z}$ defines a 1-cocycle on $\Gamma_0(N)$ with values in $\frac{1}{12}\mathbb{Z}/\mathbb{Z}$. In this work we generalise the cocycle properties satisfied by the rational function P_2 and its smoothed version $P_2^{(N)}$ to the collection of higher Bernoulli rational functions which we introduce (see Theorems 0.2 and 0.3).

0.3 Kronecker's second limit formula and elliptic units

We now briefly review the two main results on the smoothed elliptic units given by (0.14), that is the smoothed version of Kronecker's second limit formula they satisfy and the fact that they are algebraic S-units (see [Sie80], [Sta80] and [Ram64], [Rob73] for versions of these results).

Theorem 0.7 [Kronecker's second limit formula (a smoothed version)] : *Let $\mathbb{K}$ be an imaginary quadratic field. Fix an integral ideal $\mathfrak{f}\neq\mathcal{O}_{\mathbb{K}}$. Set $q\mathbb{Z}=\mathfrak{f}\cap\mathbb{Z}$ for some positive integer q. Fix an integral ideal $\mathfrak{b}$ representing a class in the class group at $\mathfrak{f}$. Let $\mathfrak{a}$ be an integral ideal of prime norm N such that $(N,q)=1$. Let μ,ν be elements of $\mathbb{K}$ such that $\mathfrak{f}\mathfrak{b}^{-1}=\mathbb{Z}\mu+\mathbb{Z}\nu$ and $\mathfrak{f}(\mathfrak{a}\mathfrak{b})^{-1}=\mathbb{Z}\mu+\mathbb{Z}\nu/N$ with $\mu/\nu=\tau\in\mathbb{K}\cap\mathbb{H}$. Assume further that $\nu/q\equiv 1 \bmod \mathfrak{f}\mathfrak{b}^{-1}$. Then*

$$N.\zeta_{\mathfrak{f}}'(\mathfrak{b},0)-\zeta_{\mathfrak{f}}'(\mathfrak{a}\mathfrak{b},0)=-\frac{1}{w_{\mathfrak{f}}}\log\left|\frac{\theta\left(\frac{1}{q},\tau\right)^N}{\theta\left(\frac{N}{q},N\tau\right)}\right|^2$$

where $w_{\mathfrak{f}}$ is the number of roots of unity in $\mathcal{O}_{\mathbb{K}}$ which are congruent to 1 mod $\mathfrak{f}$.

Proof :

The partial zeta function $\zeta_{\mathfrak{f}}(\mathfrak{b},s)$ is defined by:

$$\zeta_{\mathfrak{f}}(\mathfrak{b},s)=\sum_{\mathfrak{b}'\sim\mathfrak{b}}\mathcal{N}(\mathfrak{b}')^{-s}$$

where the sum ranges over integral ideals in the same class as $\mathfrak{b}$ in the ray class group at $\mathfrak{f}$. Using Siegel's trick one may rewrite this as:

$$\zeta_{\mathfrak{f}}(\mathfrak{b}, s) = \frac{1}{w_{\mathfrak{f}}}\mathcal{N}(\mathfrak{b})^{-s} \sum_{\mu \in 1+\mathfrak{f}\mathfrak{b}^{-1}} |\mu|^{-s}$$

where w is the number of roots of unity in $\mathbb{K} = \mathbb{Q}(\tau)$. Since $[\mu, \nu]$ is a $\mathbb{Z}$-basis of the fractional ideal $\mathfrak{f}\mathfrak{b}^{-1}$ we obtain the expression:

$$\zeta_{\mathfrak{f}}(\mathfrak{b}, s) = \frac{1}{w_{\mathfrak{f}}}\mathcal{N}(\mathfrak{b})^{-s} \sum_{m,n \in \mathbb{Z}} |m\mu + n\nu + 1|^{-s}.$$

Since $\nu/q \equiv 1 \bmod \mathfrak{f}$ this can be rewritten as:

$$\zeta_{\mathfrak{f}}(\mathfrak{b}, s) = \frac{1}{w_{\mathfrak{f}}}\mathcal{N}(\mathfrak{b})^{-s} \sum_{m,n \in \mathbb{Z}} |m\mu + n\nu + \nu/q|^{-s}.$$

Let us define as in [Sta80] the function

$$G(s) = \pi^{-s}\Gamma(s) \sum_{m,n \in \mathbb{Z}} |m\mu + n\nu + \nu/q|^{-s}.$$

This function can be analytically continued to $\mathbb{C}-\{1\}$ and it follows from [[Sta80], formula (10)] that:

$$G(0) = -\log\left|\theta(u\tau + v, \tau)e^{2i\pi\left(\frac{u(u\tau+v)}{2} + \frac{\tau}{12} - \frac{u\tau+v}{2} + \frac{1}{4}\right)}\right|^2$$

where $\tau = \mu/\nu$ and u, v are the unique rational number satisfying $u.\mu + v.\nu = \nu/q$, that is $u = 0$ and $v = 1/q$. On the other hand, since $G(s) = \pi^{-s}\Gamma(s).w_{\mathfrak{f}}.\mathcal{N}(\mathfrak{b})^s\zeta_{\mathfrak{f}}(\mathfrak{b}, s)$ and

$$\lim_{s\to 0} \zeta_{\mathfrak{f}}(\mathfrak{b}, s)\Gamma(s) = \zeta_{\mathfrak{f}}'(\mathfrak{b}, 0)$$

we obtain the equality:

$$\zeta_{\mathfrak{f}}'(\mathfrak{b}, 0) = -\frac{1}{w_{\mathfrak{f}}}\log\left|\theta\left(\frac{1}{q}, \tau\right) e^{2i\pi\left(\frac{\tau}{12} - \frac{1}{2q} + \frac{1}{4}\right)}\right|^2.$$

The same argument can be applied to the determination of $\zeta_{\mathfrak{f}}'(\mathfrak{a}\mathfrak{b}, 0)$. Indeed, since $\nu/q \equiv 1 \bmod \mathfrak{f}\mathfrak{b}^{-1}$ it is also true that $\nu/q \equiv 1 \bmod \mathfrak{f}(\mathfrak{a}\mathfrak{b})^{-1}$ and

$$\zeta_{\mathfrak{f}}(\mathfrak{b}, s) = \frac{1}{w_{\mathfrak{f}}}\mathcal{N}(\mathfrak{a}\mathfrak{b})^{-s} \sum_{m,n \in \mathbb{Z}} \left|m\mu + n\frac{\nu}{N} + 1\right|^{-s} = \frac{1}{w_{\mathfrak{f}}}\mathcal{N}(\mathfrak{a}\mathfrak{b})^{-s} \sum_{m,n \in \mathbb{Z}} \left|m\mu + n\frac{\nu}{N} + \frac{\nu}{q}\right|^{-s}.$$

It follows once again from [[Sta80], formula (10)] that:

$$\zeta_{\mathfrak{f}}'(\mathfrak{a}\mathfrak{b}, 0) = -\frac{1}{w_{\mathfrak{f}}}\log\left|\theta\left(\frac{N}{q}, N\tau\right) e^{2i\pi\left(\frac{N\tau}{12} - \frac{N}{2q} + \frac{1}{4}\right)}\right|^2$$

since $0.\mu + (N/q).(\nu/N) = \nu/q$. Putting both statements together yields:

$$N.\zeta_{\mathfrak{f}}'(\mathfrak{b}, 0) - \zeta_{\mathfrak{f}}'(\mathfrak{a}\mathfrak{b}, 0) = -\frac{1}{w_{\mathfrak{f}}}\log\left|\frac{\theta\left(\frac{1}{q}, \tau\right)^N}{\theta\left(\frac{N}{q}, N\tau\right)} e^{i\pi(N-1)/2}\right|^2.$$

The number $e^{i\pi(N-1)/2}$ is a root of unity of order ≤ 4 and its modulus is then 1, thus we obtain the desired result. $\square$

We may illustrate this formula on the following example: set $\mathbb{K} = \mathbb{Q}(e^{2i\pi/3})$ and $\mathfrak{f} = 13\mathcal{O}_{\mathbb{K}} + (10 + e^{2i\pi/3})\mathcal{O}_{\mathbb{K}}$ a prime ideal of norm $q = 13$. Fix $\mathfrak{b} = \mathcal{O}_{\mathbb{K}}$ and $\mathfrak{a} = 7\mathcal{O}_{\mathbb{K}} + (3+e^{2i\pi/3})\mathcal{O}_{\mathbb{K}}$ a prime ideal of norm $N = 7$. Then $\mu = -3+e^{2i\pi/3}$ and $\nu = -26+13e^{2i\pi/3}$ are such that $\mathfrak{f}\mathfrak{b}^{-1} = \mathbb{Z}\mu + \mathbb{Z}\nu$ and $\mathfrak{f}(\mathfrak{a}\mathfrak{b}^{-1}) = \mathbb{Z}\mu + \mathbb{Z}(\nu/7)$ with $\nu/13 \equiv 1 \bmod \mathfrak{f}$. Setting $\tau = \mu/\nu = \frac{10+e^{2i\pi/3}}{91}$, we may check the Kronecker limit formula as:

$$13\zeta_{\mathfrak{f}}'(\mathfrak{b}, 0) - \zeta_{\mathfrak{f}}'(\mathfrak{a}\mathfrak{b}, 0) = -\log \left| \theta\left(\frac{7}{91}, \frac{10+e^{2i\pi/3}}{91}\right)^7 \theta\left(\frac{7}{13}, \frac{10+e^{2i\pi/3}}{13}\right)^{-1} \right|^2 \approx 0.8916656...$$

The evaluation of a smoothed θ function involved in this Kronecker limit formula also produces algebraic units:

Theorem 0.8 [Algebraicity of elliptic units] : *Let $\mathbb{K}$ be an imaginary quadratic field. Fix an integral ideal $\mathfrak{f} \neq \mathcal{O}_{\mathbb{K}}$. Set $q\mathbb{Z} = \mathfrak{f} \cap \mathbb{Z}$ for some positive integer q. Fix an integral ideal $\mathfrak{b}$ representing a class in the class group at $\mathfrak{f}$. Let $\mathfrak{a}$ be an integral ideal of prime norm N such that $(N, 6.q) = 1$. Let μ, ν be elements of $\mathbb{K}$ such that $\mathfrak{f}\mathfrak{b}^{-1} = \mathbb{Z}\mu + \mathbb{Z}\nu$ and $\mathfrak{f}(\mathfrak{a}\mathfrak{b})^{-1} = \mathbb{Z}\mu + \mathbb{Z}\nu/N$ with $\mu/\nu = \tau \in \mathbb{K} \cap \mathbb{H}$. Assume further that $\nu/q \equiv 1 \bmod \mathfrak{f}\mathfrak{b}^{-1}$. Then the complex number*

$$\frac{\theta\left(\frac{1}{q}, \tau\right)^N}{\theta\left(\frac{N}{q}, N\tau\right)}$$

is the image in $\mathbb{C}$ of an algebraic q-unit $u_{\mathfrak{f},\mathfrak{b},\mathfrak{a}}$ in the ray class field at $\mathfrak{f}$. In addition, the explicit reciprocity law is given by $\sigma_{\mathfrak{b}'}(u_{\mathfrak{f},\mathfrak{b},\mathfrak{a}}) = u_{\mathfrak{f},\mathfrak{b}\mathfrak{b}',\mathfrak{a}}$ where $\mathfrak{b}' \to \sigma_{\mathfrak{b}'}$ is Artin's map.

Proof :

This is essentially a smoothed version of [[Ram64], Theorem 5] (see also [[Rob73], Theorem 1]). $\square$

The example we have already used illustrates this theorem as:

$$u_1 = \theta\left(\frac{7}{91}, \frac{10+e^{2i\pi/3}}{91}\right)^7 \theta\left(\frac{7}{13}, \frac{10+e^{2i\pi/3}}{13}\right)^{-1} = \frac{-3-7\sqrt{-3}}{4} + \sqrt{\frac{-41+17\sqrt{-3}}{8}}$$

is a 13-unit inside the ray class field $\mathbb{K}^+(\mathfrak{f})$ above $\mathbb{K} = \mathbb{Q}(e^{2i\pi/3})$ of modulus $\mathfrak{f} = 13\mathcal{O}_{\mathbb{K}} + (10 + e^{2i\pi/3})\mathcal{O}_{\mathbb{K}}$.

In the rest of this work, we generalise the properties of the pair of functions (θ, P_2) to the pair of higher functions (G_{n-2}, B_n) for the special linear group $\mathrm{SL}_n(\mathbb{Z})$ and construct higher analogues of the elliptic units above number fields with exactly one complex place of degree $n \geq 3$. These higher elliptic units are expected to be algebraic units inside specific abelian extensions of the base field where they are evaluated and they are expected to satisfy a Kronecker limit formula in the flavour of Theorem 0.7 (see Conjecture 0.5).

Chapter I

Geometric families of multiple elliptic Gamma functions and cocycle properties

I.1 Introduction to chapter I

In this chapter we study the collection of multiple elliptic Gamma functions and the associated collection of Bernoulli rational functions. The first function in the hierarchy of multiple elliptic Gamma functions is the θ function defined on $\mathbb{C} \times \mathbb{H}$ by:

$$\theta(z, \tau) = \prod_{m \geq 0} (1 - e^{2i\pi(m+1)\tau} e^{-2i\pi z})(1 - e^{2i\pi m\tau} e^{2i\pi z}).$$

(see chapter 0 for the properties of the θ function). Recall in particular the modular property:

$$\theta\left(\frac{z}{\tau}, \frac{-1}{\tau}\right) = \theta(z, \tau) e^{\frac{2i\pi}{\tau} P_2(z,\tau)}$$

where the rational function $P_2(z, \tau)$ is given by:

$$P_2(z, \tau) = \frac{z^2 + z - z\tau}{2} - \frac{\tau}{4} + \frac{\tau^2 - 1}{12}.$$

More generally, to any matrix $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ we can associate a rational function $P_{2,\gamma}(z, \tau) \in \mathbb{Q}(z, \tau)$ such that:

$$\theta\left(\frac{z}{c\tau + d}, \frac{a\tau + b}{c\tau + d}\right) = \theta(z, \tau) e^{2i\pi P_{2,\gamma}(z,\tau)}.$$

This may be interpreted as a coboundary relation and shows that $\gamma \to P_{2,\gamma}$ is a 1-cocycle on $\mathrm{SL}_2(\mathbb{Z})$ with values in rational functions and that it is split by the θ function. In this chapter, we show that the collection of pairs of functions (G_{n-2}, B_n) behaves similarly to the pair $(\theta, P_{2,\gamma})$ under an action of a special linear group $\mathrm{SL}_n(\mathbb{Z})$ of higher dimension $n \geq 3$ on rank n lattices and give insight on how they might be used to compute invariants in number theory.

For $n = 3$, it was shown in the 2000s by Felder and Varchenko [FV00] that the elliptic Gamma function introduced by Ruijsenaars [Rui97] enjoyed modular transformation properties for $\mathrm{SL}_3(\mathbb{Z})$ similar to those of the θ function for $\mathrm{SL}_2(\mathbb{Z})$. We recall that the elliptic Gamma function is defined on $\mathbb{C} \times \mathbb{H} \times \mathbb{H}$ by:

$$\Gamma(z, \tau, \sigma) = \prod_{m,n \geq 0} \left(\frac{1 - \exp(2i\pi((m+1)\tau + (n+1)\sigma - z))}{1 - \exp(2i\pi(m\tau + n\sigma + z))} \right).$$

This function may then be associated with a 2-cocycle on $\mathrm{SL}_3(\mathbb{Z})$ with values in rational functions (see section I.2). In 2001, Nishizawa [Nis01] introduced a whole hierarchy of multiple elliptic Gamma functions which encompass both the θ function and the elliptic Γ function. Recall that they are multivariate analytic functions defined for all $r \in \mathbb{N}$ by:

$$G_r(z, \tau_0, \ldots, \tau_r) = \prod_{m_0,\ldots,m_r \geq 0} \left(1 - e^{2i\pi(-z + \sum_{j=0}^{r}(m_j+1)\tau_j)}\right) \left(1 - e^{2i\pi(z + \sum_{j=0}^{r} m_j \tau_j)}\right)^{(-1)^r} \tag{I.1}$$

This definition recovers both the θ and the elliptic Γ functions as $\theta = G_0$ and $\Gamma = G_1$. The G_r functions share similar transformation properties under an action of $\mathrm{SL}_{r+2}(\mathbb{Z})$ on the abstract lattice generated by $1, \tau_0, \ldots, \tau_r$ (see section I.2) and it is natural to study the entire collection obtained when varying $r \in \mathbb{N}$. In particular, when $r \geq 1$, the following pseudo-periodicity relation for the G_r function involves a lower degree G_{r-1} function as:

$$G_r(z + \tau_j, \tau_0, \ldots, \tau_r) = G_{r-1}(z, \tau_0, \ldots, \widehat{\tau_j}, \ldots, \tau_r)\, G_r(z, \tau_0, \ldots, \tau_r)$$

where as usual the notation $\widehat{\tau_j}$ indicates that the variable τ_j should be omited. In [FHRZ08] Felder, Henriques, Rossi and Zhu enriched the theory of the elliptic Gamma function by introducing geometric families of elliptic Gamma functions associated to arbitrary rank 3 lattices, offering a comprehensive perspective of the underlying geometric phenomena. Namely, if L is a rank 3 lattice and $a, b \in \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$ are two primitive linear forms on L, they define general functions $\Gamma_{a,b} : \mathbb{C} \times \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{C}) \to \mathbb{C}$ using Ruijsenaars' elliptic Γ function as a building block and show that the collection of functions obtained when varying a, b enjoys modular transformation properties under an action of $\mathrm{SL}_3(\mathbb{Z})$ on L. The two most important properties in that regard are the so-called modular and equivariance properties. For any linearly independent $a, b, c \in \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$, there is a rational function $P_{a,b,c} \in \mathbb{Q}[w](x)$ such that the equality:

$$\Gamma_{a,b}(w, x)\Gamma_{b,c}(w, x)\Gamma_{c,a}(w, x) = \exp(2i\pi P_{a,b,c}(w, x)) \tag{I.2}$$

holds for (w, x) in a dense open subset of $\mathbb{C} \times \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{C}) \simeq \mathbb{C} \times \mathbb{C}^3$. Additionally, both $\Gamma_{a,b}$ and $P_{a,b,c}$ are *equivariant* under an action of $\mathrm{SL}_3(\mathbb{Z})$, which means that for all $g \in \mathrm{SL}_3(\mathbb{Z})$:

$$\Gamma_{g\cdot a, g\cdot b}(w, g \cdot x) = \Gamma_{a,b}(w, x)$$
$$P_{g\cdot a, g\cdot b, g\cdot c}(w, g \cdot x) = P_{a,b,c}(w, x)$$

These equivariance properties play a key role in arithmetic applications, as the functions $\Gamma_{a,b}$ and $P_{a,b,c}$ will be evaluated on homology classes associated to specific tori in $\mathrm{SL}_3(\mathbb{Z})$.

The construction of the functions $\Gamma_{a,b}$ may already be viewed as the generalisation of a well-known construction for the θ function. For a rank 2 lattice L, and a linear form $a \in \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z}) = \Lambda$, we may also define an equivariant function $\theta_a : \mathbb{C} \times \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{C}) \to \mathbb{C}$ such that for any pair of linearly independent $a, b \in \Lambda$ there is an equivariant rational function $Q_{a,b} \in \mathbb{Q}[w](x)$ which may be expressed in terms of Dedekind sums and satisfying:

$$\theta_a(w, x)\theta_b(w, x)^{-1} = \exp(2i\pi Q_{a,b}(w, x)) \tag{I.3}$$

In the first part of this chapter, we upgrade the construction carried out in [FHRZ08] to higher degree G_r functions (see section I.2). For a lattice $\Lambda = \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$ of rank $n = r + 2 \geq 2$ we introduce a collection of geometric functions $G_{n-2,a_1,\dots,a_{n-1}}$ attached to families of $n-1$ linear forms $a_1, \dots, a_{n-1} \in \Lambda$ (see Proposition I.7) which are built using Nishizawa's G_{n-2} functions. For rank $n = 2, 3$ lattices, we recover the functions $\theta_a = G_{0,a}(0)$ and $\Gamma_{a,b} = G_{1,a,b}(0)$ respectively. In section I.2 we show that the collection of $G_{n-2,a_1,\dots,a_{n-1}}$ functions obtained when varying the linear forms $a_1, \dots, a_{n-1}$ satisfy modular and equivariance properties similar to those satisfied by the collections of θ_a and $\Gamma_{a,b}$ functions. Namely, our first main result is a general version of formulae (I.2) and (I.3) for the $G_{n-2,a_1,\dots,a_{n-1}}$ functions involving a collection of higher degree Bernoulli rational functions $B_{n,a_1,\dots,a_n}$ (see Definition I.8).

Theorem I.1: *Let L be an oriented lattice of rank $n \geq 2$. Let $a_1, \dots, a_n \in \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$ be a family of n linearly independent primitive linear forms on L. Fix $v \in V/L \simeq \mathbb{Q}^n/\mathbb{Z}^n$ where $V = L \otimes_{\mathbb{Z}} \mathbb{Q}$.*

1. *[Modular property] For (w, x) in a dense open subset of $\mathbb{C} \times \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{C}) \simeq \mathbb{C} \times \mathbb{C}^n$:*

$$\prod_{j=1}^{n} G_{n-2,a_1,\dots,\widehat{a_j},\dots,a_n}(v)(w,x)^{(-1)^{j+1}} = \exp(2i\pi B_{n,a_1,\dots,a_n}(v)(w,x)) \tag{I.4}$$

2. *[Equivariance relations] For any $g \in \mathrm{SL}_n(\mathbb{Z})$, the following equalities hold in the space of functions $V/L \times \mathbb{C} \times \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{C}) \to \mathbb{C}$:*

$$G_{n-2,g\cdot a_1,\dots,g\cdot a_{n-1}}(g\cdot v)(w, g\cdot x) = G_{n-2,a_1,\dots,a_{n-1}}(v)(w,x)$$
$$B_{n,g\cdot a_1,\dots,g\cdot a_n}(g\cdot v)(w, g\cdot x) = B_{n,a_1,\dots,a_n}(v)(w,x).$$

Theorem I.1 gives a more comprehensive perspective on the numerous properties of the G_{n-2} functions (see I.2.1.1). In particular, formula (I.4) may be understood as a collection of coboundary relations for $\mathrm{SL}_n(\mathbb{Z})$ as follows. For two sets A and B write $\mathcal{F}(A, B)$ for the set of functions on A with values in B. Fix a primitive linear form $a \in \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$ as a base point and define the two functions:

$$\psi_{n,a} := \begin{cases} \mathrm{SL}_n(\mathbb{Z})^{n-2} & \to \mathcal{F}(V/L \times \mathbb{C} \times \mathrm{Hom}_{\mathbb{Z}}(L,\mathbb{C}), \mathbb{C}) \\ (g_1, \dots, g_{n-2}) & \to \big((v,w,x) \to G_{n-2,a,g_1\cdot a,\dots,(g_1\dots g_{n-2})\cdot a}(v)(w,x)\big) \end{cases}$$

$$\phi_{n,a} := \begin{cases} \mathrm{SL}_n(\mathbb{Z})^{n-1} & \to \mathcal{F}(V/L, \mathbb{Q}[w](x)) \\ (g_1, \dots, g_{n-1}) & \to B_{n,a,g_1\cdot a,(g_1 g_2)\cdot a,\dots,(g_1\dots g_{n-1})\cdot a}(v)(w,x) \end{cases}$$

When $a, g_1 \cdot a, \dots, (g_1 \dots g_{n-1}) \cdot a$ are linearly independent, the first point of Theorem I.1 implies that these functions satisfy the multiplicative coboundary relation:

$$\partial^{\times}\psi_{n,a}(g_1, \dots, g_{n-1})(v)(w,x) = \exp(2i\pi\phi_{n,a}(g_1, \dots, g_{n-1})(v)(w,x))$$

for any $v \in V/L$ and for (w, x) in a dense open subset of $\mathbb{C} \times \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{C})$. In section I.4 and chapter III we shall derive arithmetic applications of *both* collections of functions $\psi = (\psi_{n,a})_{n,a}$ and $\phi = (\phi_{n,a})_{n,a}$ to the computation of class field invariants in number fields.

The second part of this chapter is devoted to the study of the collection of Bernoulli rational functions $B_{n,a_1,\dots,a_n}$ together with the attached collection of $(n-1)$-cocycles $\phi_{n,a}$, as the restriction of $\phi_{n,a}$ to specific tori will be used in arithmetic applications to compute partial zeta functions in totally real number fields of degree n at $s=0$. Shifting the focus on the $B_{n,a_1,\dots,a_n}$ functions we get as a consequence of Theorem I.1 the additive cocycle relation:

$$\sum_{j=0}^{n}(-1)^j B_{n,a_0,\dots,\widehat{a_j},\dots,a_n}(v)(w,x)\in\mathbb{Z}$$

for any family $a_0,\dots,a_n$ of primitive linear forms in general position, i.e. such that for any $0\le j\le n$, $\mathrm{rk}(a_0,\dots,\widehat{a_j},\dots,a_n)=n$. We improve this result in section I.3 by showing that the stronger additive cocycle relation:

$$\sum_{j=0}^{n}(-1)^j B_{n,a_0,\dots,\widehat{a_j},\dots,a_n}(v)(w,x)=0 \tag{I.5}$$

holds for almost all configurations of $a_0,\dots,a_n$ in the rank n lattice Λ (see the bad position condition (BP) in section I.3). To prove (I.5) we show that the rational functions $B_{n,a_1,\dots,a_n}$ are given by a coefficient extraction in the generating series associated to a rational polyhedral cone and that the cocycle relation they satisfy may be obtained as a specialisation of a cocycle relation satisfied by some indicator functions of closed polyhedral cones. For ease of presentation, we will consider cones in $\mathbb{Q}$-vector spaces but most results we prove on cones may be readily transposed to cones in $\mathbb{F}$-vector spaces where $\mathbb{F}$ is any ordered field, say $\mathbb{R}$ for instance. A polyhedral cone in a $\mathbb{Q}$-vector space V is a set

$$C=\mathbb{Q}_{\ge 0}v_1+\cdots+\mathbb{Q}_{\ge 0}v_p+\mathbb{Q}_{>0}v'_1+\cdots+\mathbb{Q}_{>0}v'_q$$

where the v_i's and the v'_j's are non-zero vectors in V. Let us denote by $\mathcal{K}(V)$ the $\mathbb{Q}$-algebra generated by the indicator functions of such cones and by $\mathcal{L}(V)$ the subspace of $\mathcal{K}(V)$ generated by the indicator functions of those cones containing some line $\mathbb{Q}v$. We prove a cocycle relation for specific cones which we now define. For non-zero linear forms $a_0,\dots,a_m\in V^\vee$ define:

$$c^\vee(a_0,\dots,a_m)(v):=\begin{cases}1 \text{ if } \forall\, 0\le j\le m, a_j(v)\ge 0\\ 0 \text{ otherwise}\end{cases}$$

In section I.3 we prove our second main result which is the technical heart of this chapter and might be of independent interest.

Theorem I.2: *Let V be a $\mathbb{Q}$-vector space of dimension $n\ge 1$ and let $a_0,\dots,a_n$ be $n+1$ non-zero linear forms on V which generate $V^\vee$. For all $0\le j\le n$, denote $\varepsilon_j=(-1)^j\mathrm{sign}\det(a_0,\dots,\widehat{a_j},\dots,a_n)\in\{-1,0,1\}$.*

(i) *If there are coefficients $\lambda_0>0,\dots,\lambda_n>0$ such that $\sum_{j=0}^n\lambda_j a_j=0$ then the signs ε_j are all equal to a common sign ε and*

$$\sum_{j=0}^{n}\varepsilon_j c^\vee(a_0,\dots,\widehat{a_j},\dots,a_n)\equiv\varepsilon\delta \mod \mathcal{L}(V)$$

where δ is the Dirac function at 0.

(ii) *If there is a relation* $\sum_{j=0}^{n} \lambda_j a_j = 0$ *with at least one positive and one negative coefficient among* $\lambda_0, \ldots, \lambda_n$*, then:*

$$\sum_{j=0}^{n} \varepsilon_j c^\vee(a_0, \ldots, \widehat{a_j}, \ldots, a_n) \equiv 0 \bmod \mathcal{L}(V).$$

This theorem may be viewed as a dual theorem to [[CDG15], Theorem 1.1] from which it is inspired. It is interesting to note that for $n = 2$ this theorem allows us to recover a result in a recent article by Sharifi and Venkatesh [SV24] (see section I.3.2). Our main goal however, is to deduce from this theorem the following corollary on the cocycle relation (I.5) satisfied by the Bernoulli rational functions (see Proposition I.14 for more details) for almost all configurations of the parameters.

Corollary I.2: *Let* $a_0, \ldots, a_n$ *be* $n+1$ *linear forms on* V*. Suppose that either* $\mathrm{rk}(a_0, \ldots, a_n) < n$*, or* $a_0, \ldots, a_n$ *generate* $V^\vee$ *and there is a relation* $\sum_{j=0}^{n} \lambda_j a_j = 0$ *with at least one positive and one negative coefficient among* $\lambda_0, \ldots, \lambda_n$*. Then:*

$$\sum_{j=0}^{n} (-1)^j B_{n,a_0,\ldots,\widehat{a_j},\ldots,a_n}(v)(w, x) = 0. \tag{I.6}$$

In applications the linear forms a_j will be taken of the form $g_j \cdot a$ for some base point $a \in V^\vee$ and the invertible matrix g_j in some subgroup U of $\mathrm{SL}_n(\mathbb{Z})$. At the end of section I.3 we give examples of specific subgroups U of $\mathrm{SL}_n(\mathbb{Z})$ for which any family $g_0, \ldots, g_n \in U$ is such that the family $g_0 \cdot a, \ldots, g_n \cdot a$ satisfies the hypothesis of Corollary I.1 for any base point $a \in V^\vee$. The collection of functions $(\phi_{n,a})_a$ thus reduces to a collection of homogeneous $(n-1)$-cocycles for U. In the arithmetic applications we have in mind to the computation of partial zeta values in totally real number fields at $s = 0$, the group U will arise from the group of totally positive units of a given totally real number field. Following [Shi76], [Col88], [DyDF14] and [CDG15] we will prove that the partial zeta values in totally real number fields may be expressed as combinations of values of the geometric Bernoulli functions $B_{n,a_1,\ldots,a_n}$ which appear in the study of the collection of $G_{n-2,a_1,\ldots,a_{n-1}}$ functions:

Theorem I.3: *Let* $\mathbb{F}$ *be a totally real number field of degree* n*. Let* $\mathfrak{f}$ *be an integral ideal in* $\mathcal{O}_\mathbb{F}$ *and fix an integral ideal* $\mathfrak{b}$ *representing a class in the narrow ray class group* $\mathrm{Cl}^+(\mathfrak{f})$*. Then there are explicitly computable cones* $c_\rho = c^\vee(a_{1,\rho}, \ldots, a_{n,\rho})$ *parametrised by* $\rho \in \mathfrak{S}_{n-1}$ *and signs* $\nu_\rho \in \{-1, 0, +1\}$ *such that:*

$$\zeta_\mathfrak{f}(\mathfrak{b}, 0) = \frac{1}{n} \sum_{k=1}^{n} \sum_{\rho \in \mathfrak{S}_{n-1}} \nu_\rho B_{n,a_{1,\rho},\ldots,a_{n,\rho}}(1_\mathbb{F})(0, -\sigma_k)$$

where $\sigma_1, \ldots, \sigma_n$ *are the real embeddings of* $\mathbb{F}$ *and the* $a_{i,\rho}$*'s are* $\mathbb{Q}$*-linear forms on the* n*-dimensional* $\mathbb{Q}$*-vector space* $\mathbb{F}$*.*

We remark that Theorem I.3 expresses partial zeta values at $s = 0$ in a totally real number field $\mathbb{F}$ as traces of algebraic numbers in $\mathbb{F}$. These values are therefore rational numbers, as was already known from the theorem of Klingen and Siegel, and already obtained by Shintani using his method. In section I.4 we will give two explicit examples of such computations for real cubic fields.

This chapter is organised as follows. In section I.2 we define both collections of $G_{n-2,a_1,\ldots,a_{n-1}}$ and $B_{n,a_1,\ldots,a_n}$ functions and prove Theorem I.1. In section I.3 we study a cocycle relation for indicator functions of cones and prove Theorem I.2 which is the technical heart of this chapter. We then deduce Corollary I.1 via a coefficient extraction in the generating series attached to cones and show that the $\phi_{n,a}$ functions form a collection of $(n-1)$-cocycles when restricted to specific tori in $\mathrm{SL}_n(\mathbb{Z})$ arising from unit groups in number fields. In section I.4 we show that the Bernoulli rational functions $B_{n,a_1,\ldots,a_n}$ may be used to express the values of partial zeta functions at $s=0$ in totally real number fields and give the proof of Theorem I.3.

I.2 Geometric families of multiple elliptic Gamma functions

In this section, we recall the properties of the multiple elliptic Gamma functions (the G_r functions) and construct geometric families $G_{r,a_1,\ldots,a_{r+1}}$ of these functions attached to a family of $r+1$ linear forms $a_1,\ldots,a_{r+1}$, upgrading the G_r functions to collections of equivariant functions for $\mathrm{SL}_{r+2}(\mathbb{Z})$ by adapting the construction of the $\Gamma_{a,b}$ functions in [FHRZ08] to higher degrees.

I.2.1 The G_r functions

I.2.1.1 The definition of the G_r functions

We review the definition and properties of the G_r functions inspired by one of Jacobi's θ functions and the elliptic Γ function of Ruijsenaars. The θ function is defined for $z\in\mathbb{C},\tau\in\mathbb{H}$ by:

$$\theta(z,\tau)=\prod_{m\geq 1}(1-e^{2i\pi(m+1)\tau}e^{-2i\pi z})(1-e^{2i\pi m\tau}e^{2i\pi z})$$

It is an infinite product which is absolutely convergent and holomorphic over $\mathbb{C}\times\mathbb{H}$ enjoying transformation properties under the action of $\mathrm{SL}_2(\mathbb{Z})$ on the upper half-plane. Indeed, for $z\in\mathbb{C},\tau\in\mathbb{H}$:

$$\begin{aligned}\theta(z+1,\tau)&=\theta(z,\tau)=\theta(z,\tau+1)\\ \theta(z+\tau,\tau)&=-e^{-2i\pi z}\theta(z,\tau)\\ \theta\left(\frac{z}{\tau},\frac{-1}{\tau}\right)&=\theta(z,\tau)e^{\frac{2i\pi}{\tau}P_2(z,\tau)}\end{aligned}$$

where the polynomial P_2 is explicitly given by:

$$P_2(z,\tau)=\frac{z^2+z-z\tau}{2}-\frac{\tau}{4}+\frac{\tau^2-1}{12}.$$

The elliptic Γ function which was introduced by Ruijsenaars [Rui97] is defined by:

$$\Gamma(z,\tau,\sigma)=\prod_{m,n\geq 0}\left(\frac{1-\exp(2i\pi((m+1)\tau+(n+1)\sigma-z))}{1-\exp(2i\pi(m\tau+n\sigma+z))}\right)\tag{I.7}$$

As pointed out by Spiridonov (see [Spi04]) this function was already studied by Jackson in 1905 and implicitly studied in theoretical physics under other names since the 70s. The infinite product (I.7) is absolutely convergent and holomorphic in both $\tau \in \mathbb{H}$ and $\sigma \in \mathbb{H}$ and it is meromorphic in $z \in \mathbb{C}$ with poles at points in $\mathbb{Z} + \mathbb{Z}_{\leq 0}\tau + \mathbb{Z}_{\leq 0}\sigma$. In [[FV00], Theorems 3.1 and 4.1] Felder and Varchenko proved that the elliptic Γ function enjoys properties under an action of $\mathrm{SL}_3(\mathbb{Z})$ involving the θ function. In particular, for any z, τ, σ in the domain of convergence of $\Gamma(z, \tau, \sigma)$:

$$\begin{aligned}\Gamma(z,\tau,\sigma) &= \Gamma(z,\sigma,\tau)\\ \Gamma(z+1,\tau,\sigma) = \Gamma(z,\tau+1,\sigma) &= \Gamma(z,\tau,\sigma+1) = \Gamma(z,\tau,\sigma)\\ \Gamma(z+\tau,\tau,\sigma) &= \theta(z,\sigma)\Gamma(z,\tau,\sigma)\\ \Gamma(z+\tau+\sigma,\tau,\sigma) &= \Gamma(-z,\tau,\sigma)^{-1}\end{aligned}$$

Finally, if $\sigma/\tau \notin \mathbb{R}$ then:

$$\Gamma(z,\tau,\sigma)^{-1}\Gamma\left(\frac{z}{\tau},\frac{-1}{\tau},\frac{\sigma}{\tau}\right)\Gamma\left(\frac{z-\tau}{\sigma},-\frac{\tau}{\sigma},-\frac{1}{\sigma}\right)^{-1} = \exp(2i\pi P_3(z,\tau,\sigma)) \qquad \text{(I.8)}$$

where

$$\begin{aligned}P_3(z,\tau,\sigma) = \frac{z^3}{6\tau\sigma} - \frac{\tau+\sigma-1}{4\tau\sigma}z^2 + \frac{\tau^2+\sigma^2+3\tau\sigma-3\tau-3\sigma+1}{12\tau\sigma}z\\ +\frac{1}{24}(\tau+\sigma-1)\left(\frac{1}{\tau}+\frac{1}{\sigma}-1\right)\end{aligned}$$

The G_r functions introduced by Nishizawa [Nis01] using q-polylogarithms generalise both the θ and elliptic Γ functions and are defined for $z \in \mathbb{C}$ and parameters $\tau_0, \ldots, \tau_r \in \mathbb{H}$ by:

$$G_r(z,\underline{\tau}) = \prod_{m_0,\ldots,m_r\geq 0}\left(1-e^{2i\pi(-z+\sum_{j=0}^r(m_j+1)\tau_j)}\right)\left(1-e^{2i\pi(z+\sum_{j=0}^r m_j\tau_j)}\right)^{(-1)^r}$$

where $\underline{\tau} = (\tau_0, \ldots, \tau_r)$. This infinite product is absolutely convergent and holomorphic in each parameter $\tau_j \in \mathbb{H}$ and it is either holomorphic in $z \in \mathbb{C}$ if r is even or meromorphic in $z \in \mathbb{C}$ with poles at points of $\mathbb{Z} + \sum_{j=0}^r \mathbb{Z}_{\leq 0}\tau_j$ if r is odd. For $r = 0, 1$ we recover the definition of the θ and Γ functions so that $G_0 = \theta$ and $G_1 = \Gamma$. The range of the parameters τ_j can be extended from $\mathbb{H}$ to $\mathbb{C} - \mathbb{R}$ by using nicer expressions of the G_r functions as the exponentials of infinite sums involving sinuses and cosines (see [[FV00], formula (15)] and [[Nis01], Proposition 3.6]), namely:

$$G_r(z,\underline{\tau}) = \begin{cases}\exp\left(\sum_{j\geq 1}\frac{1}{(2i)^r j}\frac{\sin(\pi j(2z-(\tau_0+\cdots+\tau_r)))}{\prod_{k=0}^r\sin(\pi j\tau_k)}\right) & \text{if } r \text{ is odd}\\ \exp\left(\sum_{j\geq 1}\frac{2}{(2i)^{r+1} j}\frac{\cos(\pi j(2z-(\tau_0+\cdots+\tau_r)))}{\prod_{k=0}^r\sin(\pi j\tau_k)}\right) & \text{if } r \text{ is even}\end{cases} \qquad \text{(I.9)}$$

This expression is valid provided that $\underline{\tau} \in (\mathbb{C} - \mathbb{R})^{r+1}$ and $|\Im(2z - (\tau_0 + \cdots + \tau_r))| < \sum_{j=0}^r |\Im(\tau_j)|$ and allows us to extend the range of parameters to $\underline{\tau} \in (\mathbb{C} - \mathbb{R})^{r+1}$ by putting:

$$G_r(z,\tau_0,\ldots,\tau_{j-1},-\tau_j,\tau_{j+1},\ldots,\tau_r) = G_r(z+\tau_j,\tau_0,\ldots,\tau_{j-1},\tau_j,\tau_{j+1},\ldots,\tau_r)^{-1} \qquad \text{(I.10)}$$

The G_r functions satisfy relations similar to that of the θ and elliptic Γ functions, as proven by Nishizawa in [Nis01] and later by Narukawa in [Nar04]. First, the G_r functions are 1-periodic in each of their arguments and they satisfy:

$$\begin{aligned} G_r(z+\tau_0+\cdots+\tau_r,\underline{\tau}) &= G_r(-z,\underline{\tau})^{(-1)^r} \\ G_r(-z,-\underline{\tau}) &= G_r(z,\underline{\tau})^{-1} \end{aligned} \tag{I.11}$$

Furthermore, if $r \geq 1$, the G_r functions are almost periodic in z with periods τ_j for $0 \leq j \leq r$ with a correction factor involving a lower degree function:

$$G_r(z+\tau_j,\underline{\tau}) = G_{r-1}(z,\tau_0,\ldots,\tau_{j-1},\widehat{\tau_j},\tau_{j+1},\ldots,\tau_r)\, G_r(z,\underline{\tau})$$

where the notation $\widehat{\tau_j}$ indicates that the variable τ_j should be omitted. We now recall the modular property for the G_r functions.

I.2.1.2 Modular property and Bernoulli polynomials

The modular property for the G_r functions was later proved by Narukawa [Nar04]. To state their theorem, we first need to introduce the multiple Bernoulli polynomials which were implicitly used in the study of the Barnes' multiple Γ function [Bar04]. We adopt the following conventions regarding Bernoulli numbers:

$$\frac{t}{e^t-1} = \sum_{k\geq 0} B_k \frac{t^k}{k!} \tag{I.12}$$

Consider an integer $n \geq 1$. Let $\underline{\omega} = (\omega_1,\ldots,\omega_n) \in (\mathbb{C}-\{0\})^n$. We define the multiple Bernoulli polynomials $B^*_{n,m}(z,\underline{\omega})$ with the following generating function:

$$e^{zt}\prod_{j=1}^{n}\frac{\omega_j t}{e^{\omega_j t}-1} = \sum_{m\geq 0} B^*_{n,m}(z,\underline{\omega})\frac{t^m}{m!} \tag{I.13}$$

These polynomials may be expressed explicitly using Bernoulli numbers as:

$$\frac{1}{m!}B^*_{n,m}(z,\underline{\omega}) = \sum_{l=0}^{m}\frac{z^l}{l!}\left(\sum_{\substack{k_1+\cdots+k_n=m-l\\ k_j\geq 0}}\left(\prod_{1\leq j\leq n}\frac{B_{k_j}\omega_j^{k_j}}{k_j!}\right)\right)$$

For $\underline{\omega} \in (\mathbb{C}-\{0\})^n$, $B^*_{n,m}(z,\underline{\omega})$ is a degree m homogeneous polynomial in $n+1$ variables with rational coefficients, which is symmetric in the n variables of $\underline{\omega}$. In [Nar04], Narukawa used the rescaled homogeneous rational functions $B_{n,m}(z,\underline{\omega}) = (\prod_{j=1}^{n}\omega_j^{-1})B^*_{n,m}(z,\underline{\omega})$. These obey many relations which can easily be obtained from the properties of the generating function (see [Nar04]). We will be most interested by the diagonal polynomials $B^*_{n,n}$. For instance, the polynomial $B^*_{2,2}$ is given by:

$$(\omega_1\omega_2)B_{2,2}(z,\omega_1,\omega_2) = B^*_{2,2}(z,\omega_1,\omega_2) = z^2 - z(\omega_1+\omega_2) + \frac{\omega_1^2+\omega_2^2+3\omega_1\omega_2}{6}.$$

Narukawa's theorem (see [[Nar04], Theorem 7]) can be stated as follows. Let $n \geq 2$. Fix $\underline{\omega} \in (\mathbb{C}-\{0\})^n$ and suppose that $\omega_j/\omega_k \in \mathbb{C}-\mathbb{R}$ for all $1 \leq j \neq k \leq n$. Then for $z \in \mathbb{C}$ outside the discrete set of poles of the left-hand side:

$$\prod_{j=1}^{n} G_{n-2}\left(\frac{z}{\omega_j},\left(\frac{\omega_k}{\omega_j}\right)_{k\neq j}\right) = \exp\left(\frac{-2i\pi}{n!}B_{n,n}(z,\underline{\omega})\right)\,. \tag{I.14}$$

I.2.1.3 Distribution relations

The arithmetic applications we have in mind for the G_r functions are inspired by the construction of Siegel units using the $\theta = G_0$ function and the construction of associated cocycles (see for instance [DPV24]). To extend the analogy, we end this section by proving that the G_r functions satisfy distribution relations similar to the distribution relations satisfied by Siegel units.

Proposition I.4: *Consider an integer $N \geq 2$. Then the following distribution relations hold:*

$$\prod_{k=0}^{N-1} G_r\left(z + \frac{k}{N}, \underline{\tau}\right) = G_r(Nz, N\underline{\tau})$$

$$\prod_{k=0}^{N-1} G_r\left(z + \frac{k}{N}\tau_l, \tau_0, \ldots, \tau_l, \ldots, \tau_r\right) = G_r\left(z, \tau_0, \ldots, \frac{\tau_l}{N}, \ldots, \tau_r\right)$$

These two relations together give the complete distribution relation:

$$\prod_{k,k_0,\ldots,k_r=0}^{N-1} G_r\left(z + \frac{k + k_0\tau_0 + \cdots + k_r\tau_r}{N}, \tau_0, \ldots, \tau_r\right) = G_r(Nz, \tau_0, \ldots, \tau_r)$$

Proof :
The first relation follows from the standard cyclotomic relation:

$$\prod_{k=0}^{N-1}(1 - e^{2i\pi k/N}y) = 1 - y^N$$

applied here to both $y_0 = -z + \sum_{j=0}^{r}(m_j + 1)\tau_j$ and $y_1 = z + \sum_{j=0}^{r} m_j\tau_j$ for all $m_0, \ldots, m_r \geq 0$.

The second relation is obtained by straightforward computation using the set identities:

$$\mathbb{Z}_{>0} - \{0, 1/N, \ldots, (N-1)/N\} = \frac{1}{N}\mathbb{Z}_{>0},$$
$$\mathbb{Z}_{\geq 0} + \{0, 1/N, \ldots, (N-1)/N\} = \frac{1}{N}\mathbb{Z}_{\geq 0}.$$

□

I.2.2 Geometric $G_{r,\underline{a}}$ functions

In this section we upgrade the construction carried out by Felder, Henriques, Rossi and Zhu to higher degree and define geometric families $G_{r,a_1,\ldots,a_{r+1}}$ parametrised by $r+1$ linear forms of the G_r functions that encompass the geometric families $\Gamma_{a,b}$ of the elliptic Gamma function. We then prove Theorem I.1, i.e. the modular property for the $G_{r,a_1,\ldots,a_{r+1}}$ functions and their equivariance property under the action of $\mathrm{SL}_{r+2}(\mathbb{Z})$. The construction of our $G_{r,a_1,\ldots,a_{r+1}}$ functions is adapted from the construction of the elliptic $\Gamma_{a,b}$ functions as we use a generalised version of the alternative definition given by [[FHRZ08], Proposition 3.5] and then revert the computations to prove that the definition is indeed valid.

We start by giving a precise geometric setup. Consider a $\mathbb{Q}$-vector space V of dimension n and a rank n lattice $L \subset V$. Denote $\Lambda := \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z}) \subset V^\vee = \mathrm{Hom}_{\mathbb{Q}}(V, \mathbb{Q})$. The lattice L is then canonically isomorphic to $\mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z})$. Fix a $\mathbb{Z}$-basis $B = [e_1, \ldots, e_n]$ of L and denote by $B_\Lambda = [f_1, \ldots, f_n]$ the dual $\mathbb{Z}$-basis of Λ such that for all $1 \leq j, k \leq n$, $f_j(e_k) = \delta_{jk}$ where δ_{jk} is Kronecker's symbol. This fixes orientation forms on L and Λ given by $\det_B$ and $\det_{B_\Lambda}$ respectively. When there is no risk of confusion we drop the subscripts and write as usual det for the orientation forms. This also fixes an action of $\mathrm{SL}_n(\mathbb{Z})$ on L by left multiplication, i.e. $g \cdot \alpha = g\alpha$ and the contragredient action of $\mathrm{SL}_n(\mathbb{Z})$ on Λ given by $g \cdot a = ag^{-1}$. In particular, the pairing:

$$\begin{cases} \Lambda \times L & \to \mathbb{Z} \\ (a, \alpha) & \to a(\alpha) \end{cases}$$

is equivariant under the action of $\mathrm{SL}_n(\mathbb{Z})$, which means that for any $g \in \mathrm{SL}_n(\mathbb{Z})$ and any $(a, \alpha) \in \Lambda \times L$, $(g \cdot a)(g \cdot \alpha) = a(\alpha)$. Note that the basis C is also a basis of the $\mathbb{C}$-vector space $\mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{C}) \simeq \mathbb{C}^n$ which induces an action of $\mathrm{SL}_n(\mathbb{Z})$ on $\mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{C})$ extending the action of $\mathrm{SL}_n(\mathbb{Z})$ on Λ. We will now consider families of linearly independent primitive linear forms $a_1, \ldots, a_{n-1}$ in Λ and define functions $G_{n-2, a_1, \ldots, a_{n-1}}$ attached to theses families. Recall that an element w in a $\mathbb{Z}$-module Λ is called primitive if for all $(n, w') \in \mathbb{Z}_{\geq 1} \times \Lambda$, $w = nw' \Rightarrow n = 1$. In order to define the functions $G_{n-2, a_1, \ldots, a_{n-1}}$ properly, we now define positive dual families which will be used in the rest of this work.

Definition I.5: *Let $(a_1, \ldots, a_m)$ be a family of m linearly independent elements in a lattice $\Lambda \simeq \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$. We call $(\alpha_1, \ldots, \alpha_m) \in L$ a positive dual family to $\underline{a} = (a_1, \ldots, a_m)$ if for all $1 \leq j \leq m$ the following holds:*

$$a_j(\alpha_j) > 0, \qquad a_k(\alpha_j) = 0, \forall\, k \neq j$$

The two important cases in this work will be those where $m = n - 1$ and $m = n$ in the lattice Λ of rank n. The following lemma shows that in these cases two positive dual families to the same family $\underline{a}$ are closely related.

Lemma I.6: *Let Λ be a lattice of rank n with an orientation form* det.

(i) *Let $\underline{a} = (a_1, \ldots, a_{n-1})$ be a family of $n-1$ linearly independent elements in Λ. If $(\alpha_1, \ldots, \alpha_{n-1})$ and $(\alpha'_1, \ldots, \alpha'_{n-1})$ are two positive dual families to $\underline{a}$ in $L = \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z})$ then there are rational numbers $t_1, \ldots, t_{n-1}$ such that:*

$$a_j(\alpha'_j)\alpha_j = a_j(\alpha_j)\alpha'_j + t_j \det(a_1, \ldots, a_{n-1}, \cdot)$$

(ii) *A family $\underline{a} = (a_1, \ldots, a_n)$ of n linearly independent elements in Λ has exactly one positive dual family $\underline{\alpha} \in L = \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z})$ containing only primitive vectors. We call this family the primitive positive dual family to $\underline{a}$. Any other positive dual family $\beta_1, \ldots, \beta_n$ to $\underline{a}$ satisfies for all $1 \leq j \leq n, \beta_j = m_j \alpha_j$ for some integer $m_j > 0$.*

Proof :

(i) For any $1 \leq j \leq n-1$, set $\gamma_j = a_j(\alpha'_j)\alpha_j - a_j(\alpha_j)\alpha'_j$. Then for all $1 \leq k \leq n-1$, we get $\gamma_j(a_k) = 0$. This means that either $\gamma_j = 0$ and then $t_j = 0$ or the $\mathbb{Q}$-linear forms γ_j and $\det(a_1, \ldots, a_{n-1}, \cdot)$ defined on the $\mathbb{Q}$-vector space $V^\vee$ share the same kernel, and therefore, they must be linearly dependent. This proves the first claim.

(ii) Suppose that $\underline{\alpha}$, $\underline{\alpha}'$ are two positive dual families to $a_1, \ldots, a_n$ in L consisting of primitive vectors. Consider once again $\gamma_j = a_j(\alpha'_j)\alpha_j - a_j(\alpha_j)\alpha'_j$ such that for all $1 \leq k \leq n$, $\gamma_j(a_k) = 0$. The family $(a_1, \ldots, a_n)$ is a basis of the $\mathbb{Q}$-vector space $V^\vee$, so $\gamma_j = 0$ for all $1 \leq j \leq n$. This means that $a_j(\alpha'_j)\alpha_j = a_j(\alpha_j)\alpha'_j$ for all $1 \leq j \leq n$. There are only two opposite primitive vectors in the line $\mathbb{Q}\alpha_j$ so if both α_j and α'_j are primitive it must be that $\alpha_j = \alpha'_j$ because both lie in the half-plane $\{v \in V \mid a_j(v) > 0\}$. Thus, if such a family exists, it is unique. Regarding the existence, we may write $a_j = \sum_{k=1}^{n} a_{j,k} f_k$ and define the matrix:

$$A = \begin{pmatrix} a_{1,1} & \ldots & a_{1,n} \\ \vdots & \vdots & \vdots \\ a_{n,1} & \ldots & a_{n,n} \end{pmatrix}$$

If ϵ is the sign of $\det(A)$ then the matrix $B = \epsilon.\mathrm{com}(A)^T$ defines a positive dual family to $\underline{a}$ which we use to compute the primitive positive dual family to $\underline{a}$. Define $\Delta_{i,j} = (-1)^{(i+j)}\det(a_{u,v})_{u \neq i, v \neq j}$ and $\delta_j = \gcd(\Delta_{1,j}, \ldots, \Delta_{n,j})$. Then the primitive positive dual family $\underline{\alpha}$ to $\underline{a}$ is given explicitly by:

$$\alpha_k = \sum_{j=1}^{n} \frac{\epsilon \Delta_{k,j}}{\delta_j} e_k$$

If $\beta_1, \ldots, \beta_n$ is any other positive dual family to $\underline{a}$ then there are unique integers $m_1, \ldots, m_n > 0$ such that $\beta_1/m_1, \ldots, \beta_n/m_n$ are primitive vectors in L. The family $\beta_1/m_1, \ldots, \beta_n/m_n$ is a primitive positive dual family to $\underline{a}$ in L, therefore it must be equal to $\alpha_1, \ldots, \alpha_n$. This shows that $\beta_j = \alpha_j m_j$ for all $1 \leq j \leq n$. □

We may now define the geometric variants of the G_r functions using lemma I.6 and adapting [[FHRZ08], Proposition 3.5] to higher rank lattices:

Proposition I.7: *Let $n \geq 2$ and set $r = n-2$. Let $\underline{a} = (a_1, \ldots, a_{r+1})$ be a family of $r+1$ linearly independent primitive vectors in the oriented lattice Λ of rank $n = r+2$. There is a unique integer $s > 0$ and a unique primitive element $\gamma \in L = \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z})$ such that* $\det(a_1, \ldots, a_{r+1}, \cdot) = s\gamma$. *Fix a vector $v \in V/L$. For any choice of positive dual family $\underline{\alpha} = (\alpha_1, \ldots, \alpha_{r+1})$ to $\underline{a}$ in L the function defined by the finite product:*

$$G_{r,\underline{a}}^{\underline{\alpha}}(v)(w,x) := \prod_{\delta \in F(\underline{a},\underline{\alpha},v)/\mathbb{Z}\gamma} G_r\left(\frac{w + x(\delta)}{x(\gamma)}, \frac{1}{x(\gamma)} x(\underline{\alpha})\right) \tag{I.15}$$

where

$$F(\underline{a}, \underline{\alpha}, v) = \{\delta \in v + L \mid \forall\, 1 \leq j \leq r+1,\ 0 \leq a_j(\delta) < a_j(\alpha_j)\} \tag{I.16}$$

is well defined for (w, x) in a dense open set of the $\mathbb{C}$-vector space $\mathbb{C} \times \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{C}) \simeq \mathbb{C} \times \mathbb{C}^{r+2}$. Furthermore, it is independent of the choice of $\underline{\alpha}$.

Proof :

Consider the following open subset of $\mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{C})$:

$$U(\underline{a}, \underline{\alpha}) = \{x \in \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{C}) \mid x(\gamma) \neq 0,\ x(\alpha_j)/x(\gamma) \notin \mathbb{R}, \forall\, 1 \leq j \leq r+1\}$$

The set $U(\underline{a},\underline{\alpha})$ is the complementary set of a finite union of $\mathbb{R}$-vector spaces of dimension $\leq 2r+3$ in the $\mathbb{R}$-vector space $\mathrm{Hom}_{\mathbb{Z}}(L,\mathbb{C}) \simeq \mathbb{R}^{2r+4}$, therefore it is a dense open subset of $\mathrm{Hom}_{\mathbb{Z}}(L,\mathbb{C})$ endowed with the finite dimensional $\mathbb{R}$-vector space topology. For any $x \in U(\underline{a},\underline{\alpha})$, define:

$$S(\underline{a},\underline{\alpha},v,x) = \bigcap_{\delta \in F(\underline{a},\underline{\alpha},v)} \left\{ w \in \mathbb{C} \mid {}^{w+x(\delta)\notin \mathbb{Z}x(\gamma)+\sum_{j=1}^{r+1}\mathbb{Z}_{>0}x(\alpha_j)}_{w+x(\delta)\notin \mathbb{Z}x(\gamma)+\sum_{j=1}^{r+1}\mathbb{Z}_{\leq 0}x(\alpha_j)} \right\}$$

Then the right-hand side of (I.15) is well-defined and non-zero on:

$$\Omega(\underline{a},\underline{\alpha},v) = \bigcup_{x \in U(\underline{a},\underline{\alpha})} \{(w,x) \mid w \in S(\underline{a},\underline{\alpha},v,x)\} \tag{I.17}$$

which is a dense open subset of $\mathbb{C} \times \mathrm{Hom}_{\mathbb{Z}}(L,\mathbb{C})$ endowed with the finite dimensional $\mathbb{C}$-vector space topology.

Let us now prove that the right-hand side of (I.15) is indeed independent of the choice of positive dual family $\underline{\alpha}$. To achieve this, we will show that for any other choice of positive dual family $\underline{\alpha}'$ to $\underline{a}$, the functions $G^{\underline{\alpha}}_{r,\underline{a}}$ and $G^{\underline{\alpha}'}_{r,\underline{a}}$ coincide. Let us first compute the right-hand side of (I.15) explicitly using the definition of the G_r function. Consider $(w,x) \in \Omega(\underline{a},\underline{\alpha},v)$. Put for $1 \leq j \leq r+1$, $d_j = \pm 1$ such that $d_j x(\alpha_j)/x(\gamma) \in \mathbb{H}$. Put also $D = \sum_{j=1}^{r+1}(d_j-1)/2$. Then using the inversion relation (I.10) we get:

$$G^{\underline{\alpha}}_{r,\underline{a}}(v)(w,x)^{(-1)^D} = \prod_{\delta \in F(\underline{a},\underline{\alpha},v)/\mathbb{Z}\gamma} G_r\left(\frac{w+x(\delta)}{x(\gamma)} + \sum_{j=1}^{r+1}\frac{d_j-1}{2}\frac{x(\alpha_j)}{x(\gamma)}, \left(\frac{1}{x(\gamma)}d_j x(\alpha_j)\right)_{1\leq j\leq r+1}\right)$$

which by definition of the ordinary G_r functions is given by:

$$G^{\underline{\alpha}}_{r,\underline{a}}(v)(w,x)^{(-1)^D} = \prod_{\delta \in F(\underline{a},\underline{\alpha},v)/\mathbb{Z}\gamma} \prod_{\underline{m}\geq 0} \Bigg[\left(1 - e^{2i\pi\left(\sum_{j=1}^{r+1}\frac{(d_j m_j + (1+d_j)/2)x(\alpha_j)}{x(\gamma)} - \frac{w+x(\delta)}{x(\gamma)}\right)}\right) \times \left(1 - e^{2i\pi\left(\sum_{j=1}^{r+1}\frac{(d_j m_j + (d_j-1)/2)x(\alpha_j)}{x(\gamma)} + \frac{w+x(\delta)}{x(\gamma)}\right)}\right)^{(-1)^r} \Bigg]$$

Let us denote by $C^+(\underline{a},\underline{\alpha},v,x)$ the set of $\delta' \in v+L$ satisfying for all $1 \leq j \leq r+1$:

$$\begin{cases} a_j(\delta') \geq 0, & \text{if } d_j = 1 \\ a_j(\delta') < 0, & \text{if } d_j = -1 \end{cases}$$

and similarly denote by $C^-(\underline{a},\underline{\alpha},v,x)$ the set of $\delta' \in v+L$ satisfying for all $1 \leq j \leq r+1$:

$$\begin{cases} a_j(\delta') \geq 0, & \text{if } d_j = -1 \\ a_j(\delta') < 0, & \text{if } d_j = 1. \end{cases}$$

Consider $\delta' \in C^+(\underline{a},\underline{\alpha},v,x)$. If $d_j = 1$ then $a_j(\delta') \geq 0$ so that performing Euclidian division by $a_j(\alpha_j)$ gives a unique integer $m_j \geq 0$ satisfying $0 \leq a_j(\delta' - m_j\alpha_j) < a_j(\alpha_j)$.

On the contrary, if $d_j = -1$ then $a_j(\delta') < 0$ and there exists a unique integer $m_j > 0$ such that $0 \le a_j(\delta' + m_j\alpha_j) < a_j(\alpha_j)$. Then for all $1 \le k \le r+1$:

$$0 \le a_k \left(\delta' - \sum_{j=1}^{r+1} d_j m_j \alpha_j \right) < a_k(\alpha_k)$$

and thus $\delta' - \sum_{j=1}^{r+1} d_j m_j \alpha_j \in F(\underline{a}, \underline{\alpha}, v)$. This shows that the cone $C^+(\underline{a}, \underline{\alpha}, v, x)$ can be written as a disjoint union:

$$C^+(\underline{a}, \underline{\alpha}, v, x) = \bigcup_{\delta \in F(\underline{a}, \underline{\alpha}, v)} \bigcup_{\underline{m} \ge 0} \left\{ \delta + \sum_{j=1}^{r+1} (d_j m_j + (d_j - 1)/2)\alpha_j \right\}.$$

A similar argument applied to $C^-(\underline{a}, \underline{\alpha}, v, x)$ gives the decomposition:

$$C^-(\underline{a}, \underline{\alpha}, v, x) = \bigcup_{\delta \in F(\underline{a}, \underline{\alpha}, v)} \bigcup_{\underline{m} \ge 0} \left\{ \delta - \sum_{j=1}^{r+1} (d_j m_j + (d_j + 1)/2)\alpha_j \right\}.$$

Thus, the expression $G^{\underline{\alpha}}_{r,\underline{a}}(v)(w,x)^{(-1)^D}$ is equal to:

$$\prod_{\delta' \in C^-(\underline{a}, \underline{\alpha}, v, x)/\mathbb{Z}\gamma} \left(1 - e^{-2i\pi \left(\frac{w + x(\delta')}{x(\gamma)} \right)} \right) \prod_{\delta' \in C^+(\underline{a}, \underline{\alpha}, v, x)/\mathbb{Z}\gamma} \left(1 - e^{2i\pi \left(\frac{w + x(\delta')}{x(\gamma)} \right)} \right)^{(-1)^r}$$

Then, we only need to show that the sets $C^{\pm}(\underline{a}, \underline{\alpha}, v, x)$ are independent of the choice for $\underline{\alpha}$. Consider another positive dual family $\underline{\alpha}'$ to $\underline{a}$. Write $a_j(\alpha_j) = s_j > 0$ and $a_j(\alpha'_j) = s'_j > 0$. Then from lemma I.6 there is a rational number t_j such that:

$$s_j \alpha'_j = s'_j \alpha_j + t_j \gamma$$

which gives

$$d_j s_j \frac{x(\alpha'_j)}{x(\gamma)} = d_j s'_j \frac{x(\alpha_j)}{x(\gamma)} + d_j t_j \in \mathbb{H}$$

This shows that the signs d_j (and therefore also D) are independent of the choice for $\underline{\alpha}$. By construction we get $C^{\pm}(\underline{a}, \underline{\alpha}, v, x) = C^{\pm}(\underline{a}, \underline{\alpha}', v, x)$ and the definition of the geometric $G^{\underline{\alpha}}_{r,\underline{a}}$ function is independent of the choice for $\underline{\alpha}$. Notice that the sets $U(\underline{a}) = U(\underline{a}, \underline{\alpha}), S(\underline{a}, v, x) = S(\underline{a}, \underline{\alpha}, v, x), \Omega(\underline{a}, v) = \Omega(\underline{a}, \underline{\alpha}, v)$ are all independent of the choice for $\underline{\alpha}$. □

From now on we denote by $G_{r,\underline{a}} := G^{\underline{\alpha}}_{r,\underline{a}}$ the geometric G_r function associated to $\underline{a}$ for any suitable choice of $\underline{\alpha}$. By convention, when $a_1, \ldots, a_{r+1}$ are not linearly independent, we define $G_{r,\underline{a}}$ to be the constant function equal to 1. When $r = 0$ and $r = 1$ we recover the geometric variants of the θ and elliptic Γ function as:

$$\theta_a(w, x) = G_{0,a}(0)(w, x), \quad \Gamma_{a,b}(w, x) = G_{1,a,b}(0)(w, x).$$

Thus, we will often write $\theta_a(v)(w,x)$ for $G_{0,a}(v)(w,x)$ as well as $\Gamma_{a,b}(v)(w,x)$ for $G_{1,a,b}(v)(w,x)$. We remark that Proposition I.7 gives a definition of our $G_{r,a_1,\ldots,a_{r+1}}$ functions as G_r functions for specific cones in a rank $r+2$ lattice. This construction may be compared to Winding's construction of G_r functions attached to cones in a rank $r+1$ lattice [Win18].

We argue that we may think of our $G_{r,a_1,\dots,a_{r+1}}$ functions as objects in projective geometry whereas Winding's construction belongs to affine geometry and both should be related in some sense.

To express the transformation properties of the function $G_{r,\underline{a}}$ under the action of $\mathrm{SL}_{r+2}(\mathbb{Z})$ we need to introduce the family of geometric Bernoulli rational functions which encompass both $Q_{a,b}$ and $P_{a,b,c}$ appearing in formulae (I.3) and (I.2) respectively. Recall that we have defined the polynomials $B^*_{n,n}$ in section I.2.1.2.

Definition I.8: *Let $a_1,\dots,a_n$ be a family of n linearly independent primitive vectors in the oriented lattice Λ of rank n and $\alpha_1,\dots,\alpha_n$ be its primitive positive dual family in L (see lemma I.6). Let ϵ be the sign of $(-1)^n\det(a_1,\dots,a_n)$. We define the geometric Bernoulli polynomial attached to $a_1,\dots,a_n$ and to $v\in V/L$ on $\mathbb{C}\times\mathrm{Hom}_{\mathbb{Z}}(L,\mathbb{C})\simeq\mathbb{C}\times\mathbb{C}^n$ by the finite sum:*

$$B^*_{n,a_1,\dots,a_n}(v)(w,x) := \frac{\epsilon}{n!}\sum_{\delta\in F(\underline{a},v)} B^*_{n,n}(w+x(\delta),x(\alpha_1),\dots,x(\alpha_n))$$

where

$$F(\underline{a},v)=\{\delta\in v+L, 0\le a_j(\delta)<a_j(\alpha_j),\forall\, 1\le j\le n\}$$

is a finite set. The geometric Bernoulli polynomial $B^*_{n,a_1,\dots,a_n}(v)$ is a degree n homogeneous polynomial in $n+1$ variables on $\mathbb{C}\times\mathrm{Hom}_{\mathbb{Z}}(L,\mathbb{C})\simeq\mathbb{C}\times\mathbb{C}^n$, with rational coefficients depending on $a_1,\dots,a_n$ and v.

Here we give a simple example and recover the Bernoulli polynomials $B^*_{n,n}$ from section I.2.1.2. Suppose that the a_j are the vectors of the basis C, which means in coordinates that $\forall\, 1\le j\le n$, $a_j=(0,\dots,0,1,0,\dots,0)$ with a 1 at the j-th position. The corresponding primitive positive dual family is given by the vectors of the basis B, which in coordinates gives $\alpha_j=(0,\dots,0,1,0,\dots,0)^T$ with a 1 at the j-th position. Suppose further $v=0$, so that $F=F(\underline{a},0)=\{0\}$. Denote by τ_j the value of x on α_j. Then:

$$B^*_{n,a_1,\dots,a_n}(0)(w,x) := \frac{(-1)^n}{n!}B^*_{n,n}(w,\tau_1,\dots,\tau_n).$$

The modularity property for the geometric $G_{n-2,\underline{a}}$ functions will involve the rescaled degree 0 homogeneous rational functions

$$B_{n,a_1,\dots,a_n}(v)(w,x)=\left(\prod_{j=1}^n x(\alpha_j)\right)^{-1} B^*_{n,a_1,\dots,a_n}(v)(w,x). \tag{I.18}$$

If $a_1,\dots,a_n$ are linearly dependent, we set by convention $B_{n,\underline{a}}=0$. For $n=2$ and $n=3$ the rational functions $B_{2,a,b}(0)$ and $B_{3,a,b,c}(0)$ may be identified with the rational functions $Q_{a,b}$ and $P_{a,b,c}$ appearing in formulae (I.3) and (I.2) respectively and we may now generalise these formulae by proving Theorem I.1.

Proof of Theorem I.1:

Let $a_1,\dots,a_n$ be a family of n linearly independent elements in the rank n lattice Λ and let $\alpha_1,\dots,\alpha_n$ be the primitive positive dual family to $a_1,\dots,a_n$ in $L=\mathrm{Hom}_{\mathbb{Z}}(\Lambda,\mathbb{Z})$. Fix $v\in V/L$.

1. We first show that the modular property for the geometric $G_{n-2,\underline{a}}$ functions is a consequence of the modular property (I.14) for the ordinary G_{n-2} function. Write ϵ for

the sign of $(-1)^n\det(a_1,\dots,a_n)$. Then for all $1\leq j\leq n$ there is a positive integer s_j such that:

$$\det(a_1,\dots,\widehat{a_j},\dots,a_n,\cdot)=s_j.\epsilon.(-1)^j\alpha_j$$

Then, by definition, for any $1\leq j\leq n$:

$$G_{n-2,((a_k)_{k\neq j})}(v)(w,x)=\prod_{\delta\in F_j/\mathbb{Z}\alpha_j}G_{n-2}\left(\frac{w+x(\delta)}{\epsilon(-1)^jx(\alpha_j)},\left(\frac{x(\alpha_k)}{\epsilon(-1)^jx(\alpha_j)}\right)_{k\neq j}\right)$$

where

$$F_j=\{\delta\in v+L,0\leq a_k(\delta)<a_k(\alpha_k),\forall\,1\leq k\neq j\leq n\}$$

and using the inversion relation (I.11) we get:

$$G_{n-2,((a_k)_{k\neq j})}(v)(w,x)=\prod_{\delta\in F_j/\mathbb{Z}\alpha_j}G_{n-2}\left(\frac{w+x(\delta)}{x(\alpha_j)},\left(\frac{x(\alpha_k)}{x(\alpha_j)}\right)_{k\neq j}\right)^{(-1)^j.\epsilon}.$$

This equality holds in the dense open subset $\Omega_j=\Omega(a_1,\dots,\widehat{a_j},\dots,a_n;v)$ of $\mathbb{C}\times\mathrm{Hom}_{\mathbb{Z}}(L,\mathbb{C})$ (see formula (I.17)). Consequently, the following equality holds on the dense open subset $\Omega=\cap_{j=1}^n\Omega_j$ of $\mathbb{C}\times\mathrm{Hom}_{\mathbb{Z}}(L,\mathbb{C})$ for the whole product:

$$\prod_{j=1}^nG_{r,((a_k)_{k\neq j})}(v)(w,x)^{(-1)^{j+1}}=\prod_{j=1}^n\prod_{\delta\in F_j/\mathbb{Z}\alpha_j}G_{n-2}\left(\frac{w+x(\delta)}{x(\alpha_j)},\left(\frac{x(\alpha_k)}{x(\alpha_j)}\right)_{k\neq j}\right)^{-\epsilon}.$$

Put $F=\{\delta\in v+L,0\leq a_k(\delta)<a_k(\alpha_k),\forall\,1\leq k\leq n\}$. Then we may write uniformly $F\simeq F_j/\mathbb{Z}\alpha_j$ for all $1\leq j\leq n$. Using Narukawa's theorem (see (I.14)) for each δ in the finite set F yields:

$$\prod_{j=1}^nG_{n-2,((a_k)_{k\neq j})}(v)(w,x)^{(-1)^{j+1}}=\prod_{\delta\in F}\exp\left(\frac{2i\pi\epsilon}{n!}B_{n,n}(w+x(\delta),x(\underline{\alpha}))\right).$$

The identification of the right-hand side of the formula above with the definition of the rational function $B_{n,a_1,\dots,a_n}(v)(w,x)$ (see Definition I.8) gives the conclusion:

$$\prod_{j=1}^nG_{n-2,((a_k)_{k\neq j})}(v)(w,x)^{(-1)^{j+1}}=\exp(2i\pi B_{n,a_1,\dots,a_n}(v)(w,x)).$$

2. Consider $g\in\mathrm{SL}_n(\mathbb{Z})$. Remember that the actions of $\mathrm{SL}_n(\mathbb{Z})$ on $\mathrm{Hom}_{\mathbb{Z}}(L,\mathbb{C})$ and L satisfy $(g\cdot a)(g\cdot\alpha)=a(\alpha)$ for any $(a,\alpha)\in\mathrm{Hom}_{\mathbb{Z}}(L,\mathbb{C})\times L$. The action on L further extends to $V=L\otimes\mathbb{Q}$ and passes to the quotient V/L. Thus, in the construction of $G_{n-2,a_1,\dots,a_{n-1}}$ or $B_{n,a_1,\dots,a_n}$ replacing $\underline{a}$ with $g\cdot\underline{a}$ and v with $g\cdot v$ replaces $\underline{\alpha}$ with $g\cdot\underline{\alpha}$, γ with $g\cdot\gamma$ and $F=F(\underline{a},\underline{\alpha},v)$ with $F_g=F(g\cdot\underline{a},g\cdot\underline{\alpha},g\cdot v)=g\cdot F$ while ϵ is left unchanged. Therefore we may write:

$$G_{n-2,g\cdot\underline{a}}(g\cdot v)(w,g\cdot x)=\prod_{\delta\in F_g/\mathbb{Z}g\cdot\gamma}G_{n-2}\left(\frac{w+(g\cdot x)(\delta)}{(g\cdot x)(g\cdot\gamma)},\frac{1}{(g\cdot x)(g\cdot\gamma)}(g\cdot x)(g\cdot\underline{\alpha})\right).$$

Then, identifying $F_g=g\cdot F$ and putting $\delta'=g\cdot\delta$ gives

$$G_{n-2,g\cdot\underline{a}}(g\cdot v)(w,g\cdot x)=\prod_{\delta'\in F/\mathbb{Z}\gamma}G_{n-2}\left(\frac{w+x(\delta')}{x(\gamma)},\frac{1}{x(\gamma)}x(\underline{\alpha})\right)$$

which gives the conclusion:

$$G_{n-2,g\cdot\underline{a}}(g\cdot v)(w,g\cdot x)=G_{n-2,\underline{a}}(v)(w,x)$$

As for $B_{n,g\cdot a_1,\dots,g\cdot a_n}(g\cdot v)$, the set $F=F(\underline{a},\underline{\alpha},v)$ is once again replaced by $g\cdot F$ so that:

$$B_{n,g\cdot a_1,\dots,g\cdot a_n}(g\cdot v)(w,g\cdot x)=\frac{\epsilon}{n!}\sum_{\delta\in g\cdot F}B_{n,n}(w+(g\cdot x)(\delta),(g\cdot x)(g\cdot\underline{\alpha})).$$

Put once again $\delta'=g\cdot\delta$, which gives:

$$B_{n,g\cdot a_1,\dots,g\cdot a_n}(g\cdot v)(w,g\cdot x)=\frac{\epsilon}{n!}\sum_{\delta'\in F}B_{n,n}(w+x(\delta'),x(\underline{\alpha})).$$

Identify the right-hand side to conclude that:

$$B_{n,g\cdot a_1,\dots,g\cdot a_n}(g\cdot v)(w,g\cdot x)=B_{n,a_1,\dots,a_n}(v)(w,x).$$

□

Remark: The set $\mathcal{F}(V/L\times\mathbb{C}\times\mathrm{Hom}_{\mathbb{Z}}(L,\mathbb{C}),\mathbb{C})$ is naturally endowed with an action of $\mathrm{SL}_n(\mathbb{Z})$ given by $(g\cdot f)(v)(w,x)=f(g^{-1}\cdot v)(w,g^{-1}\cdot x)$. The second part of Theorem I.1 may then be restated as:

$$\begin{aligned}g\cdot G_{n-2,a_1,\dots,a_{n-1}}&=G_{n-2,g\cdot a_1,\dots,g\cdot a_{n-1}}\\ g\cdot B_{n,a_1,\dots,a_n}&=B_{n,g\cdot a_1,\dots,g\cdot a_n}\end{aligned}$$

We also add that both functions $G_{n-2,\underline{a}}$ and $B_{n,\underline{a}}$ behave nicely under permutation of vectors, namely for any permutation $\sigma\in\mathfrak{S}_{n-1}$, $G_{n-2,\sigma(\underline{a})}=G_{n-2,\underline{a}}^{\mathrm{sgn}(\sigma)}$ where $\mathrm{sgn}(\sigma)$ is the signature of the permutation σ and for any permutation $\sigma\in\mathfrak{S}_n$, $B_{n,\sigma(\underline{a})}=\mathrm{sgn}(\sigma)B_{n,\underline{a}}$. The modular property (I.4) may be restated as a partial coboundary relation between two collections of functions. Indeed, fixing a non zero primitive linear form $a\in\Lambda$ as a base point we may define:

$$\psi_{n,a}:=\begin{cases}\mathrm{SL}_n(\mathbb{Z})^{n-2} & \to\mathcal{F}(V/L\times\mathbb{C}\times\mathrm{Hom}_{\mathbb{Z}}(L,\mathbb{C}),\mathbb{C})\\ (g_1,\dots,g_{n-2}) & \to\big((v,w,x)\to G_{n-2,a,g_1\cdot a,\dots,(g_1\dots g_{n-2})\cdot a}(v)(w,x)\big)\end{cases}$$

$$\phi_{n,a}:=\begin{cases}\mathrm{SL}_n(\mathbb{Z})^{n-1} & \to\mathcal{F}(V/L,\mathbb{Q}[w](x))\\ (g_1,\dots,g_{n-1}) & \to B_{n,a,g_1\cdot a,(g_1g_2)\cdot a,\dots,(g_1\dots g_{n-1})\cdot a}(v)(w,x)\end{cases}$$

When the linear forms $a,g_1\cdot a,(g_1g_2)\cdot a,\dots,(g_1\dots g_{n-1})\cdot a$ are linearly independent, we may rewrite formula $(I.4)$ as a relation between the multiplicative coboundary of $\psi_{n,a}$ and $\phi_{n,a}$ on a dense open subset of $\mathbb{C}\times\mathrm{Hom}_{\mathbb{Z}}(L,\mathbb{C})$:

$$\partial^{\times}\psi_{n,a}(g_1,\dots,g_{n-1})(v)(w,x)=\exp(2i\pi\phi_{n,a}(g_1,\dots,g_{n-1})(v)(w,x))\qquad\text{(I.19)}$$

We end this section by proving that the geometric families of $G_{r,\underline{a}}$ functions satisfy distribution relations which generalise Proposition I.4.

Proposition I.9: *Consider an integer $N\geq 2$. Then the following distribution relations hold:*

$$\prod_{Nv'\equiv v\mod L}G_{r,\underline{a}}(v')(w,x)=G_{r,\underline{a}}(v)(Nw,x).\qquad\text{(I.20)}$$

Proof :
We will rewrite the left-hand side of (I.20) in order to use the results from Proposition I.4. Indeed:

$$\prod_{Nv'\equiv v \bmod L} G_{r,\underline{a}}(v')(w,x) = \prod_{\beta\in L/NL} G_{r,\underline{a}}\left(\frac{v+\beta}{N}\right)(w,x).$$

Using the definition of $G_{r,\underline{a}}$ we get:

$$\prod_{Nv'\equiv v \bmod L} G_{r,\underline{a}}(v')(w,x) = \prod_{\beta\in L/NL}\ \prod_{\delta'\in F(\underline{a},\underline{\alpha},(v+\beta)/N)/\mathbb{Z}\gamma} G_r\left(\frac{w+x(\delta')}{x(\gamma)}, \frac{1}{x(\gamma)}x(\underline{\alpha})\right).$$

Define $F_N = \sqcup_{\beta\in L/NL} F(\underline{a},\underline{\alpha},(v+\beta)/N)$. Using the definition of $F(\underline{a},\underline{\alpha},\cdot)$ it is clear that:

$$F_N = \sqcup_{\beta\in L/NL}\left\{\delta' \in \frac{v+\beta}{N} + L \mid \forall 1\le j\le r+1,\ 0\le a_j(\delta) < a_j(\alpha_j)\right\}$$
$$F_N = \left\{\delta' \in \frac{v+L}{N} \mid \forall 1\le j\le r+1,\ 0\le a_j(\delta) < a_j(\alpha_j)\right\}.$$

Consider $\delta'\in F_N$. Then $N\delta'$ belongs to

$$\{\delta\in v+L \mid \forall 1\le j\le r+1,\ 0\le a_j(\delta) < Na_j(\alpha_j)\}$$

and there are unique integers $0\le k_1,\dots,k_{r+1} < N$ such that

$$N\delta' - \sum_{j=1}^{r+1} k_j\alpha_j \in \{\delta\in v+L \mid \forall 1\le j\le r+1,\ 0\le a_j(\delta) < a_j(\alpha_j)\} = F(\underline{a},\underline{\alpha},v).$$

Thus we get a bijection:

$$f := \begin{cases} F_N/\mathbb{Z}\gamma & \to \{0,1,\dots,N-1\}^{r+2}\times F(\underline{a},\underline{\alpha},v)\\ \delta' & \to ((k,k_1,\dots,k_{r+1}),\delta)\end{cases}$$

defined by $N\delta' \equiv k\gamma + \sum_{j=1}^{r+1} k_j\alpha_j + \delta \bmod \mathbb{Z}\gamma$. It follows that:

$$\prod_{Nv'\equiv v \bmod L} G_{r,\underline{a}}(v')(w,x) = \prod_{\delta'\in F_N/\mathbb{Z}\gamma} G_r\left(\frac{w+x(\delta')}{x(\gamma)}, \frac{1}{x(\gamma)}x(\underline{\alpha})\right)$$
$$\prod_{Nv'\equiv v \bmod L} G_{r,\underline{a}}(v')(w,x) = \prod_{\delta\in F(\underline{a},\underline{\alpha},v)/\mathbb{Z}\gamma}\ \prod_{k,k_1,\dots,k_{r+1}=0}^{N-1} G_r\left(\frac{w+x\left(\frac{\delta+k\gamma+k_1\alpha_1+\dots+k_{r+1}\alpha_{r+1}}{N}\right)}{x(\gamma)}, \frac{1}{x(\gamma)}x(\underline{\alpha})\right)$$

Using the third relation from Proposition I.4 we get:

$$\prod_{Nv'\equiv v \bmod L} G_{r,\underline{a}}(v')(w,x) = \prod_{\delta\in F(\underline{a},\underline{\alpha},v)/\mathbb{Z}\gamma} G_r\left(\frac{Nw+x(\delta)}{x(\gamma)}, \frac{1}{x(\gamma)}x(\underline{\alpha})\right)$$
$$\prod_{Nv'\equiv v \bmod L} G_{r,\underline{a}}(v')(w,x) = G_{r,\underline{a}}(v)(Nw,x)$$

which is the desired result. □

In chapter III, we use the $G_{r,\underline{a}}$ functions to construct conjectural elliptic units above number fields with exactly one complex place which should behave as the Siegel units, and these distribution relations already show some of these similarities.

For the rest of this chapter we shift our focus from the $G_{r,\underline{a}}$ functions to the collection of Bernoulli rational functions $B_{n,a_1,\dots,a_n}$ and the associated collection of $(n-1)$-cocycles $\phi_{n,a}$. In particular, we will show that formula (I.19) holds under less restrictive conditions on the g_i's and that $\phi_{n,a}$ truly becomes a cocycle on specific subgroups of $\mathrm{SL}_n(\mathbb{Z})$.

I.3 Cocycle properties for the collection of $B_{n,\underline{a}}$ functions

The goal of this section is to show that the additive cocycle relation:

$$\sum_{j=0}^{n}(-1)^j B_{n,a_0,\dots,\widehat{a_j},\dots,a_n}(v)(w,x) \in \mathbb{Z} \tag{I.21}$$

which holds for linear forms $a_0,\dots,a_n$ in general position in a rank n lattice Λ as a consequence of Theorem I.1 may be improved to a finer cocycle relation:

$$\sum_{j=0}^{n}(-1)^j B_{n,a_0,\dots,\widehat{a_j},\dots,a_n}(v)(w,x) = 0 \tag{I.22}$$

which holds for a wider range of configurations of $a_0,\dots,a_n$ inside the rank n lattice Λ. This will be achieved in Proposition I.14 as we show that this relation may be obtained as the specialisation of a cocycle relation for indicator functions of closed cones.

I.3.1 A cocycle relation for closed cones

We now introduce several notations for cones in a $\mathbb{Q}$-vector space V of finite dimension n. These notations as well as the strategies for the proofs by induction on the dimension are inspired by [Hil07], [CDG15], and are typical of the theory of polyhedral cones. We focus on $\mathbb{Q}$-vector spaces for ease of presentation, but we wish to highlight that these results would hold for a vector space over $\mathbb{R}$ or over any ordered field. For a more complete presentation on the theory of convex cones we refer to [Bar02].

Let V be a $\mathbb{Q}$-vector space of dimension n. A convex cone in V is any convex set C satisfying $\forall x,y \in C, x+y \in C$ and $\forall \lambda > 0, \lambda.x \in C$. A convex cone C is said to be polyhedral if there are two (possibly empty) sets of vectors $v_1,\dots,v_p$ and $v'_1,\dots,v'_q$ in V such that $C = \mathbb{Q}_{\geq 0}v_1 + \dots + \mathbb{Q}_{\geq 0}v_p + \mathbb{Q}_{>0}v'_1 + \dots \mathbb{Q}_{>0}v'_q$. The vectors v_i and v'_j are called generators of the cone C. Note that by convention $\{0\}$ is a polyhedral cone with empty set of generators. Let $\mathcal{K}(V)$ be the $\mathbb{Q}$-algebra of $\mathbb{Q}$-valued functions on V generated by the indicator functions of polyhedral cones. Denote by $\mathcal{L}(V)$ the subspace of $\mathcal{K}(V)$ generated by the indicator functions of the closed polyhedral cones containing a line $\mathbb{Q}v$ for some non zero vector $v \in V$. Note that $\mathcal{L}(V)$ is not stable under multiplication. Most statements in the theory of cones may be proved by induction on the dimension and rest on the fact that when V' is a subspace of V there is a natural inclusion:

$$\begin{cases} \mathcal{K}(V') & \to \mathcal{K}(V) \\ f & \to \tilde{f} : v \to \begin{cases} f(v) & \text{if } v \in V' \\ 0 & \text{otherwise} \end{cases} \end{cases} \tag{I.23}$$

and this inclusion sends $\mathcal{L}(V')$ to $\mathcal{L}(V)$.

We now focus on closed cones. For any $v_1, \ldots, v_m \in V$, we denote by $c(v_1, \ldots, v_m)$ the indicator function of the closed polyhedral cone $\mathbb{Q}_{\geq 0}v_1 + \cdots + \mathbb{Q}_{\geq 0}v_m$. In this article, we also use a dual representation for the closed polyhedral cones. Namely, for linear forms $a_1, \ldots, a_m \in V^\vee$ the set

$$\cap_{i=1}^m \{v \in V \mid a_i(v) \geq 0\}$$

is a closed polyhedral cone in V and we denote by $c^\vee(a_1, \ldots, a_m)$ its indicator function. Note that by convention we may put $c(\emptyset) = \delta$ where δ is the Dirac function at 0 whereas $c^\vee(\emptyset) = 1$ is the indicator function of V. This dual representation of a cone already allows us to shed a different light on the notion of positive dual family (see Definition I.5). Indeed, fix a basis $B = [e_1, \ldots, e_n]$ of V and denote by L the $\mathbb{Z}$-lattice $\oplus_{j=1}^n \mathbb{Z}e_j$. It is clear that generators of cones may be rescaled, so that any polyhedral cone in V admits a set of generators which lie inside L. It is also true that any closed polyhedral cone in V admits a dual representation with linear forms $a_1, \ldots, a_m \in \Lambda = \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$ for some integer $m \geq 0$. In the specific case where $m = n$ and $a_1, \ldots, a_n \in \Lambda$ are linearly independent, the primitive positive dual family $\alpha_1, \ldots, \alpha_n \in L$ to $a_1, \ldots, a_n$ satisfies $c^\vee(a_1, \ldots, a_n) = c(\alpha_1, \ldots, \alpha_n)$. Lastly, we say that a family $(v_1, \ldots, v_m) \in V^m$ is in general position in V if any of its subfamilies of size at most n is free.

In [[CDG15], §1] Charollois, Dasgupta and Greenberg describe a cocycle relation for indicator functions of polyhedral cones using previous work by Hill [Hil07] on open polyhedral cones. Using the Solomon-Hu pairing (see [HS01]) they construct Shintani $(n-1)$-cocycles for $\mathrm{SL}_n(\mathbb{Z})$ with values in some spaces of rational functions which are cohomologous to Sczech cocycles [Scz93]. Our goal is to describe another cocycle relation for indicator functions $c^\vee(a_1, \ldots, a_m)$ of closed polyhedral cones in a dual setting and show how formula (I.22) may be deduced via the Solomon-Hu pairing. We start by stating some basic properties which will be very useful.

Lemma I.10: *Let V be a $\mathbb{Q}$-vector space of dimension n and set $V^\vee$.*

(i) *If $a_1, \ldots, a_m \in V^\vee$ do not generate $V^\vee$ (in particular if $m < n$) then $c^\vee(a_1, \ldots, a_m) \in \mathcal{L}(V)$.*

(ii) *For any $a_1, \ldots, a_m \in V^\vee$, $c^\vee(a_1, \ldots, a_m) = \prod_{j=1}^m c^\vee(a_j)$.*

(iii) *If $a, a' \in V^\vee$ satisfy $a = \lambda a'$ for some $\lambda > 0$ then $c^\vee(a) = c^\vee(a')$.*

(iv) *For any $a \in V^\vee$, $c^\vee(a, -a)$ is the indicator function of $\ker(a)$ and $c^\vee(a, -a) + 1 = c^\vee(a) + c^\vee(-a)$.*

(v) *If $a_1, \ldots, a_m \in V^\vee$ generate $V^\vee$ and if there are positive coefficients $\lambda_1, \ldots, \lambda_m$ such that $\sum_{j=1}^m \lambda_j a_j = 0$, then $c^\vee(a_1, \ldots, a_m) = \delta$ is the Dirac function at 0.*

(vi) *If $a_1, \ldots, a_{m+1} \in V^\vee$ satisfy $\sum_{j=1}^{m+1} \lambda_j a_j = 0$ with $\lambda_{m+1} < 0$ and $\lambda_j \geq 0$ for $1 \leq j \leq m$ then $c^\vee(a_1, \ldots, a_{m+1}) = c^\vee(a_1, \ldots, a_m)$.*

Proof :

(i) If $a_1, \dots, a_m$ do not generate $V^\vee$ then $\cap_{j=1}^m \ker(a_j) \neq \{0\}$. For any $v \in \cap_{j=1}^m \ker(a_j) - \{0\}$, and any $\lambda \in \mathbb{Q}$, it is clear that $c^\vee(a_1, \dots, a_m)(\lambda.v) = 1$. Therefore the cone described by $c^\vee(a_1, \dots, a_m)$ contains the line $\mathbb{Q}v$ and $c^\vee(a_1, \dots, a_m) \in \mathcal{L}(V)$.

(ii) The cone described by $c^\vee(a_1, \dots, a_m)$ is naturally defined as the intersection of the cones described by $c^\vee(a_1), \dots, c^\vee(a_m)$, therefore $c^\vee(a_1, \dots, a_m) = \prod_{j=1}^m c^\vee(a_j)$.

(iii) If $a = \lambda a'$ with $\lambda > 0$ then for any $v \in V$, $a(v) \geq 0 \Leftrightarrow a'(v) \geq 0$, which gives $c^\vee(a) = c^\vee(a')$.

(iv) For $a \in V^\vee$ and $v \in V$, $c^\vee(a, -a)(v) = 1$ if and only if $a(v) \geq 0$ and $-a(v) \geq 0$ i.e. if and only if $a(v) = 0$. The equality $c^\vee(a, -a)(v) + 1 = c^\vee(a)(v) + c^\vee(-a)(v)$ is easily computed in all three cases $a(v) > 0$, $a(v) = 0$ and $a(v) < 0$.

(v) Suppose that $a_1, \dots, a_m$ generate $V^\vee$ and that there are coefficients $\lambda_j > 0$ such that $\sum_{j=1}^m \lambda_j a_j = 0$. Suppose that there is a non-zero vector $v \in V$ such that $c^\vee(a_1, \dots, a_m) = 1$. Then $a_j(v) \geq 0$ for $1 \leq j \leq m$. As $a_1, \dots, a_m$ generate $V^\vee$ and $v \neq 0$, there is an index $1 \leq l \leq m$ such that $a_l(v) > 0$. The sum $\sum_{j=1}^m \lambda_j a_j(v)$ is equal to 0 and contains only non-negative terms, therefore all terms must be zero, which contradicts $\lambda_l a_l(v) > 0$. Then we only check that $c^\vee(a_1, \dots, a_m)(0) = 1$ and conclude that $c^\vee(a_1, \dots, a_m) = \delta$ is the Dirac function at 0.

(vi) Suppose $\sum_{j=1}^{m+1} \lambda_j a_j = 0$ with $\lambda_j \geq 0$ for $1 \leq j \leq m$ and $\lambda_{m+1} < 0$. Then $a_1(v) \geq 0, \dots, a_m(v) \geq 0 \Rightarrow a_{m+1}(v) \geq 0$ which gives

$$\cap_{i=1}^m \{v \in V \mid a_i(v) \geq 0\} = \cap_{i=1}^{m+1} \{v \in V \mid a_i(v) \geq 0\}$$

and therefore $c^\vee(a_1, \dots, a_{m+1}) = c^\vee(a_1, \dots, a_m)$. □

We now give a crucial definition which will be used in the proof of Theorem I.2.

Definition I.11: *Let V be a $\mathbb{Q}$-vector space and let $a_0, \dots, a_m$ be $m+1$ non-zero linear forms on V such that $\mathrm{rk}(a_0, \dots, a_m) = m$. There is a unique linear combination $\sum_{j=0}^m \lambda_j a_j = 0$ with coefficients $\lambda_j \in \mathbb{Q}$ satisfying:*

- $(\lambda_0, \dots, \lambda_m) \neq (0, \dots, 0)$
- $\#\{0 \leq j \leq m \mid \lambda_j < 0\} \leq \#\{0 \leq j \leq m \mid \lambda_j > 0\}$
- *if $\#\{0 \leq j \leq m \mid \lambda_j < 0\} = \#\{0 \leq j \leq m \mid \lambda_j > 0\}$, the first non-zero coefficient λ_l is negative*
- *the first non-zero coefficient λ_l has absolute value* 1.

This linear combination will be refered to as the standard non-trivial relation among $a_0, \dots, a_m$. In this situation, we define $k^-(a_0, \dots, a_m) = \#\{0 \leq j \leq m \mid \lambda_j < 0\}$.

In particular, when V has finite dimension n, Definition I.11 applies to any family $a_0, \dots, a_n \in V^\vee$ which generates $V^\vee$. In this situation, we may also define $k^0(a_0, \dots, a_m)$ (resp. $k^+(a_0, \dots, a_m)$) the number of coefficients $\lambda_j = 0$ (resp. $\lambda_j > 0$) in the relation, but these won't be much needed. Before giving the proof of Theorem I.2, we discuss the configurations of linear forms to which it applies in the light of Definition I.11. Indeed, we have carefully avoided some configurations of the linear forms a_j which we will refer to as "bad position" or (BP) for short. Namely, in the vector space $V^\vee$ of dimension n, a set of non-zero linear forms $a_0, \dots, a_n$ is in bad position (BP) if:

$$\operatorname{rk}(a_0, \dots, a_n) = n \text{ and } k^-(a_0, \dots, a_n) = 0 \text{ and } k^0(a_0, \dots, a_n) > 0. \tag{BP}$$

Theorem I.2 applies to all other configurations of the linear forms $a_0, \dots, a_n$ with a specific treatment when $\operatorname{rk}(a_0, \dots, a_n) = n$ and $k^+(a_0, \dots, a_n) = n+1$. The specific behaviour associated to this configuration regarding cocycle relations was already observed in [[Hil07], Proposition 2].

We are now ready to prove Theorem I.2, showing that the functions $c^\vee(\cdot)$ satisfy some cocycle relations. These are inspired by the cocycle relations described by Hill [[Hil07], Proposition 2] for the indicator functions $c^\circ(v_1, \dots, v_m)$ of open cones $\mathbb{R}_{>0}v_1 + \dots + \mathbb{R}_{>0}v_m$ and by Charollois, Dasgupta and Greenberg [[CDG15], Theorem 1.1] for variants of the functions $c^\circ(\cdot)$ for which some boundaries are included depending on a so-called Q-perturbation process. We argue that Theorem I.2 is simpler as there is no need to select the boundary pieces to add or remove with such a process, yet the proof uses similar ideas as those developed in the proof of [[CDG15], Theorem 1.1].

Proof of Theorem I.2:

Let V be a $\mathbb{Q}$-vector space of dimension n. Consider $n+1$ linear forms $a_0, \dots, a_n \in V^\vee$ which generate $V^\vee$ and are not (BP). Define $q(k)$ by $q(0) = 1$ and $q(k) = 0$ otherwise. We wish to prove that:

$$\sum_{j=0}^{n} \varepsilon_j c^\vee(a_0, \dots, \widehat{a_j}, \dots, a_n) \equiv q(k^-(a_0, \dots, a_n))\varepsilon_0 \delta \bmod \mathcal{L}(V)$$

where $\varepsilon_j = (-1)^j \operatorname{sign}\det(a_0, \dots, \widehat{a_j}, \dots, a_n)$ and δ is the Dirac at 0. The proof is split into three parts. We first treat the case where $a_0, \dots, a_n$ are in general position with $k^-(a_0, \dots, a_n) = 0$. Then, using this first result we treat the case where $a_0, \dots, a_n$ are in general position with $k^-(a_0, \dots, a_n) > 0$ by double induction on the dimension n and the value of $k^-(a_0, \dots, a_n)$. Finally, using this second result, we treat the case where $a_0, \dots, a_n$ are not in general position and $k^-(a_0, \dots, a_n) > 0$ by single induction on n.

First case: $a_0, \dots, a_n$ are in general position and $k^-(a_0, \dots, a_n) = 0$:

Let $\sum_{j=0}^n \lambda_j a_j = 0$ be the standard non-trivial relation among $a_0, \dots, a_n$. Because the coefficients λ_j are all positive, the signs ε_j are all equal to ε_0. Indeed, if $j \geq 1$ then:

$$\begin{aligned}
\varepsilon_j &= (-1)^j \operatorname{sign}\det(-\lambda_j a_j/\lambda_0, \dots, \widehat{a_j}, \dots, a_n) \\
\varepsilon_j &= -(-1)^j \operatorname{sign}(\lambda_j/\lambda_0) \operatorname{sign}\det(a_j, a_1, \dots, \widehat{a_j}, \dots, a_n) \\
\varepsilon_i &= -(-1)^j (-1)^{j+1} \operatorname{sign}\det(a_1, \dots, a_n) \\
\varepsilon_j &= \varepsilon_0
\end{aligned}$$

Let us denote by ε the common sign of the $\varepsilon_0 = \dots = \varepsilon_n = \varepsilon$. Consider now $f(a_0, \dots, a_n) = \prod_{j=0}^n (c^\vee(a_j) - 1)$. Expanding the product gives:

$$\begin{aligned}
f(a_0, \dots, a_n) &= \sum_{P \subset [|0,n|]} (-1)^{n+1-\#P} \prod_{j \in P} c^\vee(a_j) \\
f(a_0, \dots, a_n) &= \sum_{P \subset [|0,n|]} (-1)^{n+1-\#P} c^\vee(a_j, j \in P)
\end{aligned}$$

The last line follows from lemma I.10 (ii). For any $P \subset [|0,n|]$ with $\#P < n$, the function $c^\vee(a_j, j \in P)$ belongs to $\mathcal{L}(V)$ (lemma I.10 (i)). Thus:

$$f(a_0,\dots,a_n) - c^\vee(a_0,\dots,a_n) + \sum_{j=0}^{n} c^\vee(a_0,\dots,\widehat{a_j},\dots,a_n) \in \mathcal{L}(V)$$

Since $\sum_{j=0}^{n} \varepsilon_j c^\vee(a_0,\dots,\widehat{a_j},\dots,a_n) = \varepsilon \sum_{j=0}^{n} c^\vee(a_0,\dots,\widehat{a_j},\dots,a_n)$ this gives:

$$\varepsilon f(a_0,\dots,a_n) - \varepsilon c^\vee(a_0,\dots,a_n) + \sum_{j=0}^{n} \varepsilon_j c^\vee(a_0,\dots,\widehat{a_j},\dots,a_n) \in \mathcal{L}(V)$$

Now, since the coefficients λ_i are all positive by assumption, $c^\vee(a_0,\dots,a_n)$ is the dirac at 0 by lemma I.10 (v). On the other hand:

$$f(a_0,\dots,a_n) = \prod_{j=0}^{n} (c^\vee(a_j) - 1) = 0$$

The last equality holds because for all $v \in V$, $\sum_{j=0}^{n} \lambda_j a_j(v) = 0$ so at least one of the $a_j(v)$ must be non-negative. Therefore we may conclude that $\sum_{j=0}^{n} \varepsilon_j c^\vee(a_0,\dots,\widehat{a_j},\dots,a_n) \equiv \varepsilon\delta \bmod \mathcal{L}(V)$ as claimed.

Second case: $a_0,\dots,a_n$ are in general position and $k^-(a_0,\dots,a_n) > 0$: the case $n = 1$ is immediate as in this case $\lambda_0 a_0 + \lambda_1 a_1 = 0$ with $\lambda_0 = -1$ and $\lambda_1 > 0$ which gives $c^\vee(a_0) - c^\vee(a_1) = c^\vee(a_0) - c^\vee(a_0) = 0 \in \mathcal{L}(V)$ by lemma I.10 (iii). We are now ready to perform double induction on both n and k^-. Suppose that the result holds for any family $a_0,\dots,a_{n-1}$ of linear forms on a $n-1$ dimensional $\mathbb{Q}$-vector space V' generating $V'^\vee$ with $k^-(a_0,\dots,a_{n-1}) > 0$ and that it holds in dimension n whenever $a_0,\dots,a_n$ are in general position in $V^\vee$ with $k^-(a_0,\dots,a_n) = k^-$. Suppose now that $a_0,\dots,a_n$ are in general position in $V^\vee$ with $k^-(a_0,\dots,a_n) = k^- + 1$. We aim to prove that:

$$\sum_{j=0}^{n} \varepsilon_j c^\vee(a_0,\dots,\widehat{a_j},\dots,a_n) \in \mathcal{L}(V)$$

where as before $\varepsilon_j = (-1)^j \operatorname{sign}\det(a_0,\dots,\widehat{a_j},\dots,a_n)$. Denote as before $\sum_{j=0}^{n} \lambda_j a_j = 0$ the standard non-trivial relation among $a_0,\dots,a_n$. By assumption, $k^-(a_0,\dots,a_n) > 0$ so there is at least one index l such that $\lambda_l < 0$. We fix any such index l and we will show that the desired result may be deduced from the result for the families $a_0,\dots,a_{l-1},-a_l,a_{l+1},\dots,a_n$ in $V^\vee$ and $a_{0|\ker a_l},\dots,\widehat{a_{l|\ker a_l}},\dots,a_{n|\ker a_l}$ in $(\ker a_l)^\vee$ as follows. For simplicity we use once again the auxiliary function defined by $q(k) = 1$ if $k = 0$ and $q(k) = 0$ otherwise. The family $(a'_0,\dots,a'_n) = (a_0,\dots,a_{l-1},-a_l,a_{l+1},\dots,a_n)$ is in general position in $V^\vee$ with $k^-(a'_0,\dots,a'_n) = k^-$. By induction hypothesis on k^-, fixing any index $m \neq l$:

$$\sum_{j=0}^{n} \varepsilon'_j c^\vee(a'_0,\dots,\widehat{a'_j},\dots,a'_n) \equiv q(k^-)\varepsilon'_m \delta \bmod \mathcal{L}(V) \tag{I.24}$$

where $\varepsilon'_j = (-1)^j \operatorname{sign}\det(a'_0,\dots,\widehat{a'_j},\dots,a'_n)$. It is clear that $\varepsilon'_j = -\varepsilon_j$ when $j \neq l$ and $\varepsilon'_l = \varepsilon_l$ so that (I.24) reads:

$$\varepsilon_l c^\vee(a_0,\dots,\widehat{a_l},\dots,a_n) + \sum_{\substack{j=0\\ j\neq l}}^{n} (-\varepsilon_j) c^\vee(a'_0,\dots,\widehat{a'_j},\dots,a'_n) \equiv -q(k^-)\varepsilon_m \delta \bmod \mathcal{L}(V). \tag{I.25}$$

It is then sufficient to prove that:

$$\left(\sum_{\substack{j=0\\j\neq l}}^{n}\varepsilon_j(c^\vee(a_0,\dots,\widehat{a_j},\dots,a_n)+c^\vee(a'_0,\dots,\widehat{a'_j},\dots,a'_n))\right)\equiv q(k^-)\varepsilon_m\delta \bmod \mathcal{L}(V)$$

Using lemma I.10 (ii) then (iv), we obtain:

$$\begin{aligned}
&\sum_{\substack{j=0\\j\neq l}}^{n}\varepsilon_j(c^\vee(a_0,\dots,\widehat{a_j},\dots,a_n)+c^\vee(a'_0,\dots,\widehat{a'_j},\dots,a'_n))\\
&=\sum_{\substack{j=0\\j\neq l}}^{n}\varepsilon_j(c^\vee(a_0,\dots,\widehat{a_j},\dots,a_n)+c^\vee(a_0,\dots,-a_l,\dots,\widehat{a_j},\dots,a_n))\\
&=\sum_{\substack{j=0\\j\neq l}}^{n}\varepsilon_j c^\vee(a_0,\dots,\widehat{a_l},\dots,\widehat{a_j},\dots,a_n)(c^\vee(a_l)+c^\vee(-a_l))\\
&=\sum_{\substack{j=0\\j\neq l}}^{n}\varepsilon_j c^\vee(a_0,\dots,\widehat{a_l},\dots,\widehat{a_j},\dots,a_n)(1+\ker(a_l))
\end{aligned}$$

where $\ker(a_l)$ is the indicator function of the kernel of a_l. For any $j\neq l$, the function $c^\vee(a_0,\dots,\widehat{a_l},\dots,\widehat{a_j},\dots,a_n)$ lies in $\mathcal{L}(V)$ by lemma I.10 (i), thus we only need to prove that:

$$\sum_{\substack{j=0\\j\neq l}}^{n}\varepsilon_j c^\vee(a_0,\dots,\widehat{a_l},\dots,\widehat{a_j},\dots,a_n)\ker(a_l)\equiv q(k^-)\varepsilon_m\delta \bmod \mathcal{L}(V)$$

This step of the proof makes use of the induction hypothesis on n. Denote $a''_0,\dots,a''_n$ the restrictions of $a_0,\dots,a_n$ on $\ker(a_l)$. Then the natural inclusion described by (I.23) gives:

$$\sum_{\substack{j=0\\j\neq l}}^{n}\varepsilon_j c^\vee(a_0,\dots,\widehat{a_l},\dots,\widehat{a_j},\dots,a_n)\ker(a_l)=\sum_{\substack{j=0\\j\neq l}}^{n}\varepsilon_j c^\vee_{\ker(a_l)}(a''_0,\dots,\widehat{a''_l},\dots,\widehat{a''_j},\dots,a''_n)$$

The elements $a''_0,\dots,\widehat{a''_l},\dots,a''_n$ are in general position in the dual $\ker(a_l)^\vee$ of $\ker(a_l)$. This space may be oriented using the form:

$$\mathcal{O}_l(x_0,\dots,\widehat{x_l},\dots,x_n)=\det(\tilde{x}_0,\dots,\tilde{x}_{l-1},a_l,\tilde{x}_{l+1},\dots,\tilde{x}_n)$$

for any lifts $\tilde{x}_j$ of the x_j to $V^\vee$ satisfying $\tilde{x}_{j|_{\ker a_l}}=x_j$. Lastly, if $\sum_{j=0}^{n}\lambda_j a_j=0$ is the standard non-trivial relation among $a_0,\dots,a_n$ in $V^\vee$ then $\sum_{j\neq l}\lambda_j a''_j=0$ is the standard non-trivial relation among $a''_0,\dots,\widehat{a''_l},\dots,a''_n$ in $\ker(a_l)^\vee$ and $k^-(a''_0,\dots,\widehat{a''_l},\dots,a''_n)=k^-(a_0,\dots,a_l,\dots,a_n)-1=k^-$. Therefore, using the induction hypothesis on n, we obtain:

$$\sum_{\substack{j=0\\j\neq l}}^{n}\varepsilon_j c^\vee_{\ker(a_l)}(a''_0,\dots,\widehat{a''_l},\dots,\widehat{a''_j},\dots,a''_n)\equiv q(k^-)\varepsilon_m\delta_{\ker(a_l)} \bmod \mathcal{L}(\ker(a_l))$$

where $\delta_{\ker(a_l)}$ is the Dirac function at 0 on $\ker(a_l)$. This last expression can be lifted back to V using the inclusion (I.23) as:

$$\sum_{\substack{j=0\\ j\neq l}}^{n} \varepsilon_j c^\vee(a_0,\ldots,\widehat{a_l},\ldots,\widehat{a_j},\ldots,a_n)\ker(a_l) \equiv q(k^-)\varepsilon_m\delta \bmod \mathcal{L}(V) \tag{I.26}$$

where it is clear that the inclusion map sends $\delta_{\ker(a_l)}$ to δ. We now piece (I.25) and (I.26) together and find:

$$\begin{aligned}
&\sum_{j=0}^{n}(-1)^j \operatorname{sign}\det(a_0,\ldots,\widehat{a_j},\ldots,a_n)c^\vee(a_0,\ldots,\widehat{a_j},\ldots,a_n)\\
&\quad=\left(\sum_{j=0}^{n}(-1)^j \operatorname{sign}\det(a'_0,\ldots,\widehat{a'_j},\ldots,a'_n)c^\vee(a'_0,\ldots,\widehat{a'_j},\ldots,a'_n)+q(k^-)\varepsilon_m\delta\right)\\
&\qquad+\left(\sum_{\substack{j=0\\ j\neq l}}^{n}\varepsilon_j c^\vee(a_0,\ldots,\widehat{a_l},\ldots,\widehat{a_j},\ldots,a_n)\ker(a_l)-q(k^-)\varepsilon_m\delta\right)
\end{aligned}$$

Both terms in the right-hand side belong to $\mathcal{L}(V)$, therefore the left hand side does too, as claimed.

Third case: $a_0,\ldots,a_n$ are not in general position and $k^-(a_0,\ldots,a_n)>0$:
We prove this case by induction over $n\geq 2$ because it can't occur for $n=1$. For $n=2$ it may only happen when $a_j=\lambda a_l$ for some $j\neq l$ and some $\lambda\in\mathbb{Q}_{>0}$. Without loss of generality, assume $a_0=\lambda a_1$. In that case, it follows from I.10 (ii) and (iii) that:

$$\operatorname{sign}\det(a_1,a_2)c^\vee(a_1,a_2)-\operatorname{sign}\det(a_0,a_2)c^\vee(a_0,a_2)+0=0$$

which proves the case $n=2$. Let us now assume the result holds in dimension $n-1$, where $n\geq 3$. Consider V a $\mathbb{Q}$-vector space of dimension n and suppose that $a_0,\ldots,a_n$ are non-zero linear forms on V which generate $V^\vee$. Assume that $a_0,\ldots,a_n$ are not in general position in $V^\vee$ and that $k^-(a_0,\ldots,a_n)>0$. Without loss of generality we may assume that a_n lies outside the span of $a_0,\ldots,a_{n-1}$. Choose a vector $v_n\in\cap_{j=0}^{n-1}\ker(a_j)$ such that $a_n(v_n)>0$. This is possible because $a_0,\ldots,a_{n-1}$ do not generate $V^\vee$. Let us now use the projection $\pi:V\to\ker(a_n)$ with kernel $\mathbb{Q}v_n$ corresponding to the decomposition $V=\ker(a_n)\oplus\mathbb{Q}v_n$. For $0\leq j\leq n$, denote a'_j the restriction of a_j to $\ker(a_n)$. Write as before $\varepsilon_j=(-1)^j\operatorname{sign}\det(a_0,\ldots,\widehat{a_j},\ldots,a_n)$. As $a_0,\ldots,a_{n-1}$ are linearly dependent, $\varepsilon_n=0$, and we may compute:

$$\sum_{j=0}^{n}\varepsilon_j c^\vee(a_0,\ldots,\widehat{a_j},\ldots,a_n)=\sum_{j=0}^{n-1}\varepsilon_j c^\vee(a_n)c^\vee(a'_0,\ldots,\widehat{a'_j},\ldots,a'_{n-1})(\pi(\cdot))$$

Let $\sum_{j=0}^{n}\lambda_j a_j=0$ be the standard non-trivial relation among $a_0,\ldots,a_n$. The assumption that a_n lies outside the span of $a_0,\ldots,a_{n-1}$ is equivalent to $\lambda_n=0$. Therefore, the standard non-trivial relation among $a'_0,\ldots,a'_{n-1}$ is $\sum_{j=0}^{n-1}\lambda_j a'_j=0$ and $k^-(a'_0,\ldots,a'_{n-1})=k^-(a_0,\ldots,a_n)>0$. If the linear forms $a'_0,\ldots,a'_{n-1}$ are in general position in $(\ker a_n)^\vee$, we may use the result proven in the second case, and otherwise use the induction hypothesis as $a'_0, \ldots, a'_{n-1}$ are not in general position in $\ker(a_n)^\vee$. In both cases, we obtain that:

$$\sum_{j=0}^{n-1} \varepsilon_j c^\vee(a'_0, \ldots, \widehat{a'_j}, \ldots, a'_{n-1}) \in \mathcal{L}(V')$$

and remark that $\forall f \in \mathcal{L}(\ker(a_n))$, $c^\vee(a_n) f(\pi(\cdot)) \in \mathcal{L}(V)$ for any linear projection $\pi : V \to \ker(a_n)$. Indeed, if f is the characteristic function of a cone containing a line $\mathbb{Q}v$ in $\ker(a_n)$, then $f(\pi(\cdot))$ is the characteristic function of a cone containing the line $\mathbb{Q}v$ in V. For $\lambda \in \mathbb{Q}$, $a_n(\lambda.v) = 0$ therefore $c^\vee(a_n)(\lambda.v) f(\pi(\lambda.v)) = f(\lambda.v) = 1$ and $c^\vee(a_n) f(\pi(\cdot))$ is the characteristic function of a cone containing the line $\mathbb{Q}v$. Thus:

$$\sum_{j=0}^{n-1} \varepsilon_j c^\vee(a_n) c^\vee(a'_0, \ldots, \widehat{a'_j}, \ldots, a'_{n-1})(\pi(\cdot)) \in \mathcal{L}(V)$$

from which we conclude that $\sum_{j=0}^{n} \varepsilon_j c^\vee(a_0, \ldots, \widehat{a_j}, \ldots, a_n) \in \mathcal{L}(V)$ as claimed. □

Remark: In the case where $a_0, \ldots, a_n$ do not span $V^\vee$ it is clearly also true that:

$$\sum_{j=0}^{n} (-1)^j \operatorname{sign}\det(a_0, \ldots, \widehat{a_j}, \ldots, a_n) c^\vee(a_0, \ldots, \widehat{a_j}, \ldots, a_n) = 0$$

as each term in the left hand side is zero.

Unfortunately, the case where $a_0, \ldots, a_n$ are not in general position with $k^-(a_0, \ldots, a_n) = 0$ (this corresponds to the "bad position" condition (BP)) already fails in dimension 2. Indeed, if a_0, a_1, a_2 satisfy $\operatorname{rk}(a_0, a_1, a_2) = 2$ and $\lambda_0 a_0 + \lambda_1 a_1 = 0$ with $\lambda_0 > 0$ and $\lambda_1 > 0$ then:

$$\begin{aligned}
&\operatorname{sign}\det(a_1, a_2) c^\vee(a_1, a_2) - \operatorname{sign}\det(a_0, a_2) c^\vee(a_0, a_2) + \operatorname{sign}\det(a_0, a_1) c^\vee(a_0, a_1) \\
&= \operatorname{sign}\det(-a_0, a_2) c^\vee(-a_0, a_2) - \operatorname{sign}\det(a_0, a_2) c^\vee(a_0, a_2) + 0 \\
&= -\operatorname{sign}\det(a_0, a_2) c^\vee(a_2)(c^\vee(-a_0) + c^\vee(a_0)) \\
&= -\operatorname{sign}\det(a_0, a_2) c^\vee(a_2)(1 + \ker(a_0)) \notin \mathcal{L}(V) \oplus \mathbb{Q}\delta
\end{aligned}$$

Indeed, the function $c^\vee(a_2) \in \mathcal{L}(V)$ by lemma I.10 (i), however, the function $c^\vee(a_2)\ker(a_0)$ is the indicator function of the cone $\mathbb{Q}_{\geq 0} v_2$ where v_2 is the vector defined by $a_0(v_2) = 0$ and $a_2(v_2) = 1$.

The first part of the proof of Theorem I.2 may be slightly adjusted to prove that when $k^-(a_0, \ldots, a_n) = 0$ and $a_0, \ldots, a_n$ are not in general position, the following holds:

$$\sum_{i=0}^{n} (-1)^i \operatorname{sign}\det(a_0, \ldots, \widehat{a_i}, \ldots, a_n) c^\vee(a_0, \ldots, \widehat{a_i}, \ldots, a_n) \equiv \pm \prod_{\lambda_i = 0} c^\vee(a_i)[\cap_{\lambda_i > 0} \ker(a_i)] \mod \mathcal{L}(V)$$

It is not hard to see that the right-hand side of this formula doesn't belong to $\mathcal{L}(V) \oplus \mathbb{Q}\delta$. We will now introduce the generating functions of cones and relate this cocycle for indicator functions of cones to our geometric Bernoulli cocycle.

I.3.2 Generating functions of cones and the $B_{n,\underline{a}}$ functions

In this section we briefly recall some results on the generating functions associated to cones in $\mathbb{R}^n$. Let $V_\mathbb{R}$ be a $\mathbb{R}$-vector space of finite dimension n and fix an isomorphism $V_\mathbb{R} \simeq \mathbb{R}^n$ which defines $V_\mathbb{Q} = V \simeq \mathbb{Q}^n$ and $L = V_\mathbb{Z} \simeq \mathbb{Z}^n$. It is well-known that a generating function may be associated to any rational polyhedral cone in $V \simeq \mathbb{R}^n$ which does not contain any line by the formula:

$$g(C,v)(y) = \sum_{\delta \in C\cap(v+L)} y^\delta$$

where $y^\delta = y_1^{\delta_1} y_2^{\delta_2} \dots y_n^{\delta_n}$ and $v \in V_\mathbb{R}$. This is well-defined for y in a certain open subset of $\mathbb{C}^n$ and it is possible to extend this function by analytic continuation. Indeed, if $C = \mathbb{R}_{\geq 0}\alpha_1 \dots, \mathbb{R}_{\geq 0}\alpha_n$ with primitive vectors $\alpha_1, \dots, \alpha_m \in L$ then the generating series associated to C is in fact defined by:

$$g(C,v)(y) = \frac{\sum_{\delta \in P\cap(v+L)} y^\delta}{(1-y^{\alpha_1})\dots(1-y^{\alpha_m})}$$

where $P = P(\alpha_1, \dots, \alpha_m) = \{\sum_{i=1}^m \mu_i\alpha_i \mid 0 \leq \mu_i < 1, \forall i\}$. This formula is given by the standard decomposition of a rational polyhedral cone:

$$C \cap (v+L) = \sqcup_{\delta \in P\cap(v+L)} \delta + \mathbb{Z}_{\geq 0}\alpha_1 + \dots + \mathbb{Z}_{\geq 0}\alpha_m \tag{I.27}$$

We highlight that Definitions I.7 and I.8 used this decomposition implicitly with the set $F(\underline{a}, \underline{\alpha}, v) = P(\underline{\alpha}) \cap (v+L)$. The key result regarding generating series associated to cones is the fact that the function g may be extended to the subspace $\mathcal{K}_\mathbb{Q}(V_\mathbb{R})$ of $\mathcal{K}(V_\mathbb{R})$ spanned by the indicator functions of rational polyhedral cones by linearity, as was proven independently by Khovanskii and Pukhlikov, and by Lawrence. A polyhedral cone in $V_\mathbb{R}$ is said to be rational if it admits a set of generators inside $V_\mathbb{Q}$, in which case it also admits a set of generators inside $L = V_\mathbb{Z}$. The identification $\mathcal{K}_\mathbb{Q}(V_\mathbb{R}) \simeq \mathcal{K}(V) \otimes_\mathbb{Q} \mathbb{R}$ where $V = V_\mathbb{Q}$ thus shows that the generating series function g may be extended to $\mathcal{K}(V)$. In particular, for any $f \in \mathcal{L}_\mathbb{Q}(V_\mathbb{R}) \simeq \mathcal{L}(V) \otimes_\mathbb{Q} \mathbb{R}, g(f,\cdot) = 0$. The Solomon-Hu [HS01] pairing is then defined for $(f,v,x) \in \mathcal{K}(V)/\mathcal{L}(V) \times V/L \times \mathrm{Hom}_\mathbb{Z}(L,\mathbb{C})$ by $h(f,v)(x) = g(f,v)(e^x)$. We use this pairing to express the geometric Bernoulli rational functions via a coefficient extraction.

Definition I.12: *The map*

$$h_0 := \begin{cases} \mathcal{K}(V) & \to \mathcal{F}(V/L, \mathbb{Q}[w](x)) \\ f & \to \mathrm{coeff}[t^0]\left(e^{wt}h(f,v)(t.x)\right) \end{cases}$$

is $\mathbb{Q}$-linear and vanishes on $\mathcal{L}(V)$.

It is also clear that $h_0(\delta, v)$ is the constant function equal to 1 if $v \in L$ and 0 otherwise. We shall now use this function to express the geometric Bernoulli rational functions (see Definition I.8) associated to non-zero integral linear forms $a_1, \dots, a_n \in \Lambda = \mathrm{Hom}_\mathbb{Z}(L,\mathbb{Z})$.

Lemma I.13: *For non-zero linear forms $a_1, \dots, a_n \in \Lambda$ in the rank n oriented lattice Λ, for a vector $v \in V/L$ and for (w,x) in a dense open subset of $\mathbb{C} \times \mathrm{Hom}_\mathbb{Z}(L,\mathbb{C})$:*

$$B_{n,a_1,\dots,a_n}(v)(w,x) = h_0(\mathrm{sign}\det(a_1,\dots,a_n)c^\vee(a_1,\dots,a_n), v)(w,x).$$

Proof :
If $a_1, \dots, a_n$ are linearly dependent, then both sides are zero so the equality holds. If $a_1, \dots, a_n$ are linearly independent, the proof follows from the definition of $B_{n,a_1,\dots,a_n}$: let $\alpha_1, \dots, \alpha_n$ be the primitive positive dual family to $a_1, \dots, a_n$ in $L \simeq \mathbb{Z}^n$. Denote $\epsilon = \operatorname{sign}\det(a_1, \dots, a_n)$. Then identifying $F(\underline{a}, \underline{\alpha}, v) = P(\underline{\alpha}) \cap (v + L)$ we get:

$$\begin{aligned}
B_{n,a_1,\dots,a_n}(w,x) &= \operatorname{coeff}[t^0]\left((-1)^n \epsilon \sum_{\delta \in F(\underline{a},\underline{\alpha},v)} \frac{e^{wt} e^{x(\delta)t}}{\prod_{j=1}^n (e^{x(\alpha_j)t} - 1)}\right) \\
B_{n,a_1,\dots,a_n}(w,x) &= \operatorname{coeff}[t^0]\left(\epsilon \sum_{\delta \in P(\underline{\alpha}) \cap (v+L)} \frac{e^{wt} e^{x(\delta)t}}{\prod_{j=1}^n (1 - e^{x(\alpha_j)t})}\right) \\
B_{n,a_1,\dots,a_n}(w,x) &= \operatorname{coeff}[t^0]\left(\epsilon e^{wt} h(c(\alpha_1, \dots, \alpha_n), v)(t.x)\right) \\
B_{n,a_1,\dots,a_n}(w,x) &= \operatorname{coeff}[t^0]\left(\epsilon e^{wt} h(c^\vee(a_0, \dots, a_{r+1}), v)(t.x)\right)
\end{aligned}$$

where we use that $c(\alpha_1, \dots, \alpha_n) = c^\vee(a_1, \dots, a_n)$ by definition of $\alpha_1, \dots, \alpha_n$. The identification of the right-hand side with $h_0(\epsilon c^\vee(a_1, \dots, a_n), v)(w, x)$ proves the claim. □

Using this lemma, we may finally describe the cocycle relations satisfied by the Bernoulli rational functions $B_{n,a_1,\dots,a_n}$.

Proposition I.14: *Let $a_0, \dots, a_n$ be $n+1$ non-zero linear forms in Λ. Fix $v \in V/L$.*

- *Suppose* $\operatorname{rk}(a_0, \dots, a_n) \leq n-1$, *or* $\operatorname{rk}(a_0, \dots, a_n) = n$ *and* $k^-(a_0, \dots, a_n) > 0$. *Then the following equality holds in $\mathbb{Q}[w](x)$:*

$$\sum_{j=0}^n (-1)^j B_{n,a_0,\dots,\widehat{a_j},\dots,a_n}(v) = 0$$

- *Suppose $a_0, \dots, a_n$ are in general position with $k^-(a_0, \dots, a_n) = 0$. Then in $\mathbb{Q}[w](x)$:*

$$\sum_{j=0}^n (-1)^j B_{n,a_0,\dots,\widehat{a_j},\dots,a_n}(v) = \begin{cases} \operatorname{sign}\det(a_1, \dots, a_n) & \textit{if } v \in L \\ 0 & \textit{otherwise} \end{cases}$$

Proof :
If $\operatorname{rk}(a_0, \dots, a_n) \leq n-1$ then the equality is trivial as all terms in the left-hand side vanish. Suppose now that $\operatorname{rk}(a_0, \dots, a_n) = n$. We combine the results from lemma I.13 with the results from Theorem I.2. Denote as before $\varepsilon_j = (-1)^j \operatorname{sign}\det(a_0, \dots, \widehat{a_j}, \dots, a_n)$. From lemma I.13 we get the equality between rational functions in $\mathbb{Q}[w](x)$:

$$\sum_{j=0}^n (-1)^j B_{n,a_0,\dots,\widehat{a_j},\dots,a_n}(v) = h_0\left(\sum_{j=0}^n \varepsilon_j c^\vee(a_0, \dots, \widehat{a_j}, \dots, a_n), v\right)$$

If $k^-(a_0, \dots, a_n) > 0$ then $\sum_{j=0}^n \varepsilon_j c^\vee(a_0, \dots, \widehat{a_j}, \dots, a_n) \in \mathcal{L}_{\mathbb{Q}}(V)$ by Theorem I.2, therefore the right-hand side vanishes. If $a_0, \dots, a_n$ are in general position with $k^-(a_0, \dots, a_n) =$

0 then $\sum_{j=0}^{n} \varepsilon_j c^\vee(a_0,\ldots,\widehat{a_j},\ldots,a_n) - \varepsilon_0\delta \in \mathcal{L}(V)$, therefore:

$$\sum_{j=0}^{n}(-1)^j B_{n,a_0,\ldots,\widehat{a_j},\ldots,a_n}(v) = \varepsilon_0 h_0(\delta,v) = \begin{cases} \varepsilon_0 & \text{if } v \in L \\ 0 & \text{otherwise} \end{cases}$$

which proves the claim. □

Unfortunately, the case where $a_0,\ldots,a_n$ span Λ yet are not in general position with $k^-(a_0,\ldots,a_n)=0$ doesn't give good results. Indeed, already for $n=2$, if a,b are linearly independent with $\det(a,b)=\pm 1$, then:

$$B_{2,-a,b}(0)(w,x) - B_{2,a,b}(0)(w,x) + B_{2,a,-a}(0)(w,x) = \operatorname{sign}\det(a,b)\left(\frac{w}{x(\beta)} - \frac{1}{2}\right) \neq 0$$

where α,β is the primitive positive dual family to a,b in L. The right-hand side depends on b and there is no possible value we could have chosen by convention for $B_{2,a,-a}$ which would have given:

$$B_{2,-a,b}(v)(w,x) - B_{2,a,b}(v)(w,x) + B_{2,a,-a}(v)(w,x) = 0$$

for any non-zero primitive $b \in \Lambda$.

We end this section by showing that in dimension 2 we may use Theorem I.2 in conjonction with the Solomon-Hu pairing to recover formula (5.16) in a recent article by Sharifi and Venkatesh [SV24] which is a key ingredient in the lifting of a cocycle carried out in [[SV24], Proposition 5.4.1]. Let us write $\theta_{SV}(l_1,l_2)$ for the function $\theta_L(l_1,l_2)$ they define in §5.3.1 to avoid confusion with our notations. When $l_1,l_2 \in S^1$ with $\epsilon_{12} = \det(l_1,l_2) \neq 0$ the function $\theta_{SV}(l_1,l_2)$ may be expressed using [[SV24], Lemma 5.3.1] in our notations as:

$$\theta_{SV}(l_1,l_2) = \frac{1-\epsilon_{12}}{2} + \epsilon_{12} h(c^\vee(a_1,a_2),0)(-x) \tag{I.28}$$

where $a_i = \langle \cdot, l_i\rangle$ and x is the linear form corresponding to the formal coordinates (u_1,u_2) defined in [[SV24], §5.3.1]. Let us consider $\kappa(l_1,l_2,l_3) = \theta_{SV}(l_1,l_2)+\theta_{SV}(l_2,l_3)-\theta_{SV}(l_1,l_3)$. We wish to show that if l_1,l_2,l_3 are in general position then $\kappa(l_1,l_2,l_3) = \delta_{SV}(l_1,l_2,l_3)$ where the function $\delta_{SV}(l_1,l_2,l_3)$ is defined to be 0 if l_2 lies on the counterclockwise portion of S^1 joining l_1 to l_3 and 1 otherwise. To achieve this we must translate our notion of configurations based on the value of $k^-(a_1,a_2,a_3)$ in terms of the function δ_{SV}. Let us then rewrite $\kappa(l_1,l_2,l_3)$ when l_1,l_2,l_3 are in general position as:

$$\kappa(l_1,l_2,l_3) = \frac{1-\epsilon_{12}-\epsilon_{23}+\epsilon_{13}}{2} + h(\epsilon_{12}c^\vee(a_1,a_2)+\epsilon_{23}c^\vee(a_2,a_3)-\epsilon_{13}c^\vee(a_1,a_3),0)(-x).$$

where $\epsilon_{ij} = \det(l_i,l_j) = \det(a_i,a_j)$. It follows from Theorem I.2 that:

$$\epsilon_{12}c^\vee(a_1,a_2)+\epsilon_{23}c^\vee(a_2,a_3)-\epsilon_{13}c^\vee(a_1,a_3) \equiv \epsilon_{12} q(k^-(a_1,a_2,a_3))\delta \bmod \mathcal{L}(V)$$

where $q(k)=0$ if $k>0$ and $q(0)=1$ and δ is the Dirac function at 0. Using the linearity of h and its vanishing on $\mathcal{L}(V)$ we get:

$$\kappa(l_1,l_2,l_3) = \frac{1-\epsilon_{12}-\epsilon_{23}+\epsilon_{13}}{2} + \epsilon_{12} q(k^-(a_1,a_2,a_3)) \tag{I.29}$$

since $h(\delta,0)(-x)=1$. To compute the last term in (I.29) we consider $\sum_{j=1}^{3}\lambda_j a_j = 0$ the standard non-trivial relation among a_1,a_2,a_3 (see Definition I.11) for which it is also true that $\sum_{j=1}^{3}\lambda_j l_j = 0$. Put $\kappa_1(l_1,l_2,l_3) = (1-\epsilon_{12}-\epsilon_{23}+\epsilon_{13})/2$ and $\kappa_2(l_1,l_2,l_3) = \epsilon_{12}q(k^-(a_1,a_2,a_3))$ so that $\kappa = \kappa_1+\kappa_2$. Then we treat 8 cases separately according to the signs of the ϵ_{ij}'s and gather the results in the following table:

ϵ_{12}	ϵ_{13}	ϵ_{23}	λ_1	λ_2	λ_3	k^-	κ_1	κ_2	κ	δ_{SV}
$+$	$+$	$+$	$+$	$-$	$+$	1	0	0	0	0
$+$	$+$	$-$	$+$	$+$	$-$	1	1	0	1	1
$+$	$-$	$+$	$+$	$+$	$+$	0	-1	1	0	0
$+$	$-$	$-$	$-$	$+$	$+$	1	0	0	0	0
$-$	$+$	$+$	$-$	$+$	$+$	1	1	0	1	1
$-$	$+$	$-$	$+$	$+$	$+$	0	2	-1	1	1
$-$	$-$	$+$	$+$	$+$	$-$	1	0	0	0	0
$-$	$-$	$-$	$+$	$-$	$+$	1	1	0	1	1

This completes the proof of [[SV24], formula (5.16)] when l_1, l_2, l_3 are in general position. A similar formula holds for the coefficient of degree 0 in the expansion of $\theta_{SV}(l_1, l_2)$ near the origin and it may be obtained using Proposition I.14 for $n = 2$. Interesting future work in this direction would be to use Theorem I.2 in dimension $n \geq 3$ to prove analogues of [[SV24], formula (5.16)] for higher degree analogues of the function $\theta_{SV}(l_1, l_2)$ which appear in recent work by Xu (see [Xu25]).

I.3.3 The cocycle $\phi_{n,a}$ for unit groups

We now define the cocycle properly on specific subgroups of $\mathrm{SL}_n(\mathbb{Z})$. Fix a primitive linear form $a \in \Lambda = \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$. We consider subgroups U of $\mathrm{SL}_n(\mathbb{Z})$ satisfying the following property: $\forall\, m \geq 2, \forall\, g_1, \ldots, g_m \in U, \forall \mu_1, \ldots, \mu_m \in \mathbb{Z}_{\geq 0}$,

$$\sum_{j=1}^{m} \mu_j g_j \cdot a = 0 \Rightarrow \mu_1 = 0, \ldots, \mu_m = 0 \tag{I.30}$$

This property guarantees that we avoid families of vectors in bad position (see (BP)) in what follows and we give examples of such groups down below. Recall the definition of the two functions:

$$\psi_{n,a} := \begin{cases} \mathrm{SL}_n(\mathbb{Z})^{n-2} & \to \mathcal{F}(V/L \times \mathbb{C} \times \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{C}), \mathbb{C}) \\ (g_1, \ldots, g_{n-2}) & \to \big((v, w, x) \to G_{n-2, a, g_1 \cdot a, \ldots, (g_1 \cdots g_{n-2}) \cdot a}(v)(w, x)\big) \end{cases}$$

$$\phi_{n,a} := \begin{cases} \mathrm{SL}_n(\mathbb{Z})^{n-1} & \to \mathcal{F}(V/L, \mathbb{Q}[w](x)) \\ (g_1, \ldots, g_{n-1}) & \to B_{n, a, g_1 \cdot a, (g_1 g_2) \cdot a, \ldots, (g_1 \cdots g_{n-1}) \cdot a}(v)(w, x) \end{cases}$$

It follows from Proposition I.14 that $\phi_{n,a}$ is an additive $(n-1)$-cocycle for U as:

$$\partial \phi_{n,a}(g_1, \ldots, g_{n+1}) = 0$$

for any $g_1, \ldots, g_{n+1} \in U$. Furthermore, from Theorem I.1 we deduce that the multiplicative cocycle $\exp(2i\pi\phi_{n,a})$ is partially split by $\psi_{n,a}$. Namely, when $a, g_1 \cdot a, \ldots, (g_1 \ldots g_{n-1}) \cdot a$ are linearly independent, the multiplicative coboundary of $\psi_{n,a}$ is given by:

$$\partial^{\times} \psi_{n,a}(g_1, \ldots, g_n) = \exp(2i\pi\phi_{n,a}(g_1, \ldots, g_n))$$

We expect this splitting property to be true more generally, as we expect that the modular property:

$$\prod_{j=1}^{n} G_{n-2, (a_k)_{k \neq j}}(v)(w, x)^{(-1)^{j+1}} = \exp(2i\pi B_{n, a_1, \ldots, a_n}(v)(w, x))$$

holds whenever $a_1, \ldots, a_n$ are not (BP), but the strategy of proof we have in mind to extend the domain of validity of Theorem I.1 would probably carry us too far from the matter at hand. It should be noted that there are two easier configurations of the a_i's for which the result is true for purely cohomological reasons, and this already covers all cases for $n = 2, 3$. In chapter II, we will focus more on this splitting relation and show that it indeed holds for the subgroups U of $\mathrm{SL}_n(\mathbb{Z})$ satisfying (I.30).

We now give several examples of such groups U. A simple example is given by the subgroup of $\mathrm{SL}_n(\mathbb{Z})$ consisting of matrices which stabilise the set $\{\alpha \in L \mid a(\alpha) \geq 0\}$. This corresponds to the case where linear forms are on the same side of some hyperplane through the origin in [[FHRZ08], Lemma 3.9]. It is clear that such a group satisfies condition (I.30) and that it is isomorphic to the subgroup:

$$U^0 := \left\{ g \in \mathrm{SL}_n(\mathbb{Z}) \mid g = \begin{pmatrix} 1 & * & \ldots & * \\ 0 & * & \ldots & * \\ \vdots & \vdots & \vdots & \vdots \\ 0 & * & \ldots & * \end{pmatrix} \right\}$$

of $\mathrm{SL}_n(\mathbb{Z})$. This simple example is not very interesting for us as it reduces $\phi_{n,a}$ to a $(n-1)$-cocycle for $\mathrm{SL}_{n-1}(\mathbb{Z})$ and we now give another set of examples which motivated this work. Consider a number field $\mathbb{K}$ of degree $n \geq 2$ with at least one real embedding $\sigma_{\mathbb{R}}$. Denote by $\mathcal{O}_{\mathbb{K}}$ the ring of integers of $\mathbb{K}$ and $\mathcal{O}_{\mathbb{K}}^{\times}$ the group of units of $\mathcal{O}_{\mathbb{K}}$. By Dirichlet's unit theorem, $\mathcal{O}_{\mathbb{K}}^{\times}$ is a free abelian group of rank $r_1 + r_2 - 1$ where r_1 is the number of real embeddings of $\mathbb{K}$ and r_2 is the number of complex places of $\mathbb{K}$. Consider now a finite index subgroup $\mathcal{U}$ of $\mathcal{O}_{\mathbb{K}}^{\times}$ such that $\forall \varepsilon \in \mathcal{U}, \sigma_{\mathbb{R}}(\varepsilon) > 0$. In most examples, $\mathcal{U}$ will be the group of totally positive units $\mathcal{O}_{\mathbb{K}}^{+,\times}$ of $\mathcal{O}_{\mathbb{K}}$. Suppose that L is a lattice of rank n in $\mathbb{K}$ which is stable under multiplication by elements of $\mathcal{U}$. Fix a $\mathbb{Z}$-basis $B = [e_1, \ldots, e_n]$ of L. Then $\mathcal{U}$ may be identified with a commutative subgroup U of $\mathrm{SL}_n(\mathbb{Z})$ which satisfies condition (I.30) for any non-zero primitive linear form $a \in \Lambda$. Indeed, for any integer $m \geq 2$, if $\varepsilon_1, \ldots, \varepsilon_m \in \mathcal{U} \simeq U$ and $\mu_1 \geq 0, \ldots, \mu_m \geq 0$ then since:

$$\left(\sum_{j=1}^{m} \mu_j (\varepsilon_j \cdot a) \right) (\cdot) = a \left(\sum_{j=m}^{n} \mu_j \varepsilon_j^{-1} \cdot \right)$$

it follows that

$$\sum_{j=1}^{m} \mu_j (\varepsilon_j \cdot a) = 0 \Rightarrow a \left(\sum_{j=1}^{m} \mu_j \varepsilon_j^{-1} \cdot \right) = 0 \Rightarrow \sum_{j=1}^{m} \mu_j \varepsilon_j^{-1} = 0 \Rightarrow \sum_{j=1}^{m} \mu_j \sigma_{\mathbb{R}}(\varepsilon_j^{-1}) = 0$$

as non-zero elements of $\mathbb{K}$ give bijections of $\mathbb{K}$ by multiplication. The latter expression is a sum of non-negative numbers which is equal to zero, from which we conclude that $\mu_j = 0, \forall\, 1 \leq j \leq m$. Thus $\phi_{n,a}$ is a true $(n-1)$-cocycle for the group $U \simeq \mathcal{U}$.

The arithmetic applications in the next section make use of the cocycle properties of $\phi_{n,a}$ for these groups of totally positive units to compute partial zeta values at $s = 0$ in totally real number fields following Shintani's method.

I.4 Application to the computation of partial zeta functions at $s = 0$ for totally real number fields

In this last section we express partial zeta values in totally real number fields at $s = 0$ in terms of the Bernoulli rational functions $B_{n,a_1,\ldots,a_n}$. This gives a connection between the arithmetic of totally real number fields and cocycles extracted from the multiple elliptic Gamma functions. Our approach will use the tools developed by Shintani in [Shi76].

I.4.1 Shintani's method

Let $\mathbb{F}$ be a totally real number field of degree n and denote by $\mathcal{O}_{\mathbb{F}}$ the ring of integers of $\mathbb{F}$. Fix an integral ideal $\mathfrak{f}$ of $\mathcal{O}_{\mathbb{F}}$ which will be the finite part of the class field modulus. For any integral ideal $\mathfrak{b}$ coprime to $\mathfrak{f}$ which represents a class in the narrow ray class group mod $\mathfrak{f}$, the partial zeta function attached to $\mathfrak{f}$ and $\mathfrak{b}$ is defined by:

$$\zeta_{\mathfrak{f}}(\mathfrak{b}, s) := \sum_{\mathfrak{a} \sim \mathfrak{b}} \mathcal{N}(\mathfrak{a})^{-s}$$

Following Siegel [Sie80] we may rewrite this as:

$$\zeta_{\mathfrak{f}}(\mathfrak{b}, s) = \mathcal{N}(\mathfrak{b})^{-s} \sum_{\mu \in (1+\mathfrak{f}\mathfrak{b}^{-1})^{+}/\mathcal{O}_{\mathfrak{f}}^{+,\times}} \mathcal{N}(\mu)^{-s}$$

where $\mathcal{O}_{\mathfrak{f}}^{+,\times}$ is the group of totally positive units of $\mathcal{O}_{\mathbb{F}}$ congruent to 1 mod $\mathfrak{f}$ and $(1+\mathfrak{f}\mathfrak{b}^{-1})^{+}$ is the set of totally positive elements $v \in \mathbb{F}$ such that $v - 1 \in \mathfrak{f}\mathfrak{b}^{-1}$. The fractional ideal $L = \mathfrak{f}\mathfrak{b}^{-1}$ is a lattice of rank n inside $\mathbb{F}$. Shintani's strategy to express the values of these partial zeta functions at integers $k \leq 0$ revolves around finding a fundamental domain for the action of $\mathcal{O}_{\mathfrak{f}}^{+,\times}$ on $(1 + L)^{+}$ which can be decomposed in rational polyhedral cones. Shintani then associates to each of these cones a partial zeta function which admits a meromorphic continuation over $\mathbb{C} - \{1\}$ with values at integers $k \leq 0$ given by specific Bernoulli polynomials. The zeta function associated to a rational polyhedral cone $C \subset \mathbb{F}^{+} \cup \{0\}$ where $\mathbb{F}^{+}$ is the set of totally positive elements in $\mathbb{F}$ and to a vector $v \in \mathbb{F}/L - \{0\}$:

$$\zeta(C, L, v, s) := \sum_{\mu \in C \cap (v+L)} \mathcal{N}(\mu)^{-s}$$

We now prove that the values at $s = 0$ of these zeta functions associated to rational polyhedral cones may be expressed as linear combination of the Bernoulli rational functions $B_{n,a_1,\ldots,a_n}$.

Proposition I.15: *Let $\alpha_1, \ldots, \alpha_m$ be m linearly independent primitive vectors in L where $1 \leq m \leq n$. Suppose that the cone $C = \sum_{j=1}^{m} \mathbb{Q}_{\geq 0}\alpha_j$ is included in $\mathbb{F}^{+} \cup \{0\}$. Fix $v \in \mathbb{F}/L - \{0\}$. Denote by $\sigma_1, \ldots, \sigma_n$ the real embeddings of $\mathbb{F}$. Then:*

$$\zeta(C, L, v, 0) = \frac{1}{n} \sum_{k=1}^{n} h_0(C, v)(0, -\sigma_k)$$

In particular, if $n = m$ and $c^{\vee}(a_1, \ldots, a_n) = C$ with $\operatorname{sign}\det(a_1, \ldots, a_n) = 1$ *then:*

$$\zeta(C, L, v, 0) = \frac{1}{n} \sum_{k=1}^{n} B_{n,a_1,\ldots,a_n}(v)(0, -\sigma_k)$$

Furthermore, this number is the trace of an element in $\mathbb{F}$ and therefore belongs to $\mathbb{Q}$.

Proof :

We follow closely Shintani's original proof of [[Shi76], Theorem 1]. The norm on $v + L$

is a product of affine linear forms with positive coefficients. Namely, we fix a $\mathbb{Z}$-basis $B = [e_1, \ldots, e_n]$ of L consisting of positive elements, and we fix an ordering on the real embeddings $\sigma_1, \ldots, \sigma_n$ of $\mathbb{F}$. If we write $l_{jk} = \sigma_j(e_k) > 0$ then for any $\mu = \sum_{k=1}^{n} \mu_k e_k \in v + L$ we get:

$$\mathcal{N}(\mu) = \prod_{j=1}^{n} \sigma_j(\mu) = \prod_{j=1}^{n} \sum_{k=1}^{n} l_{jk} \mu_k$$

For the rest of this section, $\Gamma(s)$ will denote Euler's Γ function. Following Shintani's original proof [Shi76] we get for any rational polyhedral cone $C \subset \mathbb{F}^+$:

$$\Gamma(s)^n \zeta(C, L, v, s) = \int_0^{+\infty} \cdots \int_0^{+\infty} h(C, v) \left(-\sum_{j=1}^{n} u_j \sigma_j \right) (u_1 \ldots u_n)^{s-1} du_1 \ldots du_n$$

Consider Shintani's decomposition of the positive orthant $\mathbb{R}^n_{>0}$ given by the sets $D_k = \{(u_1, \ldots, u_n) \in \mathbb{R}^n_{\geq 0} \mid u_i \leq u_k, \forall 1 \leq i \leq n\}$. We write the above integrals as integrals over the sets D_k which we handle separately:

$$\Gamma(s)^n \zeta(C, L, v, s) = \sum_{k=1}^{n} \int_{D_k} h(C, v) \left(-\sum_{j=1}^{n} u_j \sigma_j \right) (u_1 \ldots u_n)^{s-1} du_1 \ldots du_n$$

Let us now introduce variables $t, x_1, \ldots, x_n$ such that $t = u_k \in [0, \infty)$ and $t x_i = u_i$ for $i \neq k$ with $x_i \in [0, 1]$. For convenience write $d_k(x, s) = \prod_{j \neq k} x_j^{s-1} dx_j$. Then by a change of variables we get:

$$\Gamma(s)^n \zeta(C, L, v, s) = \sum_{k=1}^{n} \int_0^{+\infty} t^{ns-1} \int_0^1 \cdots \int_0^1 h(C, v) \left(-t \left(\sigma_k + \sum_{\substack{j=1 \\ j \neq k}}^{n} x_j \sigma_j \right) \right) dt d_k(x, s)$$

Let us now isolate the integral over t. Define for $y \in \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{R})$ such that $y(\mathbb{F}^+) \subset \mathbb{R}_{>0}$:

$$\chi(s, y) := \frac{1}{\Gamma(s)} \int_0^{+\infty} h(C, v)(-t.y) t^{ns-1} dt$$

It follows from lemma 3.1 in [Col88] that at $s = 0$:

$$\chi(0, y) = \frac{1}{n} h_0(C, v)(0, -y)$$

(see Definition I.12 for the definition of h_0). The value of $\zeta(C, L, v, s)$ at $s = 0$ is then given by:

$$\zeta(C, L, v, 0) = \lim_{s \to 0} \frac{\Gamma(s)^{-(n-1)}}{n} \sum_{k=1}^{n} \int_0^1 \ldots, \int_0^1 h_0(C, v) \left(0, -\sigma_k - \sum_{j \neq k} x_j \sigma_j \right) d_k(x, s)$$

Now, by definition of h_0 we have :

$$h_0(C, v)(0, -\sigma_k - \sum_{j \neq k} x_j \sigma_j) = \frac{\text{polynomial}}{\prod_{l=1}^{m} (-\sigma_k - \sum_{j \neq k} x_j \sigma_j)(\alpha_l)}$$

and the denominator does not vanish on the integration domain as each α_l belongs to $\mathbb{F}^+$. Therefore, using a variant of [[Col88], Lemma 3.2] we get:

$$\zeta(C, L, v, 0) = \frac{1}{n} \sum_{k=1}^{n} h_0(C, v)(0, -\sigma_k) \tag{I.31}$$

By linearity, this result also applies to open rational polyhedral cones. In the particular case where $n = m$ and $c^\vee(a_1, \ldots, a_n) = C$ with $\operatorname{sign}\det(a_1, \ldots, a_n) = 1$ we may rewrite this result using lemma I.13 as:

$$\zeta(c^\vee(a_1, \ldots, a_n), L, v, 0) = \frac{1}{n} \sum_{k=1}^{n} B_{n,a_1,\ldots,a_n}(v)(0, -\sigma_k) \tag{I.32}$$

It is clear that right-hand side of either (I.31) or (I.32) is the trace of an algebraic number inside $\mathbb{F}$ and therefore lies in $\mathbb{Q}$, which was already obtained by Shintani. □

From this point forward, we may take two slightly different approaches to the computation of the full partial zeta functions at $s = 0$, one of them following closely Shintani's original strategy [Shi76] and the other following the "signed fundamental domain" strategy in [DyDF14].

I.4.2 Shintani's fundamental domain

The first approach to the computation of partial zeta values at $s = 0$ using Proposition I.15 follows closely Shintani's original strategy carried out in [Shi76]. We introduce the set:

$$\overline{D} = \{x \in \mathbb{F}^+ \mid \forall\, u \in \mathcal{O}_{\mathfrak{f}}^{+,\times}, \operatorname{Tr}((u-1)x) \geq 0\}$$

which may be written in our notation as $\overline{D} = c^\vee(a_u, u \in \mathcal{O}_{\mathfrak{f}}^{+,\times})$ where $a_u = \operatorname{Tr}((u-1)\cdot)$. In essence, Shintani proved that there is a finite set $E \subset \mathcal{O}_{\mathfrak{f}}^{+,\times}$ such that in our notation $\overline{D} = c^\vee(a_u, u \in E) c^\vee(a_{u^{-1}}, u \in E)$ and such that the set D whose indicator function is $c^\vee(a_u, u \in E) \prod_{u \in E}(1 - c^\vee(-a_{u^{-1}}))$ constitutes a fundamental domain for the action of $\mathcal{O}_{\mathfrak{f}}^{+,\times}$ on $\mathbb{F}^+$. This already gives the equality:

$$\zeta_{\mathfrak{f}}(\mathfrak{b}, 0) = \zeta(D, \mathfrak{f}\mathfrak{b}^{-1}, 1_{\mathbb{F}}, 0)$$

Then the set D may be decomposed as a finite disjoint union of open rational polyhedral cones $C_j, 1 \leq j \leq m$ which all admit a set of linearly independent generators $\{\alpha_{j,1}, \ldots, \alpha_{j,n_j}\} \subset \mathbb{F}^+$ with $1 \leq n_j \leq n$. Putting this together with the results from Proposition I.15 gives:

$$\zeta_{\mathfrak{f}}(\mathfrak{b}, 0) = \frac{1}{n} \sum_{k=1}^{n} \sum_{j=1}^{m} h_0(C_j, 1_{\mathbb{F}})(0, -\sigma_k). \tag{I.33}$$

Most of the cones C_j are not full-dimensional (case $n_j < n$) and we would like to obtain a result using only full-dimensional cones. To achieve this, we may use any algebraic manipulations on the right-hand side of (I.33) using the properties of h_0 on $\mathcal{K}(V)$ where here $V = L \otimes \mathbb{Q} = \mathbb{F}$. Indeed, if $f \in \mathcal{K}(V)$ is any function congruent to the indicator function of D modulo $\mathcal{L}(V)$, then we may use the linearity of h_0 and its vanishing on $\mathcal{L}(V)$ to conclude that:

$$\zeta_{\mathfrak{f}}(\mathfrak{b},0) = \frac{1}{n}\sum_{k=1}^{n} h_0(f,1_{\mathbb{F}})(0,-\sigma_k)$$

In particular, in favorable cases we may linearise the indicator function of D and show that it belongs to the set $\mathcal{L}(V)+\mathcal{K}^n(V)$ where $\mathcal{K}^n(V)$ is the subspace of $\mathcal{K}(V)$ spanned by the functions $c^\vee(a_1,\dots,a_n)$ for linearly independent $a_1,\dots,a_n$ in Λ. This linearisation expresses the indicator function of D modulo $\mathcal{L}(V)$ in terms of functions $c^\vee(a_{j,1},\dots,a_{j,n})$ where each of the $a_{j,l}$'s are equal to some $\pm a_k$. This gives the following:

Proposition I.16: *There is an integer $N\geq 1$ and there are units $u_{j,l}\in E$ as well as signs $\epsilon_{j,l},\eta_{j,l}\in\{\pm 1\}$ for $1\leq j\leq N$ and $1\leq l\leq n$ such that:*

$$\zeta_{\mathfrak{f}}(\mathfrak{b},0) = \frac{1}{n}\sum_{k=1}^{n}\sum_{j=1}^{N} \operatorname{sign}\det(a_{j,1},\dots,a_{j,n}) B_{n,a_{j,1},\dots,a_{j,n}}(1_{\mathbb{F}})(0,-\sigma_k)$$

where $a_{j,l}=\epsilon_{j,l}\operatorname{Tr}((u_{j,l}^{\eta_{j,l}}-1)\cdot)$.

Example: Consider the real quadratic field $\mathbb{F}=\mathbb{Q}(\sqrt{19})$. Fix $\mathfrak{f}=(13)$ and $\mathfrak{b}=(1)$. Denote by σ_1,σ_2 the real embeddings of $\mathbb{F}$. The group of totally positive units congruent to 1 mod $\mathfrak{f}$ is generated by $\varepsilon=170+39\sqrt{19}$. A possible totally positive basis for $L=\mathfrak{f}\mathfrak{b}^{-1}$ is given by $B=[13,65+13\sqrt{19}]$. In this situation we have $D=c^\vee(a_1)(1-c^\vee(a_{-1}))$ where $a_j=\operatorname{Tr}((\varepsilon^j-1)\cdot)$. Reducing modulo $\mathcal{L}(V)$ gives $D\equiv -c^\vee(a_1,a_{-1})$ mod $\mathcal{L}(V)$. Therefore:

$$\zeta_{\mathfrak{f}}(\mathfrak{b},0) = -\frac{1}{2}\sum_{k=1}^{2}\operatorname{sign}\det(a_1,a_{-1})B_{2,a_1,a_{-1}}(1_{\mathbb{F}})(0,-\sigma_k)$$

We compute explicitly in the basis B the coordinates of the linear forms a_1 and a_{-1}:

$$a_1 = 338.[13,122],\quad a_{-1}=338.[13,8]$$

where we have factored in the gcd's of the coefficients. We carried out the computations using the computer software Pari/GP [The24] and found:

$$-\frac{1}{2}\sum_{k=1}^{2}\operatorname{sign}\det(a_1,a_{-1})B_{2,a_1,a_{-1}}(1_{\mathbb{F}})(0,-\sigma_k) = \operatorname{Tr}\left(\frac{33}{104}\right) = \frac{33}{52}$$

which we may check is the value of $\zeta_{(13)}((1),0)$ using for instance Pari/GP's **bnrL1** command.

Proposition I.16 is already a great way to express the partial zeta values in terms of these Bernoulli polynomials $B_{n,a_1,\dots,a_n}$ related to the multiple elliptic Gamma functions. However, the explicit computation of this decomposition is quite tedious in general. In addition, the linear forms involved are not quite of the form we hoped for following the discussion at the end of section I.3.3. Indeed, we would like to evaluate the Bernoulli polynomials on a cycle whose shape would resemble the cycle described in [[CDG15], section 2.6] which corresponds to the sign fundamental domain decomposition in [DyDF14]. This is the second approach which we carry out in the next section.

I.4.3 Signed fundamental domains

We now prove Theorem I.3 using the second approach following [Col88], [DyDF14]. This approach has two benefits: on the one hand the "signed fundamental domain" decomposition described in [DyDF14] is easier to compute than the Shintani decomposition. On the other hand, the cones involved may be interpreted in terms of algebraic cycles (see [CDG15]).

Proof of Theorem I.3:

Recall that $\mathbb{F}$ is a totally real number field of degree n, that $\mathfrak{f}$ is an integral ideal in $\mathcal{O}_{\mathbb{F}}$ and that the integral ideal $\mathfrak{b}$ represents a class in the narrow ray class group at $\mathfrak{f}$. Denote by $\sigma_1, \dots, \sigma_n$ the real embeddings of $\mathbb{F}$. We wish to prove that:

$$\zeta_{\mathfrak{f}}(\mathfrak{b}, 0) = \frac{1}{n} \sum_{k=1}^{n} \sum_{\rho \in \mathfrak{S}_{n-1}} \nu_\rho B_{n, a_{1,\rho}, \dots, a_{n,\rho}}(1_{\mathbb{F}})(0, -\sigma_k)$$

where the ν_ρ's are signs in $\{-1, 0, +1\}$ and the $a_{j,\rho}$'s are $\mathbb{Q}$-linear forms on $\mathbb{F}$. To achieve this we will recall the notations from [DyDF14] to describe their signed fundamental domain for the action of $\mathcal{O}_{\mathfrak{f}}^{+,\times}$ on $L = \mathfrak{f}\mathfrak{b}^{-1}$. Let $\varepsilon_1, \dots, \varepsilon_{n-1}$ be fundamental units for $\mathcal{O}_{\mathfrak{f}}^{+,\times}$. For any permutation $\rho \in \mathfrak{S}_{n-1}$ and any index $1 \leq i \leq n$ define the elements

$$f_{i,\rho} = \prod_{j=1}^{i-1} \varepsilon_{\rho(j)}.$$

We use cones C_ρ with generators $f_{1,\rho}, \dots, f_{n,\rho}$, and we now describe how the boundaries of the cones are chosen. Let us consider the set S of permutations $\rho \in \mathfrak{S}_{n-1}$ such that $f_{1,\rho}, \dots, f_{n,\rho}$ are linearly independent over $\mathbb{Q}$. Let us consider the canonical embedding of $\mathbb{F}$ into $\mathbb{R}^n$ given by:

$$\sigma := \begin{cases} \mathbb{F} & \to \mathbb{R}^n \\ v & \to (\sigma_1(v), \dots, \sigma_n(v)) \end{cases}$$

For any permutation $\rho \in S$ and any index $1 \leq i \leq n$ we define the signs $\mu_{i,\rho} \in \{-1, +1\}$ by the formula:

$$\mu_{i,\rho} = \frac{\det(\sigma(f_{1,\rho}), \dots, \sigma(f_{i-1,\rho}), e_n, \sigma(f_{i+1,\rho}), \dots, \sigma(f_{n,\rho}))}{\det(\sigma(f_{1,\rho}), \dots, \sigma(f_{n,\rho}))}$$

where the determinants are taken in the canonical basis of $\mathbb{R}^n$ and $e_n = [0, \dots, 0, 1]^t$ is the last vector of this basis. Then we may define as in [DyDF14] the sets:

$$\mathbb{R}_{i,\rho} := \begin{cases} [0, +\infty) & \text{if } \mu_{i,\rho} > 0 \\ (0, +\infty) & \text{if } \mu_{i,\rho} < 0 \end{cases}$$

Let us then define the cones $C_\rho = \sum_{i=1}^{n} \mathbb{R}_{i,\rho} f_{i,\rho}$ for $\rho \in S$ and $C_\rho = \sum_{i=1}^{n} \mathbb{R}_{\geq 0} f_{i,\rho}$ if $\rho \notin S$. We will denote by c_ρ the indicator function of the set C_ρ. It follows from [[DyDF14], Theorem 1] that there are explicit signs $w_\rho \in \{-1, 0, 1\}$ for $\rho \in S$ such that the following equality holds in $\mathcal{K}(V)$:

$$\sum_{\rho \in S} w_\rho \sum_{u \in \mathcal{O}_{\mathfrak{f}}^{+,\times}} c_\rho(u \cdot) = \chi_{\mathbb{F}^+}$$

where $V = \mathbb{F} \simeq \mathbb{Q}^n$ and $\chi_{\mathbb{F}^+}$ is the indicator function of $\mathbb{F}^+$. This already gives the relation for the partial zeta function:

$$\zeta_{\mathfrak{f}}(\mathfrak{b}, s) = \mathcal{N}(\mathfrak{b})^{-s} \sum_{\rho \in S} w_\rho \, \zeta(C_\rho, L, 1_{\mathbb{F}}, s)$$

and specialising at $s = 0$ yields by Proposition I.15:

$$\zeta_{\mathfrak{f}}(\mathfrak{b}, 0) = \frac{1}{n} \sum_{k=1}^{n} \sum_{\rho \in S} w_\rho \, h_0(C_\rho, 1_{\mathbb{F}})(0, -\sigma_k).$$

We may rephrase this by describing C_ρ in terms of linear forms. Indeed, let us define for any $\rho \in S$ and any index $1 \leq i \leq n$:

$$b_{i,\rho} = \frac{\det(f_{1,\rho}, \ldots, \widehat{f_{i,\rho}}, \ldots, f_{n,\rho})}{\det(f_{1,\rho}, \ldots, f_{i,\rho}, \ldots, f_{n,\rho})}$$

where the determinant is taken relative to any $\mathbb{Q}$-basis of $\mathbb{F}$. In other words, $b_{i,\rho}$ is the linear form on $\mathbb{F}$ satisfying $b_{i,\rho}(f_{i,\rho}) = 1$ and $b_{i,\rho}(f_{j,\rho}) = 0$ if $j \neq i$. For any $\rho \in S$ we may split the indices $i \in \{1, \ldots, n\}$ into two sets depending on the value of $\mu_{i,\rho}$ by setting $I_\rho = \{1 \leq i \leq n \mid \mu_{i,\rho} > 0\}$ and $J_\rho = \{1, \ldots, n\} - I_\rho$. This gives the following expression for c_ρ:

$$c_\rho = \prod_{i \in I_\rho} c^\vee(b_{i,\rho}) \prod_{j \in J_\rho} (1 - c^\vee(-b_{j,\rho}))$$

It is then easy to see that we have the following reduction modulo $\mathcal{L}(V)$:

$$c_\rho \equiv (-1)^{\# J_\rho} \prod_{i \in I_\rho} c^\vee(b_{i,\rho}) \prod_{j \in J_\rho} c^\vee(-b_{j,\rho}) \bmod \mathcal{L}(V).$$

We therefore define $a_{i,\rho}$ to be unique primitive element in $\Lambda = \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$ such that $a_{i,\rho} = \lambda_{i,\rho} \mu_{i,\rho} b_{i,\rho}$ with $\lambda_{i,\rho} \in \mathbb{Q}_{>0}$. Using the linearity of h_0 and its vanishing on $\mathcal{L}(V)$ we get the following relation:

$$\zeta_{\mathfrak{f}}(\mathfrak{b}, 0) = \frac{1}{n} \sum_{k=1}^{n} \sum_{\rho \in S} w_\rho (-1)^{\# J_\rho} h_0(c^\vee(a_{1,\rho}, \ldots, a_{n,\rho}), 1_{\mathbb{F}})(0, -\sigma_k).$$

Identifying the right-hand side using lemma I.13 gives the desired result:

$$\zeta_{\mathfrak{f}}(\mathfrak{b}, 0) = \frac{1}{n} \sum_{k=1}^{n} \sum_{\rho \in \mathfrak{S}_{n-1}} \nu_\rho B_{n, a_{1,\rho}, \ldots, a_{n,\rho}}(1_{\mathbb{F}})(0, -\sigma_k)$$

where $\nu_\rho = w_\rho (-1)^{\# J_\rho} \mathrm{sign} \det(a_{1,\rho}, \ldots, a_{n,\rho})$ if $\rho \in S$ and $\nu_\rho = 0$ otherwise. Note that the signs w_ρ and the cones C_ρ appearing in the signed fundamental domain decomposition in [DyDF14] are explicitly computable and so are the linear forms $a_{i,\rho}$ as well as the sets J_ρ. Using the explicit definition of the $B_{n, a_{1,\rho}, \ldots, a_{n,\rho}}$ we may rewrite the right-hand side as the trace of an element in $\mathbb{F}$ which implies that $\zeta_{\mathfrak{f}}(\mathfrak{b}, 0) \in \mathbb{Q}$, as was previously known from the theorem of Klingen and Siegel. □

This second expression of the partial zeta functions using the signed fundamental domain from [DyDF14] is closer to what we had in mind in the discussion carried out in section I.3.3 as the cones C_ρ admit generators $f_{i,\rho}$ which may be described in terms of an algebraic cycle (see for instance [CDG15]). Yet, we would like the linear forms $a_{i,\rho}$ to be described in terms of similar algebraic cycles. Interesting future work would be to somehow construct a "dual cycle" to the cycle presented in [[Scz93], lemma 5] to be evaluated against the Bernoulli polynomials cocycle. Namely, we wish to construct a linear form $a \in \Lambda$ and define the corresponding cones:

$$c'_\rho = c^\vee(f_{1,\rho} \cdot a, \dots, f_{n,\rho} \cdot a)$$

for any permutation $\rho \in \mathfrak{S}_{n-1}$ where we recall that the action on $V^\vee$ is defined by $(g \cdot a)(v) = a(g^{-1}v)$. It would hopefully then be possible to find coefficients $\nu'_\rho \in \mathbb{Q}$ such that the linear combination $\sum_{\rho \in \mathfrak{S}_{n-1}} \nu'_\rho c'_\rho$ is congruent to a signed fundamental domain modulo $\mathcal{L}(V)$. This would then lead to a relation of the form:

$$\zeta_{\mathfrak{f}}(\mathfrak{b}, 0) = \frac{1}{n} \sum_{k=1}^{n} \sum_{\rho \in \mathfrak{S}_{n-1}} \nu'_\rho B_{n, f_{1,\rho} \cdot a, \dots, f_{n,\rho} \cdot a}(1_{\mathbb{F}})(0, -\sigma_k)$$

which we may rewrite using the cocycle $\phi_{n,a}$ on the subgroup of $\mathrm{SL}_n(\mathbb{Z})$ corresponding to the torus $\mathcal{O}_{\mathfrak{f}}^{+,\times}$ as:

$$\zeta_{\mathfrak{f}}(\mathfrak{b}, 0) = \frac{1}{n} \sum_{k=1}^{n} \sum_{\rho \in \mathfrak{S}_{n-1}} \nu'_\rho \phi_{n,a}(\varepsilon_{\rho(1)}, \dots, \varepsilon_{\rho(n)})(1_{\mathbb{F}})(0, -\sigma_k).$$

In the real quadratic case we may carry out this last approach by setting $a = \det(1_{\mathbb{F}}, \cdot)/\det(1_{\mathbb{F}}, \varepsilon)$ where ε is a generator for $\mathcal{O}_{\mathfrak{f}}^{+,\times}$. This gives the expression:

$$\zeta_{\mathfrak{f}}(\mathfrak{b}, 0) = -\frac{1}{2} \sum_{k=1}^{2} \phi_{2,a}(\varepsilon)(1_{\mathbb{F}})(0, -\sigma_k)$$

and therefore the partial zeta values at $s = 0$ are given by the evaluation of a partial 1-cocycle for $\mathrm{SL}_2(\mathbb{Z})$ against a 1-cycle arising from $\mathcal{O}_{\mathfrak{f}}^{+,\times}$.

As a last remark on Theorem I.3, we stress that it is a reformulation of Shintani's result, borrowing ideas from [Col88], [DyDF14] and [CDG15], and that there have been many other approaches to the computation of partial zeta values at non-positive integers in totally real fields (see [Bek24] or [CGS00] for instance). We argue that the novelty of our work lies in the extraction of arithmetic quantities such as partial zeta values in totally real fields at $s = 0$ from the study of the geometric families of $G_{n-2,a_1,\dots,a_{n-1}}$ functions via the associated geometric families of Bernoulli rational functions $B_{n,a_1,\dots,a_n}$. We will now give two examples of computations of partial zeta values at $s = 0$ in real cubic fields following the procedure given in the proof of Theorem I.3. All the computations were carried out using the computer software Pari/GP [The24].

I.4.4 Cubic examples

I.4.4.1 First real cubic example

We now carry out our procedure to recover an example from [CGS00]. Consider the real cubic field $\mathbb{F} = \mathbb{Q}(z)$ where z is a root of the polynomial $x^3 - x^2 - 4x - 1$. Fix $\mathfrak{f} = (5)$

and $\mathfrak{b} = (1)$. Fix the basis $B = [5, 5z + 10, 5z^2 - 5z]$ of $L = \mathfrak{f}\mathfrak{b}^{-1}$. A possible choice of fundamental units for $\mathcal{O}_{\mathfrak{f}}^{+,\times}$ is given by:

$$\varepsilon_1 = 15z^2 + 25z + 6 \ \ ; \ \ \varepsilon_2 = -15z^2 + 20z + 56$$

Write $\mathfrak{S} = \{Id, (12)\}$ and compute as in the proof of Theorem I.3 the signs $\mu_{i,\rho}$:

$\mu_{1,\mathrm{Id}} = -1$	$\mu_{2,\mathrm{Id}} = +1$	$\mu_{3,\mathrm{Id}} = -1$
$\mu_{1,(12)} = +1$	$\mu_{2,(12)} = -1$	$\mu_{3,(12)} = +1$

This readily gives $J_{\mathrm{Id}} = \{1, 3\}$ and $J_{(12)} = \{2\}$. The explicit cones described in the proof of Theorem I.3 are:

$$\begin{aligned} C_{\mathrm{Id}} &= \mathbb{R}_{>0}1_{\mathbb{F}} + \mathbb{R}_{\geq 0}\varepsilon_1 + \mathbb{R}_{>0}\varepsilon_1\varepsilon_2 = (1 - c^{\vee}(-b_{1,\mathrm{Id}}))c^{\vee}(b_{2,\mathrm{Id}})(1 - c^{\vee}(-b_{3,\mathrm{Id}})) \\ C_{(12)} &= \mathbb{R}_{\geq 0}1_{\mathbb{F}} + \mathbb{R}_{>0}\varepsilon_2 + \mathbb{R}_{\geq 0}\varepsilon_1\varepsilon_2 = c^{\vee}(b_{1,(12)})(1 - c^{\vee}(-b_{2,(12)}))c^{\vee}(b_{3,(12)}) \end{aligned}$$

where the linear forms $b_{i,\rho}$ are given on the basis B by:

$b_{1,\mathrm{Id}} = \frac{1}{7}[35, 22, 114]$	$b_{2,\mathrm{Id}} = \frac{1}{7}[0, -10, 29]$	$b_{3,\mathrm{Id}} = \frac{1}{7}[0, 3, -8]$
$b_{1,(12)} = \frac{1}{97}[485, 302, 1588]$	$b_{2,(12)} = \frac{1}{97}[0, 10, -29]$	$b_{3,(12)} = \frac{1}{97}[0, 3, 1]$

The corresponding signs w_ρ given in [DyDF14] are $w_{\mathrm{Id}} = w_{(12)} = 1$. The complete signs ν_ρ are $\nu_{\mathrm{Id}} = -1$ and $\nu_{(12)} = 1$. Following the proof of Theorem I.3 we set $a_{i,\mathrm{Id}} = 7\mu_{i,\mathrm{Id}}b_{i,\mathrm{Id}}$ and $a_{i,(12)} = 97\mu_{i,(12)}b_{i,(12)}$ for $1 \leq i \leq 3$. We may then compute using formula (I.18):

$$R_1 = -\sum_{k=1}^{3} B_{3,a_{1,\mathrm{Id}},a_{2,\mathrm{Id}},a_{3,\mathrm{Id}}}(1_{\mathbb{F}})(0, -\sigma_k) = \mathrm{Tr}_{\mathbb{F}/\mathbb{Q}}\left(\frac{1975z^2 - 4525z - 1424}{120}\right) = \frac{4489}{60}$$

$$R_2 = \sum_{k=1}^{3} B_{3,a_{1,(12)},a_{2,(12)},a_{3,(12)}}(1_{\mathbb{F}})(0, -\sigma_k) = \mathrm{Tr}_{\mathbb{F}/\mathbb{Q}}\left(\frac{-1975z^2 + 4525z + 1448}{120}\right) = \frac{-4453}{60}$$

It follows from Theorem I.3 that

$$\zeta_{\mathfrak{f}}(\mathfrak{b}, 0) = \frac{1}{3}(R_1 + R_2) = \frac{1}{5}$$

which recovers the result in [CGS00].

I.4.4.2 Second real cubic example

We now study an example with a modulus $\mathfrak{f}$ which is not of the form $N\mathcal{O}_{\mathbb{F}}$ for some rational integer $N \geq 1$. Consider the real cubic field $\mathbb{F} = \mathbb{Q}(z)$ where z is a root of the polynomial $x^3 - x^2 - 6x + 3$. Fix $\mathfrak{f} = (1-z)\mathcal{O}_{\mathbb{F}}$ the unramified prime ideal above 3 in $\mathcal{O}_{\mathbb{F}}$ and $\mathfrak{b} = (1)$. Fix the basis $B = [3, x+5, x^2+2]$ of $L = \mathfrak{f}\mathfrak{b}^{-1}$. A possible choice of fundamental units for $\mathcal{O}_{\mathfrak{f}}^{+,\times}$ is given by:

$$\varepsilon_1 = -4z^2 + z + 28 \quad ; \quad \varepsilon_2 = -3z^2 + 3z + 22$$

The explicit cones described in the proof of Theorem I.3 are:

$$\begin{aligned} C_{\mathrm{Id}} &= \mathbb{R}_{\geq 0}1_{\mathbb{F}} + \mathbb{R}_{>0}\varepsilon_1 + \mathbb{R}_{\geq 0}\varepsilon_1\varepsilon_2 = c^{\vee}(b_{1,\mathrm{Id}})(1 - c^{\vee}(-b_{2,\mathrm{Id}}))c^{\vee}(-b_{3,\mathrm{Id}}) \\ C_{(12)} &= \mathbb{R}_{>0}1_{\mathbb{F}} + \mathbb{R}_{\geq 0}\varepsilon_2 + \mathbb{R}_{>0}\varepsilon_1\varepsilon_2 = (1 - c^{\vee}(-b_{1,(12)}))c^{\vee}(b_{2,(12)})(1 - c^{\vee}(-b_{3,(12)})) \end{aligned}$$

We directly give the signs $\nu_{\mathrm{Id}} = 1$ and $\nu_{(12)} = -1$ and the expression of the $a_{i,\rho}$'s on the basis B as:

$a_{1,\mathrm{Id}} = [108, 280, 349]$	$a_{2,\mathrm{Id}} = [0, 25, 13]$	$a_{3,\mathrm{Id}} = [0, 4, 1]$
$a_{1,(12)} = [-432, -395, -1019]$	$a_{2,(12)} = [0, 25, 13]$	$a_{3,(12)} = [0, 1, 1]$

Thus we may compute using formula (I.18) :

$$\begin{aligned} R_1 &= \sum_{k=1}^{3} B_{3,a_{1,\mathrm{Id}},a_{2,\mathrm{Id}},a_{3,\mathrm{Id}}}(1_{\mathbb{F}})(0, -\sigma_k) = \mathrm{Tr}_{\mathbb{F}/\mathbb{Q}}\left(\frac{3z^2 - 7}{6}\right) = 3 \\ R_2 &= -\sum_{k=1}^{3} B_{3,a_{1,(12)},a_{2,(12)},a_{3,(12)}}(1_{\mathbb{F}})(0, -\sigma_k) = \mathrm{Tr}_{\mathbb{F}/\mathbb{Q}}\left(\frac{-3z^2 + 11}{6}\right) = -1 \end{aligned}$$

It follows from Theorem I.3 that

$$\zeta_{\mathfrak{f}}(\mathfrak{b}, 0) = \frac{1}{3}(R_1 + R_2) = \frac{2}{3}$$

which can be verified using Pari/GP's **bnrL1** command for instance.

I.5 Discussion of chapter I

In this section we discuss some aspects of chapter I which will be useful for the remaining chapters.

I.5.1 G_r functions with a few real algebraic parameters

In chapter III we will evaluate the higher elliptic Gamma functions at points in a degree $r+2$ number field with exactly one complex place (see (III.2)). In most cases, it will be clear that these evaluations are well-defined, but when the number field is of even degree and contains a real subfield, it might happen that some of the parameters τ_j lie in $\mathbb{R}$. In this section, we describe how to handle this case. Namely, we prove the following:

Proposition I.17: *Consider $r+1$ non-zero complex numbers $\tau_0, \ldots, \tau_r$. Assume that, up to rearrangement of the terms, $\tau_0, \ldots, \tau_l$ belong to $\mathbb{C} - \mathbb{R}$ and that $\tau_{l+1}, \ldots, \tau_r$ are real algebraic irrational numbers, where $1 \leq l \leq r$. Then the function*

$$G_r(z, \tau_0, \ldots, \tau_r) = \begin{cases} \exp\left(\sum_{j\geq 1} \frac{1}{(2i)^r j} \frac{\sin(\pi j(2z-(\tau_0+\cdots+\tau_r)))}{\prod_{k=0}^r \sin(\pi j \tau_k)}\right) & \text{if } r \text{ is odd} \\ \exp\left(\sum_{j\geq 1} \frac{2}{(2i)^{r+1} j} \frac{\cos(\pi j(2z-(\tau_0+\cdots+\tau_r)))}{\prod_{k=0}^r \sin(\pi j \tau_k)}\right) & \text{if } r \text{ is even} \end{cases} \tag{I.34}$$

is well-defined when $|\Im(2z - (\tau_0 + \cdots + \tau_r))| < \sum_{j=0}^r |\Im(\tau_j)|$.

Proof :

The result is an application of Liouville's classic theorem on the "bad approximation" of real algebraic irrational numbers. Let us denote by n_k the degree of each of the τ_k for $l+1 \leq k \leq r$. Then there is a constant $C_k > 0$ such that for any rational number p/q:

$$\left|\tau_k - \frac{p}{q}\right| \geq \frac{C_k}{q^{n_k}}.$$

In particular, for any $j \geq 1$ and any integer $p \in \mathbb{Z}_{>0}$,

$$|\pi j \tau_k - \pi p| \geq \pi \frac{C_k}{j^{n_k - 1}}$$

and thus

$$\frac{1}{\sin \pi j \tau_k} = O(j^{n_k - 1})$$

as $j \to \infty$. Let us write $y = \exp(i\pi(2z - (\tau_0 + \cdots + \tau_r)))$, so that (I.34) may be written as:

$$G_r(z, \tau_0, \ldots, \tau_r) = \exp\left(\sum_{j\geq 1} \frac{1}{(2i)^{r+1} j} \frac{y + (-1)^r y^{-1}}{\prod_{k=0}^r \sin(\pi j \tau_k)}\right).$$

The general term of this infinite sum

$$b_j = \frac{1}{(2i)^{r+1} j} \frac{y^j + (-1)^r y^{-j}}{\prod_{k=0}^r \sin(\pi j \tau_k)}$$

satisfies

$$b_j = O\left(j^{-1} \prod_{k=l+1}^r j^{n_k - 1} \frac{\max(|y|, |y|^{-1})^j}{\prod_{k=0}^l \sin(\pi j \tau_k)}\right).$$

Since for $0 \leq k \leq l$, $\sin \pi j \tau_k = O(e^{i\pi|\Im(\tau_k)|})$ we get the estimation:

$$b_j = O\left(j^{n'} \exp\left(\pi j \left(|\Im(2z - (\tau_0 + \cdots + \tau_r))| - \sum_{j=0}^r |\Im(\tau_j)|\right)\right)\right)$$

where $n' = -1 + \sum_{k=l+1}^r (n_k - 1)$. By assumption, the exponential factor is of absolute value less than 1, therefore the sum $\sum_{j\geq 1} b_j$ is absolutely convergent and the complex number $G_r(z, \tau_0, \ldots, \tau_r)$ is well-defined. □

We add, as a general remark, that using the properties of the G_r functions (see section I.2.1.1) one may extend the domain in which the function $G_r(z, \tau_0, \ldots, \tau_r)$ is defined in this case where some of the parameters τ_j are real algebraic irrational numbers, that is for z outside of the range $|\Im(2z - (\tau_0 + \cdots + \tau_r))| < \sum_{j=0}^r |\Im(\tau_j)|$ (and of course z avoiding the poles of the function).

Chapter II

Smoothed functions and cocycle properties for congruence subgroups in $\mathrm{SL}_n(\mathbb{Z})$

II.1 Introduction to chapter II

In this chapter, we are interested in the smoothing operation on the G_r functions which generalises the smoothed $\theta^{(N)}$ function

$$\theta^{(N)} = \frac{\theta(1/q,\tau)^N}{\theta(N/q,N\theta)}.$$

Recall that this function satisfied the modular property:

$$\theta^{(N)}\left(\frac{z}{c\tau+d},\frac{a\tau+b}{c\tau+d}\right) = \theta^{(N)}(z,\tau).e^{2i\pi P_{2,\gamma}^{(N)}(z,\tau)} \tag{II.1}$$

for any $\gamma \in \Gamma_0(N)$ where

$$P_{2,\gamma}^{(N)}(z,\tau) = \begin{cases} 0 & \text{if } c=0 \text{ and } d=1 \\ \frac{1-N}{2} & \text{if } c=0 \text{ and } d=-1 \\ \mathrm{sign}(c)\left(s\left(\frac{|c|}{N},d\right) - Ns(|c|,d) + \frac{1-N}{4}\right) & \text{if } c\neq 0 \end{cases}$$

is valued in $\frac{1}{12}\mathbb{Z}$ and depends on $\gamma \in \Gamma_0(N)$ but not on z and τ (see chapter 0).

Our goal in this chapter is to generalise this smoothing operation to our higher elliptic Gamma functions and prove that a similar simpler smoothed modular property holds. To state the main theorem in this chapter we now introduce some notations relative to this smoothing operation. For the rest of this chapter, we fix a rank $n \geq 2$ lattice L with a $\mathbb{Z}$-basis $B = [e_1,\ldots,e_n]$ as well as an integer $N \geq 2$. The lattice L' generated by the $\mathbb{Z}$-basis $B' = [Ne_1, e_2, \ldots, e_n]$ is called the smoothing lattice. We denote by Λ (resp. Λ') the dual space $\mathrm{Hom}_{\mathbb{Z}}(L,\mathbb{Z})$ (resp. $\mathrm{Hom}_{\mathbb{Z}}(L',\mathbb{Z})$) of L (resp. L') and define

$$\Lambda_N = \{a \in \Lambda \mid a_{|L'} \text{ is primitive in } \Lambda'\}$$

where we recall that an element $a \in \Lambda$ is primitive if $a/d \in \Lambda$ for some integer d implies $d = \pm 1$. The set Λ_N is naturally endowed with an action of the following congruence subgroup in $\mathrm{SL}_n(\mathbb{Z})$:

$$\Gamma_0(N,n) = \left\{ g \in \mathrm{SL}_n(\mathbb{Z}) \;\middle|\; g \equiv \begin{pmatrix} * & * & \dots & * \\ 0 & * & \dots & * \\ \vdots & \vdots & \vdots & \vdots \\ 0 & * & \dots & * \end{pmatrix} \bmod N \right\}, \tag{II.2}$$

the action being given by right multiplication by the inverse as $g \cdot a = a \times g^{-1}$. The congruence group $\Gamma_0(N,n)$ also acts on L and L' by left multiplication. When the linear forms $a_1, \dots, a_n$ belong to Λ_N it makes sense to introduce the smoothed functions:

$$G_{n-2,a_1,\dots,a_{n-1}}(v)(w,x,L,L') = \frac{G_{n-2,a_1,\dots,a_{n-1}}(v)(w,x,L')^N}{G_{n-2,a_1,\dots,a_{n-1}}(v)(w,x,L)}, \tag{II.3}$$

$$B_{n,a_1,\dots,a_n}(v)(w,x,L,L') = NB_{n,a_1,\dots,a_n}(v)(w,x,L') - B_{n,a_1,\dots,a_n}(v)(w,x,L). \tag{II.4}$$

We deduce immediately from Theorem I.1 that these functions are equivariant under the action of $\Gamma_0(N,n)$ as for all $g \in \Gamma_0(N,n)$:

$$G_{n-2,g\cdot a_1,\dots,g\cdot a_{n-1}}(g\cdot v)(w,g\cdot x,L,L') = G_{n-2,a_1,\dots,a_{n-1}}(v)(w,x,L,L') \tag{II.5}$$

$$B_{n,g\cdot a_1,\dots,g\cdot a_n}(g\cdot v)(w,g\cdot x,L,L') = B_{n,a_1,\dots,a_n}(v)(w,x,L,L'). \tag{II.6}$$

It also follows at once from Theorem I.1 that these smoothed functions satisfy the coboundary relation:

$$\prod_{j=1}^{n} G_{n-2,a_1,\dots,\widehat{a_j},\dots,a_n}(v)(w,x,L,L')^{(-1)^{j+1}} = \exp(2i\pi B_{n,a_1,\dots,a_n}(v)(w,x,L,L')) \tag{II.7}$$

for linearly independent $a_1, \dots, a_n \in \Lambda_N$. From Corollary I.1 we also obtain the cocycle relation:

$$\sum_{j=0}^{n} (-1)^j B_{n,a_0,\dots,\widehat{a_j},\dots,a_n}(v)(w,x,L,L') = 0 \tag{II.8}$$

for most configurations of the linear forms $a_0, \dots, a_n \in \Lambda_N$.

In this chapter we derive from relation (II.7) a cocycle relation for the smoothed geometric functions $G_{n-2,a_1,\dots,a_{n-1}}(v)(w,x,L,L')$, turning the function

$$a_1, \dots, a_{n-1} \to G_{n-2,a_1,\dots,a_{n-1}}(v)(w,x,L,L')$$

into a partial modular symbol for $\Gamma_0(N,n)$. The main result in this chapter is expressed under two conditions on the positions of the linear forms $a_1, \dots, a_n$ in Λ relative to the smoothing lattice L'. First, we shall say that $a_1, \dots, a_n$ are well placed (in $V^\vee$) if either $\mathrm{rk}(a_1, \dots, a_n) \neq n-1$ or if $\mathrm{rk}(a_1, \dots, a_n) = n-1$ and 0 is not a barycenter of $a_1, \dots, a_n$ in $V^\vee$ (see Definition II.4 for more details on this hypothesis). The second condition concerns the position of the lattice L' in the case where $\mathrm{rk}(a_1, \dots, a_n) = n$. When $a_1, \dots, a_n$ are linearly independent, there are unique primitive elements $\alpha_1, \dots, \alpha_n$ in L such that

$$a_j(\alpha_j) > 0 \quad \text{and} \quad a_j(\alpha_k) = 0, \forall\, k \neq j.$$

We shall say in the spirit of [Das08] that the index N smoothing lattice L' is *good* for the linear forms $a_1, \dots, a_n$ if and only if $a_1, \dots, a_n \in \Lambda_N$ and either $\mathrm{rk}(a_1, \dots, a_n) < n$ or

$\mathrm{rk}(a_1, \ldots, a_n) = n$ and for any $1 \leq j \leq n$, $\alpha_j \bmod L'$ is a generator of the cyclic group L/L' (see Definition II.18). Our main result is the following:

Theorem II.1: *Suppose that $a_1, \ldots, a_n \in \Lambda$ are non-zero linear forms which are well placed in $V^\vee$ and assume that the smoothing lattice L' is good for $a_1, \ldots, a_n$ where $n, N \geq 2$. Then there is an integer $b = b(a_1, \ldots, a_n, v) \in \mathbb{Z}$ which depends on the linear forms $a_1, \ldots, a_n$ and on the class of v in V/L' but not on $w, x \in \mathbb{C} \times \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{C})$ such that for all $(v, w, x) \in V/L' \times \mathbb{C} \times \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{C})$:*

$$\prod_{j=1}^{n} G_{n-2,a_1,\ldots,\widehat{a_j},\ldots,a_n}(v)(w, x, L, L')^{(-1)^{j+1}} = \exp\left(\frac{2i\pi b}{\mathcal{D}(N, n)}\right) \tag{II.9}$$

where $\mathcal{D}(N, n) = \prod_{p|N} p^{\lfloor \frac{n}{p-1} \rfloor}$.

Formula (II.9) is an analogue for general $n \geq 2$ of formula (0.15), which we may recover as follows. Fix $L = \mathbb{Z}e_1 + \mathbb{Z}e_2$ and $x \in \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{C})$ satisfying $x(e_1) = \tau$ and $x(e_2) = 1$. Write $a_1 = [1, 0]$ for the linear form on L satisfying $a_1(e_1) = 1$ and $a_1(e_2) = 0$, as well as $a_2 = [d, -c]$ for the linear form satisfying $a_2(e_1) = d$ and $a_2(e_2) = -c$. Then, identifying $\theta = G_0$ we may write in our notations for $v_0 = 0$:

$$\theta_{[1,0]}(v_0)(z, x) = \theta(z, \tau) \quad \text{and} \quad \theta_{[d,-c]}(v_0)(z, x) = \theta\left(\frac{z}{c\tau + d}, \frac{a\tau + b}{c\tau + d}\right)$$

and the integer $b(a_1, a_2, v_0) = b([1, 0], [d, -c], 0)$ is given by the formula:

$$P_{2,\gamma}^{(N)} = \frac{-b(a_1, a_2, v_0)}{\mathcal{D}(N, 2)}$$

where $\gamma = \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$. In arithmetic applications, it is often useful to restrict the possible values of z to the field $\mathbb{Q}(\tau)$ when τ is a real quadratic or imaginary quadratic number, in which case the regime $w = 0$, $z = v_1\tau + v_2$ with $v_1, v_2 \in \mathbb{Q}$ may be used (see for instance the determination of values of L-functions at $s = 1$ for real quadratic fields in [[Sie80], Chapter II, §6]).

From Theorem II.1 we immediately deduce partial multiplicative cocycle relations for the smoothed $G_{n-2,a_1,\ldots,a_{n-1}}(v)(w, x, L, L')$ raised to the power $\mathcal{D}(N, n)$ as:

Corollary II.1: *Under the same hypothesis as in Theorem II.1:*

$$\left(\prod_{j=1}^{n} G_{n-2,a_1,\ldots,\widehat{a_j},\ldots,a_n}(v)(w, x, L, L')^{(-1)^{j+1}}\right)^{\mathcal{D}(N,n)} = 1.$$

We note that when the dimension n is fixed, the integers $\mathcal{D}(N, n)$ are also uniformly bounded by the integer $\mathcal{D}(n)$ defined by:

$$\mathcal{D}(n) = \prod_{p \leq n+1} p^{\lfloor \frac{n}{p-1} \rfloor}.$$

For instance, when $n = 2$ we recover $\mathcal{D}(2) = 12$ so that for all $N \geq 2$ and for all $\gamma \in \Gamma_0(N)$, $\mathcal{D}(2).P_{2,\gamma}^{(N)} \in \mathbb{Z}$. The general bound $\mathcal{D}(n)$ appears in the study of certain higher dimensional Dedekind sums related to the smoothed functions $B_{n,a_1,\ldots,a_n}(v)(w, x, L, L')$

(see section II.3.3) and it is a classical bound in the study of such objects (see [Zag73] where other versions of higher dimensional Dedekind sums have denominators uniformly bounded for even n by $\mu_{n/2} = 2^{-n}\mathcal{D}(n)$ in our notation).

Let us now give an outline of this chapter, which is divided into two main parts. In section II.2 prove that the modularity property (I.4) holds for almost all configurations of the linear forms $a_1, \ldots, a_n$, thus expanding the range of Theorem I.1:

Theorem II.2: *Let $a_1, \ldots, a_n$ be non-zero linear forms in Λ which are well placed in $V^\vee$. Then:*

$$\prod_{j=1}^{n} G_{n-2,a_1,\ldots,\widehat{a_j},\ldots,a_n}(v)(w,x)^{(-1)^{j+1}} = \exp(2i\pi B_{n,a_1,\ldots,a_n}(v)(w,x)).$$

In chapter I we already proved the case where $\mathrm{rk}(a_1, \ldots, a_n) = n$ and the case where $\mathrm{rk}(a_1, \ldots, a_n) \leq n-2$ is trivial. Thus, in this chapter, we only prove the case where $\mathrm{rk}(a_1, \ldots, a_n) = n-1$ and 0 is not a barycenter of $a_1, \ldots, a_n$ (see Proposition II.5). Theorem II.2 is a first step in the direction of Theorem II.1.

Next, in section II.3 we apply to the $B_{n,a_1,\ldots,a_n}$ functions a standard smoothing operation inspired by [CD14] and prove the following theorem:

Theorem II.3: *Let $a_1, \ldots, a_n \in \Lambda$ be linearly independent and suppose that the smoothing lattice L' is good for $a_1, \ldots, a_n$. Let $\alpha_1, \ldots, \alpha_n$ be the primitive positive dual basis to $a_1, \ldots, a_n$ in L. Let $\alpha_{1,j} = \langle \alpha_j, e_1 \rangle$, $s_j = a_j(\alpha_j)$ and $v = \sum_{j=1}^{n} v_j \alpha_j / s_j$. Fix any set of representatives $\mathcal{F}$ for L/M where $M = \oplus_{j=1}^{n} \mathbb{Z}\alpha_j$. Then:*

$$B_{n,a_1,\ldots,a_n}(v)(w,x,L,L') = \epsilon \sum_{\delta \in \mathcal{F}} \sum_{d \mid N, d \neq 1} \mathrm{Tr}_{\mathbb{Q}(\zeta_d)/\mathbb{Q}} \left(\prod_{j=1}^{n} \left(\frac{\zeta_d^{-\alpha_{1,j} \lfloor \frac{v_j+\delta_j}{s_j} \rfloor}}{\zeta_d^{\alpha_{1,j}} - 1} \right) \right) \tag{II.10}$$

where $\epsilon = \mathrm{sign}\det(a_1, \ldots, a_n)$, $\zeta_d = \exp(2i\pi/d)$ and $\delta = \sum_{j=1}^{n} \delta_j \alpha_j / s_j$ for any $\delta \in \mathcal{F}$.

This theorem states that for fixed $a_1, \ldots, a_n \in \Lambda$ and fixed $v \in V/L'$ the function $(w,x) \to B_{n,a_1,\ldots,a_n}(v)(w,x,L,L')$ is actually constant, provided that the smoothing lattice L' is *good* for $a_1, \ldots, a_n$. In addition, for any $v \in V/L'$, the rational number $B_{n,a_1,\ldots,a_n}(v)(w,x,L,L')$ is expressed as a sum of traces of algebraic numbers in cyclotomic fields whose denominators are well understood (see [Das08] or [Zag73] for instance). It follows from a detailed analysis of these algebraic numbers that:

$$B_{n,a_1,\ldots,a_n}(v)(w,x,L,L') \in \mathcal{D}(N,n)^{-1}\mathbb{Z}$$

which generalises the well-known integrality result $P_{2,\gamma}^{(N)} \in \mathcal{D}(N,2)^{-1}\mathbb{Z}$. Theorem II.1 shall then be obtained as a consequence of this result together with Theorem II.2.

This chapter is organised as follows: in section II.2 we prove Theorem II.2 by a careful analysis of the cones involved in the definition of the smoothed $G_{n-2,a_1,\ldots,a_{n-1}}(v)(w,x,L,L')$ functions. In section II.3 we perform the smoothing operation on the Bernoulli rational functions and prove Theorem II.3. Then, at the end of section II.3 we prove Theorem II.1 as a consequence of Theorem II.2 together with Theorem II.3. Lastly, in section II.4 we give cohomological interpretations of our main results.

II.2 The modular property

In this section we recall some definitions given in chapter I and recall the general geometric setup for our geometric G_{n-2} functions (see section II.2.1) and then we give a five-step proof of Theorem II.2 in section II.2.2.

II.2.1 Geometric setup

In this section we fix the geometric setup and recall a few important definitions given in chapter I. Let V be a $\mathbb{Q}$-vector space of finite dimension n and L be a rank n lattice in V. We fix a $\mathbb{Z}$-basis $B = [e_1, \ldots, e_n]$ of L. Define $\Lambda = \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$ and fix $B_\Lambda = [f_1, \ldots, f_n]$ the basis of Λ satisfying $f_j(e_k) = \delta_{jk}$ where δ_{jk} is Kronecker's symbol. This fixes determinant forms on both L and Λ, as well as actions of $\mathrm{SL}_n(\mathbb{Z})$ on L by left multiplication and on Λ by inverse right multiplication such that $\forall\,(a, \alpha) \in \Lambda \times L$, $\forall\, g \in \mathrm{SL}_n(\mathbb{Z})$, $(g \cdot a)(g \cdot \alpha) = a(\alpha)$.

Let us now recall the alternative definition of the functions $G_{n-2,a_1,\ldots,a_{n-1}}$ given in the proof of Proposition I.7 and which we will use in this section. Let us fix linearly independent primitive linear forms $a_1, \ldots, a_{n-1} \in V^\vee$ and let us denote as usual by γ the unique primitive vector in L satisfying $s.\gamma = \det(a_1, \ldots, a_{n-1}, \cdot)$ for some positive integer s. Let us fix a positive dual family $\alpha_1, \ldots, \alpha_{n-1}$ to $a_1, \ldots, a_{n-1}$ in L. Fix $x \in \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{C})$ such that $x(\alpha_j)/x(\gamma) \notin \mathbb{R}$ for all $1 \leq j \leq n-1$. Then define the signs $d_j = \mathrm{sign}(\Im(x(\alpha_j)/x(\gamma))) \in \{-1, +1\}$ and set $D = \sum_{j=1}^{n-1}(d_j - 1)/2$. Finally, define the two cones:

$$C^+(\underline{a}, x) = \{\delta \in V \mid \forall\, 1 \leq j \leq n-1, a_j(\delta) \geq 0 \text{ if } d_j = 1, a_j(\delta) < 0 \text{ if } d_j = -1\},$$
$$C^-(\underline{a}, x) = \{\delta \in V \mid \forall\, 1 \leq j \leq n-1, a_j(\delta) \geq 0 \text{ if } d_j = -1, a_j(\delta) < 0 \text{ if } d_j = 1\}.$$

These two cones are independent of the choice of $\alpha_1, \ldots, \alpha_{n-1}$ and are invariant by translation along γ. It follows from the proof of Proposition I.7 that:

$$G_{n-2,a_1,\ldots,a_{n-1}}(v)(w, x)^{(-1)^D} = \prod_{\delta \in (v+L) \cap C^-(\underline{a}, x)/\mathbb{Z}\gamma} \left(1 - e^{-2i\pi\left(\frac{w + x(\delta)}{x(\gamma)}\right)}\right) \times \prod_{\delta \in (v+L) \cap C^+(\underline{a}, x)/\mathbb{Z}\gamma} \left(1 - e^{2i\pi\left(\frac{w + x(\delta)}{x(\gamma)}\right)}\right)^{(-1)^n}. \quad \text{(II.11)}$$

This is the formulation which we use for the proof of Theorem II.2 in section II.2.2.

Another remark we wish to make before moving on to the proof of Theorem II.2 is a remark on the hypothesis that the linear forms $a_1, \ldots, a_n$ are well placed in $V^\vee$. In particular, when $\mathrm{rk}(a_1, \ldots, a_n) = n-1$, this condition can be read on the standard non-trivial relation among $a_1, \ldots, a_n$ introduced in chapter I (see Definition I.11).

Definition II.4: *Let $a_1, \ldots, a_n$ be n linear forms on V. We say that $a_1, \ldots, a_n$ are well placed in $V^\vee$ if either $\mathrm{rk}(a_1, \ldots, a_n) \neq n-1$ or if $\mathrm{rk}(a_1, \ldots, a_n) = n-1$ and 0 is not a barycenter of $a_1, \ldots, a_n$. Equivalently, $a_1, \ldots, a_n$ are well placed in $V^\vee$ if $\mathrm{rk}(a_1, \ldots, a_n) \neq n-1$ or if $\mathrm{rk}(a_1, \ldots, a_n) = n-1$ and $k^-(a_1, \ldots, a_n) > 0$.*

Notice that this condition is very similar to the "good position" (= not bad) condition in chapter I for $n+1$ linear forms under which the cocycle relation (I.6) holds. The rest of section II.2 is devoted to the proof of Theorem II.2 in the case where $\mathrm{rk}(a_1, \ldots, a_n) = n-1$ and $k^-(a_1, \ldots, a_n) > 0$ and will make use of the standard non-trivial relation among $a_1, \ldots, a_n$ which we have recalled.

II.2.2 Proof of the modular property

In this section we give the proof of Theorem II.2. We once again highlight that when $\mathrm{rk}(a_1, \dots, a_n) = n$, the theorem was proven in chapter I using Narukawa's theorem (see [Nar04]), and that when $\mathrm{rk}(a_1, \dots, a_n) \leq n-2$ the statement is trivial. We therefore only need to prove the following.

Proposition II.5: *Let $a_1, \dots, a_n \in \Lambda$ be n non-zero primitive linear forms such that $\mathrm{rk}(a_1, \dots, a_n) = n-1$ and $k^-(a_1, \dots, a_n) > 0$. Then for all $v, w, x \in V/L \times \mathbb{C} \times \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{C})$:*

$$\prod_{j=1}^{n} G_{n-2,a_1,\dots,\widehat{a_j},\dots,a_n}(v)(w,x)^{(-1)^{j+1}} = 1. \tag{II.12}$$

Remark: If we remove the assumption that $k^-(a_1, \dots, a_n) > 0$ the result does not generally hold. To see this we analyse the simple case where $a_1 = -a_2$ and $\mathrm{rk}(a_2, \dots, a_n) = n-1$ with $n \geq 3$. In this case, the left-hand side of (II.12) reduces to:

$$\prod_{j=1}^{n} G_{n-2,a_1,\dots,\widehat{a_j},\dots,a_n}(v)(w,x)^{(-1)^{j+1}} = \frac{G_{n-2,-a_2,a_3,\dots,a_n}(v)(w,x)}{G_{n-2,a_2,a_3,\dots,a_n}(v)(w,x)}.$$

For simplicity, fix a $\mathbb{Z}$-basis $B = [e_1, \dots, e_n]$ of L and consider the linear form a_j satisfying $a_j(e_k) = 0$ if $k \neq j$, $a_j(e_j) = 1$ for $2 \leq j \leq n$. If we assume further that $F(\underline{a}, \underline{\alpha}, v) = \{v\}$ where $\underline{a} = (a_2, \dots, a_n)$ and $\underline{\alpha} = (e_2, \dots, e_n)$ (see (I.16) for the definition of $F(\underline{a}, \underline{\alpha}, v)$) and that $a_2(v) = 0$ then:

$$\frac{G_{n-2,-a_2,a_3,\dots,a_n}(v)(w,x)}{G_{n-2,a_2,a_3,\dots,a_n}(v)(w,x)} = \frac{G_{n-2}\left(\frac{w+x(v)}{x(-e_1)}, \frac{x(-e_2)}{x(-e_1)}, \frac{x(e_3)}{x(-e_1)}, \dots, \frac{x(e_n)}{x(-e_1)}\right)}{G_{n-2}\left(\frac{w+x(v)}{x(e_1)}, \frac{x(e_2)}{x(e_1)}, \frac{x(e_3)}{x(e_1)}, \dots, \frac{x(e_n)}{x(e_1)}\right)}.$$

Using [[Nis01], Proposition 3.2] we get:

$$\frac{G_{n-2,-a_2,a_3,\dots,a_n}(v)(w,x)}{G_{n-2,a_2,a_3,\dots,a_n}(v)(w,x)} = G_{n-3}\left(\frac{w+x(v)}{x(e_1)}, \frac{x(e_3)}{x(e_1)}, \dots, \frac{x(e_n)}{x(e_1)}\right)$$

and this is not identically equal to 1. For $n = 2$ we get under the same assumptions the simpler form

$$\theta_{-a}(v)(w,x)\theta_a(w,x)^{-1} = \exp\left(-2i\pi\left(\frac{w+x(v)}{x(e_1)} - \frac{1}{2}\right)\right)$$

which is also not identically equal to 1.

We organise the proof of Proposition II.5 into five main steps:

- Step 1: we first show in section II.2.2.1 that we can order the linear forms $a_1, \dots, a_n$ at will. To simplify the notations, we shall choose an ordering on $a_1, \dots, a_n$ such that the standard non-trivial relation among $a_1, \dots, a_n$ is $\sum_{j=1}^{n} \lambda_j a_j = 0$ with

$$\begin{cases} \lambda_j = 0 & \text{for } 1 \leq j < l \\ \lambda_j < 0 & \text{for } l \leq j < m \\ \lambda_j > 0 & \text{for } m \leq j \leq n \end{cases}$$

  for some $1 \leq l < m \leq n$.

- Step 2: In section II.2.2.2, we shall rewrite each term in the right-hand side of (II.12) using formula (II.11) and reorganise the factors. More precisely, we define two families of cones $(C_j^1)_j$ and $(C_j^2)_j$ for $l \leq j \leq n$ (see Definition II.7) and prove in Lemma II.8 that:

$$\prod_{j=1}^{n} G_{n-2,a_1,\dots,\widehat{a_j},\dots,a_n}^{(-1)^{j+1}}(v)(w,x) = \prod_{j=l}^{n} \prod_{\delta\in(v+L)\cap C_j^1/\mathbb{Z}\gamma} \left(1-e^{2i\pi\left(\frac{w+x(\delta)}{x(\gamma)}\right)}\right)^{\mu_j} \times \prod_{\delta\in(v+L)\cap C_j^2/\mathbb{Z}\gamma} \left(1-e^{-2i\pi\left(\frac{w+x(\delta)}{x(\gamma)}\right)}\right)^{\mu_j(-1)^n} \tag{II.13}$$

where γ is the primitive vector in L satisfying $\det(a_1,\dots,a_{n-1},\cdot) = s_n\gamma$ for some positive integer s_n and the μ_j's are explicit signs in $\{-1,+1\}$. Denoting by c_j^1 and c_j^2 the indicator functions associated to these cones we may define two functions $f^1, f^2 : V/\mathbb{Q}\gamma \to \mathbb{Z}$ by:

$$f^1 = \sum_{j=l}^{n} \mu_j c_j^1 \quad \text{and} \quad f^2 = \sum_{j=l}^{n} \mu_j c_j^2 \times (-1)^n.$$

Then formula (II.13) may be rewritten as:

$$\prod_{j=1}^{n} G_{n-2,a_1,\dots,\widehat{a_j},\dots,a_n}(v)(w,x)^{(-1)^{j+1}} = \prod_{\delta\in v+L/\mathbb{Z}\gamma} \left(1-e^{2i\pi\left(\frac{w+x(\delta)}{x(\gamma)}\right)}\right)^{f^1(\delta)} \left(1-e^{-2i\pi\left(\frac{w+x(\delta)}{x(\gamma)}\right)}\right)^{f^2(\delta)}. \tag{II.14}$$

- Step 3: The remainder of the proof is devoted to showing that $f^1 = f^2 = 0$. The proof that $f^2 = 0$ is exactly the same as the proof that $f^1 = 0$ on which we now focus. This is done by a technical combinatorial analysis of the cones C_j^1. In section II.2.2.3 we give an example for $n = 4$ where we show how the table of the signs of the linear forms a_k on the cones C_j^1 contains the relevant information for the proof that $f^1 = 0$. We shall show that this table of signs must obey certain rules (see Lemma II.9), for instance the signs of a_k on C_j^1 must be related to the sign of a_j on C_k^1 for $l \leq j \neq k \leq n$. We then deduce from this set of rules that if $l \leq j, j', j'' \leq n$ are three distinct indices then the triple intersection $C_j^1 \cap C_{j'}^1 \cap C_{j''}^1$ is empty (see Lemma II.11).

- Step 4: In section II.2.2.4 we show that any vector $\delta \in V$ belongs to exactly 0 or 2 of the cones $C_l^1, \dots, C_n^1$. The third step in the proof guarantees that any $\delta \in V$ belongs to either 0, 1 or 2 of these cones, so we only need to show (see Lemma II.12) that a vector $\delta \in V$ cannot belong to exactly one of these cones. This is by far the most technical part of the proof, relying on a technical property of the sign table (see Lemma II.9, (iii)).

- Step 5: In section II.2.2.5 we complete the proof by showing that if $\delta \in C_j^1 \cap C_{j'}^1$ for some $j \neq j'$ then $f^1(\delta) = 0$. This is done by showing that in this case $\mu_j = -\mu_{j'}$ (see Lemma II.17).

II.2.2.1 Invariance under permutation

In this section we justify that Proposition II.5 holds for the linear forms $a_1, \dots, a_n$ if and only if it holds for any permutation $a_{\sigma(1)}, \dots, a_{\sigma(n)}$ of $a_1, \dots, a_n$ where $\sigma \in \mathfrak{S}_n$ using the following result.

Lemma II.6: *Let $a_1, \dots, a_n \in \Lambda$ be n non-zero linear forms. For any permutation $\sigma \in \mathfrak{S}_n$:*

$$\prod_{j=1}^{n} G^{(-1)^{j+1}}_{n-2,a_{\sigma(1)},\dots,\widehat{a_{\sigma(j)}},\dots,a_{\sigma(n)}} = \left(\prod_{j=1}^{n} G^{(-1)^{j+1}}_{n-2,a_1,\dots,\widehat{a_j},\dots,a_n}\right)^{\mathrm{sgn}(\sigma)}$$

Proof :

As the transpositions generate $\mathfrak{S}_n$ it is sufficient to prove this statement for transpositions. Fix $\sigma = (kl)$ the transposition switching k, l with $k < l$. We wish to prove that:

$$\prod_{j=1}^{n} G^{(-1)^{j+1}}_{n-2,a_{\sigma(1)},\dots,\widehat{a_{\sigma(j)}},\dots,a_{\sigma(n)}} = \left(\prod_{j=1}^{n} G^{(-1)^{j+1}}_{n-2,a_1,\dots,\widehat{a_j},\dots,a_n}\right)^{-1}$$

We will repeatedly use the fact that if $b_1, \dots, b_{n-1}$ are non-zero linear forms then for any permutation $\rho \in \mathfrak{S}_{n-1}$:

$$G_{n-2,b_{\rho(1)},\dots,b_{\rho(n-1)}} = G^{\mathrm{sgn}(\rho)}_{n-2,b_1,\dots,\widehat{b_j},\dots,b_n} \tag{II.15}$$

which is clear from the definition (see (I.15)). Consider first an index $1 \leq j \leq n$ such that $j \neq k$ and $j \neq l$. Then σ reduces to the transposition (kl) on the set $\{1, \dots, j-1, j+1, \dots, n\}$ and it follows from (II.15) that:

$$G_{n-2,a_{\sigma(1)},\dots,\widehat{a_{\sigma(j)}},\dots,a_{\sigma(n)}} = G^{-1}_{n-2,a_1,\dots,\widehat{a_j},\dots,a_n}.$$

Suppose now that $j = k$. Then

$$G_{n-2,a_{\sigma(1)},\dots,\widehat{a_{\sigma(k)}},\dots,a_{\sigma(n)}} = G_{n-2,a_1,\dots,a_{k-1},a_{k+1},\dots,a_{l-1},a_k,a_{l+1},\dots,a_n} = G_{n-2,a_{\rho(1)},\dots,\widehat{a_{\rho(l)}},\dots,a_{\rho(n)}}$$

where ρ is the cycle $(k, l, l-1, \dots, k+1)$ which has signature $(-1)^{l+k+1}$. Therefore, formula (II.15) implies that

$$G^{(-1)^{k+1}}_{n-2,a_{\sigma(1)},\dots,\widehat{a_{\sigma(k)}},\dots,a_{\sigma(n)}} = G^{(-1)^{l}}_{n-2,a_1,\dots,\widehat{a_l},\dots,a_n}$$

and we may prove similarly if $j = l$ that:

$$G^{(-1)^{l+1}}_{n-2,a_{\sigma(1)},\dots,\widehat{a_{\sigma(l)}},\dots,a_{\sigma(n)}} = G^{(-1)^{k}}_{n-2,a_1,\dots,\widehat{a_k},\dots,a_n}$$

This gives the desired result:

$$\prod_{j=1}^{n} G^{(-1)^{j+1}}_{n-2,a_{\sigma(1)},\dots,\widehat{a_{\sigma(j)}},\dots,a_{\sigma(n)}} = \left(\prod_{j=1}^{n} G^{(-1)^{j+1}}_{n-2,a_1,\dots,\widehat{a_j},\dots,a_n}\right)^{-1}.$$

□

In particular to show Proposition II.5 we may switch the ordering of the linear forms $a_1, \dots, a_n$. Thus, for the remainder of section II.2.2 we assume that $a_1, \dots, a_n$ are non-zero linear forms such that $\mathrm{rk}(a_1, \dots, a_n) = n-1$ and such that the standard non-trivial relation $\sum_{j=1}^n \lambda_j a_j = 0$ among $a_1, \dots, a_n$ (see [[Mor25], Definition 11]) satisfies:

$$\begin{cases} \lambda_j = 0 & \text{for } 1 \leq j < l \\ \lambda_j < 0 & \text{for } l \leq j < m \\ \lambda_j > 0 & \text{for } m \leq j \leq n \end{cases} \tag{II.16}$$

for some $1 \leq l < m \leq n$. We end this section by making the following remark: there are two simpler cases corresponding to $l = n-1$ and to $l = 1$. If $l = n-1$ then the right-hand side of (II.12) is

$$G_{n-2,a_1,\dots,a_{n-1}}(v)(w,x)^{(-1)^n} G_{n-2,a_1,\dots,a_{n-1}}(v)(w,x)^{(-1)^{n+1}} = 1.$$

In the case where $l = 1$, formula (II.12) may be obtained by introducing a linear form a_{n+1} such that $\mathrm{rk}(a_1, \dots, a_{n+1}) = n$ and by applying directly [[Mor25], Theorem 1] in conjunction with [[Mor25], Corollary 2] to the families $a_1, \dots, \widehat{a_j}, \dots, a_{n+1}$ for $1 \leq j \leq n$. This strategy however doesn't generalise to the case $1 < l < n-1$ for which we need the proof presented in this section.

II.2.2.2 Definition of the cones C_j^1 and C_j^2

We now explicitly describe the product in the left-hand side of (II.12) in the case where the coefficients λ_j satisfying $\sum_{j=1}^n \lambda_j a_j = 0$ verify (II.16). We remark that whenever $1 \leq j < l$ we get $\mathrm{rk}(a_1, \dots, \widehat{a_j}, \dots, a_n) = n-2$ therefore $G_{n-2,a_1,\dots,\widehat{a_j},\dots,a_n} = 1$ by definition. In that case, formula (II.12) which we aim to prove reduces to:

$$\prod_{j=l}^n G_{n-2,a_1,\dots,\widehat{a_j},\dots,a_n}(v)(w,x)^{(-1)^{j+1}} = 1$$

Let us now describe each of the non-trivial terms in the left-hand side of (II.12), using the alternative definition (II.11) for the function $G_{n-2,a_1,\dots,\widehat{a_j},\dots,a_n}$. Indeed, for any fixed $l \leq j \leq n$ define $\gamma^{(j)}$ to be the unique primitive vector in L such that $\det(a_1, \dots, \widehat{a_j}, \dots, a_n, \cdot) = s^{(j)}\gamma^{(j)}$ for some positive integer $s^{(j)}$. Let $(\alpha_k^{(j)})_{k \neq j}$ be a fixed positive dual family to $a_1, \dots, \widehat{a_j}, \dots, a_n$ in L, i.e. a family satisfying for all $1 \leq k \leq n$, $k \neq j$:

$$a_k(\alpha_{k'}^{(j)}) = 0 \text{ for all } 1 \leq k' \leq n, k' \neq k, j \text{ and } a_k(\alpha_k^{(j)}) > 0.$$

Next we fix $x \in \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{C})$ such that for all $l \leq j \leq n$ and all $1 \leq k \leq n$, $k \neq j$, we have $x(\alpha_k^{(j)})/x(\gamma^{(j)}) \notin \mathbb{R}$. Let us then define the linear form $y^{(j)} : V \to \mathbb{R}$ by $y^{(j)}(v) = \Im(x(v)/x(\gamma^{(j)}))$. For $1 \leq k \leq n$, $k \neq j$, define

$$d_k^{(j)} = \mathrm{sign}(y^{(j)}(\alpha_k^{(j)})) \in \{\pm 1\} \tag{II.17}$$

and set $D_j = \sum_{k \neq j}(d_k^{(j)} - 1)/2$. Finally, recall from section II.2.1 the cones:

$$C_j^{\pm} = \{\delta \in V \mid \forall 1 \leq k \leq n, k \neq j, a_k(\delta) \geq 0 \text{ if } \pm d_k^{(j)} = 1, a_k(\delta) < 0 \text{ if } \pm d_k^{(j)} = -1\}. \tag{II.18}$$

It follows from formula (II.11) that for any $l \leq j \leq n$:

$$G_{n-2,a_1,\dots,\widehat{a_j},\dots,a_n}(v)(w,x) = \prod_{\delta\in(v+L)\cap C_j^-/\mathbb{Z}\gamma^{(j)}} \left(1 - e^{-2i\pi\left(\frac{w+x(\delta)}{x(\gamma^{(j)})}\right)}\right)^{(-1)^{D_j}}$$

$$\times \prod_{\delta\in(v+L)\cap C_j^+/\mathbb{Z}\gamma^{(j)}} \left(1 - e^{2i\pi\left(\frac{w+x(\delta)}{x(\gamma^{(j)})}\right)}\right)^{(-1)^{D_j+n}} \quad \text{(II.19)}$$

Let us now fix $\gamma = \gamma^{(n)}$, so that for any $l \leq j \leq n$, $\gamma^{(j)} = (-1)^{j+n}\text{sign}(\lambda_j)\gamma$, where we recall that $\sum_{j=l}^{n} \lambda_j a_j = 0$ is the standard non-trivial relation among $a_1, \dots, a_n$. For simplicity, we shall define the sign

$$\varepsilon_j = (-1)^{j+n}\text{sign}(\lambda_j) \in \{-1, +1\}. \quad \text{(II.20)}$$

The reorganisation of the terms in (II.19) will be made by relabeling the cones C_j^+ and C_j^- depending on the value of ε_j:

Definition II.7: *For $l \leq j \leq n$, define:*

$$(C_j^1, C_j^2) = \begin{cases} (C_j^+, C_j^-) & \text{if } \varepsilon_j = 1 \\ (C_j^-, C_j^+) & \text{if } \varepsilon_j = -1 \end{cases}$$

and denote by $c_j^1, c_j^2 : V \to \{0,1\}$ their indicator functions.

Note that we have explicitly:

$$C_j^1 = \{\delta \in V \mid \forall 1 \leq k \leq n, k \neq j, a_k(\delta) \geq 0 \text{ if } \varepsilon_j d_k^{(j)} = 1, a_k(\delta) < 0 \text{ if } \varepsilon_j d_k^{(j)} = -1\}$$
$$C_j^2 = \{\delta \in V \mid \forall 1 \leq k \leq n, k \neq j, a_k(\delta) \geq 0 \text{ if } \varepsilon_j d_k^{(j)} = -1, a_k(\delta) < 0 \text{ if } \varepsilon_j d_k^{(j)} = 1\}.$$

Now, since $a_k(\gamma) = 0$ for any $1 \leq k \leq n$, it is clear that $c_j^1(v + m\gamma) = c_j^1(v)$ and $c_j^2(v+m\gamma) = c_j^2(v)$ for any $v \in V$ and any $m \in \mathbb{Q}$. Therefore, both functions c_j^1, c_j^2 reduce to functions on the quotient space $V/\mathbb{Q}\gamma$. Finally we define the signs:

$$\mu_j = \begin{cases} (-1)^{j+1+D_j+n} & \text{if } \varepsilon_j = 1 \\ (-1)^{j+1+D_j} & \text{if } \varepsilon_j = -1 \end{cases} \quad \text{(II.21)}$$

We are now ready to give a simple form for the left-hand side of formula (II.12) which we will use for the rest of the proof.

Lemma II.8: *With notations as above:*

$$\prod_{j=l}^{n} G_{n-2,a_1,\dots,\widehat{a_j},\dots,a_n}(v)(w,x)^{(-1)^{j+1}} = \prod_{\delta\in(v+L)/\mathbb{Z}\gamma} \left(1 - e^{2i\pi\left(\frac{w+x(\delta)}{x(\gamma)}\right)}\right)^{f^1(\delta)} \left(1 - e^{-2i\pi\left(\frac{w+x(\delta)}{x(\gamma)}\right)}\right)^{f^2(\delta)}$$

where the functions $f^1, f^2 : V/\mathbb{Q}\gamma \to \mathbb{Z}$ are defined by:

$$f^1 = \sum_{j=l}^{n} \mu_j c_j^1 \quad \textit{and} \quad f^2 = \sum_{j=l}^{n} \mu_j c_j^2 \times (-1)^n.$$

Proof :

Let us briefly denote by $J^{\pm}$ the set of indices $l \leq j \leq n$ satisfying $\varepsilon_j = \pm 1$. Let us rewrite the terms in formula (II.19) for each $j \in J^+$ as:

$$\prod_{j\in J^+} G_{n-2,a_1,\dots,\widehat{a_j},\dots,a_n}(v)(w,x)^{(-1)^{j+1}} = \prod_{j\in J^+} \prod_{\delta\in(v+L)\cap C_j^-/\mathbb{Z}\gamma} \left(1-e^{-2i\pi\left(\frac{w+x(\delta)}{x(\gamma)}\right)}\right)^{\mu_j(-1)^n} \times \prod_{\delta\in(v+L)\cap C_j^+/\mathbb{Z}\gamma} \left(1-e^{2i\pi\left(\frac{w+x(\delta)}{x(\gamma)}\right)}\right)^{\mu_j}.$$

On the other hand, since for all $j \in J^-$, $\gamma^{(j)} = -\gamma$, the product over $j \in J^-$ is:

$$\prod_{j\in J^-} G_{n-2,a_1,\dots,\widehat{a_j},\dots,a_n}(v)(w,x)^{(-1)^{j+1}} = \prod_{j\in J^-} \prod_{\delta\in(v+L)\cap C_j^-/\mathbb{Z}\gamma} \left(1-e^{2i\pi\left(\frac{w+x(\delta)}{x(\gamma)}\right)}\right)^{\mu_j} \times \prod_{\delta\in(v+L)\cap C_j^+/\mathbb{Z}\gamma} \left(1-e^{-2i\pi\left(\frac{w+x(\delta)}{x(\gamma)}\right)}\right)^{\mu_j(-1)^n}.$$

Putting everything together and using the relabeled cones C_j^1 and C_j^2 gives:

$$\prod_{j=l}^{n} G_{n-2,a_1,\dots,\widehat{a_j},\dots,a_n}(v)(w,x)^{(-1)^{j+1}} = \prod_{j=l}^{n} \prod_{\delta\in(v+L)\cap C_j^1/\mathbb{Z}\gamma} \left(1-e^{2i\pi\left(\frac{w+x(\delta)}{x(\gamma)}\right)}\right)^{\mu_j} \times \prod_{\delta\in(v+L)\cap C_j^2/\mathbb{Z}\gamma} \left(1-e^{-2i\pi\left(\frac{w+x(\delta)}{x(\gamma)}\right)}\right)^{\mu_j(-1)^n}$$

Thus the functions f^1 and f^2 are defined precisely so that:

$$\prod_{j=l}^{n} G_{n-2,a_1,\dots,\widehat{a_j},\dots,a_n}(v)(w,x)^{(-1)^{j+1}} = \prod_{\delta\in(v+L)/\mathbb{Z}\gamma} \left(1-e^{2i\pi\left(\frac{w+x(\delta)}{x(\gamma)}\right)}\right)^{f^1(\delta)} \left(1-e^{-2i\pi\left(\frac{w+x(\delta)}{x(\gamma)}\right)}\right)^{f^2(\delta)}$$

which is the desired result. □

II.2.2.3 Sign tables and emptiness of triple intersections

The rest of the proof of Proposition II.5 consists in proving that $f^1 = 0$ and $f^2 = 0$. Both statements are proven similarly so we focus on the proof that $f^1 = 0$.

Let us briefly explain the general idea on a simple example in the four-dimensional case. Let us fix:

$$a_1 = [1,0,0,0],\ a_2 = [0,1,0,0],\ a_3 = [0,0,-1,0],\ a_4 = [0,1,1,0].$$

These are four linear forms on $\mathbb{Z}^4$ such that $\mathrm{rk}(a_1,a_2,a_3,a_4) = 3$ and $0\cdot a_1 - a_2 + a_3 + a_4 = 0$ so that $l = 2$ and $m = 3$. The vector γ is given by $\gamma = [0,0,0,-1]^T$. Let us also fix $x = [2i,3i,5i,-1]$ so that $x(\gamma) = 1$. We wish to describe explicitly the cones C_j^1 for

$2 \leq j \leq 4$, therefore we need to describe the signs $\varepsilon_j d_k^{(j)}$ for $2 \leq j \leq 4$ and $1 \leq k \leq 4$, $k \neq j$. We may compute a choice of elements $\alpha_k^{(j)}$ as follows:

k \ j	2	3	4
1	$\alpha_1^{(2)} = [1,0,0,0]^T$	$\alpha_1^{(3)} = [1,0,0,0]^T$	$\alpha_1^{(4)} = [1,0,0,0]^T$
2		$\alpha_2^{(3)} = [0,1,-1,0]^T$	$\alpha_2^{(4)} = [0,1,0,0]^T$
3	$\alpha_3^{(2)} = [0,1,-1,0]^T$		$\alpha_3^{(4)} = [0,0,-1,0]^T$
4	$\alpha_4^{(2)} = [0,1,0,0]^T$	$\alpha_4^{(3)} = [0,0,1,0]^T$	

Table II.1: Example of a table of the $\alpha_k^{(j)}$s

This gives the computation of the signs $\varepsilon_j d_k^{(j)}$:

k \ j	2	3	4
1	$+$	$+$	$+$
2		$-$	$+$
3	$-$		$-$
4	$+$	$+$	

Table II.2: Example of a sign table containing the $\varepsilon_j d_k^{(j)}$s

From this sign table we deduce that $C_3^1 \cap C_4^1 = \emptyset$ as the conditions on $\varepsilon_2 d_3^{(2)} = -\varepsilon_2 d_4^{(2)}$ are incompatible. If we set $H_j^+ = \{\delta \in V \mid a_j(\delta) \geq 0\}$ and $H_j^- = V - H_j^+$ then it follows from this table that:

$$C_2^1 \cap C_3^1 = C_2^1 \cap H_2^- = C_3^1 \cap H_3^-$$

and

$$C_2^1 \cap C_4^1 = C_2^1 \cap H_2^+ = C_4^1 \cap H_4^+$$

In addition, the relation $a_2 = a_3 + a_4$ implies that $C_3^1 \cap H_3^+ = \emptyset$ and $C_4^1 \cap H_4^- = \emptyset$. Therefore, $C_2^1 = C_3^1 \sqcup C_4^1$ and we may check that $f^1 = -c_2^1 + c_3^1 + c_4^1 = 0$. This is the general idea of the proof and we prove the general case in what follows.

As showcased by the previous example, we need to study the cones C_j^1 and therefore the signs $\varepsilon_j d_k^{(j)}$ for $k \neq j$. We start by proving a crucial lemma on the relations between these signs that govern the sign tables (see Table II.2). This will be useful for the last three steps of the proof of Proposition II.5. As a corollary, we will prove that any intersection of three of the C_j^1s is empty.

There are three main relations between the signs $\varepsilon_j d_k^{(j)}$ among which the first two are quite simple. The third one is more technical and to state it we need to introduce for $l \leq j \leq n$ and $1 \leq k \leq n$, $k \neq j$ the following positive real numbers:

$$u_k^{(j)} = \left| \Im \left(\frac{x(\alpha_k^{(j)})}{\lambda_k a_k(\alpha_k^{(j)}) x(\gamma)} \right) \right| \tag{II.22}$$

which are independent of the choice of $\alpha_k^{(j)}$. We are now ready to state the crucial technical lemma:

Lemma II.9: *The signs* $\varepsilon_j d_k^{(j)}$ *obey the following relations:*

(i) $\forall\, l \leq j, j' \leq n, \forall\, 1 \leq k < l, \varepsilon_j d_k^{(j)} = \varepsilon_{j'} d_k^{(j')}$. *In other words, the rows* $1 \leq k < l$ *in the sign table are constant and can be ignored.*

(ii) $\forall\, l \leq k, j \leq n, k \neq j, \varepsilon_j d_k^{(j)} = -\mathrm{sign}(\lambda_j \lambda_k) \varepsilon_k d_j^{(k)}$. *In other words, the sign table is completely determined by its upper triangular portion.*

(iii) $\forall\, l \leq j, k, k' \leq n$, $k \neq k' \neq j$, *if* $u_k^{(j)} \leq u_{k'}^{(j)}$ *then* $\varepsilon_k d_{k'}^{(k)} = \varepsilon_j d_{k'}^{(j)}$. *This technical point says that the knowledge of a single column is enough to determine the entire sign table, the particular sign relations being given by the relative positions of the* $u_k^{(j)}$ *for* $k \neq j$ *on the real axis.*

Proof :

(i) To prove the first relation, one only needs to notice that when $1 \leq k < l$ it is possible to choose $\alpha_k^{(j)} = \alpha_k^{(n)}$ for any $l \leq j \leq n$ which leads to $\varepsilon_j d_k^{(j)} = \varepsilon_n d_k^{(n)}$. The desired relation follows.

(ii) To prove the second relation we remark that when $k \neq j$, $a_j(\alpha_k^{(j)}) = -\lambda_k a_k(\alpha_k^{(j)})/\lambda_j$ and therefore $a_j(-\mathrm{sign}(\lambda_j \lambda_k)\alpha_k^{(j)}) > 0$. Since for all $k' \neq j, k$, $a_{k'}(\alpha_k^{(j)}) = 0$, we may replace $\alpha_j^{(k)}$ in the positive dual family to $a_1, \ldots, \widehat{a_k}, \ldots, a_n$ by $-\mathrm{sign}(\lambda_j \lambda_k)\alpha_k^{(j)}$. Thus, the sign $d_j^{(k)}$ satisfies by definition :

$$d_j^{(k)} \frac{x(-\mathrm{sign}(\lambda_j \lambda_k)\alpha_k^{(j)})}{\varepsilon_k x(\gamma)} \in \mathbb{H}$$

which we compare to

$$d_k^{(j)} \frac{x(\alpha_k^{(j)})}{\varepsilon_j x(\gamma)} \in \mathbb{H}$$

This gives exactly the relation $\varepsilon_j d_k^{(j)} = -\mathrm{sign}(\lambda_j \lambda_k) \varepsilon_k d_j^{(k)}$.

(iii) This last point is more subtle and will only be used in section II.2.2.4. For any three distinct indices $l \leq j, k, k' \leq n$, let us set:

$$v_{k,j'}^{(j)} = \lambda_k^2 a_k(\alpha_k^{(j)}) \alpha_{k'}^{(j)} - \lambda_k \lambda_{k'} a_{k'}(\alpha_{k'}^{(j)}) \alpha_k^{(j)}$$

Then clearly $a_i(v_{k,j'}^{(j)}) = 0$ whenever $i \neq j, k, k'$ and, by construction, it is also true that $a_j(v_{k,j'}^{(j)}) = 0$. Indeed, using the fact that $\sum_{i=l}^n \lambda_i a_i = 0$ with $\lambda_j \neq 0$ we get:

$$\begin{aligned}
\lambda_j a_j(v_{k,j'}^{(j)}) &= \lambda_k^2 \lambda_j a_k(\alpha_k^{(j)}) a_j(\alpha_{k'}^{(j)}) - \lambda_k \lambda_{k'} \lambda_j a_{k'}(\alpha_{k'}^{(j)}) a_j(\alpha_k^{(j)}) \\
\lambda_j a_j(v_{k,j'}^{(j)}) &= -\lambda_k^2 a_k(\alpha_k^{(j)}) \sum_{i \neq j} \lambda_i a_i(\alpha_{k'}^{(j)}) + \lambda_k \lambda_{k'} a_{k'}(\alpha_{k'}^{(j)}) \sum_{i \neq j} \lambda_i a_i(\alpha_k^{(j)}) \\
\lambda_j a_j(v_{k,j'}^{(j)}) &= -\lambda_k^2 \lambda_{k'} a_k(\alpha_k^{(j)}) a_{k'}(\alpha_{k'}^{(j)}) + \lambda_k^2 \lambda_{k'} a_{k'}(\alpha_{k'}^{(j)}) a_k(\alpha_k^{(j)}) \\
\lambda_j a_j(v_{k,j'}^{(j)}) &= 0.
\end{aligned}$$

Moreover, $a_{k'}(v_{k,j'}^{(j)}) = \lambda_k^2 a_k(\alpha_k^{(j)}) a_{k'}(\alpha_{k'}^{(j)}) > 0$. Thus, a possible choice for $\alpha_{k'}^{(k)}$ is $\alpha_{k'}^{(k)} = v_{k,j'}^{(j)}$ and $x(v_{k,j'}^{(j)}) \in \mathbb{C} - \mathbb{R}$. It follows that the sign $d_{k'}^{(k)}$ satisfies:

$$d_{k'}^{(k)} \frac{x(v_{k,j'}^{(j)})}{\varepsilon_k x(\gamma)} \in \mathbb{H}.$$

Replacing $v_{k,j'}^{(j)}$ by its expression gives:

$$d_{k'}^{(k)}\left(\frac{\lambda_k^2 a_k(\alpha_k^{(j)})x(\alpha_{k'}^{(j)})}{\varepsilon_k x(\gamma)} - \frac{\lambda_k\lambda_{k'}a_{k'}(\alpha_{k'}^{(j)})x(\alpha_k^{(j)})}{\varepsilon_k x(\gamma)}\right) \in \mathbb{H}$$

We may now rewrite this expression in terms of elements related to the values $u_k^{(j)}$ and $u_{k'}^{(j)}$:

$$\varepsilon_k\lambda_k^2\lambda_{k'}d_{k'}^{(k)}a_k(\alpha_k^{(j)})a_{k'}(\alpha_{k'}^{(j)})\left(\frac{x(\alpha_{k'}^{(j)})}{\lambda_{k'}a_{k'}(\alpha_{k'}^{(j)})x(\gamma)} - \frac{x(\alpha_k^{(j)})}{\lambda_k a_k(\alpha_k^{(j)})x(\gamma)}\right) \in \mathbb{H}. \qquad \text{(II.23)}$$

Let us denote by $U_k^{(j)}$ the complex number

$$U_k^{(j)} = \frac{x(\alpha_k^{(j)})}{\lambda_k a_k(\alpha_k^{(j)})x(\gamma)}$$

so that (II.23) may be rewritten as:

$$\varepsilon_k\lambda_{k'}d_{k'}^{(k)}(U_{k'}^{(j)} - U_k^{(j)}) \in \mathbb{H}.$$

By definition $u_k^{(j)} = |\Im(U_k^{(j)})|$ and $u_{k'}^{(j)} = |\Im(U_{k'}^{(j)})|$, thus the sign of $\varepsilon_k\lambda_{k'}d_{k'}^{(k)}$ depends on which of the two values $u_k^{(j)}, u_{k'}^{(j)}$ is the largest. If $u_k^{(j)} \leq u_{k'}^{(j)}$ then the sign $d_{k'}^{(k)}$ satisfies:

$$d_{k'}^{(k)}\frac{x(\alpha_{k'}^{(j)})}{a_{k'}(\alpha_{k'}^{(j)})\varepsilon_k x(\gamma)} \in \mathbb{H}$$

which we compare to the sign $d_{k'}^{(j)}$ satisfying:

$$d_{k'}^{(j)}\frac{x(\alpha_{k'}^{(j)})}{a_{k'}(\alpha_{k'}^{(j)})\varepsilon_j x(\gamma)} \in \mathbb{H}.$$

This leads to the desired equality $\varepsilon_k d_{k'}^{(k)} = \varepsilon_j d_{k'}^{(j)}$. Note that if $u_k^{(j)} = u_{k'}^{(j)}$ then the two complex numbers $U_k^{(j)}$ and $U_{k'}^{(j)}$ must lie in opposite half-planes in $\mathbb{C} - \mathbb{R}$ and the sign equality holds. □

We are now ready to prove as a corollary that any intersection of three of the cones C_j^1 is empty. In fact, we prove something slightly stronger using the following definition.

Definition II.10: *Suppose that $l \leq j, j' \leq n$ are two distinct indices. We shall say that the two cones C_j^1 and $C_{j'}^1$ are compatible if the columns associated to the indices j and j' in the sign table are compatible. Explicitly, C_j^1 and $C_{j'}^1$ are compatible if and only if for all $1 \leq k \leq n$ (or $l \leq k \leq n$ by lemma II.9, (i)), $k \neq j, j'$ implies $\varepsilon_j d_k^{(j)} = \varepsilon_{j'} d_k^{(j')}$.*

One may think of this definition as providing a necessary (but not sufficient) condition for two cones C_j^1 and $C_{j'}^1$ to have a non-empty intersection. We may now prove a lemma on the compatibility relations between the C_j^1's.

Lemma II.11:

(i) *For any two distinct indices* $l \leq j, j' \leq n$*, if* $C_j^1 \cap C_{j'}^1 \neq \emptyset$ *then the cones* C_j^1 *and* $C_{j'}^1$ *are compatible.*

(ii) *For any three distinct indices* $l \leq j, j', j'' \leq n$*, if* $\varepsilon_j d_{j''}^{(j)} = \varepsilon_{j'} d_{j''}^{(j')}$ *and* $\varepsilon_{j'} d_j^{(j')} = \varepsilon_{j''} d_j^{(j'')}$ *then* $\varepsilon_j d_{j'}^{(j)} = -\varepsilon_{j''} d_{j'}^{(j'')}$*.*

(iii) *For any three distinct indices* $l \leq j, j', j'' \leq n$*, if on the one hand the cones* C_j^1 *and* $C_{j'}^1$ *are compatible and on the other hand the cones* $C_{j'}^1$ *and* $C_{j''}^1$ *are compatible, then the cones* C_j^1 *and* $C_{j''}^1$ *are not compatible.*

(iv) *For any three distinct indices* $l \leq j, j', j'' \leq n$*,* $C_j^1 \cap C_{j'}^1 \cap C_{j''}^1 = \emptyset$*.*

Proof :

(i) Suppose that the two distinct indices $l \leq j, j' \leq n$ are such that C_j^1 and $C_{j'}^1$ are not compatible. This means that there is an index $l \leq k \leq n$ distinct from j, j' such that $\varepsilon_j d_k^{(j)} = -\varepsilon_{j'} d_k^{(j')}$. Assume without loss of generality that $\varepsilon_j d_k^{(j)} = +1$. Suppose that there is a vector $\delta \in C_j^1 \cap C_{j'}^1$. Then on the one hand, $a_k(\delta) > 0$ since $\varepsilon_j d_k^{(j)} = +1$ and $\delta \in C_j^1$, while on the other hand, $a_k(\delta) \leq 0$ since $\varepsilon_{j'} d_k^{(j')} = -1$ and $\delta \in C_{j'}^1$. This yields a contradiction.

(ii) Consider three indices $l \leq j \neq j' \neq j'' \leq n$. Using lemma II.9, (i) we get the following three relations:

$$\varepsilon_j d_{j'}^{(j)} = -\mathrm{sign}(\lambda_j \lambda_{j'}) \varepsilon_{j'} d_j^{(j')} \tag{II.24}$$

$$\varepsilon_j d_{j''}^{(j)} = -\mathrm{sign}(\lambda_j \lambda_{j''}) \varepsilon_{j''} d_j^{(j'')} \tag{II.25}$$

$$\varepsilon_{j'} d_{j''}^{(j')} = -\mathrm{sign}(\lambda_{j'} \lambda_{j''}) \varepsilon_{j''} d_{j'}^{(j'')}. \tag{II.26}$$

Suppose that:

$$\varepsilon_j d_{j''}^{(j)} = \varepsilon_{j'} d_{j''}^{(j')} \tag{II.27}$$

$$\varepsilon_{j'} d_j^{(j')} = \varepsilon_{j''} d_j^{(j'')} \tag{II.28}$$

We wish to prove that $\varepsilon_j d_{j'}^{(j)} = -\varepsilon_{j''} d_{j'}^{(j'')}$. We start by rewriting (II.24) using (II.28) as:

$$\begin{aligned}
\varepsilon_j d_{j'}^{(j)} &= -\mathrm{sign}(\lambda_j \lambda_{j'}) \varepsilon_{j''} d_j^{(j'')} \\
\varepsilon_j d_{j'}^{(j)} &= \mathrm{sign}(\lambda_j \lambda_{j'}) \mathrm{sign}(\lambda_j \lambda_{j''}) \varepsilon_j d_{j''}^{(j)} \\
\varepsilon_j d_{j'}^{(j)} &= \mathrm{sign}(\lambda_j^2 \lambda_{j'} \lambda_{j''}) \varepsilon_{j'} d_{j''}^{(j')} \\
\varepsilon_j d_{j'}^{(j)} &= -\mathrm{sign}(\lambda_j^2 \lambda_{j'}^2 \lambda_{j''}^2) \varepsilon_{j''} d_{j'}^{(j'')}
\end{aligned}$$

where we used (II.25) in the second line, (II.27) in the first line and (II.28) in the final line. Thus $\varepsilon_j d_{j'}^{(j)} = -\varepsilon_{j''} d_{j'}^{(j'')}$.

(iii) Consider three indices $l \leq j \neq j' \neq j'' \leq n$ and suppose that C_j^1 is compatible with $C_{j'}^1$ while $C_{j'}^1$ is compatible with $C_{j''}^1$. Then in particular $\varepsilon_j d_{j''}^{(j)} = \varepsilon_{j'} d_{j''}^{(j')}$ and $\varepsilon_{j'} d_j^{(j')} = \varepsilon_{j''} d_j^{(j'')}$ therefore by (ii) we get $\varepsilon_j d_{j'}^{(j)} = -\varepsilon_{j''} d_{j'}^{(j'')}$ which guarantees that C_j^1 and $C_{j''}^1$ are incompatible.

(iv) Consider three indices $l \leq j \neq j' \neq j'' \leq n$ and suppose that $C_j^1 \cap C_{j'}^1 \cap C_{j''}^1 \neq \emptyset$. In particular, each of the two-ways intersections are non-empty, and therefore by (i), the of cones C_j^1 and $C_{j'}^1$) are compatible, and the same is true for the cones $C_{j'}^1$ and $C_{j''}^1$, as well as for the cones $C_{j''}^1$ and C_j^1). This contradicts (iii), therefore we must have an empty intersection $C_j^1 \cap C_{j'}^1 \cap C_{j''}^1 = \emptyset$. □

It follows from lemma II.11, (iv) that any vector $\delta \in V$ belongs to at most two of the cones C_j^1. The fourth and next step in the proof of Proposition II.5 is showing that a vector $\delta \in V$ cannot belong to exactly one of the cones C_j^1.

II.2.2.4 Any vector $\delta \in V$ is in none or exactly two of the cones C_j^1

In this section, we prove that a vector $\delta \in V$ belongs to either either 0 or 2 of the C_j^1's. From lemma II.11, (iv) we already know that a vector δ belongs to at most 2 of the cones C_j^1, thus it will suffice to prove that δ cannot belong to exactly one cone C_j^1. The result will be proven in the following form:

Lemma II.12: *For all $l \leq j \leq n$:*

$$C_j^1 \subset \bigcup_{k \neq j} C_k^1$$

where the union ranges on indices $l \leq k \leq n$, $k \neq j$.

This is the most technical part of the proof and it will use most results of section II.2.2.3. It will follow from this result that if $\delta \in V$ belongs to the cone C_j^1 then, since $C_j^1 \subset \cup_{k \neq j} C_k^1$, there is an index $k \neq j$ such that $\delta \in C_k^1$. Hence δ cannot belong to exactly one of the cones C_j^1. Let us now sketch the proof of Lemma II.12. Define for all $l \leq j \leq n$ the set of indices:

$$\mathcal{I}(j) = \{l \leq j' \leq n \mid j' \neq j \text{ and the cones } C_j^1, C_{j'}^1 \text{ are compatible}\}. \tag{II.29}$$

From Lemma II.11 it is clear that we only need to prove that for all $l \leq j \leq n$:

$$C_j^1 \subset \bigcup_{j' \in \mathcal{I}(j)} C_{j'}^1. \tag{II.30}$$

To do so, we shall first prove (II.30) when $\#\mathcal{I}(j) \geq 2$. Then, we prove that for any $l \leq j \leq n$, $\#\mathcal{I}(j) \geq 1$, and lastly, we prove (II.30) when $\#\mathcal{I}(j) = 1$. The hypothesis that $k^-(a_1, \ldots, a_n) > 0$ will only be used in this last step.

Let us now prove that for any $l \leq j \leq n$, if $\#\mathcal{I}(j) \geq 2$ then (II.30) holds. To this end we introduce the following partition of the cones C_j^1. Write as before $H_j^+ = \{v \in V \mid a_j(v) > 0\}$ and $H_j^- = V - H_j^+$. Then the two cones $C_j^1 \cap H_j^+$ and $C_j^1 \cap H_j^-$ form a partition of C_j^1. Our claim is that whenever some C_j^1 intersects some other $C_{j'}^1$ for $j \neq j'$, the intersection must be exactly one of these components. More precisely:

Lemma II.13:

(i) *Suppose that the cones C_j^1 and $C_{j'}^1$ are compatible for some distinct indices $l \leq j, j' \leq n$. Then:*

$$C_j^1 \cap C_{j'}^1 = C_j^1 \cap H_j^{\varepsilon_{j'} d_j^{(j')}} = C_{j'}^1 \cap H_{j'}^{\varepsilon_j d_{j'}^{(j)}}.$$

(ii) *As a consequence, for any $l \leq j \leq n$, if $\mathcal{I}(j)$ contains at least two distinct indices $l \leq j', j'' \leq n$, $j \neq j', j''$ then $C_j^1 \subset C_{j'}^1 \cup C_{j''}^1$.*

Proof :

(i) The cones C_j^1 and $C_{j'}^1$ are compatible, so by definition we get that for all index $1 \leq k \leq n$ distinct from j and j', $\varepsilon_j d_k^{(j)} = \varepsilon_{j'} d_k^{(j')}$. Recall that the half-spaces $H_k^{\pm}$ are defined for $1 \leq k \leq n$ by $H_k^+ = \{\delta \in V \mid a_k(\delta) > 0\}$ and $H_k^- = V - H_k^+$. For simplicity, let us denote by H_k the half-space H_k^+ if $\varepsilon_j d_k^{(j)} = 1$ or H_k^- if $\varepsilon_j d_k^{(j)} = -1$ for $1 \leq k \leq n$, $k \neq j$. Denote also by H_j the half-space H_j^+ if $\varepsilon_{j'} d_j^{(j')} = 1$ or H_j^- if $\varepsilon_{j'} d_j^{(j')} = -1$. Then

$$C_j^1 = \bigcap_{k \neq j} H_k \qquad C_j^1 = \bigcap_{k \neq j'} H_k$$

so that

$$C_j^1 \cap C_{j'}^1 = \left(\bigcap_{k \neq j, j'} H_k \right) \cap H_j \cap H_{j'} = C_j^1 \cap H_j = C_{j'}^1 \cap H_{j'}$$

which is the desired result.

(ii) Let us now suppose that $\mathcal{I}(j)$ contains at least two distinct indices j', j''. Then by definition the cones C_j^1 and $C_{j'}^1$ are compatible and so are the cones C_j^1 and $C_{j'}^1$. It follows from the Lemma II.11, (iii) that the cones $C_{j'}^1$ and $C_{j''}^1$ are not compatible, and more precisely, it follows from of Lemma II.11, (ii) that:

$$\varepsilon_{j'} d_j^{(j')} = -\varepsilon_{j''} d_j^{(j'')}$$

Up to switching j' and j'' we may assume without loss of generality that $\varepsilon_{j'} d_j^{(j')} = +1$ so that $\varepsilon_{j''} d_j^{(j'')} = -1$. Then, using (i) we get:

$$C_j^1 \cap C_{j'}^1 = C_j^1 \cap H_j^+$$
$$C_j^1 \cap C_{j''}^1 = C_j^1 \cap H_j^-.$$

Therefore:

$$C_j^1 \cap (C_{j'}^1 \cup C_{j''}^1) = \left(C_j^1 \cap C_{j'}^1\right) \cup \left(C_j^1 \cap C_{j''}^1\right) = (C_j^1 \cap H_j^+) \cup (C_j^1 \cap H_j^-) = C_j^1$$

and $C_j^1 \subset C_{j'}^1 \cup C_{j''}^1$. This proves (II.30) in the case where $\#\mathcal{I}(j) \geq 2$. □

The remainder of this section is devoted to the proof of (II.30) in the case where $\mathcal{I}(j)$ contains no more than one element. In fact, we shall first prove using lemma II.9, (iii) that the set $\mathcal{I}(j)$ cannot be empty and then handle the case where $\mathcal{I}(j)$ contains exactly one element.

Lemma II.14: *Fix an index $l \leq j \leq n$. Let $l \leq j' \leq n$ be an index satisfying $j' \neq j$ and for all $l \leq k \leq n$, $k \neq j, j'$, $u_{j'}^{(j)} \leq u_k^{(j)}$ (see (II.22) for the definition of the positive real numbers $u_k^{(j)}$). Then the cones C_j^1 and $C_{j'}^1$ are compatible and $j' \in \mathcal{I}(j)$. In particular, $\mathcal{I}(j)$ cannot be empty.*

Proof :

By definition of j' we get that for all indices $l \leq k \leq n$ distinct from both j and j',

$u^{(j)}_{j'} \leq u^{(j)}_{k}$. Using lemma II.9, (iii) we get that for all such indices $l \leq k \leq n$ distinct from both j and j', $\varepsilon_j d^{(j)}_k = \varepsilon_{j'} d^{(j')}_k$. Thus, C^1_j and $C^1_{j'}$ are compatible and $j' \in \mathcal{I}(j)$. □

In the last part of the proof of lemma II.12 we need to treat the case where $\mathcal{I}(j)$ contains only one index j'. In that case, we wish to prove that one of the two components $C^1_j \cap H^+_j$, $C^1_j \cap H^-_j$ is empty. To achieve this, we first give a sufficient condition for one of these components to be empty under the condition that $k^-(a_1, \ldots, a_n) > 0$.

Lemma II.15: *Fix a sign $\nu \in \{-1, 1\}$ and an index $l \leq j \leq n$. Assume that for all index $l \leq k \leq n$ distinct from j, $\nu\lambda_j\lambda_k\varepsilon_j d^{(j)}_k > 0$. Then $C^1_j \cap H^\nu_j = \emptyset$.*

Proof :
Here we use the hypothesis that $k^-(a_1, \ldots, a_n) > 0$. Recall that $\sum_{k=l}^n \lambda_k a_k = 0$ is the standard non-trivial relation among $a_1, \ldots, a_n$ with the coefficients λ_k satisfying (II.16). In particular, $\lambda_l < 0$ and $\lambda_n > 0$. Suppose that $C^1_j \cap H^\nu_j \neq \emptyset$ and fix some $\delta \in C^1_j \cap H^\nu_j$. By definition of C^1_j, for all index $1 \leq k \leq n$ distinct from j, the vector δ satisfies $a_k(\delta) > 0$ if $\varepsilon_j d^{(j)}_k = 1$ and $a_k(\delta) \leq 0$ if $\varepsilon_j d^{(j)}_k = -1$. In addition, by definition of H^ν_j, this element δ must satisfy $a_j(\delta) > 0$ if $\nu = 1$ and $a_j(\delta) \leq 0$ if $\nu = -1$. In particular, $\nu\lambda_j^2 a_j(\delta) \geq 0$ in either case. Thus, the assumption that $\nu\lambda_j\lambda_k\varepsilon_j d^{(j)}_k > 0$ for all $l \leq k \leq n$, $k \neq j$ leads to $\nu\lambda_j\lambda_k a_k(\delta) \geq 0$ for all $l \leq k \leq n$. Using the relation $\sum_{k=l}^n \lambda_k a_k = 0$ we get the equality:

$$\sum_{k=l}^{n} \nu\lambda_j\lambda_k a_k(\delta) = 0$$

which is the vanishing of a sum of non-negative terms. Therefore, for each $l \leq k \leq n$ we get $\nu\lambda_j\lambda_k a_k(\delta) = 0$ and thus $a_k(\delta) = 0$. From the vanishing of $a_j(\delta)$ we obtain that $\nu = -1$ while the vanishing of $a_k(\delta)$ for $k \neq j$ leads to $\varepsilon_j d^{(j)}_k = -1$ for $l \leq k \leq n$, $k \neq j$. Thus the assumption $\nu\lambda_j\lambda_k\varepsilon_j d^{(j)}_k > 0$ for $l \leq k \leq n$, $k \neq j$ gives $\lambda_j\lambda_k > 0$ for all $l \leq k \leq n$ and all the coefficients $\lambda_l, \ldots, \lambda_n$ must share the same sign, which contradicts the fact that $\lambda_l < 0$ and $\lambda_n > 0$. Therefore we must conclude that $C^1_j \cap H^\nu_j = \emptyset$ as claimed. □

Using lemma II.15 we shall finally prove that if $\mathcal{I}(j)$ contains only one index j', then $C^1_j \subset C^1_{j'}$. This will complete the proof of lemma II.12.

Lemma II.16: *Suppose that the index $l \leq j \leq n$ is such that $\mathcal{I}(j)$ contains exactly one index j'. Then:*

(i) *For all $l \leq k \leq n$, $k \neq j$, the sign* $\text{sign}(\lambda_k) d^{(j)}_k$ *is equal to* $\text{sign}(\lambda_{j'}) d^{(j)}_{j'}$.

(ii) *As a consequence, $C^1_j \subset C^1_{j'}$.*

Proof :
(i) Let us fix a bijection $\sigma : \{1, \ldots, n-l\} \to \{l \leq k \leq n \mid k \neq j\}$ such that $u^{(j)}_{\sigma(k)} \leq u^{(j)}_{\sigma(k')}$ whenever $1 \leq k < k' \leq n - l$. In particular, it follows from lemma II.14 that $j' = \sigma(1)$. Let us then prove by induction on $2 \leq \kappa \leq n - l$ that $\text{sign}(\lambda_{\sigma(\kappa)}) d^{(j)}_{\sigma(\kappa)} = \text{sign}(\lambda_{j'}) d^{(j)}_{j'}$.

<u>First case:</u> $\kappa = 2$. We set $k = \sigma(\kappa)$. By definition of σ we get that for $3 \leq \kappa' \leq n - l$, $u^{(j)}_k \leq u^{(j)}_{\sigma(\kappa')}$. From lemma II.9, (iii) we get $\varepsilon_j d^{(j)}_{\sigma(\kappa')} = \varepsilon_k d^{(k)}_{\sigma_j(\kappa')}$ for any $3 \leq \kappa' \leq n - l$.

Thus $k \notin \mathcal{I}(j)$ if and only if $\varepsilon_j d^{(j)}_{j'} = -\varepsilon_k d^{(k)}_{j'}$ where once again $j' = \sigma(1)$. Using lemma II.9, (ii) this gives:

$$\varepsilon_j d^{(j)}_{j'} = \operatorname{sign}(\lambda_{j'}\lambda_k)\varepsilon_{j'} d^{(j')}_k = \operatorname{sign}(\lambda_{j'}\lambda_k)\varepsilon_j d^{(j)}_k.$$

where the last equality holds as the cones C^1_j and $C^1_{j'}$ are compatible by assumption. Hence, $\operatorname{sign}(\lambda_{j'})d^{(j)}_{j'} = \operatorname{sign}(\lambda_k)d^{(j)}_k$ and the case $\kappa = 2$ is proven.

Induction: Assume that the result holds for $k = \sigma(2), \ldots, \sigma(\kappa-1)$. Let $k = \sigma(\kappa)$. For any $\kappa < \kappa' \leq n-l$, by definition of σ and lemma II.9, (iii) we get $\varepsilon_j d^{(j)}_{\sigma_j(k')} = \varepsilon_k d^{(k)}_{\sigma_j(k')}$. Therefore, $k \notin \mathcal{I}_j$ is equivalent to the existence of some $k'' = \sigma(\kappa'')$ with $\kappa'' < \kappa$ such that $\varepsilon_j d^{(j)}_{k''} = -\varepsilon_k d^{(k)}_{k''}$. Using once again lemma II.9, (ii) we rewrite this as:

$$\varepsilon_j d^{(j)}_{k''} = \operatorname{sign}(\lambda_k\lambda_{k''})\varepsilon_{k''} d^{(k'')}_k = \operatorname{sign}(\lambda_k\lambda_{k''})\varepsilon_j d^{(j)}_k$$

where the last equality holds by lemma II.9, (iii) as $u^{(j)}_{k''} \leq u^{(j)}_k$. Thus we get $\operatorname{sign}(\lambda_{k''})d^{(j)}_{k''} = \operatorname{sign}(\lambda_k)d^{(j)}_k$. It then follows from the induction hypothesis $\operatorname{sign}(\lambda_{k''})d^{(j)}_{k''} = \operatorname{sign}(\lambda_{j'})d^{(j)}_{j'}$ that $\operatorname{sign}(\lambda_{j'})d^{(j)}_{j'} = \operatorname{sign}(\lambda_k)d^{(j)}_k$. This completes the proof by induction.

(ii) Let $\nu = \operatorname{sign}(\lambda_j\lambda_{j'})\varepsilon_j d^{(j)}_{j'}$. For all index $l \leq k \leq n$ distinct from j, since by (i) $\operatorname{sign}(\lambda_k)d^{(j)}_k = \operatorname{sign}(\lambda_{j'})d^{(j)}_{j'}$ we get:

$$\begin{aligned}
\nu\lambda_j\lambda_k\varepsilon_j d^{(j)}_k &= |\lambda_j|\operatorname{sign}(\lambda_{j'})d^{(j)}_{j'}\lambda_k\varepsilon_j^2 d^{(j)}_k \\
\nu\lambda_j\lambda_k\varepsilon_j d^{(j)}_k &= |\lambda_j|\operatorname{sign}(\lambda_k)\lambda_k(d^{(j)}_k)^2 \\
\nu\lambda_j\lambda_k\varepsilon_j d^{(j)}_k &= |\lambda_j\lambda_k| > 0.
\end{aligned}$$

It then follows from lemma II.15 that $C^1_j \cap H^\nu_j = \emptyset$ and since $C^1_j = (C^1_j \cap H^{-\nu}_j) \cup (C^1_j \cap H^\nu_j)$ we conclude that $C^1_j = C^1_j \cap H^{-\nu}_j$. From lemma II.9, (ii) we get $\nu = -\varepsilon_{j'} d^{(j')}_j$, therefore $C^1_j = C^1_j \cap H^{-\nu}_j = C^1_j \cap C^1_{j'}$ by lemma II.13. This gives the conclusion $C^1_j \subset C^1_{j'}$. □

Let us now piece all these results together to prove lemma II.12.

Proof of lemma II.12:
Consider $l \leq j \leq n$. Then $\#\mathcal{I}(j) \geq 1$ by lemma II.14. If $\#\mathcal{I}(j) = 1$ then $\mathcal{I}(j)$ contains exactly one element $j' \neq j$ and $C^1_j \subset C^1_{j'}$ by lemma II.16, (ii). If $\#\mathcal{I}(j) \geq 2$ then $\mathcal{I}(j)$ contains at least two distinct elements $j', j'' \neq j$ and $C^1_j \subset C^1_{j'} \cup C^1_{j''}$ by lemma II.13, (ii). Thus in any case, $C^1_j \subset \cup_{k\neq j} C^1_k$ and the proof is complete. □

II.2.2.5 Vanishing of f^1 and proof of Proposition II.5

In this section we prove that f^1 vanishes. From the previous sections, we already know that any vector $\delta \in V$ belongs to either 0 or 2 of the cones C^1_j. In the case where δ belongs to none of the C^1_j's, it is clear that $f^1(\delta) = 0$ by definition. Thus, it will suffice to prove that when δ belongs to the intersection of any two cones C^1_j and $C^1_{j'}$ for $j \neq j'$, $f^1(\delta) = 0$.

Lemma II.17: *Assume $\delta \in C^1_j \cap C^1_{j'}$ for some distinct indices $l \leq j, j' \leq n$. Then $f^1(\delta) = 0$.*

Proof :
It follows from lemma II.11, (iv) that for any $j'' \neq j, j'$, $\delta \notin C^1_{j''}$. Thus by definition of f^1 (see lemma II.8):

$$f^1(\delta) = (-1)^{j+1+D_j+n\frac{1+\varepsilon_j}{2}} + (-1)^{j'+1+D_{j'}+n\frac{1+\varepsilon_{j'}}{2}}.$$

Let $g(j,j') = (j-j') + (D_j - D_{j'}) + n\frac{\varepsilon_j - \varepsilon_{j'}}{2}$. The statement that $f^1(\delta) = 0$ is equivalent to $g(j,j') \equiv 1 \bmod 2$ as

$$f^1(\delta) = (-1)^{j+1+D_j+n\frac{1-\varepsilon_j}{2}}(1 + (-1)^{g(j,j')}).$$

Let us then prove that $g(j,j') \equiv 1 \bmod 2$. Write D_j explicitly as:

$$D_j = \sum_{k\neq j} \frac{d_k^{(j)} - 1}{2}.$$

Since $C^1_j \cap C^1_{j'} \neq \emptyset$, it follows from lemma II.11, (i) that for all index $1 \leq k \leq n$ distinct from both j and j', $\varepsilon_j d_k^{(j)} = \varepsilon_{j'} d_k^{(j')}$. Thus:

$$D_j - D_{j'} = \sum_{k\neq j} \frac{d_k^{(j)} - 1}{2} - \sum_{k\neq j'} \frac{d_k^{(j')} - 1}{2}$$

$$D_j - D_{j'} = \frac{d_{j'}^{(j)} - d_j^{(j')}}{2} + \sum_{k\neq j,j'} \frac{d_k^{(j)} - d_k^{(j')}}{2}$$

$$D_j - D_{j'} = \frac{d_{j'}^{(j)} - d_j^{(j')}}{2} + \frac{(1 - \varepsilon_j\varepsilon_{j'})}{2} \sum_{k\neq j,j'} d_k^{(j)}.$$

The sum over $k \neq j, j'$ contains $n-2$ terms which are either $+1$ or -1 therefore we already obtain

$$D_j - D_{j'} \equiv \frac{d_{j'}^{(j)} - d_j^{(j')}}{2} + (n-2)\frac{1 - \varepsilon_j\varepsilon_{j'}}{2} \equiv \frac{d_{j'}^{(j)} - d_j^{(j')}}{2} + (n-2)\frac{\varepsilon_j - \varepsilon_{j'}}{2} \bmod 2$$

and therefore

$$g(j,j') \equiv (j-j') + \frac{d_{j'}^{(j)} - d_j^{(j')}}{2} + (2n-2)\frac{\varepsilon_j - \varepsilon_{j'}}{2} \bmod 2$$

$$g(j,j') \equiv (j-j') + \frac{d_{j'}^{(j)} - d_j^{(j')}}{2} \bmod 2.$$

Recall that lemma II.9, (ii) gives the equality $\varepsilon_j d_{j'}^{(j)} = -\mathrm{sign}(\lambda_j\lambda_{j'})\varepsilon_{j'} d_j^{(j')}$ and that by definition $\varepsilon_j = (-1)^{j+n}\mathrm{sign}(\lambda_j)$. Thus, $(-1)^j d_{j'}^{(j)} = -(-1)^{j'} d_j^{(j')}$ and:

$$g(j,j') \equiv (j-j') + d_{j'}^{(j)} \frac{1 + (-1)^{j+j'}}{2} \bmod 2.$$

There are only two cases to treat, depending on the parity of $j - j'$. First, suppose that $j - j'$ is even. Then $1 + (-1)^{j+j'} = 2$ and $g(j,j') \equiv d_{j'}^{(j)} \equiv 1 \bmod 2$. Now, suppose that

$j - j'$ is odd. Then $1 + (-1)^{j+j'} = 0$ and $g(j, j') \equiv j - j' \equiv 1 \mod 2$. In each of these two cases, $g(j, j') \equiv 1 \mod 2$ and thus $f^1(\delta) = 0$. This completes the proof. □

Finally, we may piece together all the results from sections II.2.2.1 to II.2.2.5 to give the proof of Proposition II.5 and thus deduce Theorem II.2.

Proof of Proposition II.5:

Let $a_1, \dots, a_n \in \Lambda$ be primitive integral linear forms on V satisfying $\mathrm{rk}(a_1, \dots, a_n) = n - 1$ and such that 0 is not a barycenter of $a_1, \dots, a_n$. This means that there is a relation $\sum_{j=1}^n \lambda_j a_j = 0$ with at least one positive and one negative coefficient among the λ_j's. Using the results from sections II.2.2.1 and II.2.2.2 we may suppose that the coefficients $\lambda_1, \dots, \lambda_n$ in the standard non-trivial relation among $a_1, \dots, a_n$ satisfy the relations (II.16) and that

$$\prod_{j=l}^n G_{n-2,a_1,\dots,\widehat{a_j},\dots,a_n}(v)(w,x)^{(-1)^{j+1}} = \prod_{\delta \in (v+L)/\mathbb{Z}\gamma} \left(1 - e^{2i\pi\left(\frac{w+x(\delta)}{x(\gamma)}\right)}\right)^{f^1(\delta)} \left(1 - e^{-2i\pi\left(\frac{w+x(\delta)}{x(\gamma)}\right)}\right)^{f^2(\delta)}.$$

Now, suppose that $f^1 \neq 0$. Then there exists some $\delta \in V$ such that $f^1(\delta) \neq 0$. Therefore, by definition of f^1, there is some index $l \leq j \leq n$ such that $\delta \in C_j^1$. By lemma II.12 the cone C_j^1 is a subset of $\cup_{k \neq j} C_k^1$ and therefore there is an index $l \leq j' \leq n$ distinct from j such that $\delta \in C_{j'}^1$. Then, since $\delta \in C_j^1 \cap C_{j'}^1$, using lemma II.17 we get that $f^1(\delta) = 0$ which is a contradiction. Therefore $f^1 = 0$ and similarly $f^2 = 0$, which gives the desired conclusion:

$$\prod_{j=1}^n G_{n-2,a_1,\dots,\widehat{a_j},\dots,a_n}(v)(w,x)^{(-1)^{j+1}} = 1.$$

□

Remark: Note that to prove the similar case $f^2 = 0$ one may reuse most of the work carried out in sections II.2.2.1 to II.2.2.5. Indeed, it is clear that the sign table governing the cones C_j^1 also governs the cones C_j^2 and that two cones C_j^2 and $C_{j'}^2$ are compatible if and only if C_j^1 and $C_{j'}^1$ are compatible. Thus, it is clear that lemma II.11, (iv) may be adapted for the cones C_j^2 as:

For any three distinct indices $l \leq j, j', j'' \leq n, C_j^2 \cap C_{j'}^2 \cap C_{j''}^2 = \emptyset$.

Next, there is only a small adaptation to make to express a version of lemma II.13, (i) for the cones C_j^2. Indeed, if C_j^2 and $C_{j'}^2$ are compatible then:

$$C_j^2 \cap C_{j'}^2 = C_j^2 \cap H_j^{-\varepsilon_{j'} d_j^{(j')}} = C_{j'}^2 \cap H_{j'}^{-\varepsilon_j d_{j'}^{(j)}}.$$

Another adaptation is in order for lemma II.15 as the condition $\nu \lambda_j \lambda_k \varepsilon_j d_k^{(j)} > 0$ for all index $l \leq k \leq n$ distinct from j implies that $C_j^1 \cap H_j^{-\nu} = \emptyset$. Lemma II.12 may then be directly adapted as for all $l \leq j \leq n$:

$$C_j^2 \subset \bigcup_{k \neq j} C_k^2.$$

Finally, a computation similar to the one carried out in the proof of lemma II.17 shows that $f^2 = 0$.

II.3 A classic smoothing operation

In this section we are interested in the smoothed versions of both $G_{n-2,a_1,\ldots,a_{n-1}}$ and $B_{n,a_1,\ldots,a_n}$ functions which are defined by:

$$G_{n-2,a_1,\ldots,a_{n-1}}(v)(w,x,L,L') = \frac{G_{n-2,a_1,\ldots,a_{n-1}}(v)(w,x,L')^N}{G_{n-2,a_1,\ldots,a_{n-1}}(v)(w,x,L)}.$$

and

$$B_{n,a_1,\ldots,a_n}(v)(w,x,L,L') = NB_{n,a_1,\ldots,a_n}(v)(w,x,L') - B_{n,a_1,\ldots,a_n}(v)(w,x,L)$$

for linear forms $a_1,\ldots,a_n$ which are primitive on both L and L' where $L/L' \simeq \mathbb{Z}/N\mathbb{Z}$. In this section we prove Theorem II.3 and explain how we deduce Theorem II.1. Let us now give an overview of this section. In section II.3.1 we give the geometric setup needed for the rest of the proof and give an explicit formula for the smoothed $B_{n,a_1,\ldots,a_n}(v)(w,x,L,L')$ when the linear forms $a_1,\ldots,a_n$ are linearly independent and the smoothing lattice L' is *good* for $a_1,\ldots,a_n$. Then, in section II.3.2 we prove that the function $B_{n,a_1,\ldots,a_n}(v)(w,x,L,L')$ is in fact a rational-valued function which depends only on the linear forms $a_1,\ldots,a_n$ and on the class of v in V/L' but not on $w,x \in \mathbb{C} \times \mathrm{Hom}_{\mathbb{Z}}(L,\mathbb{C})$. Finally, in section II.3.3 we use Fourier analysis following [CD14] to prove that the rational numbers $B_{n,a_1,\ldots,a_n}(v)(w,x,L,L')$ may be expressed in terms of traces of cyclotomic units from which we may deduce a bound on its denominator in terms of the dimension n and the smoothing index N. At the end of this section, we shall explain how to derive Theorem II.1 from this last result.

II.3.1 Geometric setup

In this section we consider the situation where V is a $\mathbb{Q}$-vector space of dimension n and L is a lattice of rank n in V with a $\mathbb{Z}$-basis $B = [e_1,\ldots,e_n]$. We fix an integer $N \geq 2$ and the smoothing lattice $L' = N\mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \cdots \oplus \mathbb{Z}e_n$. Denote by $\Lambda = \mathrm{Hom}_{\mathbb{Z}}(L,\mathbb{Z})$ and $\Lambda' = \mathrm{Hom}_{\mathbb{Z}}(L',\mathbb{Z})$ the dual spaces attached to L and L' respectively. Define $C = [f_1,\ldots,f_n]$ the $\mathbb{Z}$-basis of Λ dual to B such that $\forall\, 1 \leq j,k \leq n, f_j(e_k) = \delta_{jk}$. Similarly, the $\mathbb{Z}$-basis C' of Λ' dual to B' is given by $C' = [f_1/N, f_2,\ldots,f_n]$. One may view $\Lambda \subset \Lambda'$ as rank n lattices in the dual space $V^\vee = \mathrm{Hom}_{\mathbb{Q}}(V,\mathbb{Q})$. Let us now define $\Lambda_N \subset V^\vee$ to be the set of linear forms $a \in V^\vee$ which restrict to primitive integral linear forms on both L and L'. Explicitly:

$$\Lambda_N = \left\{ \sum_{k=1}^{n} \mu_k f_k \mid \mu_1,\ldots,\mu_n \in \mathbb{Z},\ \gcd(\mu_1 N, \mu_2,\ldots,\mu_n) = 1 \right\}.$$

The set Λ_N is endowed with an action of the congruence subgroup $\Gamma_0(N,n) \subset \mathrm{SL}_n(\mathbb{Z})$ (see (II.2)) given by $g\cdot(\mu_1,\mu_2,\ldots,\mu_n) = (\mu_1,\mu_2,\ldots,\mu_n)\times g^{-1}$. For the rest of this section we fix non-zero linear forms $a_1,\ldots,a_n$ in Λ_N which are linearly independent. This fixes a unique family of primitive vectors $\alpha_1,\ldots,\alpha_n \in L$ such that for all $1 \leq j \leq n$:

$$a_j(\alpha_k) = 0, \forall\, 1 \leq k \neq j \leq n \quad \text{and} \quad a_j(\alpha_j) = s_j > 0$$

This is the primitive positive dual basis to $a_1,\ldots,a_n$ in L in the sense of [[Mor25], lemma 6]. This definition comes from the theory of rational polyhedral cones as the cone:

$$C := \{\delta \in V \mid \forall 1 \leq j \leq n, a_j(\delta) \geq 0\}$$

can be expressed in terms of generators as:

$$C = \mathbb{Q}_{\geq 0}\alpha_1 + \cdots + \mathbb{Q}_{\geq 0}\alpha_n.$$

We now recall the definition of a *good* smoothing lattice.

Definition II.18: *Suppose that $a_1, \ldots, a_n \in \Lambda$ are linearly independent primitive linear forms and let $\alpha_1, \ldots, \alpha_n$ be the primitive positive dual basis to $a_1, \ldots, a_n$ in L. The smoothing lattice L' of index N in L is said to be good for $a_1, \ldots, a_n$ if $a_1, \ldots, a_n \in \Lambda_N$ and the primitive positive dual basis $\alpha'_1, \ldots, \alpha'_n$ to $a_1, \ldots, a_n$ in L' is precisely $N\alpha_1, \ldots, N\alpha_n$.*

Notice that in general if k_j is the order of α_j mod L' in the cyclic group L/L' then it is clear that $\alpha'_j = k_j\alpha_j$, thus the condition α_j mod L' generates L/L' is equivalent to $\alpha'_j = N\alpha_j$. In the case where $a_1, \ldots, a_n$ are not linearly independent, we say that the smoothing lattice L' is *good* for $a_1, \ldots, a_n$ if $a_1, \ldots, a_n \in \Lambda_N$. Definition (II.18) is inspired by the definition of a good smoothing ideal in [Das08]. Notice that in our case we do not suppose that N is prime. Let us now give an equivalent formulation of this statement as a condition on the coordinates of $\alpha_1, \ldots, \alpha_n$ in the basis $B = [e_1, \ldots, e_n]$ which will be useful in the proof of Theorem II.3.

Lemma II.19: *Let $a_1, \ldots, a_n \in \Lambda$ be linearly independent. Let $\alpha_1, \ldots, \alpha_n$ be the primitive positive dual family to $a_1, \ldots, a_n$ in L. For all $1 \leq j \leq n$, write*

$$\alpha_j = \sum_{j=1}^{n} \alpha_{k,j} e_k$$

with $\alpha_{k,j} \in \mathbb{Z}$. The order of α_j mod L' in L/L' is precisely $n_j = N/\gcd(N, \alpha_{1,j})$. In particular, the smoothing lattice L' is good for $a_1, \ldots, a_n$ if and only if $\gcd(N, \alpha_{1,j}) = 1$ for all $1 \leq j \leq n$.

Proof :
Let $\alpha'_1, \ldots, \alpha'_n$ be the primitive positive dual family to $a'_1, \ldots, a'_n$ in L'. Then $\alpha'_1, \ldots, \alpha'_n$ is also a positive dual family to $a_1, \ldots, a_n$ in L. It follows from lemma I.6 that there are positive integers $m_1, \ldots, m_n$ such that $\alpha'_j = m_j\alpha_j$ for all $1 \leq j \leq n$ where $\alpha_1, \ldots, \alpha_n$ is the primitive positive dual family to $a_1, \ldots, a_n$ in L. Write for all $1 \leq j \leq n$:

$$\alpha_j = \sum_{k=1}^{n} \alpha_{k,j} e_k$$

with $\alpha_{k,j} \in \mathbb{Z}$ and $\gcd(\alpha_{1,j}, \ldots, \alpha_{n,j}) = 1$. By definition of the integer $n_j = N/\gcd(N, \alpha_{1,j}) = N/l_j$, the vector

$$n_j\alpha_j = \frac{\alpha_{1,j}}{l_j}(Ne_1) + \sum_{k=2}^{n} \frac{N\alpha_{k,j}}{l_j} e_k$$

belongs to L' and $\gcd(\alpha_{1,j}/l_j, N\alpha_{2,j}/l_j, \ldots, N\alpha_{n,j}/l_j) = 1$. In particular, $n_j\alpha_j$ is a primitive vector in L' and $\alpha'_j = (m_j/n_j)(n_j\alpha_j)$ is also a primitive vector in L'. Therefore $m_j = n_j$ and $\alpha'_j = n_j\alpha_j$. □

For the rest of section II.3 we suppose that the smoothing lattice L' is indeed *good* for the linear forms $a_1, \ldots, a_n$ and we focus on the case where $a_1, \ldots, a_n$ are linearly independent. The goal of this section is to give an explicit formulation for the rational function $B_{n,a_1,\dots,a_n}(v)(w,x,L,L')$ in terms of periodic Bernoulli polynomials. Indeed, let us recall the definition of the classic Bernoulli polynomials using the generating series:

$$\frac{e^{Xz}}{e^z-1}=\sum_{k\geq 0}B_k(X)\frac{z^{k-1}}{k!}. \tag{II.31}$$

We may introduce the periodic versions of the Bernoulli polynomials $b_k(x)=B_k(x-\lfloor x\rfloor)$ for $x\in\mathbb{R}$. In this section we shall prove the following:

Proposition II.20: *Assume that $a_1,\dots,a_n\in\Lambda_N$ are linearly independent and that the smoothing lattice L' is good for $a_1,\dots,a_n$. Let $\alpha_1,\dots,\alpha_n$ be the positive dual basis to $a_1,\dots,a_n$. Fix a set $\mathcal{F}$ of representatives for L/M where $M=\oplus_{j=1}^n\mathbb{Z}\alpha_j$. Then there are explicit integers $r_j(\delta)\in\mathbb{Z}$ for all $1\leq j\leq n$ and all $\delta=\sum_{j=1}^n\delta_j\alpha_j/s_j\in\mathcal{F}$ such that:*

$$B_{n,a_1,\dots,a_n}(v)(w,x,L,L')=\epsilon\sum_{m=0}^n\frac{w^m}{m!}\sum_{k_1+\dots+k_n=n-m}\sum_{\delta\in\mathcal{F}}Y(k_1,\dots,k_n,v,\delta)\prod_{j=1}^n\frac{x(\alpha_j)^{k_j-1}}{k_j!}$$

where $\epsilon=\operatorname{sign}\det(a_1,\dots,a_n)$, $v=\sum_{j=1}^n v_j\alpha_j/s_j$,

$$Y(k_1,\dots,k_n,v,\delta)=N\sum_{q\in Q}\prod_{j=1}^n b_{k_j}\left(\frac{v_j+\delta_j+(r_j(\delta)+q_j)s_j}{Ns_j}\right)N^{k_j-1}-\prod_{j=1}^n b_{k_j}\left(\frac{v_j+\delta_j}{s_j}\right) \tag{II.32}$$

and

$$Q=\left\{(q_1,\dots,q_n)\in(\mathbb{Z}/N\mathbb{Z})^n\mid\sum_{j=1}^n q_j\alpha_{1,j}\equiv 0 \bmod N\right\}. \tag{II.33}$$

To prove Proposition II.20 we shall first give an explicit description of both functions $B_{n,a_1,\dots,a_n}(v)(w,x,L)$ and $B_{n,a_1,\dots,a_n}(v)(w,x,L')$ individually in terms of periodic Bernoulli polynomials. Then, we shall show how to express any set of representatives $\mathcal{F}'$ of L'/M' in terms of a fixed set of representatives $\mathcal{F}$ of L/M and in terms of the set Q given by (II.33). Let us start with the explicit description in terms of periodic Bernoulli polynomials:

Lemma II.21: *Fix a set of representatives $\mathcal{F}$ (resp. $\mathcal{F}'$) for L/M (resp. L'/M'). Then:*

$$B_{n,a_1,\dots,a_n}(v)(w,x,L)=\epsilon\sum_{m=0}^n\frac{w^m}{m!}\sum_{k_1+\dots+k_n=n-m}\sum_{\delta\in\mathcal{F}}\prod_{j=1}^n\frac{b_{k_j}(\frac{v_j+\delta_j}{s_j})x(\alpha_j)^{k_j-1}}{k_j!}$$

$$B_{n,a_1,\dots,a_n}(v)(w,x,L')=\epsilon\sum_{m=0}^n\frac{w^m}{m!}\sum_{k_1+\dots+k_n=n-m}\sum_{\delta'\in\mathcal{F}'}\prod_{j=1}^n\frac{b_{k_j}(\frac{v_j+\delta'_j}{Ns_j})N^{k_j-1}x(\alpha_j)^{k_j-1}}{k_j!}$$

where $\epsilon=\operatorname{sign}\det(a_1,\dots,a_n)$, $v=\sum_{j=1}^n v_j\alpha_j/s_j$ and the sums range over integers $k_1\geq 0,\dots,k_n\geq 0$.

Proof :

We recall that by definition (see (I.18) and (I.13)) :

$$B_{n,a_1,\dots,a_n}(v)(w,x,L)=\epsilon\times\operatorname{coeff}[t^0]\left(\sum_{\delta\in(v+L)\cap P(\underline{a})}\frac{e^{wt}e^{x(\delta)t}}{\prod_{j=1}^n(1-e^{x(\alpha_j)t})}\right)$$

$$B_{n,a_1,\dots,a_n}(v)(w,x,L') = \epsilon \times \mathrm{coeff}[t^0] \left(\sum_{\delta' \in (v+L') \cap N.P(\underline{a})} \frac{e^{wt} e^{x(\delta')t}}{\prod_{j=1}^n (1 - e^{x(N\alpha_j)t})} \right)$$

where $\epsilon = \mathrm{sign}\det(a_1, \dots, a_n)$. Let us define for simplicity $F = \{\delta \in L \mid , v + \delta \in P(\underline{a})\}$ and $F' = \{\delta' \in L' \mid , v + \delta' \in N.P(\underline{a})\}$. It is clear that the set F (resp. F') is the unique set of representatives for L/M (resp. L'/M') such that $v + F \subset P(\underline{a})$ (resp. $v + F' \subset N.P(\underline{a})$). Let us write these sets explicitly as:

$$F = \left\{ \sum_{j=1}^n \frac{\delta_j \alpha_j}{s_j} \in L \mid \forall 1 \le j \le n, 0 \le v_j + \delta_j < s_j, \delta_j \in \mathbb{Z} \right\}$$
$$F' = \left\{ \sum_{j=1}^n \frac{\delta'_j . N\alpha_j}{Ns_j} \in L' \mid \forall 1 \le j \le n, 0 \le v_j + \delta'_j < Ns_j, \delta'_j \in \mathbb{Z} \right\}$$

so that using the definition of the classic Bernoulli polynomials (see (II.31)) we get:

$$B_{n,a_1,\dots,a_n}(v)(w,x,L) = \epsilon \sum_{m=0}^n \frac{w^m}{m!} \sum_{k_1+\dots+k_n=n-m} \sum_{\delta \in F} \prod_{j=1}^n \frac{B_{k_j}(\frac{v_j+\delta_j}{s_j}) x(\alpha_j)^{k_j-1}}{k_j!}$$
$$B_{n,a_1,\dots,a_n}(v)(w,x,L') = \epsilon \sum_{m=0}^n \frac{w^m}{m!} \sum_{k_1+\dots+k_n=n-m} \sum_{\delta' \in F'} \prod_{j=1}^n \frac{B_{k_j}(\frac{v_j+\delta'_j}{Ns_j}) N^{k_j-1} x(\alpha_j)^{k_j-1}}{k_j!}$$

Since for all $\delta \in F$ (resp. $\delta' \in F'$) and all $1 \le j \le n$, $0 \le (v_j + \delta_j)/s_j < 1$ (resp. $0 \le (v_j + \delta_j)/(Ns_j) < 1$) we may rewrite this using the periodic Bernoulli polynomials and then replace the sets F and F' with any sets of representatives $\mathcal{F}$ and $\mathcal{F}'$ for L/M and L'/M' respectively. This gives:

$$B_{n,a_1,\dots,a_n}(v)(w,x,L) = \epsilon \sum_{m=0}^n \frac{w^m}{m!} \sum_{k_1+\dots+k_n=n-m} \sum_{\delta \in \mathcal{F}} \prod_{j=1}^n \frac{b_{k_j}(\frac{v_j+\delta_j}{s_j}) x(\alpha_j)^{k_j-1}}{k_j!}$$
$$B_{n,a_1,\dots,a_n}(v)(w,x,L') = \epsilon \sum_{m=0}^n \frac{w^m}{m!} \sum_{k_1+\dots+k_n=n-m} \sum_{\delta' \in \mathcal{F}'} \prod_{j=1}^n \frac{b_{k_j}(\frac{v_j+\delta'_j}{Ns_j}) N^{k_j-1} x(\alpha_j)^{k_j-1}}{k_j!}.$$

Indeed, if $\tilde{\delta} = \sum_{j=1}^n \tilde{\delta}_j \alpha_j / s_j \in L$ represents the same class in L/M as δ then there are integers $m_1, \dots, m_n$ such that $\tilde{\delta} = \delta + \sum_{j=1}^n m_j \alpha_j$. Therefore:

$$b_{k_j}\left(\frac{v_j + \tilde{\delta}_j}{s_j}\right) = b_{k_j}\left(\frac{v_j + \delta_j}{s_j} + m_j\right) = b_{k_j}\left(\frac{v_j + \delta_j}{s_j}\right)$$

and the explicit description above do not depend on the choice of representatives for L/M and L'/M'. This completes the proof. □

Let us now give an explicit link between the quotient sets L/M and L'/M'. Fix sets of representatives $\mathcal{F}$ and $\mathcal{F}'$ for L/M and L'/M' respectively. Let us remark that the identifications $\mathcal{F} \simeq L/M$ and $\mathcal{F}' \simeq L'/M'$ induce group structures on both $\mathcal{F}$ and $\mathcal{F}'$ defined respectively by:

$$\delta_1 * \delta_2 \equiv \delta_1 + \delta_2 \bmod M, \quad \delta'_1 * \delta'_2 \equiv \delta'_1 + \delta'_2 \bmod M'$$

The neutral elements of $\mathcal{F}$ and $\mathcal{F}'$ correspond to the representative for the trivial classes M and M' in L/M and L'/M'. We now relate $\mathcal{F}$ and $\mathcal{F}'$ for a *good* smoothing lattice L'.

Lemma II.22: *Suppose that the smoothing lattice L' is good for the linear forms $a_1, \ldots, a_n$. Consider the map*

$$f := \begin{cases} \mathcal{F}' & \to \mathcal{F} \\ \sum_{j=1}^n \frac{\delta'_j \alpha_j}{s_j} & \to \sum_{j=1}^n \frac{\delta_j \alpha_j}{s_j} \bmod M \end{cases}$$

where the δ_j's are given by Euclidian division as $\delta'_j = q_j s_j + \delta_j$ with $0 \le v_j + \delta_j < s_j$ and $\delta_j \in \mathbb{Z}$. Here it is understood that $f(\delta')$ is the representative in $\mathcal{F}$ for the class $\sum_{j=1}^n \delta_j \alpha_j / s_j \bmod M$. The map f is a N^{n-1} to 1 surjective group morphism and its kernel is isomorphic to the group:

$$Q = \left\{ (q_1, \ldots, q_n) \in \mathbb{Z}/N\mathbb{Z}^n \mid \sum_{j=1}^n q_j \alpha_{1,j} \equiv 0 \bmod N \right\}.$$

Proof :

Let us first prove that the map f is surjective. Consider $\delta = \sum_{j=1}^n \frac{\delta_j \alpha_j}{s_j} \in \mathcal{F}$ and write in coordinates:

$$\delta = \sum_{j=1}^n \frac{\delta_j}{s_j} \alpha_{1,j} e_1 + \sum_{k=2}^n \sum_{j=1}^n \frac{\delta_j}{s_j} \alpha_{k,j} e_k$$

The assumption that $\delta \in L$ is equivalent to $\sum_{j=1}^n \frac{\delta_j}{s_j} \alpha_{k,j} \in \mathbb{Z}$ for all $1 \le k \le n$. We now wish to find integers $r_1, \ldots, r_n \in \{0, \ldots, N-1\}^n$ such that

$$\sum_{j=1}^n \frac{\delta_j}{s_j} \alpha_{1,j} + \sum_{j=1}^n r_j \alpha_{1,j} \in N\mathbb{Z}.$$

It follows from lemma II.19 that the integer $\alpha_{1,1}$ is coprime to N so there is an integer $\beta \in \mathbb{Z}$ such that $\beta \alpha_{1,1} \equiv 1 \bmod N$. Let r_1 be the remainder in the Euclidian division of the integer $-\beta \sum_{j=1}^n \frac{\delta_j}{s_j} \alpha_{1,j}$ by N satisfying $0 \le r_1 < N$. Fix $r_2 = \cdots = r_n = 0$. Then:

$$\sum_{j=1}^n \frac{\delta_j}{s_j} \alpha_{1,j} + \sum_{j=1}^n r_j \alpha_{1,j} \equiv \left(\sum_{j=1}^n \frac{\delta_j}{s_j} \alpha_{1,j} \right) (1 - \beta \alpha_{1,1}) \equiv 0 \bmod N$$

This shows that $\tilde{\delta} = \delta + \sum_{j=1}^n r_j \alpha_j \in L'$. Let δ' be the representative in $\mathcal{F}'$ for the class $[\tilde{\delta}]$, that is $\delta' = \delta + \sum_{j=1}^n (r_j + m_j N) \alpha_j$ for some integers $m_1, \ldots, m_n$. Then it is clear that $f(\delta') = \delta$. Thus f is surjective. In addition, the preimage $f^{-1}(\delta)$ is explicitly given by:

$$f^{-1}(\delta) = \left\{ \left[\delta + \sum_{j=1}^n r_j \alpha_j + \sum_{j=1}^n q_j \alpha_j \right] \bmod M' \mid (q_1, \ldots, q_n) \in Q \right\}.$$

Indeed, if $f(\delta') = \delta$ then for all $1 \le j \le n$, $\delta'_j = (r_j + q_j) s_j + \delta_j$ for some $q_j \in \mathbb{Z}$. The condition that $\delta'_j \in L'$ is equivalent to $\sum_{j=1}^n (\delta_j + s_j(r_j + q_j)) \alpha_{1,j} \in N\mathbb{Z}$ which by definition of the r_j's is equivalent to $\sum_{j=1}^n q_j \alpha_{1,j} \in N\mathbb{Z}$ i.e. $(q_1, \ldots, q_n) \in Q$. In particular, $\ker f \simeq Q$.

Let us now prove that f is a group morphism. Consider $\delta_1', \delta_2' \in \mathcal{F}'$, define $\delta_3' = \delta_1' * \delta_2'$ and write for $k = 1, 2, 3$:

$$\delta_k' = \sum_{j=1}^{n} \frac{\delta_{k,j}'\alpha_j}{s_j}$$

The definition of the group law gives that

$$\delta_{3,j}' = (\delta_{1,j}' + \delta_{2,j}') + Ns_j m_j' \text{ for some } m_j' \in \mathbb{Z} \tag{II.34}$$

Define for $k = 1, 2, 3$ and $1 \le j \le n$ the unique integers $\delta_{k,j}$ satisfying:

$$0 \le \delta_{k,j} < a_j(\alpha_j), \quad \delta_{k,j}' = q_{k,j}s_j + \delta_{k,j}, q_{k,j} \in \mathbb{Z}$$

so that $f(\delta_k') = \sum_{j=1}^{n} \frac{\delta_{k,j}\alpha_j}{s_j}$ for $k = 1, 2, 3$. The definition of the group law on $\mathcal{F}$ gives:

$$f(\delta_1') * f(\delta_2') \equiv \sum_{j=1}^{n} \frac{\delta_{1,j} + \delta_{2,j}\alpha_j}{s_j} \bmod M$$

whereas $f(\delta_3') \equiv \sum_{j=1}^{n} \frac{\delta_{3,j}\alpha_j}{s_j} \bmod M$. We shall prove that $\delta_{1,j} + \delta_{2,j} - \delta_{3,j} \in s_j\mathbb{Z}$ for $1 \le j \le n$. Indeed, by definition of $\delta_{k,j}$ we get:

$$\delta_{1,j} + \delta_{2,j} = \delta_{1,j}' - q_{1,j}s_j + \delta_{2,j}' - q_{2,j}s_j$$

Using (II.34) we get:

$$\delta_{1,j} + \delta_{2,j} = \delta_{3,j}' + Ns_j m_j' - q_{1,j}s_j - q_{2,j}s_j = \delta_{3,j} + s_j(q_{3,j} + Nm_j' - q_{1,j} - q_{2,j}) \equiv \delta_{3,j} \bmod s_j\mathbb{Z}$$

This show that $\delta_1 + \delta_2 \equiv \delta_3 \bmod M$ and so that:

$$f(\delta_1' * \delta_2') = f(\delta_3') = f(\delta_1') * f(\delta_2').$$

Thus f is a group morphism. Next, we prove that f is an N^{n-1} to 1 map. This is given by the snake lemma for the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & \ker f & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & M' & \to & L' & \to & \mathcal{F}' & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & M & \to & L & \to & \mathcal{F} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & (\mathbb{Z}/N\mathbb{Z})^n & & \mathbb{Z}/N\mathbb{Z} & & 0 & &
\end{array}$$

The snake lemma gives a connecting group morphism $\ker f \to (\mathbb{Z}/N\mathbb{Z})^n$ such that the following sequence is exact:

$$0 \to 0 \to \ker f \to (\mathbb{Z}/N\mathbb{Z})^n \to \mathbb{Z}/N\mathbb{Z} \to 0$$

As a consequence, $\# \ker f = N^{n-1}$ and f is an N^{n-1} to 1 map. □

As an immediate consequence of this lemma we obtain the following corollary:

Corollary II.23: *For any set of representatives $\mathcal{F}$ for L/M, and any set of representatives $\mathcal{Q}$ for Q, the set*

$$\mathcal{F}' := \{\delta + \sum_{j=1}^{n}(r_j(\delta) + q_j)\alpha_j \mid (q_1, \ldots, q_n) \in \mathcal{Q}\}$$

is a set of representatives for L'/M', where the $r_j(\delta)$'s are defined in the proof of lemma II.22.

We may now prove Proposition II.20 with this particular choice of sets of representatives for both L/M and L'/M'.

Proof of Proposition II.20:

Let us fix two sets of representatives $\mathcal{F}$ and $\mathcal{Q}$ for L/M and Q respectively. Denote by $\mathcal{F}'$ the set of representatives for L'/M' given by Corollary II.23. It follows from lemma II.21 that:

$$B_{n,a_1,\ldots,a_n}(v)(w,x,L') = \epsilon \sum_{m=0}^{n} \frac{w^m}{m!} \sum_{k_1+\cdots+k_n=n-m} \sum_{\delta' \in \mathcal{F}'} \prod_{j=1}^{n} \frac{b_{k_j}(\frac{v_j+\delta'_j}{Ns_j})N^{k_j-1}x(\alpha_j)^{k_j-1}}{k_j!}$$

where $\epsilon = \operatorname{sign}\det(a_1, \ldots, a_n)$. Using Corollary II.23 we may rewrite this using a double sum on $\mathcal{F}$ and $\mathcal{Q}$ as:

$$B_{n,a_1,\ldots,a_n}(v)(w,x,L') = \epsilon \sum_{m=0}^{n} \frac{w^m}{m!} \sum_{k_1+\cdots+k_n=n-m} \sum_{\delta \in \mathcal{F}} \sum_{q \in \mathcal{Q}} \prod_{j=1}^{n} \frac{b_{k_j}(\frac{v_j+\delta_j+(r_j(\delta)+q_j)s_j}{Ns_j})N^{k_j-1}x(\alpha_j)^{k_j-1}}{k_j!}$$

Note that this expression is independent of the choice of representative set $\mathcal{Q}$ for Q. Thus, by definition of the smoothed function $B_{n,a_1,\ldots,a_n}(v)(w,x,L,L')$ (see (II.4)) and using once again lemma II.21:

$$B_{n,a_1,\ldots,a_n}(v)(w,x,L,L') = \epsilon \sum_{m=0}^{n} \frac{w^m}{m!} \sum_{k_1+\cdots+k_n=n-m} \sum_{\delta \in \mathcal{F}} Y(k_1,\ldots,k_n,v,\delta) \prod_{j=1}^{n} \frac{x(\alpha_j)^{k_j-1}}{k_j!}$$

where for all $k_1 \geq 0, \ldots, k_n \geq 0$ such that $\sum_{j=1}^{n} k_j \leq n$ and for all $\delta \in \mathcal{F}$:

$$Y(k_1,\ldots,k_n,v,\delta) = N \sum_{q \in Q} \prod_{j=1}^{n} b_{k_j}\left(\frac{v_j+\delta_j+(r_j(\delta)+q_j)s_j}{Ns_j}\right) N^{k_j-1} - \prod_{j=1}^{n} b_{k_j}\left(\frac{\delta_j}{s_j}\right).$$

□

II.3.2 Smoothed Bernoulli rational functions are rational-valued

The goal of this section is to prove the following rationality statement for the smoothed $B_{n,a_1,\ldots,a_n}(v)(w,x,L,L')$ functions.

Proposition II.24: *Assume that $a_1, \ldots, a_n \in \Lambda_N$ are linearly independent and that the smoothing lattice L' is good for $a_1, \ldots, a_n$. Let $\alpha_1, \ldots, \alpha_n$ be the positive dual basis to*

$a_1, \dots, a_n$. Put as before $\epsilon = \operatorname{sign}\det(a_1, \dots, a_n)$ and fix a set $\mathcal{F}$ of representatives for L/M. Then:

$$B_{n,a_1,\dots,a_n}(v)(w,x,L,L') = \epsilon \sum_{\delta\in\mathcal{F}} \left(N \sum_{q\in Q} \prod_{j=1}^{n} b_1 \left(\frac{v_j + \delta_j + (r_j(\delta)+q_j)s_j}{Ns_j} \right) - \prod_{j=1}^{n} b_1 \left(\frac{v_j+\delta_j}{s_j} \right) \right)$$

where the integers $r_j(\delta)$ are given in the proof of lemma II.22 and Q is defined in Proposition II.20. In particular $B_{n,a_1,\dots,a_n}(v)(w,x,L,L')$ is a rational number which depends only on the linear forms $a_1, \dots, a_n$ and on the class of v in V/L' but not on $w, x \in \mathbb{C} \times \operatorname{Hom}_{\mathbb{Z}}(L, \mathbb{C})$.

This Proposition essentially expresses the smoothed function $B_{n,a_1,\dots,a_n}(v)(w,x,L,L')$ in terms of a smoothed higher Dedekind sum. To prove this statement we shall prove that each term $Y(k_1, \dots, k_n, v, \delta)$ vanishes, unless $k_1 = k_2 = \dots = k_n = 1$. This is exactly the claim of the following crucial lemma:

Lemma II.25: *Suppose that $k_1, \dots, k_n$ are non-negative integers such that $\sum_{j=1}^n k_j \le n$. Suppose that $(k_1, \dots, k_n) \neq (1, \dots, 1)$. Then $Y(k_1, \dots, k_n, v, \delta) = 0$ for any $\delta \in \mathcal{F}$.*

Proof :

Let us fix non-negative integers $k_1, \dots, k_n$ such that $\sum_{j=1}^n k_j \le n$. The condition $(k_1, \dots, k_n) \neq (1, \dots, 1)$ is equivalent to the existence of an index $1 \le j \le n$ such that $k_j = 0$. Define $J = \{1 \le j \le n \mid k_j \neq 0\}$ and $J^c = \{1, \dots, n\} - J \neq \emptyset$. Let us define a map:

$$g_J := \begin{cases} Q & \to \prod_{j\in J} \mathbb{Z}/N\mathbb{Z} = Q(J) \\ (q_1, \dots, q_n) & \to (q_j)_{j\in J} \end{cases}$$

This map is clearly a group morphism, and we shall prove that it is surjective. Indeed, by assumption, J^c is not empty so we may fix an index $j' \in J^c$. The condition that the smoothing lattice L' is good for the linear forms $a_1, \dots, a_n$ implies that $\alpha_{1,j'}$ is coprime to N. Therefore, for any element $(q_j)_{j\in J} \in Q(J)$ there is an integer $q_{j'} \in \mathbb{Z}/N\mathbb{Z}$ such that $\sum_{j\in J} q_j \alpha_{1,j} + q_{j'}\alpha_{1,j'} \equiv 0 \bmod N$. Setting $q_{j''} = 0$ for all $j'' \in J^c - \{j'\}$ gives $(q_1, \dots, q_n) \in Q$ and $g_J(q_1, \dots, q_n) = (q_j)_{j\in J}$. Therefore, g_J is surjective. We shall use the function g_J to switch the sum and product in the expression of $Y(k_1, \dots, k_n, \delta)$. First, let us rewrite $Y(k_1, \dots, k_n, \delta)$ in terms of the set J:

$$Y(k_1, \dots, k_n, v, \delta) = N^{1-n} \sum_{q\in Q} \prod_{j=1}^{n} b_{k_j} \left(\frac{v_j + \delta_j + (r_j(\delta)+q_j)s_j}{Ns_j} \right) N^{k_j} - \prod_{j=1}^{n} b_{k_j} \left(\frac{v_j+\delta_j}{s_j} \right)$$

$$Y(k_1, \dots, k_n, v, \delta) = N^{1-n} \sum_{q\in Q} \prod_{j\in J} b_{k_j} \left(\frac{v_j + \delta_j + (r_j(\delta)+q_j)s_j}{Ns_j} \right) N^{k_j} - \prod_{j\in J} b_{k_j} \left(\frac{v_j+\delta_j}{s_j} \right)$$

where we have used the fact that if $j \notin J$ then $b_{k_j} = b_0$ is the constant function equal to 1. From this expression it is clear that the term

$$\prod_{j\in J} b_{k_j} \left(\frac{v_j + \delta_j + (r_j(\delta)+q_j)s_j}{Ns_j} \right) N^{k_j}$$

only depends on the image of $g(q) \in Q_J$ of q, therefore:

$$Y(k_1, \dots, k_n, v, \delta) = N^{1-n} \# \ker(g_J) \sum_{q_J\in Q(J)} \prod_{j\in J} b_{k_j} \left(\frac{v_j + \delta_j + (r_j(\delta)+q_j)s_j}{Ns_j} \right) N^{k_j} - \prod_{j\in J} b_{k_j} \left(\frac{v_j+\delta_j}{s_j} \right)$$

We may now use the fact that $\#\ker(g_j) = N^{n-1-\#J}$ and then the fact that $Q_J = \prod_{j\in J}\mathbb{Z}/N\mathbb{Z}$ to switch sum and product in the expression of $Y(k_1,\ldots,k_n,\delta)$ to obtain:

$$Y(k_1,\ldots,k_n,v,\delta) = \sum_{q_J\in Q(J)}\prod_{j\in J} b_{k_j}\left(\frac{v_j+\delta_j+(r_j(\delta)+q_j)s_j}{Ns_j}\right)N^{k_j-1} - \prod_{j\in J} b_{k_j}\left(\frac{v_j+\delta_j}{s_j}\right)$$
$$Y(k_1,\ldots,k_n,v,\delta) = \prod_{j\in J}\sum_{q_j\in\mathbb{Z}/N\mathbb{Z}} b_{k_j}\left(\frac{v_j+\delta_j+(r_j(\delta)+q_j)s_j}{Ns_j}\right)N^{k_j-1} - \prod_{j\in J} b_{k_j}\left(\frac{v_j+\delta_j}{s_j}\right)$$

It follows from the well-known distribution relation:

$$\sum_{k\in\mathbb{Z}/N\mathbb{Z}} b_m\left(\frac{x+k}{N}\right)N^{m-1} = b_m(x) \tag{II.35}$$

applied here to $x = (v_j+\delta_j+r_j(\delta)s_j)/s_j$ that:

$$Y(k_1,\ldots,k_n,v,\delta) = \prod_{j\in J} b_{k_j}\left(\frac{v_j+\delta_j+r_j(\delta)s_j}{s_j}\right) - \prod_{j\in J} b_{k_j}\left(\frac{v_j+\delta_j}{s_j}\right).$$

Since the b_{k_j} functions are 1-periodic and $r_j\in\mathbb{Z}$ we get the desired conclusion

$$Y(k_1,\ldots,k_n,v,\delta) = 0.$$

□

We are now ready to deduce Proposition II.24 from lemma II.25.

Proof of Proposition II.24:
It follows from Proposition II.20 that:

$$B_{n,a_1,\ldots,a_n}(v)(w,x,L,L') = \epsilon\sum_{m=0}^{n}\frac{w^m}{m!}\sum_{k_1+\cdots+k_n=n-m}\sum_{\delta\in\mathcal{F}} Y(k_1,\ldots,k_n,v,\delta)\prod_{j=1}^{n}\frac{x(\alpha_j)^{k_j-1}}{k_j!}$$

where $\epsilon = \operatorname{sign}\det(a_1,\ldots,a_n)$. Since for all $(k_1,\ldots,k_n)\neq(1,\ldots,1)$ and all $\delta\in\mathcal{F}$, $Y(k_1,\ldots,k_n,v,\delta)=0$ the sum reduces to:

$$B_{n,a_1,\ldots,a_n}(v)(w,x,L,L') = \epsilon\sum_{\delta\in\mathcal{F}} Y(1,\ldots,1,v,\delta)$$

which gives exactly the desired expression when replacing $Y(1,\ldots,1,v,\delta)$ by its definition:

$$B_{n,a_1,\ldots,a_n}(v)(w,x,L,L') = \epsilon\sum_{\delta\in\mathcal{F}}\left(N\sum_{q\in Q}\prod_{j=1}^{n} b_1\left(\frac{v_j+\delta_j+(r_j(\delta)+q_j)s_j}{Ns_j}\right) - \prod_{j=1}^{n} b_1\left(\frac{v_j+\delta_j}{s_j}\right)\right).$$

It is then clear that $B_{n,a_1,\ldots,a_n}(v)(w,x,L,L')$ is a rational number depending only on the linear forms $a_1,\ldots,a_n$ and on the class of v in V/L' but not on $w,x\in\mathbb{C}\times\operatorname{Hom}_{\mathbb{Z}}(L,\mathbb{C})$.
□

II.3.3 Smoothed Bernoulli rational functions have bounded denominators

In this section we carry out the proof of Theorem II.3 by expressing the smoothed functions $B_{n,a_1,\ldots,a_n}(v)(w,x,L,L')$ in terms of traces of cyclotomic units, thus proving a uniform bound on their denominator in terms of the dimension n and the smoothing index N. To achieve this, we will use the Fourier transformation on the finite group $\mathbb{Z}/N\mathbb{Z}$ to rewrite the expression obtained in Proposition II.24 borrowing ideas from [CD14]. Let us denote by $\zeta = \zeta_N = \exp(2i\pi/N)$ a primitive N-th root of unity. Introduce the auxiliary function $\chi : (\mathbb{Z}/N\mathbb{Z})^n \to \{0, N\}$ defined by:

$$\chi(q) = \sum_{k\in\mathbb{Z}/N\mathbb{Z}} \zeta^{k.\sum_{j=1}^n q_j\alpha_{1,j}} = \begin{cases} 0 \text{ if } q \notin Q \\ N \text{ if } q \in Q \end{cases}$$

where $\alpha_{1,j} = \langle \alpha_j, e_1\rangle$. We may then write following Proposition II.24 and using the auxiliary function χ:

$$B_{n,a_1,\ldots,a_n}(v)(w,x,L,L') = \epsilon \sum_{\delta\in\mathcal{F}} \left(Z(\delta) - \prod_{j=1}^n b_1\left(\frac{v_j+\delta_j}{s_j}\right)\right) \tag{II.36}$$

where

$$Z(\delta) := \sum_{q\in\mathbb{Z}/N\mathbb{Z}} \chi(q) \prod_{j=1}^n b_1\left(\frac{v_j+\delta_j+(r_j(\delta)+q_j)s_j}{Ns_j}\right).$$

Using the definition of χ we may rewrite this auxiliary function $Z(\delta)$ as:

$$Z(\delta) = \sum_{q\in\mathbb{Z}/N\mathbb{Z}^n}\sum_{k\in\mathbb{Z}/N\mathbb{Z}}\prod_{j=1}^n \zeta^{q_j\alpha_{1,j}k} b_1\left(\frac{v_j+\delta_j+(r_j(\delta)+q_j)s_j}{Ns_j}\right)$$

and we remark that the sum over $q \in \mathbb{Z}/N\mathbb{Z}^n$ may be then be inverted with the product over $1 \le j \le n$ so:

$$Z(\delta) = \sum_{k\in\mathbb{Z}/N\mathbb{Z}}\prod_{j=1}^n\sum_{q_j\in\mathbb{Z}/N\mathbb{Z}} \zeta^{q_j\alpha_{1,j}k} b_1\left(\frac{v_j+\delta_j+(r_j(\delta)+q_j)s_j}{Ns_j}\right). \tag{II.37}$$

The proof of Theorem II.3 will essentially follow from the following lemma which is a reformulation of [[CD14], Lemma 2.13].

Lemma II.26: *Suppose $N \ge 2$ is an integer. If $x \in \mathbb{R}$ and $y \in \mathbb{F}_N - \{0\}$:*

$$\sum_{q\in\mathbb{Z}/N\mathbb{Z}} \zeta^{yq} b_1\left(\frac{x+q}{N}\right) = \frac{\zeta^{-y\lfloor x\rfloor}}{\zeta^y - 1}$$

where $\zeta = \exp(2i\pi/N)$.

This lemma allows us to write the expression $Z(\delta)$ in terms of traces of cyclotomic units, and we may now prove Theorem II.3.

Proof of Theorem II.3:

Let us first treat the term

$$\sum_{q_j\in\mathbb{Z}/N\mathbb{Z}} \zeta^{q_j\alpha_{1,j}k} b_1\left(\frac{v_j+\delta_j+(r_j(\delta)+q_j)s_j}{Ns_j}\right)$$

in expression (II.37) for $k=0$ and $1\leq j\leq n$ using the distribution relation (II.35). This gives:

$$\sum_{q_j\in\mathbb{Z}/N\mathbb{Z}} b_1\left(\frac{v_j+\delta_j+(r_j(\delta)+q_j)s_j}{Ns_j}\right)=N^0 b_1\left(\frac{v_j+\delta_j+r_j(\delta)s_j}{s_j}\right)=b_1\left(\frac{v_j+\delta_j}{s_j}\right)$$

since $r_j(\delta)\in\mathbb{Z}$ and b_1 is 1-periodic. Thus the term for $k=0$ cancels with the term $\prod_{j=1}^n b_1\left(\frac{v_j+\delta_j}{s_j}\right)$ in expression (II.36) and:

$$B_{n,a_1,\ldots,a_n}(v)(w,x,L,L')=\epsilon\sum_{\delta\in\mathcal{F}}\sum_{k=1}^{N-1}\prod_{j=1}^{n}\sum_{q_j\in\mathbb{Z}/N\mathbb{Z}}\zeta^{q_j\alpha_{1,j}k}b_1\left(\frac{v_j+\delta_j+(r_j(\delta)+q_j)s_j}{Ns_j}\right) \tag{II.38}$$

Let us now apply lemma II.26 to each term in expression (II.38) to obtain:

$$\sum_{q_j\in\mathbb{Z}/N\mathbb{Z}}\zeta^{q_j\alpha_{1,j}k}b_1\left(\frac{v_j+\delta_j+(r_j(\delta)+q_j)s_j}{Ns_j}\right)=\left(\frac{\zeta^{-\alpha_{1,j}k\lfloor\frac{v_j+\delta_j+r_j(\delta)s_j}{s_j}\rfloor}}{\zeta^{\alpha_{1,j}k}-1}\right).$$

Since $r_j(\delta)\in\mathbb{Z}$ we get $\lfloor\frac{v_j+\delta_j+r_j(\delta)s_j}{s_j}\rfloor=\lfloor\frac{v_j+\delta_j}{s_j}\rfloor$ and therefore:

$$B_{n,a_1,\ldots,a_n}(v)(w,x,L,L')=\epsilon\sum_{\delta\in\mathcal{F}}\left(\sum_{k=1}^{N-1}\prod_{j=1}^{n}\left(\frac{\zeta^{-\alpha_{1,j}k\lfloor\frac{v_j+\delta_j}{s_j}\rfloor}}{\zeta^{\alpha_{1,j}k}-1}\right)\right)$$

This may be written as a sum of traces of cyclotomic units using the well-known bijection:

$$\mathbb{Z}/N\mathbb{Z}-\{0\}=\sqcup_{d|N,d\neq 1}\mathbb{Z}/d\mathbb{Z}^\times$$

which gives:

$$B_{n,a_1,\ldots,a_n}(v)(w,x,L,L')=\epsilon\sum_{\delta\in\mathcal{F}}\sum_{d|N,d\neq1}\mathrm{Tr}_{\mathbb{Q}(\zeta_d)/\mathbb{Q}}\left(\prod_{j=1}^{n}\left(\frac{\zeta_d^{-\alpha_{1,j}\lfloor\frac{v_j+\delta_j}{s_j}\rfloor}}{\zeta_d^{\alpha_{1,j}}-1}\right)\right)$$

where for all $d|N$, $\zeta_d=\exp(2i\pi/d)$ and $\mathrm{Tr}_{\mathbb{Q}(\zeta_d)/\mathbb{Q}}$ is the trace from $\mathbb{Q}(\zeta_d)$ to $\mathbb{Q}$. This is the desired relation and this completes the proof of Theorem II.3. □

We end this section by using Theorem II.3 to obtain a uniform bound on the denominators of all $B_{n,a_1,\ldots,a_n}(v)(w,x,L,L')$ functions in terms of n and N, thus proving Theorem II.1. Results of this type are now classic and some may be found in [Zag73], [Das08] and [CD14].

Proof of Theorem II.1:

We first make the remark that when $a_1,\ldots,a_n\in\Lambda_N$ are linearly dependent and in good position in $V^\vee$, the function $B_{n,a_1,\ldots,a_n}$ is identically 0 and we may set $b(a_1,\ldots,a_n,v)=0$ identically in that case. Let us now suppose that $a_1,\ldots,a_n\in\Lambda_N$ are linearly independent and that the smoothing lattice L' is *good* for $a_1,\ldots,a_n$. Let us study each term $\mathrm{Tr}_{\mathbb{Q}(\zeta_d)/\mathbb{Q}}(u_d)$ appearing in Theorem II.3, where

$$u_d=\prod_{j=1}^{n}\left(\frac{\zeta_d^{-\alpha_{1,j}\lfloor\frac{v_j+\delta_j}{s_j}\rfloor}}{\zeta_d^{\alpha_{1,j}}-1}\right)$$

for $d|N, d \neq 1$. On the one hand, if d is divisible by two distinct primes, as $\alpha_{1,j}$ is coprime to d, it is well-known that $\zeta_d^{\alpha_{1,j}} - 1$ is a unit inside $\mathbb{Q}(\zeta_d)$ which implies that u_d is a unit in the ring of integers $\mathcal{O}_{\mathbb{Q}(\zeta_d)}$ of $\mathbb{Q}(\zeta_d)$ and $\mathrm{Tr}_{\mathbb{Q}(\zeta_d)/\mathbb{Q}}(u_d) \in \mathbb{Z}$. On the other hand, when $d = p^\nu$ is a power of the prime p, the algebraic integer $\zeta_d^{\alpha_{1,j}} - 1$ is a generator for the unique prime ideal $\mathfrak{P}_d$ above p in $\mathbb{Q}(\zeta_d)$. This cyclotomic extension is totally ramified at p, therefore $\mathfrak{P}_d^{\varphi(d)} = (p)$ where $\varphi(d) = p^{\nu-1}(p-1)$ is Euler's totient function evaluated at $d = p^\nu$. Thus u_d is a generator of the ideal $\mathfrak{P}_d^{-n}$. Let us introduce the different ideal $\mathfrak{D}$ of $\mathbb{Q}(\zeta_d)$ defined by:

$$\mathfrak{D}^{-1} = \{x \in \mathbb{Q}(\zeta_d) \mid \forall y \in \mathcal{O}_{\mathbb{Q}(\zeta_d)}, \mathrm{Tr}_{\mathbb{Q}(\zeta_d)/\mathbb{Q}}(xy) \in \mathbb{Z}\}.$$

In particular, any element u in $\mathfrak{D}^{-1}$ satisfies $\mathrm{Tr}_{\mathbb{Q}(\zeta_d)/\mathbb{Q}}(u) = \mathrm{Tr}_{\mathbb{Q}(\zeta_d)/\mathbb{Q}}(u \times 1) \in \mathbb{Z}$. It follows from [[Neu99], Lemma 10.1] that in this particular situtation $\mathfrak{P}_d^m$ divides exactly $\mathfrak{D}$ where $m = p^{\nu-1}(p\nu - \nu - 1)$. If we find an integer k such that $p^k u_d \in \mathfrak{D}^{-1}$ then we will obtain $\mathrm{Tr}_{\mathbb{Q}(\zeta_d)/\mathbb{Q}}(p^k u_d) \in \mathbb{Z}$ and therefore $\mathrm{Tr}_{\mathbb{Q}(\zeta_d)/\mathbb{Q}}(u_d) \in p^{-k}\mathbb{Z}$. To determine the minimal such integer k let us remark that:

$$(p^k u_d) = \mathfrak{P}^{k\varphi(p^\nu)-n}$$

therefore $p^k u_d \in \mathfrak{D}^{-1}$ if and only if $k\varphi(p^\nu) - n \geq -p^{\nu-1}(p\nu - \nu - 1)$ which gives the condition:

$$k \geq \frac{n}{\varphi(p^\nu)} - 1 + \frac{1}{p-1}.$$

Therefore:

$$\mathrm{Tr}_{\mathbb{Q}(\zeta_d)/\mathbb{Q}}(u_d) \in p^{-\left\lceil \frac{n}{\varphi(d)} - 1 + \frac{1}{p-1} \right\rceil}\mathbb{Z}$$

where $\lceil x \rceil$ is the ceiling function satisfying $\lceil x \rceil - 1 < x \leq \lceil x \rceil$ and $\lceil x \rceil \in \mathbb{Z}$. We can then conclude that

$$B_{n,a_1,\ldots,a_n}(v)(w,x,L,L') \in \sum_{p|N} \sum_{\nu=1}^{v_p(N)} p^{-\left\lceil \frac{n}{(p-1)p^{\nu-1}} - 1 + \frac{1}{p-1} \right\rceil}\mathbb{Z}$$

where $v_p(N)$ is the p-adic valuation of N. For all prime divisor p of N the term for $\nu = 1$ is dominant therefore we get the simpler relation:

$$B_{n,a_1,\ldots,a_n}(v)(w,x,L,L') \in \sum_{p|N} p^{-\left\lceil \frac{n+1}{p-1} - 1 \right\rceil}\mathbb{Z}.$$

It is not hard to check that $\left\lceil \frac{n+1}{p-1} - 1 \right\rceil = \lfloor \frac{n}{p-1} \rfloor$ for any $n \geq 2, p \geq 2$, therefore we may set $\mathcal{D}(N,n) = \prod_{p|N} p^{\lfloor \frac{n}{p-1} \rfloor}$, which uniformly bounds the denominators of all values $B_{n,a_1,\ldots,a_n}(v)(w,x,L,L')$:

$$B_{n,a_1,\ldots,a_n}(v)(w,x,L,L') \in \mathcal{D}(N,n)^{-1}\mathbb{Z}.$$

Lastly, define $b(a_1,\ldots,a_n,v)$ to be precisely the integer $B_{n,a_1,\ldots,a_n}(v)(w,x,L,L').\mathcal{D}(N,n)$. It follows from Proposition II.24 that $b(a_1,\ldots,a_n,v)$ does not depend on the choice of $w, x \in \mathbb{C} \times \mathrm{Hom}_{\mathbb{Z}}(L,\mathbb{C})$. Putting this together with Theorem II.2 shows that when $a_1,\ldots,a_n$ are well placed in $V^\vee$ and the smoothing lattice L' is *good* for $a_1,\ldots,a_n$:

$$\left(\prod_{j=1}^{n} G_{n-2,a_1,\ldots,\widehat{a_j},\ldots,a_n}(v)(w,x,L,L')^{(-1)^{j+1}} \right) = \exp\left(\frac{2i\pi b(a_1,\ldots,a_n,v)}{\mathcal{D}(N,n)} \right)$$

which is the desired result. □

II.4 Cohomological interpretation

In this last section we give a cohomological interpretation of the results presented in this chapter. We first recall some functions introduced in chapter I which satisfy cocycle and coboundary relations as a consequence of formulae (I.4) and (I.6). Then, we introduce smoothed versions of these functions and show how the smoothing operation affects their properties. Lastly, we restrict these functions to tori associated to groups of units in number fields, yielding proper cocycles for subgroups of $\Gamma_0(N,n)$.

In chapter I, we defined two collections of functions $\psi_{n,a}$ and $\phi_{n,a}$ attached to a primitive linear form $a \in \Lambda$ defined by:

$$\psi_{n,a} := \begin{cases} \mathrm{SL}_n(\mathbb{Z})^{n-2} & \to \mathcal{F}(V/L \times \mathbb{C} \times \mathrm{Hom}_{\mathbb{Z}}(L,\mathbb{C}),\mathbb{C}) \\ (g_1,\dots,g_{n-2}) & \to \big((v,w,x) \to G_{n-2,a,g_1\cdot a,\dots,(g_1\dots g_{n-2})\cdot a}(v)(w,x)\big) \end{cases}$$

$$\phi_{n,a} := \begin{cases} \mathrm{SL}_n(\mathbb{Z})^{n-1} & \to \mathcal{F}(V/L, \mathbb{Q}[w](x)) \\ (g_1,\dots,g_{n-1}) & \to B_{n,a,g_1\cdot a,(g_1g_2)\cdot a,\dots,(g_1\dots g_{n-1})\cdot a}(v)(w,x) \end{cases}.$$

The modular property (I.4) was then rephrased by saying that when the linear forms $a, g_1\cdot a,\dots,(g_1\dots g_{n-1})\cdot a$ were linearly independent, the multiplicative coboundary of $\psi_{n,a}$ was given by $\exp(2i\pi\phi_{n,a})$ as:

$$\partial^{\times}\psi_{n,a}(g_1,\dots,g_{n-1}) = \exp(2i\pi\phi_{n,a}(g_1,\dots,g_{n-1})). \tag{II.39}$$

Theorem II.2 implies that the coboundary relation (II.39) holds for any $a_1,\dots,a_n$ which are well placed in $V^\vee$, that is whenever $\mathrm{rk}(a_1,\dots,a_n) \neq n-1$ or whenever $\mathrm{rk}(a_1,\dots,a_n) = n-1$ and 0 is not a barycenter of $a_1,\dots,a_n$ in $V^\vee$. On the other hand, the cocycle relation (I.6) gives the partial cocycle relation:

$$\partial\phi_{n,a}(g_1,\dots,g_n) = 0 \tag{II.40}$$

for any $g_1,\dots,g_n \in \mathrm{SL}_n(\mathbb{Z})$ such that $a, g_1\cdot a,\dots,(g_1\dots g_n)\cdot a$ are not in bad position in $V^\vee$ in the sense of (BP) in chapter I.

Under the smoothing operation we introduced in this chapter we may define smoothed versions of these functions for a primitive linear form $a \in \Lambda_N$:

$$\psi^{(N)}_{n,a} := \begin{cases} \Gamma_0(N,n)^{n-2} & \to \mathcal{F}(V/L' \times \mathbb{C} \times \mathrm{Hom}_{\mathbb{Z}}(L,\mathbb{C}),\mathbb{C}) \\ (g_1,\dots,g_{n-2}) & \to \big((v,w,x) \to G_{n-2,a,g_1\cdot a,\dots,(g_1\dots g_{n-2})\cdot a}(v)(w,x,L,L')\big) \end{cases}$$

$$\phi^{(N)}_{n,a} := \begin{cases} \Gamma_0(N,n)^{n-1} & \to \mathcal{F}(V/L', \mathbb{Q}[w](x)) \\ (g_1,\dots,g_{n-1}) & \to B_{n,a,g_1\cdot a,(g_1g_2)\cdot a,\dots,(g_1\dots g_{n-1})\cdot a}(v)(w,x,L,L'). \end{cases}$$

Let us now rephrase the main results of this chapter in terms of these smoothed functions. It follows from (II.39) that they satisfy the coboundary relation

$$\partial^{\times}\psi^{(N)}_{n,a}(g_1,\dots,g_{n-1}) = \exp(2i\pi\phi^{(N)}_{n,a}(g_1,\dots,g_{n-1})) \tag{II.41}$$

whenever $a, g_1\cdot a,\dots,(g_1\dots g_{n-1})\cdot a$ are well placed in $V^\vee$. The cocycle relation (II.40) also directly gives the cocycle relation

$$\partial\phi^{(N)}_{n,a}(g_1,\dots,g_n) = 0 \tag{II.42}$$

for any $g_1, \dots, g_n \in \Gamma_0(N,n)$ such that $a, g_1 \cdot a, \dots, (g_1 \dots g_n) \cdot a$ are not in bad position in $V^\vee$ in the sense of (BP). In addition, Theorem II.3 implies that whenever the smoothing lattice L' is *good* for the linear forms $a, g_1 \cdot a, \dots, (g_1 \dots g_{n-1}) \cdot a$, the value of the function $\phi_{n,a}^{(N)}(g_1, \dots, g_{n-1})$ does not depend on $w, x \in \mathbb{C} \times \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{C}) \simeq \mathbb{C} \times \mathbb{C}^n$ and thus the function $\phi_{n,a}^{(N)}(g_1, \dots, g_{n-1})$ may be viewed as an element of $\mathcal{F}(V/L', \mathcal{D}(N,n)^{-1}\mathbb{Z})$ which is essentially a smoothed higher Dedekind sum. As for the function $\psi_{n,a}^{(N)}$, it follows from Corollary II.1 that for any $g_1, \dots, g_{n-1} \in \Gamma_0(N,n)$:

$$\partial^\times \left(\psi_{n,a}^{(N)}\right)^{\mathcal{D}(N,n)} = 1$$

whenever the linear forms $a, g_1 \cdot a, \dots, (g_1 \dots g_{n-1}) \cdot a$ satisfy the hypothesis of Theorem II.1.

Let us now consider specific subgroups U of $\Gamma_0(N,n)$ satisfying condition (I.30): $\forall m \geq 2, \forall g_1, \dots, g_m \in U, \forall \mu_1, \dots, \mu_m \in \mathbb{Z}_{\geq 0}$,

$$\sum_{j=1}^{m} \mu_j (g_j \cdot a) = 0 \Rightarrow \mu_1 = 0, \dots, \mu_m = 0.$$

In particular, this condition is satisfied by certain unit groups in number fields as explained in section I.3.3. Indeed, if $\mathbb{K}$ is a number field of degree n with at least one real place $\sigma_{\mathbb{R}}$, we may consider subgroups $\mathcal{U}$ of the unit group $\mathcal{O}_{\mathbb{K}}^\times$ satisfying $\forall\, \varepsilon \in \mathcal{U}, \sigma_{\mathbb{R}}(\varepsilon) > 0$. We may also consider two lattices L and L' in $\mathbb{K}$ corresponding to fractional ideals of $\mathbb{K}$ such that $L' \subset L$ and $L/L' \simeq \mathbb{Z}/N\mathbb{Z}$. In particular, these fractional ideals are stabilised by elements of $\mathcal{U}$ so we may identify each element $\varepsilon \in \mathcal{U}$ with the matrix M_ε corresponding to the multiplication by ε in a basis $B' = [Ne_1, e_2, \dots, e_n]$ of L' such that $B = [e_1, e_2, \dots, e_n]$ is a $\mathbb{Z}$-basis of L. The group $U = \{M_\varepsilon \mid \varepsilon \in \mathcal{U}\} \simeq \mathcal{U}$ is then an abelian subgroup of $\Gamma_0(N,n)$ satisfying (I.30). It follows from the discussion in section I.3.3 that the application $\phi_{n,a}^{(N)}$ is a $(n-1)$-cocycle on the group $U \simeq \mathcal{U}$ with values in $\mathcal{F}(V/L', \mathbb{Q}[w](x))$ and Theorem II.3 implies that $\phi_{n,a}^{(N)}(g_1, \dots, g_{n-1}) \in \mathcal{F}(V/L', \mathcal{D}(N,n)^{-1}\mathbb{Z})$ whenever the smoothing lattice is *good* for $a, g_1 \cdot a, \dots, (g_1, \dots, g_{n-1}) \cdot a$. Under hypothesis (I.30) it is also true that the splitting relation (II.41) holds for all $g_1, \dots, g_{n-1} \in U$:

$$\partial^\times \psi_{n,a}^{(N)}(g_1, \dots, g_{n-1}) = \exp(2i\pi \phi_{n,a}^{(N)}(g_1, \dots, g_{n-1}))$$

as (I.30) guarantees that the linear forms $a, g_1 \cdot a, \dots, (g_1 \dots g_{n-1}) \cdot a$ are well placed in $V^\vee$.

It would be interesting to find some conditions on the linear form $a \in \Lambda_N$ and on the unit group $\mathcal{U}$ such that for a fixed smoothing lattice $L' \subset L$, the lattice L' is *good* for any family $g_1 \cdot a, \dots, g_n \cdot a$, where $g_1, \dots, g_n \in \mathcal{U}$ (i.e. the smoothing lattice is "uniformly *good* for a and $\mathcal{U}$"). Indeed, if the smoothing lattice L' were to be uniformly *good* for a and $\mathcal{U}$ then the restriction of the smoothed function $\phi_{n,a}^{(N)}$ to $U \simeq \mathcal{U}$ would give a cocycle in $H^{n-1}(\mathcal{U}, \mathcal{F}(V/L', \mathcal{D}(N,n)^{-1}\mathbb{Z}))$. In addition, the restriction of $(\psi_{n,a}^{(N)})^{\mathcal{D}(N,n)}$ to $U \simeq \mathcal{U}$ would yield a multiplicative cocycle in $H^{n-2}(\mathcal{U}, \mathcal{F}(V/L' \times \mathbb{C} \times \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{C}), \mathbb{C}))$. In chapter III we construct conjectural higher elliptic units above number fields with exactly one complex place by evaluating the $(n-2)$-cocycles $\psi_{n,a}^{(N)}$ against some $(n-2)$-cycles on groups of totally positive units.

II.5 Discussion of chapter II

II.5.1 Smooth partial zeta functions in totally real number fields

In chapter I we gave a link between values of partial zeta functions at $s = 0$ in totally real number fields and values of our higher Bernoulli rational functions. In this section, we revisit this statement with the smoothed versions of these functions and recover a classic integrality result on the values at $s = 0$ of smoothed partial zeta functions in totally real number fields (see [DR80], [CN79], [CD14]). Indeed, let us fix a totally real number field $\mathbb{F}$, an integral ideal $\mathfrak{f}$ and an integral ideal $\mathfrak{b}$ coprime to $\mathfrak{f}$ representing a class in the narrow ray class group at $\mathfrak{f}$ in $\mathbb{F}$. Fix an auxiliary ideal $\mathfrak{a}$ called the smoothing ideal, of norm N, such that N is coprime to $\mathfrak{fb}$. Assume further that $\mathcal{O}_{\mathbb{K}}/\mathfrak{a}$ is cyclic and let us fix a $\mathbb{Z}$-basis $B = [e_1, \ldots, e_n]$ of $L = \mathfrak{fb}^{-1}\mathfrak{a}^{-1}$ such that $B' = [Ne_1, e_2, \ldots, e_n]$ is a $\mathbb{Z}$-basis of $L' = \mathfrak{fb}^{-1}$. Let us denote by $\sigma_1, \ldots, \sigma_n$ the embeddings of $\mathbb{F}$. It follows from Theorem I.3 that

$$\zeta_{\mathfrak{f}}(\mathfrak{b}, 0) = \frac{1}{n}\sum_{k=1}^{n}\sum_{\rho\in\mathfrak{S}_{n-1}} \nu_\rho B_{n,a_{1,\rho},\ldots,a_{n,\rho}}(1_{\mathbb{F}})(0, -\sigma_k, \mathfrak{fb}^{-1})$$

$$\zeta_{\mathfrak{f}}(\mathfrak{ab}, 0) = \frac{1}{n}\sum_{k=1}^{n}\sum_{\rho\in\mathfrak{S}_{n-1}} \nu_\rho B_{n,a_{1,\rho},\ldots,a_{n,\rho}}(1_{\mathbb{F}})(0, -\sigma_k, \mathfrak{fb}^{-1}\mathfrak{a}^{-1})$$

for some linear forms $a_{j,\rho} \in \mathrm{Hom}_{\mathbb{Q}}(\mathbb{F}, \mathbb{Q})$ and some signs $\nu_\rho \in \{-1, 0, +1\}$ which are independent of the choice of ideals $\mathfrak{b}$ and $\mathfrak{a}$. Putting both results together gives the smoothed relation:

$$N\zeta_{\mathfrak{f}}(\mathfrak{b}, 0) - \zeta_{\mathfrak{f}}(\mathfrak{ab}, 0) = \frac{1}{n}\sum_{k=1}^{n}\sum_{\rho\in\mathfrak{S}_{n-1}} \nu_\rho B_{n,a_{1,\rho},\ldots,a_{n,\rho}}(1_{\mathbb{F}})(0, -\sigma_k, \mathfrak{fb}^{-1}, \mathfrak{fb}^{-1}\mathfrak{a}^{-1}).$$

Using this relation and Theorem II.3 we would like to conclude that:

$$N\zeta_{\mathfrak{f}}(\mathfrak{b}, 0) - \zeta_{\mathfrak{f}}(\mathfrak{ab}, 0) \in \mathcal{D}(N, n)^{-1}\mathbb{Z}.$$

However, it is not true that the smoothing lattice L' is *good* for each set of linear forms $a_{1,\rho}, \ldots, a_{n,\rho}$ given in the proof of Theorem I.3. To accomodate for this, we may change the underlying signed fundamental domain we use. Indeed, if $\alpha_{1,\rho}, \ldots, \alpha_{n,\rho}$ is the positive dual family to $a_{1,\rho}, \ldots, a_{n,\rho}$ in $\mathfrak{fb}^{-1}$, then we set

$$d = \prod_{\rho\in\mathfrak{S}_{n-2}}\prod_{j=1}^{n}\mathcal{N}(\alpha_{j,\rho}).$$

Let us suppose that the smoothing index N is coprime to d, which certainly leaves infinitely many possible values for N. We may consider the linear forms $b_{j,\rho} = (y \to a_{j,\rho}(e_1^{-1}y))$. This change of linear forms corresponds to a change of the underlying signed fundamental domain $\mathcal{D}$ to $e_1\mathcal{D}$. The positive dual family $\beta_{1,\rho}, \ldots, \beta_{n,\rho}$ to $b_{1,\rho}, \ldots, b_{n,\rho}$ in $L = \mathfrak{fb}^{-1}\mathfrak{a}^{-1}$ is given by $\beta_{j,\rho} = e_1\alpha_{j,\rho}/m_{j,\rho}$ where $m_{j,\rho}$ is a positive integer coprime to N. It is clear that since N is coprime to d, the positive dual family $\beta'_{1,\rho}, \ldots, \beta'_{n,\rho}$ to $b_{1,\rho}, \ldots, b_{n,\rho}$ in $L' = \mathfrak{fb}^{-1}$ is given by $\beta'_{j,\rho} = N\beta_{j,\rho}$. Thus, the smoothing lattice L' is *good* for each set of linear forms $b_{1,\rho}, \dots, b_{n,\rho}$ (see Definition II.18). It then follows from Theorem I.3 that:

$$\zeta_{\mathfrak{f}}(\mathfrak{b}, 0) = \frac{1}{n} \sum_{k=1}^{n} \sum_{\rho \in \mathfrak{S}_{n-1}} \nu_\rho B_{n, b_{1,\rho}, \dots, b_{n,\rho}}(1_{\mathbb{F}})(0, -\sigma_k, \mathfrak{f}\mathfrak{b}^{-1})$$

$$\zeta_{\mathfrak{f}}(\mathfrak{a}\mathfrak{b}, 0) = \frac{1}{n} \sum_{k=1}^{n} \sum_{\rho \in \mathfrak{S}_{n-1}} \nu_\rho B_{n, b_{1,\rho}, \dots, b_{n,\rho}}(1_{\mathbb{F}})(0, -\sigma_k, \mathfrak{f}\mathfrak{b}^{-1}\mathfrak{a}^{-1})$$

Thus, using our smoothed version of the $B_{n,\underline{a}}$ functions we get the formula for smoothed zeta functions:

$$N\zeta_{\mathfrak{f}}(\mathfrak{b}, 0) - \zeta_{\mathfrak{f}}(\mathfrak{a}\mathfrak{b}, 0) = \frac{1}{n} \sum_{k=1}^{n} \sum_{\rho \in \mathfrak{S}_{n-1}} \nu_\rho B_{n, b_{1,\rho}, \dots, b_{n,\rho}}(1_{\mathbb{F}})(0, -\sigma_k, \mathfrak{f}\mathfrak{b}^{-1}, \mathfrak{f}\mathfrak{b}^{-1}\mathfrak{a}^{-1}).$$

Since the smoothing lattice is *good* for each of the families $b_{1,\rho}, \dots, b_{n,\rho}$ we obtain the uniform bound on values of smooth partial zeta functions at $s = 0$:

$$N\zeta_{\mathfrak{f}}(\mathfrak{b}, 0) - \zeta_{\mathfrak{f}}(\mathfrak{a}\mathfrak{b}, 0) \in \mathcal{D}(N, n)^{-1}\mathbb{Z}$$

where $\mathcal{D}(N, n) = \prod_{p|N} p^{\lfloor \frac{n}{p-1} \rfloor}$. Such integrality properties are known to hold at negative integers as the denominator of the rational number

$$N^{1+k}\zeta_{\mathfrak{f}}(\mathfrak{b}, -k) - \zeta_{\mathfrak{f}}(\mathfrak{a}\mathfrak{b}, -k)$$

is well-controlled (see [DR80], [CN79], [CD14]) and it would be interesting future work to try to prove this statement using our framework.

II.5.2 Smoothing operation and correction factors

In this chapter, we have shown that the smoothed versions of the higher G_{n-2} and B_n functions enjoy much nicer transformation properties for congruence subgroups in $\mathrm{SL}_n(\mathbb{Z})$ then the corresponding non-smoothed functions. This is a general phenomenon in the study of elliptic functions (see chapter 0). Another general way of fixing functions which are almost modular is to introduce an adequate exponential prefactor. For instance, the η function of Dedekind has nicer transformation properties than the function $\Phi(q) = \prod_{n \geq 1}(1 - q^n)$ thanks to the prefactor $q^{1/24}$ (see (0.5)). Similarly, the basic ingredient for Siegel units given in (0.13) comes with an exponential prefactor which ensures that the function has better modular transformation properties. In general, we might wish to identify a rational function $R \in \mathbb{Q}(z, \tau_0, \dots, \tau_r)$ such that the function

$$(z, \tau_0, \dots, \tau_r) \to e^{2i\pi R(z, \tau_0, \dots, \tau_r)} G_r(z, \tau_0, \dots, \tau_r)$$

enjoys simpler transformation properties under the action of $\mathrm{SL}_n(\mathbb{Z})$. Such a rational function would need to satisfy $R(Nz, N\tau_0, \dots, N\tau_r) = NR(z, \tau_0, \dots, \tau_r)$ so that it disappears when performing the smoothing operation we described in this chapter. Some work on the determination of a prefactor of this flavour has been carried out by Paşol and Zudilin (see [PZ18]) for the elliptic Gamma function in the regime $\tau = \sigma$ and it will be interesting future work to try to find the general adequate prefactor $\exp(2i\pi R)$.

Chapter III

Higher elliptic units

III.1 Introduction to chapter III

In this chapter, we construct conjectural higher elliptic units above number fields with exactly one complex place using our higher elliptic Gamma functions. Our approach is based on the recent article by Bergeron, Charollois and García [BCG23] in which they construct conjectural higher elliptic units above complex cubic fields using the elliptic Gamma function. We upgrade their construction to general number fields with exactly one complex place and formulate a conjecture on the algebraicity of these special values of higher elliptic Gamma functions.

When $\mathbb{K}$ is a number field of degree n with exactly one complex place, sometimes also called an Almost Totally Real field (ATR for short), the higher elliptic units we compute should be given by products of the form

$$\prod_{j=1}^{\kappa} \frac{G_{n-2}(z_j, \tau_{1,j}, \ldots, \tau_{n-1,j})^N}{G_{n-2}(Nz_j, N\tau_{1,j}, \ldots, N\tau_{n-1,j})}$$

where N is a choice of smoothing index and the z_j's and the $\tau_{l,j}$'s should be carefully chosen elements in $\mathbb{K}$. Most of this chapter is dedicated to the construction of a detailed geometric setup which gives profound insights on how these parameters should be chosen.

The elliptic units we aim to compute above an ATR field $\mathbb{K}$ are expected to belong to certain abelian extensions of $\mathbb{K}$. To make this statement precise, we shall need some vocabulary from class field theory. First, we say that an element of $\mathbb{K}$ is totally positive if its image under any real embedding of $\mathbb{K}$ belongs to $\mathbb{R}_{>0}$. We now move on to the definition of ray class groups. If $\mathfrak{f}$ is an integral ideal of $\mathbb{K}$, we shall denote by $\mathrm{Cl}^+(\mathfrak{f})$ the narrow ray class group at $\mathfrak{f}$, that is the quotient group $I(\mathfrak{f})/P^+(\mathfrak{f})$ where $I(\mathfrak{f})$ is the set of fractional ideals in $\mathbb{K}$ which are coprime to $\mathfrak{f}$ and $P^+(\mathfrak{f})$ is the subset of $I(\mathfrak{f})$ consisting of principal fractional ideals which admit a totally positive generator $\beta \in \mathbb{K}^\times$ satisfying $\beta \equiv 1 \bmod \mathfrak{f}$. The wide class group at $\mathfrak{f}$ is the group $\mathrm{Cl}(\mathfrak{f}) = I(\mathfrak{f})/P(\mathfrak{f})$ where $P(\mathfrak{f})$ is the set of principal fractional ideals coprime to $\mathfrak{f}$ generated by some $\beta \equiv 1 \bmod \mathfrak{f}$. When $\mathfrak{f} = \mathcal{O}_{\mathbb{K}}$, the group $\mathrm{Cl}^+(\mathfrak{f}) = \mathrm{Cl}^+(\mathbb{K})$ is the usual narrow Hilbert class group of $\mathbb{K}$ and $\mathrm{Cl}(\mathfrak{f}) = \mathrm{Cl}(\mathbb{K})$ is the usual class group of $\mathbb{K}$. By general class field theory, these class groups may be associated to certain abelian extensions of $\mathbb{K}$ which are called class fields. We shall denote by $\mathbb{K}^+(\mathfrak{f})$ the narrow ray class field at $\mathfrak{f}$ corresponding to the narrow ray class group at $\mathfrak{f}$. It is an extension of $\mathbb{K}$ which is unramified outside of $\mathfrak{f}$ and outside of the real places of $\mathbb{K}$ such that the extension $\mathbb{K}^+(\mathfrak{f})/\mathbb{K}$ is abelian with Galois group $\mathrm{Gal}(\mathbb{K}^+(\mathfrak{f})/\mathbb{K}) \simeq \mathrm{Cl}^+(\mathfrak{f})$.

In what follows, we shall fix an integral ideal $\mathfrak{f} \neq \mathcal{O}_{\mathbb{K}}$ in an ATR field $\mathbb{K}$ and construct conjectural units in $\mathbb{K}^+(\mathfrak{f})$. These units are built on the model of elliptic units above imaginary quadratic fields (see Theorem 0.8) and they should also satisfy a "Kronecker limit formula" similar to the one satisfied by elliptic units (see Theorem 0.7), relating the logarithm of their moduli to values of derivatives of partial zeta functions at $s = 0$. For an integral ideal $\mathfrak{f}$ and a class $\mathfrak{c}$ in $\mathrm{Cl}^+(\mathfrak{f})$ the partial zeta function at $\mathfrak{f}$ for the class $\mathfrak{c}$ is defined by the infinite sum:

$$\zeta_{\mathfrak{f}}(\mathfrak{c}, s) = \sum_{\mathfrak{b} \in \mathfrak{c}} \mathcal{N}(\mathfrak{b})^{-s}$$

where the sum ranges over integral ideals $\mathfrak{b}$ in the class $\mathfrak{c}$. This is a holomorphic function of the variable s which is well-defined for $\Re(s) > 1$ and it can be analytically continued to a meromorphic function over $\mathbb{C}$ with a single pole at $s = 1$. Since the complex place of $\mathbb{K}$ splits in $\mathbb{K}^+(\mathfrak{f})$, this function vanishes at order ≥ 1 at $s = 0$. When the order of vanishing at $s = 0$ is 1, that is $\zeta_{\mathfrak{f}}'(\mathfrak{c}, 0) \neq 0$, it is expected that this value is given by the logarithm of the absolute value of some unit in $\mathbb{K}^+(\mathfrak{f})$ called the Stark unit (see section III.4.2.5 for a brief discussion of the Stark conjectures for ATR fields). The conjectural higher elliptic units we construct by evaluating the multiple elliptic Gamma functions at points in $\mathbb{K}$ are expected to give an analytic description of the Stark unit in the spirit of Hilbert's 12th problem.

This problem, which was formulated by Hilbert in 1900 (see [Hil02]) as part of his famous list of 23 problems, asks for the construction of the abelian extensions of a general number field using analytic functions in the spirit of the Theorem of Kronecker and Weber. Indeed, this theorem states that all abelian extensions of $\mathbb{Q}$ are cyclotomic, that is they are obtained by evaluating the function $z \to \exp(2i\pi z)$ at points in $\mathbb{Q}$. This problem has been solved for imaginary quadratic fields using the elliptic units and more generally using the results of the theory of *Complex Multiplication* on elliptic curves. The general case remains open in the archimedian setting. Recent progress was made on a p-adic analogue of this problem (where the constructions of abelian extensions may be done using p-adic analytic functions) by Dasgupta and Kakde [DK24] who construct the abelian extensions of totally real number fields using Brumer-Stark units for which they give a p-adic analytic description. For number fields with exactly one complex place, a conjectural solution to Hilbert's 12th problem for complex cubic fields is proposed in [BCG23] using the elliptic Gamma function, and, if proven, our general conjecture would give a solution to Hilbert's 12th problem for general ATR fields.

This chapter is organised as follows. In section III.2 we review the construction of conjectural higher elliptic units above complex cubic fields carried out in [BCG23]. In section III.3 we give a detailed geometric setup for the construction of higher elliptic units above general ATR number fields and explain how to evaluate the higher elliptic Gamma functions to obtain them. In section III.4 we formulate our main conjecture and discuss various aspects of the construction of higher elliptic units.

III.2 Review of the complex cubic case

Our study of higher elliptic units is based on the treatment of the complex cubic case by Bergeron, Charollois and García in [BCG23]. In their article, they give a construction for conjectural elliptic units above a complex cubic field $\mathbb{K}$ by evaluating the elliptic Gamma function at points in $\mathbb{K}$. They relate their evaluations to the values of derivatives of partial zeta functions in $\mathbb{K}$ at $s = 0$, thus proving an analogue for complex cubic fields of Kronecker's second limit formula. In this section, we briefly review their construction and their results which motivated our work.

Let us fix $\mathbb{K}$ a complex cubic field. Let us denote by $\sigma_{\mathbb{R}} = \sigma_1$ the real embedding of $\mathbb{K}$ and by $\sigma_{\mathbb{C}} = \sigma_2 = \overline{\sigma_3}$ one of the complex embeddings of $\mathbb{K}$. Fix an integral ideal $\mathfrak{f} \neq \mathcal{O}_{\mathbb{K}}$ which will be the finite part of the class field modulus. We denote by $\mathbb{K}^+(\mathfrak{f})$ the narrow ray class field at $\mathfrak{f}$, which we assume to be totally complex. We denote by q the positive integer satisfying $q\mathbb{Z} = \mathfrak{f} \cap \mathbb{Z}$. Fix an integral ideal $\mathfrak{b}$ of $\mathcal{O}_{\mathbb{K}}$ representing a class in $\mathrm{Cl}^+(\mathfrak{f})$. Fix an integral ideal $\mathfrak{a}$ of $\mathcal{O}_{\mathbb{K}}$ such that $\mathcal{O}_{\mathbb{K}}/\mathfrak{a}$ is cyclic of order $N = \mathcal{N}(\mathfrak{a})$ coprime to $q\mathfrak{b}$. Such an ideal $\mathfrak{a}$ is called a smoothing ideal and N is called the smoothing index (see Definition III.3). This is coherent with the smoothing operation carried out in chapter II. Denote by $\mathcal{O}_{\mathfrak{f}}^{+,\times}$ the group of totally positive units in $\mathcal{O}_{\mathbb{K}}$ which are congruent to 1 mod $\mathfrak{f}$. This is a free abelian group of rank 1 by Dirichlet's unit theorem. Therefore, there is a unique element $\varepsilon \in \mathcal{O}_{\mathbb{K}}^{\times}$ such that $\mathcal{O}_{\mathfrak{f}}^{+,\times} = \varepsilon^{\mathbb{Z}}$ and $\sigma_{\mathbb{R}}(\varepsilon) > 1$.

Let us now consider a primitive vector h in $L = \mathfrak{f}\mathfrak{b}^{-1}$ such that $h/q \equiv 1 \bmod L$ and h/N is a generator of the cyclic group $L/\mathfrak{a}^{-1}L$. Such a vector is called an admissible base point. Let us fix a $\mathbb{Z}$-basis $B_L = [e_0, e_1, e_2]$ of L such that $e_0 = h$ and $[e_0/N, e_1, e_2]$ is a $\mathbb{Z}$-basis of $\mathfrak{a}^{-1}L$. We further assume that the real number

$$i \cdot \det(\sigma_j(e_k))_{1\leq j\leq 3, 0\leq k\leq 2}$$

is positive to lift any orientation ambiguity. Let us denote by $a = a_h$ the integral linear form defined on L by $a(y) = \det_{B_L}(h, \varepsilon h, y)$ and denote by b the integral linear form on L defined by $b(y) = a(\varepsilon y)$. Bergeron, Charollois and García define an arithmetic evaluation of the elliptic Gamma function as follows:

$$\Gamma_{\mathfrak{f},\mathfrak{b},\mathfrak{a}}(\varepsilon, h) = \frac{\Gamma_{a,b}(0)\left(\sigma_{\mathbb{C}}\left(\frac{h}{q}\right), \sigma_{\mathbb{C}}, L\right)^N}{\Gamma_{a,b}(0)\left(\sigma_{\mathbb{C}}\left(\frac{h}{q}\right), \sigma_{\mathbb{C}}, \mathfrak{a}^{-1}L\right)}.$$

In the notations of chapter II, this can be written simply as:

$$\Gamma_{\mathfrak{f},\mathfrak{b},\mathfrak{a}}(\varepsilon, h) = \Gamma_{a,b}(0)\left(\sigma_{\mathbb{C}}\left(\frac{h}{q}\right), \sigma_{\mathbb{C}}, \mathfrak{a}^{-1}L, L\right).$$

These evaluations of the elliptic Gamma function are expected to yield algebraic numbers in the abelian extension $\mathbb{K}^+(\mathfrak{f})$ of $\mathbb{K}$, as explained precisely by the following conjecture:

Conjecture III.1 [[BCG23], Conjecture] : *Let $\mathbb{K}$ be a complex cubic field and $\mathfrak{f} \neq \mathcal{O}_{\mathbb{K}}$ be an integral ideal in $\mathcal{O}_{\mathbb{K}}$. Fix a complex embedding σ of $\mathbb{K}^+(\mathfrak{f})$ which extends the fixed complex embedding $\sigma_{\mathbb{C}}$ of $\mathbb{K}$. Fix a class $\mathfrak{c}$ in $\mathrm{Cl}^+(\mathfrak{f})$ and an integral ideal $\mathfrak{b}$ in the class $\mathfrak{c}$. Fix a smoothing ideal $\mathfrak{a}$ of norm N such that N is coprime to $6.q.\mathfrak{b}$ where $q\mathbb{Z} = \mathfrak{f} \cap \mathbb{Z}$. Then, the complex number $\Gamma_{\mathfrak{f},\mathfrak{b},\mathfrak{a}}(\varepsilon, h)$ is independent of the choice of admissible base point h as well as from the choice of ideal $\mathfrak{b}$ and it is the image in $\mathbb{C}$ of an algebraic unit $u_{\mathfrak{f},\mathfrak{c},\mathfrak{a}}$ in $\mathbb{K}^+(\mathfrak{f})$ under the complex embedding σ of $\mathbb{K}^+(\mathfrak{f})$. Moreover:*

(i) Every embedding of $\mathbb{K}^+(\mathfrak{f})$ extending the real embedding of $\mathbb{K}$ maps $u_{\mathfrak{f},\mathfrak{c},\mathfrak{a}}$ to the unit circle.

(ii) If $\mathfrak{c} \to \sigma_{\mathfrak{c}}$ is the Artin map, the explicit reciprocity law is given by $\sigma_{\mathfrak{c}'}(u_{\mathfrak{f},\mathfrak{c},\mathfrak{a}}) = u_{\mathfrak{f},\mathfrak{c}\mathfrak{c}',\mathfrak{a}}$ for any class $\mathfrak{c}'$ in the narrow ray class group at $\mathfrak{f}$.

If proven, this conjecture would give a positive answer to Hilbert's 12th problem for complex cubic fields, as any finite abelian extension of $\mathbb{K}$ may be embedded in a totally complex narrow ray class field of $\mathbb{K}$.

Let us briefly present an example supporting Conjecture III.1. A more detailed presentation of this example is given in section IV.2.1.1. Consider the field $\mathbb{K} = \mathbb{Q}(z)$ where $z = e^{2i\pi/3}13^{1/3}$ is the root of the polynomial $x^3 - 13$ in the upper half-plane. Fix $\mathfrak{f}$ the unique prime ideal in $\mathcal{O}_{\mathbb{K}}$ of norm 3, $\mathfrak{b} = (1)$ and $\mathfrak{a}$ the unique prime ideal of norm 5 in $\mathcal{O}_{\mathbb{K}}$. The fundamental unit $\varepsilon = 17z^2 + 40z + 94$ satisfies $\sigma_{\mathbb{R}}(\varepsilon) > 1$. Then, for the admissible base point $h = -(21z^2 + 42z + 114)$ we compute

$$\Gamma_{\mathfrak{f},\mathfrak{b},\mathfrak{a}}(\varepsilon, h) = \frac{\Gamma\left(\frac{1}{3}, -\frac{\varepsilon+5348}{1965}, -\frac{\varepsilon^{-1}+467}{1965}\right)^5}{\Gamma\left(\frac{5}{3}, -\frac{\varepsilon+5348}{393}, -\frac{\varepsilon^{-1}+467}{393}\right)} \approx -0.0660917... + i \cdot 0.0932299...$$

with 1000 digits of precision. This special value of the elliptic Gamma function coincides up to this precision with a root of the polynomial

$$\begin{aligned} P_{\text{abs}} = & x^{18} + 384x^{17} + 2310x^{16} - 10646490x^{15} + 1596241353x^{14} + 18608357181x^{13} \\ & + 156933809421x^{12} + 215098256580x^{11} + 381407365338x^{10} + 338205493469x^9 \\ & + 381407365338x^8 + 215098256580x^7 + 156933809421x^6 + 18608357181x^5 \\ & + 1596241353x^4 - 10646490x^3 + 2310x^2 + 384x + 1 \end{aligned}$$

which defines an absolute equation of $\mathbb{K}^+(\mathfrak{f})$ over $\mathbb{Q}$.

A remark we can make in view of the results of chapter II is that Conjecture III.1 should still hold for N not coprime to 6 if one is willing to replace the complex number $\Gamma_{\mathfrak{f},\mathfrak{b},\mathfrak{a}}(\varepsilon, h)$ by its power $\Gamma_{\mathfrak{f},\mathfrak{b},\mathfrak{a}}(\varepsilon, h)^{\mathcal{D}(N,3)}$.

To support their conjecture, Bergeron, Charollois and García proved a key unconditionnal result, that is a version of Kronecker's limit formula for complex cubic fields:

Theorem III.2 [[BCG23], Theorem 3.2] : *The modulus of the complex number $\Gamma_{\mathfrak{f},\mathfrak{b},\mathfrak{a}}(\varepsilon, h)$ is independent of the choice of admissible base point h and it satisfies the Kronecker limit formula:*

$$\mathcal{N}(\mathfrak{a})\zeta_{\mathfrak{f}}'([\mathfrak{b}], 0) - \zeta_{\mathfrak{f}}'([\mathfrak{a}\mathfrak{b}], 0) = \log\left|\Gamma_{\mathfrak{f},\mathfrak{b},\mathfrak{a}}(\varepsilon, h)\right|^2.$$

Conjecture III.1 together with Theorem III.2 imply that these higher elliptic units $u_{\mathfrak{f},\mathfrak{c},\mathfrak{a}}$ should be smoothed versions of Stark units (see section III.4.2.5 for a discussion of the relations with the rank one abelian Stark conjectures). In the cubic case, Conjecture III.1 has been tested numerically on hundreds of examples and in the following sections we generalise this conjecture to ATR fields of degree $n \geq 4$ in some specific cases supported by numerical evidence.

III.3 Construction of higher elliptic units

In this section we generalise the construction of higher elliptic units carried out in [BCG23] to higher degree ATR number fields. We shall first give a very broad description of the shape of our elliptic units in section III.3.1. Then, we give a very precise description of our geometric setup in sections III.3.2 to III.3.5, allowing for the formulation of a precise conjecture on the algebraicity of our higher elliptic units (see section III.4).

III.3.1 Evaluation of the G_r functions against a r-cycle

In this section we give the basic geometric setup to construct higher elliptic units using higher elliptic Gamma functions. Consider an ATR number field $\mathbb{K}$ of degree $n \geq 3$. Fix a complex embedding $\sigma_{\mathbb{C}}$ of $\mathbb{K}$ and fix an ordering $\sigma_1, \ldots, \sigma_{n-2}, \sigma_{n-1} = \sigma_{\mathbb{C}}, \sigma_n = \overline{\sigma_{\mathbb{C}}}$ on the embeddings of $\mathbb{K}$. We fix a global orientation of $\mathbb{K}$ as follows: for any basis $B = [e_1, \ldots, e_n]$ of the $\mathbb{Q}$-vector space $\mathbb{K}$, the number $i \cdot \det((\sigma_j(e_k))_{1 \leq j,k \leq n})$ is non zero and real, and we say that the basis B is positive if this number is positive. This applies in particular to any $\mathbb{Z}$-basis of a fractional ideal in $\mathbb{K}$. Consider an integral ideal $\mathfrak{f} \neq \mathcal{O}_{\mathbb{K}}$ and set q to be the positive integer satisfying $q\mathbb{Z} = \mathfrak{f} \cap \mathbb{Z}$. We shall assume for the rest of this dissertation that the narrow ray class field at $\mathfrak{f}$ is totally complex and that the ideal $\mathfrak{f}$ satisfies the following simplifying hypothesis:

There are no units in $\mathcal{O}_{\mathbb{K}}^{\times}$ of negative norm which are congruent to 1 mod $\mathfrak{f}$. (H1)

We shall discuss the role of this hypothesis in section III.4. Let us now fix an integral ideal $\mathfrak{b}$ coprime to $\mathfrak{f}$ and set $L = \mathfrak{f}\mathfrak{b}^{-1}$. The integral ideal $\mathfrak{b}$ represents a class in the narrow ray class group $\mathrm{Cl}^+(\mathfrak{f})$ at $\mathfrak{f}$. Let us now define the notion of smoothing ideal:

Definition III.3: *An integral ideal $\mathfrak{a}$ is a smoothing ideal for $\mathfrak{f}$ (resp. for $\mathfrak{f}$ and $\mathfrak{b}$) if the quotient $\mathcal{O}_{\mathbb{K}}/\mathfrak{a}$ is a cyclic abelian group of order N such that N is coprime to $\mathfrak{f}$ (resp. N is coprime to $\mathfrak{f}$ and $\mathfrak{b}$). The integer N is then called the smoothing index.*

This definition is coherent with the smoothing operation described in chapter II for the pair of lattices $L \subset \mathfrak{a}^{-1}L$. Let us then fix a smoothing ideal $\mathfrak{a}$ for $\mathfrak{f}$ and $\mathfrak{b}$. We consider specific base points in L associated to the data $\mathfrak{f}, \mathfrak{b}, \mathfrak{a}$ which we will use to evaluate geometric G_{n-2} functions.

Definition III.4: *A base point $h \in L = \mathfrak{f}\mathfrak{b}^{-1}$ is said to be weakly admissible (for the data $\mathfrak{f}, \mathfrak{b}, \mathfrak{a}$) if there is an integer $k \in \mathbb{Z}/q\mathbb{Z}^{\times}$ and a unit $\varepsilon \in \mathcal{O}_{\mathbb{K}}^{\times}$ such that $h/q \equiv k\varepsilon \bmod L$ and h/N generates the cyclic group $\mathfrak{a}^{-1}L/L \simeq \mathbb{Z}/N\mathbb{Z}$. The vector h is said to be strongly admissible if $h/q \equiv 1 \bmod L$ and h/N generates the cyclic group $\mathfrak{a}^{-1}L/L \simeq \mathbb{Z}/N\mathbb{Z}$.*

In this section we fix a weakly admissible base point $h \in L$ and we set m to be the maximal integer satisfying $h/m \in L$. Let us also fix a positive $\mathbb{Z}$-basis B_L of the $\mathbb{Z}$-module L. By Dirichlet's unit theorem, the unit group

$$\mathcal{O}_{\mathfrak{f}}^{+,\times} = \{\varepsilon \in \mathcal{O}_{\mathbb{K}}^{\times}, \varepsilon \equiv 1 \bmod \mathfrak{f}, \sigma(\varepsilon) > 0, \text{ for all real embedding } \sigma \text{ of } \mathbb{K}\} \tag{III.1}$$

is a free $\mathbb{Z}$-module of rank $r = n - 2$. This group acts on L by multiplication, and it may be identified with an algebraic torus of rank r in $\mathrm{SL}_n(\mathbb{Z})$ by identifying any unit $\varepsilon \in \mathcal{O}_{\mathfrak{f}}^{+,\times}$ with the matrix of multiplication by ε in the basis B_L. The group $\mathcal{O}_{\mathfrak{f}}^{+,\times}$ naturally acts on the dual space $\Lambda = \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$ by inverse multiplication: for any $b \in \Lambda$, $\varepsilon \cdot b = b(\varepsilon^{-1} \cdot)$. In particular, for any $b \in \Lambda$ and any $\beta \in L$, we get $(\varepsilon \cdot b)(\varepsilon \cdot \beta) = b(\beta)$. Let us fix a set of fundamental units $u_1, \ldots, u_r$ for $\mathcal{O}_{\mathfrak{f}}^{+,\times}$ and let us set $u_0 = 1$. We assume further that:

The family $u_0 = 1, u_1, \ldots, u_r$ is a free family in the $\mathbb{Q}$-vector space $\mathbb{K} \simeq \mathbb{Q}^n$. (H2)

This hypothesis is independent of the choice for the base point h and guarantees that the linear form $f = f(u_1, \ldots, u_r, h) := (y \to \det_{B_L}(h, u_1 h, \ldots, u_r h, y))$ is non-zero for any $h \in \mathbb{K}^{\times}$. Let us denote by $a = a(u_1, \ldots, u_r, h)$ the unique primitive element in Λ and by $\lambda = \lambda(u_1, \ldots, u_r, h)$ the unique positive integer such that $\lambda m^{r+1} a = f$. The basic ingredient for our construction of higher elliptic units consists of evaluations of smoothed G_r functions of the form:

$$G^{\pm}_{r,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(u_1,\ldots,u_r;h) = \frac{G_{r,\pm(a,au_1,\ldots,au_r)}(0)\left(\sigma_{\mathbb{C}}\left(\frac{h}{q}\right),\sigma_{\mathbb{C}},L\right)^N}{G_{r,\pm(a,au_1,\ldots,au_r)}(0)\left(\sigma_{\mathbb{C}}\left(\frac{h}{q}\right),\sigma_{\mathbb{C}},\mathfrak{a}^{-1}L\right)} \tag{III.2}$$

(see Proposition I.7 for the definition of the geometric G_r function) where the $\pm$ depends on some orientation choice which we discuss later on. Here au_j stands for the function $y \to a(u_j y)$ which is also $u_j^{-1}\cdot a$. In the notations of chapter II, this evaluation can be written in terms of a smoothed G_r function as:

$$G^{\pm}_{r,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(u_1,\ldots,u_r;h) = G_{r,\pm(a,au_1,\ldots,au_r)}(0)\left(\sigma_{\mathbb{C}}\left(\frac{h}{q}\right),\sigma_{\mathbb{C}},\mathfrak{a}^{-1}L,L\right).$$

It will follow from lemma III.7 that the evaluation (III.2) is well-defined. Let us note that if we replace $\sigma_{\mathbb{C}}$ by the other complex embedding of $\mathbb{K}$ which is $\overline{\sigma_{\mathbb{C}}}$ we get essentially the same information as:

$$G^{\pm}_{r,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(u_1,\ldots,u_r;h,\overline{\sigma_{\mathbb{C}}}) = \overline{G^{\pm}_{r,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(u_1,\ldots,u_r;h,\sigma_{\mathbb{C}})}^{(-1)^r}$$

by (I.11). This justifies our focus on only one of the two complex embeddings of $\mathbb{K}$. In the special cases $r=0$, $r=1$ (that is the cases where $\mathbb{K}$ is imaginary quadratic and complex cubic respectively) we will identify as before:

$$\theta^{\pm}_{\mathfrak{f},\mathfrak{b},\mathfrak{a}}(h) = G^{\pm}_{0,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(\emptyset;h)$$
$$\Gamma^{\pm}_{\mathfrak{f},\mathfrak{b},\mathfrak{a}}(\varepsilon;h) = G^{\pm}_{1,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(\varepsilon;h).$$

The definition of an arithmetic G_r function given by (III.2) provides us with a way to evaluate the multiplicative r-cocycle built from smoothed G_r functions against an r-cycle $\Upsilon_{\mathfrak{f}} \in H_r(\mathcal{O}_{\mathfrak{f}}^{+,\times},\mathbb{Z})$. The cycle we use is an adaptation of Sczech's cycle for totally real number fields [Scz93] to our setting of ATR number fields. Explicitly, if we fix a set $\varepsilon_1,\ldots,\varepsilon_r$ of fundamental units for $\mathcal{O}_{\mathfrak{f}}^{+,\times}$, then the cycle $\Upsilon_{\mathfrak{f}} = \Upsilon_{\mathfrak{f}}(\varepsilon_1,\ldots,\varepsilon_r)$ is given by:

$$\Upsilon_{\mathfrak{f}} = \sum_{\rho\in\mathfrak{S}_r}\operatorname{sgn}(\rho)[\varepsilon_{\rho(1)}|\ldots|\varepsilon_{\rho(r)}]$$

where

$$[\varepsilon_1|\ldots|\varepsilon_r] = \left(\varepsilon_1,\ldots,\prod_{j=1}^{k}\varepsilon_j,\ldots,\prod_{j=1}^{r}\varepsilon_j\right).$$

It follows from [[Scz93], Lemma 5] that the class of $\Upsilon_{\mathfrak{f}}$ in $H_r(\mathcal{O}_{\mathfrak{f}}^{+,\times},\mathbb{Z})$ is independent of the choice of fundamental units for $\mathcal{O}_{\mathfrak{f}}^{+,\times}$. The evaluation $\langle G_r,\Upsilon_{\mathfrak{f}}\rangle$ of the arithmetic G_r function against this cycle $\Upsilon_{\mathfrak{f}}$ takes the form:

$$I_{r,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(\varepsilon_1,\ldots,\varepsilon_r;\underline{h},\underline{\mu},\underline{\nu}) = \prod_{\rho\in\mathfrak{S}_r} G^{\mu_\rho}_{r,\mathfrak{f},\mathfrak{b},\mathfrak{a}}([\varepsilon_{\rho(1)}|\ldots|\varepsilon_{\rho(r)}];h_\rho)^{\nu_\rho} \tag{III.3}$$

where for any permutation $\rho\in\mathfrak{S}_r$, h_ρ is a strongly admissible base point and μ_ρ,ν_ρ are some orientation signs in $\{\pm1\}$. This is well-defined provided that for any $\rho\in\mathfrak{S}_r$, the unit system $1, \varepsilon_{\rho(1)}, \dots, \prod_{j=1}^{r} \varepsilon_{\rho(j)}$ forms a free family of $\mathbb{K}$ viewed as a $\mathbb{Q}$-vector space (see (H2)). We are now ready to give a vague form of our general conjecture:

Conjecture III.5 (Vague form) : *Let $\mathbb{K}$ be an ATR field of degree $n = r + 2 \geq 3$. Suppose that $\mathfrak{f}, \mathfrak{b}, \mathfrak{a}$ are as above. Let $\varepsilon_1, \dots, \varepsilon_r$ be a set of fundamental units for $\mathcal{O}_{\mathfrak{f}}^{+,\times}$ such that for any $\rho \in \mathfrak{S}_r$, the unit system $[\varepsilon_{\rho(1)}| \dots |\varepsilon_{\rho(r)}]$ satisfies (H2). Then there are admissible base points h_ρ for $\rho \in \mathfrak{S}_r$ and orientation signs $\mu_\rho, \nu_\rho \in \{\pm 1\}$ such that the complex number*

$$I_{r,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(\varepsilon_1, \dots, \varepsilon_r; \underline{h}, \underline{\mu}, \underline{\nu})$$

satisfies:

1. *An algebraicity statement: the complex number $I_{r,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(\varepsilon_1, \dots, \varepsilon_r; \underline{h}, \underline{\mu}, \underline{\nu})^{\mathcal{D}(N,n)}$ is the image in $\mathbb{C}$ of an algebraic unit $u_{\mathfrak{f},\mathfrak{b},\mathfrak{a}}$ in $\mathbb{K}^+(\mathfrak{f})$ under a complex embedding $\sigma'_{\mathbb{C}}$ extending $\sigma_{\mathbb{C}}$ (see section II.1 for the definition of the integer $\mathcal{D}(N, n)$).*

2. *A Kronecker limit formula of the form:*

$$\mathcal{N}(\mathfrak{a})\zeta'_{\mathfrak{f}}([\mathfrak{b}], 0) - \zeta'_{\mathfrak{f}}([\mathfrak{a}\mathfrak{b}], 0) = \log \left| I_{r,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(\varepsilon_1, \dots, \varepsilon_r; \underline{h}, \underline{\mu}, \underline{\nu}) \right|^2. \tag{III.4}$$

In our main conjecture (see III.37) we shall make these statements precise and give a precise form for the explicit reciprocity law. We now briefly discuss this vague form of the conjecture, and in particular we discuss the question of the choice for the base points $h_\rho, \rho \in \mathfrak{S}_r$. Indeed, we believed originally that any set of strongly admissible base points would give a positive result (i.e. an algebraic number $I_{r,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(\varepsilon_1, \dots, \varepsilon_r; \underline{h}, \underline{\mu}, \underline{\nu})^{\mathcal{D}(N,n)}$ satisfying a Kronecker limit formula), however, the computations we have done for elliptic units above ATR fields of degree $n \geq 4$ show that set of base points $\underline{h} = (h_\rho)_\rho$ should satisfy a crucial compatibility condition which we discuss in section III.3.4.4. In section IV.2.6.1 we shall give an example of computation where we choose incompatible base points and where the complex number we obtain is not algebraic and does not satisfy a Kronecker limit formula.

A second point we need to discuss in this vague form of the conjecture is the choice of orientations μ_ρ, ν_ρ. On the computational side, because there are a finite number of these choices, we may check which orientations give the Kronecker limit formula (III.4). On the theoretical side, these orientations should depend on some explicit signed fundamental domain for the action of the unit group $\mathcal{O}_{\mathfrak{f}}^{+,\times}$ on $\mathbb{K}$ (see [Esp14]). In practice, we can make a good guess on the orientations that should be chosen in simple examples, following [Col88] and [DyDF14]. Indeed, we define the sign of the unit system $\varepsilon_1, \dots, \varepsilon_r$ by:

$$\mathrm{sign}(\varepsilon_1, \dots, \varepsilon_r) = \mathrm{sign}(\det(\log(\sigma_j(\varepsilon_k))_{1 \leq j,k \leq r}))$$

which means that $\mathrm{sign}(\varepsilon_1, \dots, \varepsilon_r)$ is the sign of the regulator of $\varepsilon_1, \dots, \varepsilon_r$ for the fixed ordering $\sigma_1, \dots, \sigma_n$ on the embeddings of $\mathbb{K}$. Then, in most simple examples, the orientations may be taken to be:

$$\mu_\rho = \nu_\rho = \mathrm{sign}(\varepsilon_1, \dots, \varepsilon_r).\mathrm{sgn}(\rho). \tag{III.5}$$

Lastly, and this is perhaps the most important point regarding explicit computations, we note that the evaluation (III.3) may be extremely complicated if the fundamental units for $\mathcal{O}_{\mathfrak{f}}^{+,\times}$ are huge and their associated cycle $\Upsilon_{\mathfrak{f}}$ is poorly placed (see section III.3.2

for a discussion of this matter). This is typically the case whenever $\mathfrak{f}$ is large. We can simplify the evaluation by considering instead a cycle $\Upsilon = \Upsilon(\varepsilon_1, \ldots, \varepsilon_r) \in H_r(\mathcal{O}_{\mathbb{K}}^{+,\times}, \mathbb{Z})$ associated to fundamental units in the group $\mathcal{O}_{\mathbb{K}}^{+,\times}$ of totally positive units in $\mathbb{K}$. This cycle is define similarly to $\Upsilon_{\mathfrak{f}}$ by:

$$\Upsilon = \sum_{\rho \in \mathfrak{S}_r} \operatorname{sgn}(\rho)[\varepsilon_{\rho(1)}| \ldots |\varepsilon_{\rho(r)}]$$

where $\varepsilon_1, \ldots, \varepsilon_r$ are fundamental units for $\mathcal{O}_{\mathbb{K}}^{+,\times}$ and its class in $H_r(\mathcal{O}_{\mathbb{K}}^{+,\times}, \mathbb{Z})$ does not depend on the particular choice of fundamental units $\varepsilon_1, \ldots, \varepsilon_r$. The evaluation of a r-cocycle ψ against the cycle $\Upsilon_{\mathfrak{f}}$ can then be done by averaging evaluations of the r-cocycle ψ against Υ over the finite group $\mathcal{O}_{\mathbb{K}}^{+,\times}/\mathcal{O}_{\mathfrak{f}}^{+,\times}$ as follows:

$$\langle \psi, \Upsilon_{\mathfrak{f}} \rangle = \sum_{\varepsilon \in \mathcal{O}_{\mathbb{K}}^{+,\times}/\mathcal{O}_{\mathfrak{f}}^{+,\times}} \langle \varepsilon \cdot \psi, \Upsilon \rangle \tag{III.6}$$

(see for instance [[GS03], Proposition 7.4]). We shall give evaluations in this form as it considerably simplifies the formulas which define higher elliptic units.

III.3.2 Detailed geometric setup

In this section we give a very detailed description of the formalism introduced in section III.3.1 for the construction of higher elliptic units.

III.3.2.1 The linear forms $a, au_1, \ldots, au_r$ and their positive dual family

In this section we give a detailed geometric setup to understand how the base points h_ρ, $\rho \in \mathfrak{S}_r$ should be chosen for our evaluations. To this end we shall express explicitly the arithmetic G_r function (III.2) and analyse the properties of the underlying geometric G_r function using the explicit definitions given in chapters I and II. Let $\mathbb{K}$ be an ATR field of degree $n = r + 2$ and let $\mathfrak{f}$, $\mathfrak{b}$, $\mathfrak{a}$ be given as in section III.3.1. Recall that we have fixed an ordering on the embeddings of $\mathbb{K}$ and a global orientation of $\mathbb{K}$ (see section III.3.1). Put $L = \mathfrak{f}\mathfrak{b}^{-1}$. Fix a system of fundamental units $(\varepsilon_1, \ldots, \varepsilon_r)$ of $\mathcal{O}_{\mathbb{K}}^{+,\times}$. Let us fix a permutation $\rho \in \mathfrak{S}_r$ and write $(u_1, \ldots u_r) = [\varepsilon_{\rho(1)}| \ldots |\varepsilon_{\rho(r)}]$. For convenience, we will always write $u_0 = 1$. Choose $h \in L$ a weakly admissible base point (see Definition III.4). Let us denote by m the maximal integer satisfying $h/m \in L$. We recall that the linear form $a = a_h$ is the unique primitive linear form on L satisfying:

$$\lambda m^{r+1} a = \det_{B_L}(h, u_1 h, \ldots, u_r h, \cdot) \tag{III.7}$$

for some positive integer λ. This linear form does not depend on the choice of positive basis of L, therefore we may assume that the chosen basis B_L is given by $B_L = [e_0 = h/m, e_1, \ldots, e_{r+1}]$ such that for any $1 \leq j \leq r$, $u_j h = \sum_{k=0}^{j} m c_{jk0} e_k$ where the coefficients c_{jk0} are integers and $c_{jj0} > 0$. This is done by computing the Hermite Normal Form of the matrix expressing $(u_0 h, \ldots, u_r h)$ in any positive basis of L (see for instance [[Coh93], Theorem 2.4.3]). Explicitly, $(h, u_1 h, \ldots, u_r h) = m.[e_0, \ldots, e_{r+1}].\mathfrak{U}$ where $\mathfrak{U}$ is the $(r+2)\times(r+1)$ matrix:

$$\mathfrak{U} = \begin{pmatrix} 1 & c_{100} & \dots & c_{j00} & \dots & c_{r00} \\ 0 & c_{110} & \dots & c_{j10} & \dots & c_{r10} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & c_{jj0} & \dots & c_{rj0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & \dots & c_{rr0} \\ 0 & 0 & \dots & 0 & \dots & 0 \end{pmatrix}. \tag{III.8}$$

In this basis B_L we identify the units u_j with the matrix $(c_{jkl})_{k,l} \in \mathrm{SL}_n(\mathbb{Z})$ such that $u_j e_l = \sum_{k=0}^{r+1} c_{jkl} e_k$. Fix the dual basis $B_\Lambda = (f_0, \dots, f_{r+1})$ of $\Lambda = \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$ such that $f_j(e_k) = \delta_{jk}$ where δ_{jk} is the Kronecker symbol. In this basis the linear form $a = a_h$ defined in section III.3.1 is exactly f_{r+1}. For $1 \le j \le r$, the composition of a with multiplication by u_j is written $au_j = \sum_{l=1}^{r+1} c_{j(r+1)l} f_l$. It follows from hypothesis (H2) that the family of linear forms $(a, au_1, \dots, au_r)$is free in $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{K}, \mathbb{Q}) \simeq \mathbb{Q}^n$ therefore we may denote as before by γ the unique primitive vector in L such that $\det_{B_\Lambda}(a, au_1, \dots, au_r, \cdot) = s\gamma$ for some positive integer s. Since for $0 \le j \le r$, $au_j(\gamma) = 0$ and $au_j(h) = 0$, we get

$$\bigcap_{j=0}^{r} \ker(au_j) = \mathbb{Q}\gamma = \mathbb{Q}h$$

and therefore there is a sign $\eta \in \{-1, +1\}$ such that $\eta m \gamma = h$.

Let us now describe how we construct a positive dual family $\alpha_0, \dots, \alpha_r$ to $a, au_1, \dots, au_r$ in L (see Definition I.5). To achieve this, we first identify the family $\underline{a} = (a = au_0, au_1, \dots, au_r)$ with the matrix obtained by concatenation of the coefficients of the linear forms au_i in the basis B_Λ:

$$\underline{a} = \begin{pmatrix} au_0 \\ au_1 \\ \vdots \\ au_i \\ \vdots \\ au_r \end{pmatrix} = \begin{pmatrix} 0 & 0 & \dots & 0 & \dots & 1 \\ 0 & c_{1(r+1)1} & \dots & c_{1(r+1)k} & \dots & c_{1(r+1)(r+1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & c_{i(r+1)1} & \dots & c_{i(r+1)k} & \dots & c_{i(r+1)(r+1)} \\ \vdots & \vdots & \vdots & \vdots & & \\ 0 & c_{r(r+1)1} & \dots & c_{r(r+1)k} & \dots & c_{r(r+1)(r+1)} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathcal{A} \end{pmatrix} \tag{III.9}$$

This matrix has a first column filled with zeroes and the submatrix $\mathcal{A}$ is a square matrix of size $r+1$. It follows from (H2) that $\mathcal{A}$ is invertible. We shall now use the Smith normal form of $\mathcal{A}$ (see for instance [[Coh93], Theorem 2.4.12]). There are invertible matrices $U, V \in \mathrm{GL}_{r+1}(\mathbb{Z})$ and a unique diagonal matrix $S(\mathcal{A}) \in M_{r+1}(\mathbb{Z})$ such that $U\mathcal{A}V = S(\mathcal{A})$ and

$$S(\mathcal{A}) = \begin{pmatrix} A_r & 0 & \dots & 0 \\ 0 & A_{r-1} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & A_0 \end{pmatrix}$$

where $A_0, \dots, A_{r+1} \in \mathbb{Z}$ are the elementary divisors of $\mathcal{A}$ such that $A_i \mid A_{i+1}$ for all $0 \le i \le r$. Since A_0 is the gcd of all coefficients in $\mathcal{A}$ and $\mathcal{A}$ contains a 1, it follows that $A_0 = 1$. From this writing we may identify

$$\lambda(u_1, \dots, u_r; h) = \lambda = \prod_{j=1}^{r} c_{jj0} \tag{III.10}$$

$$s(u_1,\ldots,u_r;h) = s = \prod_{j=1}^{r} A_j \tag{III.11}$$

where λ is defined in section III.3.1 and $\det_{B_\Lambda}(au_0,\ldots,au_r,\cdot) = s\gamma$. We also define

$$t(u_1,\ldots,u_r;h) = t = A_r. \tag{III.12}$$

which will be a very important parameter in our construction.

Next, to obtain a positive dual family $\underline{\alpha}(u_1,\ldots,u_r;h) = \underline{\alpha} = (\alpha_0,\ldots,\alpha_r)$ to $a, au_1,\ldots,au_r$ in L (see Definition I.5) we set $\mathcal{B} = (\prod_{i=1}^{r} A_i^{-1})\mathrm{com}(\mathcal{A})^T$ where $\mathrm{com}(\mathcal{A})$ is the comatrix of $\mathcal{A}$. We claim that the columns of the integral matrix $\mathcal{B}$ give a positive dual family $\alpha_0,\ldots,\alpha_r$ to $a, au_1,\ldots,au_r$ in L. Indeed, if we write the coefficients of the matrix $\mathcal{B}$ as $\mathcal{B} = (b_{ij})_{1\le i,j\le r+1}$ then the family of vectors $\alpha_0,\ldots,\alpha_r$ defined by:

$$\alpha_j = \sum_{i=1}^{r+1} b_{ij} e_i \tag{III.13}$$

forms a positive dual family to $a, au_1,\ldots,au_r$ in L satisfying:

$$au_j(\alpha_j) = t, \quad au_k(\alpha_j) = 0,\ \forall\, k \neq j. \tag{III.14}$$

We shall often write this as the multiplication of two matrices $\underline{a}\cdot\underline{\alpha} = tI_{r+1}$. We say that this choice of $\underline{\alpha}$ is a uniform positive dual family to $\underline{a}$ because the value $au_j(\alpha_j) = t$ is independent of j. We argue that this choice is minimal amongst all uniform positive dual families to $\underline{a}$ in L, as explained by the following lemma.

Lemma III.6: *Let $\mathcal{A}$ be the matrix defined in (III.9).*

(i) Let $\mathcal{B}' \in M_{r+1}(\mathbb{Z})$ be a square matrix of size $r+1$ such that $\mathcal{A}\cdot\mathcal{B}' = dI_{r+1}$ for some integer d. Then t divides d.

(ii) Any uniform positive dual family $\underline{\alpha}'$ to $\underline{a}$ in L satisfies $\underline{a}\cdot\underline{\alpha}' = dI_{r+1}$ where t divide d.

(iii) If $\underline{\alpha}, \underline{\alpha}'$ are uniform positive dual families to $\underline{a}$ in L satisfying $\underline{a}\cdot\underline{\alpha} = dI_{r+1}$ and $\underline{a}\cdot\underline{\alpha}' = d'I_{r+1}$ then for all $0\le j\le r$, $d'\alpha_j - d\alpha'_j \in \gcd(d,d')\mathbb{Z}\gamma$.

Proof :

(i) Consider the matrices $U, V \in \mathrm{GL}_{r+1}(\mathbb{Z})$ such that $U\mathcal{A}V = S(\mathcal{A})$ is the Smith normal form of $\mathcal{A}$. Then

$$dI_{r+1} = \mathcal{A}\mathcal{B}' = (U\mathcal{A}V)\cdot\left(V^{-1}\mathcal{B}'U^{-1}\right) = \begin{pmatrix} t & 0 & \ldots & 0 \\ 0 & A_{r-1} & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & \ldots & 1 \end{pmatrix} V^{-1}\mathcal{B}'U^{-1}.$$

Since $V^{-1}\mathcal{B}'U^{-1} \in M_{r+1}(\mathbb{Z})$, the first row of $S(\mathcal{A}) \times V^{-1}\mathcal{B}'U^{-1}$ is divisible by t and therefore t divides d.

(ii) Let $\underline{\alpha}' = (\alpha'_0,\ldots,\alpha'_r)$ be a uniform positive dual family to $\underline{a}$ in L such that $\underline{a}\cdot\underline{\alpha}' = dI_{r+1}$. Write the coordinates of the α'_j in the basis B_L as $\alpha'_j = \sum_{i=0}^{r+1} b'_{ij} e_i$. Then the matrix $\mathcal{B}' = (b'_{ij})_{1\le i,j\le r+1}$ satisfies $\mathcal{A}\cdot\mathcal{B}' = dI_{r+1}$ and thus, by (i), t divides d.

(iii) Suppose that $\underline{a} \cdot \underline{\alpha} = dI_{r+1}$ and $\underline{a} \cdot \underline{\alpha}' = d'I_{r+1}$. Write as before the coordinates of $\alpha_0, \dots, \alpha_r$ and $\alpha'_0, \dots, \alpha'_r$ as:

$$\alpha_j = \sum_{i=0}^{r+1} b_{ij} e_i \quad \text{and} \quad \alpha'_j = \sum_{i=0}^{r+1} b'_{ij} e_i.$$

Then the matrices $\mathcal{B} = (b_{ij})_{1 \le i,j \le r+1} \in M_{r+1}(\mathbb{Z})$ and $\mathcal{B}' = (b'_{ij})_{1 \le i,j \le r+1} \in M_{r+1}(\mathbb{Z})$ satisfy $\mathcal{A}\mathcal{B} = dI_{r+1}$ and $\mathcal{A}\mathcal{B}' = d'I_{r+1}$. Since $\mathcal{A}$ is invertible, it follows that $d'\mathcal{B} = d\mathcal{B}'$. Thus, for any $0 \le j \le r$:

$$d'\alpha_j - d\alpha'_j = (d'b_{0j} - db'_{0j})e_0 = (d'b_{0j} - db'_{0j})\eta\gamma \in \gcd(d, d')\mathbb{Z}\gamma$$

as claimed.

□

From now on we denote by $\underline{\alpha} = \underline{\alpha}(u_1, \dots, u_r; h)$ the minimal uniform positive dual family $\underline{\alpha}$ to $\underline{a}$ given above. Recalling the construction in section III.3.1 and the definition of the geometric $G_{r,\underline{a}}$ function (see Definition I.7) we may explicitly write the evaluation (III.2) as:

$$G^{+}_{r,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(u_1, \dots, u_r; h) = \prod_{\delta \in F(\underline{a}, \underline{\alpha}, 0)/\mathbb{Z}\gamma} \frac{G_r\left(\frac{h+q\delta}{q\gamma}, \frac{\alpha_0}{\gamma}, \dots, \frac{\alpha_r}{\gamma}\right)^N}{G_r\left(\frac{N(h+q\delta)}{q\gamma}, \frac{N\alpha_0}{\gamma}, \dots, \frac{N\alpha_r}{\gamma}\right)} \tag{III.15}$$

where we identify the elements of $\mathbb{K}$ with their image in $\mathbb{C}$ under the embedding $\sigma_{\mathbb{C}}$ to lighten notations. A similar formula holds for the term $G^{-}_{r,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(u_1, \dots, u_r; h)$ associated with the orientation $\mu = -1$ (see lemma III.27). Let us now justify that the evaluation (III.15) is well-defined.

Lemma III.7:

(i) If $\mathbb{K}$ does not contain a proper subfield $\mathbb{Q} \subsetneq \mathbb{L} \subsetneq \mathbb{K}$ then each of the complex numbers $\sigma(\alpha_0/\gamma), \dots, \sigma(\alpha_r/\gamma)$ belongs to $\mathbb{C} - \mathbb{R}$.

(ii) Suppose that $\mathbb{K}$ contains a proper subfield $\mathbb{Q} \subsetneq \mathbb{L} \subsetneq \mathbb{K}$. Assume further that $\mathbb{L}$ is the maximal such subfield. Then $\mathbb{K}$ must be of degree $n \ge 4$, $\mathbb{L}$ must be totally real of absolute degree $n' \le n/2$. In addition, at most $n' - 1$ of the complex numbers $\sigma(\alpha_0/\gamma), \dots, \sigma(\alpha_r/\gamma)$ are real, in which case they are real algebraic irrational numbers of degree $\le n'$.

(iii) In either case, the arithmetic evaluation (III.2) is well-defined.

Proof :
(i) The elements $\gamma, \alpha_0, \dots, \alpha_r$ form a $\mathbb{Q}$-basis of $\mathbb{K}$ and therefore the element $1, \alpha_0/\gamma, \dots, \alpha_r/\gamma$ do too. When $\mathbb{K}$ does not contain a subfield $\mathbb{Q} \subsetneq \mathbb{L} \subsetneq \mathbb{K}$, since $\sigma_{\mathbb{C}}(\mathbb{K}) \cap \mathbb{R} = \mathbb{Q} = \mathbb{Q}.\sigma_{\mathbb{C}}(1)$, the complex numbers $\sigma_{\mathbb{C}}(\alpha_0/\gamma), \dots, \sigma_{\mathbb{C}}(\alpha_r/\gamma)$ all lie in $\mathbb{C} - \mathbb{R}$.

(ii) Let $\mathbb{L}$ be the maximal proper subfield of $\mathbb{K}$ which we assume to be distinct from $\mathbb{Q}$. If $\mathbb{L}$ had a complex place then $\mathbb{K}$ would have at least two, therefore $\mathbb{L}$ is totally real and one of its real places ramifies in $\mathbb{K}$. The degree of $\mathbb{K}$ is $n \ge 4$ because quadratic and cubic fields cannot have proper subfields.

The elements $1, \alpha_0/\gamma, \ldots, \alpha_r/\gamma$ form as in (i) a $\mathbb{Q}$-basis of $\mathbb{K}$. If $\sigma_{\mathbb{C}}(\alpha_{j_1}/\gamma), \ldots, \sigma_{\mathbb{C}}(\alpha_{j_k}/\gamma)$ all belong to $\sigma_{\mathbb{C}}(\mathbb{K}) \cap \mathbb{R} = \sigma_{\mathbb{C}}(\mathbb{L})$ then $k \leq n'-1$ as the elements $\sigma_{\mathbb{C}}(1), \sigma_{\mathbb{C}}(\alpha_{j_1}/\gamma), \ldots, \sigma_{\mathbb{C}}(\alpha_{j_k}/\gamma)$ must form a free family in the $\mathbb{Q}$-vector space $\sigma_{\mathbb{C}}(\mathbb{L})$ of dimension n'. In particular, at least $n/2 \geq 2$ of the parameters $\sigma_{\mathbb{C}}(\alpha_0/\gamma), \ldots, \sigma_{\mathbb{C}}(\alpha_r/\gamma)$ belong to $\mathbb{C} - \mathbb{R}$.

(iii) When $\mathbb{K}$ contains no proper subfield, the parameters $\sigma_{\mathbb{C}}(\alpha_0/\gamma), \ldots, \sigma_{\mathbb{C}}(\alpha_r/\gamma)$ belong to $\mathbb{C} - \mathbb{R}$ by (i), therefore the right-hand side of (III.15) is well-defined. When $\mathbb{K}$ contains a proper subfield, the parameters $\sigma_{\mathbb{C}}(\alpha_0/\gamma), \ldots, \sigma_{\mathbb{C}}(\alpha_r/\gamma)$ satisfy the hypothesis of Proposition (I.17) by (ii), therefore the right-hand side of (III.15) is well-defined using the appropriate formulation (I.34). □

In most cases, the ATR field $\mathbb{K}$ does not contain a proper subfield, and it is clear that the evaluation (III.2) is well-defined. This lemma essentially explains that the evaluation still makes sense when $\mathbb{K}$ contains a proper subfield if we are careful when choosing which formula we use in the computations. This last case is showcased in example IV.2.2.4.

To end this section on the general explicit setup to compute (III.2), we say a few words on the size of the set $F(\underline{a}, \underline{\alpha}, v = 0)/\mathbb{Z}\gamma$. Indeed, we may identify the set $F(\underline{a}, \underline{\alpha}, 0)/\mathbb{Z}\gamma$ with the quotient space L/M where $M = M(u_1, \ldots, u_r; h) = \mathbb{Z}\gamma \oplus (\oplus_{j=0}^{r} \mathbb{Z}\alpha_j)$. As this sublattice M of L has index $\det(\mathcal{B}) = t^{r+1}/s$ in L, it follows that the geometric function $G_{r,\underline{a}}$ is a product of t^{r+1}/s ordinary elliptic G_r functions and so is the evaluation (III.2). In the best case scenario, $t^{r+1} = s$ and $G_{r,\underline{a}}$ is a single ordinary elliptic G_r function, whereas in the worst case scenario $t = s$ and $G_{r,\underline{a}}$ is a product of $t^r = s^r$ ordinary elliptic G_r functions. In what follows, we will try to find the best choice for the base point h and especially we will describe how to construct a base point h for which $t = 1$ and $s = 1$. This will be the subject of sections III.3.2.2 to III.3.4.4, and the choice of base point h is fully explained in Proposition III.25.

III.3.2.2 Dependence on $u_1, \ldots, u_r$ and h

In this section we analyse the dependence of the quantities defined in section III.3.2.1 on the choice of units $u_1, \ldots, u_r$ and on the choice of base point h. To this end we make the following important remark: the linear form $a = a(u_1, \ldots, u_r; h)$ may be easily expressed in terms of a linear form $\tilde{a} = \tilde{a}(u_1, \ldots, u_r)$ which depends on $u_1, \ldots, u_r$ but not on h. Indeed, let us fix $\tilde{B}$ a positive $\mathbb{Z}$-basis of $\mathcal{O}_{\mathbb{K}}$ and let us consider the linear form:

$$\tilde{f} := (y \to \det_{\tilde{B}}(1, u_1, \ldots, u_r, y)) \,. \tag{III.16}$$

If $\tilde{\lambda}$ is the unique positive integer such that $\tilde{f}$ maps $\mathcal{O}_{\mathbb{K}}$ to $\tilde{\lambda}.\mathbb{Z}$ then we may define the primitive linear form $\tilde{a}$ by:

$$\tilde{a} := \left(y \to \tilde{\lambda}^{-1} \det_{\tilde{B}}(1, u_1, \ldots, u_r, y)\right) = \tilde{\lambda}^{-1} \tilde{f}. \tag{III.17}$$

Using this auxiliary function we may write:

$$a = \left(y \to \tilde{a}\left(\frac{\tilde{\lambda}}{\lambda} \frac{\mathcal{N}(h/m)}{\mathcal{N}(L)} \frac{m}{h} y\right)\right). \tag{III.18}$$

Our claim is that most of the geometric setup introduced in section III.3.2.1 for the linear form $a = a(u_1, \ldots, u_r, h)$ has a direct counterpart for the linear form $\tilde{a} = \tilde{a}(u_1, \ldots, u_r)$ and that the two are closely related. In what follows, we will write with a $\sim$ the counterpart of these quantities. For the rest of this section, we assume that the basis $\tilde{B} = [\tilde{e}_0 = 1, \tilde{e}_1, \dots, \tilde{e}_{r+1}]$ is such that $u_j = \sum_{k=0}^{j} \tilde{c}_{jk0}\tilde{e}_k$ with $\tilde{c}_{jj0} > 0$ for $1 \le j \le r$. This is once again done by computing the Hermite Normal Form of a matrix representing $1, u_1, \dots, u_r$ in another positive $\mathbb{Z}$-basis of $\mathcal{O}_{\mathbb{K}}$. Explicitly, $(1, u_1, \dots, u_r) = [\tilde{e}_0, \dots, \tilde{e}_{r+1}].\widetilde{\mathfrak{U}}$ where $\widetilde{\mathfrak{U}}$ is the $(r+2) \times (r+1)$ matrix:

$$\widetilde{\mathfrak{U}} = \begin{pmatrix} 1 & \tilde{c}_{100} & \dots & \tilde{c}_{j00} & \dots & \tilde{c}_{r00} \\ 0 & \tilde{c}_{110} & \dots & \tilde{c}_{j10} & \dots & \tilde{c}_{r10} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \tilde{c}_{jj0} & \dots & \tilde{c}_{rj0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & \dots & \tilde{c}_{rr0} \\ 0 & 0 & \dots & 0 & \dots & 0 \end{pmatrix}. \tag{III.19}$$

Let us fix $B^\vee = [\tilde{f}_0, \dots, \tilde{f}_{r+1}]$ the $\mathbb{Z}$-basis of the dual space $\mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}_{\mathbb{K}}, \mathbb{Z})$ such that $\tilde{f}_j(\tilde{e}_k) = \delta_{jk}$. For any $1 \le j \le r$, the unit u_j may be identified with the matrix $(\tilde{c}_{jkl})_{k,l} \in \mathrm{SL}_n(\mathbb{Z})$ representing multiplication by u_j in $\mathcal{O}_{\mathbb{K}}$, i.e. such that $u_j\tilde{e}_l = \sum_{k=0}^{r+1} \tilde{c}_{jkl}\tilde{e}_k$ for any $0 \le l \le r+1$. In the basis $B^\vee$ of $\mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}_{\mathbb{K}}, \mathbb{Z})$ we may identify $a = \tilde{f}_{r+1}$ and $\tilde{a}u_j = \sum_{l=1}^{r+1} \tilde{c}_{j(r+1)l}\tilde{f}_l$. The assumption (H2) guarantees that $\mathrm{rk}(a, au_1, \dots, au_r) = r+1$ and it is clear that $\bigcap_{j=0}^{r} \ker(au_j) = \mathbb{Q}$ where as before we write $u_0 = 1$. Let us then denote by $\tilde{s}$ the unique positive integer and $\tilde{\eta}$ the sign in $\{-1, +1\}$ such that $\det_{B^\vee}(\tilde{a}, \tilde{a}u_1, \dots, \tilde{a}u_r, \cdot) = \tilde{\eta}\tilde{s}$.

We may now describe a positive dual family $\underline{\tilde{\alpha}} = (\tilde{\alpha}_0, \dots, \tilde{\alpha}_r)$ to the family $\underline{\tilde{a}} = (\tilde{a}, \tilde{a}u_1, \dots, \tilde{a}u_r)$ in $\mathcal{O}_{\mathbb{K}}$ as follows. We associate to the family $\underline{\tilde{a}}$ a square matrix $\tilde{\mathcal{A}}$ of size $r+1$ as above for $\underline{a}$ such that:

$$\underline{\tilde{a}} = \begin{pmatrix} \tilde{a}u_0 \\ \tilde{a}u_1 \\ \vdots \\ \tilde{a}u_i \\ \vdots \\ \tilde{a}u_r \end{pmatrix} = \begin{pmatrix} 0 & 0 & \dots & 0 & \dots & 1 \\ 0 & \tilde{c}_{1(r+1)1} & \dots & \tilde{c}_{1(r+1)k} & \dots & \tilde{c}_{1(r+1)(r+1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \tilde{c}_{i(r+1)1} & \dots & \tilde{c}_{i(r+1)k} & \dots & \tilde{c}_{i(r+1)(r+1)} \\ \vdots & \vdots & \vdots & \vdots & & \\ 0 & \tilde{c}_{r(r+1)1} & \dots & \tilde{c}_{r(r+1)k} & \dots & \tilde{c}_{r(r+1)(r+1)} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \tilde{\mathcal{A}} \end{pmatrix} \tag{III.20}$$

Let us write as above the elementary divisors of this matrix $\tilde{\mathcal{A}}$ associated to the linear forms $\tilde{a}, \tilde{a}u_1, \dots, \tau_r$ as $[\tilde{A}_r, \dots, \tilde{A}_1, \tilde{A}_0 = 1]$ such that $\tilde{A}_i | \tilde{A}_{i+1}$. Then we get as before:

$$\tilde{\lambda}(u_1, \dots, u_r) = \tilde{\lambda} = \prod_{j=1}^{r} \tilde{c}_{jj0}, \tag{III.21}$$

$$\tilde{s}(u_1, \dots, u_r) = \tilde{s} = \prod_{j=1}^{r} \tilde{A}_j \tag{III.22}$$

and if we set

$$\tilde{t} = \tilde{A}_r \tag{III.23}$$

we may construct as above a uniform positive dual family $\underline{\tilde{\alpha}}$ to $\underline{\tilde{a}}$ in $\mathcal{O}_{\mathbb{K}}$ satisfying $\underline{\tilde{a}} \cdot \underline{\tilde{\alpha}} = \tilde{t}I_{r+1}$ by computing the square matrix $\tilde{\mathcal{B}} = (\prod_{i=1}^{r} \tilde{A}_i^{-1})\mathrm{com}(\tilde{\mathcal{A}})^T$ and extracting its columns. Explicitly, if $\widetilde{\mathcal{B}} = (\tilde{b}_{ij})_{0 \le i,j \le r+1}$ we set

$$\tilde{\alpha}_j = \sum_{i=1}^{r+1} \tilde{b}_{ij}e_i \tag{III.24}$$

for $0 \leq j \leq r$, so that the elements $\tilde{\alpha}_0, \dots, \tilde{\alpha}_r$ form a positive dual family to $\tilde{a}, \tilde{a}u_1, \dots, \tilde{a}u_r$ in $\mathcal{O}_{\mathbb{K}}$ satisfying for all $0 \leq j \leq r$:

$$\tilde{a}u_j(\tilde{\alpha}_j) = \tilde{t}, \quad \tilde{a}u_k(\tilde{\alpha}_j) = 0, \ \forall\, k \neq j. \tag{III.25}$$

In the case $r = 1$ (that is $\mathbb{K}$ is a complex cubic field and $u_1 = \varepsilon$ is the fundamental unit of $\mathbb{K}$), $\tilde{\lambda}$ is the maximal integer satisfying $\varepsilon \in \mathbb{Z} + \tilde{\lambda}\mathcal{O}_{\mathbb{K}}$ and $\det(1, \varepsilon, \varepsilon^2) = \tilde{\lambda}^3\tilde{t}$. If we consider the set E of elements $\beta \in \mathcal{O}_{\mathbb{K}}$ for which the $\mathbb{Z}$-module $\mathcal{O}(\beta) = \mathbb{Z} + \mathbb{Z}\varepsilon + \mathbb{Z}\beta$ is a ring, and if for each $\beta \in E$ we denote by $\mathrm{ct}(\beta)$ the content of the ring $\mathcal{O}(\beta)$ in the sense of [[Bha04], Definition 14], then $\tilde{\lambda} = \max\{\mathrm{ct}(\beta) \mid \beta \in E\}$ as $\mathcal{O}(\tilde{\lambda}\tilde{e}_2) = \mathbb{Z} + \tilde{\lambda}\mathcal{O}_{\mathbb{K}}$. On the other hand, when $\tilde{\lambda} = 1$, the integer $\tilde{t}$ is the index of the ring $\mathbb{Z}[\varepsilon]$ in $\mathcal{O}_{\mathbb{K}}$. Therefore in general we call $\tilde{\lambda} = \tilde{\lambda}(u_1, \dots, u_r)$ the content of the unit system $u_1, \dots, u_r$ and we call $\tilde{t} = \tilde{t}(u_1, \dots, u_r)$ the overflow of this unit system. We insist that these quantities only depend on the unit system $(u_1, \dots, u_r)$. In what follows, we shall relate the quantities $\lambda, \tilde{\lambda}, t, \tilde{t}, s, \tilde{s}$ and more generally the two construction from sections III.3.2.1 and III.3.2.2.

III.3.2.3 Linking the two constructions

In this section we give a first fundamental link between the two constructions associated to $a = a_h$ and to $\tilde{a}$. In the next sections, this link will be explored in more detail using generalised different ideals. Let us start by defining a specific integer ℓ which will correspond in some sense to the level of the computation associated to a base point h.

Definition III.8: *The level $\ell = \ell(u_1, \dots, u_r; h)$ associated to the unit system $u_1, \dots, u_r$ and to the weakly admissible base point h is defined as the maximal integer satisfying*

$$\tilde{\alpha}_0 h, \dots, \tilde{\alpha}_r h \in m\ell L + \mathbb{Z}h.$$

We immediately make the following remark: if in the basis B_L the coordiantes of $\tilde{\alpha}_j h$ are given by $\tilde{\alpha}_j h = \sum_{k=0}^{r+1} \langle \tilde{\alpha}_j h, e_k \rangle e_k$ then ℓ is the gcd of the coefficients $\langle \tilde{\alpha}_j h, e_k \rangle$ for $0 \leq j \leq r$ and $1 \leq k \leq r+1$. Thus, in particular, if d is any integer satisfying

$$\tilde{\alpha}_0 h, \dots, \tilde{\alpha}_r h \in m.d.L + \mathbb{Z}h$$

then d must divide all the coefficients $\langle \tilde{\alpha}_j h, e_k \rangle$ for $0 \leq j \leq r$ and $1 \leq k \leq r+1$ and $d \mid \ell$.

The level ℓ naturally appears when comparing the families $\alpha_0, \dots, \alpha_r$ and $\tilde{\alpha}_0 h, \dots, \tilde{\alpha}_r h$. We shall now make this comparison and deduce a fundamental relation between the quantities $\lambda, \tilde{\lambda}, t, \tilde{t}$ and ℓ.

Proposition III.9: *Let $\epsilon = \mathrm{sign}(\mathcal{N}(h)) \in \{-1, +1\}$.*

(i) The family $\epsilon\underline{\tilde{\alpha}}h = (\epsilon\tilde{\alpha}_0 h, \dots, \epsilon\tilde{\alpha}_r h)$ is a uniform positive dual family to $\underline{a}$ in L such that $\underline{a} \cdot (\epsilon\underline{\tilde{\alpha}}h) = m\ell t I_{r+1}$.

(ii) The following fundamental relation holds in $\mathbb{Z}$:

$$\lambda \ell t = \frac{|\mathcal{N}(h/m)|}{\mathcal{N}(L)} \tilde{\lambda}\tilde{t}. \tag{III.26}$$

(iii) For any $0 \leq j \leq r$ there is some integer $m_j \in \mathbb{Z}$ such that

$$\alpha_j = \frac{\tilde{\alpha}_j h + m_j h}{\epsilon m \ell}. \tag{III.27}$$

Proof :
We shall prove simultaneously the three points of this proposition. We use formula (III.18) which relates the linear forms a and $\tilde{a}$ in conjunction with (III.7) and (III.16) to evaluate $\lambda m^{r+1} a(u_k \tilde{\alpha}_j h)$ for any $0 \leq j, k \leq r$. Indeed:

$$\lambda m^{r+1} a(u_k \tilde{\alpha}_j h) = \det_{B_L}(h, u_1 h, \ldots, u_r h, u_k \tilde{\alpha}_j h).$$

Using standard linear algebra and the definition of the norm of the fractional ideal L we may rewrite this equality using $\det_{\tilde{B}}$ instead of $\det_{B_L}$ as

$$\lambda m^{r+1} a(u_k \tilde{\alpha}_j h) = \det_{B_L}(\tilde{B}) \det_{\tilde{B}}(h, u_1 h, \ldots, u_r h, u_k \tilde{\alpha}_j h)$$
$$\lambda m^{r+1} a(u_k \tilde{\alpha}_j h) = \frac{1}{\mathcal{N}(L)} \det_{\tilde{B}}(h, u_1 h, \ldots, u_r h, u_k \tilde{\alpha}_j h).$$

Then, using the definition of the norm $\mathcal{N}(h)$ of h we get

$$\lambda m^{r+1} a(u_k \tilde{\alpha}_j h) = \frac{\mathcal{N}(h)}{\mathcal{N}(L)} \det_{\tilde{B}}(1, u_1, \ldots, u_k \tilde{\alpha}_j)$$
$$\lambda m^{r+1} a(u_k \tilde{\alpha}_j h) = \frac{\mathcal{N}(h)}{\mathcal{N}(L)} \tilde{\lambda} \tilde{a}(u_k \tilde{\alpha}_j).$$

We now use formula (III.24) to conclude that for $k \neq j$, $a(u_k \tilde{\alpha}_j h) = 0$ and for $k = j$:

$$\lambda m^{r+1} a(u_j \tilde{\alpha}_j h) = \frac{\mathcal{N}(h)}{\mathcal{N}(L)} \tilde{\lambda} \tilde{t}$$

which we may rewrite as:

$$a u_j \tilde{\alpha}_j h = \frac{\mathcal{N}(h)}{\mathcal{N}(L)} \frac{\tilde{\lambda}}{\lambda m^{r+1}} \tilde{t} \neq 0. \tag{III.28}$$

The lattice L is a fractional ideal and $\tilde{\alpha}_0, \ldots, \tilde{\alpha}_r \in \mathcal{O}_{\mathbb{K}}$, therefore $\tilde{\alpha}_0 h, \ldots, \tilde{\alpha}_r h \in L$ and since $a \in \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$, the rational number

$$d = \frac{\mathcal{N}(h)}{\mathcal{N}(L)} \frac{\tilde{\lambda}}{\lambda m^{r+1}} \tilde{t}$$

is actually a non-zero integer whose sign is $\epsilon = \epsilon_h = \mathrm{sign}(\mathcal{N}(h))$. We have proven that $\epsilon \underline{\tilde{\alpha}} h$ is a uniform positive dual family to $\underline{a}$ in L such that

$$\underline{a} \cdot (\epsilon \underline{\tilde{\alpha}} h) = \epsilon d I_{r+1} = |d| I_{r+1}.$$

Let us now rewrite this using the level ℓ (see Definition III.8). By definition, there are integers $m_0, \ldots, m_r$ and elements $\beta_0, \ldots, \beta_r \in L$ such that for any $0 \leq j \leq r$:

$$\tilde{\alpha}_j h = \epsilon(\ell m \beta_j - m_j h).$$

Then (III.28) may be written in terms of β_j as:

$$a(u_k\beta_j) = \begin{cases} 0 & \text{if } j \neq k \\ \frac{|d|}{m\ell} = \frac{|\mathcal{N}(h/m)|}{\mathcal{N}(L)} \frac{\tilde{\lambda}}{\lambda} \frac{\tilde{t}}{\ell} & \text{if } j = k. \end{cases}$$

The elements $u_k\beta_j$ belong to L therefore $a(u_k\beta_j) \in \mathbb{Z}$ for any $0 \leq j, k \leq r$ and $\beta_0, \ldots, \beta_r$ is a uniform positive dual family to $a, au_1, \ldots, au_r$ in L. It follows from lemma III.6 that t divides the integer $d/(m\ell)$. We now prove that $tm\ell = |d|$. Indeed, it follows from lemma III.6, (iii) that for any $0 \leq j \leq r$:

$$\frac{|d|}{m\ell}\alpha_j - t\beta_j \in t\mathbb{Z}\gamma.$$

Replacing β_j by its definition we obtain:

$$\frac{d}{m\ell t}\alpha_j - \frac{(\tilde{\alpha}_j + m_j)h}{m\ell} \in \mathbb{Z}\gamma$$

and thus:

$$\tilde{\alpha}_j h \in \frac{d}{t}\alpha_j - m_j h + m\ell\mathbb{Z}\gamma$$

This shows in particular that

$$\tilde{\alpha}_0 h, \ldots, \tilde{\alpha}_r h \in \frac{|d|}{t}L + \mathbb{Z}\gamma$$

and therefore the inequality $|d|/t \leq m\ell$ holds. This may be written as $|d| \leq m\ell t$ and the fact that $m\ell t$ divides $|d|$ with $|d| > 0$ implies that $|d| \geq m\ell t$. Thus the two quantities must be equal. We have thus proven that $\underline{a} \cdot (\epsilon\underline{\tilde{\alpha}}h) = |d|I_{r+1} = m\ell t I_{r+1}$ and that the fundamental equality:

$$\lambda\ell t = \frac{|\mathcal{N}(h/m)|}{\mathcal{N}(L)}\tilde{\lambda}\tilde{t}$$

holds in $\mathbb{Z}$. The identification of β_j with α_j gives the desired relation between the families $\underline{\alpha}$, $\underline{\tilde{\alpha}}$:

$$\alpha_j = \frac{\tilde{\alpha}_j h + m_j h}{\epsilon m\ell}.$$

□

This Proposition is fundamental in two different ways for the construction of higher elliptic units. On the one hand, the relation (III.26) between the integers $\lambda, \tilde{\lambda}, \ell, t, \tilde{t}$ is crucial to understand how the choice of base point h should be made to obtain interesting evaluations of the higher elliptic Gamma functions (and evaluations that can be computed in a small amount of time). On the other hand, the relation between the families $\underline{\alpha}$ and $\underline{\tilde{\alpha}}$ show that the base point h impacts the parameters $m\alpha_0/h, \ldots, m\alpha_r/h$ of the G_r functions essentially through the level ℓ. For instance, in the cubic case, the functions $\Gamma^{\pm}_{\mathfrak{f},\mathfrak{b},\mathfrak{a}}$ are essentially evaluated at points

$$\tau = \pm\frac{\varepsilon + n_0}{\ell\tilde{\lambda}} = \frac{\tau_0}{\ell}, \sigma = \pm\frac{\varepsilon^{-1} + n_1}{\ell\tilde{\lambda}} = \frac{\sigma_0}{\ell}$$

for some integers n_0, n_1, where ε is a fundamental unit for $\mathcal{O}_{\mathfrak{f}}^{+,\times}$ (or a fundamental unit for $\mathcal{O}_{\mathbb{K}}^{+,\times}$ following remark III.6). This also means that when comparing two different choices of base points h and h', we only need to compare similar functions for different levels ℓ and ℓ'.

Let us now end this section with some refinements on the fundamental relation III.26, obtained by studying the matrix of multiplication by h in the bases B_L and $\tilde{B}$.

Lemma III.10:

(i) Let H be the matrix representing in the bases B_L and $\tilde{B}$ the integral linear map $\mathrm{mul}_h : \mathcal{O}_{\mathbb{K}} \to L$ corresponding to multiplication by h. Then H is an upper triangular matrix with coefficients in $m\mathbb{Z}$ of the form:

$$H = \begin{pmatrix} m & * & \dots & * & \dots & * & * \\ 0 & \frac{mc_{110}}{\tilde{c}_{110}} & \dots & * & \dots & * & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \frac{mc_{jj0}}{\tilde{c}_{jj0}} & \dots & * & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & \dots & \frac{mc_{rr0}}{\tilde{c}_{rr0}} & * \\ 0 & 0 & \dots & 0 & \dots & 0 & \epsilon m\ell t/\tilde{t} \end{pmatrix}$$

where $\epsilon = \mathrm{sign}(\mathcal{N}(h))$.

(ii) As a consequence, for any $1 \le j \le r$, $\tilde{c}_{jj0} \mid c_{jj0}$ and thus $\tilde{\lambda} \mid \lambda$. Moreover, $\tilde{t} \mid \ell t$.

Proof :

(i) We shall first prove by induction on $0 \le j \le r$ that $h\tilde{e}_j$ belongs to the $\mathbb{Z}$-span of $e_0, \dots, e_j$ and that $\langle h\tilde{e}_j, e_j\rangle = m\frac{c_{jj0}}{\tilde{c}_{jj0}}$, where we set by convention $c_{000} = 1$ and $\tilde{c}_{000} = 1$.

Case $j = 0$: Since $\tilde{e}_0 = 1$ and $e_0 = h/m$ we get $h\tilde{e}_0 = me_0 = \frac{mc_{000}}{\tilde{c}_{000}}e_0$.

Inductive step: Suppose that $1 \le j \le r$ and that for any $0 \le k \le j-1$, $h\tilde{e}_k$ belongs to the $\mathbb{Z}$-span of $e_0, \dots, e_k$ with $\langle h\tilde{e}_k, e_k\rangle = m\frac{c_{kk0}}{\tilde{c}_{kk0}}$. Since $u_j = \sum_{k=0}^{j} \tilde{c}_{jk0}\tilde{e}_k$ we may write

$$h\tilde{e}_j = \frac{h}{\tilde{c}_{jj0}}\left(u_j - \sum_{k=0}^{j-1} \tilde{c}_{jk0}\tilde{e}_k\right)$$

By induction hypothesis, the term $\sum_{k=0}^{j-1} \frac{\tilde{c}_{jk0}}{\tilde{c}_{jj0}} h\tilde{e}_k$ belongs to the $\mathbb{Q}$-span of $e_0, \dots, e_{j-1}$. In addition, since $hu_j = \sum_{k=0}^{j} mc_{jk0}e_k$ we get

$$h\tilde{e}_j = \frac{mc_{jj0}}{\tilde{c}_{jj0}}e_j + \sum_{k=0}^{j-1} \frac{c_{jk0}e_k - \tilde{c}_{jk0}\tilde{e}_k}{\tilde{c}_{jj0}}.$$

Thus $h\tilde{e}_j$ belongs to the intersection of the lattice L with the $\mathbb{Q}$-span of $e_0, \dots, e_j$, that is $h\tilde{e}_j$ belongs to the $\mathbb{Z}$-span of $e_0, \dots, e_j$ and $\langle h\tilde{e}_j, e_j\rangle = \frac{mc_{jj0}}{\tilde{c}_{jj0}}$. This proves the result by induction.

Let us now show that the coefficient $\langle h\tilde{e}_{r+1}, e_{r+1}\rangle = \epsilon m\ell t/\tilde{t}$. Since $h/m \in L$, the image of the map mul_h is a subset of $m.L$ and all coefficients of H are divisible by m. Let us write temporarily $mR = \langle h\tilde{e}_{r+1}, e_{r+1}\rangle$. Computing the determinant of H which is equal to $\mathcal{N}(h)/\mathcal{N}(L)$ we get:

$$\frac{\mathcal{N}(h)}{\mathcal{N}(L)} = \det(H) = m^{r+2} \prod_{j=1}^{r} \frac{c_{jj0}}{\tilde{c}_{jj0}} \times R.$$

Using the formula for λ and $\tilde{\lambda}$ given by (III.10) and (III.21) respectively this gives:

$$\frac{\tilde{\lambda}}{\lambda}\frac{\mathcal{N}(h/m)}{\mathcal{N}(L)} = R.$$

The desired result $R = \epsilon\ell t/\tilde{t}$ is then obtained by using formula (III.26).

(ii) The matrix H/m is still integral as it corresponds to the multiplication by $h/m \in L$ in the basis B_L and $\tilde{B}$. Thus we get $\tilde{c}_{jj0} \mid c_{jj0}$ for any $1 \leq j \leq r$ and the content $\tilde{\lambda}$ divides λ. In addition, the overflow $\tilde{t}$ divides ℓt. □

This last point, that the overflow $\tilde{t}$ divides ℓt is crucial to understand what sort of base point h we shall be looking for in general. One of our primary goals for our evaluations is to get the lowest possible value for t and if possible to get $t = 1$. This comes with the tradeoff that the overflow $\tilde{t}$ must divide the level ℓ. In the next section, we revisit formula (III.26) in terms of generalised different ideals.

III.3.3 Generalised different ideals

III.3.3.1 Definition of generalised different ideals

In this section we define generalised different ideals which we shall use to give some insight on the value of the overflow $\tilde{t}$.

Definition III.11: *Let $f : \mathbb{K} \to \mathbb{Q}$ be a non-zero $\mathbb{Q}$-linear form on $\mathbb{K}$. Let I be a fractional ideal of $\mathbb{K}$. The fractional ideal $\mathfrak{D}(f, I)$ defined by:*

$$\mathfrak{D}(f, I)^{-1} = \{x \in \mathbb{K} \mid \forall\, y \in I, f(xy) \in \mathbb{Z}\}$$

is called the different ideal of f on I by analogy with the usual different ideal $\mathfrak{d}$ of $\mathbb{K}$ given by $\mathfrak{d} = \mathfrak{D}(\mathrm{Tr}, \mathcal{O}_{\mathbb{K}})$.

We start by proving some straightforward basic facts about these different ideals.

Lemma III.12:

(i) If $I' = \beta I$ for some $\beta \in \mathbb{K}^\times$ then $\mathfrak{D}(f, I') = \beta.\mathfrak{D}(f, I)$.

(ii) If $I = I_1 + I_2$ then $\mathfrak{D}(f, I) = \mathfrak{D}(f, I_1) + \mathfrak{D}(f, I_2)$.

(iii) For any fractional ideal I of $\mathbb{K}$, $\mathfrak{D}(f, I) = I.\mathfrak{D}(f, \mathcal{O}_{\mathbb{K}})$.

(iv) If $f' = (y \to f(\beta y))$ for some $\beta \in \mathbb{K}^\times$ then $\mathfrak{D}(f', I) = \beta.\mathfrak{D}(f, I)$.

(v) For any $f \in \mathrm{Hom}_{\mathbb{Q}}(\mathbb{K}, \mathbb{Q}) - \{0\}$, there is a unique element $\xi \in \mathbb{K}^\times$ such that $f = (y \to \mathrm{Tr}(\xi y))$. In particular:

$$\mathfrak{D}(f, I) = (\xi) \times I \times \mathfrak{d}.$$

(vi) For a fixed fractional ideal I, all different ideals $\mathfrak{D}(f, I)$ belong to the class of $I \times \mathfrak{d}$ in the class group of $\mathbb{K}$.

(vii) If $f(I) \subset \mathbb{Z}$ then $\mathfrak{D}(f, I) \subset \mathcal{O}_{\mathbb{K}}$. In addition, if $f(I) = \mathbb{Z}$ then $\mathfrak{D}(f, I)$ is a primitive integral ideal.

Proof :

(i) Let I and I' be two fractional ideals such that $I' = \beta I$ for some $\beta \in \mathbb{K}^\times$. Let us rewrite the definition of $\mathfrak{D}(f, I')$ as:

$$\mathfrak{D}(f, I')^{-1} = \{x \in \mathbb{K} \mid \forall\, y \in I', f(xy) \in \mathbb{Z}\}$$
$$\mathfrak{D}(f, I')^{-1} = \{x \in \mathbb{K} \mid \forall\, y \in I, f(x\beta y) \in \mathbb{Z}\}$$
$$\mathfrak{D}(f, I')^{-1} = \{x \in \mathbb{K} \mid x\beta \in \mathfrak{D}(f, I)^{-1}\}$$

It follows that $\mathfrak{D}(f, I')^{-1} = \beta^{-1}\mathfrak{D}(f, I)^{-1}$ and therefore $\mathfrak{D}(f, I') = \beta.\mathfrak{D}(f, I)$.

(ii) Since $(I + J)^{-1} = I^{-1} \cap J^{-1}$ for any fractional ideals I, J we only need to prove that

$$\mathfrak{D}(f, I_1 + I_2)^{-1} = \mathfrak{D}(f, I_1)^{-1} \cap \mathfrak{D}(f, I_2)^{-1}.$$

Let $x \in \mathfrak{D}(f, I)^{-1}$ where $I = I_1 + I_2$. This means that for $y \in I_1 + I_2$, $f(xy) \in \mathbb{Z}$. Since $I_1 \subset I$ and $I_2 \subset I$ we get $f(xy) \in \mathbb{Z}$ for all $y \in I_1$ and for all $y \in I_2$. Thus $x \in \mathfrak{D}(f, I_1)^{-1} \cap \mathfrak{D}(f, I_2)^{-1}$. On the other hand, if x is in this intersection, then for any $y = y_1 + y_2 \in I_1 + I_2$ we get $f(xy) = f(xy_1) + f(xy_2) \in \mathbb{Z} + \mathbb{Z} = \mathbb{Z}$. Thus $x \in \mathfrak{D}(f, I_1 + I_2)^{-1}$ as claimed.

(iii) Any fractional ideal is the sum of two principal fractional ideals. Thus, if $I = I_1 + I_2$ with $I_j = \beta_j \mathcal{O}_\mathbb{K}$ for some $\beta_j \in \mathbb{K}^\times$, $j = 1, 2$ then by (ii),

$$\mathfrak{D}(f, I) = \mathfrak{D}(f, I_1) + \mathfrak{D}(f, I_2).$$

Since $I_j = \beta_j \mathcal{O}_\mathbb{K}$ we get $\mathfrak{D}(f, I_j) = \beta_j.\mathfrak{D}(f, \mathcal{O}_\mathbb{K})$ by (i). Thus $\mathfrak{D}(f, I) = (\beta_1\mathcal{O}_\mathbb{K} + \beta_2\mathcal{O}_\mathbb{K}).\mathfrak{D}(f, \mathcal{O}_\mathbb{K})$. Since $\beta_1\mathcal{O}_\mathbb{K} + \beta_2\mathcal{O}_\mathbb{K} = I$, this leads to $\mathfrak{D}(f, I) = I.\mathfrak{D}(f, \mathcal{O}_\mathbb{K})$ as claimed.

(iv) This is essentially the same as the proof for (i). Indeed, by definition:

$$\mathfrak{D}((y \to f(\beta y)), I)^{-1} = \{x \in \mathbb{K} \mid \forall\, y \in I, f(x\beta y) \in \mathbb{Z}\}.$$

It follows from the proof of (i) that $\mathfrak{D}((y \to f(\beta y), I)^{-1} = \beta^{-1}\mathfrak{D}(f, I)^{-1}$ and therefore $\mathfrak{D}((y \to f(\beta y), I) = \beta.\mathfrak{D}(f, I)$.

(v) The existence of the unique ξ is a standard fact in linear algebra. If $f = (y \to \mathrm{Tr}(\xi y))$ then it follows from (iii) that $\mathfrak{D}(f, I) = I.\mathfrak{D}(f, \mathcal{O}_\mathbb{K})$ and it follows from (iv) that $\mathfrak{D}(f, \mathcal{O}_\mathbb{K}) = \xi.\mathfrak{D}(\mathrm{Tr}, \mathcal{O}_\mathbb{K}) = \xi.\mathfrak{d}$. Thus we get $\mathfrak{D}(f, I) = (\xi) \times I \times \mathfrak{d}$.

(vi) This is a consequence of (v) as $\mathfrak{D}(f, I) = (\xi) \times I \times \mathfrak{d}$ for any $f = (y \to \mathrm{Tr}(\xi y))$.

(vii) Suppose that $f(I) \subset \mathbb{Z}$. Let $x \in \mathcal{O}_\mathbb{K}$. Since I is a fractional ideal, multiplication by x maps I to a subset of I and therefore for any $y \in I$, $f(xy) \in f(I) \subset \mathbb{Z}$. Thus $x \in \mathfrak{D}(f, I)^{-1}$. This gives $\mathcal{O}_\mathbb{K} \subset \mathfrak{D}(f, I)^{-1}$ and by inversion we get $\mathfrak{D}(f, I) \subset \mathcal{O}_\mathbb{K}$. Suppose now that $f(I) = \mathbb{Z}$. Suppose that $\mathfrak{D}(f, I) \subset d\mathcal{O}_\mathbb{K}$ for some integer $d \geq 1$. Then $d^{-1}\mathcal{O}_\mathbb{K} \subset \mathfrak{D}(f, I)^{-1}$ and thus $\forall y \in I$, $f(y) = d.f(y/d) \in d\mathbb{Z}$. Thus, by assumption, since $f(I) = \mathbb{Z}$ we conclude that $d = 1$. This proves that $\mathfrak{D}(f, I)$ is a primitive integral ideal. □

On the computational side, we may compute the ideal $\mathfrak{D}(f, I)$ using lemma III.12, (v) which only requires finding the element $\xi \in \mathbb{K}^\times$ satisfying $f = (y \to \mathrm{Tr}(\xi y))$. This may be done by solving a linear system as if $B' = [e'_0, \ldots, e'_{r+1}]$ is a $\mathbb{Z}$-basis of $\mathcal{O}_\mathbb{K}$ and $T = (\mathrm{Tr}(e'_i e'_j))_{0 \leq i,j \leq r+1}$ then ξ is expressed as a vector on the basis B' by:

$$\xi = T^{-1} \begin{pmatrix} f(e'_0) \\ \vdots \\ f(e'_{r+1}) \end{pmatrix}.$$

Another approach is to generalise directly [[Coh93], Proposition 4.8.19] as follows:

Lemma III.13: *For $B = [e_0, \ldots, e_{r+1}]$ a $\mathbb{Z}$-basis of I and $B' = [e'_0, \ldots, e'_{r+1}]$ any $\mathbb{Q}$-basis of $\mathbb{K}$ we may define $\mathcal{M} \in M_{r+2}(\mathbb{Q})$ to be the matrix whose coefficients are the rational numbers $f(e_i e'_j)$ for $0 \leq i, j \leq r+1$. Then $\mathfrak{D}(f, I)^{-1}$ is explicitly parametrised as the image of $\mathbb{Z}^{r+2}$ under the $\mathbb{Q}$-linear isomorphism $\iota : \mathbb{Q}^{r+2} \to \mathbb{K}$ defined by:*

$$\iota \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{r+1} \end{pmatrix} = (e'_0, \ldots, e'_{r+1}) \times \mathcal{M}^{-1} \times \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{r+1} \end{pmatrix}.$$

Proof :
The proof is essentially the same as the proof given in [[Coh93], Proposition 4.8.19] for the usual different ideal. Let $\mathcal{M}$ be the matrix $(f(e_i e'_j))_{0 \leq i,j \leq r+1}$. Let us denote by m_{ij} the coefficients of $\mathcal{M}^{-1}$ so that:

$$\sum_{k=0}^{r+1} f(e_j e'_k) m_{kl} = \delta_{jl}$$

where δ_{jl} is Kronecker's symbol. Let $x = \iota((x_0, \ldots, x_{r+1})^T)$ and $y = \sum_{j=0}^{r+1} y_j e_j$ where the x_i's are rational numbers and the y_i's are integers. Then:

$$f(xy) = f\left(\left(\sum_{j=0}^{r+1} y_j e_j\right) \times \left(\sum_{k=0}^{r+1} e'_k \sum_{l=0}^{r+1} m_{kl} x_l\right)\right)$$

$$f(xy) = \sum_{j=0}^{r+1} y_j \sum_{l=0}^{r+1} \left(\sum_{k=0}^{r+1} f(e_j e'_k) m_{kl}\right) x_l$$

$$f(xy) = \sum_{j=0}^{r+1} y_j x_j$$

Therefore, $f(xy) \in \mathbb{Z}$ for all $y \in I$ if and only if $x_j \in \mathbb{Z}$ for all $0 \leq j \leq r+1$ and $\mathfrak{D}(f, I)^{-1} = \iota(\mathbb{Z}^{r+2})$. □

In what follows, we shall be interested in the different ideals $\mathfrak{D}(a, L)$ and $\mathfrak{D}(\tilde{a}, \mathcal{O}_{\mathbb{K}})$ associated to the linear forms a and $\tilde{a}$. In particular, we may show a straightforward relation between these two fractional ideals which is closely related to (III.26).

Lemma III.14: *The different ideals $\mathfrak{D}(a, L)$ and $\mathfrak{D}(\tilde{a}, \mathcal{O}_{\mathbb{K}})$ are primitive integral ideals related by:*

$$\lambda \mathfrak{D}(a, L) = \tilde{\lambda} \frac{\mathcal{N}(h/m)}{\mathcal{N}(L)} \frac{m}{h} L.\mathfrak{D}(\tilde{a}, \mathcal{O}_{\mathbb{K}}).$$

Proof :
First, since $a(L) = \mathbb{Z}$ and $\tilde{a}(\mathcal{O}_{\mathbb{K}}) = \mathbb{Z}$, it follows from lemma III.12 that these different ideals are primitive integral. Secondly, recall from (III.18) that

$$a = \left(y \to \tilde{a}\left(\frac{\tilde{\lambda}}{\lambda} \frac{\mathcal{N}(h/m)}{\mathcal{N}(L)} \frac{m}{h} y\right)\right).$$

Thus, it follows from lemma III.12, (iv) that

$$\mathfrak{D}(a, L) = \frac{\tilde{\lambda}}{\lambda}\frac{\mathcal{N}(h/m)}{\mathcal{N}(L)}\frac{m}{h}\mathfrak{D}(\tilde{a}, L)$$

and using lemma III.12, (iii) we get the desired relation:

$$\lambda\mathfrak{D}(a, L) = \tilde{\lambda}\frac{\mathcal{N}(h/m)}{\mathcal{N}(L)}\frac{m}{h}L.\mathfrak{D}(\tilde{a}, \mathcal{O}_{\mathbb{K}}).$$

□

We immediately note that the fundamental relation (III.26) gives:

$$\frac{\mathfrak{D}(a, L)}{t} = \ell\frac{m}{h}L.\frac{\mathfrak{D}(\tilde{a}, \mathcal{O}_{\mathbb{K}})}{\tilde{t}}.$$

We claim that the values of $\tilde{s}$ and of the overflow $\tilde{t}$ essentially depend on $\mathfrak{D}(\tilde{a}, \mathcal{O}_{\mathbb{K}})$ and that this different ideal plays an important role in our construction. To prove precise statements on this, we shall need to do some more work on these generalised different ideals.

III.3.3.2 Explicit description of the different ideals associated to a and $\tilde{a}$

To describe precisely the different ideals associated to a and $\tilde{a}$ and the links with other quantities we introduced, we will use lemma III.13. Recall that we have fixed $B_L = [e_0, \dots, e_{r+1}]$ a positive $\mathbb{Z}$-basis of L and $\tilde{B} = [\tilde{e}_0, \dots, \tilde{e}_{r+1}]$ a positive $\mathbb{Z}$-basis of $\mathcal{O}_{\mathbb{K}}$ in section III.3.2. Let us now define the matrix $\mathcal{M}$ (resp. $\widetilde{\mathcal{M}}$) to be the matrix whose coefficients are $(a(e_i\tilde{e}_j))_{0\leq i,j\leq r+1}$ (resp. the matrix $(\tilde{a}(\tilde{e}_i\tilde{e}_j))_{0\leq i,j\leq r+1}$). Since $a(L) \subset \mathbb{Z}$ and $\tilde{a}(\mathcal{O}_{\mathbb{K}}) \subset \mathbb{Z}$ these matrices are integral. It follows from lemma III.13 that $\mathcal{M}^{-1}$ represents the inverse different ideal $\mathfrak{D}(a, L)$ in the basis B_L and that $\widetilde{\mathcal{M}}^{-1}$ represents $\mathfrak{D}(\tilde{a}, \mathcal{O}_{\mathbb{K}})$ in the basis $\tilde{B}$. In particular, the norms of these ideals are related to the determinants of these matrices as:

$$|\det(\mathcal{M})| = \mathcal{N}(\mathfrak{D}(a, L)) \quad \text{and} \quad |\det(\widetilde{\mathcal{M}})| = \mathcal{N}(\mathfrak{D}(\tilde{a}, \mathcal{O}_{\mathbb{K}})).$$

We shall now express a link between these two matrices and the matrices $\mathcal{A}$ and $\widetilde{\mathcal{A}}$ defined in (III.9) and (III.20).

Lemma III.15: *The two following matrix equalities hold:*

$$(\mathbf{0}\ \mathcal{A}) = \widetilde{\mathfrak{U}}^T\mathcal{M}^T, \quad \textit{and} \quad (\mathbf{0}\ \widetilde{\mathcal{A}}) = \widetilde{\mathfrak{U}}^T\widetilde{\mathcal{M}}^T$$

where ^{T} *stands for transposition and* $\widetilde{\mathfrak{U}}$ *is the matrix defined in (III.19). As a consequence,* $\tilde{\lambda} \mid s$ *and* $\tilde{\lambda} \mid \tilde{s}$.

Proof :

Recall that the matrix $(\mathbf{0}\ \mathcal{A})$ is given by its coefficients $a(u_je_l)$ for $0 \leq j \leq r$ and $0 \leq l \leq r+1$. Replacing u_j by its expression $u_j = \sum_{k=0}^{j}\tilde{c}_{jk0}\tilde{e}_k$ gives $a(u_je_l) = \sum_{k=0}^{j}\tilde{c}_{jk0}a(\tilde{e}_ke_l)$. Thus:

$$(\mathbf{0}\ \mathcal{A}) = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ \tilde{c}_{100} & \tilde{c}_{110} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \tilde{c}_{r00} & \tilde{c}_{r10} & \tilde{c}_{r20} & \dots & \tilde{c}_{rr0} & 0 \end{pmatrix} \begin{pmatrix} a(\tilde{e}_0e_0) & \dots & a(\tilde{e}_0e_l) & \dots & a(\tilde{e}_0e_{r+1}) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a(\tilde{e}_je_0) & \dots & a(\tilde{e}_je_l) & \dots & a(\tilde{e}_je_{r+1}) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a(\tilde{e}_{r+1}e_0) & \dots & a(\tilde{e}_{r+1}e_l) & \dots & a(\tilde{e}_{r+1}e_{r+1}) \end{pmatrix}$$

which is exactly the statement that $(\mathbf{0}\ \mathcal{A}) = \widetilde{\mathfrak{U}}^T \mathcal{M}^T$. Let us now analyse this relation more precisely, by introducing two auxiliary square matrices $\widetilde{\mathfrak{U}}_0$ and $\mathcal{M}_0$ of size $r+1$ such that

$$\widetilde{\mathfrak{U}} = \left(\begin{array}{c} \\ \widetilde{\mathfrak{U}}_0 \\ \\ \hline 0 \ \ldots \ 0 \end{array}\right) \text{ and } \mathcal{M} = \left(\begin{array}{ccccc|c} 0 & 0 & \ldots & 0 & 0 & 1 \\ \hline & & & & & * \\ & & \mathcal{M}_0 & & & \vdots \\ & & & & & * \end{array}\right).$$

Then the relation $(\mathbf{0}\ \mathcal{A}) = \widetilde{\mathfrak{U}}^T \mathcal{M}^T$ reduces to $\mathcal{A} = \widetilde{\mathfrak{U}}_0^T \mathcal{M}_0^T$ and taking determinants gives

$$s = \det(\mathcal{A}) = \det(\widetilde{\mathfrak{U}}_0^T) \times \det(\mathcal{M}_0^T) = \tilde{\lambda}.\det(\mathcal{M}_0) \in \tilde{\lambda}.\mathbb{Z}.$$

Thus $\tilde{\lambda} \mid s$. The proof that $(\mathbf{0}\ \widetilde{\mathcal{A}}) = \widetilde{\mathfrak{U}}^T \widetilde{\mathcal{M}}^T$ and that $\tilde{\lambda} \mid \tilde{s}$ is essentially the same. □

We may now improve this result by considering a slight modification of the matrices $\mathcal{A}$, $\widetilde{\mathcal{A}}$ and $\widetilde{\mathfrak{U}}$. Indeed, let us define the square matrices:

$$\mathcal{A}_1 = \begin{pmatrix} au_0 \\ au_1 \\ \vdots \\ au_r \\ a\tilde{e}_{r+1} \end{pmatrix} \quad \text{and} \quad \widetilde{\mathcal{A}}_1 = \begin{pmatrix} \tilde{a}u_0 \\ \tilde{a}u_1 \\ \vdots \\ \tilde{a}u_r \\ \tilde{a}\tilde{e}_{r+1} \end{pmatrix} \quad \text{and} \quad \widetilde{\mathfrak{U}}_1 = \begin{pmatrix} \widetilde{\mathfrak{U}} & \widetilde{X} \end{pmatrix}$$

where $\widetilde{X} = (0, \ldots, 0, 1)^T$. Then, on the one hand it is clear that the proof of lemma III.15 may be adapted to show that:

$$\mathcal{A}_1 = \widetilde{\mathfrak{U}}_1^T \mathcal{M}^T, \quad \text{and} \quad \widetilde{\mathcal{A}}_1 = \widetilde{\mathfrak{U}}_1^T \widetilde{\mathcal{M}}^T. \tag{III.29}$$

On the other hand, the matrices $\mathcal{A}_1$ and $\widetilde{\mathcal{A}}_1$ are explicitly given by:

$$\mathcal{A}_1 = \left(\begin{array}{c|ccc} 0 & & & \\ \vdots & & \mathcal{A} & \\ 0 & & & \\ \hline a(e_0\tilde{e}_{r+1}) & a(e_1\tilde{e}_{r+1}) & \ldots & a(e_{r+1}\tilde{e}_{r+1}) \end{array}\right) = \left(\begin{array}{c|c} \mathbf{0} & \mathcal{A} \\ \hline a(e_0\tilde{e}_{r+1}) & \mathcal{X}_1 \end{array}\right) \tag{III.30}$$

and by:

$$\widetilde{\mathcal{A}}_1 = \left(\begin{array}{c|ccc} 0 & & & \\ \vdots & & \widetilde{\mathcal{A}} & \\ 0 & & & \\ \hline \tilde{a}(\tilde{e}_0\tilde{e}_{r+1}) & \tilde{a}(\tilde{e}_1\tilde{e}_{r+1}) & \ldots & \tilde{a}(\tilde{e}_{r+1}\tilde{e}_{r+1}) \end{array}\right) = \left(\begin{array}{c|c} \mathbf{0} & \widetilde{\mathcal{A}} \\ \hline \tilde{a}(\tilde{e}_0\tilde{e}_{r+1}) & \widetilde{\mathcal{X}}_1 \end{array}\right) \tag{III.31}$$

where $\mathcal{X}_1 = (a(e_1\tilde{e}_{r+1}), \ldots, a(e_{r+1}\tilde{e}_{r+1}))$ and $\widetilde{\mathcal{X}}_1 = (\tilde{a}(\tilde{e}_1\tilde{e}_{r+1}), \ldots, \tilde{a}(\tilde{e}_{r+1}\tilde{e}_{r+1}))$. Thus we already obtain two new relations:

Lemma III.16: *Taking determinants in (III.29) gives the relations in $\mathbb{Z}_{>0}$:*

$$s.\ell.t = \tilde{\lambda}.\tilde{t}.\mathcal{N}(\mathfrak{D}(a, L)) \tag{III.32}$$

$$\tilde{s} = \tilde{\lambda}.\mathcal{N}(\mathfrak{D}(\tilde{a}, \mathcal{O}_{\mathbb{K}})). \tag{III.33}$$

Proof :

(i) Let us take determinants in the matrix equalities given in (III.29). The matrix $\widetilde{\mathfrak{U}}_1$ is an upper triangular matrix so its determinant is the product of the diagonal coefficients, i.e. $\det(\widetilde{\mathfrak{U}}) = \prod_{j=1}^{r} \tilde{c}_{jj0} = \tilde{\lambda}$. From the definition of $\mathcal{A}_1$ it is clear that $\det(\mathcal{A}_1) = \det(\mathcal{A}) \times a(e_0\tilde{e}_{r+1}) = s \times a(h\tilde{e}_{r+1}/m)$ as $s = \det(\mathcal{A})$ by definition. Now, since $|\langle h\tilde{e}_{r+1}, e_{r+1}\rangle| = m\ell t/\tilde{t}$ (see lemma III.10) we get:

$$|\det(\mathcal{A}_1)| = s \times \ell t/\tilde{t}.$$

This gives the desired equality:

$$s.\ell.t = \tilde{\lambda}.\tilde{t}.\mathcal{N}(\mathfrak{D}(a, L)).$$

(ii) The determinant of $\widetilde{\mathcal{A}}_1$ is by definition $\det(\widetilde{\mathcal{A}}_1) = \det(\widetilde{\mathcal{A}}) \times \tilde{a}(\tilde{e}_0\tilde{e}_{r+1}) = \tilde{s}$ as $\tilde{a}(\tilde{e}_0\tilde{e}_{r+1}) = \tilde{a}(\tilde{e}_{r+1}) = 1$. Thus taking determinants in (III.29) gives:

$$\tilde{s} = \tilde{\lambda}.\mathcal{N}(\mathfrak{D}(\tilde{a}, \mathcal{O}_{\mathbb{K}}))$$

as claimed. □

This lemma shows in particular that the value of $\tilde{s}$ and therefore the value of the overflow $\tilde{t}$ depends essentially on $\tilde{\lambda}$ and $\mathfrak{D}(\tilde{a}, \mathcal{O}_{\mathbb{K}})$. We shall now define two matrices $\mathcal{B}_1$ and $\widetilde{\mathcal{B}}_1$ by:

$$\mathcal{B}_1 = t \times \mathcal{A}_1^{-1} \quad \text{and} \quad \widetilde{\mathcal{B}}_1 = \tilde{t} \times \widetilde{\mathcal{A}}_1^{-1}. \tag{III.34}$$

These matrices expand the matrices $\mathcal{B}$ and $\widetilde{\mathcal{B}}$ defined in section III.3.2 as we shall show now, and they are related to the families $\alpha_0, \ldots, \alpha_r$ and $\tilde{\alpha}_0, \ldots, \tilde{\alpha}_r$ introduced in (III.13) and (III.24).

Lemma III.17: *Let $\mathcal{X}_1$ and $\widetilde{\mathcal{X}}_1$ be as defined in (III.30) and (III.31). Then:*

$$\mathcal{B}_1 = \left(\begin{array}{c|c} -\mathcal{X}_1\mathcal{B} \times (\tilde{t}/(\ell t)) & \tilde{t}/\ell \\ \hline \mathcal{B} & \mathbf{0} \end{array}\right) \quad \textit{and} \quad \widetilde{\mathcal{B}}_1 = \left(\begin{array}{c|c} -\widetilde{\mathcal{X}}_1\widetilde{\mathcal{B}} & \tilde{t} \\ \hline \widetilde{\mathcal{B}} & \mathbf{0} \end{array}\right).$$

In particular, the matrix $\widetilde{\mathcal{B}}_1$ is integral.

Proof :

The lemma follows from the inversion formula for the matrices

$$\mathcal{A}_1 = \left(\begin{array}{c|c} \mathbf{0} & \mathcal{A} \\ \hline \ell t/\tilde{t} & \mathcal{X}_1 \end{array}\right) \quad \text{and} \quad \widetilde{\mathcal{A}}_1 = \left(\begin{array}{c|c} \mathbf{0} & \widetilde{\mathcal{A}} \\ \hline 1 & \widetilde{\mathcal{X}}_1 \end{array}\right)$$

together with the fact that by definition $\mathcal{A} \cdot \mathcal{B} = tI_{r+1}$ and $\widetilde{\mathcal{A}} \cdot \widetilde{\mathcal{B}} = \tilde{t}I_{r+1}$. □

In the next section, we shall use the matrix $\widetilde{\mathcal{B}}_1$ to gain additional information on the family $\tilde{\alpha}_0, \ldots, \tilde{\alpha}_r$ under some simplifying hypothesis.

III.3.4 The choice of base points $(h_\rho)_{\rho\in\mathfrak{S}_r}$

In this section, we explain how the base points $(h_\rho)_{\rho\in\mathfrak{S}_r}$ should be chosen in Conjecture III.5. The choice of these base points is the major difficulty we need to overcome in order to formulate a precise version of our main conjecture. There are two important aspects we keep in mind:

- Computations show (see section IV.2.6.1) that for fields of degree $n \geq 4$, the base points $(h_\rho)_\rho$ cannot be chosen independently. Hence, we must define a notion of compatible sets of base points $(h_\rho)_\rho$ for which a precise form of Conjecture III.5 holds (see Definition III.24).
- To perform efficient computations, and to get the simplest formulas for higher elliptic units, the values of the parameters t_ρ should be the smallest possible, and, after that, the value of the levels ℓ_ρ should also be the smallest possible.

With these two points in mind, we shall explain how to choose the set of base points $(h_\rho)_\rho$ under some specific conditions (see (H3), (H4) and (H5) below). Let us fix once again a permutation $\rho \in \mathfrak{S}_r$ and a unit system $u_1, \ldots, u_r = [\varepsilon_{\rho(1)}|\ldots|\varepsilon_{\rho(r)}]$ which we assume to satisfy (H2). Let us discuss the role of the content $\tilde{\lambda}$ in the evaluation (III.2). By definition, the integers s and t share the same prime factors, therefore $t = 1 \Leftrightarrow s = 1$. It follows from lemma III.15 that the content $\tilde{\lambda}$ divides s, thus if $\tilde{\lambda} \neq 1$ it is impossible to achieve the best case scenario $t = s = 1$. We shall therefore assume that the unit system $u_1, \ldots, u_r$ satisfies the additionnal condition:

$$\text{The content } \tilde{\lambda} = \tilde{\lambda}(u_1, \ldots, u_r) \text{ of the unit system } u_1, \ldots, u_r \text{ is equal to 1.} \qquad \text{(H3)}$$

It is clear that this condition makes sense only when the unit system $u_1, \ldots, u_r$ satisfies (H2) and that this condition does not depend on the choice of base point h. In view of the results from section III.3.3, this hypothesis simplifies considerably the situation as the basis $\tilde{B}$ may be taken to be $[1, u_1, \ldots, u_r, \tilde{e}_{r+1}]$. We shall show that under this hypothesis we can describe more precisely the values of λ, ℓ and t associated to the base point h (see Proposition III.20). Then, we show (see Proposition III.25) that under some assumptions, we can always choose a set of *compatible* base points h_ρ such that for any permutation ρ, $t_\rho = t_{\rho,h_\rho} = 1$ by using *helper ideals* (see Definitions III.24 and III.21).

The general idea is that for any $\rho \in \mathfrak{S}_r$, the strongly admissible base point h_ρ should be chosen if possible as a generator of the ideal $\frac{qN}{\mathfrak{ab}}\mathfrak{D}(\tilde{a}_\rho, \mathcal{O}_\mathbb{K})$. However, there is no guarantee that this ideal is principal, nor that it is generated by a strongly admissible base point. To deal with this problem we use ideals of the form $m.\mathfrak{H}$ where $m > 0$ is an integer coprime to q and $\mathfrak{H}$ is an ideal of the satisfying $\mathcal{O}_\mathbb{K}/\mathfrak{H} \simeq (\mathbb{Z}/p_\mathfrak{H}\mathbb{Z})^{r+1}$ where $\mathfrak{H} \cap \mathbb{Z} = p_\mathfrak{H}\mathbb{Z}$. We show that it is always possible to choose the base points as generators of ideals

$$\frac{qN}{\mathfrak{ab}}\mathfrak{D}(\tilde{a}_\rho, \mathcal{O}_\mathbb{K}).m_\rho\mathfrak{H} = h_\rho\mathcal{O}_\mathbb{K}$$

for some integers $m_\rho > 0$ and some ideal $\mathfrak{H}$ independent of ρ which satisfies the above property. The presence of the ideal $m_\rho\mathfrak{H}$ as a divisor of the ideal generated by h_ρ only has a little and well-controlled impact on the geometric setup we have presented in the previous sections. The fact that the ideal $\mathfrak{H}$ should not depend on the permutation ρ is crucial for the validity of our construction of higher elliptic units and it gives rise to the compatibility condition expressed in Definition III.24.

III.3.4.1 The family $\tilde{\alpha}_0, \ldots, \tilde{\alpha}_r$ and the different ideal $\mathfrak{D}(\tilde{a}, \mathcal{O}_{\mathbb{K}})$

In this section we work once again with fixed ρ and drop the subscripts to gain in clarity. We shall assume that the unit system $u_1, \ldots, u_r = [\varepsilon_{\rho(1)}| \ldots |\varepsilon_{\rho(r)}]$ satisfies (H2) and is such that the content $\tilde{\lambda}$ is equal to 1 (see hypothesis (H3)). We start by explaining how this affects the results from sections III.3.2 and III.3.3. Since $\tilde{\lambda} = 1$ we may assume that the basis $\tilde{B}$ is given by $\tilde{B} = [1, u_1, \ldots, u_r, \tilde{e}_{r+1}]$ in which case the matrix $\widetilde{\mathfrak{U}}$ defined in (III.19) is given by:

$$\widetilde{\mathfrak{U}} = \begin{pmatrix} 1 & 0 & \ldots & 0 & \ldots & 0 \\ 0 & 1 & \ldots & & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & \ldots & 1 \\ 0 & 0 & \ldots & 0 & \ldots & 0 \end{pmatrix} = \begin{pmatrix} I_{r+1} \\ \hline \mathbf{0} \end{pmatrix}.$$

In addition, the relations (III.26), (III.32) and (III.33) become:

$$\begin{aligned} \lambda \ell t &= \frac{|\mathcal{N}(h/m)|}{\mathcal{N}(L)} \tilde{t} \\ s.\ell.t &= \tilde{t}.\mathcal{N}(\mathfrak{D}(a, L)) \\ \tilde{s} &= \mathcal{N}(\mathfrak{D}(\tilde{a}, \mathcal{O}_{\mathbb{K}})). \end{aligned}$$

More importantly, we get the matrix equalities:

$$\widetilde{\mathcal{A}}_1 = \widetilde{\mathcal{M}}^T \quad \text{and} \quad \mathcal{A}_1 = \mathcal{M}^T$$

(see section III.3.3.2 for the definition of these matrices). We shall now use these equalities to give a description of the family $\tilde{\alpha}_0, \ldots, \tilde{\alpha}_r$ in terms of the different ideal $\mathfrak{D}(\tilde{a}, \mathcal{O}_{\mathbb{K}})$ (see Definition III.11).

Lemma III.18: *Assume that $u_1, \ldots, u_r$ satisfies (H2) and (H3). Let $\tilde{x}_0, \ldots, \tilde{x}_r, \tilde{t} \in \mathbb{Z}$ be the coefficients of the first row of $\widetilde{\mathcal{B}}_1$. Then the family $(\tilde{\alpha}_0 + \tilde{x}_0, \ldots, \tilde{\alpha}_r + \tilde{x}_r, \tilde{t})$ is a $\mathbb{Z}$-basis of the fractional ideal $\tilde{t}.\mathfrak{D}(\tilde{a}, \mathcal{O}_{\mathbb{K}})^{-1}$. As a consequence, this fractional ideal is integral and $\tilde{t} \in \mathfrak{D}(\tilde{a}, \mathcal{O}_{\mathbb{K}}) \cap \mathbb{Z}$.*

Proof :

Under hypothesis (H3) the matrix $\widetilde{\mathcal{B}}_1$ satisfies $\widetilde{\mathcal{B}}_1 \widetilde{\mathcal{M}}^T = \tilde{t} I_n$, which we can rewrite as $\tilde{t}^{-1} \widetilde{\mathcal{B}}_1 = (\widetilde{\mathcal{M}}^T)^{-1}$. The matrix $\widetilde{\mathcal{M}}$ is symmetric, so we get the simpler equality $\tilde{t}^{-1} \widetilde{\mathcal{B}}_1 = \widetilde{\mathcal{M}}^{-1}$. It follows from lemma III.13 that the columns of the matrix $\widetilde{\mathcal{M}}^{-1}$ define a $\mathbb{Z}$-basis of $\mathfrak{D}(\tilde{a}, \mathcal{O}_{\mathbb{K}})$. Using the formula for $\widetilde{\mathcal{B}}_1$ given by lemma III.17 gives:

$$\tilde{t}.\widetilde{\mathcal{M}}^{-1} = \left(\begin{array}{ccc|c} \tilde{x}_0 & \ldots & \tilde{x}_r & \tilde{t} \\ \hline & & & 0 \\ & \widetilde{\mathcal{B}} & & \vdots \\ & & & 0 \end{array} \right)$$

and since the columns of the matrix $\widetilde{\mathcal{B}}$ define $\tilde{\alpha}_0, \ldots, \tilde{\alpha}_r$ (see (III.24)) the family

$$\left(\frac{\tilde{\alpha}_0 + \tilde{x}_0}{\tilde{t}}, \ldots, \frac{\tilde{\alpha}_r + \tilde{x}_r}{\tilde{t}}, 1 \right)$$

is a $\mathbb{Z}$-basis of $\mathfrak{D}(\tilde{a}, \mathcal{O}_{\mathbb{K}})^{-1}$. Thus, the family $(\tilde{\alpha}_0 + \tilde{x}_0, \dots, \tilde{\alpha}_r + \tilde{x}_r, \tilde{t})$ is a $\mathbb{Z}$-basis of the fractional ideal $\tilde{t}.\mathfrak{D}(\tilde{a}, \mathcal{O}_{\mathbb{K}})^{-1}$ as claimed. The elements $\tilde{\alpha}_j + \tilde{x}_j$ belong to $\mathcal{O}_{\mathbb{K}}$ for $0 \leq j \leq r$ therefore the fractional ideal $\tilde{t}.\mathfrak{D}(\tilde{a}, \mathcal{O}_{\mathbb{K}})^{-1}$ is integral. Lastly, if $\mathfrak{D}(\tilde{a}, \mathcal{O}_{\mathbb{K}}) \cap \mathbb{Z} = d.\mathbb{Z}$ then

$$\tilde{t}.d^{-1} \in \tilde{t}.\mathfrak{D}(\tilde{a}, \mathcal{O}_{\mathbb{K}})^{-1} \cap \mathbb{Z} \subset \mathcal{O}_{\mathbb{K}} \cap \mathbb{Z} \subset \mathbb{Z}$$

which gives $d \mid \tilde{t}$ and $\tilde{t} \in \mathfrak{D}(\tilde{a}, \mathcal{O}_{\mathbb{K}})$. □

III.3.4.2 Target ideals

Using the characterisation of the elements $\tilde{\alpha}_0, \dots, \tilde{\alpha}_r$ given in Lemma III.18, we may now give some insight on how to choose the base point h for the unit system $u_1, \dots, u_r$ to achieve the value $t = 1$ in the computations. Recall from Lemma III.10 that the overflow $\tilde{t}$ divides the product $\ell.t$, therefore $t = 1$ may only happen if $\tilde{t} \mid \ell$. We now give a sufficient condition on the ideal generated by h for this to happen. Indeed, the condition that h is a weakly admissible base point implies that the fractional ideal $h\mathcal{O}_{\mathbb{K}}$ may be decomposed as $m\frac{qN}{\mathfrak{ab}}\mathfrak{I}$ for some integer $m > 0$ coprime to q and some primitive integral ideal $\mathfrak{I}$ coprime to $\mathfrak{f} \times \mathfrak{a}$ (recall that an integral ideal is primitive if $\mathfrak{I} \subset d\mathcal{O}_{\mathbb{K}}$ for some integer $d > 0$ implies $d = 1$). The choice of h then essentially boils down to the choice of integer m and to the choice of fractional ideal $\mathfrak{I}$. We argue that when the primitive integral ideal $\mathfrak{D}(\tilde{a}, \mathcal{O}_{\mathbb{K}})$ divides the ideal $\mathfrak{I}$, the desired division $\tilde{t} \mid \ell$ holds. Unfortunately, in general it can happen that $\mathfrak{D}(\tilde{a}, \mathcal{O}_{\mathbb{K}})$ is not coprime to $\mathfrak{f}$ or $\mathfrak{a}$. Thus, we shall assume the following:

The different ideal $\mathfrak{D}(\tilde{a}, \mathcal{O}_{\mathbb{K}})$ is coprime to q. (H4)

This hypothesis makes sense when the unit system $u_1, \dots, u_r$ satisfies (H2) and it depends only on $u_1, \dots, u_r$ and $\mathfrak{f}$. In particular, this hypothesis is independent of (H3).

A comment we wish to make on this assumption is that we will also need to assume that $\mathfrak{D}(\tilde{a}, \mathcal{O}_{\mathbb{K}})$ is coprime to N. However, the different ideal $\mathfrak{D}(\tilde{a}, \mathcal{O}_{\mathbb{K}})$ does not depend on $\mathfrak{a}$ and in general the choice of smoothing ideal $\mathfrak{a}$ is flexible, therefore to ensure that $\mathfrak{D}(\tilde{a}, \mathcal{O}_{\mathbb{K}})$ is coprime to N we will make sure to choose $\mathfrak{a}$ such that N is coprime to $\tilde{t}$ (see Definition III.26). We are now ready to prove the following:

Lemma III.19: *Assume that the unit system $u_1, \dots, u_r$ satisfies (H2), (H3) and (H4). Assume further that N is coprime to $\mathfrak{D}(\tilde{a}, \mathcal{O}_{\mathbb{K}})$. Suppose that the weakly admissible base point h satisfies $h\mathcal{O}_{\mathbb{K}} = m\frac{qN}{\mathfrak{ab}}\mathfrak{I}$ with $m > 0$ coprime to q and $\mathfrak{I}$ a primitive integral ideal coprime to $\mathfrak{f}.\mathfrak{a}$ such that $\mathfrak{D}(\tilde{a}, \mathcal{O}_{\mathbb{K}})$ divides $\mathfrak{I}$. Then $\tilde{t} \mid \ell$ and $\tilde{s} \mid \lambda$.*

Proof :

(i) Let us start by proving that $\tilde{t} \mid \ell$. It follows from lemma III.18 that for $0 \leq j \leq r$, $\tilde{\alpha}_j + \tilde{x}_j \in \tilde{t}.\mathfrak{D}(\tilde{a}, \mathcal{O}_{\mathbb{K}})^{-1}$ for some integers $\tilde{x}_j$. Let us denote by $\mathfrak{I}'$ the integral ideal satisfying $\mathfrak{D}(\tilde{a}, \mathcal{O}_{\mathbb{K}})\mathfrak{I}' = \mathfrak{I}$. Then for any $0 \leq j \leq r$ we get:

$$(\tilde{\alpha}_j + \tilde{x}_j)h \in m\frac{qN}{\mathfrak{ab}} \times \tilde{t}.\mathfrak{I}' \subset m\tilde{t}L.$$

Thus $\tilde{\alpha}_j h \in m\tilde{t}L + \mathbb{Z}h$ for all $0 \leq j \leq r$ and by definition of ℓ (see Definition III.8) this implies that $\tilde{t} \mid \ell$.

(ii) Let us now prove that $\tilde{s} \mid \lambda$. To do this, we introduce an auxiliary $\mathbb{Z}$-basis $B'' = [e''_0, \dots, e''_{r+1}]$ of $\mathcal{O}_{\mathbb{K}}$ such that $\tilde{a}u_j e''_k = 0$ whenever $k > j$ and $\tilde{a}u_j e''_k = c''_{jk} \in \mathbb{Z}$

otherwise. This is once again done by computing a HNF representation of the matrix $\widetilde{\mathcal{A}}_1^T$. Indeed, there is a matrix $P \in \mathrm{GL}_n(\mathbb{Z})$ and an upper trianguler integral matrix $\mathcal{A}''$ such that $\widetilde{\mathcal{A}}_1^T = P\mathcal{A}''$. Therefore, $\widetilde{\mathcal{A}}_1 \cdot (P^T)^{-1} = (\mathcal{A}'')^T$ and

$$\mathcal{A}'' = \begin{pmatrix} 1 & c''_{10} & \dots & c''_{j0} & \dots & c''_{r0} & * \\ 0 & c''_{11} & \dots & c''_{j1} & \dots & c''_{r1} & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & c''_{jj} & \dots & c''_{rj} & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & \dots & c''_{rr} & * \\ 0 & 0 & \dots & 0 & \dots & 0 & 1 \end{pmatrix}$$

In particular, the value of $\tilde{s} = \det(\widetilde{\mathcal{A}}_1)$ is given by $\tilde{s} = \prod_{j=1}^r c''_{jj}$. Using this representation we obtain a series of identities of the form $(y \to \tilde{a}(u'_j y)) \in \mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}_{\mathbb{K}}, \mathbb{Z})$ where for all $1 \leq j \leq r$:

$$u'_j = \frac{u_j + \sum_{k=0}^{j-1} q_k^{(j)} u_k}{c''_{jj}}$$

for some rational number $q_k^{(j)}$. This shows that the element u'_j belongs to $\mathfrak{D}(\tilde{a}, \mathcal{O}_{\mathbb{K}})^{-1}$ and since $\mathfrak{D}(\tilde{a}, \mathcal{O}_{\mathbb{K}})$ divides the integral ideal $h.L^{-1}$, the element $u'_j h$ belongs to L. Therefore, for any $y \in L$:

$$\begin{aligned} \lambda m^{r+1} a(y) &= \det_{B_L}(h, u_1 h, \dots, u_r h, y) \\ \lambda m^{r+1} a(y) &= \det_{B_L}\left(u_0 h, (u_1 + q_0^{(j)} u_0)h, \dots, \left(u_r + \sum_{k=0}^{r-1} q_k^{(r)} u_k\right) h, y\right) \\ \lambda a(y) &= \prod_{j=1}^r c''_{jj} \times \det_{B_L}\left(u_0 h/m, u'_1 h/m, \dots, u'_r h/m, y\right) \end{aligned}$$

Thus, if y satisfies $a(y) = 1$, since $\prod_{j=1}^r c''_{jj} = \tilde{s}$ we obtain:

$$\lambda = \tilde{s} \times \det_{B_L}\left(u_0 h/m, u'_1 h/m, \dots, u'_r h/m\right) \in \tilde{s}.\mathbb{Z}$$

and therefore $\tilde{s} \mid \lambda$. ☐

Remark: in the case where (H4) does not hold, i.e. essentially in the case where q and $\mathfrak{D}(\tilde{a}, \mathcal{O}_{\mathbb{K}})$ are not coprime we must define the auxiliary ideal $\mathfrak{D}_{\hat{q}}(\tilde{a}, \mathcal{O}_{\mathbb{K}})$ to be the largest divisor of $\mathfrak{D}(\tilde{a}, \mathcal{O}_{\mathbb{K}})$ which is coprime to q. If we also define $\tilde{s}_{\hat{q}}$ (resp. $\tilde{t}_{\hat{q}}$) to be the largest divisor of $\tilde{s}$ (resp. $\tilde{t}$) which is coprime to q, then Lemma III.19 may be adapted to give the following statement: if $h\mathcal{O}_{\mathbb{K}} = m.\frac{qN}{\mathfrak{ab}}.\mathfrak{I}$ for some $m > 0$ coprime to q and $\mathfrak{I}$ a primitive integral ideal coprime to $\mathfrak{f}.\mathfrak{a}$ such that $\mathfrak{D}_{\hat{q}}(\tilde{a}, \mathcal{O}_{\mathbb{K}})$ divides $\mathfrak{I}$, then $\tilde{s}_{\hat{q}} \mid \lambda$ and $\tilde{t}_{\hat{q}} \mid \ell$.

Let us now make one final assumption on $\mathfrak{f}$ that ensures we can describe explicitly the parameters λ, ℓ and t. This hypothesis concerns the shape of the quotient $\mathcal{O}_{\mathbb{K}}/\mathfrak{f}$ and plays a role in the determination of the valuations of λ, ℓ and t at primes dividing q. We assume the following:

$$\text{The quotient } \mathcal{O}_{\mathbb{K}}/\mathfrak{f} \text{ is a cyclic abelian group.} \tag{H5}$$

This condition is equivalent to $\mathfrak{f} \cap \mathbb{Z} = \mathcal{N}(\mathfrak{f})\mathbb{Z}$ and we now give a second equivalent definition when $\mathfrak{f} \neq (1)$ (which we always assume). Let $q = \prod_{j=1}^{k} q_j^{n_j}$ be the prime factorisation of q, where $q\mathbb{Z} = \mathfrak{f} \cap \mathbb{Z}$. Then $\mathfrak{f} \neq (1)$ satisfies (H5) if and only if

$$\mathfrak{f} = \prod_{j=1}^{k} \mathfrak{q}_j^{n_j}$$

where for $1 \leq j \leq k$, $\mathfrak{q}_j$ is an ideal of norm q_j. Note that hypothesis (H5) depends only on $\mathfrak{f}$ and not on the unit system $u_1, \ldots, u_r$. In particular, it is independent of the hypotheses (H2), (H3) and (H4). We are now ready to prove the following proposition on the values of λ, ℓ, t when the base point h is carefully chosen with respect to the different ideal $\mathfrak{D}(\tilde{a}, \mathcal{O}_{\mathbb{K}})$.

Proposition III.20: *Assume (H3), (H4) and (H5). Assume further that the ideal $\mathfrak{D}(\tilde{a}, \mathcal{O}_{\mathbb{K}})$ is coprime to N. Suppose that the weakly admissible base point h satisfies $h\mathcal{O}_{\mathbb{K}} = m\frac{qN}{\mathfrak{ab}}\mathfrak{D}(\tilde{a}, \mathcal{O}_{\mathbb{K}})$ with $m > 0$ coprime to q. Then*

$$\lambda = q^r.N^r.\tilde{s} \quad \textit{and} \quad \ell = q.N.\tilde{t} \quad \textit{and} \quad t = 1.$$

Proof :

The proof contains 3 main steps consisting of the analysis of the divisibilities by q, by N and by the factors of $\tilde{t}$.

Divisibility by q: Let us first treat the case of q. Under assumption (H5) any unit u_j is congruent to an integer modulo $\mathfrak{f}$, therefore there are integers m'_j for $1 \leq j \leq r$ such that $u_j + m'_j \in \mathfrak{f}$. Since $h \in qL/\mathfrak{f}$, this gives $(u_j + m'_j)h \in mqL$ for any $1 \leq j \leq r$ and therefore $q \mid c_{jj0}$ (see (III.8) for the definition of the integers c_{jj0}) for any $1 \leq j \leq r$. Thus, $q^r \mid \lambda$. On the other hand, for any $0 \leq j \leq r$, there are integers m''_j such that $\tilde{\alpha}_j + m''_j \in \mathfrak{f}$. Therefore, $(\tilde{\alpha}_j + m''_j)h \in mqL$ and by definition of the level ℓ we get $q \mid \ell$.

Divisibility by N: This case is exactly proven in the same way as the divisibility by q since the smoothing ideal $\mathfrak{a}$ is such that $\mathcal{O}_{\mathbb{K}}/\mathfrak{a}$ is cyclic and $h \in NL/\mathfrak{a}$.

Factors of $\tilde{t}$: It follows from Lemma III.19 that $\tilde{t} \mid \ell$ and $\tilde{s} = \mathcal{N}(\mathfrak{D}(\tilde{a}, \mathcal{O}_{\mathbb{K}})) \mid \lambda$ under assumptions (H3), (H4).

Let us now put everything together and prove the desired result using lemma III.9. First, as q, N and $\tilde{t}$ are pairwise coprime (and q, N and $\tilde{s}$ are also pairwise coprime) as a result of hypothesis (H4), one gets $q^r.N^r.\tilde{s} \mid \lambda$ and $q.N.\tilde{t} \mid \ell$. Now, we use the fundamental relation:

$$\lambda \ell t = \frac{|\mathcal{N}(h/m)|}{\mathcal{N}(L)}\tilde{t}$$

given in Proposition (III.9). Since $|\mathcal{N}(h/m)|/\mathcal{N}(L) = q^{r+1}.N^{r+1}.\tilde{s}$ we get:

$$\frac{\lambda}{q^r.N^r.\tilde{s}}.\frac{\ell}{q.N.\tilde{t}}.t = 1.$$

As all three terms on the left-hand side are positive integers, we must conclude that they are all equal to 1. Therefore

$$\lambda = q^r.N^r.\tilde{s} \quad \text{and} \quad \ell = q.N.\tilde{t} \quad \text{and} \quad t = 1$$

as claimed. □

This Proposition shows that the base point h should be chosen if possible to satisfy $h\mathcal{O}_{\mathbb{K}} = m\frac{qN}{\mathfrak{ab}}\mathfrak{D}(\tilde{a}, \mathcal{O}_{\mathbb{K}})$. There are three remarks we wish to make about this choice of base point h. First, there is no guarantee that the ideal $m\frac{qN}{\mathfrak{ab}}\mathfrak{D}(\tilde{a}, \mathcal{O}_{\mathbb{K}})$ is principal in $\mathcal{O}_{\mathbb{K}}$, nor that it possesses a generator h which is a weakly admissible base point. This will be adressed in the following section, where we use helper ideals to circumvent this problem. The second remark is that if we do not assume the collection of hypotheses (H3), (H4) and (H5), the best choice for h should still be a base point satisfying $h = m\frac{qN}{\mathfrak{ab}}\mathfrak{D}_{\hat{q}}(\tilde{a}, \mathcal{O}_{\mathbb{K}})$ where as before $\mathfrak{D}_{\hat{q}}(\tilde{a}, \mathcal{O}_{\mathbb{K}})$ is the coprime to q part of $\mathfrak{D}(\tilde{a}, \mathcal{O}_{\mathbb{K}})$ (see section IV.2.6 for some examples in this case). As a last remark, we point out that in the language of [Mor24], the ideal $\mathfrak{D}(\tilde{a}, \mathcal{O}_{\mathbb{K}})$ (or rather $\mathfrak{D}_{\hat{q}}(\tilde{a}, \mathcal{O}_{\mathbb{K}})$ in general) is the product of all the *target ideals* for the unit system $u_1, \ldots, u_r$. Thus we might refer to $\mathfrak{D}(\tilde{a}, \mathcal{O}_{\mathbb{K}})$ as the target ideal for the unit system $u_1, \ldots, u_r$ since the determination of this ideal is crucial for computations.

III.3.4.3 Helper ideals

In this section we show how to treat the case where the ideal $\frac{qN}{\mathfrak{ab}}\mathfrak{D}(\tilde{a}, \mathcal{O}_{\mathbb{K}})$ is not generated by a weakly admissible base point h. To this end we introduce the notion of *helper ideals*: these are certain integral ideals $\mathfrak{H}$ which don't influence the value of t and such that the ideal $\frac{qN}{\mathfrak{ab}}\mathfrak{D}(\tilde{a}, \mathcal{O}_{\mathbb{K}}).\mathfrak{H}$ is generated by a weakly admissible base point (see lemma III.22). These ideals are defined as follows:

Definition III.21: *An integral ideal $\mathfrak{H}$ is a helper ideal if either $\mathfrak{H} = \mathcal{O}_{\mathbb{K}}$ or $\mathfrak{H}$ is a primitive ideal coprime to $q.N.\tilde{t}$ such that $p_{\mathfrak{H}}/\mathfrak{H}$ is a cyclic group of order $p_{\mathfrak{H}}$, where $\mathfrak{H} \cap \mathbb{Z} = p_{\mathfrak{H}}\mathbb{Z}$. An extended helper ideal is an ideal of the form $m.\mathfrak{H}$ where m is a positive integer coprime to q and $\mathfrak{H}$ is a helper ideal.*

These ideals are interesting because they do not impact the value of t as explained by the following lemma.

Lemma III.22: *Assume that the unit system $u_1, \ldots, u_r$ satisfies (H2) (H3) and (H4). Assume that $\mathcal{O}_{\mathbb{K}}/\mathfrak{f}$ is cyclic (see (H5)). Assume further that the ideal $\mathfrak{D}(\tilde{a}, \mathcal{O}_{\mathbb{K}})$ is coprime to N. Suppose that $\mathfrak{H}$ is a helper ideal and that h is a weakly admissible base point such that $h\mathcal{O}_{\mathbb{K}} = m\frac{qN}{\mathfrak{ab}}\mathfrak{D}(\tilde{a}, \mathcal{O}_{\mathbb{K}}).\mathfrak{H}$ with $m > 0$ coprime to q. Set $p_{\mathfrak{H}}\mathbb{Z} = \mathfrak{H} \cap \mathbb{Z}$ with $p_{\mathfrak{H}} > 0$ and assume that $p_{\mathfrak{H}}$ is coprime to $q.N.\tilde{t}$. Then*

$$\lambda = q^r.N^r.\tilde{s}.p_{\mathfrak{H}}^r \quad \textit{and} \quad \ell = q.N.\tilde{t}.p_{\mathfrak{H}} \quad \textit{and} \quad t = 1.$$

Proof :
The proof is very similar to the proof of Proposition III.20 and it is naturally split into three parts: the determination of λ, then ℓ, then the proof that $t = 1$ using (III.9).

Determination of λ: It follows from the proof of Proposition III.20 that $q^r.N^r.\tilde{s} \mid \lambda$. To prove that $p_{\mathfrak{H}}^r \mid \lambda$ we remark as before that since $p_{\mathfrak{H}}/\mathfrak{H}$ is cyclic, there are integers $m_j'(\mathfrak{H})$ such that $u_j + m_j'(\mathfrak{H}) \in p_{\mathfrak{H}}/\mathfrak{H}$ for $1 \leq j \leq r$. Thus $(u_j + m_j'(\mathfrak{H}))h \in mp_{\mathfrak{H}}L$ and thus $p_{\mathfrak{H}} \mid c_{jj0}$ for any $1 \leq j \leq r$. This gives $p_{\mathfrak{H}}^r \mid \lambda$. The fact that $p_{\mathfrak{H}}$ is coprime to $q.N.\tilde{t}$ and thus to $q.N.\tilde{s}$ implies that $q^r.N^r.\tilde{s}.p_{\mathfrak{H}}^r \mid \lambda$.

Determination of ℓ: It follows from the proof of Proposition III.20 that $q.N.\tilde{t} \mid \ell$. To prove that $p_{\mathfrak{H}} \mid \ell$ we remark as before that since $p_{\mathfrak{H}}/\mathfrak{H}$ is cyclic, there are integers $m_j''(\mathfrak{H})$ such that $\tilde{\alpha}_j + m_j''(\mathfrak{H}) \in p_{\mathfrak{H}}/\mathfrak{H}$ for $1 \leq j \leq r$. Thus $(\tilde{\alpha}_j + m_j''(\mathfrak{H}))h \in mp_{\mathfrak{H}}L$ and by definition of ℓ one gets $p_{\mathfrak{H}} \mid \ell$. Once again, $p_{\mathfrak{H}}$ is coprime to $q.N.\tilde{t}$ so $q.N.\tilde{t}.p_{\mathfrak{H}} \mid \ell$.

Determination of t: We use once again formula (III.26):

$$\lambda \ell t = \frac{|\mathcal{N}(h/m)|}{\mathcal{N}(L)} \tilde{t}.$$

Since $|\mathcal{N}(h/m)|/\mathcal{N}(L) = q^{r+1}.N^{r+1}.\tilde{s}.p_{\mathfrak{H}}^{r+1}$ we get:

$$\frac{\lambda}{q^r.N^r.\tilde{s}.p_{\mathfrak{H}}^r}.\frac{\ell}{q.N.\tilde{t}.p_{\mathfrak{H}}}.t = 1.$$

As before, we must conclude that each of the three positive integers on the left-hand side is equal to 1, that is:

$$\lambda = q^r.N^r.\tilde{s}.p_{\mathfrak{H}}^r \quad \text{and} \quad \ell = q.N.\tilde{t}.p_{\mathfrak{H}} \quad \text{and} \quad t = 1$$

as claimed. □

The use of helper ideals gives us a bit more room to find a weakly admissible base point h, at the cost of an increase of the level ℓ by a factor $p_{\mathfrak{H}}$. Luckily, the use of helper ideals is enough to guarantee that we find a suitable ideal, as we explain in the following lemma:

Lemma III.23: *For any class $\mathfrak{c}$ in the wide or narrow ray class group at $\mathfrak{f}$ and any integer k, there is a helper ideal $\mathfrak{H}$ coprime to $k\mathcal{O}_{\mathbb{K}}$ and an integer $m > 0$ coprime to q such that $m\mathfrak{H} \in \mathfrak{c}$. In other words, the set of extended helper ideals coprime to $k\mathcal{O}_{\mathbb{K}}$ generates the wide or narrow ray class group at $\mathfrak{f}$. In addition, the set of helper ideals which are coprime to k generates the usual class group of $\mathbb{K}$.*

Proof :

Fix a class $\mathfrak{c}$ in the wide or narrow ray class group at $\mathfrak{f}$. By Cebotarev's density theorem (see [[Cox22], Theorem 8.17]), there are infinitely many degree one prime ideals in the class $\mathfrak{c}^{-1}$, that is to say there are infinitely many pairs $(p, \mathfrak{P})$ where p is a rational prime and $\mathfrak{P}$ is a prime ideal of norm p in $\mathcal{O}_{\mathbb{K}}$ such that $\mathfrak{P} \in \mathfrak{c}^{-1}$. Fix any of these pairs such that p is coprime to $q.k$. Fix a positive integer m such that $mp \equiv 1 \bmod q$. Then $\mathfrak{H} = p/\mathfrak{P}$ is a helper ideal such that $m\mathfrak{H} \in \mathfrak{c}$. This completes the proof of the first point. For the second point, if we fix a class $\mathfrak{c}'$ in the class group of $\mathbb{K}$, then there is a pair $(p, \mathfrak{P})$ where p is a rational prime coprime to k and $\mathfrak{P}$ is a prime ideal of norm p such that $\mathfrak{P} \in \mathfrak{c}'^{-1}$. Then $\mathfrak{H} = p/\mathfrak{P}$ is a helper ideal coprime to k which belongs to $\mathfrak{c}'$. □

In particular, this lemma implies that there is a helper ideal $\mathfrak{H}$ coprime to $q.N.\tilde{t}$ and an integer $m > 0$ such that the ideal

$$m\frac{N}{\mathfrak{ab}}\mathfrak{D}(\tilde{a}, \mathcal{O}_{\mathbb{K}}).\mathfrak{H}$$

belongs to the trivial class in the wide class group at $\mathfrak{f}$. This ideal is then generated by some h/q where h is a strongly admissible base point in the sense of Definition III.4.

III.3.4.4 Using target and helper ideals to choose the base points $(h_\rho)_{\rho \in \mathfrak{S}_r}$

In this section we gather the results on both target ideals and helper ideals to explain how to choose the base points $(h_\rho)_{\rho \in \mathfrak{S}_r}$. These should be chosen to be weakly admissible generators of ideals of the form

$$h_\rho \mathcal{O}_{\mathbb{K}} = m_\rho \frac{qN}{\mathfrak{ab}} \widetilde{\mathfrak{D}}_\rho.\mathfrak{H}_\rho$$

where $\widetilde{\mathfrak{D}}_\rho = \mathfrak{D}(\tilde{a}_\rho, \mathcal{O}_\mathbb{K})$ is the target ideal associated to $[\varepsilon_{\rho(1)}|\dots|\varepsilon_{\rho(r)}]$, m_ρ is a positive integer coprime to q and $\mathfrak{H}_\rho$ is a helper ideal (see Definition III.21). However, for ATR fields of degree $n \geq 4$, the computations show that choosing these base points independently does not yield higher elliptic units in general (see section IV.2.6.1). This leads us to defining the notion of a compatible set of base points $(h_\rho)_\rho$.

Definition III.24: *A set $\underline{h} = (h_\rho)_\rho$ of base points is said to be weakly compatible (for the data $\mathfrak{f}, \mathfrak{b}, \mathfrak{a}$ and for the choice of fundamental units $\varepsilon_1, \dots, \varepsilon_r$) if each of the base points h_ρ is weakly admissible in the sense of Definition III.4 and if for each $\rho \in \mathfrak{S}_r$ the fractional ideal $h_\rho\mathcal{O}_\mathbb{K}$ may be decomposed as:*

$$h_\rho \mathcal{O}_\mathbb{K} = m_\rho \frac{qN}{\mathfrak{ab}} \widetilde{\mathfrak{D}}_\rho . \mathfrak{H}$$

where $\mathfrak{H}$ is a helper ideal independent of ρ and m_ρ is a positive integer coprime to q. The set $\underline{h}$ is said to be strongly compatible if, in addition, each of the base points h_ρ is strongly admissible in the sense of Definition III.4.

This definition is crucial, as it guarantees that our computation of higher elliptic units, which is already a mixed-level computation in some sense, has the correct levels ℓ_ρ when ρ varies in $\mathfrak{S}_r$. We gather all the results of this section in the following Proposition:

Proposition III.25: *Suppose that $\mathfrak{f}$ satisfies (H5). Fix a set of fundamental units $\varepsilon_1, \dots, \varepsilon_r$ for $\mathcal{O}_\mathbb{K}^{+,\times}$ and assume that for each $\rho \in \mathfrak{S}_r$, the unit system $[\varepsilon_{\rho(1)}|\dots|\varepsilon_{\rho(r)}]$ satisfies (H2), (H3) and (H4). Fix a smoothing ideal $\mathfrak{a}$ of norm N such that N is coprime to $\prod_{\rho\in\mathfrak{S}_r} \tilde{t}_\rho$. Fix an integral ideal $\mathfrak{b}$ representing a class in the narrow ray class group at $\mathfrak{f}$. Then there is a helper ideal $\mathfrak{H}$ independent of ρ and there are positive integers $(m_\rho)_\rho$ coprime to q such that all the ideals*

$$m_\rho \frac{N}{\mathfrak{ab}} \widetilde{\mathfrak{D}}_\rho . \mathfrak{H}$$

belong to the trivial class of the wide class group at $\mathfrak{f}$. In addition, we may fix a generator h_ρ for each of the ideals $m_\rho \frac{qN}{\mathfrak{ab}} \widetilde{\mathfrak{D}}_\rho . \mathfrak{H}$ so that the set of base points $\underline{h} = (h_\rho)_\rho$ is strongly compatible for $\varepsilon_1, \dots, \varepsilon_r$.

Proof :

It follows from lemma III.12, (vi) that the different ideals $\widetilde{\mathfrak{D}}_\rho$ all belong to the same class in the usual class group of $\mathbb{K}$, that is the class of the usual different ideal $\mathfrak{d}$. We now need to prove that there are integers m'_ρ coprime to q such that the different ideals $m'_\rho\widetilde{\mathfrak{D}}_\rho$ all belong to the same class in the wide ray class group at $\mathfrak{f}$. Let us fix $\widetilde{\mathfrak{D}}_1$ one of these different ideals. Then for each $\rho \in \mathfrak{S}_r$, the fractional ideal $\widetilde{\mathfrak{D}}_\rho/\widetilde{\mathfrak{D}}_1$ is principal and coprime to q by (H4), therefore it admits a generator ξ_ρ which is coprime to q. Since $\mathcal{O}_\mathbb{K}/\mathfrak{f}$ is cyclic by (H5), there are positive integers d_ρ coprime to q such that for each ρ, $\xi_\rho - d_\rho \in \mathfrak{f}$. Let us set for each $\rho \in \mathfrak{S}_r$ a positive integer m'_ρ satisfying $m'_\rho d_\rho \equiv 1 \bmod q$. Then $m'_\rho\xi_\rho \equiv 1 \bmod \mathfrak{f}$ for each $\rho \in \mathfrak{S}_r$. Thus the ideals $m'_\rho\widetilde{\mathfrak{D}}_\rho$ all belong to the same class in the wide class group at $\mathfrak{f}$. It then follows from Lemma III.23 that there is a helper ideal $\mathfrak{H}$ coprime to $q.N.\tilde{t}$ and a positive integer m'' coprime to q, both independent of ρ such that for all $\rho \in \mathfrak{S}_r$, the ideal

$$m'' m'_\rho \frac{N}{\mathfrak{ab}} \widetilde{\mathfrak{D}}_\rho . \mathfrak{H}$$

belongs to the trivial class in the wide class group at $\mathfrak{f}$. This gives the desired conclusion with $m_\rho = m''m'_\rho$. The ideals $m_\rho \frac{N}{\mathfrak{ab}} \widetilde{\mathfrak{D}}_\rho.\mathfrak{H}$ then all possess generators g_ρ satisfying $g_\rho \equiv 1 \bmod L$ and we may set $h_\rho = qg_\rho$ to obtain a strongly compatible set of base points $(h_\rho)_\rho$. □

Remarks:

- This proposition is our main tool to produce sets of base points $(h_\rho)_\rho$ algorithmically to check our conjectures and this is the main setting we shall adopt in the following sections.
- When some of the hypotheses (H3), (H4), (H5) are not satisfied, Proposition III.25 may still be used to produce compatible sets of base points in a broader sense which we haven't completely determined yet. Some examples to showcase what sort of computations remain possible are presented in section IV.2.6.
- When choosing a helper ideal, one should try to minimise the value of $p_\mathfrak{H}$ where $\mathfrak{H} \cap \mathbb{Z} = p_\mathfrak{H}\mathbb{Z}$. Indeed, as we explain in section IV.1.1.2, the complexity of our computations is linear in ℓ and $p_\mathfrak{H}$ divides exactly ℓ.

We end this section by discussing briefly the hypothesis that the norm N of the smoothing ideal is coprime to $\prod_\rho \tilde{t}_\rho$. To make use of the results of chapter II we must make sure that the smoothed G_r functions we compute are such that the corresponding smoothing lattice is *good* for the involved linear forms. Thus we define the notion of a *good* smoothing ideal $\mathfrak{a}$ following [Das08] as:

Definition III.26: *Let $\mathfrak{a}$ be a smoothing ideal for $\mathfrak{f}$ in the sense of Definition III.3. Fix a set of fundamental units $\varepsilon_1, \ldots, \varepsilon_r$ for $\mathcal{O}_\mathbb{K}^{+,\times}$ such that for each $\rho \in \mathfrak{S}_r$, the unit system $[\varepsilon_{\rho(1)}| \ldots |\varepsilon_{\rho(r)}]$ satisfies (H2). We say that $\mathfrak{a}$ is a good smoothing ideal (for the data $\mathfrak{f}, (\varepsilon_1, \ldots, \varepsilon_r)$) if the smoothing index $N = \mathcal{N}(\mathfrak{a})$ is coprime to*

$$\mathcal{N}(\mathfrak{f}). \prod_{\rho \in \mathfrak{S}_r} \tilde{t}_\rho$$

where $\tilde{t}_\rho$ is the overflow of the unit system $[\varepsilon_{\rho(1)}| \ldots |\varepsilon_{\rho(r)}]$ defined in (III.23).

In practice, when the smoothing ideal $\mathfrak{a}$ is *good* , it is guaranteed that for each $\rho \in \mathfrak{S}_r$ the minimal uniform positive dual family $\alpha_{\rho,0}, \ldots, \alpha_{\rho,r}$ to $a_\rho, a_\rho u_{\rho,1}, \ldots, a_\rho u_{\rho,r}$ in L is such that the $\alpha_{\rho,0}/N, \ldots, \alpha_{\rho,r}/N$ is the minimal uniform positive dual family to $a_\rho, a_\rho u_{\rho,1}, \ldots, a_\rho u_{\rho,r}$ in $\mathfrak{a}^{-1}L$. This corresponds to the *good* condition for the smoothing lattice introduced in chapter II.

III.3.5 Invariance under change of base points $(h_\rho)_\rho$

In this section, we study the invariance of our construction under a change of the base points forming the compatible set $\underline{h} = (h_\rho)_\rho$. Indeed, such invariance properties our key to proving that we compute meaningful objects and are useful to prove parts of the algebraicity conjectures we have in mind. From Definition III.24 we see that in order to change the set of base points $\underline{h}$ we may either change the helper ideal uniformly for each of the h_ρ's or change individually some of the m_ρ's. The first change impacts the overall level of the computations and is therefore particularly difficult to treat (see section III.4.2.6

for a discussion on the subject). In this section we discuss the second change possibility and show that an invariance property holds, as long as we perform an averaging process on some unit group.

III.3.5.1 Action of units and integers

In this section we analyse how the arithmetic evaluation $G^{\mu}_{r,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(u_1,\ldots,u_r;h)^{\nu}$ defined in (III.2) changes when we change the base point h chosen in Proposition III.25. More precisely, we discuss the case where h and h' are two strongly admissible base points satisfying

$$h\mathcal{O}_{\mathbb{K}} = m\frac{qN}{\mathfrak{ab}}\mathfrak{D}(\tilde{a},\mathcal{O}_{\mathbb{K}}).\mathfrak{H} = \frac{m}{m'}h'\mathcal{O}_{\mathbb{K}}.$$

In particular, this means that h' can replace h in any strongly compatible set of base points for $\varepsilon_1,\ldots,\varepsilon_r$. We will analyse the replacement of h by h' in three steps. First, we shall treat the case where $h' = dh$ for some positive integer $d \equiv 1 \bmod q$, then the case where $h' = \varepsilon h$ for some unit $\varepsilon \in \mathcal{O}_{\mathbb{K}}^{\times}$. In the next section, we treat the general case where $mh' = \xi m' h$ for some positive integers m, m' coprime to q and some unit $\xi \in \mathcal{O}_{\mathbb{K}}^{\times}$ such that $m \equiv \xi m' \bmod \mathfrak{f}$.

For the rest of this section it will be useful to recall some notations. Recall that $\lambda m^{r+1}a = \det_{B_L}(h, u_1h,\ldots,u_rh,\cdot)$ and that $\det_{B_\Lambda}(a, u_1a,\ldots,u_ra,\cdot) = s\gamma$ with γ primitive in L and $s>0$. We denote by $\alpha_0,\ldots,\alpha_r$ the uniform positive dual family defined in (III.13) and by $F = F(\underline{a},\underline{\alpha}, v=0)/\mathbb{Z}\gamma$ the set defined in Proposition I.7. We introduce the auxiliary function Ω defined on integers coprime to q by:

$$\Omega(k) = \prod_{\delta\in F}\frac{G_r\left(\frac{kh+q\delta}{q\gamma},\frac{\alpha_0}{\gamma},\ldots,\frac{\alpha_r}{\gamma}\right)^N}{G_r\left(\frac{N(kh+q\delta)}{q\gamma},\frac{N\alpha_0}{\gamma},\ldots,\frac{N\alpha_r}{\gamma}\right)}. \tag{III.35}$$

In particular, it is clear that by definition (see (III.2)):

$$G^{+}_{r,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(u_1,\ldots,u_r;h) = \Omega(1).$$

Let us now prove a simple lemma expressing the value $G^{\mu}_{r,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(u_1,\ldots,u_r;k.h)^{\nu}$ when k is a positive integer coprime to q in terms of the auxiliary function Ω.

Lemma III.27: *For general signs $\mu,\nu\in\{\pm 1\}$ and for any positive integer k coprime to q:*

$$G^{\mu}_{r,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(u_1,\ldots,u_r;k.h)^{\nu} = \Omega(\mu.k)^{\nu\mu^r}.$$

Proof :

We only need to treat the case where $\mu=-1$. This means that in the construction the linear form a is replaced with $-a$, thus the elements α_j are replaced by $-\alpha_j$, the set F is replaced by $-F$ and the vector γ is replaced by $(-1)^{r+1}\gamma$. Thus

$$G^{-}_{r,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(u_1,\ldots,u_r;h) = \prod_{\delta\in -F}\frac{G_r\left(\frac{k.h+q\delta}{(-1)^{r+1}q\gamma},\frac{-\alpha_0}{(-1)^{r+1}\gamma},\ldots,\frac{-\alpha_r}{(-1)^{r+1}\gamma}\right)^N}{G_r\left(\frac{N(k.h+q\delta)}{(-1)^{r+1}q\gamma},\frac{-N\alpha_0}{(-1)^{r+1}\gamma},\ldots,\frac{-N\alpha_r}{(-1)^{r+1}\gamma}\right)}.$$

Recalling property (I.11) which states that in general

$$G_r(-z, -\tau_0, \dots, -\tau_r) = G_r(z, \tau_0, \dots, \tau_r)^{-1}$$

we obtain

$$G^{-}_{r,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(u_1, \dots, u_r; h) = \prod_{\delta \in F} \frac{G_r\left(\frac{-k.h+q\delta}{q\gamma}, \frac{\alpha_0}{\gamma}, \dots, \frac{\alpha_r}{\gamma}\right)^{N.(-1)^r}}{G_r\left(\frac{N(-k.h+q\delta)}{q\gamma}, \frac{N\alpha_0}{\gamma}, \dots, \frac{N\alpha_r}{\gamma}\right)^{(-1)^r}}.$$

The identification of the right-hand side gives exactly:

$$G^{-}_{r,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(u_1, \dots, u_r; k.h) = \Omega(-k)^{(-1)^r}.$$

which is the desired result. □

Let us now move on to the analysis of the replacement of h by some other base point h'. We start with the following remark: if we change h to dh for some positive integer d such that $d \equiv 1 \mod q$, then the computation of (III.2) does not change:

Lemma III.28: *Let h be a weakly admissible base point for $\mathfrak{f}, \mathfrak{b}, \mathfrak{a}$. Fix a system $u_1, \dots, u_r$ of fundamental units satisfying (H2). Consider a positive integer d congruent to 1 mod q. Then*

$$G^{\mu}_{r,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(u_1, \dots, u_r; dh) = G^{\mu}_{r,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(u_1, \dots, u_r; h).$$

This is equivalent to the statement that Ω is q-periodic.

Proof :

Let us prove that Ω is q-periodic. Indeed, since $h = \eta m \gamma$ for some positive integer m coprime to q and $\eta = \pm 1$,

$$\Omega(k) = \prod_{\delta \in F} \frac{G_r\left(\frac{\eta.m.k}{q} + \frac{\delta}{\gamma}, \frac{\alpha_0}{\gamma}, \dots, \frac{\alpha_r}{\gamma}\right)^{N}}{G_r\left(\frac{N.\eta.m.k}{q} + \frac{N\delta}{\gamma}, \frac{N\alpha_0}{\gamma}, \dots, \frac{N\alpha_r}{\gamma}\right)}.$$

and since the G_r function is 1-periodic in its first argument, the function Ω is q-periodic. It follows that

$$G^{\mu}_{r,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(u_1, \dots, u_r; dh) = \Omega(\mu.d)^{\mu^r} = \Omega(\mu)^{\mu^r} = G^{\mu}_{r,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(u_1, \dots, u_r; h)$$

since $d \equiv 1 \mod q$. □

Let us now carry out the second step, where we analyse what happens when h is replaced by εh for some unit $\varepsilon \in \mathcal{O}_{\mathbb{K}}^{\times}$.

Lemma III.29: *Let h be a weakly admissible base point for $\mathfrak{f}, \mathfrak{b}, \mathfrak{a}$. Fix a system $u_1, \dots, u_r$ of fundamental units satisfying (H2). Then for all orientations μ, ν and for all unit $\varepsilon \in \mathcal{O}_{\mathbb{K}}^{\times}$:*

$$G^{\mu}_{r,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(u_1, \dots, u_r; \varepsilon h)^{\nu} = \Omega(\mu\mathcal{N}(\varepsilon))^{\nu\mu^r\mathcal{N}(\varepsilon)^{r+1}} = G^{\mu\mathcal{N}(\varepsilon)}_{r,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(u_1, \dots, u_r; h)^{\nu\mathcal{N}(\varepsilon)}.$$

Proof :

We analyse how the construction carried out to evaluate (III.2) changes when the base point h is replaced with εh. First, the linear forms a_h and $a_{\varepsilon h}$ are related by:

$$\begin{aligned}\lambda a_{\varepsilon h} &= \det_{B_L}(\varepsilon h, u_1\varepsilon h, \dots, u_r\varepsilon h, \cdot)\\ \lambda a_{\varepsilon h} &= \mathcal{N}(\varepsilon)\det_{B_L}(h, u_1 h, \dots, u_r h, \varepsilon^{-1}\cdot)\\ \lambda a_{\varepsilon h} &= \lambda\mathcal{N}(\varepsilon)a_h(\varepsilon^{-1}\cdot)\end{aligned}$$

This implies directly that for all $0 \leq j \leq r$ we have $\alpha_{\varepsilon h,j} = \mathcal{N}(\varepsilon)\varepsilon\alpha_{h,j}$ and

$$\begin{aligned}s\gamma_{\varepsilon h} &= \det_{B_\Lambda}(a_{\varepsilon h}, u_1 a_{\varepsilon h}, \dots, u_r a_{\varepsilon_h}, \cdot)\\ s\gamma_{\varepsilon h} &= \mathcal{N}(\varepsilon)^{r+1}\det_{B_\Lambda}(a_h(\varepsilon^{-1}\cdot), u_1 a_h(\varepsilon^{-1}\cdot), \dots, u_r a_h(\varepsilon^{-1}\cdot), \cdot)\\ s\gamma_{\varepsilon h} &= s\mathcal{N}(\varepsilon)^r\varepsilon\gamma_h\end{aligned}$$

The set F_h is replaced by $F_{\varepsilon h} = \mathcal{N}(\varepsilon)\varepsilon F_h$. Thus, using $\gamma = \gamma_h$, $\alpha_j = \alpha_{h,j}$, $F = F_h$ we get:

$$G^{\mu}_{r,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(u_1, \dots, u_r; \varepsilon h)^{\nu} = \prod_{\delta\in F}\frac{G_r\left(\frac{\mu\varepsilon h + q\mathcal{N}(\varepsilon)\varepsilon\delta}{q\mathcal{N}(\varepsilon)^r\varepsilon\gamma}, \frac{\mathcal{N}(\varepsilon)\varepsilon\alpha_0}{\mathcal{N}(\varepsilon)^r\varepsilon\gamma}, \dots, \frac{\mathcal{N}(\varepsilon)\varepsilon\alpha_r}{\mathcal{N}(\varepsilon)^r\varepsilon\gamma}\right)^{N.\nu\mu^r}}{G_r\left(\frac{N(\mu\varepsilon h + q\mathcal{N}(\varepsilon)\varepsilon\delta)}{q\mathcal{N}(\varepsilon)^r\varepsilon\gamma}, \frac{N\mathcal{N}(\varepsilon)\varepsilon\alpha_0}{\mathcal{N}(\varepsilon)^r\varepsilon\gamma}, \dots, \frac{N\mathcal{N}(\varepsilon)\varepsilon\alpha_r}{\mathcal{N}(\varepsilon)^r\varepsilon\gamma}\right)^{\nu\mu^r}}.$$

Using once again (I.11) we get:

$$G^{\mu}_{r,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(u_1, \dots, u_r; \varepsilon h)^{\nu} = \prod_{\delta\in F}\frac{G_r\left(\frac{\mu\mathcal{N}(\varepsilon)h + q\delta}{q\gamma}, \frac{\alpha_0}{\gamma}, \dots, \frac{\alpha_r}{\gamma}\right)^{N.\nu\mu^r\mathcal{N}(\varepsilon)^{r+1}}}{G_r\left(\frac{N(\mu\mathcal{N}(\varepsilon)h + q\delta)}{q\gamma}, \frac{N\alpha_0}{\gamma}, \dots, \frac{N\alpha_r}{\gamma}\right)^{\nu\mu^r\mathcal{N}(\varepsilon)^{r+1}}}.$$

From the definition of Ω we recognise this as:

$$G^{\mu}_{r,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(u_1, \dots, u_r; \varepsilon h)^{\nu} = \Omega(\mu\mathcal{N}(\varepsilon))^{\nu\mu^r\mathcal{N}(\varepsilon)^{r+1}}.$$

On the other hand, it follows from lemma III.27 that

$$G^{\mu\mathcal{N}(\varepsilon)}_{r,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(u_1, \dots, u_r; h)^{\nu\mathcal{N}(\varepsilon)} = \Omega(\mu\mathcal{N}(\varepsilon))^{\nu\mu^r\mathcal{N}(\varepsilon)^{r+1}}$$

which gives the desired result. □

Let us now prove a general result on the action of units and integers on the base point h:

Proposition III.30: *Consider an integer k coprime to q and ε a unit in $\mathcal{O}_{\mathbb{K}}^{\times}$. Then for any orientations μ, ν:*

$$G^{\mu}_{r,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(u_1, \dots, u_r; k.\varepsilon.h)^{\nu} = \Omega(\mu.\mathrm{sign}(k)^r.\mathcal{N}(\varepsilon).|k|)^{\nu\mu^r\mathcal{N}(\varepsilon)^{r+1}}.$$

Proof :

Let us rewrite the term $G^{\mu}_{r,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(u_1, \dots, u_r; k.\varepsilon.h)^{\nu}$ in terms of $|k|$ as:

$$G^{\mu}_{r,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(u_1, \dots, u_r; k.\varepsilon.h)^{\nu} = G^{\mu}_{r,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(u_1, \dots, u_r; |k|.(\mathrm{sign}(k)\varepsilon).h)^{\nu}.$$

Then it follows from lemma III.29 that

$$G^{\mu}_{r,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(u_1,\ldots,u_r;k.\varepsilon.h)^{\nu} = G^{\mu\mathcal{N}(\mathrm{sign}(k).\varepsilon)}_{r,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(u_1,\ldots,u_r;|k|.h)^{\nu\mathcal{N}(\mathrm{sign}(k).\varepsilon)}.$$

and from lemma III.27 that

$$G^{\mu}_{r,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(u_1,\ldots,u_r;k.\varepsilon.h)^{\nu} = \Omega(\mu.\mathcal{N}(\mathrm{sign}(k).\varepsilon).|k|)^{\nu\mu^r\mathcal{N}(\mathrm{sign}(k).\varepsilon)^{r+1}}.$$

Since $\mathcal{N}(\mathrm{sign}(k)) = \mathrm{sign}(k)^{r+2} = \mathrm{sign}(k)^r$ we get the desired result:

$$G^{\mu}_{r,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(u_1,\ldots,u_r;k.\varepsilon.h)^{\nu} = \Omega(\mu.\mathrm{sign}(k)^r.\mathcal{N}(\varepsilon).|k|)^{\nu\mu^r\mathcal{N}(\varepsilon)^{r+1}}.$$

□

III.3.5.2 The unit group $\mathcal{Z}^1_{\mathfrak{f}}$

In this section, we analyse the general situation where h, h' are two possible base points associated to the unit system $[\varepsilon_{\rho(1)}|\ldots|\varepsilon_{\rho(r)}]$ in Proposition III.25, for the same choice of helper ideal $\mathfrak{H}$. This means that there is a fractional ideal J of $\mathbb{K}$ such that $(h) = mJ$ and $(h') = m'J$ for some integers $m, m' \in \mathbb{Z}_{>0}$ coprime to q. The ideal mJ is then generated by mh'/m' so there exists a unit $\xi \in \mathcal{O}^{\times}_{\mathbb{K}}$ such that $h = m\xi h'/m'$. Moreover, because $h, h' \equiv 1 \bmod \mathfrak{f}$ we get $m\xi/m' \equiv 1 \bmod \mathfrak{f}$. This leads us to define the following unit group:

Definition III.31:

$$\mathcal{Z}^1_{\mathfrak{f}} = \{(k,\varepsilon) \in \mathbb{Z}/q\mathbb{Z}^{\times} \times \mathcal{O}^{\times}_{\mathbb{K}}/\mathcal{O}^{\times}_{\mathfrak{f}} \mid k\varepsilon \equiv 1 \bmod \mathfrak{f}\}.$$

It is easy to see that $\mathcal{Z}^1_{\mathfrak{f}}$ is a finite group which contains at least $(1,1)$ and $(q-1,-1)$ and that these two elements differ unless $\mathbb{K}$ is of even degree and $q = 2$. In addition, since by assumption (H1) there is no unit of negative norm congruent to one modulo $\mathfrak{f}$, the set $\mathcal{Z}^1_{\mathfrak{f}}$ is naturally embedded in $(\mathcal{O}_{\mathbb{K}}/\mathfrak{f})^{\times}$ by the classic class field theory long exact sequence:

$$1 \to \mathcal{O}^{\times}_{\mathfrak{f}} \to \mathcal{O}^{\times}_{\mathbb{K}} \to (\mathcal{O}_{\mathbb{K}}/\mathfrak{f})^{\times} \to \mathrm{Cl}^+(\mathfrak{f}) \to \mathrm{Cl}(\mathbb{K}) \to 1.$$

In the case where $\mathcal{N}(f) = q$ (see hypothesis (H5)) we get the simpler isomorphism $\mathcal{Z}^1_{\mathfrak{f}} \simeq \mathcal{O}^{\times}_{\mathbb{K}}/\mathcal{O}^{\times}_{\mathfrak{f}}$. Lastly, notice that the unit -1 gives an involution of $\mathcal{Z}^1_{\mathfrak{f}}$ given by $(k,\varepsilon) \to (q-k,-\varepsilon)$.

Under assumption (H1) there are no units in $\mathcal{O}_{\mathbb{K}}$ of negative norm which are congruent to 1 modulo $\mathfrak{f}$, therefore we may define the character $\chi_{\mathfrak{f}}$ on $\mathcal{Z}^1_{\mathfrak{f}}$ by:

$$\chi_{\mathfrak{f}}(k) = \chi_{\mathfrak{f}}(k,\varepsilon) = \mathcal{N}(\varepsilon) \in \{\pm 1\}. \tag{III.36}$$

Indeed, if any two units $\varepsilon, \varepsilon'$ satisfy $k\varepsilon \equiv k\varepsilon' \equiv 1 \bmod \mathfrak{f}$ then $\varepsilon\varepsilon'^{-1} \equiv 1 \bmod \mathfrak{f}$ and therefore $\mathcal{N}(\varepsilon\varepsilon'^{-1}) = 1$. The value of $\chi_{\mathfrak{f}}(k) = \mathcal{N}(\varepsilon) = \mathcal{N}(\varepsilon')$ is then well-defined. It follows from assumption (H1) and lemma III.29 that the product

$$\prod_{(k,\varepsilon)\in\mathcal{Z}^1_{\mathfrak{f}}} G^{\mu}_{r,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(u_1,\ldots,u_r;k\varepsilon h)^{\nu} = \prod_{(k,\varepsilon)\in\mathcal{Z}^1_{\mathfrak{f}}} G^{\mu\chi_{\mathfrak{f}}(k)}_{r,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(u_1,\ldots,u_r;kh)^{\nu\chi_{\mathfrak{f}}(k)} \tag{III.37}$$

is well-defined for any orientations μ,ν where it is understood that $(k,\varepsilon)\in\mathcal{Z}_{\mathfrak{f}}^1$ always comes with $k>0$. We are now ready to prove that the averaging process over the unit group $\mathcal{Z}_{\mathfrak{f}}^1$ given by (III.37) is invariant when replacing the base point h by h' in a strongly compatible set of base points.

Proposition III.32: *Assume (H1). Consider two base points h,h' associated to the unit system $(u_1,\ldots,u_r)=[\varepsilon_{\rho(1)}|\ldots,\varepsilon_{\rho(r)}]$ in Proposition III.25 with the same helper ideal $\mathfrak{H}$. Denote by $m,m'>0$ the integers such that h/m and h'/m' are primitive in L and denote by ξ the unit in $\mathcal{O}_{\mathbb{K}}^{\times}$ satisfying $m'h=\xi mh'$. Then, for any orientations μ,ν:*

(i) If $m''>0$ is an integer satisying $m'm''\equiv 1 \bmod q$ then $(mm'',\xi)\in\mathcal{Z}_{\mathfrak{f}}^1$ and:

$$G^{\mu}_{r,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(u_1,\ldots,u_r;h)^{\nu}=G^{\mu}_{r,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(u_1,\ldots,u_r;mm''\xi h')^{\nu}.$$

(ii) The following equality holds by rearrangement of the terms:

$$\prod_{(k,\varepsilon)\in\mathcal{Z}_{\mathfrak{f}}^1}G^{\mu}_{r,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(u_1,\ldots,u_r;k\varepsilon h)^{\nu}=\prod_{(k,\varepsilon)\in\mathcal{Z}_{\mathfrak{f}}^1}G^{\mu}_{r,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(u_1,\ldots,u_r;k\varepsilon h')^{\nu}.$$

Proof :

(i) Write once again $h=m\xi h'/m'$ where $m,m'>0$ are coprime to q and $m\xi\equiv m' \bmod \mathfrak{f}$. This gives:

$$G^{\mu}_{r,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(u_1,\ldots,u_r;h)^{\nu}=G^{\mu}_{r,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(u_1,\ldots,u_r;m\xi h'/m')^{\nu}.$$

Consider a positive integer m'' such that $m'm''\equiv 1 \bmod q$. Then $mm''\xi\equiv m'm''\equiv 1 \bmod \mathfrak{f}$, so $(mm'',\xi)\in\mathcal{Z}_{\mathfrak{f}}^1$ and it follows from lemma III.28 that

$$G^{\mu}_{r,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(u_1,\ldots,u_r;m\xi h'/m')^{\nu}=G^{\mu}_{r,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(u_1,\ldots,u_r;mm''\xi h')^{\nu}.$$

This proves the desired equality:

$$G^{\mu}_{r,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(u_1,\ldots,u_r;h)^{\nu}=G^{\mu}_{r,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(u_1,\ldots,u_r;mm''\xi h')^{\nu}.$$

(ii) Write once again $h=m\xi h'/m'$ with $m\xi\equiv m' \bmod \mathfrak{f}$ and fix a positive integer m'' such that $mm''\equiv 1 \bmod q$. Then for all $(k,\varepsilon)\in\mathcal{Z}_{\mathfrak{f}}^1$, we have $k\varepsilon h=(mk)(\xi\varepsilon)h'/m'$ and (i) implies that:

$$G^{\mu}_{r,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(u_1,\ldots,u_r;k\varepsilon h)^{\nu}=G^{\mu}_{r,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(u_1,\ldots,u_r;(mm''k)(\xi\varepsilon)h')^{\nu}.$$

Using this equality on the product over $\mathcal{Z}_{\mathfrak{f}}^1$ gives:

$$\prod_{(k,\varepsilon)\in\mathcal{Z}_{\mathfrak{f}}^1}G^{\mu}_{r,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(u_1,\ldots,u_r;k\varepsilon h)^{\nu}=\prod_{(k,\varepsilon)\in\mathcal{Z}_{\mathfrak{f}}^1}G^{\mu}_{r,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(u_1,\ldots,u_r;(mm''k)(\xi\varepsilon)h')^{\nu}$$

with $(mm''k)(\xi\varepsilon)\equiv mm''\xi\equiv m'm''\equiv 1 \bmod \mathfrak{f}$. Multiplication by (mm'',ξ) thus induces a bijection of $\mathcal{Z}_{\mathfrak{f}}^1$ and if we set $(k',\varepsilon')=(mm''k,\xi\varepsilon)$ then we get:

$$\prod_{(k,\varepsilon)\in\mathcal{Z}_{\mathfrak{f}}^1}G^{\mu}_{r,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(u_1,\ldots,u_r;k\varepsilon h)^{\nu}=\prod_{(k',\varepsilon')\in\mathcal{Z}_{\mathfrak{f}}^1}G^{\mu}_{r,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(u_1,\ldots,u_r;k'\varepsilon' h')^{\nu}$$

which is the desired result. □

We have successfully proven that this averaging process on the group $\mathcal{Z}_{\mathfrak{f}}^1$ gives some invariance property under the choice of base point h in Proposition III.25, essentially because changing the base point h by a positive rational number switches the order of the terms in the product (III.37). There is still work to be done concerning the independence of our global computation (see the Main Conjecture III.37) on the choice of the base point h and more specifically concerning the invariance by simultaneous change of the helper ideal in a compatible set of base points. This shall be discussed in section III.4.2.6.

We end this section by giving a concrete expression for the product (III.37) which is useful for computations. We first make a remark on the link between the character $\chi_{\mathfrak{f}}$ and the norm-type characters on the narrow ray class group at $\mathfrak{f}$. Indeed, if χ is a norm-type character on $\mathrm{Cl}^+(\mathfrak{f})$ then there exists a character χ_{finite} on $(\mathcal{O}_{\mathbb{K}}/\mathfrak{f})^\times$ such that for all $y \in \mathbb{K}$ coprime to $\mathfrak{f}$: $\chi(y\mathcal{O}_{\mathbb{K}}) = \chi_{\text{finite}}(y \bmod \mathfrak{f}).\mathrm{sign}(\mathcal{N}(y))$. In particular, if $(k, \varepsilon) \in \mathcal{Z}_{\mathfrak{f}}^1$, this equality gives:

$$\chi(k\mathcal{O}_{\mathbb{K}}) = \chi(k\varepsilon\mathcal{O}_{\mathbb{K}}) = \chi_{\text{finite}}(k\varepsilon \bmod \mathfrak{f})\mathcal{N}(\varepsilon) = \mathcal{N}(\varepsilon) = \chi_{\mathfrak{f}}(k)$$

as $k\varepsilon \equiv 1 \bmod \mathfrak{f}$. Therefore the character $\chi_{\mathfrak{f}}$ introduced in (III.36) is the common value of all norm-type characters on $\mathrm{Cl}^+(\mathfrak{f})$ for the ideal $k\mathcal{O}_{\mathbb{K}}$. This is especially interesting as the functional equation of Hecke L-functions in ATR fields implies that if $L'(0, \chi) \neq 0$ for a character χ of the narrow ray class group $\mathrm{Cl}^+(\mathfrak{f})$ then χ is of norm-type (see for instance [Neu99] for the functional equation). It is worth noting that $\chi_{\mathfrak{f}}(1) = 1$ and $\chi_{\mathfrak{f}}(q-1) = (-1)^n$ where n is the degree of $\mathbb{K}$. We use this character to write using lemma III.29:

$$\prod_{(k,\varepsilon)\in\mathcal{Z}_{\mathfrak{f}}^1} G^{\mu}_{r,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(u_1, \ldots, u_r; k\varepsilon h)^{\nu} = \prod_{(k,\varepsilon)\in\mathcal{Z}_{\mathfrak{f}}^1} G^{\mu\chi_{\mathfrak{f}}(k)}_{r,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(u_1, \ldots, u_r; kh)^{\chi_{\mathfrak{f}}(k)\nu} \tag{III.38}$$

The right-hand side of formula (III.38) may be explicitly written as

$$\prod_{(k,\varepsilon)\in\mathcal{Z}_{\mathfrak{f}}^1} G^{\mu\chi_{\mathfrak{f}}(k)}_{r,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(u_1, \ldots, u_r; kh)^{\chi_{\mathfrak{f}}(k)\nu} =$$

$$\prod_{(k,\varepsilon)\in\mathcal{Z}_{\mathfrak{f}}^1} \prod_{\delta\in F} \left(\frac{G_r\left(\frac{\mu\chi_{\mathfrak{f}}(k)kh+q\delta}{q\gamma}, \frac{\alpha_0}{\gamma}, \ldots, \frac{\alpha_r}{\gamma}\right)^N}{G_r\left(\frac{\mu\chi_{\mathfrak{f}}(k)kNh+Nq\delta}{q\gamma}, \frac{N\alpha_0}{\gamma}, \ldots, \frac{N\alpha_r}{\gamma}\right)} \right)^{\nu\mu^r\chi_{\mathfrak{f}}(k)^{r+1}} \tag{III.39}$$

where $\underline{\alpha}, \gamma, M$ are defined in section III.3 and in Proposition I.7. Most of the computation of this double product can be done at once for all terms since the parameters $\alpha_0/\gamma, \ldots, \alpha_r/\gamma$ are fixed and the imaginary parts of $\mu\chi_{\mathfrak{f}}(k)kh/q\gamma$ for $(k, \varepsilon) \in \mathcal{Z}_{\mathfrak{f}}^1$ are all equal. Remember that in the context of the hypotheses of Proposition III.25, this double product reduces to a simple product as $F = \{0\}$ and we get:

$$\prod_{(k,\varepsilon)\in\mathcal{Z}_{\mathfrak{f}}^1} G^{\mu\chi_{\mathfrak{f}}(k)}_{r,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(u_1, \ldots, u_r; kh)^{\chi_{\mathfrak{f}}(k)\nu} =$$

$$\prod_{(k,\varepsilon)\in\mathcal{Z}_{\mathfrak{f}}^1} \left(\frac{G_r\left(\frac{\mu\chi_{\mathfrak{f}}(k)kh}{q\gamma}, \frac{\alpha_0}{\gamma}, \ldots, \frac{\alpha_r}{\gamma}\right)^N}{G_r\left(\frac{\mu\chi_{\mathfrak{f}}(k)kNh}{q\gamma}, \frac{N\alpha_0}{\gamma}, \ldots, \frac{N\alpha_r}{\gamma}\right)} \right)^{\nu\mu^r\chi_{\mathfrak{f}}(k)^{r+1}}. \tag{III.40}$$

This last formulation is the one we use most commonly to compute the higher elliptic units.

III.3.5.3 The unit group $\mathcal{Z}_{\mathfrak{f}}^{1,+}$

As seen in the previous section, the process of averaging on the finite group $\mathcal{Z}_{\mathfrak{f}}^{1}$ is essential to producing higher elliptic units which don't depend on the particular choice of base points h. In view of the discussion carried out at the end of section III.3.1 the averaging process over $\mathcal{Z}_{\mathfrak{f}}^{1}$ should be replaced in general by an averaging process over $\mathcal{O}_{\mathbb{K}}^{+,\times}/\mathcal{O}_{\mathfrak{f}}^{+,\times}$. This is also apparent in the computations as the averaging process over $\mathcal{Z}_{\mathfrak{f}}^{1}$ tends to count values multiple times. Thus we define the smaller group:

Definition III.33:

$$\mathcal{Z}_{\mathfrak{f}}^{1,+} = \{(k,\varepsilon) \in \mathbb{Z}/q\mathbb{Z}^{\times} \times \mathcal{O}_{\mathbb{K}}^{+,\times}/\mathcal{O}_{\mathfrak{f}}^{+,\times} \mid k\varepsilon \equiv 1 \bmod \mathfrak{f}\}.$$

Under hypothesis (H5) this is isomorphic to $\mathcal{O}_{\mathbb{K}}^{+,\times}/\mathcal{O}_{\mathfrak{f}}^{+,\times}$. Note that this group may be embedded inside $\mathcal{Z}_{\mathfrak{f}}^{1}$ as the class of (k,ε) in $\mathcal{Z}_{\mathfrak{f}}^{1,+}$ may be sent to the class of (k,ε) in $\mathcal{Z}_{\mathfrak{f}}^{1}$. In particular the set $\mathcal{Z}_{\mathfrak{f}}^{1,+}$ acts on $\mathcal{Z}_{\mathfrak{f}}^{1}$ by multiplication. Let us denote by $Z_1,\dots,Z_g$ the orbits of $\mathcal{Z}_{\mathfrak{f}}^{1}$ under this action. Then, we conjecture the following:

Conjecture III.34: *Let $Z_1,\dots,Z_g$ be the orbits in $\mathcal{Z}_{\mathfrak{f}}^{1}$ under the action of $\mathcal{Z}_{\mathfrak{f}}^{1,+}$, where g is the index of $\mathcal{Z}_{\mathfrak{f}}^{1,+}$ in $\mathcal{Z}_{\mathfrak{f}}^{1}$. Assume that the set $\underline{h} = (h_\rho)_\rho$ of strongly compatible base points is such that the class of h_ρ/q in $\mathrm{Cl}^{+}(\mathfrak{f})$ is independent of $\rho \in \mathfrak{S}_r$. Then for any $1 \le j \le g$, if $\mathcal{W}_j$ is the complex number defined by:*

$$\mathcal{W}_j = \prod_{\rho\in\mathfrak{S}_r}\prod_{(k,\varepsilon)\in Z_j} G^{\mu}_{r,\mathfrak{f},\mathfrak{b},\mathfrak{a}}([\varepsilon_{\rho(1)}|\dots|\varepsilon_{\rho(r)}];k\varepsilon h_\rho)^{\nu}$$

then $\mathcal{W}_1 = \dots = \mathcal{W}_g$ and

$$\prod_{\rho\in\mathfrak{S}_r}\prod_{(k,\varepsilon)\in\mathcal{Z}_{\mathfrak{f}}^{1}} G^{\mu}_{r,\mathfrak{f},\mathfrak{b},\mathfrak{a}}([\varepsilon_{\rho(1)}|\dots|\varepsilon_{\rho(r)}];k\varepsilon h_\rho)^{\nu} = \left(\prod_{\rho\in\mathfrak{S}_r}\prod_{(k,\varepsilon)\in\mathcal{Z}_{\mathfrak{f}}^{1,+}} G^{\mu}_{r,\mathfrak{f},\mathfrak{b},\mathfrak{a}}([\varepsilon_{\rho(1)}|\dots|\varepsilon_{\rho(r)}];k\varepsilon h_\rho)^{\nu}\right)^{g}.$$

This conjecture is out of our reach for the moment, but it is supported by numerical evidence and it should follow from standard arguments in the study of the cohomology of arithmetic groups. Of course, being able to replace the averaging over $\mathcal{Z}_{\mathfrak{f}}^{1}$ by an averaging over $\mathcal{Z}_{\mathfrak{f}}^{1,+}$ makes for a more elegant conjecture, but unfortunately it is more difficult to prove the invariance from the choice of base points h_ρ on this writing. Indeed, the values $\mathcal{W}_j$ are expected to be equal for $1 \le j \le g$ but it does not always happen as a result of some reorganisation of the terms (see section IV.2.5.1 for a more detailed discussion on the subject).

We also note that to state Conjecture III.34 we had to make the assumption that a strongly compatible set $\underline{h} = (h_\rho)_\rho$ of base points exists for which the class of h_ρ/q in $\mathrm{Cl}^{+}(\mathfrak{f})$ is independent of $\rho \in \mathfrak{S}_r$. It is not clear that such a set always exists, and we would need to prove the following conjecture on the classes of the involved generalised different ideals.

Conjecture III.35: *Let $\mathbb{K}$ be an ATR field of degree $n = r+2 \ge 3$. Let $\varepsilon_1,\dots,\varepsilon_r$ be fundamental units for $\mathcal{O}_{\mathbb{K}}^{+,\times}$ such that for each $\rho \in \mathfrak{S}_r$ the unit system $[\varepsilon_{\rho(1)}|\dots|\varepsilon_{\rho(r)}]$*

satisfies (H2), (H3). Let $\mathfrak{D}_\rho = \mathfrak{D}(\tilde{a}_\rho, \mathcal{O}_\mathbb{K})$ be the generalised different ideal of $\tilde{a}_\rho = \tilde{a}([\varepsilon_{\rho(1)}|\ldots|\varepsilon_{\rho(r)}])$. Then the class of $\mathfrak{D}_\rho$ in the narrow Hilbert class group of $\mathbb{K}$ is independent of the permutation $\rho \in \mathfrak{S}_r$.

This conjecture might be proven using similar ideas to those used by Hecke to prove that the class of the usual different ideal $\mathfrak{d}$ is a square in the wide class group of $\mathbb{K}$.

III.4 The main conjecture

In this section we present a conjecture on the special values of G_r functions evaluated at points in a degree $n = r+2$ ATR field which generalises Conjecture III.1. We shall then discuss some links with Hilbert's 12th problem and the rank one abelian Stark conjecture for ATR fields.

III.4.1 Formulation of the conjecture

In this section we finally give a precise formulation for our main conjecture on the special values of multiple elliptic Gamma functions over ATR fields. To this end we gather the hypotheses (H2), (H3) and (H4) we introduced along the way in section III.3.2 and introduce the following notion of a set of fundamental units which is *adapted* to $\mathfrak{f}$.

Definition III.36: *Let $\varepsilon_1, \ldots, \varepsilon_r$ be fundamental units for the group $\mathcal{O}_\mathbb{K}^{+,\times}$ of totally positive units in $\mathbb{K}$. We say that the fundamental units $\varepsilon_1, \ldots, \varepsilon_r$ are adapted to $\mathfrak{f}$ if for each $\rho \in \mathfrak{S}_r$, the unit system $[\varepsilon_{\rho(1)}|\ldots|\varepsilon_{\rho(r)}]$ satisfies (H2), (H3) and the different ideal $\mathfrak{D}(\tilde{a}, \mathcal{O}_\mathbb{K})$ associated to the unit system $[\varepsilon_{\rho(1)}|\ldots|\varepsilon_{\rho(r)}]$ is coprime to $\mathcal{N}(\mathfrak{f})$ (see (H4)).*

Note that in this definition, the only hypothesis which depends on $\mathfrak{f}$ is (H4) and that for a fixed set of fundamental units $\varepsilon_1, \ldots, \varepsilon_r$ a positive proportion of all integral ideals in $\mathcal{O}_\mathbb{K}$ satisfy (H4). Thus, once we have found a set of fundamental units $\varepsilon_1, \ldots, \varepsilon_r$ which are *adapted* to some integral ideal $\mathfrak{f}$, there are many other integral ideals for which the same set of fundamental units is *adapted* . We may now present our main conjecture:

Main Conjecture III.37: *Suppose that $\mathbb{K}$ is an ATR field of degree $n = r+2 \geq 3$. Fix an integral ideal $\mathfrak{f} \neq \mathcal{O}_\mathbb{K}$ satisfying (H1) and (H5) such that $\mathbb{K}^+(\mathfrak{f})$ is totally complex. Let $\sigma_\mathbb{C}$ be one of the two complex embeddings of $\mathbb{K}$ and fix a complex embedding σ of $\mathbb{K}^+(\mathfrak{f})$ which extends $\sigma_\mathbb{C}$. Fix a set of fundamental units $\varepsilon_1, \ldots, \varepsilon_r$ for $\mathcal{O}_\mathbb{K}^{+,\times}$ which are adapted to $\mathfrak{f}$. Fix a smoothing ideal $\mathfrak{a}$ of norm N which is good in the sense of Definition III.26. Fix a class $\mathfrak{c}$ in the narrow ray class group at $\mathfrak{f}$. Fix an integer $k_0 > 0$ coprime to $q = \mathcal{N}(\mathfrak{f})$ and an integral ideal $\mathfrak{b}$ coprime to $\mathfrak{f}$ such that the integral ideal $k_0.\mathfrak{b}$ belongs to the class $\mathfrak{c}$. Let $\underline{h}$ be a compatible set of strongly admissible base points for the data $\mathfrak{f}, \mathfrak{b}, \mathfrak{a}$ and for the fundamental units $\varepsilon_1, \ldots, \varepsilon_r$, as given by Proposition III.25. Then, there are orientation signs $\mu_\rho = \mu_\rho(\varepsilon_1, \ldots, \varepsilon_r, \underline{h}), \nu_\rho = \nu_\rho(\varepsilon_1, \ldots, \varepsilon_r, \underline{h}) \in \{-1, +1\}$ such that the complex number*

$$\mathcal{V}_{\mathfrak{f},k_0,\mathfrak{b},\mathfrak{a}}(\varepsilon_1, \ldots, \varepsilon_r) = \prod_{(k,\varepsilon)\in\mathcal{Z}_\mathfrak{f}^1} I_{r,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(\varepsilon_1, \ldots, \varepsilon_r; k_0.k.\varepsilon.\underline{h}, \underline{\mu}, \underline{\nu})^{\mathcal{D}(N,n)} \qquad \text{(III.41)}$$

is independent of the choice for the compatible set of strongly admissible base points $\underline{h}$ in Proposition III.25 (see section III.3.5.2 for the definition of the set $\mathcal{Z}_\mathfrak{f}^1$ and (III.3) for the definition of the evaluation $I_{r,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(\varepsilon_1, \ldots, \varepsilon_r; k_0.n.\varepsilon.\underline{h}, \underline{\mu}, \underline{\nu})$). Furthermore:

1. *The complex number $\mathcal{V}_{\mathfrak{f},\mathfrak{c},\mathfrak{a}} = \mathcal{V}_{\mathfrak{f},k_0,\mathfrak{b},\mathfrak{a}}(\varepsilon_1, \ldots, \varepsilon_r)$ is independent of the choice of fundamental units $\varepsilon_1, \ldots, \varepsilon_r$ adapted to $\mathfrak{f}$ and from the representation of the class $\mathfrak{c}$ by the integral ideal $k_0.\mathfrak{b}$.*

2. *The complex number $\mathcal{V}_{\mathfrak{f},\mathfrak{c},\mathfrak{a}}$ is the image in $\mathbb{C}$ of an algebraic unit $u_{\mathfrak{f},\mathfrak{c},\mathfrak{a}}$ in the class field $\mathbb{K}^+(\mathfrak{f})$ under the complex embedding σ of $\mathbb{K}^+(\mathfrak{f})$.*

3. *Any embedding of $\mathbb{K}^+(\mathfrak{f})$ above a real embedding of $\mathbb{K}$ maps $u_{\mathfrak{f},\mathfrak{c},\mathfrak{a}}$ to the unit circle.*

4. *If $\mathfrak{c}'$ is a class in the narrow ray class group at $\mathfrak{f}$ then the explicit reciprocity law is given by*
$$\sigma_{\mathfrak{c}'}(u_{\mathfrak{f},\mathfrak{c},\mathfrak{a}}) = u_{\mathfrak{f},\mathfrak{c}\mathfrak{c}',\mathfrak{a}}$$
where $\mathfrak{c} \to \sigma_{\mathfrak{c}}$ is the Artin map.

5. *The complex number $\mathcal{V}_{\mathfrak{f},\mathfrak{c},\mathfrak{a}}$ satisfies the following Kronecker limit formula:*
$$\mathcal{N}(\mathfrak{a})\zeta_{\mathfrak{f}}'(\mathfrak{c}, 0) - \zeta_{\mathfrak{f}}'(\mathfrak{a}\mathfrak{c}, 0) = \frac{\#\mathcal{Z}_{\mathfrak{f}}^{1,+}}{\mathcal{D}(N, n)\#\mathcal{Z}_{\mathfrak{f}}^{1}} \log |\mathcal{V}_{\mathfrak{f},\mathfrak{c},\mathfrak{a}}|^2 . \qquad \text{(III.42)}$$

Remarks :

1. In the case $n = 2$, that is for imaginary quadratic fields, it is already known (see chapter 0) that the values
$$\theta^{\pm}_{\mathfrak{f},\mathfrak{b},\mathfrak{a}}(h)^{\mathcal{D}(N,2)}$$
are q-units in abelian extensions of imaginary quadratic fields as a consequence of the theory of Complex Multiplication and that they satisfy the second limit formula due to Kronecker (see chapter 0).

2. In the case $n = 3$, the Kronecker limit formula is already proven in [BCG23] whereas the algebraicity property remains conjectural.

3. The orientations in the conjecture can be computed by checking which ones among the $2.2^{r!}$ possible choices give the desired Kronecker limit formula (III.42).

4. The condition (H1) which asserts that there are no units of negative norm in $\mathcal{O}_{\mathbb{K}}^{\times}$ which are congruent to 1 mod $\mathfrak{f}$ is not very restrictive. Indeed, if such a unit exists, none of the characters associated with the abelian extension $\mathbb{K}^+(\mathfrak{f})/\mathbb{K}$ may be of norm type (see section III.3.5.2 for a discussion of norm-type characters) and therefore all the Dirichlet L-functions attached to the abelian extension $\mathbb{K}^+(\mathfrak{f})/\mathbb{K}$ (and therefore all the associated partial zeta functions) vanish at $s = 0$ with order ≥ 2 (see for instance [Neu99] for the functional equation of L-functions). The left-hand side of (III.42) is then identically 0 and with a bit more work it is possible to show that the right-hand side vanishes too.

5. This conjecture is formulated with restrictions, but it has been successfully tested in many other cases outside of the hypotheses (H3), (H4) and (H5). It is our aim to complete the formulation of this conjecture outside of these restrictions and to extend the definition of compatible sets of admissible base points $\underline{h}$ for which the conjecture holds. In particular, it is expected that the conjecture may be extend to any set of fundamental units $\varepsilon_1, \ldots, \varepsilon_r$ for $\mathcal{O}_{\mathbb{K}}^{+,\times}$ and that the value of $\mathcal{V}_{\mathfrak{f},\mathfrak{c},\mathfrak{a}}$ should indeed remain independent of this choice of fundamental units.

6. As the value of $\mathcal{D}(N,n)$ is identically 1 if N is supported at primes $p > n+1$ we make the difference in the computations between "small smoothings" (which are divisible by some small prime $p \leq n+1$) and "big smoothings" which are not. The interest of big smoothings is that the minimal polynomial of the value $\mathcal{V}_{\mathfrak{f},\mathfrak{c},\mathfrak{a}}$ should have smaller coefficients and it should therefore be more easily recognizable. The tradeoff however is that the computation time is linear in the smoothing index and therefore computations with "big smoothings" may take more time.

Let us now illustrate the conjecture for a simple quartic ATR field using a small smoothing. Consider the quartic field $\mathbb{K} = \mathbb{Q}(z)$ where z is the root of the polynomial $x^4 - 6x^3 - x^2 - 3x + 1$ in the upper half-plane and fix the ordering on the real embeddings of $\mathbb{K}$ such that $\sigma_1(z) < \sigma_2(z)$. Take $\mathfrak{f}$ the unique degree one prime above $q = 2$ in $\mathbb{K}$. A possible choice for the fundamental units is given by $\varepsilon_1 = (-2z^3 + 13z^2 - z + 3)/7,\ \varepsilon_2 = (-5z^3 + 29z^2 + 15z + 18)/7$. We choose $\mathfrak{a}$ the degree one prime above $N = 5$ in $\mathbb{K}$, $\mathfrak{c}$ the trivial class of the narrow ray class group at $\mathfrak{f}$ represented by $k_0 = 1$ and $\mathfrak{b} = \mathcal{O}_{\mathbb{K}}$. Here $q = 2$ therefore $\mathcal{Z}^1_{\mathfrak{f}} = \{(1,1)\} = \mathcal{Z}^{1,+}_{\mathfrak{f}}$. A possible choice of compatible base points h_1 and h_2 as given in Proposition III.25 is $h_1 = h_2 = (-18z^3 + 96z^2 + 82z + 62)/7$. We may compute for the orientations $\underline{\mu} = \underline{\nu} = [-1,1]$ the two quotients

$$v_1 = \frac{G_2\left(\frac{1}{2}, \frac{5z^3-29z^2-15z-81}{70}, \frac{6z^3-39z^2+10z+5}{70}, \frac{-2z^3+13z^2-z+24}{70}\right)^{-5}}{G_2\left(\frac{5}{2}, \frac{5z^3-29z^2-15z-81}{14}, \frac{6z^3-39z^2+10z+5}{14}, \frac{-2z^3+13z^2-z-24}{14}\right)^{-1}},$$

$$v_2 = \frac{G_2\left(\frac{-1}{2}, \frac{-2z^3+13z^2-z+24}{70}, \frac{-5z^3+29z^2+15z+81}{70}, \frac{2z^3-13z^2-6z-101}{70}\right)^{5}}{G_2\left(\frac{-5}{2}, \frac{-2z^3+13z^2-z+24}{14}, \frac{-5z^3+29z^2+15z+81}{14}, \frac{2z^3-13z^2-6z-101}{14}\right)}$$

The values obtained are respectively $v_1 \approx -2.0167576... - i \cdot 5.8008598...$ and $v_2 \approx -0.4159958... + i \cdot 0.0018434...$ Their product $v_1 v_2 \approx 0.8496565... - i \cdot 2.4094157...$ is not an algebraic integer in $\mathbb{K}^+(\mathfrak{f})$. Yet the *fifth* power $(v_1 v_2)^5 \approx 108.0070738... - i \cdot 13.4979021...$ of this product coincides up to at least 1000 digits with a root of the polynomial

$$x^8 - 215x^7 + 11629x^6 + 11941x^5 + 3913x^4 + 11941x^3 + 11629x^2 - 215x + 1$$

which defines an absolute equation of $\mathbb{K}^+(\mathfrak{f})$ over $\mathbb{Q}$. The constant term of this polynomial is 1, so its roots are units inside $\mathbb{K}^+(\mathfrak{f})$. We may also check the Kronecker limit formula (III.42) up to 1000 digits as:

$$\mathcal{N}(\mathfrak{a})\zeta'_{\mathfrak{f}}([\mathfrak{b}], 0) - \zeta'_{\mathfrak{f}}([\mathfrak{ab}], 0) \approx \log |v_1 v_2|^2 \approx 1.8759781...$$

We will present an example in the same conditions except for the smoothing which will be a "big smoothing" in section IV.2.2.1.

III.4.2 Discussion of the conjecture

In this section we discuss various aspects of the Main Conjecture III.37, including an alternative formulation using the subgroup $\mathcal{Z}^{1,+}_{\mathfrak{f}}$ of $\mathcal{Z}^1_{\mathfrak{f}}$ and a discussion of the link with the rank one abelian Stark conjectures.

III.4.2.1 Alternative formulation

In this section we discuss the possibility of using Conjecture III.34 to simplify our main conjecture. Indeed, if we assume Conjecture III.34, then:

$$\prod_{(k,\varepsilon)\in\mathcal{Z}_{\mathfrak{f}}^1} I_{r,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(\varepsilon_1,\dots,\varepsilon_r;k_0.k.\varepsilon.\underline{h},\underline{\mu},\underline{\nu}) = \left(\prod_{(k,\varepsilon)\in\mathcal{Z}_{\mathfrak{f}}^{1,+}} I_{r,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(\varepsilon_1,\dots,\varepsilon_r;k_0.k.\varepsilon.\underline{h},\underline{\mu},\underline{\nu})\right)^g$$

where g is the index of $\mathcal{Z}_{\mathfrak{f}}^{1,+}$ in $\mathcal{Z}_{\mathfrak{f}}^1$, provided that the strongly compatible set of base points $\underline{h}$ is such that the class of h_ρ/q in $\mathrm{Cl}^+(\mathfrak{f})$ is independent of $\rho\in\mathfrak{S}_r$. The complex number

$$\mathcal{V}_{\mathfrak{f},\mathfrak{c},\mathfrak{a}} = \prod_{(k,\varepsilon)\in\mathcal{Z}_{\mathfrak{f}}^1} I_{r,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(\varepsilon_1,\dots,\varepsilon_r;k_0.k.\varepsilon.\underline{h},\underline{\mu},\underline{\nu})^{\mathcal{D}(N,n)}$$

would be then be the g-th power of the complex number

$$\mathcal{W}_{\mathfrak{f},\mathfrak{c},\mathfrak{a}} = \prod_{(k,\varepsilon)\in\mathcal{Z}_{\mathfrak{f}}^{1,+}} I_{r,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(\varepsilon_1,\dots,\varepsilon_r;k_0.k.\varepsilon.\underline{h},\underline{\mu},\underline{\nu})^{\mathcal{D}(N,n)}$$

and we would get the simpler Kronecker limit formula:

$$\mathcal{N}(\mathfrak{a})\zeta_{\mathfrak{f}}'(\mathfrak{c},0)-\zeta_{\mathfrak{f}}'(\mathfrak{a}\mathfrak{c},0)=\frac{1}{\mathcal{D}(N,n)}\log\left|\mathcal{W}_{\mathfrak{f},\mathfrak{c},\mathfrak{a}}\right|^2.$$

From this we would conjecture in general that the unit $u_{\mathfrak{f},\mathfrak{c},\mathfrak{a}}$ is already a g-th power in $\mathbb{K}^+(\mathfrak{f})$. We therefore give the following alternative formulation for the conjecture:

Alternative Conjecture III.38: *Suppose that $\mathbb{K}$ is an ATR field of degree $n=r+2\geq 3$. Fix an integral ideal $\mathfrak{f}\neq\mathcal{O}_{\mathbb{K}}$ satisfying (H1) and (H5) such that $\mathbb{K}^+(\mathfrak{f})$ is totally complex. Let $\sigma_{\mathbb{C}}$ be one of the two complex embeddings of $\mathbb{K}$ and fix a complex embedding σ of $\mathbb{K}^+(\mathfrak{f})$ which extends $\sigma_{\mathbb{C}}$. Fix a set of fundamental units $\varepsilon_1,\dots,\varepsilon_r$ for $\mathcal{O}_{\mathbb{K}}^{+,\times}$ which are adapted to $\mathfrak{f}$. Fix a smoothing ideal $\mathfrak{a}$ of norm N which is good in the sense of Definition III.26. Fix a class $\mathfrak{c}$ in the narrow ray class group at $\mathfrak{f}$. Fix an integer $k_0>0$ coprime to $q=\mathcal{N}(\mathfrak{f})$ and an integral ideal $\mathfrak{b}$ coprime to $\mathfrak{f}$ such that the integral ideal $k_0.\mathfrak{b}$ belongs to the class $\mathfrak{c}$. Let $\underline{h}$ be a compatible set of strongly admissible base points for the data $\mathfrak{f},\mathfrak{b},\mathfrak{a}$ and for the fundamental units $\varepsilon_1,\dots,\varepsilon_r$, as given by Proposition III.25. Assume further that $\underline{h}$ is such that the class of h_ρ/q in $\mathrm{Cl}^+(\mathfrak{f})$ is independent of $\rho\in\mathfrak{S}_r$ (it is always possible to find such a set if Conjecture III.35 holds). Then, there are orientation signs $\mu_\rho=\mu_\rho(\varepsilon_1,\dots,\varepsilon_r,h),\nu_\rho=\nu_\rho(\varepsilon_1,\dots,\varepsilon_r,h)\in\{-1,+1\}$ such that the complex number*

$$\mathcal{W}_{\mathfrak{f},k_0,\mathfrak{b},\mathfrak{a}}(\varepsilon_1,\dots,\varepsilon_r)=\prod_{(k,\varepsilon)\in\mathcal{Z}_{\mathfrak{f}}^{1,+}} I_{r,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(\varepsilon_1,\dots,\varepsilon_r;k_0.k.\varepsilon.\underline{h},\underline{\mu},\underline{\nu})^{\mathcal{D}(N,n)} \tag{III.43}$$

is independent of the choice for the compatible set of strongly admissible base points $\underline{h}$ in Proposition III.25. Furthermore:

1. *The complex number $\mathcal{W}_{\mathfrak{f},\mathfrak{c},\mathfrak{a}}=\mathcal{W}_{\mathfrak{f},k_0,\mathfrak{b},\mathfrak{a}}(\varepsilon_1,\dots,\varepsilon_r)$ is independent of the choice of fundamental units $\varepsilon_1,\dots,\varepsilon_r$ adapted to $\mathfrak{f}$ and from the representation of the class $\mathfrak{c}$ by the integral ideal $k_0.\mathfrak{b}$.*

2. *The complex number* $\mathcal{W}_{\mathfrak{f},\mathfrak{c},\mathfrak{a}}$ *is the image in* $\mathbb{C}$ *of an algebraic unit* $w_{\mathfrak{f},\mathfrak{c},\mathfrak{a}}$ *in the class field* $\mathbb{K}^+(\mathfrak{f})$ *under the complex embedding* σ *of* $\mathbb{K}^+(\mathfrak{f})$.

3. *Any embedding of* $\mathbb{K}^+(\mathfrak{f})$ *above a real embedding of* $\mathbb{K}$ *maps* $w_{\mathfrak{f},\mathfrak{c},\mathfrak{a}}$ *to the unit circle.*

4. *If* $\mathfrak{c}'$ *is a class in the narrow ray class group at* $\mathfrak{f}$ *then the explicit reciprocity law is given by*
$$\sigma_{\mathfrak{c}'}(w_{\mathfrak{f},\mathfrak{c},\mathfrak{a}}) = w_{\mathfrak{f},\mathfrak{c}\mathfrak{c}',\mathfrak{a}}$$
where $\mathfrak{c} \to \sigma_{\mathfrak{c}}$ *is the Artin map.*

5. *The complex number* $\mathcal{W}_{\mathfrak{f},\mathfrak{c},\mathfrak{a}}$ *satisfies the following Kronecker limit formula:*
$$\mathcal{N}(\mathfrak{a})\zeta_{\mathfrak{f}}'(\mathfrak{c},0) - \zeta_{\mathfrak{f}}'(\mathfrak{a}\mathfrak{c},0) = \frac{1}{\mathcal{D}(N,n)} \log |\mathcal{W}_{\mathfrak{f},\mathfrak{c},\mathfrak{a}}|^2 . \qquad \text{(III.44)}$$

This second formulation of the main conjecture is also coherent with the discussion carried out at the end of section III.3.1. All the examples we present in section IV.2 satisfy this second (and more elegant) formulation of the conjecture, thus we shall present them with this alternative version of the conjecture in mind. It relies on an extra assumption on the classes represented by the elements h_ρ/q in $\mathrm{Cl}^+(\mathfrak{f})$ and in future work we shall prove that we may always find such sets of base points (see Conjecture III.35).

III.4.2.2 Optimal setting

In this section we describe a special setting that gives the simplest examples of computations of higher elliptic units. This is the setting in which we carried out most of our computations as it simplifies greatly the research of ATR fields together with their class field moduli for which the hypotheses of the main conjecture are satisfied. We shall say that the pair $(\mathbb{K},\mathfrak{f})$ is an optimal setting (where $\mathbb{K}$ is an ATR field and $\mathfrak{f} \neq (1)$ is an integral ideal in $\mathcal{O}_{\mathbb{K}}$) if $\mathcal{O}_{\mathbb{K}}^{+,\times} = \mathcal{O}_{\mathfrak{f}}^{+,\times}$. This implies in particular that $\mathcal{Z}_{\mathfrak{f}}^{1,+} = \{(1,1)\}$. Finding an optimal setting gives us the best practical chances of finding fundamental units *adapted* to $\mathfrak{f}$ and therefore of having a working example. However, this optimal setting condition is very restrictive, as explained by the following straightforward lemma:

Lemma III.39: *Let* $\mathbb{K}$ *be an ATR field and let* $\varepsilon_1,\ldots,\varepsilon_r$ *be fundamental units for* $\mathcal{O}_{\mathbb{K}}^{+,\times}$. *The integral ideal*
$$J(\mathbb{K}) = \sum_{j=1}^{r}(\varepsilon_j - 1)\mathcal{O}_{\mathbb{K}} = \sum_{\varepsilon\in\mathcal{O}_{\mathbb{K}}^{+,\times}} (\varepsilon - 1)\mathcal{O}_{\mathbb{K}}$$
does not depend on the choice of fundamental units for $\mathcal{O}_{\mathbb{K}}^{+,\times}$ *and an integral ideal* $\mathfrak{f}$ *satisfies* $\mathcal{O}_{\mathfrak{f}}^{+,\times} = \mathcal{O}_{\mathbb{K}}^{+,\times}$ *if and only if* $\mathfrak{f}$ *divides* $J(\mathbb{K})$. *In particular, as there are only finitely many divisors of* $J(\mathbb{K})$, *there can only be finitely many moduli* $\mathfrak{f}$ *in* $\mathbb{K}$ *for which* $(\mathbb{K},\mathfrak{f})$ *is an optimal setting.*

Proof :
The equality $\mathcal{O}_{\mathfrak{f}}^{+,\times} = \mathcal{O}_{\mathbb{K}}^{+,\times}$ is equivalent to the statement that $\varepsilon - 1 \in \mathfrak{f}$ for any $\varepsilon \in \mathcal{O}_{\mathbb{K}}^{+,\times}$. Thus $\mathcal{O}_{\mathfrak{f}}^{+,\times} = \mathcal{O}_{\mathbb{K}}^{+,\times}$ implies that $\mathfrak{f} \mid \sum_{j=1}^{r}(\varepsilon_j - 1)\mathcal{O}_{\mathbb{K}} = J(\mathbb{K})$. Suppose now that $\mathfrak{f} \mid J(\mathbb{K})$. Then any $\varepsilon \in \mathcal{O}_{\mathbb{K}}^{+,\times}$ may be written in the form $\varepsilon = \prod_{j=1}^{r} \varepsilon_j^{n_j}$ for some integers $n_1,\ldots,n_r$ and $\varepsilon \equiv 1 \bmod \mathfrak{f}$ as each of the ε_j are congruent to 1 mod $\mathfrak{f}$. Therefore $\mathcal{O}_{\mathfrak{f}}^{+,\times} = \mathcal{O}_{\mathbb{K}}^{+,\times}$. □

This result is interesting for computations as it gives us a way of finding optimal settings by computing the ideal $J(\mathbb{K})$ and checking for each of the divisors of $J(\mathbb{K})$ if the hypotheses of the main conjecture hold (see section IV.1.4).

III.4.2.3 Expanding the conjecture

In this section we briefly discuss some of the hypotheses we made to state our main conjecture and discuss how the main conjecture could be expanded to cases where some of these hypotheses are not satisfied.

1. As mentioned in section III.4.1, the hypothesis (H1) is there to ensure that the result is non-trivial. If it is not satisfied, then all partial zeta functions vanish at $s = 0$ with order ≥ 2 and the invariant we compute is expected to be equal to 1 (which is already a non-trivial statement about the vanishing of the right-hand side of (III.42)). The hypothesis that $\mathbb{K}^+(\mathfrak{f})$ is totally complex serves the same purpose.

2. The hypothesis that $\mathcal{O}_{\mathbb{K}}/\mathfrak{f}$ is cyclic (see (H5)) is quite restrictive, but it gives better conditions for the computations. We would need to construct more examples where this is not satisfied to expand the conjecture, and these examples would require a lot more time to compute.

3. The hypotheses that the different ideals $\mathfrak{D}(\tilde{a}_\rho, \mathcal{O}_{\mathbb{K}})$ are coprime to q and that all the contents $\tilde{\lambda}_\rho$ are equal to 1 (see (H4), (H3)) give better conditions for the computations, but as shown in section IV.2.6 we may compute higher elliptic units in conditions where they are not satisfied. The notion of a strongly compatible set of base points $\underline{h}$ would need to be adapted in a way we do not forsee and the existence of such a set would need to be proven, as the arguments used in the proof of Proposition III.25 are not suited to this case.

4. The hypothesis that the involved unit systems are non-degenerate, that is they satisfy (H2) is important to perform the computations of non-degenerate G_r functions. It would be interesting to investigate the case where some of these unit systems $u_1, \ldots, u_r$ are such that $1, u_1, \ldots, u_r$ is not a free family of the $\mathbb{Q}$-vector space $\mathbb{K}$, and see if the evaluation still seems to yield an algebraic unit, where the term corresponding to the unit system $u_1, \ldots, u_r$ is set to 1 by convention.

III.4.2.4 Hilbert's 12th problem for ATR fields

Hilbert's 12th problem asks for the construction of the abelian extensions of general number fields using analytic functions. It is solved for the field of rational numbers using the exponential function and for imaginary quadratic fields using the elliptic units built from the θ function. If proven, our main conjecture III.37 would constitute progress towards a solution Hilbert's 12th problem for general number fields with exactly one complex place. Indeed, when the left-hand side of (III.42) does not vanish, the conjectural elliptic unit $u_{\mathfrak{f},\mathfrak{c},\mathfrak{a}}$ we compute by evaluating the multiple elliptic Gamma functions is expected to generate a non-trivial extension of $\mathbb{K}$ and in many cases the whole class field $\mathbb{K}^+(\mathfrak{f})$. It would then suffice to prove statements on the exact extensions of $\mathbb{K}$ generated by these units and possibly some of their roots to obtain a general construction of $\mathbb{K}^+(\mathfrak{f})$ using analytic functions.

III.4.2.5 Rank one abelian Stark conjectures

To end this section, we discuss the link between our conjecture and the rank one abelian Stark conjecture for ATR fields (see [Sta80], [DG11]). This conjecture is expressed as follows:

Conjecture III.40 [Rank one abelian Stark conjecture for ATR fields] : *Write $e_{\mathfrak{f}}$ for the number of roots of unity in $\mathbb{K}^+(\mathfrak{f})$. Fix a complex embedding σ of $\mathbb{K}^+(\mathfrak{f})$ extending a fixed complex embedding $\sigma_{\mathbb{C}}$ of $\mathbb{K}$. There is a unit u_{Stark} in $\mathbb{K}^+(\mathfrak{f})$ such that*

- *For any class $\mathfrak{c}$ in $\mathrm{Cl}^+(\mathfrak{f})$, if $\sigma_{\mathfrak{c}} \in \mathrm{Gal}(\mathbb{K}^+(\mathfrak{f})/\mathbb{K})$ is the image of $\mathfrak{c}$ under Artin's map then:*
$$\zeta_{\mathfrak{f}}'(\mathfrak{c},0) = -\frac{1}{e_{\mathfrak{f}}}\log|\sigma\left(u_{Stark}^{\sigma_{\mathfrak{b}}}\right)|^2.$$
- *Every complex embedding of $\mathbb{K}^+(\mathfrak{f})$ above a real embedding of $\mathbb{K}$ maps u_{Stark} to the unit circle.*
- *The number field $\mathbb{K}^+(\mathfrak{f})(u_{Stark}^{1/e_{\mathfrak{f}}})$ is an abelian extension of $\mathbb{K}$.*

If we assume both our main conjecture III.37 and the rank one abelian Stark conjecture for ATR fields, then we obtain for all classes $\mathfrak{c}$ of the narrow ray class group at $\mathfrak{f}$ the equality:
$$\frac{\#\mathcal{Z}_{\mathfrak{f}}^{1,+}}{\#\mathcal{Z}_{\mathfrak{f}}^{1}}\log|\sigma(u_{\mathfrak{f},\mathfrak{c},\mathfrak{a}}^{\sigma_{\mathfrak{c}}})|^2 = -\frac{1}{e_{\mathfrak{f}}}\log|\sigma\left((u_{Stark}^{\sigma_{\mathfrak{c}}})^{\mathcal{D}(N,n)(N-\sigma_{\mathfrak{a}})}\right)|^2.$$
In particular, up to some roots of unity of well-controled order, we should get the equality in $\mathbb{K}^+(\mathfrak{f})$:
$$u_{\mathfrak{f},\mathfrak{c},\mathfrak{a}}^{-e_{\mathfrak{f}}\#\mathcal{Z}_{\mathfrak{f}}^{1,+}} = u_{Stark}^{\mathcal{D}(N,n)(N-\sigma_{\mathfrak{a}})\#\mathcal{Z}_{\mathfrak{f}}^{1}}. \tag{III.45}$$
If we believe further in Conjecture III.34 and in the alternative conjecture III.38 then we should get the simpler formula:
$$w_{\mathfrak{f},\mathfrak{c},\mathfrak{a}}^{-e_{\mathfrak{f}}} = u_{Stark}^{\mathcal{D}(N,n)(N-\sigma_{\mathfrak{a}})} \tag{III.46}$$
up to well-controled roots of unity. Thus our higher elliptic units give a conjectural analytic formula for smoothed Stark units and not only for their modulus at places of $\mathbb{K}^+(\mathfrak{f})$ above the unique complex place of $\mathbb{K}$. Other formulas for Stark units above ATR fields were proposed by Ren and Sczech for complex cubic fields in [RS09] and more recently by Black (see [Bla25]) for general ATR number fields using generalised log-gamma functions. Another approach to the construction of Stark units specifically above ATR fields $\mathbb{K}$ of degree $n = 2n'$ containing a totally real subfield $\mathbb{F}$ of degree n' using Hilbert modular forms was presented by Charollois and Darmon in [CD08]. They give a conjectural formula for the Stark unit as the evaluation of some multiplicative n'-cocycle (their version of the "Abel-Jacobi map") against an n'-cycle in the group coholomogy of $\mathrm{SL}_2(\mathcal{O}_{\mathbb{F}})$. It is worth noting that the presence of a real subfield is often an obstacle in our construction as we need to define higher elliptic units with evaluations of G_r functions at real algebraic irrational points. Nevertheless, it would be interesting future work to try to find a relation between any of these constructions of Stark units and ours.

III.4.2.6 Independence of the choices made in the main conjecture

In this section we briefly discuss the question of proving the independence of the complex numbers $\mathcal{W}_{\mathfrak{f},\mathfrak{c},\mathfrak{a}}$ from the choices made in the conjecture. There are four important choices that were made: the choice of fundamental units $\varepsilon_1,\ldots,\varepsilon_r$, the choice of the representation of the class $\mathfrak{c}$ by an integral ideal of the form $k_0.\mathfrak{b}$, the choice of helper ideal $\mathfrak{H}$ in Proposition III.25 and the choice of the strongly compatible set of base points $\underline{h}$. In section III.3.5.2 we already discussed the independence of the evaluation $\mathcal{W}_{\mathfrak{f},\mathfrak{c},\mathfrak{a}}$ from this last choice. Let us now briefly mention what we can expect from the other three choices.

The choice of fundamental units $\varepsilon_1,\ldots,\varepsilon_r$ is the most rigid one as many hypotheses depend on the good position of the associated r-cycle $\Upsilon(\varepsilon_1,\ldots,\varepsilon_r)$, and the conjecture is formulated for a very limited number of possible choices for these fundamental units. Still, as the conjecture concerns the evaluation of a r-cocycle against the cycle $\Upsilon(\varepsilon_1,\ldots,\varepsilon_r)$ and that the class of this cycle in $H_r(\mathcal{O}_{\mathbb{K}}^{+,\times},\mathbb{Z})$ is independent of the choice of fundamental units, it should follow from standard arguments in cohomology that the evaluation of $\mathcal{W}_{\mathfrak{f},\mathfrak{c},\mathfrak{a}}$ does not depend on the choice of fundamental units $\varepsilon_1,\ldots,\varepsilon_r$.

The choice of the integral ideal $\mathfrak{b}$ is rather easy to treat. Indeed, if we replace $\mathfrak{b}$ with an integral ideal $\mathfrak{b}'$ in the same class in $\mathrm{Cl}^+(\mathfrak{f})$ then the fractional ideal $\mathfrak{b}'/\mathfrak{b}$ is generated by some totally positive $\xi\in\mathbb{K}$ satisfying $\xi\equiv 1 \bmod \mathfrak{f}$. If the set of base points $\underline{h}=(h_\rho)_\rho$ is strongly compatible for $\mathfrak{b}$ then the set of base points $\underline{h'}=(h'_\rho)_\rho=(\xi h_\rho)_\rho$ is strongly compatible for $\mathfrak{b}'$. The linear forms a_ρ attached to h_ρ are then replaced by $a'_\rho=a_\rho(\xi^{-1}\cdot)$ and the associated elements α_ρ are replaced by $\alpha'_\rho=\xi\alpha_\rho$. This gives in the evaluation $\alpha_\rho/\gamma_\rho=\alpha'_\rho/\gamma'_\rho$ as $\gamma'_\rho=\xi\gamma_\rho$ as well. Thus the construction is indeed independent of the choice of the particular integral ideal $\mathfrak{b}$ as:

$$\prod_{(k,\varepsilon)\in\mathcal{Z}^1_{\mathfrak{f}}} I_{r,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(\varepsilon_1,\ldots,\varepsilon_r;k_0.k.\varepsilon.\underline{h},\underline{\mu},\underline{\nu})=\prod_{(k,\varepsilon)\in\mathcal{Z}^1_{\mathfrak{f}}} I_{r,\mathfrak{f},\mathfrak{b}',\mathfrak{a}}(\varepsilon_1,\ldots,\varepsilon_r;k_0.k.\varepsilon.(\xi.\underline{h}),\underline{\mu},\underline{\nu}).$$

Lastly, we discuss the choice of helper ideals in Proposition III.25. It is in general very difficult to prove that the evaluations are independent of the choice of said helper ideal, as the levels of the computations are modified when changing helper ideals. We focus on a particular simple case which illustrates the following remark: in many aspects, a helper ideal of the form $p/\mathfrak{P}$ behaves as if the computation had a semi p-smoothing by the prime ideal $\mathfrak{P}$. Let us then fix the setting where the ideal $\mathfrak{a}$ belongs to the trivial class in $\mathrm{Cl}^+(\mathfrak{f})$ and $\mathfrak{H}=p/\mathfrak{P}$ for some integral ideal $\mathfrak{P}$ of norm p such that $\mathfrak{P}$ belongs to the trivial class in $\mathrm{Cl}^+(\mathfrak{f})$. Since $\mathfrak{H}$ is a helper ideal for the data $\mathfrak{f},\mathfrak{b},\mathfrak{a}$, it is also true that $N/\mathfrak{a}$ is a helper ideal for the data $\mathfrak{f},\mathfrak{b},\mathfrak{P}$. Thus the two computations may be compared, as the same strongly compatible set of base points $\underline{h}$ may be chosen in both cases. If we believe in the conjectures of this section, we should obtain

$$w_{\mathfrak{f},\mathfrak{b},\mathfrak{a}}^{-e_{\mathfrak{f}}}=u_{Stark}^{\mathcal{D}(N,n)(N-1)}$$

and

$$w_{\mathfrak{f},\mathfrak{b},\mathfrak{P}}^{-e_{\mathfrak{f}}}=u_{Stark}^{\mathcal{D}(N,n)(p-1)}$$

up to roots of unity. Thus we would expect to obtain the equality:

$$w_{\mathfrak{f},\mathfrak{b},\mathfrak{a}}^{p-1}=w_{\mathfrak{f},\mathfrak{b},\mathfrak{P}}^{N-1}$$

up to roots of unity. Proving this equality would be a first step in the direction of proving the independence of the choice of helper ideal in Proposition III.25.

Chapter IV

Algorithms and numerical examples

In this chapter we present algorithms that were used to produce numerical evidence supporting the Main Conjecture III.37. We then present examples of computations of higher elliptic units above various ATR fields of degree 3, 4, 5 and 6.

IV.1 Computing the conjecture

In this section we explain how we obtain numerical examples to support the conjecture. First, we explain how to evaluate the ordinary G_r functions and more arithmetic evaluations of the form (III.2). Then, we explain how to compute the base points h_ρ for $\rho \in \mathfrak{S}_r$ and the other parameters needed for the general computation. Our goal is to present an algorithm (see section IV.1.2.5) which computes the products

$$\mathcal{V}_{\mathfrak{f},\mathfrak{c},\mathfrak{a}} = \prod_{(k,\varepsilon)\in\mathcal{Z}_{\mathfrak{f}}^1} I_{r,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(\varepsilon_1,\dots,\varepsilon_r;k_0.k.\varepsilon.\underline{h},\underline{\mu},\underline{\nu})^{\mathcal{D}(N,n)}$$

(see (III.41)) or its counterpart $\mathcal{W}_{\mathfrak{f},\mathfrak{c},\mathfrak{a}}$ defined in (III.43) from the data $\mathfrak{f},\mathfrak{a},\varepsilon_1,\dots,\varepsilon_r$ for all classes $\mathfrak{c}$ in the narrow ray class group at $\mathfrak{f}$. To simplify notations we shall present the computations in the case where $t_\rho = 1$ for all permutation $\rho \in \mathfrak{S}_r$ (as this is the case in the main conjecture). Explicitly, the product we wish to compute is given by:

$$\mathcal{I} = \prod_{\rho\in\mathfrak{S}_\rho}\prod_{(k,\varepsilon)\in\mathcal{Z}_{\mathfrak{f}}^1}\left(\frac{G_r\left(\frac{\mu_\rho.k_0.\chi_{\mathfrak{f}}(k)k.h}{q\gamma},\frac{\alpha_0}{\gamma},\dots,\frac{\alpha_r}{\gamma}\right)^N}{G_r\left(\frac{N.\mu_\rho.k_0.\chi_{\mathfrak{f}}(k)k.h}{q\gamma},\frac{N\alpha_0}{\gamma},\dots,\frac{N\alpha_r}{\gamma}\right)}\right)^{\nu_\rho\mu_\rho^r\chi_{\mathfrak{f}}(k)^{r+1}} \tag{IV.1}$$

which is the explicit formulation for the product appearing in (III.41) given by (III.40).

IV.1.1 Computing the G_r functions

IV.1.1.1 Computing the ordinary G_r functions

Let us now explain how we compute the ordinary G_r functions, practically speaking, to verify the conjecture on numerical examples. Indeed, the definition of the G_r functions by a multi-index infinite product given by formula (I.1) is not suited for computations. In the case $n = 2$, Jacobi's triple product formula gives a beautiful expression of the θ

function in terms of an infinite sum with converging rate in $q^{n^2/2}$:

$$\theta(z,\tau) = \frac{q^{1/24}}{\eta(\tau)} \sum_{n\in\mathbb{Z}} x^n(-1)^n q^{n(n-1)/2} \tag{IV.2}$$

where $x = \exp(2i\pi z)$, $q = \exp(2i\pi\tau)$ and $\eta(\tau)$ is Dedekind's η function. This makes the computation of θ very fast when τ is not too close to the real axis, and standard techniques in the study of modular functions allow to use the modularity of θ to reduce the computation of $\theta(z,\tau)$ to the computation of $\theta(z',\tau')$ where $\Im(\tau') \geq 1/2$. Unfortunately, there is no clear generalisation of Jacobi's triple product formula for higher degree G_r functions and we need other techniques to compute the G_r functions efficiently. In the case where the parameters $\tau_0,\ldots,\tau_r$ lie in the upper half-plane, we shall make use of [[Nis01], Proposition 3.6] which we write as:

$$G_r(z,\tau_0,\ldots,\tau_r) = \exp\left(-\sum_{j\geq 1}\frac{1}{j}\frac{q_0^j\ldots q_r^j x^{-j} + (-1)^r x^j}{\prod_{k=0}^r (1-q_k^j)}\right) \tag{IV.3}$$

where $x = \exp(2i\pi z)$ and $q_k = \exp(2i\pi\tau_k)$ for $0 \leq k \leq r$. This formula is only valid for $\tau_0,\ldots,\tau_r \in \mathbb{H}^{r+1}$ and $0 < \Im(z) < \sum_{k=0}^r \Im(\tau_k)$. We call this domain the center strip. It is remarkable that the complexity in the computation of the right-hand side in (IV.3) doesn't increase drastically with r. To compute the value $G_r(z,\tau_0,\ldots,\tau_r)$ in the general case where the parameters $\tau_0,\ldots,\tau_r$ lie in $\mathbb{C}-\mathbb{R}$, we only need to make sure that we can use the properties of the G_r functions to reach this domain. This requires two steps: a reorientation step and a translation step.

The reorientation step consists in reducing the computation of any value $G_r(z,\tau_0,\ldots,\tau_r)$ with parameters $\tau_0,\ldots,\tau_r \in \mathbb{C}-\mathbb{R}$ to the computation of some $G_r(z',\tau_0',\ldots,\tau_r')$ with parameters $\tau_0',\ldots,\tau_r'$ in the upper half-plane. To achieve this, we use repeatedly the inversion property:

$$G_r(z,\tau_0,\ldots,\tau_{k-1},-\tau_k,\tau_{k+1},\ldots,\tau_r) = G_r(z+\tau_k,\tau_0,\ldots,\tau_r)^{-1}. \tag{IV.4}$$

Starting from any $\tau_0,\ldots,\tau_r \in \mathbb{C}-\mathbb{R}$, we may define $J = \{0 \leq j \leq r \mid \Im(\tau_j) < 0\}$. Then using the inversion property (IV.4) for each $k \in J$ gives:

$$G_r(z,\tau_0,\ldots,\tau_r) = G_r\left(z - \sum_{k\in J}\tau_k, \tau_0',\ldots,\tau_r'\right)^{(-1)^{\#J}}$$

where $\tau_j' = \tau_j$ if $j \notin J$ and $\tau_j' = -\tau_j$ if $j \in J$. In any case, this gives $\Im(\tau_j') > 0$.

The translation step consists in bringing back the elliptic variable z in the center strip defined by $0 < \Im(z) < \sum_{j=0}^r \Im(\tau_j)$ when the parameters $\tau_0,\ldots,\tau_r$ belong to the upper half-plane. For this step we use recursively the pseudo-periodicity property:

$$G_r(z+\tau_k,\tau_0,\ldots,\tau_r) = G_{r-1}(z,\tau_0,\ldots,\widehat{\tau_k},\ldots,\tau_r) \times G_r(z,\tau_0,\ldots,\tau_r). \tag{IV.5}$$

Doing so will require the computation of lower degree functions, that is until we reach the case $r = 0$ in which case the computation of $G_0 = \theta$ can be done quickly using (IV.2). In the general case, the number of translation steps required to reach the center strip for all involved functions, including the new ones appearing at each step is $O(\prod_{k=1}^r (|\Im(z)|/\Im(\tau_k)))$ where we have assumed that $0 < \Im(\tau_0) < \cdots < \Im(\tau_r)$.

The proof of this statement is easily done by induction, but we shall omit it as we now focus on the pratical computations we perform to test the conjecture. Indeed, it is clear that at most one translation step is required per term in the product (III.41) as the parameter $z = \mu.k_0.\chi_{\mathfrak{f}}(k)k.h/q\gamma$ belongs to $\mathbb{Q} - \mathbb{Z}$. Thus, the computation time to test the conjecture is entirely dedicated to the computation of sums of the shape (IV.3). Let us say a few words on the number of terms we must compute in this sum. Let $y = 2\pi.\min(\Im(z), \Im(\tau_0) + \cdots + \Im(\tau_r) - \Im(z))$ be the distance from z to the boundary of the center strip. Then, to compute $G_r(z, \tau_0, \ldots, \tau_r)$ with precision $\delta > 0$ we need to compute $O(\log(\delta)/y)$ terms in the sum (IV.3) (this corresponds to a convergence rate in q_y^n where $q_y = e^{-2\pi y}$. The main difficulty in performing the computations to test the conjecture comes from the fact that in practice, the imaginary parts of all the τ_k's are very small, making the parameter y even smaller.

We end this section on the computation of the ordinary G_r functions by mentioning the rare case where some of the τ_j's are real algebraic. In this case we use the formula given in Proposition I.17 which is a sum with a similar converging rate.

IV.1.1.2 Computing the arithmetic evaluation $G^{\mu}_{r,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(u_1, \ldots, u_r; h)^{\nu}$

In this section we describe how to compute the arithmetic evaluation $G^{\mu}_{r,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(u_1, \ldots, u_r; k_0.k.\varepsilon.h)^{\nu}$ for any orientations $\mu, \nu \in \{-1, 1\}$ and any $(k, \varepsilon) \in \mathcal{Z}^1_{\mathfrak{f}}$. To simplify notations we shall treat the case where $t = 1$. It follows from Proposition III.30 that this evaluation is then given by:

$$G^{\mu}_{r,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(u_1, \ldots, u_r; k_0.k.\varepsilon.h)^{\nu} = \left(\frac{G_r\left(\frac{\mu.k_0.\chi_{\mathfrak{f}}(k)k.h}{q\gamma}, \frac{\alpha_0}{\gamma}, \ldots, \frac{\alpha_r}{\gamma}\right)^N}{G_r\left(\frac{N.\mu.k_0.\chi_{\mathfrak{f}}(k)k.h}{q\gamma}, \frac{N\alpha_0}{\gamma}, \ldots, \frac{N\alpha_r}{\gamma}\right)} \right)^{\nu\mu^r\chi_{\mathfrak{f}}(k)^{r+1}}.$$

To compute the parameter γ we compute the matrices of multiplication by $u_1, \ldots, u_r$ in the basis B_L of L and compute $s.\gamma = \det(a, au_1, \ldots, au_r, \cdot)$. The parameters $\alpha_0, \ldots, \alpha_r$ are obtained by computing the rescaled comatrix of the matrix $\mathcal{A}$ (see section III.3.2.1). It follows from Proposition III.9 that these parameters satisfy

$$\alpha_j = \frac{\tilde{\alpha}_j h + m_j h}{\epsilon m \ell}$$

for $0 \leq j \leq r$ where $m_0, \ldots, m_r$ are integers, ϵ is the sign of the norm of h, $\ell = q.N.\tilde{t}.p_{\mathfrak{H}}$ is the level of the computation (see lemma III.22) and the family $\tilde{\alpha}_0, \ldots, \tilde{\alpha}_r$ is defined in (III.24). In particular, we may rewrite the term $G^{\mu}_{r,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(u_1, \ldots, u_r; k_0.k.\varepsilon.h)^{\nu}$ as:

$$G^{\mu}_{r,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(u_1, \ldots, u_r; k_0.k.\varepsilon.h)^{\nu} = \left(\frac{G_r\left(\frac{\mu.k_0.\chi_{\mathfrak{f}}(k)k.h}{q\gamma}, \frac{\tilde{\alpha}_0 h + m_0 h}{\epsilon m\ell.\gamma}, \ldots, \frac{\tilde{\alpha}_r h + m_r h}{\epsilon m\ell.\gamma}\right)^N}{G_r\left(\frac{N.\mu.k_0.\chi_{\mathfrak{f}}(k)k.h}{q\gamma}, \frac{N(\tilde{\alpha}_0 h + m_0 h)}{\epsilon m\ell.\gamma}, \ldots, \frac{N(\tilde{\alpha}_r h + m_r h)}{\epsilon m\ell.\gamma}\right)} \right)^{\nu\mu^r\chi_{\mathfrak{f}}(k)^{r+1}}.$$

This can be simplified as $h = \eta.m.\gamma$ for some sign $\eta \in \{-1, +1\}$ as:

$$G^{\mu}_{r,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(u_1, \ldots, u_r; k_0.k.\varepsilon.h)^{\nu} = \left(\frac{G_r\left(\frac{\epsilon.\mu.k_0.\chi_{\mathfrak{f}}(k)k.m}{q}, \frac{\tilde{\alpha}_0 + m_0}{\ell}, \ldots, \frac{\tilde{\alpha}_r + m_r}{\ell}\right)^N}{G_r\left(\frac{N.\epsilon.\mu.k_0.\chi_{\mathfrak{f}}(k)k.m}{q}, \frac{N(\tilde{\alpha}_0 + m_0)}{\ell}, \ldots, \frac{N(\tilde{\alpha}_r + m_r)}{\ell}\right)} \right)^{\eta\epsilon\nu\mu^r\chi_{\mathfrak{f}}(k)^{r+1}} \tag{IV.6}$$

where we have used formula (I.11) to treat the sign $\eta.\epsilon$. From this writing it is clear that the complexity of the computation essentially depends on the parameters $\tilde{\alpha}_0, \ldots, \tilde{\alpha}_r$ over which we do not have any control once $\mathfrak{f}$ and the units $\varepsilon_1, \ldots, \varepsilon_r$ are fixed, and on the level $\ell = q.N.\tilde{t}.p_{\mathfrak{H}}$. It follows from the discussion on the complexity of the computations for the ordinary G_r functions (see section IV.1.1) that the term (IV.6) may be computed with precision $\delta > 0$ with complexity $O(|\log(\delta)| \times \ell)$ where the constant depends on $\tilde{\alpha}_0, \ldots, \tilde{\alpha}_r$. This justifies the comments we made previously: to obtain fast computations one should choose a smoothing ideal with the smallest possible norm, and one should also choose a helper ideal with the smallest possible norm.

IV.1.2 Algorithms for the computation of higher elliptic units

To perform our computations we used the software Pari/GP [The24] and we made extensive use of the tools it provides for algebraic number theory. Here is a list of classic tasks in algebraic number theory that Pari/GP does and which we won't give algorithms for:

- General manipulations of fractional ideals using their Hermite Normal Form representation. This includes multiplication and inversion of fractional ideals, intersection of fractional ideals, decomposition into prime ideals, etc.
- Computations of class groups, ray class groups and fundamental units (see the **bnfinit** and **bnrinit** commands).
- Computations of classes of fractional ideals in (ray) class groups (see the **bnfisprincipal** and **bnrisprincipal** commands).
- Computations of (ray) class fields (see the **bnrclassfield** command).
- Computations of Hermite Norm Forms and the Smith Normal Forms of integral matrices (see the **mathnf** and **matsnf** commands).
- Computations of Dirichlet-Hecke L-functions at $s = 0$ in number fields.
- General manipulations on field extensions, including the verification that a given field $\mathbb{K}$ is included in a given field $\mathbb{L}$.

IV.1.2.1 Computing the different ideals $\mathfrak{D}(\tilde{a}, \mathcal{O}_{\mathbb{K}})$

In this section we briefly recall how we compute the generalised different ideals $\mathfrak{D}(\tilde{a}, \mathcal{O}_{\mathbb{K}})$, using a general version of [[Coh93], Proposition 4.8.19] (see lemma III.13). Let us assume that the unit system $u_1, \ldots, u_r = [\varepsilon_{\rho(1)}| \ldots |\varepsilon_{\rho(r)}]$ is given and that a positive $\mathbb{Z}$-basis $B_{\mathcal{O}_{\mathbb{K}}} = [e'_0 = 1, e'_1, \ldots, e'_{r+1}]$ is given.

Algorithm 1: Computation of the different ideal $\mathfrak{D}(\tilde{a}, \mathcal{O}_{\mathbb{K}})$ and associated parameters.
Input: a degree $n \geq 3$ ATR number field $\mathbb{K}$, units $u_1, \ldots, u_r$ in $\mathcal{O}_{\mathbb{K}}^{\times}$ satisfying (H2), a positive $\mathbb{Z}$-basis $B_{\mathcal{O}_{\mathbb{K}}} = [e'_0 = 1, e'_1, \ldots, e'_{r+1}]$ of $\mathcal{O}_{\mathbb{K}}$.
Output: the content $\tilde{\lambda}$, the linear form $\tilde{a}$, the different ideal $\mathfrak{D}(\tilde{a}, \mathcal{O}_{\mathbb{K}})$, the overflow $\tilde{t}$ and the elements $\tilde{\alpha}_0, \ldots, \tilde{\alpha}_r$.

1. Compute the linear form
$$\tilde{f} = (\tilde{f}(e'_j))_{0 \leq j \leq r+1} = (\det_{B_{\mathcal{O}_{\mathbb{K}}}}(1, u_1, \ldots, u_r, e'_j))_{0 \leq j \leq r+1}.$$
Set $\tilde{\lambda} = \gcd((\tilde{f}(e'_j))_{0 \leq j \leq r+1})$ and $\tilde{a} = \tilde{f}/\tilde{\lambda}$.

2. Compute the matrix $\widetilde{\mathcal{M}} = (\tilde{a}(e'_j e'_k))_{0\leq j,k\leq r+1}$. The inverse of $\widetilde{\mathcal{M}}$ represents a fractional ideal $\mathfrak{I}$ in the basis $B_{\mathcal{O}_{\mathbb{K}}}$. Set $\mathfrak{D}(\tilde{a}, \mathcal{O}_{\mathbb{K}}) = \mathfrak{I}^{-1}$.

3. Compute the matrix U_j of multiplication by u_j in the basis $B_{\mathcal{O}_{\mathbb{K}}}$ for $0 \leq j \leq r$. Set $\tilde{a}u_j$ to be the product of the linear form $\tilde{a}$ by the matrix U_j for $0 \leq j \leq r$.

4. Compute the SNF of the matrix $\widetilde{\mathcal{A}}$ obtained by concatenation of $a, au_1, \ldots, au_r$ (see (III.20)). Set $\tilde{t}$ to be the greatest elementary divisor of $\widetilde{\mathcal{A}}$.

5. Compute $\widetilde{\mathcal{B}} = \tilde{t}.\widetilde{\mathcal{A}}^{-1} = (b_{ij})_{1\leq i,j\leq r+1}$ and set $\tilde{\alpha}_j = \sum_{i=1}^{r+1} b_{ij}e'_i$.

This algorithm computes most of the geometric setup which is independent of the choice of base point h (see section III.3.2.2).

IV.1.2.2 Computing the ideals $\mathfrak{a}, \mathfrak{b}, \mathfrak{H}$

In this section we explain how we compute the possible choices for the smoothing ideal $\mathfrak{a}$, for the representative ideals $\mathfrak{b}$ and for the helper ideal $\mathfrak{H}$. We assume that the different ideals $\tilde{D}_\rho = \mathfrak{D}(\tilde{a}_\rho, \mathcal{O}_{\mathbb{K}})$ are given (or simply the overflows $\tilde{t}_\rho$), for instance as outputs of Algorithm 1. We also assume that we have an iterator over prime numbers by increasing value (see the **forprime** command) or directly an iterator over prime ideals of $\mathcal{O}_{\mathbb{K}}$ by increasing norm.

Algorithm 2: Computation of the smoothing and representative ideals.
Input: a degree $n \geq 3$ ATR number field $\mathbb{K}$, an integral ideal $\mathfrak{f} \neq (1)$, fundamental units $\varepsilon_1, \ldots, \varepsilon_r$ for $\mathcal{O}_{\mathbb{K}}^{+,\times}$, the overflows $\tilde{t}_\rho, \rho \in \mathfrak{S}_r$, an iterator over prime numbers by increasing value.
Output: A *good* smoothing ideal $\mathfrak{a}$, a set of representative integral ideals $\mathfrak{b}_1, \ldots, \mathfrak{b}_g$ for the classes in the narrow ray class group at $\mathfrak{f}$.

1. Set $q \geq 2$ such that $q\mathbb{Z} = \mathfrak{f} \cap \mathbb{Z}$. Set $d = q \cdot \prod_\rho \tilde{t}_\rho$. Compute the set $\mathbb{Z}/q\mathbb{Z}^\times$.

2. Compute the narrow ray class group at $\mathfrak{f}$ and set $g = \#\mathrm{Cl}^+(\mathfrak{f})$. Set f to be a map on $\mathrm{Cl}^+(\mathfrak{f})$ which is identically 0 and which will be updated to map the classes in $\mathrm{Cl}^+(\mathfrak{f})$ to their representative ideals. Set $c_1 = 0, c_2 = 0$.

3. For primes p do the following until $c_1 = 1, c_2 = g$:

   - If p is not coprime to d, move on to the next prime.
   - Compute the decomposition of $p\mathcal{O}_{\mathbb{K}}$ into prime ideals. If there are no prime ideals of norm p, move on to the next prime. Otherwise, set $\mathfrak{P}$ to be one of these prime ideals of norm p.
   - If $c_1 = 0$ then set $\mathfrak{a} = \mathfrak{P}$, $c_1 = 1$ and move on to the next prime. Otherwise move on to the next step.
   - If $c_2 \geq g$, end. Otherwise, compute the class $\mathfrak{c}$ of $\mathfrak{P}$ in $\mathrm{Cl}^+(\mathfrak{f})$. For each $k \in \mathbb{Z}/q\mathbb{Z}^\times$, if $f(k.\mathfrak{c}) = 0$, set $f(k.\mathfrak{c}) = \mathfrak{P}$ and increment c_2. Move on to the next prime.

The image of the map f contains the representative ideals $\mathfrak{b}_1, \dots, \mathfrak{b}_g$.

This algorithm can be slightly modified in a number of ways, either for optimisation or to add some conditions. Firstly, we might impose some extra condition on the integral ideal $\mathfrak{a}$, for instance ask that $N > n+1$ so that $\mathcal{D}(N,n) = 1$, or we might ask that $\mathfrak{a}$ belongs to the trivial class in $\mathrm{Cl}^+(\mathfrak{f})$. Secondly, for the integral ideals $\mathfrak{b}$, we might want a set of representatives which are not necessarly prime ideals with prime norm. In practice, we set the representative $\mathfrak{b}$ for the trivial class to be $\mathcal{O}_{\mathbb{K}}$. In addition, if we set $\mathcal{G}$ to be the quotient of $(\mathcal{O}_{\mathbb{K}}/\mathcal{O}_{\mathbb{F}})^\times$ by $\mathcal{Z}_{\mathfrak{f}}^1$ (see section IV.1.2.4 for the computation of $\mathcal{Z}_{\mathfrak{f}}^1$) we compute ideals $\mathfrak{b}_i$ representing the set $\mathrm{Cl}^+(\mathfrak{f})/\mathcal{G}$ and obtain a set of representatives for $\mathrm{Cl}^+(\mathfrak{f})$ by setting $\mathfrak{c}_{i,k} = k.\mathfrak{b}_i$ for $1 \leq i \leq \#\mathrm{Cl}^+(\mathfrak{f})/\mathcal{G}$ and $k \in \mathcal{G}$. Another remark is that Algorithm 2 computes $\mathfrak{a}$ before the set of representatives for $\mathrm{Cl}^+(\mathfrak{f})$, but it is also possible to compute the representatives first.

Next, we move on to the computation of extended helper ideals (see Definition III.21), which could have been done at the same time as the computation of the smoothing and representatives ideals.

Algorithm 3: Computation of the extended helper ideals.
Input: a degree $n \geq 3$ ATR number field $\mathbb{K}$, an integral ideal $\mathfrak{f} \neq (1)$, the overflows $\tilde{t}_\rho, \rho \in \mathfrak{S}_r$, a *good* smoothing ideal $\mathfrak{a}$, an iterator over prime numbers by increasing value.
Output: A set $\underline{\mathfrak{H}}$ of extended helper ideals representing all classes in the wide class group at $\mathfrak{f}$.

1. Set $q \geq 2$ such that $q\mathbb{Z} = \mathfrak{f} \cap \mathbb{Z}$. Set $d = q \cdot \prod_\rho \tilde{t}_\rho \cdot N$. Compute the set $\mathbb{Z}/q\mathbb{Z}^\times$.
2. Compute the wide ray class group at $\mathfrak{f}$ and set $g' = \#\mathrm{Cl}(\mathfrak{f})$. Set f to be a map on $\mathrm{Cl}(\mathfrak{f})$ which is identically 0 and which will be updated to map the classes of $\mathrm{Cl}(\mathfrak{f})$ to their extended helper ideals. Set $c = 0$.
3. For primes p do the following until $c = g'$:
   - If p is not coprime to d, move on to the next prime.
   - Compute the decomposition of $p\mathcal{O}_{\mathbb{K}}$ in prime ideals. If there are no prime ideals of norm p, move on to the next prime. Otherwise, set $\mathfrak{P}$ to be one of these prime ideals of norm p.
   - Compute the classes $\mathfrak{c}(k)$ of $k.p/\mathfrak{P}$ in $\mathrm{Cl}(\mathfrak{f})$ for any $k \in \mathbb{Z}/q\mathbb{Z}^\times$. For each $k \in \mathbb{Z}/q\mathbb{Z}^\times$, if $f(\mathfrak{c}(k)) = 0$, set $f(\mathfrak{c}(k)) = k.p/\mathfrak{P}$ and increment c. Move on to the next prime.

The image of the map f forms a set $\underline{\mathfrak{H}}$ of extended helper ideals representing the classes in $\mathrm{Cl}(\mathfrak{f})$.

In practice we fix the helper ideal of the trivial class in the wide class group at $\mathfrak{f}$ to be $\mathcal{O}_{\mathbb{K}}$. Also, we might allow helper ideals to be of the form $\mathfrak{H} = \prod_{i \in I} p_i/\mathfrak{P}_i$ for distinct p_i's, which should allow for a set of helpers of smaller norms. An important remark to make is that Algorithm 3 gives a set $\underline{\mathfrak{H}}$ satisfying the following property: if two classes $\mathfrak{c}$ and $\mathfrak{c}'$ in the wide class group at $\mathfrak{f}$ are such that there exists some integer k coprime to q satisfying $\mathfrak{c} = k\mathfrak{c}'$ then there is a rational number κ such that $\mathfrak{H}(\mathfrak{c}) = \kappa\mathfrak{H}(\mathfrak{c}')$ where $\mathfrak{c} \to \mathfrak{H}(\mathfrak{c})$ is the map associating a class $\mathfrak{c}$ to the extended helper ideal $\mathfrak{H}(c)$ in $\underline{\mathfrak{H}}$ which belongs to $\mathfrak{c}$. We shall say that the set $\underline{\mathfrak{H}}$ is correctly formed and we shall always assume that the sets of extended helper ideals we compute are given in this form.

IV.1.2.3 Computing the base points h_ρ

In this section we give an algorithm computing the base points h_ρ for a fixed class $\mathfrak{c}$. We assume that the different ideals $\mathfrak{D}(\tilde{a}_\rho, \mathcal{O}_\mathbb{K})$, the smoothing ideal $\mathfrak{a}$, the representation of the class $\mathfrak{c} = [k_0.\mathfrak{b}]$ and a correctly formed set $\underline{\mathfrak{H}}$ of extended helper ideals have been computed (see sections IV.1.2.1 and IV.1.2.2).

Algorithm 4: Computation of a strongly compatible set of base points $\underline{h}$.
Input: a degree $n \geq 3$ ATR number field $\mathbb{K}$, an integral ideal $\mathfrak{f} \neq (1)$, the different ideals $\mathfrak{D}(\tilde{a}_\rho, \mathcal{O}_\mathbb{K})$, a *good* smoothing ideal $\mathfrak{a}$, a class $\mathfrak{c}$ in $\mathrm{Cl}^+(\mathfrak{f})$ represented by the integral ideal $k_0.\mathfrak{b}$ and a correctly formed set $\underline{\mathfrak{H}}$ of extended helper ideals representing $\mathrm{Cl}(\mathfrak{f})$.
Output: A strongly compatible set of base points $\underline{h}$.
Set $q \geq 2$ such that $q\mathbb{Z} = \mathfrak{f} \cap \mathbb{Z}$. Set $N = \mathcal{N}(\mathfrak{a})$. For each permutation $\rho \in \mathfrak{S}_r$:

1. Set $\mathfrak{I}_\rho = \frac{N}{\mathfrak{ab}}\mathfrak{D}(\tilde{a}_\rho, \mathcal{O}_\mathbb{K})$.
2. Compute the class of the ideal $\mathfrak{I}_\rho$ in the wide class group at $\mathfrak{f}$.
3. Set $\mathfrak{H}_\rho$ to be the extended helper ideal in $\underline{\mathfrak{H}}$ which belongs to the class of the fractional ideal $\mathfrak{I}_\rho^{-1}$ in the wide class group at $\mathfrak{f}$.
4. Set g_ρ to be a generator of $\mathfrak{I}_\rho \times \mathfrak{H}_\rho$ congruent to 1 mod $\mathfrak{f}$ and set $h_\rho = q.g_\rho$.

The set $\underline{h} = (h_\rho)_{\rho \in \mathfrak{S}_r}$ is then a strongly compatible set of base points for $\mathfrak{f}, \mathfrak{b}, \mathfrak{a}$ and for the units $\varepsilon_1, \ldots, \varepsilon_r$.

The hypothesis that the set $\underline{\mathfrak{H}}$ is correctly formed guarantees that the extended helper ideals used are all of the form $\mathfrak{H}_\rho = m_\rho \times \mathfrak{H}$ for some common helper ideal $\mathfrak{H}$ and some integers $m_\rho > 0$ coprime to q (see Proposition III.25).

IV.1.2.4 Computing the sets $\mathcal{Z}_\mathfrak{f}^1$ and $\mathcal{Z}_\mathfrak{f}^{1,+}$

In this section we give some insight on the computation of unit groups such as $\mathcal{O}_\mathbb{K}^{+,\times}$, $\mathcal{O}_\mathfrak{f}^{+,\times}$, $\mathcal{Z}_\mathfrak{f}^1$, $\mathcal{Z}_\mathfrak{f}^{1,+}$. Indeed, it is easy to obtain the fundamental units $\varepsilon'_1, \ldots, \varepsilon'_r$ for $\mathcal{O}_\mathbb{K}^\times$ in Pari/GP, and some simple manipulations with these units allow for the computation of fundamental units for $\mathcal{O}_\mathbb{K}^{+,\times}$ and $\mathcal{O}_\mathfrak{f}^{+,\times}$. Indeed, every unit $\varepsilon \in \mathcal{O}_\mathbb{K}^\times$ satisfies $\varepsilon^2 \in \mathcal{O}_\mathbb{K}^{+,\times}$ and $\varepsilon^{2\kappa} \in \mathcal{O}_\mathfrak{f}^{+,\times}$ where $\kappa = \#(\mathcal{O}_\mathbb{K}/\mathfrak{f})^\times$. Thus the computation of fundamental units for $\mathcal{O}_\mathbb{K}^{+,\times}$ or $\mathcal{O}_\mathfrak{f}^{+,\times}$ amounts to the computation of the relations among $\varepsilon'_1, \ldots, \varepsilon'_r$ in the finite groups $\mathcal{O}_\mathbb{K}^\times/\mathcal{O}_\mathbb{K}^{+,\times}$ and $\mathcal{O}_\mathbb{K}^\times/\mathcal{O}_\mathfrak{f}^{+,\times}$. To compute the set $\mathcal{Z}_\mathfrak{f}^1$ (resp. $\mathcal{Z}_\mathfrak{f}^{1,+}$) we compute the image of $\mathcal{O}_\mathbb{K}^\times$ (resp. $\mathcal{O}_\mathbb{K}^{+,\times}$) in $(\mathcal{O}_\mathbb{K}/\mathfrak{f})^\times$ via the map $\varepsilon \to \varepsilon \bmod \mathfrak{f}$.

IV.1.2.5 Computing the conjecture

Let us now put everything together to give an algorithm computing the product $\mathcal{I}$ in (IV.1).

Algorithm 5: Computation of the full product $\mathcal{I}$.
Input: a degree $n \geq 3$ ATR number field $\mathbb{K}$, an integral ideal $\mathfrak{f} \neq (1)$, fundamental units $\varepsilon_1, \ldots, \varepsilon_r$ for $\mathcal{O}_\mathfrak{f}^{+,\times}$, orientations $\underline{\mu}, \underline{\nu} \in \{-1, +1\}^{r!}$, an iterator over prime numbers by increasing value.
Output: The value $\mathcal{I}$ defined in (IV.1) which is to be tested in the main conjecture III.37.

1. Set $q\mathbb{Z} = \mathfrak{f} \cap \mathbb{Z}$. For each $\rho \in \mathfrak{S}_r$ compute the different ideal $\mathfrak{D}(\tilde{a}_\rho, \mathcal{O}_\mathbb{K})$ and the overflow $\tilde{t}_\rho$ using Algorithm 1.

2. Compute a *good* smoothing ideal $\mathfrak{a}$ and representative ideals $\mathfrak{b}_1, \ldots, \mathfrak{b}_g$ for the narrow ray class group at $\mathfrak{f}$ using Algorithm 2. Set $N = \mathcal{N}(\mathfrak{a})$.

3. Compute a correctly formed set $\underline{\mathfrak{H}}$ of extended helper ideals using Algorithm 3.

4. Compute a strongly compatible set of base points $\underline{h}$ using Algorithm 4.

5. Compute the set $\mathcal{Z}_\mathfrak{f}^1$ in the form of its image in $(\mathcal{O}_\mathbb{K}/\mathfrak{f})^\times$ and the values of the character $\chi_\mathfrak{f}$ (see section IV.1.2.4).

6. Compute the linear form a defined in (III.7) and the associated matrix $\mathcal{A}$ (see (III.9)). Compute the matrix $\mathcal{B} = \mathcal{A}^{-1}$ (since $t = 1$ here). Set $\alpha_0, \ldots, \alpha_r$ to be the vectors described by the columns of $\widetilde{\mathcal{B}}$ (see III.13).

7. Compute the value $\mathcal{I}$ given by (IV.1) for the orientations $\underline{\mu}, \underline{\nu}$ (see section IV.1.1.1 for the computation of the ordinary G_r functions).

In practice, it is often useful to perform the computations for only one class $\mathfrak{b}$, as we shall explain in section IV.1.3.2. We also make the remark that when computing multiple terms of the form:

$$\frac{G_r\left(\frac{k}{q}, \frac{\alpha_0}{\gamma}, \ldots, \frac{\alpha_r}{\gamma}\right)^N}{G_r\left(\frac{N.k}{q}, \frac{N\alpha_0}{\gamma}, \ldots, \frac{N\alpha_r}{\gamma}\right)}$$

with k varying in $\mathbb{Z}/q\mathbb{Z}^\times$ these computations can be done efficiently in parallel.

IV.1.3 Putting the conjecture to the test

In this section we briefly discuss how we check the conjecture, once the value $\mathcal{I}$ defined in (IV.1) has been computed. There are two main elements to check: the Kronecker limit formula (III.42) and the algebraic nature of the value $\mathcal{I}$.

IV.1.3.1 Testing the Kronecker limit formula

To check the Kronecker limit formula (III.42) we compute the derivatives of partial zeta functions at $s = 0$. This is done by using the **bnrL1** command in Pari/GP which computes values of Dirichlet-Hecke L-functions at $s = 1$ from which it deduces equivalents of these functions at $s = 0$ using the functional equation. We then only need to perform a finite abelian Fourier transform to deduce the values taken by the derivatives of the partial zeta functions at $s = 0$. As mentioned in section III.4.1, we take advantage of this verification of the Kronecker limit formula to obtain the correct orientations $\underline{\mu}, \underline{\nu}$. In practice, we use the **lindep** command which uses a version of the LLL algorithm [LLL82] to test for linear relations between the partial zeta values and the values of $\log |\mathcal{I}(\underline{\mu}, \underline{\nu})|$ obtained when varying $\underline{\mu}$ and $\underline{\nu}$.

IV.1.3.2 Testing the algebraicity

Once the Kronecker limit formula (III.42) has been tested numerically with the correct orientations, we check that the product $\mathcal{V} = \mathcal{V}_{\mathfrak{f},\mathfrak{c},\mathfrak{a}}$ (or $\mathcal{W}_{\mathfrak{f},\mathfrak{c},\mathfrak{a}}$) we obtain is indeed an algebraic unit inside $\mathbb{K}^+(\mathfrak{f})$. There are several strategies to do so, depending on the degree $n_{\text{abs}} = n \times \#\mathrm{Cl}^+(\mathfrak{f})$ of the field extension $\mathbb{K}^+(\mathfrak{f})/\mathbb{Q}$.

For simple examples where this extension is expected to be small, we use the **algdep** command which is analoguous to the **lindep** command which tests for a linear relation between the values $\mathcal{V}^{n_{\text{abs}}}, \ldots, \mathcal{V}, 1$. The output of the **algdep** command is a polynomial P which is a good candidate for the minimal polynomial of the complex number $\mathcal{V}$ which is expected to be algebraic of degree $\leq n_{\text{abs}}$. This strategy is useful as it requires to perform the computations for only one class in the narrow ray class group at $\mathfrak{f}$, however it requires a lot of precision on the result for the **algdep** command to produce a meaningful result. Indeed, this command always produces a result, but it might be incorrect. Thus, it is important to know how to interpret the result as correct or incorrect. Because of the properties we expect for the algebraic number $\mathcal{V}$, we expect its minimal polynomial over $\mathbb{Z}$ to be monic and palindormic. If the polynomial P we obtain satisfies this property, it is a good sign that P is correct.

The definitive test however, is checking that the polynomial $P \in \mathbb{Z}[x]$ obtained defines an absolute equation over $\mathbb{Q}$ of a subfield of $\mathbb{K}^+(\mathfrak{f})$, by computing an equation of $\mathbb{K}^+(\mathfrak{f})$ in some other way (see the **bnrclassfield** command). We argue that our main conjecture gives a way of constructing this class field, therefore we might also want another verification mechanism that doesn't rely on already knowing the class field $\mathbb{K}^+(\mathfrak{f})$. The unconditionnal test that can be done in that context is as follows: compute the splitting field $\mathbb{L}$ of the polynomial P, that is $\mathbb{L} = \mathbb{Q}[x]/(P)$ and check that $\mathbb{K}$ is a subfield of $\mathbb{L}$ such that $\mathbb{L}/\mathbb{K}$ is an abelian extension unramified outside of $\mathfrak{f}$ and outside of the archimedian places of $\mathbb{K}$. This will guarantee that $\mathbb{L}/\mathbb{K}$ is a subextension of $\mathbb{K}^+(\mathfrak{f})/\mathbb{K}$ without needing to compute $\mathbb{K}^+(\mathfrak{f})$.

Let us now discuss the general case where the degree n_{abs} of $\mathbb{K}^+(\mathfrak{f})/\mathbb{Q}$ becomes too large for the command **algdep** to succeed. In that case, we compute the values $\mathcal{V}_{\mathfrak{f},\mathfrak{c},\mathfrak{a}}$ for all classes $\mathfrak{c}$ in the narrow ray class group at $\mathfrak{f}$ and we expect these values to be units in $\mathbb{K}^+(\mathfrak{f})$ which are Galois conjugates over $\mathbb{K}$. Thus, we compute the polynomial

$$P_{\text{rel}}(X) = \prod_{\mathfrak{c} \in \mathrm{Cl}^+(\mathfrak{f})} (X - \mathcal{V}_{\mathfrak{f},\mathfrak{c},\mathfrak{a}})$$

which is expected to be the minimal polynomial of each of the values $\mathcal{V}_{\mathfrak{f},\mathfrak{c},\mathfrak{a}}$ over $\mathbb{K}$. We may then attempt to identify the coefficients of the polynomial P_{rel} by expressing them on a $\mathbb{Q}$-basis of $\mathbb{K}$ using the **lindep** command. If successful, we may identify P_{rel} with a polynomial in $\mathbb{K}[X]$ and check that it lies in fact in $\mathcal{O}_{\mathbb{K}}[X]$. From this relative polynomial we may compute an absolute polynomial which should be the minimal polynomial of the values $\mathcal{V}_{\mathfrak{f},\mathfrak{c},\mathfrak{a}}$ over $\mathbb{Q}$ by setting

$$P_{\text{abs}}(x) = \prod_{j=1}^{n} \sigma_j(P_{\text{rel}}(x)) \in \mathbb{Z}[x]$$

where the σ_j are the embeddings of $\mathbb{K}$. From either of these polynomials we can check as before that it defines a subfield of $\mathbb{K}^+(\mathfrak{f})$ or an abelian extension of $\mathbb{K}$ unramified outside of $\mathfrak{f}$ and outside of the archimedian places. It is also easy to check that the polynomial P_{abs} has the correct number of roots on the unit circle by computing approximate values for the roots of P_{abs}.

IV.1.4 Algorithms to screen for simple and optimal ATR fields

Let us now describe how we obtained tables of ATR fields $\mathbb{K}$ together with moduli $\mathfrak{f}$ which offer an optimal setting for the computation of higher elliptic units (see section III.4.2.2). To search for these optimal examples, we shall assume that we are provided with some iterator over integral polynomials of fixed degree n. This iterator might be very broad, describing all polynomials of degree n with coefficients between $-A$ and A for some $A > 0$ or describe particular polynomials such as the pure cubic polynomials $x^3 - m$ for $1 \leq m \leq A$. To gain time, it might be useful to use an iterator built from a list of all pre-computed polynomials defining a number field of degree n with exactly one complex place with discriminant less than A for some $A > 0$, with possibly additional conditions on the class number for instance. Such lists may be found on the LMFDB [LMF24] for small degrees.

Algorithm 6: Search for optimal settings
Input: a degree $n \geq 3$ and an iterator $g : \mathbb{N} \to \mathbb{Z}_n[X] = \{P \in \mathbb{Z}[X] \mid \deg(P) = n\}$.
Output: a list L of vectors $[\mathbb{K}, \mathfrak{f}_{list}, \varepsilon_{list}]$ where $\mathbb{K}$ is a number field of degree n with exactly one complex place, $\mathfrak{f}_{list} = [\mathfrak{f}_1, \ldots, \mathfrak{f}_l]$ contains class field moduli and $\varepsilon_{list} = [\varepsilon_1, \ldots, \varepsilon_{n-2}]$ contains fundamental units for $\mathcal{O}_{\mathbb{K}}^{+,\times}$ such that the data $(\mathbb{K}, \mathfrak{f}_i, \varepsilon_{list})$ satisfies optimal conditions for the construction of elliptic units.
Initialise a list $L \leftarrow List()$;
Then, for each polynomial P described by the iterator g do the following:

1. If P is reducible, move on to the next polynomial.
2. Compute the number r of real roots of P. If $r \neq n - 2$, move on to the next polynomial.
3. Let $\mathbb{K} = \mathbb{Q}[X]/(P)$. Compute a $\mathbb{Z}$-basis of the maximal order $\mathcal{O}_{\mathbb{K}}$ of $\mathbb{K}$.
4. Compute a set $\varepsilon_1, \ldots, \varepsilon_{n-2}$ of LLL-reduced fundamental units for $\mathcal{O}_{\mathbb{K}}^{+,\times}$ using Buchmann's algorithm and the LLL algorithm for reduction (see the **bnfinit** command in Pari/GP for these last two steps).
5. For each permutation $\rho \in \mathfrak{S}_r$ use Algorithm 1 to compute $[\tilde{\lambda}_\rho, \tilde{a}_\rho, \mathfrak{D}(\tilde{a}_\rho, \mathcal{O}_{\mathbb{K}}), \tilde{t}, \underline{\tilde{\alpha}}]$, and, if at any point $\tilde{\lambda}_\rho > 1$, break and move on to the next polynomial.
6. Compute the prime factorisation of the ideal $J(\mathbb{K}) = \sum_{j=1}^{r} (\varepsilon_j - 1)\mathcal{O}_{\mathbb{K}}$ (see (III.39)) and keep only the prime ideals $\mathfrak{f}_1, \ldots, \mathfrak{f}_l$ dividing J for which the hypotheses of the main conjecture III.37 are satisfied. If no such prime factor may be found, move on to the next polynomial. Otherwise, add the data $[\mathbb{K}, [\mathfrak{f}_1, \ldots, \mathfrak{f}_l], [\varepsilon_1, \ldots, \varepsilon_{n-2}]]$ to the list L.

Let us make a few remarks on Algorithm 6. In practice, as the optimal conditions depend on the choice of fundamental units $\varepsilon_1, \ldots, \varepsilon_{n-2}$ for $\mathcal{O}_{\mathbb{K}}^{+,\times}$, it may sometimes be worth it to try more than one choice for $\varepsilon_1, \ldots, \varepsilon_{n-2}$ by performing small base changes around an LLL-reduced choice of fundamental units. As a second remark, the optimal setting described in section III.4.2.2 does not guarantee that the computations will be feasible in a reasonable amount of time. Indeed, the computation time for the higher elliptic units is linear in the value $\max(\tilde{t}_\rho)$ and it often happens that the overflows $\tilde{t}_\rho$ are very large and lead to unreasonable computations. For instance, the pure cubic field $\mathbb{Q}(z = e^{2i\pi/3}\sqrt[3]{93})$ has fundamental unit $\varepsilon = 15001z^2 - 64428z - 16022$ and the corresponding value for $\tilde{t}$ is 6488331019940189336786019529‌91 while $\tilde{\lambda} = 1$. Thus, in practice, we modify Algorithm 6 so that the potential examples with high values of $\tilde{t}$ are discarded.

Using this algorithm we have already obtained more than 10000 complex cubic fields, several thousands of quartic and quintic fields and a few hundreds of degree 6 fields satisfying these optimal conditions. The best scenario we found for degree 7 fields was a case where 116 out of the 120 conditions $\tilde{\lambda}_\rho = 1$ for $\rho \in \mathfrak{S}_5$ were satisfied, and in that case the values of $\tilde{t}_\rho$ were too large to hope we could compute the higher elliptic units.

IV.2 Numerical Evidence

In this section, we provide numerical examples to support our conjecture. They may be computed with high precision in a low amount of time. In what follows, we will give computation times for 1000 digits precision on a personal computer. Computations[1] were carried out using number fields found in the LMFDB database [LMF24] as well as the computer algebra system PARI/GP [The24], making extensive use of algebraic number theory tools it provides.

In what follows, we will define our fields as $\mathbb{K} = \mathbb{Q}(z)$ where z is the complex root of some polynomial $P = (x - z)(x - \bar{z}) \prod_{j=1}^{r}(x - z_j) \in \mathbb{Q}[x]$ lying in the upper half-plane (here the z_j are real numbers with $z_1 < \cdots < z_r$). Thus, to define our orientations we will fix the ordering on the real embeddings of $\mathbb{K}$ such that $\sigma_j(z) = z_j$ for $1 \leq j \leq r$. Throughout this section, we focus mainly on computations for $\mathfrak{b} = (1)$ because most of the work on the computations of higher elliptic units does not depend on $\mathfrak{b}$. For ease of presentation, we will often write prime ideals above a prime p as $\mathfrak{P}_p, \mathfrak{P}'_p, \mathfrak{P}''_p, \ldots$. In this section we provide examples for fields of degree $3 \leq n \leq 6$. We first describe examples in an optimal setting then a few quartic examples in a non-optimal setting. We end this section with examples that fall outside of the hypotheses of the Main Conjecture III.37 to show how we can handle some of those cases. For ease of presentation, all examples are presented in the form of the alternative version of the conjecture (see Conjecture III.38) where the averaging is done over the set $\mathcal{Z}_{\mathfrak{f}}^{1,+}$. We also focus on "big smoothings", that is on smoothing ideals of prime norm $N > n + 1$ for which $\mathcal{D}(N, n) = 1$. Lastly, for clarity of presentation, we will write absolute polynomials over $\mathbb{Q}$ as polynomials in the variable x and relative polynomials over $\mathcal{O}_{\mathbb{K}}$ as polynomials in the variable X.

IV.2.1 Cubic examples in optimal settings

Here we present five examples in the cubic case where the setting is optimal (see section III.4.2.2). We first present a very detailed example to showcase the general computation process. The second example is one of the simplest ones, with $\tilde{t} = 1$ and class number one. The third example is a simple example presented in the general introduction where $q = 5$ and the narrow ray class group at $\mathfrak{f}$ is represented by the classes in $(\mathcal{O}_{\mathbb{K}}/\mathfrak{f})^\times$. The fourth example showcases the work on helper ideals when the usual class group of $\mathbb{K}$ is large. In the fifth example we present a case where $q = 7$ and the narrow ray class group at $\mathfrak{f}$ is large.

[1] A short version of the code for the computation of the ordinary G_r functions is available at https://plmlab.math.cnrs.fr/pmorain/computations_of_higher_elliptic_units. A longer version of the code which implements the algorithms from section IV.1.2 will be available in the future.

IV.2.1.1 Example 1

We first discuss in full detail a cubic example. Let z be the complex root of the polynomial x^3-13 lying in the upper half-plane. The complex cubic field $\mathbb{K}=\mathbb{Q}(z)$ has usual class number 3. We choose the ideal $\mathfrak{f}$ of norm $q=3$ such that $\mathfrak{f}^3=(3)$. The corresponding narrow ray class group is $\mathrm{Cl}^+(\mathfrak{f})\simeq\mathbb{Z}/6\mathbb{Z}$. The unit group $\mathcal{O}_{\mathbb{K}}^{+,\times}=\mathcal{O}_{\mathfrak{f}}^{+,\times}$ is generated by $\varepsilon=2z^2-3z-4$. We fix the positive $\mathbb{Z}$-basis $\tilde{B}=[1,2z^2-3z-4,-z^2+z]=[1,\varepsilon,-z^2+z]$ of $\mathcal{O}_{\mathbb{K}}$.

Application of Algorithm 1: Since $\det_{\tilde{B}}(1,\varepsilon,-z^2+z)=1$ the content of the unit system $(u_1)=(\varepsilon)$ is $\tilde{\lambda}=1$ and in $\tilde{B}$ the linear form $\tilde{a}$ is expressed as $\tilde{a}=(0,0,1)$. We compute the matrix $\widetilde{\mathcal{M}}$:

$$\widetilde{\mathcal{M}}=\begin{pmatrix}0&0&1\\0&-131&57\\1&57&-29\end{pmatrix}$$

such that $\widetilde{\mathcal{M}}^{-1}$ represents the fractional ideal $\mathfrak{D}(\tilde{a},\mathcal{O}_{\mathbb{K}})^{-1}$ in $\tilde{B}$. It follows from this computation that the overflow of the unit system (ε) is $\tilde{t}=131$ and that $\tilde{s}=131$. Since there is only one integral ideal in $\mathcal{O}_{\mathbb{K}}$ of norm 131 we may identify it with $\mathfrak{D}(\tilde{a},\mathcal{O}_{\mathbb{K}})$.

Application of Algorithm 2: Here we asked for a smoothing ideal $\mathfrak{a}$ coprime to 6 and we found $\mathfrak{a}=\mathfrak{P}_5$ the unique prime ideal of norm $N=5$ in $\mathcal{O}_{\mathbb{K}}$. The narrow ray class group at $\mathfrak{f}$ satisfies the exact sequence:

$$1\to\mathbb{Z}/3\mathbb{Z}^\times\to\mathrm{Cl}^+(\mathfrak{f})\to\mathrm{Cl}^+(\mathbb{K})\to 1.$$

A possible set of representatives for the six classes in $\mathrm{Cl}^+(\mathfrak{f})$ corresponding to this exact sequence is:

$$\{\mathcal{O}_{\mathbb{K}},\mathfrak{P}_2,\mathfrak{P}_2^2,2.\mathcal{O}_{\mathbb{K}},2.\mathfrak{P}_2,2.\mathfrak{P}_2^2\}.$$

where $\mathfrak{P}_2$ is the unique prime ideal of norm 2 in $\mathcal{O}_{\mathbb{K}}$. We shall then define $\mathfrak{b}_1=\mathcal{O}_{\mathbb{K}}$, $\mathfrak{b}_2=\mathfrak{P}_2$ and $\mathfrak{b}_3=\mathfrak{P}_2^2$.

Application of Algorithms 3 and 4: Let $\mathfrak{P}_7=7\mathcal{O}_{\mathbb{K}}+(z-3)\mathcal{O}_{\mathbb{K}}$ be a prime ideal of norm 7 in $\mathcal{O}_{\mathbb{K}}$ and $\mathfrak{P}_{11}$ be the unique prime ideal of norm 11 in $\mathcal{O}_{\mathbb{K}}$. Then we may associate to all classes modulo integers a helper ideal and a strongly admissible base point h:

Classes	Helper	Admissible base point h
$\mathfrak{b}_1=\mathcal{O}_{\mathbb{K}}$	$\mathcal{O}_{\mathbb{K}}$	$h_1=-21z^2-42z-114$
$\mathfrak{b}_2=\mathfrak{P}_2$	$7/\mathfrak{P}_7$	$h_2=(87z^2+9z-477)/2$
$\mathfrak{b}_3=\mathfrak{P}_2^2$	$11/\mathfrak{P}_{11}$	$h_3=-(3z^2+309z-807)/4$

The corresponding levels are $\ell_1=3\cdot5\cdot131=1965$, $\ell_2=7\cdot\ell_1=13755$ and $\ell_3=11\cdot\ell_1=21615$. We may compute the values

$$\Gamma^+_{\mathfrak{f},(1),\mathfrak{a}}(\varepsilon,h_1)=\frac{\Gamma\left(\frac{1}{3},\frac{\varepsilon^{-1}+5348}{1965},\frac{\varepsilon+467}{1965}\right)^5}{\Gamma\left(\frac{5}{3},\frac{\varepsilon^{-1}+5348}{393},\frac{\varepsilon+467}{393}\right)}\approx-0.0660917...+i\cdot0.0932299...$$

$$\Gamma^+_{\mathfrak{f},\mathfrak{P}_2,\mathfrak{a}}(\varepsilon,h_2)=\frac{\Gamma\left(\frac{-1}{3},\frac{\varepsilon^{-1}+1418}{13755},\frac{\varepsilon-3463}{13755}\right)^5}{\Gamma\left(\frac{-5}{3},\frac{\varepsilon^{-1}+1418}{2751},\frac{\varepsilon-3463}{2751}\right)}\approx0.0059953...+i\cdot0.0047179...$$

$$\Gamma^+_{\mathfrak{f},\mathfrak{P}_2^2,\mathfrak{a}}(\varepsilon,h_3)=\frac{\Gamma\left(\frac{1}{3},\frac{\varepsilon^{-1}-547}{21615},\frac{\varepsilon+6362}{21615}\right)^5}{\Gamma\left(\frac{5}{3},\frac{\varepsilon^{-1}-547}{4323},\frac{\varepsilon+6362}{4323}\right)}\approx-289.3045814...-i\cdot127.6382732...$$

$$\Gamma^+_{\mathfrak{f},(2),\mathfrak{a}}(\varepsilon, 2.h_1) = \frac{\Gamma\left(\frac{2}{3}, \frac{\varepsilon^{-1}+5348}{1965}, \frac{\varepsilon+467}{1965}\right)^5}{\Gamma\left(\frac{10}{3}, \frac{\varepsilon^{-1}+5348}{393}, \frac{\varepsilon+467}{393}\right)} \approx -5.0606452... - i \cdot 7.1386178...$$

$$\Gamma^+_{\mathfrak{f},2.\mathfrak{P}_2,\mathfrak{a}}(\varepsilon, 2.h_2) = \frac{\Gamma\left(\frac{-2}{3}, \frac{\varepsilon^{-1}+1418}{13755}, \frac{\varepsilon-3463}{13755}\right)^5}{\Gamma\left(\frac{-10}{3}, \frac{\varepsilon^{-1}+1418}{2751}, \frac{\varepsilon-3463}{2751}\right)} \approx 103.0063956... - i \cdot 81.0605592...$$

$$\Gamma^+_{\mathfrak{f},2.\mathfrak{P}_2^2,\mathfrak{a}}(\varepsilon, 2.h_3) = \frac{\Gamma\left(\frac{2}{3}, \frac{\varepsilon^{-1}-547}{21615}, \frac{\varepsilon+6362}{21615}\right)^5}{\Gamma\left(\frac{5}{3}, \frac{\varepsilon^{-1}-547}{4323}, \frac{\varepsilon+6362}{4323}\right)} \approx -0.0028933... + i \cdot 0.0012765...$$

We may remark immediately that $\Gamma^+_{\mathfrak{f},2.\mathfrak{P}_2^{j-1},\mathfrak{a}}(\varepsilon, 2.h_j) = \Gamma^+_{\mathfrak{f},\mathfrak{P}_2^{j-1},\mathfrak{a}}(\varepsilon, h_j)^{-1}$ for $1 \leq j \leq 3$. Let us then define $\mathcal{W}_j = \Gamma^+_{\mathfrak{f},\mathfrak{P}_2^{j-1},\mathfrak{a}}(\varepsilon, h_j)$ for $1 \leq j \leq 3$. Using these values we may identify up to 1000 digits of precision the relative polynomial in $\mathcal{O}_{\mathbb{K}}[X]$:

$$\begin{aligned} P_{\text{rel}} = \prod_{j=1}^{3}(X - \mathcal{W}_j)(X - \mathcal{W}_j^{-1}) \approx{} & (X^6+1) + (-34z^2 + 26z + 128)(X^5 + X) \\ & + (1127z^2 + 8879z - 27106)(X^4 + X^2) \\ & + (40740z^2 - 16965z - 185350)X^3. \end{aligned}$$

Alternatively, using the **algdep** command shows that all values $\mathcal{W}_j$ and $\mathcal{W}_j^{-1}$ coincide up to high precision with 6 out of the 18 roots of the absolute palindromic polynomial

$$\begin{aligned} P_{\text{abs}} = {} & x^{18} + 384x^{17} + 2310x^{16} - 10646490x^{15} + 1596241353x^{14} + 18608357181x^{13} \\ & + 156933809421x^{12} + 215098256580x^{11} + 381407365338x^{10} + 338205493469x^9 \\ & + 381407365338x^8 + 215098256580x^7 + 156933809421x^6 + 18608357181x^5 \\ & + 1596241353x^4 - 10646490x^3 + 2310x^2 + 384x + 1 \end{aligned}$$

which defines the degree 6 extension $\mathbb{K}^+(\mathfrak{f})/\mathbb{K}$. The computation time for 1000 digits is 15 seconds for $\mathcal{W}_1$, 65 seconds for $\mathcal{W}_2$ and 105 seconds for $\mathcal{W}_3$. We may also check formula (III.42) as

$$5\zeta_{\mathfrak{f}}'([\mathcal{O}_{\mathbb{K}}], 0) - \zeta_{\mathfrak{f}}'([\mathfrak{a}], 0) = \log\left|\Gamma^+_{\mathfrak{f},(1),\mathfrak{a}}(\varepsilon, h_1)\right|^2 \approx -4.3382052...$$

and similarly for the other classes. In addition, we can easily check that up to high precision the roots of P_{abs} are

$$\mathcal{W}_j, \mathcal{W}_j^{-1}, \overline{\mathcal{W}_j}, \overline{\mathcal{W}_j}^{-1} \text{ for } j = 1, 2, 3$$

together with 6 roots $z_1, z_1^{-1}, z_2, z_2^{-1}, z_3, z_3^{-1}$ which lie on the unit circle. This is coherent with the third point in the Main Conjecture III.37.

IV.2.1.2 Example 2

We now discuss one of our simplest cubic examples. Let z be the complex root of the polynomial $x^3 - 2$ in the upper half-plane. Then $\mathbb{K} = \mathbb{Q}(z)$ has class number 1. We choose the ideal $\mathfrak{f}$ as the unique ideal of norm $q = 3$ in $\mathcal{O}_{\mathbb{K}}$. The corresponding narrow ray class group is $\mathrm{Cl}^+(\mathfrak{f}) \simeq \mathbb{Z}/2\mathbb{Z}$. The unit $\varepsilon = z^2 + z + 1$ is the generator for $\mathcal{O}_{\mathbb{K}}^{+,\times} = \mathcal{O}_{\mathfrak{f}}^{+,\times}$

satisfying $\sigma_{\mathbb{R}}(\varepsilon) > 1$. The associated content is $\tilde{\lambda} = 1$ and the overflow is $\tilde{t} = 1$. We choose $\mathfrak{b} = (1)$ and $\mathfrak{a}$ the degree one prime above $N = 5$ in $\mathbb{K}$. The ideal $qN/\mathfrak{a}$ has an admissible generator $h = -3z^2 + 6z + 3$. The corresponding level is $\ell = 15$. The value

$$\Gamma^{+}_{\mathfrak{f},\mathfrak{b},\mathfrak{a}}(\varepsilon, h) = \frac{\Gamma\left(\frac{1}{3}, \frac{\varepsilon+2}{15}, \frac{\varepsilon^{-1}-7}{15}\right)^5}{\Gamma\left(\frac{5}{3}, \frac{\varepsilon+2}{3}, \frac{\varepsilon^{-1}-7}{3}\right)} \approx -1.2937005... + i \cdot 1.4743341...$$

coincides up to 1000 digits of precision with a root of the polynomial

$$P_{\text{abs}} = x^6 + 3x^5 + 6x^4 + 5x^3 + 6x^2 + 3x + 1$$

which defines an absolute equation of $\mathbb{K}^{+}(\mathfrak{f})$ over $\mathbb{Q}$. The computation time for 1000 digits is 1 second. As this is a simple example we may write explicitly what the roots of the polynomial P are and we expect the following equality to hold:

$$\frac{\Gamma\left(\frac{1}{3}, \frac{\varepsilon+2}{15}, \frac{\varepsilon^{-1}-7}{15}\right)^5}{\Gamma\left(\frac{5}{3}, \frac{\varepsilon+2}{3}, \frac{\varepsilon^{-1}-7}{3}\right)} =^{?} \frac{z - 1 - \sqrt{z^2 - 2z - 3}}{2}$$

where $z = 2^{1/3} e^{2i\pi/3}$.

IV.2.1.3 Example 3

Here we explain how we have obtained the example presented in the general introduction with $q = 5$. Let $z = e^{2i\pi/3} 10^{1/3}$ be the root of the polynomial $x^3 - 10$ in the upper half-plane. Then $\mathbb{K} = \mathbb{Q}(z)$ has class number 1. We choose the ideal $\mathfrak{f}$ as the unique ideal of norm $q = 5$ in $\mathcal{O}_{\mathbb{K}}$. The corresponding narrow ray class group is $\mathrm{Cl}^{+}(\mathfrak{f}) \simeq \mathbb{Z}/4\mathbb{Z}$. The unit $\varepsilon = (2z^2 - z - 7)/3$ is a generator for $\mathcal{O}_{\mathbb{K}}^{+,\times} = \mathcal{O}_{\mathfrak{f}}^{+,\times}$ with content $\tilde{\lambda} = 1$ and overflow $\tilde{t} = 9$ and the different ideal $\mathfrak{D}(\tilde{a}, \mathcal{O}_{\mathbb{K}})$ is equal to $\mathfrak{P}_3^2$ where $\mathfrak{P}_3$ is the unramified prime of norm 3 in $\mathcal{O}_{\mathbb{K}}$. We may choose $\mathfrak{b} = (1)$ and $\mathfrak{a}$ the unique prime ideal of norm $N = 11$ in $\mathbb{K}$. The base point $h = -(35z^2 + 20z + 35)/3$ is a strongly admissible generator for the ideal $qN\mathfrak{P}_3^2/\mathfrak{a}$. The values

$$\Gamma^{+}_{\mathfrak{f},k.\mathfrak{b},\mathfrak{a}}(\varepsilon, h) = \frac{\Gamma\left(\frac{k}{5}, \frac{\varepsilon^{-1}-1751}{495}, \frac{\varepsilon-776}{495}\right)^{11}}{\Gamma\left(\frac{11k}{5}, \frac{\varepsilon^{-1}-1751}{45}, \frac{\varepsilon-776}{45}\right)} \approx \begin{cases} -27.5333588... - i \cdot 32.7146180... & \text{for } k = 1 \\ -2.2349933... - i \cdot 4.9384566... & \text{for } k = 2 \\ -0.0760627... + i \cdot 0.1680687... & \text{for } k = 3 \\ -0.0150592... + i \cdot 0.0178931... & \text{for } k = 4 \end{cases}$$

each coincide up to 1000 digits of precision with a different root of the palindromic polynomial

$$P_{\text{abs}} = x^{12} + 57x^{11} + 1956x^{10} + 4640x^9 + 35415x^8 - 109818x^7 + 150139x^6 + \ldots$$

defining an absolute equation of $\mathbb{K}^{+}(\mathfrak{f})$ over $\mathbb{Q}$. Alternatively, we may identify the relative polynomial

$$P_{\text{rel}} = \prod_{k=1}^{4} (X - \Gamma^{+}_{\mathfrak{f},k.\mathfrak{b},\mathfrak{a}}(\varepsilon, h))$$

in $\mathcal{O}_{\mathbb{K}}[X]$ as

$$P_{\text{rel}} \approx X^4 + (-7z^2 + 5z + 19)X^3 + (-19z^2 + 70z - 59)X^2 + (-7z^2 + 5z + 19)X + 1$$

using the **lindep** command. The computation time for these values for 1000 digits of precision is 7 seconds.

IV.2.1.4 Example 4

Let z be the complex root of the polynomial $x^3 - 65$ in the upper half-plane. Then $\mathbb{K} = \mathbb{Q}(z)$ has class number 18. This means that most ideals won't be principal ideals and we will need to find many helper ideals to build our strongly admissible base points h_ρ. We choose the ideal $\mathfrak{f}$ to be the unique ideal of norm $q = 3$ in $\mathcal{O}_\mathbb{K}$. The corresponding narrow ray class group is $\mathrm{Cl}^+(\mathfrak{f}) \simeq \mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$. The unit $\varepsilon = z - 4$ is a generator for $\mathcal{O}_\mathbb{K}^{+,\times} = \mathcal{O}_\mathfrak{f}^{+,\times}$ with content $\tilde{\lambda} = 1$ and overflow $\tilde{t} = 1$. We choose $\mathfrak{b} = (1), \mathfrak{b} = \mathfrak{P}_{59}, \ldots$ representatives for the 36 classes in $\mathrm{Cl}^+(\mathfrak{f})$, and $\mathfrak{a}$ the unique prime ideal of norm $N = 5$ in $\mathcal{O}_\mathbb{K}$. We must now search for 18 helper ideals spanning the wide class group at $\mathfrak{f}$. For the trivial class we may choose $\mathfrak{H} = 13/\mathfrak{P}_{13}$ where $\mathfrak{P}_{13}$ is the unique prime ideal of norm 13. For the class represented by $\mathfrak{b} = \mathfrak{P}_{59}$ we may choose $\mathfrak{H} = 41/\mathfrak{P}_{41}$ where $\mathfrak{P}_{41}$ is the unique prime ideal of norm 41, and so on for the remaining classes. The corresponding admissible generators we found for $(qN/\mathfrak{a}) \times (13/\mathfrak{P}_{13})$ and $qN/(\mathfrak{a}\mathfrak{P}_{59}) \times (41/\mathfrak{P}_{41})$ are $h_{\mathfrak{b}=(1)} = 3z^2$ and $h_{\mathfrak{b}=\mathfrak{P}_{59}} = -(72z^2 + 75z + 1590)/59$ respectively. The corresponding levels will be $\ell_{\mathfrak{b}=(1)} = 3 \cdot 5 \cdot 13 = 195$, $\ell_{\mathfrak{b}=\mathfrak{P}_{59}} = 3 \cdot 5 \cdot 41 = 615$. We may compute the values

$$\Gamma^+_{\mathfrak{f},(1),\mathfrak{a}}(\varepsilon, h_{\mathfrak{b}=(1)}) = \frac{\Gamma\left(\frac{1}{3}, \frac{\varepsilon^{-1}-211}{195}, \frac{\varepsilon-61}{195}\right)^5}{\Gamma\left(\frac{5}{3}, \frac{\varepsilon^{-1}-211}{39}, \frac{\varepsilon-61}{39}\right)} \approx -1.6691052... + i \cdot 5.7493283...$$

$$\Gamma^+_{\mathfrak{f},\mathfrak{P}_{59},\mathfrak{a}}(\varepsilon, h_{\mathfrak{b}=\mathfrak{P}_{59}}) = \frac{\Gamma\left(\frac{-1}{3}, \frac{\varepsilon^{-1}+1034}{615}, \frac{\varepsilon-91}{615}\right)^5}{\Gamma\left(\frac{-5}{3}, \frac{\varepsilon^{-1}+1034}{123}, \frac{\varepsilon-91}{123}\right)} \approx 0.0344135... - i \cdot 0.0123218...$$

and the remaining 34 out of 36 values $\mathcal{W}_{\mathfrak{f},k.\mathfrak{b},\mathfrak{a}} = \Gamma^+_{\mathfrak{f},\mathfrak{b},\mathfrak{a}}(\varepsilon, k.h_\mathfrak{b})$ attached to the 36 classes in $\mathrm{Cl}^+(\mathfrak{f})$. We may compute and identify up to high precision the relative polynomial:

$$\prod_{k=1}^{2} \prod_{\mathfrak{b}\in \mathrm{Cl}^+(\mathfrak{f})/\mathbb{Z}/3\mathbb{Z}^\times} (X - \mathcal{W}_{\mathfrak{f},k.\mathfrak{b},\mathfrak{a}}) \approx X^{36} - (az^2 + bz + c)X^{35} + \cdots + 1 \in \mathcal{O}_\mathbb{K}[X]$$

where $a = -5967373310133, b = 769211619985, c = 93377174024326$. This palindromic polynomial defines a relative equation of the class field $\mathbb{K}^+(\mathfrak{f})$ above $\mathbb{K}$ and we identify the rest of its coefficients in $\mathcal{O}_\mathbb{K}$. The computation time for 1000 digits and for all of the 36 computations is 8 minutes, which gives 13 seconds per individual computation on average.

It is interesting to note that this field as well as the field defined by $x^3 - 2$ belong to a special family parametrised by $x^3 - 2^{3k} - (-1)^k$ for $k \geq 0$ which behaves nicely compared to other pure cubic fields with respect to the values $\tilde{\lambda}, \tilde{t}$. Generally speaking, the values of these parameters vary significantly with the discriminant, but the fields in this family share interesting properties in that regard. Putting $\mathbb{L}_k = \mathbb{Q}(z_k)$ where z_k is the complex root of the polynomial $x^3 - 2^{3k} - (-1)^k$ lying in the upper half-plane, it seems that the positive fundamental unit in $\mathbb{L}_k$ is given by $\varepsilon_k = (-1)^k(z_k - 2^k)$ and its inverse by $\varepsilon_k^{-1} = z_k^2 + 2^k z_k + 2^{2k}$. When $D_k = 2^{3k} + (-1)^k$ is cube-free (at least for $0 \leq k \leq 100$ except for $k = 49, 50$), because $D_k \not\equiv \pm 1 \bmod 9$, the field $\mathbb{L}_k$ is a so-called pure cubic field of the first kind and an integral basis of $\mathcal{O}_{\mathbb{L}_k}$ is given by $(1, z_k, z_k^2/s_k)$ where $D_k = r_k s_k^2$ with r_k, s_k square-free and relatively coprime. In that case, we may compute $\tilde{\lambda} = 1$ and $\tilde{t} = s_k$. When D_k is also square-free (at least for $0 \leq k \leq 100$ except for $k = 7, 10, 21, 26, 30, 35, 63, 68, 70, 77, 78, 90, 91$) we get $\tilde{\lambda}_k = \tilde{t}_k = 1$, which gives a very simple setting for the computations. The class group grows very rapidly in this family as showcased by the following table of the class numbers of the first few of these fields: Other particularly striking examples of huge class groups are given for $\mathbb{L}_{11}$ and $\mathbb{L}_20$ as

k	0	1	2	3	4	5
P_k	x^3-2	x^3-7	x^3-65	x^3-511	x^3-4097	$x^3-32767$
$h(\mathbb{L}_k)$	1	3	2.3^2	$2^2.3^3$	$2^3.3^4$	$2^3.3^3.5^2$

$h(\mathbb{L}_{11}) = 2^2 \cdot 3^5 \cdot 5 \cdot 3191$ and $h(\mathbb{L}_{20}) = 2^6 \cdot 3^4 \cdot 5 \cdot 11 \cdot 19 \cdot 79 \cdot 863 \cdot 18047$. The fact that the class group is large renders the computation of higher elliptic units more difficult as we need to perform the computations for each class to correctly identify a relative polynomial with huge coefficients. Indeed, we have successfully computed the higher elliptic units for the fields $\mathbb{L}_k$, $k = 0, 1, 2, 3, 4$, but already for $k = 5$ the relative polynomial we wish to recognise has expected degree 10800 and huge coefficients, thus we would need a lot of precision in the computations of the higher elliptic units $\mathcal{W}_{\mathfrak{f},\mathfrak{c},\mathfrak{a}}$ to correctly identify it.

Nethertheless, it would be interesting to understand if for $k \geq 100$ there are infinitely many fields in this family for which the conditions $\tilde{\lambda} = 1$, $\tilde{t} = 1$ are satisfied for the fundamental unit. Our construction would then conjecturally allow to construct algebraic units of very high degree.

IV.2.1.5 Example 5

Here we give an example in an optimal setting $(\mathbb{K}, \mathfrak{f})$ with $q = 7$. Let z be the complex root of the polynomial $x^3 - 14$ in the upper half-plane. Then $\mathbb{K} = \mathbb{Q}(z)$ has class number 3. We choose the ideal $\mathfrak{f}$ as the unique prime ideal of norm $q = 7$ in $\mathcal{O}_{\mathbb{K}}$. The corresponding narrow ray class group is $\mathrm{Cl}^+(\mathfrak{f}) \simeq \mathbb{Z}/18\mathbb{Z}$. The unit $\varepsilon = -z^2 + 2z + 1$ is a generator for $\mathcal{O}_{\mathbb{K}}^{+,\times} = \mathcal{O}_{\mathfrak{f}}^{+,\times}$ with content $\tilde{\lambda} = 1$ and overflow $\tilde{t} = 2 \cdot 11$. We choose $\mathfrak{b} = (1)$ and $\mathfrak{a}$ the unique integral ideal of norm $N = 5$ in $\mathcal{O}_{\mathbb{K}}$. The different ideal $\mathfrak{D}(\tilde{a}, \mathcal{O}_{\mathbb{K}})$ is equal to the product $\mathfrak{P}_2\mathfrak{P}_{11}$ where $\mathfrak{P}_2$ and $\mathfrak{P}_{11}$ are the unique integral ideals of norm 2 and 11 respectively. The ideal $qN\mathfrak{P}_2\mathfrak{P}_{11}/(\mathfrak{a}\mathfrak{b})$ is unfortunately not principal so we need to look for a helper ideal. In Proposition III.25 we may take $m = 3$ and $\mathfrak{H} = 3/\mathfrak{P}_3$ where $\mathfrak{P}_3$ is the unique integral ideal of norm 3. The ideal $(3qN\mathfrak{P}_2\mathfrak{P}_{11}/(\mathfrak{a}\mathfrak{b})) \times \mathfrak{H}$ is principal with an admissible generator $h = -21z^2 + 21z - 336$. The corresponding level of the computation is $\ell = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 = 2310$. The value

$$\Gamma^+_{\mathfrak{f},\mathfrak{b},\mathfrak{a}}(\varepsilon, h) = \frac{\Gamma\left(\frac{-3}{7}, \frac{\varepsilon^{-1}-3067}{2310}, \frac{\varepsilon+1007}{2310}\right)^5}{\Gamma\left(\frac{-15}{7}, \frac{\varepsilon^{-1}-3067}{462}, \frac{\varepsilon+1007}{462}\right)} \approx -0.1700923... + i \cdot 3.8609499...$$

coincides up to high precision with a root of the palindromic polynomial

$$P_{\text{abs}} = x^{54} - 4167x^{53} + 7931535x^{52} - 259219286x^{51} + \cdots - 4167x + 1$$

which defines an absolute equation of $\mathbb{K}^+(\mathfrak{f})$ over $\mathbb{Q}$. This polynomial has very large coefficients, and we could alternatively compute the remaining 17 out of 18 values associated to the 18 classes in $\mathrm{Cl}^+(\mathfrak{f})$ to identify a relative polynomial in $\mathcal{O}_{\mathbb{K}}[X]$ instead, as we did in the previous example. The computation time for 1000 digits and for all 18 values is 18 minutes, which gives 1 minute per individual computation on average.

IV.2.2 Quartic examples in optimal settings

We now present four examples in quartic optimal settings. The first example is one of the simplest cases, for which the overflows are $\tilde{t}_1 = \tilde{t}_2 = 1$ and it is a variation of the example presented in section III.4.1 with a larger smoothing. The second example showcases the computation of different ideals with overflows $\tilde{t}_1 = 7, \tilde{t}_2 = 7129$. The third example is a variation on the second example where we improve the choice of fundamental units to obtain the smaller overflows $\tilde{t}_1 = 1$ and $\tilde{t}_2 = 25$. The fourth example is a case where the quartic field $\mathbb{K}$ contains a real quadratic field and some of the parameters $\tilde{\alpha}_j$ belong to a real subfield of $\mathbb{K}$.

IV.2.2.1 Example 6

We first revisit one of the simplest quartic examples, for which we presented a computation of a 5-smoothed higher elliptic unit in section III.4.1. This time we shall perform the computations with a "big smoothing" to illustrate the differences in behaviour. Let z be the complex root of the polynomial $x^4 - 6x^3 - x^2 - 3x + 1$ lying in the upper half-plane. Then $\mathbb{K} = \mathbb{Q}(z)$ has class number 1. We choose $\mathfrak{f} = \mathfrak{P}_2$ the unique prime ideal of norm $q = 2$ in $\mathcal{O}_{\mathbb{K}}$. The corresponding narrow ray class group is $\mathrm{Cl}^+(\mathfrak{f}) \simeq \mathbb{Z}/2\mathbb{Z}$. We choose the fundamental units

$$\varepsilon_1 = \frac{-2z^3 + 13z^2 - z + 3}{7}, \quad \varepsilon_2 = \frac{-5z^3 + 29z^2 + 15z + 18}{7}$$

for the set $\mathcal{O}_{\mathbb{K}}^{+,\times} = \mathcal{O}_{\mathfrak{f}}^{+,\times}$ of totally positive units (congruent to 1 mod $\mathfrak{f}$). We compute the contents $\tilde{\lambda}_1 = \tilde{\lambda}_2 = 1$, the overflows $\tilde{t}_1 = \tilde{t}_2 = 1$ as defined in section III.3.2. We may choose $\mathfrak{b} = (1)$ and set $\mathfrak{a}$ to be the unique integral ideal of norm $N = 13$ in $\mathcal{O}_{\mathbb{K}}$. The ideal $qN/\mathfrak{a}$ has an admissible generator $h_1 = h_2 = (44z^3 - 258z^2 - 104z - 80)/7$ (see Definition III.4). The base points h_1 and h_2 form a compatible set in the sense of Definition III.24. The corresponding levels will be $\ell_1 = \ell_2 = 2 \cdot 13 = 26$. Let us write the parameters

$$\begin{aligned}
\tau &= \varepsilon_2 - 15, & \tau' &= -6 + \frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_1\varepsilon_2} \\
\sigma &= -7 + \frac{1}{\varepsilon_2}, & \sigma' &= -\varepsilon_2 + 15 \\
\rho &= -\varepsilon_1 - 3, & \rho' &= 4\varepsilon_1 + 19 - \frac{1}{\varepsilon_2}
\end{aligned}$$

Then

$$\frac{G^+_{2,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(\varepsilon_2, \varepsilon_1\varepsilon_2; h_2)}{G^-_{2,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(\varepsilon_1, \varepsilon_1\varepsilon_2; h_1)} = \frac{G_2\left(\frac{-1}{2}, \frac{\tau}{26}, \frac{\sigma}{26}, \frac{\rho}{26}\right)^{-13}}{G_2\left(\frac{-13}{2}, \frac{\tau}{2}, \frac{\sigma}{2}, \frac{\rho}{2}\right)^{-1}} \times \frac{G_2\left(\frac{1}{2}, \frac{\tau'}{26}, \frac{\sigma'}{26}, \frac{\rho'}{26}\right)^{13}}{G_2\left(\frac{13}{2}, \frac{\tau'}{2}, \frac{\sigma'}{2}, \frac{\rho'}{2}\right)}$$

coincides up to high precision with the root $\approx 4.1210208... - i \cdot 5.0617720...$ of the polynomial

$$P_{\text{abs}} = x^8 - 7x^7 + 33x^6 + 49x^5 + 17x^4 + 49x^3 + 33x^2 - 7x + 1$$

which defines an absolute equation of $\mathbb{K}^+(\mathfrak{f})$. The computation time for 1000 digits is 6 seconds. Compared to the absolute equation of $\mathbb{K}^+(\mathfrak{f})$ obtained in section III.4.1 this polynomial has smaller coefficients, and the computation time is about the same. We may also check formula (III.42) up to 1000 digits as:

$$13\zeta_{\mathfrak{f}}'(\mathcal{O}_{\mathbb{K}}, 0) - \zeta_{\mathfrak{f}}'([\mathfrak{a}], 0) \approx \log\left|\frac{G^+_{2,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(\varepsilon_2, \varepsilon_1\varepsilon_2; h_2)}{G^-_{2,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(\varepsilon_1, \varepsilon_1\varepsilon_2; h_1)}\right|^2 \approx 3.7519563...$$

Once again, on this simple example we expect the explicit equality:

$$\frac{G^+_{2,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(\varepsilon_2, \varepsilon_1\varepsilon_2; h_2)}{G^-_{2,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(\varepsilon_1, \varepsilon_1\varepsilon_2; h_1)} =^? \frac{\xi + \sqrt{\xi^2 - 4}}{2}$$

where $\xi = (5z^3 - 22z - 57z + 3)/7$.

IV.2.2.2 Example 7

We now discuss in more detail a quartic example with specific work on the computation of the different ideals $\mathfrak{D}(\tilde{a}_\rho, \mathcal{O}_\mathbb{K})$. Let z be the complex root of the polynomial $x^4 - 19x^3 + 18x^2 + 8x + 1$ lying in the upper half-plane. Then $\mathbb{K} = \mathbb{Q}(z)$ has class number 1. We choose $\mathfrak{f} = 3\mathcal{O}_\mathbb{K} + \beta\mathcal{O}_\mathbb{K}$ where $\beta = (5z^3 - 96z^2 + 36z + 28)/9$. The ideal $\mathfrak{f}$ is an integral ideal of norm $q = 3$. The corresponding narrow ray class group is $\mathrm{Cl}^+(\mathfrak{f}) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. We choose the fundamental units

$$\varepsilon_1 = \frac{19z^3 - 366z^2 + 438z + 44}{9}, \ \varepsilon_2 = -z^3 + 19z^2 - 18z - 8$$

for $\mathcal{O}_\mathbb{K}^{+,\times} = \mathcal{O}_\mathfrak{f}^{+,\times}$. We compute two positive $\mathbb{Z}$-bases of $\mathcal{O}_\mathbb{K}$ given by:

$$\tilde{B}_1 = [1, \varepsilon_1, \varepsilon_1\varepsilon_2, \omega = 17z^3 - 330z^2 + 441z + 25]$$
$$\tilde{B}_2 = [1, \varepsilon_2, \varepsilon_1\varepsilon_2, -\omega]$$

(see section III.3.2.2). In these bases we can compute the contents $\tilde{\lambda}_1 = \tilde{\lambda}_2 = 1$, $\tilde{a}_1 = (0, 0, 0, 1)$ in $\tilde{B}_1$ and $\tilde{a}_2 = (0, 0, 0, 1)$ in $\tilde{B}_2$. Let us then compute the matrices $\widetilde{\mathcal{M}}_1$ and $\widetilde{\mathcal{M}}_2$ associated to the different ideals $\mathfrak{D}(\tilde{a}_1, \mathcal{O}_\mathbb{K})$ and $\mathfrak{D}(\tilde{a}_2, \mathcal{O}_\mathbb{K})$ in their respective bases as:

$$\widetilde{\mathcal{M}}_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 3 & 1 \\ 0 & 3 & -2 & -15 \\ 1 & 1 & -15 & 11 \end{pmatrix} \quad \widetilde{\mathcal{M}}_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 15 & -47 & -14 \\ 0 & -47 & -328 & -141 \\ 1 & -14 & -141 & -59 \end{pmatrix}.$$

This readily gives the values for the overflows as $\tilde{t}_1 = \tilde{s}_1 = 7$ and $\tilde{t}_2 = \tilde{s}_2 = 7129$. There is only one integral ideal $\mathfrak{P}_7$ of norm 7 (resp. $\mathfrak{P}_{7129}$ of norm 7129) therefore we obtain $\mathfrak{D}(\tilde{a}_1, \mathcal{O}_\mathbb{K}) = \mathfrak{P}_7$ and $\mathfrak{D}(\tilde{a}_2, \mathcal{O}_\mathbb{K}) = \mathfrak{P}_{7129}$. Let us choose $\mathfrak{b} = \mathcal{O}_\mathbb{K}$ and $\mathfrak{a}$ the unique integral ideal of norm $N = 13$ in $\mathcal{O}_\mathbb{K}$. Both ideals $qN\mathfrak{P}_7/\mathfrak{a}$ and $2.qN\mathfrak{P}_{7129}/\mathfrak{a}$ are principal with strongly admissible generators $h_1 = (-32z^3 + 606z^2 - 543z - 112)/3$ and $h_2 = (124z^3 - 2319z^2 + 1563z + 200)/3$. The base points h_1 and h_2 form a compatible set in the sense of Definition III.24. The corresponding levels of the computations are $\ell_1 = 3 \cdot 7 \cdot 13 = 273$ and $\ell_2 = 3 \cdot 13 \cdot 7129 = 278031$. Let us write the parameters

$$\tau = \frac{2\varepsilon_1\varepsilon_2 - 92\varepsilon_1 - 3}{\varepsilon_1}, \qquad \tau' = \frac{57488\varepsilon_1\varepsilon_2 + 3\varepsilon_2 + 16}{\varepsilon_1\varepsilon_2}$$
$$\sigma = \frac{925\varepsilon_1\varepsilon_2 + 367\varepsilon_1 - 47}{\varepsilon_1\varepsilon_2}, \qquad \sigma' = \frac{47\varepsilon_1\varepsilon_2 - 328\varepsilon_2 - 348694}{1}$$
$$\rho = \frac{-\varepsilon_1\varepsilon_2 - 3\varepsilon_1 + 40}{1}, \qquad \rho' = \frac{-57\varepsilon_1\varepsilon_2 - 90004\varepsilon_2 - 2}{\varepsilon_2}$$

Then

$$\mathcal{W}_{\mathfrak{f},\mathfrak{b},\mathfrak{a}} = \frac{G^+_{2,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(\varepsilon_1, \varepsilon_1\varepsilon_2; h_1)}{G^-_{2,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(\varepsilon_2, \varepsilon_1\varepsilon_2; h_2)} = \frac{G_2\left(\frac{1}{3}, \frac{\tau}{273}, \frac{\sigma}{273}, \frac{\rho}{273}\right)^{13}}{G_2\left(\frac{13}{3}, \frac{\tau}{21}, \frac{\sigma}{21}, \frac{\rho}{21}\right)} \times \frac{G_2\left(\frac{-2}{3}, \frac{\tau'}{278031}, \frac{\sigma'}{278031}, \frac{\rho'}{278031}\right)^{-13}}{G_2\left(\frac{-26}{3}, \frac{\tau'}{21387}, \frac{\sigma'}{21387}, \frac{\rho'}{21387}\right)^{-1}}$$

coincides up to high precision with the root $\approx 10.6409709... - i \cdot 5.9332732...$ of the polynomial

$$P_{\text{abs}} = x^8 - 18x^7 + 83x^6 + 396x^5 + 597x^4 + 396x^3 + 83x^2 - 18x + 1$$

which defines a subextension of $\mathbb{K}^+(\mathfrak{f})/\mathbb{K}$. The computation time for 1000 digits is 3 minutes and 20 seconds. The fact that the higher elliptic unit we compute does not generate the whole class field $\mathbb{K}^+(\mathfrak{f})$ is a rather common phenomenon when the extension $\mathbb{K}^+(\mathfrak{f})/\mathbb{K}$ is not cyclic, as the Stark unit itself is not guaranteed to generate $\mathbb{K}^+(\mathfrak{f})$.

IV.2.2.3 Example 8

Keep $\mathbb{K}$, $\mathfrak{f}$, $\mathfrak{b}$, $\mathfrak{a}$ as in example 7 and let us change our choice of fundamental units to obtain a better situation in regards to computations. We fix another set of fundamental units for $\mathcal{O}_{\mathbb{K}}^{+,\times} = \mathcal{O}_{\mathfrak{f}}^{+,\times}$:

$$\varepsilon_1 = \frac{z^3 - 21z^2 + 54z + 11}{9}, \ \varepsilon_2 = -z^3 + 19z^2 - 18z - 8$$

Then, we compute the contents $\tilde{\lambda}_1 = \tilde{\lambda}_2 = 1$ and the overflows $\tilde{t}_1 = 1$, $\tilde{t}_2 = 25$ which improves on the parameters obtained for the previous choice of fundamental units. The generalised different ideal $\mathfrak{D}(\tilde{a}_2, \mathcal{O}_{\mathbb{K}})$ is $\mathfrak{P}_5^2$ where $\mathfrak{P}_5$ is the unique integral ideal of norm 5 in $\mathcal{O}_{\mathbb{K}}$. The ideals $qN/\mathfrak{a}$ and $2.qN\mathfrak{P}_5^2/\mathfrak{a}$ are generated by the strongly admissible base points $h_1 = (-44z^3 + 843z^2 - 927z - 232)/3$, $h_2 = (76z^3 - 1449z^2 + 1470z + 344)/3$. The corresponding levels in the computation are given by $\ell_1 = 3 \cdot 13 = 39$ and $\ell_2 = 3 \cdot 5^2 \cdot 13 = 975$. Let us write the parameters

$$\tau = \frac{2\varepsilon_1\varepsilon_2 + 109\varepsilon_1 - 3}{\varepsilon_1}, \qquad \tau' = \frac{842\varepsilon_1\varepsilon_2 + \varepsilon_2 + 3}{\varepsilon_1\varepsilon_2}$$
$$\sigma = \frac{-321\varepsilon_1\varepsilon_2 - 16\varepsilon_1 + 7}{\varepsilon_1\varepsilon_2}, \qquad \sigma' = \frac{-7\varepsilon_1\varepsilon_2 + 4\varepsilon_2 - 63}{1}$$
$$\rho = \frac{-\varepsilon_1 - 47}{1}, \qquad \rho' = \frac{-3\varepsilon_1\varepsilon_2 + 566\varepsilon_2 - 2}{\varepsilon_2}.$$

Then

$$\frac{G^+_{2,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(\varepsilon_2, \varepsilon_1\varepsilon_2; h_2)}{G^-_{2,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(\varepsilon_1, \varepsilon_1\varepsilon_2; h_1)} = \frac{G_2\left(\frac{-1}{3}, \frac{\tau}{39}, \frac{\sigma}{39}, \frac{\rho}{39}\right)^{-13}}{G_2\left(\frac{-13}{3}, \frac{\tau}{3}, \frac{\sigma}{3}, \frac{\rho}{3}\right)^{-1}} \times \frac{G_2\left(\frac{2}{3}, \frac{\tau'}{975}, \frac{\sigma'}{975}, \frac{\rho'}{975}\right)^{13}}{G_2\left(\frac{26}{3}, \frac{\tau'}{75}, \frac{\sigma'}{75}, \frac{\rho'}{75}\right)}$$

coincides up to high precision with the same root $\approx 10.6409709... - i \cdot 5.9332732...$ of the same polynomial $P_{\text{abs}} = x^8 - 18x^7 + 83x^6 + 396x^5 + 597x^4 + 396x^3 + 83x^2 - 18x + 1$ as in example 7. The computation time for 1000 digits is 11 seconds. The value obtained for the higher elliptic unit is the same as for the previous example in which the choice of fundamental units was different, which supports the idea that the product in Conjecture III.37 is independent of said choice, even though computations may be longer when a poor choice of fundamental units is made.

IV.2.2.4 Example 9

This example showcases some subtle changes in the computations when the field we consider contains a real subfield, relying on preparatory work done in previous chapters (see Proposition I.17 and Lemma III.7). Let z be the complex root of the polynomial $x^4 - 12$ lying in the upper half-plane. Then $\mathbb{K} = \mathbb{Q}(z)$ has class number 1 and *contains the real quadratic field* $\mathbb{Q}(\sqrt{3})$. We choose the ideal $\mathfrak{f}$ as the unique prime ideal of norm $q = 2$ in $\mathcal{O}_{\mathbb{K}}$. The corresponding narrow ray class group is $\mathrm{Cl}^+(\mathfrak{f}) \simeq \mathbb{Z}/2\mathbb{Z}$. Let us choose the fundamental units:

$$\varepsilon_1 = \frac{z^2 + 2z + 2}{4}, \ \varepsilon_2 = \frac{-z^2 + 4}{2} = 2 + \sqrt{3}$$

for $\mathcal{O}_{\mathbb{K}}^{+,\times} = \mathcal{O}_{\mathfrak{f}}^{+,\times}$. Note that ε_2 belongs to the subfield $\mathbb{Q}(\sqrt{3})$. We may compute the contents $\tilde{\lambda}_1 = \tilde{\lambda}_2 = 1$ as well as the overflows $\tilde{t}_1 = \tilde{t}_2 = 1$. Let us fix $\mathfrak{b} = (1)$ and $\mathfrak{a} = 23\mathcal{O}_{\mathbb{K}} + \beta\mathcal{O}_{\mathbb{K}}$ an integral ideal of norm $N = 23$ in $\mathcal{O}_{\mathbb{K}}$ where $\beta = (z^2 - 2z - 26)/4$. The ideal $qN/\mathfrak{a}$ is generated by the strongly admissible base points $h_1 = h_2 = (-3z^3 - 9z^2 - 4z + 34)/2$. The parameters we obtain are:

$$\tau = \varepsilon_2 + 175 = 177 + \sqrt{3}, \qquad \tau' = 39 + \frac{1}{\varepsilon_1}$$

$$\sigma = -342 + \frac{3}{\varepsilon_2} - \frac{1}{\varepsilon_1\varepsilon_2}, \qquad \sigma' = \varepsilon_2\varepsilon_1 + \varepsilon_2 - 352$$

$$\rho = -\varepsilon_1 + 217, \qquad \rho' = 179 - \frac{1}{\varepsilon_2} = 177 + \sqrt{3}$$

In particular $\tau, \rho' \in \mathbb{R}$, $\Im(\tau'), \Im(\rho) < 0$ and $\Im(\sigma), \Im(\sigma') > 0$. In this setting, we must use formula (I.34) to perform the computations of the product:

$$\frac{G^+_{2,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(\varepsilon_1, \varepsilon_1\varepsilon_2; h_2)}{G^-_{2,\mathfrak{f},\mathfrak{b},\mathfrak{a}}(\varepsilon_2, \varepsilon_1\varepsilon_2; h_1)} = \frac{G_2\left(\frac{1}{2}, \frac{\tau}{46}, \frac{\sigma}{46}, \frac{\rho}{46}\right)^{23}}{G_2\left(\frac{23}{2}, \frac{\tau}{2}, \frac{\sigma}{2}, \frac{\rho}{2}\right)} \times \frac{G_2\left(\frac{-1}{2}, \frac{\tau'}{46}, \frac{\sigma'}{46}, \frac{\rho'}{46}\right)^{-23}}{G_2\left(\frac{-23}{2}, \frac{\tau'}{2}, \frac{\sigma'}{2}, \frac{\rho'}{2}\right)}.$$

Th complex number we obtain coincides up to 1000 digits of precision with the root $\approx 13.9102308... - i \cdot 24.0932265...$ of the polynomial

$$P_{\text{abs}} = x^8 - 28x^7 + 778x^6 - 112x^5 - 749x^4 - 112x^3 + 778x^2 - 28x + 1$$

which defines an absolute equation of $\mathbb{K}^+(\mathfrak{f})$ over $\mathbb{Q}$. We will revisit this example in section IV.2.6.3 with a different choice of fundamental units which leads to parameters outside $\tau, \sigma, \rho, \tau', \sigma', \rho' \in \mathbb{C} - \mathbb{R}$.

It is interesting to note that previous work on the determination of Stark units in the case of a quartic field $\mathbb{K}$ with exactly one complex place containing a real quadratic subfield $\mathbb{F}$ was carried out by Charollois and Darmon [CD08] using multiplicative 1-cocycles for the Hilbert modular group $\mathrm{SL}_2(\mathcal{O}_{\mathbb{F}})$. In our construction, the presence of a real subfield is often times an obstacle to the computation of these same units using 2-cocycles for $\mathrm{SL}_4(\mathbb{Z})$, showing that the two approaches complement eachother.

IV.2.3 A quintic example in an optimal setting

In this section we present one of our simplest quintic examples. Let z be the complex root of the polynomial $x^5 - x^4 - x^3 - 2x^2 + x + 1$ lying in the upper half-plane. Then $\mathbb{K} = \mathbb{Q}(z)$ has narrow class number 1. We choose $\mathfrak{f}$ to be the unique prime ideal of norm $q = 3$. The corresponding narrow ray class group is $\mathrm{Cl}^+(\mathfrak{f}) \simeq \mathbb{Z}/3\mathbb{Z}^\times \simeq \mathbb{Z}/2\mathbb{Z}$. If we fix the fundamental units

$$\varepsilon_1 = z^4 - 2z^3 - z + 3, \ \varepsilon_2 = 2z^4 - 2z^3 - 3z + 3, \ \varepsilon_3 = 2z^4 - 3z^3 - 4z + 4$$

for $\mathcal{O}_{\mathbb{K}}^{+,\times} = \mathcal{O}_{\mathfrak{f}}^{+,\times}$ and the ordering $\{\mathrm{Id}, (32), (21), (231), (312), (31)\}$ of $\mathfrak{S}_3$ (this is the ordering given by Pari/GP for instance), then all six contents $\tilde{\lambda}_1, \dots, \tilde{\lambda}_6$ are equal to 1. The different ideals and the overflows governing our construction are:

$$\begin{aligned}
&\mathfrak{D}_1 = (7 \cdot 37 \cdot 137)\mathcal{O}_{\mathbb{K}} + (z - 11310)\mathcal{O}_{\mathbb{K}} && \tilde{t}_1 = \tilde{s}_1 = 7 \cdot 37 \cdot 137 \\
&\mathfrak{D}_2 = (31 \cdot 53)\mathcal{O}_{\mathbb{K}} + (z - 488)\mathcal{O}_{\mathbb{K}} && \tilde{t}_2 = \tilde{s}_2 = 31 \cdot 53 \\
&\mathfrak{D}_3 = 491.\mathcal{O}_{\mathbb{K}} + (z - 174)\mathcal{O}_{\mathbb{K}} && \tilde{t}_3 = \tilde{s}_3 = 491 \\
&\mathfrak{D}_4 = 107.\mathcal{O}_{\mathbb{K}} + (z + 8)\mathcal{O}_{\mathbb{K}} && \tilde{t}_4 = \tilde{s}_4 = 107 \\
&\mathfrak{D}_5 = \mathcal{O}_{\mathbb{K}} && \tilde{t}_5 = \tilde{s}_5 = 1 \\
&\mathfrak{D}_6 = 145637.\mathcal{O}_{\mathbb{K}} + (z - 52183)\mathcal{O}_{\mathbb{K}} && \tilde{t}_6 = \tilde{s}_6 = 145637.
\end{aligned}$$

We may choose $\mathfrak{b} = (1)$ and $\mathfrak{a}$ the unique degree prime ideal of norm $N = 11$ in $\mathcal{O}_{\mathbb{K}}$. The ideals $qN\mathfrak{D}_j/\mathfrak{a}$ for $1 \leq j \leq 6$ admit the following strongly admissible generators

$$\begin{aligned}
h_1 &= 147z^4 - 135z^3 - 90z^2 - 234z + 72, \\
h_2 &= -15z^4 - 30z^3 + 90z^2 + 36z + 60, \\
h_3 &= 42z^4 - 15z^3 - 87z^2 - 48z + 30, \\
h_4 &= -21z^4 + 57z^3 - 6z^2 - 9z - 48, \\
h_5 &= -3z^4 - 6z^3 + 18z^2 - 6z + 12, \\
h_6 &= 108z^4 - 246z^3 + 111z^2 - 81z + 195.
\end{aligned}$$

We then compute six quotients associated to each permutation $\rho \in \mathfrak{S}_3$:

$$v_j = \frac{G_3\left(\frac{1}{3}, \frac{\tau_j}{\ell_j}, \frac{\sigma_j}{\ell_j}, \frac{\rho_j}{\ell_j}, \frac{\varpi_j}{\ell_j}\right)^{11}}{G_3\left(\frac{11}{3}, \frac{11\tau_j}{\ell_j}, \frac{11\sigma_j}{\ell_j}, \frac{11\rho_j}{\ell_j}, \frac{11\varpi_j}{\ell_j}\right)}$$

where the parameters $\tau_j, \sigma_j, \rho_j, \varpi_j$ are given in Table IV.1 below and we have defined for ease of presentation the levels $\ell_j = 3 \cdot 11 \cdot \tilde{t}_j$, that is:

$$\ell_1 = 1170939,\ \ell_2 = 54219,\ \ell_3 = 16203,\ \ell_4 = 3531,\ \ell_5 = 33,\ \ell_6 = 4806021.$$

The corresponding higher elliptic unit $\mathcal{W}_{\mathfrak{f},\mathfrak{b},\mathfrak{a}} = \frac{v_2 v_4 v_6}{v_1 v_3 v_5}$ coincides up to high precision with the root $\approx -11.6360077... + i \cdot 3.4634701...$ of the polynomial

$$P_{\mathrm{abs}} = x^{10} + 24x^9 + 164x^8 + 99x^7 - 62x^6 - 89x^5 - 62x^4 + 99x^3 + 164x^2 + 24x + 1$$

which defines an absolute equation of $\mathbb{K}^+(\mathfrak{f})$ over $\mathbb{Q}$. The computation time for 1000 digits is 1 minute and 35 seconds, but the computation time for each of the individual computations is not uniform. The fifth computation requires 1 second whereas the sixth computation requires 58 seconds because of the level difference $\ell_5 = 33$ versus $\ell_6 = 4806021$. Since $\mathbb{K}^+(\mathfrak{f})/\mathbb{K}$ is a degree 2 extension we expect the explicit equality:

$$\mathcal{W}_{\mathfrak{f},\mathfrak{b},\mathfrak{a}} =^? \frac{\xi + \sqrt{\xi^2 - 4}}{2}$$

where $\xi = z^4 - 4z^3 + 2z^2 + 4z - 1$.

Table IV.1: Parameters for the quintic example

$\tau_1 =$	$-$	$935z^4$	$+$	$19z^3$	$+$	$2927z^2$	$+$	$601z$	$-$	796987
$\sigma_1 =$		$3z^4$	$-$	$1556z^3$	$+$	$1205z^2$	$+$	$3072z$	$-$	987058
$\rho_1 =$	$-$	$3978z^4$	$+$	$5242z^3$	$-$	$1095z^2$	$+$	$7073z$	$-$	590241
$\varpi_1 =$		$2767z^4$	$-$	$4003z^3$	$-$	$389z^2$	$-$	$5232z$	$+$	2072505
$\tau_2 =$		$377z^4$	$-$	$417z^3$	$-$	$141z^2$	$-$	$556z$	$+$	19860
$\sigma_2 =$		$71z^4$	$+$	$74z^3$	$-$	$105z^2$	$-$	$449z$	$-$	71171
$\rho_2 =$	$-$	$481z^4$	$+$	$702z^3$	$-$	$330z^2$	$+$	$936z$	$+$	324847
$\varpi_2 =$	$-$	$37z^4$	$+$	$54z^3$	$+$	$101z^2$	$+$	$72z$	$-$	28978
$\tau_3 =$	$-$	$121z^4$	$+$	$180z^3$	$+$	$17z^2$	$+$	$254z$	$-$	12881
$\sigma_3 =$		$830z^4$	$-$	$1255z^3$	$-$	$214z^2$	$-$	$1580z$	$+$	326894
$\rho_3 =$	$-$	$6z^4$	$-$	$56z^3$	$+$	$82z^2$	$+$	$41z$	$-$	52774
$\varpi_3 =$	$-$	$2082z^4$	$+$	$3154z^3$	$+$	$467z^2$	$+$	$3916z$	$-$	773076
$\tau_4 =$	$-$	$42z^4$	$+$	$57z^3$	$+$	$14z^2$	$+$	$79z$	$-$	139
$\sigma_4 =$		$287z^4$	$-$	$443z^3$	$-$	$60z^2$	$-$	$522z$	$+$	2216
$\rho_4 =$	$-$	$3z^4$	$+$	$27z^3$	$+$	z^2	$-$	$2z$	$-$	522
$\varpi_4 =$	$-$	$685z^4$	$+$	$1029z^3$	$+$	$157z^2$	$+$	$1291z$	$-$	2774
$\tau_5 =$	$-$	z^4	$+$	z^3	$-$	z^2	$+$	$4z$	$-$	455
$\sigma_5 =$					$+$	$2z^2$	$-$	$3z$	$+$	574
$\rho_5 =$		$2z^4$	$-$	$3z^3$	$+$	$4z^2$	$-$	$8z$	$+$	1282
$\varpi_5 =$					$-$	z^2	$+$	z	$-$	247
$\tau_6 =$		$935z^4$	$+$	$1775z^3$	$-$	$1242z^2$	$-$	$4691z$	$+$	9783958
$\sigma_6 =$		$2269z^4$	$-$	$1923z^3$	$-$	$6285z^2$	$-$	$169z$	$-$	19885070
$\rho_6 =$	$-$	$7142z^4$	$+$	$1239z^3$	$-$	$949z^2$	$+$	$9197z$	$-$	13206973
$\varpi_6 =$		$11437z^4$	$-$	$14892z^3$	$-$	$1641z^2$	$-$	$20621z$	$+$	7776348

IV.2.4 A degree six example in an optimal setting

In this section we present our simplest computation of a higher elliptic unit above a degree 6 field. In this setting, the higher elliptic unit is given by a product of $(6-2)! = 24$ smoothed G_4 functions evaluated at 5 parameters each on the base field. Let z be the complex root of the polynomial $x^6 - 2x^5 - 3x^4 + 10x^3 + 3x^2 - 8x - 3$ lying in the upper half-plane. Then $\mathbb{K} = \mathbb{Q}(z)$ has class number 1. We choose $\mathfrak{f}$ to be the unique prime ideal of norm $q = 2$. The corresponding narrow ray class group is $\mathrm{Cl}^+(\mathfrak{f}) \simeq \mathbb{Z}/2\mathbb{Z}$. If we fix the fundamental units

$$\begin{aligned}
\varepsilon_1 &= 3z^5 - 8z^4 - 3z^3 + 31z^2 - 15z - 11 \\
\varepsilon_2 &= 2/5z^5 - z^4 - 1/5z^3 + 18/5z^2 - 13/5z - 2/5 \\
\varepsilon_3 &= -4/5z^5 + 2z^4 + 7/5z^3 - 41/5z^2 + 6/5z + 29/5 \\
\varepsilon_4 &= 2/5z^5 - z^4 - 16/5z^3 + 8/5z^2 + 22/5z + 8/5
\end{aligned}$$

for $\mathcal{O}_{\mathbb{K}}^{+,\times} = \mathcal{O}_{\mathfrak{f}}^{+,\times}$ and the ordering of $\mathfrak{S}_4$ given by Pari/GP, then we may compute the contents $\tilde{\lambda}_1 = \cdots = \tilde{\lambda}_{24} = 1$ and the associated overflows

$\tilde{t}_1 =$	3.821.85146905507	$\tilde{t}_2 =$	31.71380217.5479992107	$\tilde{t}_3 =$	2593.28232090533
$\tilde{t}_4 =$	4793.20161.11384677	$\tilde{t}_5 =$	37.18731.12207408718823	$\tilde{t}_6 =$	757.6354278197
$\tilde{t}_7 =$	5.311.4219.261707099	$\tilde{t}_8 =$	5.775267.10124654046373	$\tilde{t}_9 =$	5^2.37.1061.1321.6449
$\tilde{t}_{10} =$	5.64781598487	$\tilde{t}_{11} =$	5.9967.23337159379	$\tilde{t}_{12} =$	5.1444658023
$\tilde{t}_{13} =$	426868799166283769	$\tilde{t}_{14} =$	4950440701232129	$\tilde{t}_{15} =$	1361.7219.388087223
$\tilde{t}_{16} =$	20173.200544349	$\tilde{t}_{17} =$	103.11593.919759	$\tilde{t}_{18} =$	523.31751222611
$\tilde{t}_{19} =$	5.1759763531590697	$\tilde{t}_{20} =$	15629.3287441	$\tilde{t}_{21} =$	5.56807.354098824061
$\tilde{t}_{22} =$	5.73.79.349.73009	$\tilde{t}_{23} =$	3544640951	$\tilde{t}_{24} =$	293.11719.13931

The different ideals are computed as usual using Algorithm 1. We may choose $\mathfrak{b} = (1)$ and $\mathfrak{a}$ the unique prime ideal of norm $N = 29$ in $\mathcal{O}_{\mathbb{K}}$. We then compute 24 quotients:

$$v_j = \frac{G_4\left(\frac{1}{2}, \frac{\tau_j}{\ell_j}, \frac{\sigma_j}{\ell_j}, \frac{\rho_j}{\ell_j}, \frac{\varpi_j}{\ell_j}, \frac{\xi_j}{\ell_j}\right)^{29}}{G_4\left(\frac{29}{2}, \frac{29\tau_j}{\ell_j}, \frac{29\sigma_j}{\ell_j}, \frac{29\rho_j}{\ell_j}, \frac{29\varpi_j}{\ell_j}\right)}$$

where the parameters $\tau_j, \sigma_j, \rho_j, \varpi_j, \xi_j$ are given in Tables IV.2 to IV.7 below and we have defined for ease of presentation the levels $\ell_j = 2 \cdot 29 \cdot \tilde{t}_j$. The corresponding higher elliptic unit

$$\mathcal{W}_{\mathfrak{f},\mathfrak{b},\mathfrak{a}} = \frac{v_2 v_3 v_6 v_7 v_{10} v_{11} v_{14} v_{15} v_{18} v_{19} v_{22} v_{23}}{v_1 v_4 v_5 v_8 v_9 v_{12} v_{13} v_{16} v_{17} v_{20} v_{21} v_{24}}$$

coincides up to 1000 digits of precision with the root $\approx -555050859076374984.5110063... - i \cdot 334493188695056032.1307346...$ of the reciprocal polynomial

$$\begin{aligned} P_{\text{abs}} = & x^{12} + 1 + 1110101718152749974(x^{11} + x) \\ & + 419967149444808248584979504483010229(x^{10} + x^2) \\ & + 2090591105457346230355038086262355202(x^9 + x^3) \\ & + 4535292963058947524812988357459161338(x^8 + x^4) \\ & + 6020269972074463320492578065966460430(x^7 + x^5) \\ & + 6311227681584443751632555661724851277x^6 \end{aligned}$$

which defines an absolute equation of $\mathbb{K}^+(\mathfrak{f})$ over $\mathbb{Q}$. As $\mathbb{K}^+(\mathfrak{f})/\mathbb{K}$ is a degree 2 extension we expect the following equality:

$$\mathcal{W}_{\mathfrak{f},\mathfrak{b},\mathfrak{a}} =^? \frac{\xi + \sqrt{\xi^2 - 4}}{2}$$

where

$$\begin{aligned} \xi = & 165358511989497 82z^5 - 2445604577168278z^4 - 73350035010478060z^3 \\ & - 4800180859024974z^2 + 59609758573992978z + 20222967495499571. \end{aligned}$$

Table IV.2: Parameters for the degree 6 example (Part 1)

$5\tau_1 =$		$451483129169z^5$	$-$	$1027896279160z^4$	$-$	$1027904929692z^3$
	$+$	$4800919668041z^2$	$-$	$287249353791z$	$+$	282632366273212901
$5\sigma_1 =$	$-$	$2562069062795z^5$	$+$	$7559238893200z^4$	$+$	$2897920038705z^3$
	$-$	$31644657377825z^2$	$+$	$14725600389930z$	$-$	473413896662735405
$5\rho_1 =$		$9516079547035z^5$	$-$	$26469407301575z^4$	$-$	$8172819882825z^3$
	$+$	$102113950298935z^2$	$-$	$50712518815770z$	$-$	292553164591261820
$5\varpi_1 =$	$-$	$377559921389z^5$	$+$	$208703459620z^4$	$+$	$996782223057z^3$
	$-$	$1165668574871z^2$	$+$	$97261357731z$	$+$	997923835090996594
$5\xi_1 =$	$-$	$272818854433z^5$	$+$	$765392397170z^4$	$+$	$269035359849z^3$
	$-$	$2948594657182z^2$	$+$	$1414694411937z$	$-$	36649032748712887
$5\tau_2 =$	$-$	$1021889853862442z^5$	$+$	$2388860888138130z^4$	$+$	$2341803908240031z^3$
	$-$	$11044703365243488z^2$	$+$	$650597397813063z$	$-$	523175469027287015558
$5\sigma_2 =$		$6179656738373595z^5$	$-$	$16735123778030025z^4$	$-$	$7969024224695395z^3$
	$+$	$70653190488068515z^2$	$-$	$32016724639436130z$	$+$	2540895723814044325100
$5\rho_2 =$	$-$	$20184715302504297z^5$	$+$	$56957132770701560z^4$	$+$	$16715406622846421z^3$
	$-$	$219589406206489418z^2$	$+$	$109491099710176303z$	$+$	521923709195397260867
$5\varpi_2 =$	$-$	$1767152601692569z^5$	$-$	$454893122345405z^4$	$+$	$6479139411227672z^3$
	$+$	$1121613680872159z^2$	$-$	$4648428884119074z$	$-$	4617817571364135180021
$5\xi_2 =$		$1412894984950354z^5$	$-$	$3122706351789385z^4$	$-$	$6618867788941337z^3$
	$+$	$7489317442740766z^2$	$+$	$2822187812019419z$	$-$	10124730013747793037599
$5\tau_3 =$	$-$	$195713546602z^5$	$+$	$477479810515z^4$	$+$	$312575067241z^3$
	$-$	$1936522948718z^2$	$+$	$267388196823z$	$+$	23608826921704267
$5\sigma_3 =$		$639272697510z^5$	$-$	$1901436185585z^4$	$-$	$900048030250z^3$
	$+$	$8557507515740z^2$	$-$	$4189938413380z$	$+$	160262816426300235
$5\rho_3 =$		$95213662795z^5$	$-$	$651397955255z^4$	$+$	$1468703530055z^3$
	$-$	$1188068337625z^2$	$+$	$74681386470z$	$-$	564914393269967120
$5\varpi_3 =$		$220324382792z^5$	$-$	$268609219065z^4$	$+$	$37616336914z^3$
	$-$	$232039908342z^2$	$+$	$171920038817z$	$-$	243065751680310552
$5\xi_3 =$		$7946878394z^5$	$-$	$24083748545z^4$	$-$	$28886961752z^3$
	$+$	$94140155061z^2$	$-$	$23086902861z$	$+$	13641197503093241
$5\tau_4 =$		$1611347131786z^5$	$-$	$4073522497495z^4$	$-$	$2563068441363z^3$
	$+$	$16669126695074z^2$	$-$	$2444566366439z$	$-$	195271061589764651
$5\sigma_4 =$	$-$	$6449552305326z^5$	$+$	$17177595322125z^4$	$+$	$8029686091818z^3$
	$-$	$71975871034824z^2$	$+$	$34729368237394z$	$-$	270998917347862719
$5\rho_4 =$	$-$	$1889465484881z^5$	$+$	$9090371280310z^4$	$-$	$15749192712647z^3$
	$+$	$7217602477221z^2$	$+$	$2289170902769z$	$+$	9430537679026249296
$5\varpi_4 =$		$1411789375192z^5$	$-$	$1891627758065z^4$	$+$	$279338420114z^3$
	$-$	$612692064042z^2$	$+$	$557595527817z$	$-$	2919049715399176292
$5\xi_4 =$		$446787848858z^5$	$-$	$483308184540z^4$	$+$	$394215877386z^3$
	$-$	$1026775965773z^2$	$+$	$352662578098z$	$-$	1241817035064443863

Table IV.3: Parameters for the degree 6 example (Part 2)

$5\tau_5 =$	$-$	$3684901681164998z^5$	$+$	$8875323554885270z^4$	$+$	$7426989609867089z^3$
	$-$	$39934846078090922z^2$	$+$	$5165044077805897z$	$-$	21887542697982750832
$5\sigma_5 =$		$2218788364586322z^5$	$-$	$5847412267852995z^4$	$-$	$2277451551917896z^3$
	$+$	$21033315394365103z^2$	$-$	$3704614161272163z$	$+$	936139601289739196913
$5\rho_5 =$		$1978843777295007z^5$	$-$	$9846489287110750z^4$	$+$	$1663297318420469z^3$
	$+$	$40692806152215928z^2$	$-$	$36951869676506873z$	$-$	705483630009754019637
$5\varpi_5 =$		$1566318148869146z^5$	$-$	$3390792046686935z^4$	$-$	$2376414992661343z^3$
	$+$	$14498450430309589z^2$	$-$	$11596578847583259z$	$-$	636503095972391756296
$5\xi_5 =$	$-$	$292683377814890z^5$	$+$	$136775975268905z^4$	$+$	$3040780406260975z^3$
	$+$	$1791002060431660z^2$	$-$	$4761116193109915z$	$-$	2097360233182344094075
$5\tau_6 =$	$-$	$16281915888z^5$	$+$	$41381369235z^4$	$+$	$33136729354z^3$
	$-$	$182131074517z^2$	$+$	$27759725362z$	$+$	311838087349668
$5\sigma_6 =$		$9506357130z^5$	$-$	$21519613395z^4$	$-$	$13587588580z^3$
	$+$	$86873107315z^2$	$-$	$7892525055z$	$+$	1084335116471445
$5\rho_6 =$		$6834017089z^5$	$+$	$10083213235z^4$	$+$	$30216543618z^3$
	$+$	$22541646201z^2$	$-$	$122103094736z$	$-$	1409107497208399
$5\varpi_6 =$	$-$	$19652307406z^5$	$-$	$137111657255z^4$	$+$	$47314558673z^3$
	$+$	$326382669781z^2$	$-$	$397207914531z$	$-$	7329020669861634
$5\xi_6 =$		$3826080146z^5$	$-$	$56497642055z^4$	$-$	$30126566293z^3$
	$+$	$129663186024z^2$	$-$	$123429258499z$	$-$	2363954523642791
$5\tau_7 =$	$-$	$35692051087720z^5$	$+$	$81724519649615z^4$	$+$	$77788339842380z^3$
	$-$	$372504493767015z^2$	$+$	$20511758492810z$	$+$	6201184549098147800
$5\sigma_7 =$		$285319898537295z^5$	$-$	$682195346292880z^4$	$-$	$584339208147625z^3$
	$+$	$3075528579433425z^2$	$-$	$363451716925290z$	$-$	50381931737655187305
$5\rho_7 =$	$-$	$22367183954760z^5$	$+$	$57145505032765z^4$	$+$	$33923235806750z^3$
	$-$	$239210312409950z^2$	$+$	$68067412730320z$	$+$	8070632250210882165
$5\varpi_7 =$	$-$	$21006910304805z^5$	$+$	$51437007580510z^4$	$+$	$42542004780220z^3$
	$-$	$231386809057710z^2$	$+$	$27900311812190z$	$+$	3538609769440367225
$5\xi_7 =$		$2550307829005z^5$	$-$	$6189186011865z^4$	$-$	$4968603477585z^3$
	$+$	$27378844697680z^2$	$-$	$4145408107375z$	$-$	555253459508370910
$5\tau_8 =$		$6559398674404100z^5$	$-$	$12542375855814720z^4$	$-$	$18350309610804635z^3$
	$+$	$60382803172769930z^2$	$+$	$18290613410813535z$	$+$	7782019103416256780410
$5\sigma_8 =$	$-$	$39505909332937635z^5$	$+$	$92286137284391265z^4$	$+$	$84274836553043875z^3$
	$-$	$418790259382607275z^2$	$+$	$31483549306052970z$	$-$	37380822711788850765010
$5\rho_8 =$		$9850931706266655z^5$	$-$	$28825987231985540z^4$	$-$	$9127266513342735z^3$
	$+$	$113278492936059930z^2$	$-$	$48327005403896125z$	$-$	7279724540845983826485
$5\varpi_8 =$		$4323127686545350z^5$	$+$	$16109867604466610z^4$	$-$	$15858050920734595z^3$
	$-$	$4456261143806990z^2$	$+$	$11679315244876920z$	$+$	12580757449638439983495
$5\xi_8 =$	$-$	$2674847363419795z^5$	$+$	$5011231088220670z^4$	$+$	$16527209272818520z^3$
	$-$	$8090985528777435z^2$	$-$	$13973211794502755z$	$+$	51194154161043089683735

Table IV.4: Parameters for the degree 6 example (Part 3)

$5\tau_9 =$		$160189992470z^5$	$-$	$286932412655z^4$	$-$	$395850947100z^3$
	$+$	$1272969561725z^2$	$+$	$401687884360z$	$-$	5140494361419880
$5\sigma_9 =$	$-$	$599068674165z^5$	$+$	$1808350444660z^4$	$+$	$271811027175z^3$
	$-$	$6659337696625z^2$	$+$	$3782948878880z$	$+$	68235760134116435
$5\rho_9 =$	$-$	$20287443390z^5$	$-$	$64236591565z^4$	$+$	$293673716050z^3$
	$-$	$79737595000z^2$	$-$	$848354814670z$	$-$	35542792879390915
$5\varpi_9 =$		$33592281150z^5$	$-$	$154894719950z^4$	$+$	$64851335950z^3$
	$+$	$498185934150z^2$	$-$	$471545478625z$	$+$	7380373248530325
$5\xi_9 =$	$-$	$4497437510z^5$	$+$	$12793602965z^4$	$+$	$4541668975z^3$
	$-$	$47642329950z^2$	$+$	$20555828320z$	$+$	1005255962041740
$5\tau_{10} =$	$-$	$82635819590z^5$	$+$	$231227389120z^4$	$+$	$65686231805z^3$
	$-$	$882945500190z^2$	$+$	$451387542075z$	$+$	1777498330148930
$5\sigma_{10} =$		$801044821105z^5$	$-$	$2228959217165z^4$	$-$	$659922940585z^3$
	$+$	$8528724373955z^2$	$-$	$4267282237000z$	$-$	16648429213432760
$5\rho_{10} =$	$-$	$178200154410z^5$	$+$	$494228389910z^4$	$+$	$150294221935z^3$
	$-$	$1895081703330z^2$	$+$	$937253411860z$	$+$	3579062835733505
$5\varpi_{10} =$	$-$	$34028601195z^5$	$+$	$96665256100z^4$	$+$	$26705443835z^3$
	$-$	$369550987130z^2$	$+$	$185240762705z$	$+$	610535422917545
$5\xi_{10} =$	$-$	$3705047045z^5$	$+$	$10789175890z^4$	$+$	$2544966180z^3$
	$-$	$41420136665z^2$	$+$	$20995725235z$	$+$	55955801184225
$5\tau_{11} =$		$2628332605830z^5$	$-$	$5978682830765z^4$	$-$	$4440431024960z^3$
	$+$	$23367645089355z^2$	$+$	$940657242850z$	$-$	293044617213650310
$5\sigma_{11} =$	$-$	$7570463403065z^5$	$+$	$15624754133150z^4$	$+$	$18410923806595z^3$
	$-$	$72514909369685z^2$	$-$	$7245341154765z$	$+$	1153282210411200140
$5\rho_{11} =$		$885946918650z^5$	$+$	$6450773883390z^4$	$-$	$15895754245030z^3$
	$-$	$9913980220710z^2$	$+$	$50456631321805z$	$-$	979297902267321745
$5\varpi_{11} =$	$-$	$135465085290z^5$	$+$	$1591771446925z^4$	$-$	$2793228558530z^3$
	$-$	$2500977343160z^2$	$+$	$12026729903710z$	$-$	102948144586806035
$5\xi_{11} =$		$58125781960z^5$	$-$	$654782666955z^4$	$+$	$49635794655z^3$
	$+$	$471503549160z^2$	$-$	$2536337166775z$	$+$	$50341373134942305$5
$5\tau_{12} =$		$275234770z^5$	$-$	$700050285z^4$	$-$	$453916440z^3$
	$+$	$3028321395z^2$	$-$	$837933950z$	$+$	3513177831010
$5\sigma_{12} =$	$-$	$1231823120z^5$	$+$	$3121073100z^4$	$+$	$2018022285z^3$
	$-$	$13330389655z^2$	$+$	$3279322380z$	$-$	401816203580
$5\rho_{12} =$	$-$	$4639286235z^5$	$+$	$11125567010z^4$	$+$	$9290565460z^3$
	$-$	$49872877530z^2$	$+$	$6715451110z$	$-$	10981793441445
$5\varpi_{12} =$		$11043119160z^5$	$-$	$26647456525z^4$	$-$	$22040798030z^3$
	$+$	$119351105590z^2$	$-$	$16213553240z$	$-$	1987485189085
$5\xi_{12} =$		$2712583060z^5$	$-$	$6544594245z^4$	$-$	$5401807415z^3$
	$+$	$29337685870z^2$	$-$	$3995818555z$	$-$	58928438525

Table IV.5: Parameters for the degree 6 example (Part 4)

$5\tau_{13} =$		$1083354881437954z^5$	$-$	$270648815248435z^4$	$-$	$5109396551080057z^3$
	$+$	$3562293284637926z^2$	$+$	$14510219731794179z$	$+$	20664681746614415890451
$5\sigma_{13} =$		$906776826390780z^5$	$-$	$2766694560193745z^4$	$-$	$1105932743161470z^3$
	$+$	$11856280614335090z^2$	$-$	$5258873544222890z$	$-$	16598833422413119834345
$5\rho_{13} =$	$-$	$592128246278785z^5$	$+$	$1449103511517125z^4$	$+$	$958358404972425z^3$
	$-$	$5762694946941755z^2$	$+$	$239170294636490z$	$+$	983507257250458944390
$5\varpi_{13} =$	$-$	$367020386962859z^5$	$+$	$793529686232915z^4$	$+$	$615096797703732z^3$
	$-$	$3261723705274126z^2$	$+$	$672760768172896z$	$-$	1848830549682280921946
$5\xi_{13} =$	$-$	$2932083372483z^5$	$+$	$2274352814755z^4$	$+$	$19353293777984z^3$
	$-$	$8197419261377z^2$	$-$	$42294919112978z$	$-$	33375158182344294357
$5\tau_{14} =$	$-$	$4778545130090z^5$	$-$	$10970104442095z^4$	$+$	$35820439288175z^3$
	$+$	$28602127156460z^2$	$-$	$132232849590715z$	$-$	4016841524885592985
$5\sigma_{14} =$	$-$	$21265152992876z^5$	$+$	$56721640097925z^4$	$+$	$36621796655778z^3$
	$-$	$250065533937974z^2$	$+$	$63715452901364z$	$+$	1687006234765504521
$5\rho_{14} =$		$12507781408304z^5$	$-$	$30728262792910z^4$	$-$	$20031178051917z^3$
	$+$	$124076635099291z^2$	$-$	$13778071057666z$	$-$	4388569725287920504
$5\varpi_{14} =$		$411287604349z^5$	$+$	$10690567324655z^4$	$-$	$8612509696512z^3$
	$-$	$41504856983284z^2$	$+$	$17241820144264z$	$+$	13488571144491062926
$5\xi_{14} =$		$111473814972z^5$	$+$	$4304910738520z^4$	$-$	$2599377221686z^3$
	$-$	$15838612980767z^2$	$+$	$5983481784112z$	$+$	4229304175784463883
$5\tau_{15} =$	$-$	$16980673536601z^5$	$+$	$1605601519160z^4$	$+$	$90073296138928z^3$
	$-$	$64339979710229z^2$	$-$	$242573087269601z$	$-$	41754087197897370269
$5\sigma_{15} =$	$-$	$17011126366845z^5$	$+$	$59299683033940z^4$	$-$	$8149743266385z^3$
	$-$	$196872748350425z^2$	$+$	$143107393748280z$	$+$	27890446438746073825
$5\rho_{15} =$		$43715923584155z^5$	$-$	$112061522807055z^4$	$-$	$65272347748935z^3$
	$+$	$458771504645785z^2$	$-$	$102717645395170z$	$+$	4346748000784617150
$5\varpi_{15} =$		$11887034138056z^5$	$-$	$29018298930750z^4$	$-$	$15714825525043z^3$
	$+$	$114930767349859z^2$	$-$	$36719817241134z$	$+$	3600440024526112514
$5\xi_{15} =$		$186815677122z^5$	$-$	$395442291020z^4$	$-$	$535651033111z^3$
	$+$	$1595584676908z^2$	$+$	$388914199442z$	$+$	110844344110656393
$5\tau_{16} =$	$-$	$8161931692z^5$	$-$	$9703337650z^4$	$+$	$45808399771z^3$
	$-$	$15770307138z^2$	$-$	$123344315757z$	$-$	3330931621780558
$5\sigma_{16} =$	$-$	$20001752749z^5$	$+$	$69479601545z^4$	$+$	$5463521977z^3$
	$-$	$216455593151z^2$	$+$	$83965191436z$	$+$	5480215985583794
$5\rho_{16} =$		$61955940724z^5$	$-$	$170398381690z^4$	$-$	$112631583347z^3$
	$+$	$679225001846z^2$	$-$	$93955015046z$	$-$	6582941902240199
$5\varpi_{16} =$		$11124582914z^5$	$+$	$10594744555z^4$	$+$	$13444148528z^3$
	$+$	$30615244401z^2$	$-$	$48178691491z$	$+$	12924927956878641
$5\xi_{16} =$		$5057576214z^5$	$+$	$938377785z^4$	$-$	$7639729567z^3$
	$+$	$10290316666z^2$	$-$	$2948198081z$	$+$	3465289093558641

Table IV.6: Parameters for the degree 6 example (Part 5)

$5\tau_{17} =$	$-$	$12115739066z^5$	$+$	$63206427925z^4$	$-$	$25454155372z^3$
	$-$	$223186104769z^2$	$+$	$215816969054z$	$+$	625180023677306
$5\sigma_{17} =$		$3265703493z^5$	$-$	$17970471415z^4$	$+$	$5355063886z^3$
	$+$	$71464502337z^2$	$-$	$69395953802z$	$-$	85189558264923
$5\rho_{17} =$		$33296893526z^5$	$-$	$74091048975z^4$	$-$	$65334549948z^3$
	$+$	$310863420139z^2$	$+$	$10359133871z$	$-$	371092120270931
$5\varpi_{17} =$	$-$	$16042755054z^5$	$+$	$22030294995z^4$	$+$	$40342390687z^3$
	$-$	$96461142251z^2$	$-$	$21244326059z$	$+$	202054141485894
$5\xi_{17} =$	$-$	$9229985674z^5$	$+$	$15145451285z^4$	$+$	$20787106027z^3$
	$-$	$66285618996z^2$	$-$	$8517038109z$	$+$	172074661426559
$5\tau_{18} =$		$271734151244z^5$	$-$	$810141012030z^4$	$-$	$175399218267z^3$
	$+$	$3107589750386z^2$	$-$	$1623089840751z$	$+$	10873344530223506
$5\sigma_{18} =$	$-$	$77473171355z^5$	$+$	$223654494685z^4$	$+$	$96424321250z^3$
	$-$	$980168296555z^2$	$+$	$503551809650z$	$-$	2282491452658515
$5\rho_{18} =$	$-$	$346435742476z^5$	$+$	$762123056710z^4$	$+$	$625067589853z^3$
	$-$	$3153719289354z^2$	$+$	$165356112154z$	$-$	1886508619412059
$5\varpi_{18} =$	$-$	$47699497502z^5$	$+$	$179095038035z^4$	$+$	$56859930271z^3$
	$-$	$696836706443z^2$	$+$	$61845035813z$	$-$	3598572417914448
$5\xi_{18} =$		$23057907498z^5$	$-$	$31825162105z^4$	$-$	$57910962019z^3$
	$+$	$133630871162z^2$	$+$	$13378535233z$	$-$	2288516663187223
$5\tau_{19} =$		$208060453631z^5$	$-$	$610588191230z^4$	$-$	$1344675500503z^3$
	$+$	$6531554874994z^2$	$-$	$3531080188639z$	$-$	1754745688405877581
$5\sigma_{19} =$		$14512606220244z^5$	$-$	$35029779275070z^4$	$-$	$24383154110047z^3$
	$+$	$142690667730331z^2$	$-$	$11945282317986z$	$+$	2898060622073497506
$5\rho_{19} =$	$-$	$44152948097541z^5$	$+$	$148299217296580z^4$	$+$	$16528580312683z^3$
	$-$	$559495739573509z^2$	$+$	$300058581851804z$	$-$	$1998092430305567970 9$
$5\varpi_{19} =$	$-$	$23398571203001z^5$	$+$	$56881862039705z^4$	$+$	$28388917182763z^3$
	$-$	$218291116535449z^2$	$+$	$98675799700194z$	$-$	11910906258190646874
$5\xi_{19} =$		$901966800302z^5$	$+$	$2030674148590z^4$	$-$	$19591003348226z^3$
	$-$	$21614958628677z^2$	$+$	$33807044895112z$	$-$	41964009739540827
$5\tau_{20} =$		$128203866z^5$	$-$	$456033345z^4$	$-$	$166086648z^3$
	$+$	$1968606549z^2$	$-$	$903608989z$	$-$	3504946753681
$5\sigma_{20} =$		$677512200z^5$	$-$	$1276768770z^4$	$-$	$1477856515z^3$
	$+$	$5562069175z^2$	$+$	$967147590z$	$+$	14609024961690
$5\rho_{20} =$	$-$	$843461158z^5$	$+$	$1436164205z^4$	$+$	$814460334z^3$
	$-$	$6225744582z^2$	$+$	$3791737982z$	$+$	102521502157303
$5\varpi_{20} =$	$-$	$3107030873z^5$	$+$	$11606855225z^4$	$+$	$1317770059z^3$
	$-$	$43340784697z^2$	$+$	$20322494562z$	$-$	283764337969302
$5\xi_{20} =$	$-$	$1439029022z^5$	$+$	$4799254490z^4$	$+$	$1463810946z^3$
	$-$	$17545475513z^2$	$+$	$7487795488z$	$-$	124823430628363

Table IV.7: Parameters for the degree 6 example (Part 6)

$5\tau_{21} =$	$-$	$12548264659955z^5$	$+$	$44769624203385z^4$	$-$	$38345261830305z^3$
	$-$	$51616448215610z^2$	$+$	$37546787805665z$	$+$	$3990920957611872355O$
$5\sigma_{21} =$		$68255667137265z^5$	$+$	$9060246861650z^4$	$-$	$358996211816995z^3$
	$+$	$145417946273185z^2$	$+$	$1089641100070165z$	$-$	$4530774589996728914O$
$5\rho_{21} =$		$286264868039845z^5$	$-$	$875943960968260z^4$	$-$	$203647989333715z^3$
	$+$	$3324808593494845z^2$	$-$	$1424928543941500z$	$+$	67681267189771638185
$5\varpi_{21} =$		$79530769601080z^5$	$-$	$221227215843275z^4$	$-$	$63147894842140z^3$
	$+$	$827439885310070z^2$	$-$	$440813281107120z$	$+$	22501559795424965845
$5\xi_{21} =$	$-$	$26258260454900z^5$	$+$	$74874029106745z^4$	$+$	$55839749454535z^3$
	$-$	$247844339364680z^2$	$+$	$76433222963665z$	$+$	45231651903705332715
$5\tau_{22} =$		$835290350z^5$	$-$	$3378679265z^4$	$+$	$4181686505z^3$
	$+$	$1358150610z^2$	$-$	$3102076480z$	$+$	65050724275420
$5\sigma_{22} =$	$-$	$8832074380z^5$	$+$	$7915957000z^4$	$+$	$32644612665z^3$
	$-$	$45512970145z^2$	$-$	$78596978180z$	$-$	412509892209970
$5\rho_{22} =$	$-$	$16499040045z^5$	$+$	$35770048690z^4$	$+$	$45106111905z^3$
	$-$	$165416287465z^2$	$-$	$27188388040z$	$-$	46624952573925
$5\varpi_{22} =$		$32815031530z^5$	$-$	$74029029275z^4$	$-$	$77265988240z^3$
	$+$	$347322480620z^2$	$-$	$11580532170z$	$-$	235527220655805
$5\xi_{22} =$		$8504660240z^5$	$-$	$18579998455z^4$	$-$	$21164159085z^3$
	$+$	$86298842880z^2$	$-$	$375446045z$	$-$	46888101366075
$5\tau_{23} =$		$197276663z^5$	$-$	$487228005z^4$	$-$	$341680944z^3$
	$+$	$2047749367z^2$	$-$	$263330742z$	$+$	27104223597587
$5\sigma_{23} =$		$2760681177z^5$	$-$	$6554814560z^4$	$-$	$5714441671z^3$
	$+$	$29602273443z^2$	$-$	$3184162453z$	$+$	376367760546548
$5\rho_{23} =$	$-$	$7417067852z^5$	$+$	$17812423675z^4$	$+$	$15048710546z^3$
	$-$	$80092062628z^2$	$+$	$9901947408z$	$-$	1018354196296073
$5\varpi_{23} =$	$-$	$1323262467z^5$	$+$	$3124589385z^4$	$+$	$2815415501z^3$
	$-$	$14212314223z^2$	$+$	$1135342868z$	$-$	199971828966488
$5\xi_{23} =$		$194783218z^5$	$-$	$488400120z^4$	$-$	$341573814z^3$
	$+$	$2137801347z^2$	$-$	$509288612z$	$+$	17745311009407
$5\tau_{24} =$	$-$	$264188562z^5$	$+$	$857781370z^4$	$-$	$395743279z^3$
	$-$	$1699576658z^2$	$+$	$791276438z$	$-$	15061130239183
$5\sigma_{24} =$	$-$	$1493995205z^5$	$+$	$4609773160z^4$	$+$	$1511045675z^3$
	$-$	$19782456655z^2$	$+$	$10889343125z$	$+$	29914509515360
$5\rho_{24} =$		$5932635813z^5$	$-$	$14930576830z^4$	$-$	$9854615289z^3$
	$+$	$64970960777z^2$	$-$	$17440454052z$	$+$	23985238200167
$5\varpi_{24} =$	$-$	$250064549z^5$	$+$	$772375295z^4$	$-$	$263416973z^3$
	$-$	$3507724841z^2$	$+$	$5900091956z$	$+$	40352010686744
$5\xi_{24} =$	$-$	$638167324z^5$	$+$	$1623020590z^4$	$+$	$974202072z^3$
	$-$	$7029399681z^2$	$+$	$2952006896z$	$+$	7449821507509

IV.2.5 Quartic examples in non-optimal settings

In this section we present three quartic examples with $\#\mathcal{Z}_{\mathfrak{f}}^{1,+} > 1$. The first example is a case where $q = 5$ and $\mathcal{Z}_{\mathfrak{f}}^{1,+} \simeq \mathbb{Z}/2\mathbb{Z}$. The second example is a case where $q = 5$ and $\mathcal{Z}_{\mathfrak{f}}^{1,+} \simeq \mathbb{Z}/5\mathbb{Z}^{\times}$. The third example is a case where $q = 7$ and $\mathcal{Z}_{\mathfrak{f}}^{1,+} \simeq \mathbb{Z}/2\mathbb{Z}$.

IV.2.5.1 Example 12

We first discuss a quartic example with $q = 5$ and $\mathcal{Z}_{\mathfrak{f}}^{1,+} \simeq \mathbb{Z}/2\mathbb{Z}$. Let z be the complex root of the polynomial $x^4 - 3x + 1$ lying in the upper half-plane. Then $\mathbb{K} = \mathbb{Q}(z)$ has class number 1. We choose $\mathfrak{f} = \mathfrak{P}_5$ the unique integral ideal of norm $q = 5$ in $\mathcal{O}_{\mathbb{K}}$. The corresponding narrow ray class group is $\mathrm{Cl}^+(\mathfrak{f}) \simeq \mathbb{Z}/2\mathbb{Z}$. Let us fix a set of fundamental units

$$\varepsilon_1 = z^3 - 2z^2 + z,\ \varepsilon_2 = z^3 + 2z^2 + 3z + 1$$

for the group $\mathcal{O}_{\mathbb{K}}^{+,\times}$. They satisfy $\varepsilon_1 \equiv 1 \bmod \mathfrak{f}$ and $\varepsilon_2 \equiv -1 \bmod \mathfrak{f}$, thus $\mathcal{O}_{\mathfrak{f}}^{+,\times}$ is generated by ε_1 and ε_2^2. We may identify $\mathcal{Z}_{\mathfrak{f}}^{1,+} = \{(1,1),(4,\varepsilon_2)\}$.

The contents associated to this set of fundamental units are $\tilde{\lambda}_1 = \tilde{\lambda}_2 = 1$ and the overflows are $\tilde{t}_1 = 23$, $\tilde{t}_2 = 11$. The associated different ideal $\mathfrak{D}_1$ is the unique integral ideal of norm 23 in $\mathcal{O}_{\mathbb{K}}$ and $\mathfrak{D}_2$ is the unique ideal of norm 11 in $\mathcal{O}_{\mathbb{K}}$. We may choose $\mathfrak{b} = (1)$ and $\mathfrak{a}$ the unique integral ideal of norm $N = 7$ in $\mathbb{K}$. Both ideals $2qN\mathfrak{D}_1/\mathfrak{a}$ and $qN\mathfrak{D}_2/\mathfrak{a}$ are principal with strongly admissible generators $h_1 = 10z^3 + 110z^2 + 160z + 190$ and $h_2 = 5z^3 + 20z^2 + 10z + 25$ which are both totally positive. The corresponding levels of the computations are $\ell_1 = 5 \cdot 7 \cdot 23 = 805$ and $\ell_2 = 5 \cdot 7 \cdot 11 = 385$. Let us write the parameters

$$\begin{aligned} \tau &= 11z^3 + z^2 - 2z + 14898, & \tau' &= -4z^3 + 3z^2 + 6z + 79 \\ \sigma &= -z^3 + 2z^2 - 4z - 1737, & \sigma' &= -z^3 - 2z^2 - 4z + 567 \\ \rho &= -3z^3 + 6z^2 + 11z + 7577, & \rho' &= -z^3 - 2z^2 + 7z + 138 \end{aligned}$$

Then the two products

$$w_1 = \prod_{k=1,4} \frac{G_2\left(\frac{-2k}{5}, \frac{\tau}{805}, \frac{\sigma}{805}, \frac{\rho}{805}\right)^{-7}}{G_2\left(\frac{-14k}{5}, \frac{\tau}{35}, \frac{\sigma}{35}, \frac{\rho}{35}\right)^{-1}} \times \frac{G_2\left(\frac{k}{5}, \frac{\tau'}{385}, \frac{\sigma'}{385}, \frac{\rho'}{385}\right)^{7}}{G_2\left(\frac{7k}{5}, \frac{\tau'}{35}, \frac{\sigma'}{35}, \frac{\rho'}{35}\right)}$$

$$w_2 = \prod_{k=2,3} \frac{G_2\left(\frac{-2k}{5}, \frac{\tau}{805}, \frac{\sigma}{805}, \frac{\rho}{805}\right)^{7}}{G_2\left(\frac{-14k}{5}, \frac{\tau}{35}, \frac{\sigma}{35}, \frac{\rho}{35}\right)^{1}} \times \frac{G_2\left(\frac{k}{5}, \frac{\tau'}{385}, \frac{\sigma'}{385}, \frac{\rho'}{385}\right)^{-7}}{G_2\left(\frac{7k}{5}, \frac{\tau'}{35}, \frac{\sigma'}{35}, \frac{\rho'}{35}\right)^{-1}}$$

coincide up to 1000 digits and are close to the root $\approx 4.3174875... - i \cdot 7.2167313...$ of the polynomial

$$P_{\mathrm{abs}} = x^8 - 8x^7 + 66x^6 + 39x^5 + 49x^4 + 39x^3 + 66x^2 - 8x + 1$$

which defines an absolute equation of $\mathbb{K}^+(\mathfrak{f})$ over $\mathbb{Q}$. The computation time for 1000 digits is 46 seconds. The fact that the two products seem to be equal supports conjecture III.34. We then check formula (III.44) as:

$$\mathcal{N}(\mathfrak{a})\zeta_{\mathfrak{f}}'([\mathfrak{b}],0) - \zeta_{\mathfrak{f}}'([\mathfrak{a}\mathfrak{b}],0) \approx \log|w_1|^2 \approx 4.2587554...$$

IV.2.5.2 Example 13

We now discuss a case where $q = 5$ and $\mathcal{Z}_{\mathfrak{f}}^1 \simeq \mathcal{Z}_{\mathfrak{f}}^{1,+} \simeq \mathbb{Z}/5\mathbb{Z}^\times$. In this case all L-functions attached to $\mathbb{K}^+(\mathfrak{f})$ have a zero of order at least 2 at 0 and $\mathbb{K}^+(\mathfrak{f})$ is the narrow Hilbert class field of $\mathbb{K}$, therefore the Stark unit we aim to compute is just 1. This may happen even in the case where $\mathbb{K}^+(\mathfrak{f})$ is totally complex as the Euler factor at $\mathfrak{f}$ in the functional equation of the L-functions attached to $\mathbb{K}^+(\mathfrak{f})$ may vanish.

Let us consider z the complex root of the polynomial $x^4 - x^3 - 2x^2 - 3x - 2$ lying in the upper half-plane. Then $\mathbb{K} = \mathbb{Q}(z)$ has class number 1. We choose $\mathfrak{f} = \mathfrak{P}_5$ the unique integral ideal of norm $q = 5$ in $\mathcal{O}_{\mathbb{K}}$. The corresponding narrow ray class group is $\mathrm{Cl}^+(\mathfrak{f}) \simeq \mathbb{Z}/2\mathbb{Z}$. We choose the fundamental units

$$\varepsilon_1 = z + 1, \ \varepsilon_2 = z^3 - z^2 - 3z - 1$$

for $\mathcal{O}_{\mathbb{K}}^{+,\times}$. They satisfy $\varepsilon_1 \equiv -1 \bmod \mathfrak{f}$ and $\varepsilon_2 \equiv 3 \bmod \mathfrak{f}$, thus $\mathcal{O}_{\mathfrak{f}}^{+,\times}$ is generated by ε_1^2 and $\varepsilon_1\varepsilon_2^2$. We compute the associated contents $\tilde{\lambda}_1 = \tilde{\lambda}_2 = 1$ as well as the associated overflows $\tilde{t}_1 = \tilde{t}_2 = 1$. The different ideals $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are therefore both equal to $\mathcal{O}_{\mathbb{K}}$. We may choose $\mathfrak{b} = (1)$ and $\mathfrak{a}$ the unique integral ideal of norm $N = 7$ in $\mathcal{O}_{\mathbb{K}}$. The ideal $qN/\mathfrak{a}$ is generated by the strongly admissible base points $h_1 = h_2 = 5z^3 - 10z - 25$. The corresponding levels in the computations are $\ell_1 = \ell_2 = 5 \cdot 7 = 35$. Let us write the parameters

$$\begin{aligned} \tau &= 5z^3 - 9z^2 - 3z + 1365, & \tau' &= -z^3 + 2z^2 - 456 \\ \sigma &= -z^3 + 2z^2 + z - 219, & \sigma' &= -z^3 + z^2 + 3z + 459 \\ \rho &= -z^3 + 2z^2 - 456, & \rho' &= z^3 - 2z^2 - z + 219 \end{aligned}$$

Then

$$\prod_{k=1}^{4} \frac{G_2\left(\frac{-k}{5}, \frac{\tau}{35}, \frac{\sigma}{35}, \frac{\rho}{35}\right)^{-7}}{G_2\left(\frac{-7k}{5}, \frac{\tau}{5}, \frac{\sigma}{5}, \frac{\rho}{5}\right)^{-1}} \times \frac{G_2\left(\frac{k}{5}, \frac{\tau'}{35}, \frac{\sigma'}{35}, \frac{\rho'}{35}\right)^{7}}{G_2\left(\frac{7k}{5}, \frac{\tau'}{5}, \frac{\sigma'}{5}, \frac{\rho'}{5}\right)} \tag{IV.7}$$

is numerically ≈ 1 which is a correct Stark unit when all L-functions have a zero of order 2 at 0. The computation time for 1000 digits is 35 seconds. Computations show that the above product is not a power of one of its subproducts and it would already be a non-trivial statement to prove that the evaluation (IV.7) is equal to 1.

IV.2.5.3 Example 14

We now discuss a case where $q = 7$, $\mathcal{Z}_{\mathfrak{f}}^1 \simeq \mathcal{Z}_{\mathfrak{f}}^{1,+} \simeq \{(1,1),(6,-1)\}$. Consider z the complex root of the polynomial $x^4 - x^3 - x^2 - 5x + 1$ lying in the upper half-plane. Then $\mathbb{K} = \mathbb{Q}(z)$ has class number 1. We choose $\mathfrak{f} = \mathfrak{P}_7$ the unique integral ideal of norm $q = 7$ in $\mathcal{O}_{\mathbb{K}}$. The corresponding narrow ray class group is $\mathrm{Cl}^+(\mathfrak{f}) \simeq \mathbb{Z}/6\mathbb{Z}$. We choose the fundamental units

$$\varepsilon_1 = 3z^3 + 4z^2 + 6z - 1, \ \varepsilon_2 = -z^3 + z^2 + 3z$$

for $\mathcal{O}_{\mathbb{K}}^{+,\times}$. They satisfy $\varepsilon_1 \equiv 1 \bmod \mathfrak{f}$ and $\varepsilon_2 \equiv -1 \bmod \mathfrak{f}$, thus $\mathcal{O}_{\mathfrak{f}}^{+,\times}$ is generated by ε_1 and ε_2^2. We compute the associated contents $\tilde{\lambda}_1 = \tilde{\lambda}_2 = 1$, as well as the overflows $\tilde{t}_1 = 67$, $\tilde{t}_2 = 1$. The different ideals associated to the construction are $\mathfrak{D}_1$ the unique prime ideal of norm 67 in $\mathcal{O}_{\mathbb{K}}$ and $\mathfrak{D}_2 = \mathcal{O}_{\mathbb{K}}$. We may choose $\mathfrak{b} = (1)$ and $\mathfrak{a}$ the unique integral ideal of norm $N = 11$ in $\mathcal{O}_{\mathbb{K}}$. The ideals $qN\mathfrak{D}_1/\mathfrak{a}$ and $3qN\mathfrak{D}_2/\mathfrak{a}$ are generated by the strongly admissible base points $h_1 = -7z^3 + 49z^2 - 133z + 63$ and $h_2 = -105z^3 + 42z^2 + 84z + 483$ respectively. The corresponding levels of the computations are given by $\ell_1 = 7 \cdot 11 \cdot 67 = 5159$ and $\ell_2 = 7 \cdot 11 = 77$. Let us write the parameters

$$\begin{aligned} \tau &= 7z^3 - 13z^2 - 15z - 19098, & \tau' &= 5z^3 - 4z^2 - 6z + 2607 \\ \sigma &= 9z^3 + 12z^2 + 19z + 38971, & \sigma' &= z^3 - z^2 - z + 477 \\ \rho &= 3z^3 + 4z^2 - 16z - 15641, & \rho' &= -2z^3 + 2z^2 + 3z - 827 \end{aligned}$$

For $k = 1, 2, 3, 4, 5, 6$, write I_k for the value

$$I_k = \frac{G_2\left(\frac{k}{7}, \frac{\tau}{5159}, \frac{\sigma}{5159}, \frac{\rho}{5159}\right)^{11}}{G_2\left(\frac{11k}{7}, \frac{\tau}{469}, \frac{\sigma}{469}, \frac{\rho}{469}\right)} \times \frac{G_2\left(\frac{-3k}{7}, \frac{\tau'}{77}, \frac{\sigma'}{77}, \frac{\rho'}{77}\right)^{-11}}{G_2\left(\frac{-33k}{7}, \frac{\tau'}{7}, \frac{\sigma'}{7}, \frac{\rho'}{7}\right)^{-1}}.$$

On the one hand, $I_1 I_6 \approx 204411.4289667... + i \cdot 1113079.0424875...$ is the product in the right-hand side of formula (III.42), which we check as:

$$\mathcal{N}(\mathfrak{a})\zeta_{\mathfrak{f}}'([\mathfrak{b}], 0) - \zeta_{\mathfrak{f}}'([\mathfrak{ab}], 0) \approx \log|I_1 I_6|^2 \approx 27.8784505...$$

On the other hand, the values $I_1 I_6$, $I_2 I_5 \approx 0.0274558... - i \cdot 0.0123306...$ and $I_3 I_4 \approx 10^{-5}(1.6667587... - i \cdot 2.4168791...)$ seem to be algebraically dependent and we may identify up to high precision the relative polynomial

$$\begin{aligned} \prod_{k=1,2,3} (X - I_k I_{7-k})(X - I_k^{-1} I_{7-k}^{-1}) \approx \\ & X^6 + 1 + (-66007z^3 + 315205z^2 - 404670z + 66406)(X^5 + X) \\ & +(-2980477827z^3 - 882760955z^2 + 19370690773z - 3656696288)(X^4 + X^2) \\ & +(153504704659z^3 - 136634711339z^2 - 544162307740z + 108155656031)X^3 \end{aligned}$$

over $\mathbb{K}$. This relative polynomial defines a relative equation of $\mathbb{K}^+(\mathfrak{f})$ over $\mathbb{K}$. The computation time for 1000 digits is 6 minutes.

IV.2.6 Examples which fall outside of the scope of the conjecture

In this section we present three examples which fall outside of the scope of the Main Conjecture. First, we chose to illustrate the necessity of the compatibility condition that the base points must satisfy by considering an incompatible set of base points and showing that the statements of our Main Conjecture III.37 cannot hold in that case. Next, we illustrate two cases where some of the hypotheses (H3), (H4) are not satisfied, and show that most of the work we did in chapter III may be adapted to this case, and that we may still compute higher elliptic units in those settings.

IV.2.6.1 Example with incompatible base points

In this section we revisit our simplest quartic example (see section IV.2.2.1) to showcase what happens when an incompatible set $\underline{h}$ is chosen when attempting to compute higher elliptic units (see Definition III.24 for the definition of a compatible set). Computations show that the complex number $\mathcal{V}_{\mathfrak{f},\mathfrak{c},\mathfrak{a}}$ we compute does not seem to be algebraic and does not satisfy the Kronecker limit formula we expect.

Let z be the complex root of the polynomial $x^4-6x^3-x^2-3x+1$ lying in the upper half-plane. Then $\mathbb{K}=\mathbb{Q}(z)$ has class number 1. We choose $\mathfrak{f}=\mathfrak{P}_2$ the unique prime ideal of norm $q=2$ in $\mathcal{O}_\mathbb{K}$. The corresponding narrow ray class group is $\mathrm{Cl}^+(\mathfrak{f})\simeq\mathbb{Z}/2\mathbb{Z}$. We choose the fundamental units

$$\varepsilon_1=\frac{-2z^3+13z^2-z+3}{7},\ \varepsilon_2=\frac{-5z^3+29z^2+15z+18}{7}$$

for the set $\mathcal{O}_\mathbb{K}^{+,\times}=\mathcal{O}_\mathfrak{f}^{+,\times}$ of totally positive units (congruent to 1 mod $\mathfrak{f}$). We compute the contents $\tilde{\lambda}_1=\tilde{\lambda}_2=1$, the overflows $\tilde{t}_1=\tilde{t}_2=1$ as defined in section III.3.2. We may choose $\mathfrak{b}=(1)$ and $\mathfrak{a}$ the unique integral ideal of norm $N=13$ in $\mathcal{O}_\mathbb{K}$. The ideal $qN/\mathfrak{a}$ has an admissible generator $h_1=(44z^3-258z^2-104z-80)/7$ (see Definition III.4). The set $\underline{h}=(h_1,h_1)$ would be a compatible set in the sense of Definition III.24. Let us choose this time h_2 a generator of the ideal $qN.23/(\mathfrak{a}\mathfrak{P}_{23})$ where $\mathfrak{P}_{23}$ is the unique integral ideal of norm 23 in $\mathcal{O}_\mathbb{K}$ (that is the ideal $23/\mathfrak{P}_{23}$ is a helper ideal). The corresponding levels will be $\ell_1=2\cdot 13$ and $\ell_2=2\cdot 13\cdot 23=598$. Let us write the parameters

$$\begin{aligned}
\tau&=\frac{-5z^3+29z^2+15z+95}{7}, & \tau'&=\frac{-2z^3+13z^2-z+7668}{7}\\
\sigma&=\frac{-6z^3+39z^2-10z-47}{7}, & \sigma'&=\frac{-5z^3+29z^2+15z+14473}{7}\\
\rho&=\frac{2z^3-13z^2+z-24}{7}, & \rho'&=\frac{2z^3-13z^2-6z-24167}{7}.
\end{aligned}$$

Then

$$\begin{aligned}
\log\left|\frac{G_2\left(\frac{1}{2},\frac{\tau}{26},\frac{\sigma}{26},\frac{\rho}{26}\right)^{13}}{G_2\left(\frac{13}{2},\frac{\tau}{2},\frac{\sigma}{2},\frac{\rho}{2}\right)}\right|&\approx -4.1729132...\\
\log\left|\frac{G_2\left(\frac{1}{2},\frac{\tau'}{598},\frac{\sigma'}{598},\frac{\rho'}{598}\right)^{13}}{G_2\left(\frac{13}{2},\frac{\tau'}{46},\frac{\sigma'}{46},\frac{\rho'}{46}\right)}\right|&\approx -29.8019278...\\
13\zeta_\mathfrak{f}'(\mathcal{O}_\mathbb{K},0)-\zeta_\mathfrak{f}'([\mathfrak{a}],0)&\approx 0.3126630...
\end{aligned}$$

Therefore neither of the four possible choices of orientations give the desired Kronecker limit formula and no algebraicity property is expected for the product or the quotient of these values. This justifies the compatibility condition we imposed on the set of base points $\underline{h}$ (see Definition III.24).

IV.2.6.2 Example 16: a cubic example with $\tilde{\lambda}>1$

This example falls outside of the hypotheses of the conjecture as hypothesis (H3) is not satisfied. In the situation we describe below, $\tilde{\lambda}=2$, and we show that we can mostly ignore this fact for most of our construction, the only modification to take into account being the fact that the set $F(\underline{a},\underline{\alpha},0)$ contains 2 elements instead of only the element 0. As we show, the higher elliptic unit we compute is still expected to be algebraic and to satisfy the statements of the Main Conjecture III.37.

Let z be the complex root of the polynomial x^3-5 in the upper half-plane. Then $\mathbb{K}=\mathbb{Q}(z)$ has class number 1. We choose the ideal $\mathfrak{f}$ as the unique integral ideal of norm

$q = 3$ in $\mathcal{O}_{\mathbb{K}}$. The corresponding narrow ray class group is $\mathrm{Cl}^+(\mathfrak{f}) \simeq \mathbb{Z}/2\mathbb{Z}$. The unit $\varepsilon = 2z^2 - 4z + 1$ is a generator for $\mathcal{O}_{\mathbb{K}}^{+,\times} = \mathcal{O}_{\mathfrak{f}}^{+,\times}$ with content $\tilde{\lambda} = 2$ and overflow $\tilde{t} = 2 \times 13$. We choose $\mathfrak{b} = 1$ and $\mathfrak{a}$ the unique integral ideal of norm $N = 5$ in $\mathcal{O}_{\mathbb{K}}$. The part of the different ideal which is coprime to 2 is equal the integral ideal $\mathfrak{P}_{13} = 13\mathcal{O}_{\mathbb{K}} + (z+2)\mathcal{O}_{\mathbb{K}}$ which has norm 13. The ideal $qN\mathfrak{P}_{13}/\mathfrak{a}$ is generated by a strongly admissible base point $h = 6z^2 + 15$. The corresponding level of the computation is given by $\ell = 3 \cdot 5 \cdot 13$. The complex number $\Gamma^{-}_{\mathfrak{f},\mathfrak{b},\mathfrak{a}}(\varepsilon, h)^{-1}$ is then given by a product of two ordinary smoothed elliptic Gamma functions with parameters

$$\tau = \frac{\varepsilon^{-1} + 119}{2 \times 195}, \ \sigma = \frac{\varepsilon + 59}{2 \times 195}$$

More precisely, put $\delta = -2z^2 - z - 5$. Then the set $F(\underline{a}, \underline{\alpha}, 0)$ in formula (I.15) is exactly $\{0, \delta\}$ with $\delta/\gamma = (-4z^2 - 5z - 55)/195$. The product

$$\frac{\Gamma\left(\frac{1}{3}, \frac{\varepsilon^{-1}+119}{390}, \frac{\varepsilon+59}{390}\right)^{-5}}{\Gamma\left(\frac{5}{3}, \frac{\varepsilon^{-1}+119}{78}, \frac{\varepsilon+59}{78}\right)^{-1}} \cdot \frac{\Gamma\left(\frac{1}{3} + \frac{-4z^2-5z-55}{195}, \frac{\varepsilon^{-1}+119}{390}, \frac{\varepsilon+59}{390}\right)^{-5}}{\Gamma\left(\frac{5}{3} + \frac{-4z^2-5z-55}{39}, \frac{\varepsilon^{-1}+119}{78}, \frac{\varepsilon+59}{78}\right)^{-1}}$$

coincides up to high precision with the root $\approx 9.8439696... + i \cdot 5.1060682...$ of the palindromic polynomial

$$P_{\mathrm{abs}} = x^6 - 21x^5 + 150x^4 - 185x^3 + 150x^2 - 21x + 1$$

which defines an absolute equation of $\mathbb{K}^+(\mathfrak{f})$ over $\mathbb{Q}$. The computation time for 1000 digits is 11 seconds.

IV.2.6.3 Example with $\mathfrak{D}(\tilde{a}, \mathcal{O}_{\mathbb{K}})$ not coprime to q

In this section we revisit example IV.2.2.4 with a different choice of fundamental units which gives evaluation parameters outside of $\mathbb{R}$, at the cost of having $gcd(\tilde{t}_1, \tilde{t}_2, q = 2) > 1$. In this situation the sets F_1 and F_2 involved in the computation of the geometric G_2 functions (see Proposition I.7) both have size 4. Most of the work we did in chapter III may be adapted to this case and the higher elliptic units we compute is expected to satisfy the statements of our Main Conjecture III.37.

Let z be the complex root of the polynomial $x^4 - 12$ lying in the upper half-plane. Then $\mathbb{K} = \mathbb{Q}(z)$ has class number 1 and *contains the real quadratic field* $\mathbb{Q}(\sqrt{3})$. We choose the ideal $\mathfrak{f}$ the unique integral ideal of norm $q = 2$ in $\mathcal{O}_{\mathbb{K}}$. The corresponding narrow ray class group is $\mathrm{Cl}^+(\mathfrak{f}) \simeq \mathbb{Z}/2\mathbb{Z}$. We choose the fundamental units

$$\varepsilon_1 = \frac{z^2 + 2z + 2}{4}, \ \varepsilon_2 = \frac{-z^3 + z^2 + 4z - 2}{4}$$

for $\mathcal{O}_{\mathbb{K}}^{+,\times} = \mathcal{O}_{\mathfrak{f}}^{+,\times}$. We compute the contents $\tilde{\lambda}_1 = \tilde{\lambda}_2 = 1$ and the overflows $\tilde{t}_1 = 2 \cdot 3$, $\tilde{t}_2 = 2 \cdot 11$. We note that $4 \mid \tilde{s}_1$ and $4 \mid \tilde{s}_2$. We may choose $\mathfrak{b} = (1)$ and $\mathfrak{a} = 23\mathcal{O}_{\mathbb{K}} + \beta\mathcal{O}_{\mathbb{K}}$ an integral ideal of norm $N = 23$ in $\mathcal{O}_{\mathbb{K}}$ where $\beta = (z^2 - 2z - 26)/4$. The part of the different ideals $\mathfrak{D}_1, \mathfrak{D}_2$ coprime to $q = 2$ are given respectively by $\mathfrak{P}_3$ the unique prime ideal of norm 3 in $\mathcal{O}_{\mathbb{K}}$ and by $\mathfrak{P}_{11} = 11\mathcal{O}_{\mathbb{K}} + (z-1)\mathcal{O}_{\mathbb{K}}$. The ideals $qN\mathfrak{P}_3/\mathfrak{a}$ and $qN\mathfrak{P}_{11}/\mathfrak{a}$ are generated by the strongly admissible base points $h_1 = (-4z^3 + 11z^2 + 10z + 30)/2$ and $h_2 = (9z^3 + 4z^2 - 34z - 56)/2$. The corresponding levels of the computations are

$\ell_1 = 2 \cdot 3 \cdot 23$ and $\ell_2 = 2 \cdot 11 \cdot 23$. The value of $\mathcal{W}_{\mathfrak{f},\mathfrak{b},\mathfrak{a}}$ is given by a product of 8 ordinary smoothed G_2 functions and it coincides up to high precision with a root of the polynomial

$$P_{\text{abs}} = x^8 - 28x^7 + 778x^6 - 112x^5 - 749x^4 - 112x^3 + 778x^2 - 28x + 1$$

which defines an absolute equation of $\mathbb{K}^+(\mathfrak{f})$ over $\mathbb{Q}$. The computation time for 1000 digits is 8 seconds. The value $\mathcal{W}_{\mathfrak{f},\mathfrak{b},\mathfrak{a}}$ we obtained in this example is the same as the value obtained for example IV.2.2.4 where an alternative computation is performed and so is the polynomial P_{abs} we obtain, thus supporting the idea that the evaluation of higher elliptic units is indeed independent of the choice for the set of fundamental units for $\mathcal{O}_{\mathbb{K}}^{+,\times}$.

Conclusion and future work

In this work we constructed geometric families of multiple elliptic Gamma functions and associated Bernoulli rational functions, allowing us to rephrase the modularity properties they satisfy in terms of the cohomology of special linear groups $\mathrm{SL}_n(\mathbb{Z})$. We gave arithmetic applications of these functions by showing strong connections with partial zeta functions for both totally real and almost totally real number fields. We constructed conjectural higher elliptic units above number fields with exactly one complex place and provided the tools to verify the conjecture numerically. We also provided many examples to support this conjecture.

In the future, we wish to expand the conjecture to cover more cases outside of the simplifying hypotheses (H3), (H4) and (H5). To do this we would attempt to produce more examples in settings outside of these hypotheses, which unfortunately take more time to compute. We also wish to prove some simple cases of the conjecture (for instance in optimal settings). The strategy to prove these simple cases would be to use the cocycle properties satisfies by the higher elliptic Gamma functions to prove some relations between the complex numbers we compute which should be algebraic conjugates.

Another direction for future work is an exploration of various other number theoretic aspects of the G_r functions which we have not mentioned yet. Namely, we wish to study the arithmetic properties of the sequence of coefficients in the Fourier expansion of these functions. It is well-known that for modular forms these coefficients are integral and enjoy many interesting properties, and it is also well-known that the Fourier coefficients of θ functions are related to representations of numbers by sums of squares. It might be possible to formulate an analoguous statement for the Fourier coefficients of the multiple elliptic Gamma functions. Another important area of research regarding these functions is the determination of the action of Hecke operators for congruence subgroups of $\mathrm{SL}_n(\mathbb{Z})$ on the geometric functions $G_{n-2,a_1,\ldots,a_{n-1}}$ and $B_{n,a_1,\ldots,a_n}$.

Finally, we wish to explore the possible connections between our construction of higher elliptic units related to Stark units with the construction of Gross-Stark units in p-adic spaces. Indeed, the family of higher Bernoulli rational functions $B_{n,a_1,\ldots,a_n}$ can be used to construct a measure on a p-adic space $V_p \simeq \mathbb{Z}_p^n$ which can be seen as a higher analogue of the p-adic measure used by Darmon, Pozzi and Vonk in [DPV24] to produce Gross-Stark units above real quadratic fields. In particular, the measures built from our higher Bernoulli functions should be related in some way to the measures used by Roset and Xu in [RX25] to produce Gross-Stark units above general totally real number fields. The fact that our objects are related to both values of partial zeta functions at $s = 0$ for totally real number fields and to derivatives of partial zeta functions at $s = 0$ for ATR fields makes the existence of such a connection very likely.

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