Deformation quantization modules

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Introduction

In a few words these Notes could be considered both as an introduction to non commutative complex analytic geometry and to the study of microdifferential systems. Indeed, on a complex manifold X, we replace the structure sheaf $\mathscr{O}_X$ with a formal deformation of it, that is, a DQ-algebra, or better, a DQ-algebroid, and study modules over this ring, extending to this framework classical results of Cartan-Serre and Grauert, and also classical results on Hochschild classes and the index theorem. Here, DQ stands for "deformation quantization". But the theory of modules over DQ-algebroids is also a natural generalization of that of $\mathscr{D}$-modules. Indeed, when the Poisson structure underlying the deformation is symplectic, the study of DQ-modules naturally generalizes that of microdifferential modules, and sometimes makes it easier (see Theorem 7.2.3).

The notion of a star product is now a classical subject studied by many authors and naturally appearing in various contexts. Two cornerstones of its history are the paper [1] (see also [2, 3]) who defines $\star$-products and the fundamental result of [47] which, roughly speaking, asserts that any real Poisson manifold may be "quantized", that is, endowed with a star algebra to which the Poisson structure is associated. It is now a well-known fact (see [37, 46]) that, in order to quantize complex Poisson manifolds, sheaves of algebras are not well-suited and have to be replaced by algebroid stacks. We refer to [13, 66] for further developments.

In this paper, we consider complex manifolds endowed with DQ-algebroids, that is, algebroid stacks locally associated to sheaves of star-algebras, and study modules over such algebroids. The main results of this paper are:

- a finiteness theorem, which asserts that the convolution of two coherent kernels, satisfying a suitable properness assumption, is coherent (a kind of Grauert's theorem),
- the construction of the dualizing complex and a duality theorem, which asserts that duality commutes with convolution,
- the construction of the Hochschild class of coherent DQ-modules and

the theorem which asserts that Hochschild class commutes with convolution,

- the link between Hochschild classes and Chern classes and also with Euler classes, in the commutative case,
- the constructibility of the complex of solutions of an holonomic module into another one in the the symplectic case.

Let us describe this paper with some details.

In **Chapter 1**, we systematically study rings (*i.e.,* sheaves of rings) which are formal deformations of rings, and modules over such deformed rings. More precisely, consider a topological space X, a commutative unital ring $\mathbb{K}$ and a sheaf $\mathscr{A}$ of $\mathbb{K}[[\hbar]]$-algebras on X which is $\hbar$-complete and without $\hbar$-torsion. We also assume that there exists a base of open subsets of X, acyclic for coherent modules over $\mathscr{A}_0 := \mathscr{A}/\hbar\mathscr{A}$.

We first show how to deduce various properties of the ring $\mathscr{A}$ from the corresponding properties on $\mathscr{A}_0$. For example, $\mathscr{A}$ is a Noetherian ring as soon as $\mathscr{A}_0$ is a Noetherian ring, and an $\mathscr{A}$-module $\mathscr{M}$ is coherent as soon as it is locally finitely generated and $\hbar^n\mathscr{M}/\hbar^{n+1}\mathscr{M}$ is $\mathscr{A}_0$-coherent for all $n \geq 0$. Then, we introduce the property of being cohomologically complete for an object of the derived category $\mathrm{D}(\mathscr{A})$. We prove that this notion is local, stable by direct images and an object $\mathscr{M}$ with bounded coherent cohomology is cohomologically complete. Conversely, if $\mathscr{M}$ is cohomologically complete, it has coherent cohomology objects as soon as its graded module $\mathscr{A}_0 \overset{\mathrm{L}}{\otimes}_{\mathscr{A}} \mathscr{M}$ has coherent cohomology over $\mathscr{A}_0$ (see Theorem 1.6.4). We also give a similar criterion which ensures that an $\mathscr{A}$-module is flat.

In **Chapter 2** we consider the case where X is a complex manifold, $\mathbb{K} = \mathbb{C}$, $\mathscr{A}_0 = \mathscr{O}_X$ and $\mathscr{A}$ is locally isomorphic to an algebra $(\mathscr{O}_X[[\hbar]], \star)$ where $\star$ is a star-product. It is an algebra over $\mathbb{C}^\hbar := \mathbb{C}[[\hbar]]$. We call such an algebra $\mathscr{A}$ a DQ-algebra. We also consider DQ-algebroids, that is, $\mathbb{C}^\hbar$-algebroids (in the sense of stacks) locally equivalent to the algebroid associated with a DQ-algebra. Remark that a DQ-algebroid on a manifold X defines a Poisson structure on it. Conversely, a famous theorem of Kontsevich [47] asserts that on a real Poisson manifold there exists a DQ-algebra to which this Poisson structure is associated. In the complex case, there is a similar result using DQ-algebroids. This is a theorem of [46] after a related result of [37] in the contact case.

If $(X, \mathscr{A}_X)$ is a complex manifold X endowed with a DQ-algebroid $\mathscr{A}_X$, we denote by X^a the manifold X endowed with the DQ-algebroid $\mathscr{A}_X^{\mathrm{op}}$ opposite to $\mathscr{A}_X$.

We define the external product $\mathscr{A}_{X_1\times X_2}$ of two DQ-algebroids $\mathscr{A}_{X_1}$ and $\mathscr{A}_{X_2}$ on manifolds X_1 and X_2. There exists a canonical $\mathscr{A}_{X\times X^a}$-module $\mathscr{C}_X$ on $X\times X^a$ supported by the diagonal, which corresponds to the $\mathscr{A}_X$-bimodule $\mathscr{A}_X$.

On a complex manifold X endowed with a DQ-algebroid, we construct the $\mathbb{C}^\hbar$-algebroid $\mathscr{D}_X^{\mathscr{A}}$, a deformation quantization of the ring $\mathscr{D}_X$ of differential operators. It is a $\mathbb{C}^\hbar$-subalgebroid of $\mathscr{E}nd_{\mathbb{C}^\hbar}(\mathscr{A}_X)$. It turns out that $\mathscr{D}_X^{\mathscr{A}}$ is equivalent to $\mathscr{D}_X[[\hbar]]$. This new algebroid allows us to construct the dualizing complex $\omega_X^{\mathscr{A}}$ associated to a DQ-algebroid $\mathscr{A}_X$. This complex is the dual over $\mathscr{D}_X^{\mathscr{A}}$ of $\mathscr{A}_X$, similarly to the case of $\mathscr{O}_X$-modules. Note that the dualizing complex for DQ-algebras has already been considered in a more particular situation by [21, 22].

We also adapt to algebroids a results of [41] which allows us to replace a coherent $\mathscr{A}_X$-module by a complex of "almost free" modules, such an object being a locally finite sum $\oplus_{i\in I}(L_i)_{U_i}$, the L_i's being free $\mathscr{A}_X$-modules of finite rank defined on a neighborhood of $\overline{U_i}$. We give a similar result for algebraic manifolds.

Chapter 3. Consider three complex manifolds X_i endowed with DQ-algebroids $\mathscr{A}_{X_i}$ $(i=1,2,3)$. Let $\mathscr{K}_i\in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_{X_i\times X^a_{i+1}})$ $(i=1,2)$ be two coherent kernels and define their convolution by setting

$$\mathscr{K}_1\circ\mathscr{K}_2 := \mathrm{R}p_{14!}\big((\mathscr{K}_1\underline{\boxtimes}\mathscr{K}_2)\overset{\mathrm{L}}{\otimes}_{\mathscr{A}_{X_2\times X_2^a}}\mathscr{C}_{X_2}\big).$$

Here p_{14} denotes the projection of the product $X_1\times X_2^a\times X_2\times X_3^a$ to $X_1\times X_3^a$.

We prove in Theorem 3.2.1 that, under a natural properness hypothesis, the convolution $\mathscr{K}_1\circ\mathscr{K}_2$ belongs to $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_{X_1\times X_3^a})$ and in Theorem 3.3.3 that the convolution of kernels commutes with duality.

For further applications, it is also interesting to consider the localized algebroid $\mathscr{A}_X^{\mathrm{loc}}=\mathbb{C}^{\hbar,\mathrm{loc}}\otimes_{\mathbb{C}^\hbar}\mathscr{A}_X$, where $\mathbb{C}^{\hbar,\mathrm{loc}}=\mathbb{C}((\hbar))$. An $\mathscr{A}_X^{\mathrm{loc}}$-module $\mathscr{M}$ is good if for any relatively compact open subset U of X, there exists a coherent $\mathscr{A}_U$-module which generates $\mathscr{M}|_U$. Then we prove that there is a natural map of the Grothendieck groups $\mathrm{K}_{\mathrm{gd}}(\mathscr{A}_X^{\mathrm{loc}})\to\mathrm{K}_{\mathrm{coh}}(\mathrm{gr}_\hbar\mathscr{A}_U)$ and that this map is compatible to the composition of kernels.

Note that these theorems extend classical results of Cartan, Serre and Grauert on finiteness and duality for coherent $\mathscr{O}$-modules on complex manifolds to DQ-algebroids.

For papers related to DQ-algebras and DQ-algebroids on complex Poisson manifolds, and particularly to their classification, we refer to [5, 6, 8, 13, 51, 52, 62, 65].

Chapter 4. We introduce the Hochschild homology $\mathcal{HH}(\mathscr{A}_X)$ of the algebroid $\mathscr{A}_X$:

$$\mathcal{HH}(\mathscr{A}_X) := \mathscr{C}_{X^a} \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_{X\times X^a}} \mathscr{C}_X, \text{ an object of } \mathrm{D}^{\mathrm{b}}(\mathbb{C}_X^\hbar),$$

and, using the dualizing complex, we construct a natural convolution morphism

$$\underset{X_2}{\circ} : \mathrm{R}p_{13!}(p_{12}^{-1}\mathcal{HH}(\mathscr{A}_{X_1\times X_2^a}) \overset{\mathrm{L}}{\otimes} p_{23}^{-1}\mathcal{HH}(\mathscr{A}_{X_2\times X_3^a})) \to \mathcal{HH}(\mathscr{A}_{X_1\times X_3^a}).$$

To an object $\mathscr{M}$ of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X)$, we naturally associate its Hochschild class $\mathrm{hh}_X(\mathscr{M})$, an element of $H^0_{\mathrm{Supp}(\mathscr{M})}(X; \mathcal{HH}(\mathscr{A}_X))$. The main result of this chapter is Theorem 4.3.5 which asserts that taking the Hochschild class commutes with the convolution:

$$\mathrm{hh}_{X_1\times X_3^a}(\mathscr{K}_1 \circ \mathscr{K}_2) = \mathrm{hh}_{X_1\times X_2^a}(\mathscr{K}_1) \underset{X_2}{\circ} \mathrm{hh}_{X_2\times X_3^a}(\mathscr{K}_2). \tag{0.0.1}$$

In **Chapter 5**, we consider the case where the deformation is trivial. In this case, there is no need of the parameter $\hbar$ and we are in the well-known field of complex analytic geometry. Although the results of this chapter are considered as well-known (see in particular [34]), at least from the specialists, we have decided to include this chapter. Indeed, to our opinion, there is no satisfactory proof in the literature of the fact that the Hochschild class of coherent $\mathscr{O}_X$-modules is functorial with respect to convolution. We recall in particular the formula, in which the Todd class appears, which makes the link between Hochschild class and Chern classes. This formula was conjecturally stated by the first named author around 1991 and has only been proved very recently by Ramadoss [54] in the algebraic setting and by Grivaux [31] in the general case. For other papers closely related to this chapter, see [14, 15, 16, 34, 49, 59].

In **Chapter 6** we study Hochschild homology and Hochschild classes in the case where the Poisson structure associated to the deformation is symplectic. We prove then that the dualizing complex $\omega_X^{\mathscr{A}}$ is isomorphic to $\mathscr{C}_X$ shifted by d_X, the complex dimension of X, and we construct canonical morphisms

$$\hbar^{d_X/2}\mathbb{C}_X^\hbar\,[d_X] \to \mathcal{HH}(\mathscr{A}_X) \to \hbar^{-d_X/2}\mathbb{C}_X^\hbar\,[d_X] \tag{0.0.2}$$

whose composition is the canonical inclusion. The morphisms in (0.0.2) induce an isomorphism

$$\mathbb{C}_X^{\hbar,\mathrm{loc}}\,[d_X] \simeq \mathcal{HH}(\mathscr{A}_X^{\mathrm{loc}}). \tag{0.0.3}$$

The first morphism in (0.0.2) gives an intrinsic construction of the canonical class in $H^{-d_X}(X;\mathcal{HH}(\mathscr{A}_X))$ studied and used by several authors (see [12, 11, 26]). The isomorphism (0.0.3) allows us to associate an Euler class $\mathrm{eu}_X(\mathscr{M}) \in H^{d_X}_{\Lambda}(X;\mathbb{C}^{\hbar,\mathrm{loc}}_X)$ to any coherent $\mathscr{A}_X$-module $\mathscr{M}$ supported by a closed set Λ.

Then we show how our results apply to $\mathscr{D}$-modules. We recover in particular the Riemann-Roch theorem for $\mathscr{D}$-modules of [48] as well as the functoriality of the Euler class of $\mathscr{D}$-modules of [58].

Finally, in **Chapter 7**, we study holonomic $\mathscr{A}_X^{\mathrm{loc}}$-modules on complex symplectic manifolds. We prove that if $\mathscr{L}$ and $\mathscr{M}$ are two holonomic $\mathscr{A}_X^{\mathrm{loc}}$-modules, then the complex $\mathrm{R}\mathscr{H}om_{\mathscr{A}_X^{\mathrm{loc}}}(\mathscr{M},\mathscr{L})$ is perverse (hence, in particular, $\mathbb{C}$-constructible) over the field $\mathbb{C}^{\hbar,\mathrm{loc}}$.

If the intersection of the supports of the holonomic modules $\mathscr{L}$ and $\mathscr{M}$ is compact, formula (0.0.1) gives in particular

$$\chi(\mathrm{RHom}_{\mathscr{A}_X^{\mathrm{loc}}}(\mathscr{M},\mathscr{L})) = \int_X (\mathrm{eu}_X(\mathscr{M}) \cdot \mathrm{eu}_X(\mathscr{L})).$$

The Euler class of a holonomic module may be interpreted as a Lagrangian cycle, which makes its calculation quite easy.

If the modules $\mathscr{L}$ and $\mathscr{M}$ are simple along smooth Lagrangian submanifolds, then one can estimate the microsupport of this complex. This particular case had been already treated in [43] in the analytic framework, that is, using analytic deformations (in the sense of [55]), not formal deformations, and the proof given here is much simpler.

We also prove (Theorem 7.5.2) that if $\mathscr{L}_a$ is family of holonomic modules indexed by a holomorphic parameter a, then, under suitable geometrical hypotheses, the complex of global sections $\mathrm{RHom}_{\mathscr{A}_X^{\mathrm{loc}}}(\mathscr{M},\mathscr{L}_a)$, which belongs to $\mathrm{D}^{\mathrm{b}}_f(\mathbb{C}^{\hbar,\mathrm{loc}})$, does not depend on a. This is a kind of invariance by Hamiltonian symplectomorphism of this complex.

We have developed the theory in the framework of complex analytic manifolds. However, all along the manuscript, we explain how the results extend (and sometimes simplify) in the algebraic setting, that is on quasi-compact and separated smooth varieties over $\mathbb{C}$.

The main results of this paper, with the exception of Chapter 7, have been announced in [44, 45].

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Chapter 1

Modules over formal deformations

1.1 Preliminary

Some notations

Throughout this paper, $\mathbb{K}$ denotes a commutative unital ring.

We shall mainly follow notations of [42]. In particular, if $\mathscr{C}$ is a category, we denote by $\mathscr{C}^{\mathrm{op}}$ the opposite category. If $\mathscr{C}$ is an additive category, we denote by $\mathrm{C}(\mathscr{C})$ the category of complexes of $\mathscr{C}$ and by $\mathrm{C}^*(\mathscr{C})$ $(* = +, -, \mathrm{b})$ the full subcategory consisting of complexes bounded from below (resp. bounded from above, resp. bounded). If $\mathscr{C}$ is an abelian category, we denote by $\mathrm{D}(\mathscr{C})$ the derived category of $\mathscr{C}$ and by $\mathrm{D}^*(\mathscr{C})$ $(* = +, -, \mathrm{b})$ the full triangulated subcategory consisting of objects with bounded from below (resp. bounded from above, resp. bounded) cohomology. We denote as usual by $\tau^{\geq n}$, $\tau^{\leq n}$ etc. the truncation functors in $\mathrm{D}(\mathscr{C})$.

If A is a ring (or a sheaf of rings on a topological space X), an A-module means a left A-module. We denote by A^{op} the opposite ring of A. Hence an A^{op}-module is nothing but a right A-module. We denote by $\mathrm{Mod}(A)$ the category of A-modules. We set for short $\mathrm{D}(A) := \mathrm{D}(\mathrm{Mod}(A))$ and we write similarly $\mathrm{D}^*(A)$ instead of $\mathrm{D}^*(\mathrm{Mod}(A))$. We denote by $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(A)$ the full triangulated subcategory of $\mathrm{D}^{\mathrm{b}}(A)$ consisting of objects with coherent cohomology. If $\mathbb{K}$ is Noetherian, one denotes simply by $\mathrm{D}^{\mathrm{b}}_f(\mathbb{K})$ the full subcategory of $\mathrm{D}^{\mathrm{b}}(\mathbb{K})$ consisting of objects with finitely generated cohomology.

We denote by D'_X the duality functor for $\mathbb{K}_X$-modules:

$$\mathrm{D}'_X(\,\bullet\,) := \mathrm{R}\mathscr{H}om_{\mathbb{K}_X}(\,\bullet\,, \mathbb{K}_X). \tag{1.1.1}$$

and we simply denote by $(\bullet)^\star$ the duality functor on $\mathrm{D}^{\mathrm{b}}(\mathbb{K})$:

$$(\bullet)^\star = \mathrm{RHom}_{\mathbb{K}}(\bullet, \mathbb{K}). \tag{1.1.2}$$

If $\mathbb{K}$ is Noetherian and with finite global dimension, $(\bullet)^\star$ sends $(\mathrm{D}^{\mathrm{b}}_f(\mathbb{K}))^{\mathrm{op}}$ to $\mathrm{D}^{\mathrm{b}}_f(\mathbb{K})$.

We denote by $\{\mathrm{pt}\}$ the set with a single element.

Finiteness conditions

Let X be a topological space and let $\mathscr{A}$ be a $\mathbb{K}_X$-algebra (*i.e.*, a sheaf of $\mathbb{K}$-algebras) on X. Let us recall a few classical definitions.

- An $\mathscr{A}$-module $\mathscr{M}$ is locally finitely generated if there locally exists an exact sequence

  $$\mathscr{L}_0 \to \mathscr{M} \to 0 \tag{1.1.3}$$

  such that $\mathscr{L}_0$ is locally free of finite rank over $\mathscr{A}$.

- An $\mathscr{A}$-module $\mathscr{M}$ is locally of finite presentation if there locally exists an exact sequence

  $$\mathscr{L}_1 \to \mathscr{L}_0 \to \mathscr{M} \to 0 \tag{1.1.4}$$

  such that $\mathscr{L}_1$ and $\mathscr{L}_0$ are locally free of finite rank over $\mathscr{A}$. This is equivalent to saying that there locally exists an exact sequence

  $$0 \to \mathscr{K} \xrightarrow{u} \mathscr{N} \to \mathscr{M} \to 0 \tag{1.1.5}$$

  where $\mathscr{N}$ is locally free of finite rank and $\mathscr{K}$ is locally finitely generated. This is also equivalent to saying that there locally exists an exact sequence

  $$\mathscr{K} \to \mathscr{N} \to \mathscr{M} \to 0 \tag{1.1.6}$$

  where $\mathscr{N}$ is locally of finite presentation and $\mathscr{K}$ is locally finitely generated.

- An $\mathscr{A}$-module $\mathscr{M}$ is pseudo-coherent if for any locally defined morphism $u\colon \mathscr{N} \to \mathscr{M}$ with $\mathscr{N}$ of finite presentation, $\operatorname{Ker} u$ is locally finitely generated. This is also equivalent to saying that any locally defined $\mathscr{A}$-submodule of $\mathscr{M}$ is locally of finite presentation as soon as it is locally finitely generated.

- An $\mathscr{A}$-module $\mathscr{M}$ is coherent if it is locally finitely generated and pseudo-coherent. A ring is a coherent ring if it is so as a module over itself. One denotes by $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{A})$ the full additive subcategory of $\mathrm{Mod}(\mathscr{A})$ consisting of coherent modules. Note that $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{A})$ is a full abelian subcategory of $\mathrm{Mod}(\mathscr{A})$, stable by extension, and the natural functor $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}) \to \mathrm{Mod}(\mathscr{A})$ is exact (see [42, Exe. 8.23]).

- An $\mathscr{A}$-module $\mathscr{M}$ is Noetherian (see [38, Def. A.7]) if it is coherent, $\mathscr{M}_x$ is a Noetherian $\mathscr{A}_x$-module for any $x \in X$, and for any open subset $U \subset X$, any filtrant family of coherent submodules of $\mathscr{M}|_U$ is locally stationary. (This means that given a family $\{\mathscr{M}_i\}_{i\in I}$ of coherent submodules of $\mathscr{M}|_U$ indexed by a filtrant ordered set I, with $\mathscr{M}_i \subset \mathscr{M}_j$ for $i \le j$, there locally exists $i_0 \in I$ such that $\mathscr{M}_{i_0} \xrightarrow{\sim} \mathscr{M}_j$ for any $j \ge i_0$.) A ring is a Noetherian ring if it is so as a left module over itself.

Mittag-Leffler condition and pro-objects

We refer to [56] for the notions of ind-object and pro-object as well as to [42] for an exposition. To an abelian category $\mathscr{C}$, one associates the abelian category $\mathrm{Pro}(\mathscr{C})$ of its pro-objects. Then $\mathscr{C}$ is a full abelian subcategory of $\mathrm{Pro}(\mathscr{C})$ stable by kernel, cokernel and extension, the natural functor $\mathscr{C} \hookrightarrow \mathrm{Pro}(\mathscr{C})$ is exact, and the functor "$\varprojlim$": $\mathrm{Fct}(I^{\mathrm{op}}, \mathscr{C}) \to \mathrm{Pro}(\mathscr{C})$ is exact for any small filtrant category I. In the sequel, we identify $\mathscr{C}$ with a full subcategory of $\mathrm{Pro}(\mathscr{C})$. If $\mathscr{C}$ admits small projective limits, we denote by π the left exact functor

$$\pi\colon \mathrm{Pro}(\mathscr{C}) \to \mathscr{C}, \quad \text{“}\varprojlim_i\text{”} X_i \mapsto \varprojlim_i X_i.$$

If $\mathscr{C}$ has enough injectives, then π admits a right derived functor (loc. cit.):

$$\mathrm{R}\pi\colon \mathrm{D}^+\big(\mathrm{Pro}(\mathscr{C})\big) \to \mathrm{D}^+(\mathscr{C}).$$

Definition 1.1.1. We say that an object $M \in \mathrm{Pro}(\mathscr{C})$ satisfies the Mittag-Leffler condition if, for any $N \in \mathscr{C}$ and any morphism $M \to N$ in $\mathrm{Pro}(\mathscr{C})$, $\mathrm{Im}(M \to N)$ is representable by an object of $\mathscr{C}$.

By the definition, any quotient of an object which satisfies the Mittag-Leffler condition also satisfies the Mittag-Leffler condition.

Lemma 1.1.2. *Let $\{M_n\}_{n\in\mathbb{Z}_{\ge 1}}$ be a projective system in an abelian category $\mathscr{C}$, and set $M = \text{“}\varprojlim_n\text{”} M_n \in \mathrm{Pro}(\mathscr{C})$. Then the following conditions are equivalent:*

(i) *M satisfies the Mittag-Leffler condition,*

(ii) $\{M_n\}_{n\in\mathbb{Z}_{\geq 1}}$ *satisfies the Mittag-Leffler condition* (*that is, for any* $p \geq 1$, *the sequence* $\{\mathrm{Im}(M_n \to M_p)\}_{n\geq p}$ *is stationary*),

(iii) *there exists a projective system* $\{M'_n\}_{n\in\mathbb{Z}_{\geq 1}}$ *in* $\mathscr{C}$ *such that the morphism* $M'_{n+1} \to M'_n$ *is an epimorphism for any* $n \geq 1$ *and we have an isomorphism* $M \simeq \text{“}\varprojlim_n\text{”}\, M'_n$ *in* $\mathrm{Pro}(\mathscr{C})$.

Proof. (i) $\Rightarrow$ (ii). For any $p \geq 1$, $\mathrm{Im}(M \to M_p) \simeq \text{“}\varprojlim_{n\geq p}\text{”}\, \mathrm{Im}(M_n \to M_p)$ is representable by an object of $\mathscr{C}$. Hence, the sequence $\{\mathrm{Im}(M_n \to M_p)\}_{n\geq p}$ is stationary.

(ii) $\Rightarrow$ (iii). Set $M'_n = \mathrm{Im}(M_k \to M_n)$ for $k \gg n$. Then the morphisms $M'_n \to M_n$ induce a morphism $f\colon \text{“}\varprojlim_n\text{”}\, M'_n \to \text{“}\varprojlim_n\text{”}\, M_n$. On the other hand, for each n, $M \to M_n$ decomposes as $M \to M'_n \rightarrowtail M_n$, since taking $k \gg n$ such that $M'_n = \mathrm{Im}(M_k \to M_n)$, we have a morphism $M \to M_k \to M'_n$. These morphisms induce a morphism $g\colon \text{“}\varprojlim_n\text{”}\, M_n = M \to \text{“}\varprojlim_n\text{”}\, M'_n$. It is easy to see that f and g are inverse to each other.

(iii) $\Rightarrow$ (i). For any $N \in \mathscr{C}$ and any morphism $f\colon M \to N$ in $\mathrm{Pro}(\mathscr{C})$, there exists p such that f decomposes into $M \to M'_p \to N$. Then $\mathrm{Im}(M \to N) \simeq \text{“}\varprojlim_{n\geq p}\text{”}\, \mathrm{Im}(M'_n \to N) \simeq \mathrm{Im}(M'_p \to N)$. Q.E.D.

Note that the following lemma is well known.

Lemma 1.1.3. *Let* $\{M_n\}_{n\geq 1}$ *be a projective system of* $\mathbb{Z}$*-modules. Then* $\mathrm{R}^i\pi(\text{“}\varprojlim_n\text{”}\, M_n) \simeq 0$ *for* $i \neq 0, 1$. *If* $\{M_n\}_{n\geq 1}$ *satisfies the Mittag-Leffler condition, then* $H^1\big(\mathrm{R}\pi\, \text{“}\varprojlim_n\text{”}\, M_n\big) \simeq 0$.

Here and in the sequel, we make the following convention.

Convention 1.1.4. When we have a left exact functor $\mathscr{C} \xrightarrow{F} \mathscr{C}'$ of abelian categories and $X \in \mathrm{D}(\mathscr{C})$, the notation $\mathrm{R}^iF(X)$ stands for $H^i\big(\mathrm{R}F(X)\big)$. For example, $\mathrm{R}^i\pi\mathrm{R}\Gamma(U;\mathscr{M})$ means $H^i\big(\mathrm{R}\pi\mathrm{R}\Gamma(U;\mathscr{M})\big)$.

Lemma 1.1.5. *Let* $\mathscr{R}$ *be an algebra over a topological space* X, *and let* $\{\mathscr{M}_n\}_{n\geq 0}$ *be a projective system of* $\mathscr{R}$*-modules. Set* $\mathscr{M} = \text{“}\varprojlim_n\text{”}\, \mathscr{M}_n \in \mathrm{Pro}(\mathrm{Mod}(\mathscr{R}))$. *Let* U *be an open subset of* X *and let* $i \in \mathbb{Z}$. *Then we have an exact sequence*

$$0 \longrightarrow \mathrm{R}^1\pi\big(\text{“}\varprojlim_n\text{”}\, H^{i-1}(U;\mathscr{M}_n)\big) \longrightarrow H^i(U;\mathrm{R}\pi\mathscr{M}) \longrightarrow \varprojlim_n H^i(U;\mathscr{M}_n) \to 0.$$

Proof. We have $\mathrm{R}\Gamma(U;\mathrm{R}\pi\mathscr{M}) \simeq \mathrm{R}\pi\mathrm{R}\Gamma(U;\mathscr{M})$ and we also have $H^i(U;\mathscr{M}) \simeq$ $\text{“}\varprojlim_n\text{”}\, H^i(U;\mathscr{M}_n)$. Consider the distinguished triangle

$$\mathrm{R}\pi\tau^{<i}\mathrm{R}\Gamma(U;\mathscr{M}) \longrightarrow \mathrm{R}\pi\mathrm{R}\Gamma(U;\mathscr{M}) \longrightarrow \mathrm{R}\pi\tau^{\geq i}\mathrm{R}\Gamma(U;\mathscr{M}) \xrightarrow{+1} .$$

It gives rise to the exact sequence

$$\begin{aligned} 0 \to \mathrm{R}^i\pi\tau^{<i}\mathrm{R}\Gamma(U;\mathscr{M}) \to \mathrm{R}^i\pi\mathrm{R}\Gamma(U;\mathscr{M}) \to \mathrm{R}^i\pi\tau^{\geq i}\mathrm{R}\Gamma(U;\mathscr{M}) \\ \to \mathrm{R}^{i+1}\pi\tau^{<i}\mathrm{R}\Gamma(U;\mathscr{M}). \end{aligned}$$

Since $\mathrm{R}^k\pi \,\text{“}\varprojlim_n\text{”}\, M_n = 0$ for $k \neq 0, 1$ and any projective system $\{M_n\}_n$, we obtain $\mathrm{R}^{i+1}\pi\tau^{<i}\mathrm{R}\Gamma(U;\mathscr{M}) = 0$.

Consider the distinguished triangle

$$\tau^{<i-1}\mathrm{R}\Gamma(U;\mathscr{M}) \to \tau^{<i}\mathrm{R}\Gamma(U;\mathscr{M}) \to H^{i-1}(U;\mathscr{M})[1-i] \xrightarrow{+1} .$$

Using the isomorphism $H^{i-1}(U;\mathscr{M}) \simeq \text{“}\varprojlim_n\text{”}\, H^{i-1}(U;\mathscr{M}_n)$ and applying the functor $\mathrm{R}\pi$, we get the distinguished triangle

$$\begin{aligned} \mathrm{R}\pi\tau^{<i-1}\mathrm{R}\Gamma(U;\mathscr{M}) \to \mathrm{R}\pi\tau^{<i}\mathrm{R}\Gamma(U;\mathscr{M}) \\ \to \mathrm{R}\pi\big(\text{“}\varprojlim_n\text{”}\, H^{i-1}(U;\mathscr{M}_n)[1-i]\big) \xrightarrow{+1} . \end{aligned}$$

We obtain $\mathrm{R}^i\pi\tau^{<i}\mathrm{R}\Gamma(U;\mathscr{M}) \simeq \mathrm{R}^1\pi \,\text{“}\varprojlim_n\text{”}\, H^{i-1}(U;\mathscr{M}_n)$. Finally, we have $\mathrm{R}^i\pi\tau^{\geq i}\mathrm{R}\Gamma(U;\mathscr{M}) \simeq \varprojlim_n H^i(U;\mathscr{M}_n)$. Q.E.D.

As a corollary of this lemma, we obtain the following lemma, a slight modified version of [32, Préliminaires, Prop. (13.3.1)].

Lemma 1.1.6. *Let X be a topological space, $\{\mathscr{F}_n\}_{n\in\mathbb{Z}_{>0}}$ a projective system of abelian sheaves on X and $\mathscr{F} := \varprojlim_n \mathscr{F}_n$. Assume the following conditions:*

(a) *for any $x \in X$ and any integer i, we have*

$$\varinjlim_{x\in U} \mathrm{R}^1\pi \,\text{“}\varprojlim_n\text{”}\, H^i(U;\mathscr{F}_n) \simeq 0,$$

where U ranges over an open neighborhood system of x,

(b) *for any $x \in X$ and $i > 0$, $\varinjlim_{x\in U}\big(\varprojlim_n H^i(U;\mathscr{F}_n)\big) = 0$, where U ranges over an open neighborhood system of x,* *Then for any i, the morphism*

$$h_i \colon H^i(X;\mathscr{F}) \to \varprojlim_n H^i(X;\mathscr{F}_n)$$

is surjective. If moreover $\{H^{i-1}(X;\mathscr{F}_n)\}_n$ satisfies the Mittag-Leffler condition, then h_i is an isomorphism.

Proof. Set $\mathscr{M} = \text{“}\varprojlim_n\text{”}\, \mathscr{F}_n$. By the preceding lemma, we have an exact sequence

$$0 \longrightarrow \mathrm{R}^1\pi\big(\text{“}\varprojlim_n\text{”}\, H^{i-1}(U;\mathscr{F}_n)\big) \longrightarrow H^i(U;\mathrm{R}\pi\mathscr{M}) \longrightarrow \varprojlim_n H^i(U;\mathscr{F}_n) \to 0.$$

For any x, taking the inductive limit with respect to U in an open neighborhood system of x, we obtain $(\mathrm{R}^i\pi\mathscr{M})_x = 0$ for $i \neq 0$. Hence we conclude $\mathrm{R}\pi\mathscr{M} \simeq \mathscr{F}$. Then the exact sequence above reads as

$$0 \longrightarrow \mathrm{R}^1\pi\big(\text{“}\varprojlim_n\text{”}\, H^{i-1}(X;\mathscr{F}_n)\big) \longrightarrow H^i(X;\mathscr{F}) \longrightarrow \varprojlim_n H^i(X;\mathscr{F}_n) \to 0.$$

Hence we have the desired result. Q.E.D.

1.2 Formal deformations of a sheaf of rings

Now we consider the following situation: X is a topological space, $\mathscr{A}$ is a $\mathbb{K}$-algebra on X and $\hbar$ is a section of $\mathscr{A}$ contained in the center of $\mathscr{A}$. We set

$$\mathscr{A}_0 := \mathscr{A}/\hbar\mathscr{A}$$

Let $\mathscr{M}$ be an $\mathscr{A}$-module. We set

$$\widehat{\mathscr{M}} := \varprojlim_n \mathscr{M}/\hbar^n\mathscr{M}, \tag{1.2.1}$$

and call it the $\hbar$-completion of $\mathscr{M}$. We say that

- $\mathscr{M}$ has no $\hbar$-torsion if $\hbar\colon \mathscr{M} \to \mathscr{M}$ is injective,
- $\mathscr{M}$ is $\hbar$-separated if $\mathscr{M} \to \widehat{\mathscr{M}}$ is a monomorphism, *i.e.*, $\bigcap_{n\geq 0} \hbar^n\mathscr{M} = 0$,
- $\mathscr{M}$ is $\hbar$-complete if $\mathscr{M} \to \widehat{\mathscr{M}}$ is an isomorphism.

Lemma 1.2.1. *Let $\mathscr{M} \in \mathrm{Mod}(\mathscr{A})$ and assume that $\mathscr{M}$ has no $\hbar$-torsion. Then*

(i) *$\widehat{\mathscr{M}}$ has no $\hbar$-torsion,*

(ii) *$\mathscr{M}/\hbar^n\mathscr{M} \xrightarrow{\sim} \widehat{\mathscr{M}}/\hbar^n\widehat{\mathscr{M}}$,*

(iii) *$\widehat{\mathscr{M}} \xrightarrow{\sim} \mathscr{M}^{\wedge\wedge}$, i.e., $\widehat{\mathscr{M}}$ is $\hbar$-complete.*

Proof. (i) Consider the exact sequence

$$0 \longrightarrow \mathscr{M}/\hbar^n\mathscr{M} \xrightarrow{\hbar^a} \mathscr{M}/\hbar^{n+a}\mathscr{M} \longrightarrow \mathscr{M}/\hbar^a\mathscr{M} \longrightarrow 0.$$

Applying the left exact functor $\varprojlim_n$ we get the exact sequence

$$0 \longrightarrow \widehat{\mathscr{M}} \xrightarrow{\hbar^a} \widehat{\mathscr{M}} \longrightarrow \mathscr{M}/\hbar^a\mathscr{M},$$

which gives the result.
(ii) Consider the commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{M} \arrow[r, "\hbar^n"] \arrow[d] & \mathscr{M} \arrow[r] \arrow[d] & \mathscr{M}/\hbar^n\mathscr{M} \arrow[r] \arrow[d, "\wr"] & 0 \\
0 \arrow[r] & \widehat{\mathscr{M}} \arrow[r, "\hbar^n"] & \widehat{\mathscr{M}} \arrow[r] & \mathscr{M}/\hbar^n\mathscr{M} &
\end{tikzcd}$$

(iii) Apply the functor $\varprojlim$ to the isomorphism in (ii). Q.E.D.

In this paper, with the exception of § 1.3, we assume the following conditions:

$$(1.2.2) \quad \begin{cases} \text{(i)} & \mathscr{A} \text{ has no } \hbar\text{-torsion,} \\ \text{(ii)} & \mathscr{A} \text{ is } \hbar\text{-complete,} \\ \text{(iii)} & \mathscr{A}_0 \text{ is a left Noetherian ring,} \end{cases}$$

and

$$(1.2.3) \quad \begin{cases} \text{(iv)} & \text{there exists a base } \mathfrak{B} \text{ of open subsets of } X \text{ such that} \\ & \text{for any } U \in \mathfrak{B} \text{ and any coherent } (\mathscr{A}_0|_U)\text{-module } \mathscr{F}, \text{ we} \\ & \text{have } H^n(U;\mathscr{F}) = 0 \text{ for any } n > 0. \end{cases}$$

It follows from (1.2.2) that, for an open set U and $a_n \in \mathscr{A}(U)$ $(n \geq 0)$, $\sum_{n\geq 0} \hbar^n a_n$ is a well-defined element of $\mathscr{A}(U)$.

By (1.2.2) (ii), $\hbar\mathscr{A}_x$ is contained in the Jacobson radical of $\mathscr{A}_x$ for any $x \in X$. Indeed, for any $a \in \hbar\mathscr{A}_x$, $1-a$ is invertible in $\mathscr{A}_x$ since a is defined on an open neighborhood U of x, and $1-a$ is invertible on U.

Hence Nakayama's lemma implies the following lemma that we frequently use.

Lemma 1.2.2. *Let $\mathscr{M}$ be a locally finitely generated $\mathscr{A}$-module.*

(i) *If $\mathscr{M}$ satisfies $\mathscr{M} = \hbar\mathscr{M}$, then $\mathscr{M} = 0$.*

(ii) *Let $f\colon \mathscr{N} \to \mathscr{M}$ be a morphism of $\mathscr{A}$-modules. If the composition $\mathscr{N} \to \mathscr{M} \to \mathscr{M}/\hbar\mathscr{M}$ is an epimorphism, then f is an epimorphism.*

For $n \in \mathbb{Z}_{\geq 0}$, set $\mathscr{A}_n = \mathscr{A}/\hbar^{n+1}\mathscr{A}$. Note that there is an equivalence of categories between the category $\mathrm{Mod}(\mathscr{A}_n)$ and the full subcategory of $\mathrm{Mod}(\mathscr{A})$ consisting of modules $\mathscr{M}$ satisfying $\hbar^{n+1}\mathscr{M} \simeq 0$.

Lemma 1.2.3. *Let $n \in \mathbb{Z}_{\geq 0}$.*

(i) *An $\mathscr{A}_n$-module $\mathscr{N}$ is locally finitely generated as an $\mathscr{A}_n$-module if and only if it is so as an $\mathscr{A}$-module.*

(ii) *An $\mathscr{A}_n$-module $\mathscr{N}$ is locally of finite presentation as an $\mathscr{A}_n$-module if and only if it is so as an $\mathscr{A}$-module.*

(iii) *An $\mathscr{A}_n$-module $\mathscr{N}$ is coherent as an $\mathscr{A}_n$-module if and only if it is so as an $\mathscr{A}$-module.*

(iv) *$\mathscr{A}_n$ is a left Noetherian ring.*

Proof. Note that since we have $\mathscr{A}_n \simeq \mathscr{A}/\mathscr{A}\hbar^{n+1}$, $\mathscr{A}_n$ is an $\mathscr{A}$-module locally of finite presentation.
(i) is obvious.
(ii)-(a) Let $\mathscr{M}$ be an $\mathscr{A}_n$-module locally of finite presentation and consider an exact sequence of $\mathscr{A}_n$-modules as in (1.1.5). Then $\mathscr{K}$ is locally finitely generated as an $\mathscr{A}$-module, $\mathscr{N}$ is locally of finite presentation as an $\mathscr{A}$-module and u is $\mathscr{A}$-linear. Hence, $\mathscr{M}$ is locally of finite presentation as an $\mathscr{A}$-module.
(ii)-(b) Conversely assume that $\mathscr{M}$ is an $\mathscr{A}_n$-module which is locally of finite presentation as an $\mathscr{A}$-module. Consider an exact sequence of $\mathscr{A}$-modules as in (1.1.4). Applying the functor $\mathscr{A}_n \otimes_{\mathscr{A}} \bullet$, we find and exact sequence of $\mathscr{A}_n$-modules as in (1.1.4), which proves that $\mathscr{M}$ is locally of finite presentation as an $\mathscr{A}_n$-module.

(iii) follows from (i) and (ii) since a module is coherent if it is locally finitely generated and any submodule locally finitely generated is locally of finite presentation.
(iv) Let us prove that $\mathscr{A}_n$ is a coherent ring. Since $\mathscr{A}_0$ is a coherent ring by the assumption, $\mathscr{A}_0$ is a coherent $\mathscr{A}$-module. Using the exact sequences of $\mathscr{A}$-modules

$$0 \to \mathscr{A}_{n-1} \xrightarrow{\hbar} \mathscr{A}_n \to \mathscr{A}_0 \to 0,$$

we get by induction on n that $\mathscr{A}_n$ is a coherent $\mathscr{A}$-module. Hence (iii) implies that $\mathscr{A}_n$ is a coherent ring.

One proves similarly by induction on n that $(\mathscr{A}_n)_x$ is a Noetherian ring for all $x \in X$ and that any filtrant family of coherent $\mathscr{A}_n$-submodules of a coherent $\mathscr{A}_n$-module is locally stationary. Q.E.D.

Lemma 1.2.4. *Let $U \in \mathfrak{B}$, and $n \geq 0$.*

(i) *For any coherent $\mathscr{A}_n$-module $\mathscr{N}$, we have $H^k(U; \mathscr{N}) = 0$ for $k \neq 0$.*

(ii) *For any epimorphism $\mathscr{N} \to \mathscr{N}'$ of coherent $\mathscr{A}_n$-modules, $\mathscr{N}(U) \to \mathscr{N}'(U)$ is surjective,*

(iii) *$\mathscr{A}(U) \to \mathscr{A}_n(U)$ is surjective.*

Proof. (i) is proved by induction on n, using the exact sequence

$$0 \to \hbar\mathscr{N} \to \mathscr{N} \to \mathscr{N}/\hbar\mathscr{N} \to 0. \tag{1.2.4}$$

(ii) follows immediately from (i) and the fact that $\mathscr{A}_n$ is a coherent ring.
(iii) By (ii), $\mathscr{A}_{n+1}(U) \to \mathscr{A}_n(U)$ is surjective for any $n \geq 0$. Hence, the morphism $\varprojlim_m \mathscr{A}_m(U) \to \mathscr{A}_n(U)$ is surjective. Since the functor $\varprojlim$ commutes with the functor $\Gamma(U; \bullet)$, $\mathscr{A}(U) \xrightarrow{\sim} \varprojlim_m \mathscr{A}_m(U)$ and the result follows. Q.E.D.

Properties of $\mathscr{A}$

Recall that $\mathscr{A}$ satisfies (1.2.2) and (1.2.3).

Theorem 1.2.5. (i) *$\mathscr{A}$ is a left Noetherian ring.*

(ii) *Let $\mathscr{M}$ be a locally finitely generated $\mathscr{A}$-module. Then $\mathscr{M}$ is coherent if and only if $\hbar^n\mathscr{M}/\hbar^{n+1}\mathscr{M}$ is a coherent $\mathscr{A}_0$-module for any $n \geq 0$.*

(iii) *Any coherent $\mathscr{A}$-module $\mathscr{M}$ is $\hbar$-complete,* i.e., $\mathscr{M} \xrightarrow{\sim} \widehat{\mathscr{M}}$.

(iv) *Conversely, an $\mathscr{A}$-module $\mathscr{M}$ is coherent if and only if it is $\hbar$-complete and $\hbar^n\mathscr{M}/\hbar^{n+1}\mathscr{M}$ is a coherent $\mathscr{A}_0$-module for any $n \geq 0$.*

(v) *For any coherent $\mathscr{A}$-module $\mathscr{M}$ and any $U \in \mathfrak{B}$, we have $H^j(U;\mathscr{M}) = 0$ for any $j > 0$.*

The proof of Theorem 1.2.5 decomposes into several lemmas.

Lemma 1.2.6. *Let $\mathscr{L}$ be a locally free $\mathscr{A}$-module of finite rank and let $\mathscr{N}$ be an $\mathscr{A}$-submodule of $\mathscr{L}$. Assume that*

(a) *$(\mathscr{N} + \hbar\mathscr{L})/\hbar\mathscr{L}$ is a coherent $\mathscr{A}_0$-module,*

(b) *$\mathscr{N} \cap \hbar^n\mathscr{L} \subset \hbar\mathscr{N} + \hbar^{1+n}\mathscr{L}$ for any $n \geq 1$.*

Then we have

(i) *$\mathscr{N}$ is a locally finitely generated $\mathscr{A}$-module,*

(ii) *$\mathscr{N} \cap \hbar^n\mathscr{L} = \hbar^n\mathscr{N}$ for any $n \geq 0$,*

(iii) *$\bigcap_{n\geq 0}(\mathscr{N} + \hbar^n\mathscr{L}) = \mathscr{N}$.*

Proof. First, let us show that

$$\mathscr{N} \cap \hbar\mathscr{L} \subset \hbar\mathscr{N} + \hbar^n\mathscr{L} \quad \text{for any } n \geq 0. \tag{1.2.5}$$

Indeed, (1.2.5) is trivial for $n \leq 1$. Let us argue by induction, and let $n \geq 2$, assuming the assertion for $n-1$. We have $\mathscr{N} \cap \hbar\mathscr{L} \subset \mathscr{N} \cap (\hbar\mathscr{N} + \hbar^{n-1}\mathscr{L}) = \hbar\mathscr{N} + (\mathscr{N} \cap \hbar^{n-1}\mathscr{L}) \subset \hbar\mathscr{N} + (\hbar\mathscr{N} + \hbar^n\mathscr{L})$ by the assumption (b). This proves (1.2.5).

Set

$$\widetilde{\mathscr{N}} = \bigcap_{n\geq 0}(\mathscr{N} + \hbar^n\mathscr{L}).$$

Then $\mathscr{N} \subset \widetilde{\mathscr{N}}$ and

$$\widetilde{\mathscr{N}} \cap \hbar\mathscr{L} \subset \hbar\widetilde{\mathscr{N}}. \tag{1.2.6}$$

Indeed we have $\widetilde{\mathscr{N}} \cap \hbar\mathscr{L} \subset (\mathscr{N} + \hbar^{n+1}\mathscr{L}) \cap \hbar\mathscr{L} \subset \mathscr{N} \cap \hbar\mathscr{L} + \hbar^{n+1}\mathscr{L} \subset \hbar\mathscr{N} + \hbar^{n+1}\mathscr{L} = \hbar(\mathscr{N} + \hbar^n\mathscr{L})$ for any n.

Set

$$\bar{\mathscr{N}} = (\mathscr{N} + \hbar\mathscr{L})/\hbar\mathscr{L} = (\widetilde{\mathscr{N}} + \hbar\mathscr{L})/\hbar\mathscr{L}.$$

By the hypothesis (a), $\bar{\mathscr{N}}$ is $\mathscr{A}_0$-coherent. Hence we may assume that there exist finitely many sections s_i of $\mathscr{N}$ such that $\bar{\mathscr{N}} = \sum_i \mathscr{A}_0\overline{s}_i$, where $\overline{s}_i$ is the image of s_i in $\mathscr{L}/\hbar\mathscr{L}$.

By hypothesis (a) and Lemma 1.2.4 (ii), we have for any $U \in \mathfrak{B}$, $\bar{\mathscr{N}}(U) = \sum_i \mathscr{A}_0(U)\overline{s}_i$. Since $\mathscr{A}(U) \to \mathscr{A}_0(U)$ is surjective by Lemma 1.2.4 (iii), we have $\widetilde{\mathscr{N}}(U) \subset \sum_i \mathscr{A}(U)s_i + \hbar\mathscr{L}(U)$. Since $\widetilde{\mathscr{N}} \cap \hbar\mathscr{L} = \hbar\widetilde{\mathscr{N}}$, we have

$$\widetilde{\mathscr{N}}(U) \subset \sum_i \mathscr{A}(U)s_i + \hbar\widetilde{\mathscr{N}}(U).$$

For $v \in \widetilde{\mathscr{N}}(U)$, we shall define a sequence $\{v_n\}_{n\geq 0}$ in $\widetilde{\mathscr{N}}(U)$ and sequences $\{a_{i,n}\}_{n\geq 0}$ in $\mathscr{A}(U)$, inductively on n: set $v_0 = v$, and write

$$v_n = \sum_i a_{i,n}s_i + \hbar v_{n+1}.$$

Hence we have $\hbar^n v_n = \sum_i \hbar^n a_{i,n}s_i + \hbar^{n+1}v_{n+1}$ and we obtain

$$v = v_0 = \sum_i (\sum_{n\geq 0} \hbar^n a_{i,n})s_i.$$

Thus we have $\widetilde{\mathscr{N}} = \sum_i \mathscr{A}s_i$. Hence $\mathscr{N} = \widetilde{\mathscr{N}}$ which proves (i) and (iii).

Since $\widetilde{\mathscr{N}} \cap \hbar\mathscr{L} = \hbar\widetilde{\mathscr{N}}$ by (1.2.6), we obtain (ii) for $n = 1$. For $n \geq 1$ we have by induction $\mathscr{N} \cap \hbar^n\mathscr{L} \subset \hbar\mathscr{N} \cap \hbar^n\mathscr{L} = \hbar(\mathscr{N} \cap \hbar^{n-1}\mathscr{L}) \subset \hbar \cdot \hbar^{n-1}\mathscr{N}$.
Q.E.D.

Lemma 1.2.7. *Let $\mathscr{L}$ be a locally free $\mathscr{A}$-module of finite rank, and let $\mathscr{N}$ be an $\mathscr{A}$-submodule of $\mathscr{L}$. Assume that $(\mathscr{N} + \hbar^{n+1}\mathscr{L})/\hbar^{n+1}\mathscr{L}$ is a coherent $\mathscr{A}$-module for any $n \geq 0$. Then we have*

(i) *$\mathscr{N}$ is a locally finitely generated $\mathscr{A}$-module,*

(ii) $\bigcap_{n\geq 0}(\mathscr{N} + \hbar^n\mathscr{L}) = \mathscr{N}$,

(iii) *locally, $\hbar^n\mathscr{L} \cap \mathscr{N} \subset \hbar(\hbar^{n-1}\mathscr{L} \cap \mathscr{N})$ for $n \gg 0$,*

(iv) *$\mathscr{N}/\hbar^n\mathscr{N}$ is a coherent $\mathscr{A}$-module for any $n \geq 0$.*

Proof. We embed $\mathscr{L}$ into the $\mathscr{A}[\hbar^{-1}]$-module $\mathbb{K}[\hbar,\hbar^{-1}]\otimes_{\mathbb{K}[\hbar]}\mathscr{L} = \bigcup_{n\in\mathbb{Z}}\hbar^n\mathscr{L}$. Note that $\hbar^n$ induces an isomorphism

$$\hbar^n\colon (\mathscr{L}\cap\hbar^{-n}\mathscr{N}+\hbar\mathscr{L})/\hbar\mathscr{L} \xrightarrow{\sim} (\mathscr{N}\cap\hbar^n\mathscr{L}+\hbar^{n+1}\mathscr{L})/\hbar^{n+1}\mathscr{L}.$$

Since

$$(\mathscr{N}\cap\hbar^n\mathscr{L}+\hbar^{n+1}\mathscr{L})/\hbar^{n+1}\mathscr{L} \simeq \big((\mathscr{N}+\hbar^{n+1}\mathscr{L})/\hbar^{n+1}\mathscr{L}\big)\bigcap\big(\hbar^n\mathscr{L}/\hbar^{n+1}\mathscr{L}\big)$$

is $\mathscr{A}$-coherent, $\{(\mathscr{L}\cap\hbar^{-n}\mathscr{N}+\hbar\mathscr{L})/\hbar\mathscr{L}\}_{n\geq 0}$ is an increasing sequence of coherent $\mathscr{A}_0$-submodules of $\mathscr{L}/\hbar\mathscr{L}$. Hence it is locally stationary: locally there exists $n_0\geq 0$ such that $\mathscr{L}\cap\hbar^{-n}\mathscr{N}+\hbar\mathscr{L} = \mathscr{L}\cap\hbar^{-n_0}\mathscr{N}+\hbar\mathscr{L}$ for any $n\geq n_0$. Set

$$\mathscr{N}_0 := \mathscr{L}\cap\hbar^{-n_0}\mathscr{N}. \tag{1.2.7}$$

Then $(\mathscr{N}_0+\hbar\mathscr{L})/\hbar\mathscr{L}$ is a coherent $\mathscr{A}_0$-module and

$$\mathscr{N}_0\cap\hbar^n\mathscr{L}\subset\hbar^n(\hbar^{-n-n_0}\mathscr{N}\cap\mathscr{L})\subset\hbar^n(\mathscr{N}_0+\hbar\mathscr{L})\subset\hbar\mathscr{N}_0+\hbar^{n+1}\mathscr{L}$$

for any $n>0$. Hence by Lemma 1.2.6:

- $\mathscr{N}_0$ is locally finitely generated over $\mathscr{A}$,
- $\bigcap\limits_{n\geq 0}(\mathscr{N}_0+\hbar^n\mathscr{L}) = \mathscr{N}_0$,
- $\mathscr{N}_0\cap\hbar^n\mathscr{L} = \hbar^n\mathscr{N}_0$ for any $n\geq 0$.

(i) Since $\mathscr{N}\cap\hbar^{n_0}\mathscr{L} = \hbar^{n_0}\mathscr{N}_0$ by (1.2.7), the module $\mathscr{N}/\hbar^{n_0}\mathscr{N}_0 \simeq \mathscr{N}/(\mathscr{N}\cap\hbar^{n_0}\mathscr{L}) \simeq (\mathscr{N}+\hbar^{n_0}\mathscr{L})/\hbar^{n_0}\mathscr{L}$ is $\mathscr{A}$-coherent. Since $\hbar^{n_0}\mathscr{N}_0$ is locally finitely generated over $\mathscr{A}$, $\mathscr{N}$ is also locally finitely generated over $\mathscr{A}$.

(ii) We have

$$\begin{aligned}
\bigcap_{n\geq n_0}(\mathscr{N}+\hbar^n\mathscr{L}) &\subset (\mathscr{N}+\hbar^{n_0}\mathscr{L})\bigcap\cap_{n\geq n_0}(\mathscr{N}+\hbar^n\mathscr{L})\\
&\subset \mathscr{N}+\hbar^{n_0}\mathscr{L}\bigcap\cap_{n\geq n_0}(\mathscr{N}+\hbar^n\mathscr{L})\\
&\subset \mathscr{N}+\cap_{n\geq n_0}(\hbar^{n_0}\mathscr{L}\cap\mathscr{N}+\hbar^n\mathscr{L})\\
&\subset \mathscr{N}+\cap_{n\geq n_0}(\hbar^{n_0}\mathscr{N}_0+\hbar^n\mathscr{L})\\
&\subset \mathscr{N}+\hbar^{n_0}\mathscr{N}_0 = \mathscr{N}.
\end{aligned}$$

(iii) For $n > n_0$, we have

$$\begin{aligned}\hbar^n\mathscr{L}\cap\mathscr{N} &\subset \hbar^{n_0}(\mathscr{L}\cap\hbar^{-n_0}\mathscr{N})\cap\hbar^n\mathscr{L}\\ &\subset \hbar^{n_0}(\mathscr{N}_0\cap\hbar^{n-n_0}\mathscr{L})\\ &\subset \hbar^{n_0}\hbar^{n-n_0}\mathscr{N}_0=\hbar^n\mathscr{N}_0\\ &\subset \hbar(\mathscr{N}\cap\hbar^{n-1}\mathscr{L}).\end{aligned}$$

(iv) Since $\mathscr{N}$ has no $\hbar$-torsion, we have the exact sequence

$$0\to\mathscr{N}/\hbar^n\mathscr{N}\xrightarrow{\hbar}\mathscr{N}/\hbar^{n+1}\mathscr{N}\to\mathscr{N}/\hbar\mathscr{N}\to 0.$$

Hence, it is enough to show that $\mathscr{N}/\hbar\mathscr{N}$ is coherent. By (i), the images of $\mathscr{N}$ and $\hbar\mathscr{N}$ in $\mathscr{L}/\hbar^n\mathscr{L}$ are coherent. Since $\mathscr{N}\cap\hbar^n\mathscr{L}\subset\hbar\mathscr{N}$ for some n, by (ii), we have the exact sequence

$$\frac{\hbar\mathscr{N}}{\hbar\mathscr{N}\cap\hbar^n\mathscr{L}}\to\frac{\mathscr{N}}{\mathscr{N}\cap\hbar^n\mathscr{L}}\to\frac{\mathscr{N}}{\hbar\mathscr{N}}\to 0,$$

which implies that $\mathscr{N}/\hbar\mathscr{N}$ is coherent. Q.E.D.

Corollary 1.2.8. *Assume that $\mathscr{M}$ is a locally finitely generated $\mathscr{A}$-module. If $\mathscr{M}/\hbar^n\mathscr{M}$ is a coherent $\mathscr{A}$-module for all $n > 0$, then $\mathscr{M}$ is an $\mathscr{A}$-module locally of finite presentation and $\bigcap_{n\geq 0}\hbar^n\mathscr{M}=0$.*

Proof. We may assume that $\mathscr{M}=\mathscr{L}/\mathscr{N}$ for a locally free $\mathscr{A}$-module $\mathscr{L}$ of finite rank and $\mathscr{N}\subset\mathscr{L}$. From the exact sequence

$$0\to(\mathscr{N}+\hbar^n\mathscr{L})/\hbar^n\mathscr{L}\to\mathscr{L}/\hbar^n\mathscr{L}\to\mathscr{M}/\hbar^n\mathscr{M}\to 0,$$

we deduce that $(\mathscr{N}+\hbar^n\mathscr{L})/\hbar^n\mathscr{L}$ is coherent for any n. Hence $\mathscr{N}$ is locally finitely generated by Lemma 1.2.7, which implies that $\mathscr{M}$ is locally of finite presentation. Since $\bigcap_{n\geq 0}(\mathscr{N}+\hbar^n\mathscr{L})=\mathscr{N}$ by Lemma 1.2.7,

$$\bigcap_{n\geq 0}\hbar^n\mathscr{M}\simeq\Big(\bigcap_{n\geq 0}(\mathscr{N}+\hbar^n\mathscr{L})\Big)/\mathscr{N}$$

vanishes. Q.E.D.

Proposition 1.2.9. *$\mathscr{A}$ is coherent.*

Proof. Let $\mathscr{I}$ be a locally finitely generated $\mathscr{A}$-submodule of $\mathscr{A}$. Since

$$(\mathscr{I}+\hbar^{n+1}\mathscr{A})/\hbar^{n+1}\mathscr{A}\simeq\mathscr{I}/(\mathscr{I}\cap\hbar^{n+1}\mathscr{A})\subset\mathscr{A}/\hbar^{n+1}\mathscr{A},$$

the $\mathscr{A}$-module $\mathscr{I}/\hbar^n\mathscr{I}$ is coherent by Lemma 1.2.7 (iv). Hence Corollary 1.2.8 implies that $\mathscr{I}$ is locally of finite presentation. Q.E.D.

Lemma 1.2.10. *Any filtrant family of coherent $\mathscr{A}$-submodules of $\mathscr{A}$ is locally stationary.*

Proof. Let $\{\mathscr{I}_i\}_{i\in I}$ be a family of coherent $\mathscr{A}$-submodules of $\mathscr{A}$ indexed by a filtrant ordered set I, with $\mathscr{I}_i \subset \mathscr{I}_j$ for any $i \le j$. Then $\{(\hbar^{-k}\mathscr{I}_i \cap \mathscr{A} + \hbar\mathscr{A})/\hbar\mathscr{A}\}_{i\in I,\ k\ge 0}$ is increasing with respect to k and $i \in I$. Hence locally there exist i_0 and k_0 such that $\hbar^{-k}\mathscr{I}_i \cap \mathscr{A} + \hbar\mathscr{A} = \hbar^{-k_0}\mathscr{I}_{i_0} \cap \mathscr{A} + \hbar\mathscr{A}$ for any $i \ge i_0$ and $k \ge k_0$. Then, for $i \ge i_0$, the ideal $\mathscr{J}_i := \mathscr{A} \cap \hbar^{-k_0}\mathscr{I}_i$ satisfies

$$\mathscr{J}_i \cap \hbar^m\mathscr{A} \subset \hbar^m(\hbar^{-m-k_0}\mathscr{I}_i \cap \mathscr{A}) \subset \hbar^m(\hbar^{-k_0}\mathscr{I}_i \cap \mathscr{A} + \hbar\mathscr{A}) \subset \hbar\mathscr{J}_i + \hbar^{m+1}\mathscr{A}$$

for any $m > 0$. Hence Lemma 1.2.6 implies that $\mathscr{J}_i \cap \hbar\mathscr{A} = \hbar\mathscr{J}_i$. Since we have $\mathscr{J}_i \subset \mathscr{J}_{i_0} + \hbar\mathscr{A}$, we have $\mathscr{J}_i \subset \mathscr{J}_{i_0} + (\mathscr{J}_i \cap \hbar\mathscr{A}) \subset \mathscr{J}_{i_0} + \hbar\mathscr{J}_i$. Then Nakayama's lemma implies $\mathscr{J}_i = \mathscr{J}_{i_0}$, or equivalently, $\hbar^{-k_0}\mathscr{I}_i \cap \mathscr{A} = \hbar^{-k_0}\mathscr{I}_{i_0} \cap \mathscr{A}$ for $i \ge i_0$. Thus $\{\mathscr{I}_i \cap \hbar^{k_0}\mathscr{A}\}_i$ is locally stationary. Since $\{\mathscr{I}_i/(\mathscr{I}_i \cap \hbar^{k_0}\mathscr{A})\}_i$ is a filtrant family of coherent submodules of $\mathscr{A}_{k_0-1}$, it is also locally stationary and it follows that $\{\mathscr{I}_i\}_i$ is locally stationary. Q.E.D.

Lemma 1.2.11. *For any $x \in X$, $\mathscr{A}_x$ is a coherent ring.*

Proof. Any morphism $f\colon \mathscr{A}_x^{\oplus n} \to \mathscr{A}_x$ extends to a morphism $\tilde{f}\colon \mathscr{A}^{\oplus n}|_U \to \mathscr{A}|_U$ for some open neighborhood U of x. Since $\mathscr{N} := \operatorname{Ker}\tilde{f}$ is coherent, $\mathscr{N}_x \simeq \operatorname{Ker} f$ is a finitely generated $\mathscr{A}_x$-module. Q.E.D.

Lemma 1.2.12. *For any $x \in X$ and a finitely generated left ideal I of $\mathscr{A}_x$, $I \cap \hbar^{n+1}\mathscr{A}_x = \hbar(I \cap \hbar^n\mathscr{A}_x)$ for $n \gg 0$.*

Proof. Let us take a coherent ideal $\mathscr{I}$ of $\mathscr{A}$ defined on a neighborhood of x such that $I = \mathscr{I}_x$. Then Lemma 1.2.7 implies that $\mathscr{I} \cap \hbar^{n+1}\mathscr{A} = \hbar(\mathscr{I} \cap \hbar^n\mathscr{A})$ for $n \gg 0$. Q.E.D.

Lemma 1.2.13. *For any $x \in X$, $\mathscr{A}_x$ is a Noetherian ring.*

Proof. Set $A = \mathscr{A}_x$. Let us show that an increasing sequence $\{I_n\}_n$ of finitely generated left ideals of A is stationary. Since $\{(\hbar^{-k}I_n \cap A + \hbar A)/\hbar A\}_{n,k}$ is increasing with respect n, k, there exist n_0 and k_0 such that $\hbar^{-k}I_n \cap A + \hbar A = \hbar^{-k_0}I_{n_0} \cap A + \hbar A$ for $n \ge n_0$ and $k \ge k_0$. For any $n \ge n_0$ there exists $k \ge k_0$ such that $\hbar^{-k}I_n \cap \hbar A = \hbar(\hbar^{-k}I_n \cap A)$ by Lemma 1.2.12. Hence we have $\hbar^{-k}I_n \cap A \subset \hbar^{-k}I_n \cap (\hbar^{-k_0}I_{n_0} \cap A + \hbar A) \subset \hbar^{-k_0}I_{n_0} \cap A + (\hbar^{-k}I_n \cap \hbar A) \subset \hbar^{-k_0}I_{n_0} \cap A + \hbar(\hbar^{-k}I_n \cap A)$. Since $\hbar^{-k}I_n \cap A$ is finitely generated by Lemma 1.2.11, Nakayama's lemma implies that $\hbar^{-k}I_n \cap A = \hbar^{-k_0}I_{n_0} \cap A$. Hence $\hbar^{-k_0}I_n \cap A = \hbar^{-k_0}I_{n_0} \cap A$ for any $n \ge n_0$. Therefore $I_n \cap \hbar^{k_0}A = \hbar^{k_0}(\hbar^{-k_0}I_n \cap A)$ is stationary. Since $\{I_n/(I_n \cap \hbar^{k_0}A)\}_n$ is stationary, $\{I_n\}_n$ is stationary. Q.E.D.

Thus, we have proved that $\mathscr{A}$ is a Noetherian ring.

Lemma 1.2.14. *Let $\{\mathscr{M}_n\}_{n\geq 0}$ be a projective system of coherent $\mathscr{A}$-modules. Assume that $\hbar^{n+1}\mathscr{M}_n = 0$ and the induced morphism $\mathscr{M}_{n+1}/\hbar^{n+1}\mathscr{M}_{n+1} \to \mathscr{M}_n$ is an isomorphism for any $n \geq 0$. Then $\mathscr{M} := \varprojlim_n \mathscr{M}_n$ is a coherent $\mathscr{A}$-module and $\mathscr{M}/\hbar^{n+1}\mathscr{M} \to \mathscr{M}_n$ is an isomorphism for any $n \geq 0$.*

Proof. Since the question is local, we may assume that $X \in \mathfrak{B}$ and there exist a free $\mathbb{K}$-module V of finite rank and a morphism $V \to \mathscr{M}_0(X)$ which induces an epimorphism $\mathscr{L} := \mathscr{A} \otimes_{\mathbb{K}} V \twoheadrightarrow \mathscr{M}_0$. Since $\mathscr{M}_{n+1}(X) \to \mathscr{M}_n(X)$ is surjective and V is projective, we have a projective system of morphisms $\{V \to \mathscr{M}_n(X)\}_n$:

$$\begin{tikzcd}
 & V \arrow[d, dotted] \arrow[dr, dotted] \arrow[drrr, dotted] \arrow[drrrr] & & & & \\
\cdots \arrow[r, two heads] & \mathscr{M}_n(X) \arrow[r, two heads] & \mathscr{M}_{n-1}(X) \arrow[r, two heads] & \cdots \arrow[r, two heads] & \mathscr{M}_1(X) \arrow[r, two heads] & \mathscr{M}_0(X),
\end{tikzcd}$$

which induces a projective system of morphisms $\{\mathscr{L} \to \mathscr{M}_n\}_n$. Hence we may assume that there exists a morphism $\mathscr{L} \to \mathscr{M}$ such that the composition $\mathscr{L} \to \mathscr{M} \to \mathscr{M}_0$ is an epimorphism. Since $\mathscr{L} \to \mathscr{M}_n/\hbar\mathscr{M}_n \xrightarrow{\sim} \mathscr{M}_0$ is an epimorphism, $\mathscr{L} \to \mathscr{M}_n$ is an epimorphism by Lemma 1.2.2.

Set $\mathscr{L}_n = \mathscr{L}/\hbar^{n+1}\mathscr{L}$, and let $\mathscr{N}_n$ be the kernel of $\mathscr{L}_n \to \mathscr{M}_n$. Set $\mathscr{N} = \varprojlim_n \mathscr{N}_n$. Then we have a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{N} \arrow[r] \arrow[d] & \mathscr{L} \arrow[r] \arrow[d] & \mathscr{M} \arrow[d] & \\
0 \arrow[r] & \mathscr{N}_n \arrow[r] & \mathscr{L}_n \arrow[r] & \mathscr{M}_n \arrow[r] & 0.
\end{tikzcd}$$

In the commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \hbar^{n+1}\mathscr{L}_{n+1} \arrow[r] \arrow[d] & \hbar^{n+1}\mathscr{M}_{n+1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{N}_{n+1} \arrow[r] \arrow[d] & \mathscr{L}_{n+1} \arrow[r] \arrow[d] & \mathscr{M}_{n+1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{N}_n \arrow[r] & \mathscr{L}_n \arrow[r] \arrow[d] & \mathscr{M}_n \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

the rows and the columns are exact. Hence the left vertical arrow $\mathscr{N}_{n+1} \to \mathscr{N}_n$ is an epimorphism. Therefore, $\mathscr{N}_{n+1}(U) \to \mathscr{N}_n(U)$ is surjective for any $U \in \mathfrak{B}$, and $\mathscr{N}(U) \xrightarrow{\sim} \varprojlim_m \mathscr{N}_m(U) \to \mathscr{N}_n(U)$ is surjective. Hence $\mathscr{N} \to \mathscr{N}_n$ is an epimorphism for any $n \geq 0$, and $\{\mathscr{N}_n(U)\}_n$ satisfies the Mittag-Leffler condition.

Thus in the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{N}(U) \arrow[r] \arrow[d, "\sim"] & \mathscr{L}(U) \arrow[r] \arrow[d, "\sim"] & \mathscr{M}(U) \arrow[r] \arrow[d, "\sim"] & 0 \\
0 \arrow[r] & \varprojlim_n \mathscr{N}_n(U) \arrow[r] & \varprojlim_n \mathscr{L}_n(U) \arrow[r] & \varprojlim_n \mathscr{M}_n(U) \arrow[r] & 0,
\end{tikzcd}$$

the bottom row is exact. Hence $0 \to \mathscr{N} \to \mathscr{L} \to \mathscr{M} \to 0$ is exact. Since $\mathscr{N} \to \mathscr{N}_n$ is an epimorphism, we have $\mathscr{M}/\hbar^{n+1}\mathscr{M} \simeq \operatorname{Coker}(\mathscr{N} \to \mathscr{L}_n) \simeq \operatorname{Coker}(\mathscr{N}_n \to \mathscr{L}_n) \simeq \mathscr{M}_n$. Since $\mathscr{M}$ is locally finitely generated and $\mathscr{M}/\hbar^{n+1}\mathscr{M}$ is coherent for any $n \geq 0$, $\mathscr{M}$ is coherent by Corollary 1.2.8 and Proposition 1.2.9. Q.E.D.

Proposition 1.2.15. *Let $\mathscr{M}$ be a coherent $\mathscr{A}$-module. Then we have the following properties.*

(i) *$\mathscr{M}$ is $\hbar$-complete, i.e., $\mathscr{M} \xrightarrow{\sim} \widehat{\mathscr{M}}$,*

(ii) *for any $U \in \mathfrak{B}$, $H^k(U;\mathscr{M}) = 0$ for any $k > 0$.*

Proof. (i) Since the kernel of $\mathscr{M} \to \widehat{\mathscr{M}}$ is $\bigcap_{n\geq 0} \hbar^n\mathscr{M}$, the morphism $\mathscr{M} \to \widehat{\mathscr{M}}$ is a monomorphism by Corollary 1.2.8.

Let us show that $\mathscr{M} \to \widehat{\mathscr{M}}$ is an epimorphism. By the preceding lemma, $\widehat{\mathscr{M}}$ is a coherent $\mathscr{A}$-module, and $\widehat{\mathscr{M}}/\hbar\widehat{\mathscr{M}} \simeq \mathscr{M}/\hbar\mathscr{M}$. Hence Nakayama's lemma implies that $\mathscr{M} \to \widehat{\mathscr{M}}$ is an epimorphism.

(ii) For any $U \in \mathfrak{B}$, the map $\Gamma(U;\mathscr{M}/\hbar^{n+1}\mathscr{M}) \to \Gamma(U;\mathscr{M}/\hbar^n\mathscr{M})$ is surjective, and $H^k(U;\mathscr{M}/\hbar^n\mathscr{M}) = 0$ for any $k > 0$. Hence Lemma 1.1.6 implies (ii). Q.E.D.

Corollary 1.2.16. *Let $\mathscr{M}$ be an $\mathscr{A}$-module. If $\mathscr{M}$ satisfies the following conditions* (i) *and* (ii), *then $\mathscr{M}$ is a coherent $\mathscr{A}$-module.*

(i) *$\mathscr{M}$ is $\hbar$-complete,*

(ii) *$\hbar^n\mathscr{M}/\hbar^{n+1}\mathscr{M}$ is a coherent $\mathscr{A}_0$-module for all $n \geq 0$.*

Proof. Set $\mathscr{M}_n = \mathscr{M}/\hbar^{n+1}\mathscr{M}$. Then it is a coherent $\mathscr{A}$-module by (ii), and $\varprojlim_n \mathscr{M}_n$ is a coherent $\mathscr{A}$-module by Lemma 1.2.14. Q.E.D.

This completes the proof of Theorem 1.2.5.

Lemma 1.2.17. *Let $\mathscr{M}$ be a coherent $\mathscr{A}$-module without $\hbar$-torsion. If $\mathscr{M}/\hbar\mathscr{M}$ is a locally free $\mathscr{A}_0$-module of rank $r \in \mathbb{Z}_{\geq 0}$, then $\mathscr{M}$ is a locally free $\mathscr{A}$-module of rank r.*

Proof. We may assume that there exists a morphism of $\mathscr{A}$-modules $f\colon \mathscr{L} := \mathscr{A}^{\oplus r} \to \mathscr{M}$ such that $\mathscr{L}/\hbar\mathscr{L} \to \mathscr{M}/\hbar\mathscr{M}$ is an isomorphism. Then, Nakayama's lemma implies that f is an epimorphism. Let $\mathscr{N}$ be the kernel of f. Since $\mathscr{M}$ has no $\hbar$-torsion, we have an exact sequence $0 \to \mathscr{N}/\hbar\mathscr{N} \to \mathscr{L}/\hbar\mathscr{L} \to \mathscr{M}/\hbar\mathscr{M} \to 0$. Hence $\mathscr{N}/\hbar\mathscr{N} = 0$ and Nakayama's lemma implies $\mathscr{N} = 0$. Q.E.D.

The following proposition gives a criterion for the coherence of the projective limit of coherent modules, generalizing Lemma 1.2.14.

Proposition 1.2.18. *Let $\{\mathscr{N}_n\}_{n\geq 1}$ be a projective system of coherent $\mathscr{A}$-modules. Assume*

(a) *the pro-object "$\varprojlim_n$" $\mathscr{N}_n/\hbar\mathscr{N}_n$ is representable by a coherent $\mathscr{A}_0$-module,*

(b) *the pro-object "$\varprojlim_n$" $\mathrm{Ker}(\mathscr{N}_n \xrightarrow{\hbar} \mathscr{N}_n)$ is representable by a coherent $\mathscr{A}_0$-module.*

Then

(i) $\mathscr{N} := \varprojlim_n \mathscr{N}_n$ *is a coherent $\mathscr{A}$-module,*

(ii) $\mathscr{N}/\hbar^{k+1}\mathscr{N} \xrightarrow{\sim}$ "$\varprojlim_n$" $\mathscr{N}_n/\hbar^{k+1}\mathscr{N}_n$ *for any $k \geq 0$,*

(iii) $\mathrm{Ker}(\mathscr{N} \xrightarrow{\hbar} \mathscr{N}) \xrightarrow{\sim}$ "$\varprojlim_n$" $\mathrm{Ker}(\mathscr{N}_n \xrightarrow{\hbar} \mathscr{N}_n)$.

(iv) *Assume moreover that for each $n \geq 1$ there exists $k \geq 0$ such that $\hbar^k\mathscr{N}_n = 0$. Then the projective system $\{\mathscr{N}_n\}_n$ satisfies the Mittag-Leffler condition.*

Proof. For any $k \geq 0$, set

$$\mathscr{S}_k := \text{“}\varprojlim_n\text{”}\, \mathscr{N}_n/\hbar^{k+1}\mathscr{N}_n.$$

Then $\mathscr{S}_0$ is representable by a coherent $\mathscr{A}$-module by hypothesis (a). We shall show that $\mathscr{S}_k$ is representable by a coherent $\mathscr{A}$-module for all $k \geq 0$ by induction on k. Consider the exact sequences

$$0 \to \hbar\mathscr{N}_n/\hbar^{k+1}\mathscr{N}_n \to \mathscr{N}_n/\hbar^{k+1}\mathscr{N}_n \to \mathscr{N}_n/\hbar\mathscr{N}_n \to 0, \tag{1.2.8}$$

$$\operatorname{Ker}(\mathscr{N}_n \xrightarrow{\hbar} \mathscr{N}_n) \to \mathscr{N}_n/\hbar^k\mathscr{N}_n \xrightarrow{\hbar} \hbar\mathscr{N}_n/\hbar^{k+1}\mathscr{N}_n \to 0. \tag{1.2.9}$$

Assume that $\mathscr{S}_{k-1}$ is representable by a coherent $\mathscr{A}$-module. Applying the functor “$\varprojlim\limits_n$” to the exact sequence (1.2.9), we deduce that the object “$\varprojlim\limits_n$” $\hbar\mathscr{N}_n/\hbar^{k+1}\mathscr{N}_n$ is representable by a coherent $\mathscr{A}$-module. Then applying the functor “$\varprojlim\limits_n$” to the exact sequence (1.2.8), we deduce that $\mathscr{S}_k$ is representable by a coherent $\mathscr{A}$-module.

Since $\mathscr{N}_n \simeq \varprojlim\limits_k \mathscr{N}_n/\hbar^{k+1}\mathscr{N}_n$ by Theorem 1.2.5 (iii), we have

$$\mathscr{N} \simeq \varprojlim_{k,n} \mathscr{N}_n/\hbar^{k+1}\mathscr{N}_n \simeq \varprojlim_k \mathscr{S}_k.$$

Since $\mathscr{S}_{k+1}/\hbar^{k+1}\mathscr{S}_{k+1} \simeq \mathscr{S}_k$, Lemma 1.2.14 implies (i), (ii). The property (iii) is obvious.

Let us prove (iv). By the assumption, $\mathscr{N}_n \simeq$ “$\varprojlim\limits_k$” $\mathscr{N}_n/\hbar^k\mathscr{N}_n$. Hence

$$\text{“}\varprojlim_n\text{”}\, \mathscr{N}_n \simeq \text{“}\varprojlim_{k,n}\text{”}\, \mathscr{N}_n/\hbar^k\mathscr{N}_n \simeq \text{“}\varprojlim_k\text{”}\, \mathscr{S}_k.$$

Since $\{\mathscr{S}_k\}_k$ satisfies the Mittag-Leffler condition, $\{\mathscr{N}_n\}_n$ satisfies the Mittag-Leffler condition by Lemma 1.1.2. Q.E.D.

Remark 1.2.19. In Proposition 1.2.18 (iv), the condition $\hbar^k\mathscr{N}_n = 0$ $(k \gg 0)$ is necessary as seen by considering the projective system $\mathscr{N}_n = \hbar^n\mathscr{A}$, $(n \in \mathbb{N})$.

1.3 A variant of the preceding results

Here, we consider rings which satisfy hypotheses (1.2.2), but in which (1.2.3) is replaced with another hypothesis. Indeed, as we shall see, the ring $\mathscr{D}_X[[\hbar]]$

of differential operators on a complex manifold X has nice properties, although $\mathscr{D}_X$ does not satisfy (1.2.3). The study of modules over $\mathscr{D}_X[[\hbar]]$ is performed in [18].

We assume that X is a Hausdorff locally compact space. By a basis $\mathfrak{B}$ of compact subsets of X, we mean a family of compact subsets such that for any $x \in X$ and any open neighborhood U of x, there exists $K \in \mathfrak{B}$ such that $x \in \mathrm{Int}(K) \subset U$.

We consider a $\mathbb{K}$-algebra $\mathscr{A}$ on X and a section $\hbar$ of $\mathscr{A}$ contained in the center of $\mathscr{A}$. Set $\mathscr{A}_0 = \mathscr{A}/\hbar\mathscr{A}$. We assume the conditions (1.2.2) and

(1.3.1)
- (iv') there exist a base $\mathfrak{B}$ of compact subsets of X and a prestack $U \mapsto \mathrm{Mod}_{\mathrm{gd}}(\mathscr{A}_0|_U)$ (U open in X) such that
  - (a) for any $K \in \mathfrak{B}$ and an open subset U such that $K \subset U$, there exists $K' \in \mathfrak{B}$ such that $K \subset \mathrm{Int}(K') \subset K' \subset U$,
  - (b) $U \mapsto \mathrm{Mod}_{\mathrm{gd}}(\mathscr{A}_0|_U)$ is a full subprestack of $U \mapsto \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_0|_U)$,
  - (c) for an open subset U and $\mathscr{M} \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_0|_U)$, if $\mathscr{M}|_V$ belongs to $\mathrm{Mod}_{\mathrm{gd}}(\mathscr{A}_0|_V)$ for any relatively compact open subset V of U, then $\mathscr{M}$ belongs to $\mathrm{Mod}_{\mathrm{gd}}(\mathscr{A}_0|_U)$,
  - (d) for any open subset U of X, $\mathrm{Mod}_{\mathrm{gd}}(\mathscr{A}_0|_U)$ is stable by subobjects, quotients and extension in $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_0|_U)$,
  - (e) for any $K \in \mathscr{B}$, any open set U containing K, any $\mathscr{M} \in \mathrm{Mod}_{\mathrm{gd}}(\mathscr{A}_0|_U)$ and any $j > 0$, one has $H^j(K;\mathscr{M}) = 0$,
  - (f) for any $\mathscr{M} \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_0|_U)$, there exists an open covering $U = \bigcup_i U_i$ such that $\mathscr{M}|_{U_i} \in \mathrm{Mod}_{\mathrm{gd}}(\mathscr{A}_0|_{U_i})$,
  - (g) $\mathscr{A}_0 \in \mathrm{Mod}_{\mathrm{gd}}(\mathscr{A}_0)$.

Note that Lemmas 1.2.2 and 1.2.3 still hold.

The prestack $U \mapsto \mathrm{Mod}_{\mathrm{gd}}(\mathscr{A}_0|_U)$ being given, a coherent module which belongs to $\mathrm{Mod}_{\mathrm{gd}}(\mathscr{A}_0|_U)$ will be called a good module. Note that in view of hypothesis (iv') (f), hypothesis (iv') (g) could be deleted since all the results of this subsection will be of local nature. However, we keep it for simplicity.

Example 1.3.1. Let X be a complex manifold, $\mathscr{O}_X$ the structure sheaf and let $\mathscr{D}_X$ denote the $\mathbb{C}$-algebra of differential operators. One checks easily that, taking for $\mathfrak{B}$ the set of Stein compact subsets and for $\mathscr{A}_0$ the $\mathbb{C}$-algebra $\mathscr{D}_X$, the prestack of good $\mathscr{D}_X$-modules in the sense of [38] satisfies the hypotheses (1.3.1).

Definition 1.3.2. A coherent $\mathscr{A}$-module $\mathscr{M}$ is good if both the kernel and the cokernel of $\hbar\colon \mathscr{M} \to \mathscr{M}$ are good $\mathscr{A}_0$-modules. One denotes by $\mathrm{Mod}_{\mathrm{gd}}(\mathscr{A})$ the category of good $\mathscr{A}$-modules.

Note that an $\mathscr{A}_0$-module is good if and only if it is good as an $\mathscr{A}$-module. This allows us to state:

Definition 1.3.3. An $\mathscr{A}_n$-module $\mathscr{M}$ is good if it is good as an $\mathscr{A}$-module.

Lemma 1.3.4. *The category* $\mathrm{Mod}_{\mathrm{gd}}(\mathscr{A})$ *is a subcategory of* $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{A})$ *stable by subobjects, quotients and extension.*

Proof. First note that $\hbar^n\mathscr{M}/\hbar^{n+1}\mathscr{M}$ is a good $\mathscr{A}_0$-module for any $\mathscr{M} \in \mathrm{Mod}_{\mathrm{gd}}(\mathscr{A})$ and any integer $n \geq 0$. Indeed, it is a quotient of $\mathscr{M}/\hbar\mathscr{M}$.

For an $\mathscr{A}$-module $\mathscr{N}$, set $\mathscr{N}_\hbar := \mathrm{Ker}(\hbar\colon \mathscr{N} \to \mathscr{N})$.

We shall show that any coherent $\mathscr{A}$-submodule $\mathscr{N}$ of a good $\mathscr{A}$-module $\mathscr{M}$ is a good $\mathscr{A}$-module. It is obvious that $\mathscr{N}_\hbar$ is a good $\mathscr{A}_0$-module, because it is a coherent submodule of $\mathscr{M}_\hbar$. We shall show that $\mathscr{N}/(\hbar\mathscr{N}+\mathscr{N}\cap\hbar^{k+1}\mathscr{M})$ is a good $\mathscr{A}_0$-module for any $k \geq 0$. We argue by induction on k. For $k = 0$, it is a good $\mathscr{A}_0$-module since it is a coherent submodule of $\mathscr{M}/\hbar\mathscr{M}$. For $k > 0$, we have an exact sequence

$$
\begin{aligned}
0 \to \frac{\hbar\mathscr{N} + \mathscr{N} \cap \hbar^k\mathscr{M}}{\hbar\mathscr{N} + \mathscr{N} \cap \hbar^{k+1}\mathscr{M}} &\to \frac{\mathscr{N}}{\hbar\mathscr{N} + \mathscr{N} \cap \hbar^{k+1}\mathscr{M}} \\
&\to \frac{\mathscr{N}}{\hbar\mathscr{N} + \mathscr{N} \cap \hbar^{k}\mathscr{M}} \to 0.
\end{aligned}
\tag{1.3.2}
$$

Since $(\mathscr{N} \cap \hbar^k\mathscr{M})/(\mathscr{N} \cap \hbar^{k+1}\mathscr{M})$ is a coherent submodule of $\hbar^k\mathscr{M}/\hbar^{k+1}\mathscr{M}$, it is a good $\mathscr{A}_0$-module. Since $(\hbar\mathscr{N} + \mathscr{N} \cap \hbar^k\mathscr{M})/(\hbar\mathscr{N} + \mathscr{N} \cap \hbar^{k+1}\mathscr{M})$ is a quotient of $(\mathscr{N} \cap \hbar^k\mathscr{M})/(\mathscr{N} \cap \hbar^{k+1}\mathscr{M})$, the left term in (1.3.2) is a good $\mathscr{A}_0$-module. Hence the induction proceeds and we conclude that $\mathscr{N}/(\hbar\mathscr{N} + \mathscr{N} \cap \hbar^{k+1}\mathscr{M})$ is a good $\mathscr{A}_0$-module.

On any compact set, we have $\mathscr{N} \cap \hbar^{k+1}\mathscr{M} \subset \hbar\mathscr{N}$ for $k \gg 0$. Hence, $(\mathscr{N}/\hbar\mathscr{N})|_V$ is a good $(\mathscr{A}_0|_V)$-module for any relatively compact subset V. Hence $\mathscr{N}$ belongs to $\mathrm{Mod}_{\mathrm{gd}}(\mathscr{A})$ by (iv') (c).

Consider an exact sequence $0 \to \mathscr{M}' \to \mathscr{M} \to \mathscr{M}'' \to 0$ of coherent $\mathscr{A}$-modules. It gives rise to an exact sequence of coherent $\mathscr{A}_0$-modules

$$0 \to \mathscr{M}'_\hbar \to \mathscr{M}_\hbar \to \mathscr{M}''_\hbar \to \mathscr{M}'/\hbar\mathscr{M}' \to \mathscr{M}/\hbar\mathscr{M} \to \mathscr{M}''/\hbar\mathscr{M}'' \to 0.$$

If $\mathscr{M}$ is a good $\mathscr{A}$-module, then so is $\mathscr{M}'$. Hence the exact sequence above implies that $\mathscr{M}''_\hbar$ and $\mathscr{M}''/\hbar\mathscr{M}''$ are good $\mathscr{A}_0$-modules. This shows that $\mathrm{Mod}_{\mathrm{gd}}(\mathscr{A})$ is stable by quotients.

Finally, let us show that $\mathrm{Mod}_{\mathrm{gd}}(\mathscr{A})$ is stable by extension. If $\mathscr{M}'_\hbar$, $\mathscr{M}''_\hbar$, $\mathscr{M}'/\hbar\mathscr{M}'$ and $\mathscr{M}''/\hbar\mathscr{M}''$ are good $\mathscr{A}_0$-modules, then so are $\mathscr{M}_\hbar$ and $\mathscr{M}/\hbar\mathscr{M}$ by the exact sequence above. Q.E.D.

Lemma 1.3.5. *Let $K \in \mathfrak{B}$, and $n \geq 0$.*

(i) *For any good $\mathscr{A}_n$-module $\mathscr{N}$, we have $H^j(K;\mathscr{N}) = 0$ for $j \neq 0$.*

(ii) *For any epimorphism $\mathscr{N} \to \mathscr{N}'$ of good $\mathscr{A}_n$-modules, $\mathscr{N}(K) \to \mathscr{N}'(K)$ is surjective,*

(iii) *$\mathscr{A}(K) \to \mathscr{A}_n(K)$ is surjective.*

Proof. (i) is proved by induction on n, using the exact sequence (1.2.4).

(ii) follows immediately from (i) and the fact that the kernel of a morphism of good modules is good.

(iii) By (ii), $\mathscr{A}_{n+1}(K) \to \mathscr{A}_n(K)$ is surjective for any $n \geq 0$. Hence $\varprojlim_m \big(\mathscr{A}_m(K)\big) \to \mathscr{A}_n(K)$ is surjective.

For $s \in \mathscr{A}_n(K)$, there exist $K' \in \mathfrak{B}$ and $s' \in \mathscr{A}_n(K')$ such that $K \subset \mathrm{Int}(K')$ and $s'|_K = s$. Then s' is in the image of $\varprojlim_m \big(\mathscr{A}_m(K')\big) \to \mathscr{A}_n(K')$. Hence s is in the image of $\mathscr{A}(K) \to \mathscr{A}_n(K)$, because $\varprojlim_m \big(\mathscr{A}_m(K')\big) \to \mathscr{A}_n(K') \to \mathscr{A}_n(K)$ decomposes into

$$\varprojlim_m \big(\mathscr{A}_m(K')\big) \to \varprojlim_m \big(\mathscr{A}_m(\mathrm{Int}(K'))\big) \simeq \mathscr{A}(\mathrm{Int}(K')) \to \mathscr{A}(K) \to \mathscr{A}_n(K).$$

Q.E.D.

The proof of the following theorem is almost the same as the proof of Theorem 1.2.5, and we do not repeat it.

Theorem 1.3.6. *Assume* (1.2.2) *and* (1.3.1).

(i) *$\mathscr{A}$ is a left Noetherian ring.*

(ii) *Let $\mathscr{M}$ be a locally finitely generated $\mathscr{A}$-module. Then $\mathscr{M}$ is coherent if and only if $\hbar^n\mathscr{M}/\hbar^{n+1}\mathscr{M}$ is a coherent $\mathscr{A}_0$-module for any $n \geq 0$.*

(iii) *For any coherent $\mathscr{A}$-module $\mathscr{M}$, $\mathscr{M}$ is $\hbar$-complete, i.e., $\mathscr{M} \xrightarrow{\sim} \widehat{\mathscr{M}}$.*

(iv) *Conversely, an $\mathscr{A}$-module $\mathscr{M}$ is coherent if and only if $\mathscr{M}$ is $\hbar$-complete and $\hbar^n\mathscr{M}/\hbar^{n+1}\mathscr{M}$ is a coherent $\mathscr{A}_0$-module for any $n \geq 0$.*

(v) *For any good $\mathscr{A}$-module $\mathscr{M}$ and any $K \in \mathfrak{B}$, we have $H^j(K;\mathscr{M}) = 0$ for any $j > 0$.*

1.4 $\hbar$-graduation and $\hbar$-localization

In this section, $\mathscr{A}$ is a sheaf of algebras satisfying hypotheses (1.2.2) and either (1.2.3) or (1.3.1).

Graded modules

Let $\mathscr{R}$ be a $\mathbb{Z}[\hbar]$-algebra on a topological space X. We assume that $\mathscr{R}$ has no $\hbar$-torsion. We set

$$\mathscr{R}_0 := \mathscr{R}/\mathscr{R}\hbar.$$

Definition 1.4.1. We denote by $\mathrm{gr}_\hbar\colon \mathrm{D}(\mathscr{R}) \to \mathrm{D}(\mathscr{R}_0)$ the left derived functor of the right exact functor $\mathrm{Mod}(\mathscr{R}) \to \mathrm{Mod}(\mathscr{R}_0)$ given by $\mathscr{M} \mapsto \mathscr{M}/\hbar\mathscr{M}$. For $\mathscr{M} \in \mathrm{D}(\mathscr{R})$ we call $\mathrm{gr}_\hbar(\mathscr{M})$ the graded module associated to $\mathscr{M}$.

We have

$$\mathrm{gr}_\hbar(\mathscr{M}) \simeq \mathscr{R}_0 \overset{\mathrm{L}}{\otimes}_{\mathscr{R}} \mathscr{M} \simeq \mathbb{Z}_X \overset{\mathrm{L}}{\otimes}_{\mathbb{Z}_X[\hbar]} \mathscr{M}.$$

Lemma 1.4.2. *Let $\mathscr{M} \in \mathrm{D}(\mathscr{R})$ and let $a \in \mathbb{Z}$. Then we have an exact sequence of $\mathscr{R}_0$-modules*

$$0 \to \mathscr{R}_0 \otimes_{\mathscr{R}} H^a(\mathscr{M}) \to H^a(\mathrm{gr}_\hbar(\mathscr{M})) \to \mathscr{T}\!or_1^{\mathscr{R}}(\mathscr{R}_0, H^{a+1}(\mathscr{M})) \to 0.$$

Although this kind of results is well-known, we give a proof for the reader's convenience.

Proof. The exact sequence $0 \to \mathscr{R} \xrightarrow{\hbar} \mathscr{R} \to \mathscr{R}_0 \to 0$ gives rise to the distinguished triangle

$$\mathscr{M} \xrightarrow{\hbar} \mathscr{M} \longrightarrow \mathrm{gr}_\hbar(\mathscr{M}) \xrightarrow{+1}.$$

It induces a long exact sequence

$$H^a(\mathscr{M}) \xrightarrow{\hbar} H^a(\mathscr{M}) \longrightarrow H^a(\mathrm{gr}_\hbar(\mathscr{M})) \longrightarrow H^{a+1}(\mathscr{M}) \xrightarrow{\hbar} H^{a+1}(\mathscr{M}).$$

The result then follows from

$$\begin{aligned}\mathscr{R}_0 \otimes_{\mathscr{R}} H^a(\mathscr{M}) &\simeq \operatorname{Coker}(H^a(\mathscr{M}) \xrightarrow{\hbar} H^a(\mathscr{M})),\\ \mathscr{T}or_1^{\mathscr{R}}(\mathscr{R}_0, H^{a+1}(\mathscr{M})) &\simeq \operatorname{Ker}(H^{a+1}(\mathscr{M}) \xrightarrow{\hbar} H^{a+1}(\mathscr{M})).\end{aligned}$$

Q.E.D.

Proposition 1.4.3. (i) *Let $\mathscr{K}_1 \in \mathrm{D}(\mathscr{R}^{\mathrm{op}})$ and $\mathscr{K}_2 \in \mathrm{D}(\mathscr{R})$. Then*

$$\mathrm{gr}_\hbar(\mathscr{K}_1 \overset{\mathrm{L}}{\otimes}_{\mathscr{R}} \mathscr{K}_2) \simeq \mathrm{gr}_\hbar(\mathscr{K}_1) \overset{\mathrm{L}}{\otimes}_{\mathscr{R}_0} \mathrm{gr}_\hbar(\mathscr{K}_2). \tag{1.4.1}$$

(ii) *Let $\mathscr{K}_i \in \mathrm{D}(\mathscr{R})$ $(i = 1, 2)$. Then*

$$\mathrm{gr}_\hbar(\mathrm{R}\mathscr{H}om_{\mathscr{R}}(\mathscr{K}_1, \mathscr{K}_2)) \simeq \mathrm{R}\mathscr{H}om_{\mathscr{R}_0}(\mathrm{gr}_\hbar(\mathscr{K}_1), \mathrm{gr}_\hbar(\mathscr{K}_2)). \tag{1.4.2}$$

Proof. (i) We have

$$\begin{aligned}\mathrm{gr}_\hbar(\mathscr{K}_1 \overset{\mathrm{L}}{\otimes}_{\mathscr{R}} \mathscr{K}_2) &\simeq \mathscr{K}_1 \overset{\mathrm{L}}{\otimes}_{\mathscr{R}} \mathscr{K}_2 \overset{\mathrm{L}}{\otimes}_{\mathbb{Z}_X[\hbar]} \mathbb{Z}_X \simeq \mathscr{K}_1 \overset{\mathrm{L}}{\otimes}_{\mathscr{R}} \mathrm{gr}_\hbar(\mathscr{K}_2)\\ &\simeq \mathscr{K}_1 \overset{\mathrm{L}}{\otimes}_{\mathscr{R}} \mathscr{R}_0 \overset{\mathrm{L}}{\otimes}_{\mathscr{R}_0} \mathrm{gr}_\hbar(\mathscr{K}_2))\\ &\simeq (\mathscr{K}_1 \overset{\mathrm{L}}{\otimes}_{\mathscr{R}} \mathscr{R}_0) \overset{\mathrm{L}}{\otimes}_{\mathscr{R}_0} \mathrm{gr}_\hbar(\mathscr{K}_2)\\ &\simeq \mathrm{gr}_\hbar(\mathscr{K}_1) \overset{\mathrm{L}}{\otimes}_{\mathscr{R}_0} \mathrm{gr}_\hbar(\mathscr{K}_2).\end{aligned}$$

(ii) The proof is similar. Q.E.D.

Proposition 1.4.4. *Let $f\colon X \to Y$ be a morphism of topological spaces. Let $\mathscr{M} \in \mathrm{D}(\mathbb{Z}_X[\hbar])$ and $\mathscr{N} \in \mathrm{D}(\mathbb{Z}_Y[\hbar])$. Then*

$$\begin{aligned}\mathrm{gr}_\hbar \mathrm{R}f_* \mathscr{M} &\simeq \mathrm{R}f_* \mathrm{gr}_\hbar \mathscr{M},\\ \mathrm{gr}_\hbar f^{-1} \mathscr{N} &\simeq f^{-1} \mathrm{gr}_\hbar \mathscr{N}.\end{aligned}$$

Proof. This follows immediately from the fact that for a complex of $\mathbb{Z}_X[\hbar]$-modules $\mathscr{M}$, $\mathrm{gr}_\hbar(\mathscr{M})$ is represented by the mapping cone of $\mathscr{M} \xrightarrow{\hbar} \mathscr{M}$ and similarly for $\mathbb{Z}_Y[\hbar]$-modules. Q.E.D.

Recall that $\mathscr{A}$ is a sheaf of algebras satisfying hypotheses (1.2.2) and either (1.2.3) or (1.3.1). The functor $\mathrm{gr}_\hbar$ induces a functor (we keep the same notation):

$$\mathrm{gr}_\hbar\colon \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_0). \tag{1.4.3}$$

The following proposition is an immediate consequence of Lemma 1.4.2 and Nakayama's lemma.

Proposition 1.4.5. *Let $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A})$ and let $a \in \mathbb{Z}$. The conditions below are equivalent:*

(i) $H^a(\mathrm{gr}_\hbar(\mathscr{M})) \simeq 0$,

(ii) $H^a(\mathscr{M}) \simeq 0$ *and* $H^{a+1}(\mathscr{M})$ *has no $\hbar$-torsion.*

Corollary 1.4.6. *The functor* $\mathrm{gr}_\hbar$ *in* (1.4.3) *is conservative* (i.e., *a morphism in* $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A})$ *is an isomorphism as soon as its image by* $\mathrm{gr}_\hbar$ *is an isomorphism in* $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_0)$).

Proof. Consider a morphism $\varphi\colon \mathscr{M} \to \mathscr{N}$ in $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A})$ and assume that it induces an isomorphism $\mathrm{gr}_\hbar(\varphi)\colon \mathrm{gr}_\hbar(\mathscr{M}) \to \mathrm{gr}_\hbar(\mathscr{N})$ in $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_0)$. Let $\mathscr{M} \to \mathscr{N} \to \mathscr{L} \xrightarrow{+1}$ be a distinguished triangle. Then $\mathrm{gr}_\hbar\mathscr{L} \simeq 0$, and hence all the cohomologies of $\mathscr{L}$ vanishes by the proposition above, which means that $\mathscr{L} \simeq 0$. Q.E.D.

Homological dimension

In the sequel, for a left Noetherian $\mathbb{K}$-algebra $\mathscr{R}$, we shall say that a coherent $\mathscr{R}$-module $\mathscr{P}$ is locally projective if, for any open subset $U \subset X$, the functor

$$\mathscr{H}om_{\mathscr{R}}(\mathscr{P}, \bullet)\colon \mathrm{Mod}_{\mathrm{coh}}(\mathscr{R}|_U) \to \mathrm{Mod}(\mathbb{K}_U)$$

is exact. This is equivalent to one of the following conditions: (i) for each $x \in X$, the stalk $\mathscr{P}_x$ is projective as an $\mathscr{R}_x$-module, (ii) for each $x \in X$, the stalk $\mathscr{P}_x$ is flat as an $\mathscr{R}_x$-module, (iii) $\mathscr{P}$ is locally a direct summand of a free $\mathscr{R}$-module of finite rank.

Lemma 1.4.7. *A coherent $\mathscr{A}$-module $\mathscr{P}$ is locally projective if and only if $\mathscr{P}$ has no $\hbar$-torsion and $\mathrm{gr}_\hbar\mathscr{P}$ is a locally projective $\mathscr{A}_0$-module.*

Proof. We set for short $A := \mathscr{A}_x$ and $A_0 := (\mathscr{A}_0)_x$. Note that $A_0 \simeq \mathrm{gr}_\hbar A$.

Let P be a finitely generated A-module.

(i) Assume that P is projective. Then P is a direct summand of a free A-module. It follows that P has no $\hbar$-torsion and $\mathrm{gr}_\hbar P$ is also a direct summand of a free A_0-module.

(ii) Assume that P has no $\hbar$-torsion and $\mathrm{gr}_\hbar P$ is projective. Consider an exact sequence $0 \to N \xrightarrow{u} L \to P \to 0$ in which L is free of finite rank. Applying the functor $\mathrm{gr}_\hbar$ we find the exact sequence $0 \to \mathrm{gr}_\hbar N \xrightarrow{\mathrm{gr}_\hbar u} \mathrm{gr}_\hbar L \to \mathrm{gr}_\hbar P \to 0$ and $\mathrm{gr}_\hbar P$ being projective, there exists a map $\overline{v}\colon \mathrm{gr}_\hbar L \to \mathrm{gr}_\hbar N$ such that

$\overline{v} \circ \mathrm{gr}_\hbar u = \mathrm{id}_{\mathrm{gr}_\hbar N}$. Let us choose a map $v\colon L \to N$ such that $\mathrm{gr}_\hbar(v) = \overline{v}$. Since $\mathrm{gr}_\hbar(v \circ u) = \mathrm{id}_{\mathrm{gr}_\hbar N}$, we may write

$$v \circ u = \mathrm{id}_N - \hbar\varphi$$

where $\varphi\colon N \to N$ is an A-linear map. The map $\mathrm{id}_N - \hbar\varphi$ is invertible and we denote by ψ its inverse. Then $\psi \circ v \circ u = \mathrm{id}_N$, which proves that P is a direct summand of a free A-module. Q.E.D.

Theorem 1.4.8. *Let $d \in N$. Assume that any coherent $\mathscr{A}_0$-module locally admits a resolution of length $\leq d$ by free $\mathscr{A}_0$-modules of finite rank. Then*

(a) *for any coherent locally projective $\mathscr{A}$-module $\mathscr{P}$, there locally exists a free $\mathscr{A}$-module of finite rank $\mathscr{F}$ such that $\mathscr{P} \oplus \mathscr{F}$ is free of finite rank,*

(b) *any coherent $\mathscr{A}$-module locally admits a resolution of length $\leq d+1$ by free $\mathscr{A}$-modules of finite rank.*

Proof. (a) It is well-known (see *e.g.,* [57, Lem. B.2.2]) that the result in (a) is true when replacing $\mathscr{A}$ with $\mathscr{A}_0$. Now, let $\mathscr{P}$ be as in the statement. Then $\mathrm{gr}_\hbar\mathscr{P}$ is projective and coherent. Therefore, there exists a locally free $\mathscr{A}$-module $\mathscr{F}$ such that $\mathrm{gr}_\hbar\mathscr{P} \oplus \mathrm{gr}_\hbar\mathscr{F}$ is free of finite rank over $\mathscr{A}_0$. This implies that $\mathscr{P} \oplus \mathscr{F}$ is free of finite rank over $\mathscr{A}$ by Lemma 1.2.17.

(b)-(i) Let $\mathscr{M} \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A})$ and let us first assume that $\mathscr{M}$ has no $\hbar$-torsion. Since $\mathscr{A}$ is coherent, there exists locally an exact sequence

$$0 \to \mathscr{K} \to \mathscr{L}_{d-1} \to \cdots \to \mathscr{L}_0 \to \mathscr{M} \to 0,$$

the $\mathscr{A}$-modules $\mathscr{L}_i$ $(0 \leq i \leq d-1)$ being free of finite rank. Applying the functor $\mathrm{gr}_\hbar$, we find an exact sequence of $\mathscr{A}_0$-modules and it follows that $\mathrm{gr}_\hbar(\mathscr{K})$ is projective and finitely generated. Therefore $\mathscr{K}$ is projective and finitely generated. Let $\mathscr{F}$ be as in the statement (a). Replacing $\mathscr{K}$ and $\mathscr{L}_{d-1}$ with $\mathscr{K} \oplus \mathscr{F}$ and $\mathscr{L}_{d-1} \oplus \mathscr{F}$ respectively, the result follows in this case.

(b)-(ii) In general, any coherent $\mathscr{A}$-module $\mathscr{M}$ locally admits a resolution $0 \to \mathscr{N} \to \mathscr{L} \to \mathscr{M} \to 0$, where $\mathscr{L}$ is a free $\mathscr{A}$-module of finite rank. Since $\mathscr{N}$ has no $\hbar$-torsion, $\mathscr{N}$ admits a free resolution with length d, and the result follows. Q.E.D.

Corollary 1.4.9. *We make the hypotheses of Theorem 1.4.8. Let $\mathscr{M}^\bullet$ be a complex of $\mathscr{A}$-modules concentrated in degrees $[a,b]$ and assume that $H^i(\mathscr{M})$ is coherent for all i. Then, in a neighborhood of each $x \in X$, there exists a quasi-isomorphism $\mathscr{L}^\bullet \to \mathscr{M}^\bullet$ where $\mathscr{L}^\bullet$ is a complex of free $\mathscr{A}$-modules of finite rank concentrated in degrees $[a-d_X-1,b]$.*

Proof. The proof uses [42, Lem. 13.2.1] (or rather the dual statement). Since we do not use this result here, details are left to the reader. Q.E.D.

Localization

For a $\mathbb{Z}_X[\hbar]$-algebra $\mathscr{R}$ with no $\hbar$-torsion, we set

$$\mathscr{R}^{\mathrm{loc}} := \mathbb{Z}_X[\hbar, \hbar^{-1}] \otimes_{\mathbb{Z}_X[\hbar]} \mathscr{R}, \tag{1.4.4}$$

and we call $\mathscr{R}^{\mathrm{loc}}$ the $\hbar$-*localization* of $\mathscr{R}$. For an $\mathscr{R}$-module $\mathscr{M}$, we also set

$$\mathscr{M}^{\mathrm{loc}} := \mathscr{R}^{\mathrm{loc}} \otimes_{\mathscr{R}} \mathscr{M} \simeq \mathbb{Z}_X[\hbar, \hbar^{-1}] \otimes_{\mathbb{Z}_X[\hbar]} \mathscr{M}.$$

Lemma 1.4.10. *The algebra $\mathscr{A}^{\mathrm{loc}}$ is Noetherian.*

Proof. Let T be an indeterminate. One knows by [38, Th. A.30] that $\mathscr{A}[T]$ is Noetherian. Since $\mathscr{A}^{\mathrm{loc}} \simeq \mathscr{A}[T]/\mathscr{A}[T](T\hbar - 1)$, the result follows. Q.E.D.

1.5 Cohomologically complete modules

In order to give a criterion for the coherency of the cohomologies of a complex of modules over an algebra $\mathscr{A}$ satisfying (1.2.2) and either (1.2.3) or (1.3.1). we introduce the notion of cohomologically complete complexes.

In this section, $\mathscr{R}$ is a $\mathbb{Z}[\hbar]$-algebra satisfying

$$\mathscr{R} \text{ has no } \hbar\text{-torsion.} \tag{1.5.1}$$

Recall that $\mathscr{M}^{\mathrm{loc}} := \mathbb{Z}[\hbar, \hbar^{-1}] \otimes_{\mathbb{Z}[\hbar]} \mathscr{M}$ for an $\mathscr{R}$-module $\mathscr{M}$.

Lemma 1.5.1. *For $\mathscr{M}, \mathscr{M}' \in \mathrm{D^b}(\mathscr{R}^{\mathrm{loc}})$, we have*

$$\mathrm{R}\mathscr{H}om_{\mathscr{R}^{\mathrm{loc}}}(\mathscr{M}, \mathscr{M}') \xrightarrow{\sim} \mathrm{R}\mathscr{H}om_{\mathscr{R}}(\mathscr{M}, \mathscr{M}').$$

Proof. We have $\mathscr{R}^{\mathrm{loc}} \overset{\mathrm{L}}{\otimes}_{\mathscr{R}} \mathscr{M} \simeq \mathscr{M}$. Hence,

$$\begin{aligned} \mathrm{R}\mathscr{H}om_{\mathscr{R}^{\mathrm{loc}}}(\mathscr{M}, \mathscr{M}') &\simeq \mathrm{R}\mathscr{H}om_{\mathscr{R}^{\mathrm{loc}}}(\mathscr{R}^{\mathrm{loc}} \overset{\mathrm{L}}{\otimes}_{\mathscr{R}} \mathscr{M}, \mathscr{M}') \\ &\simeq \mathrm{R}\mathscr{H}om_{\mathscr{R}}(\mathscr{M}, \mathscr{M}'). \end{aligned}$$

Q.E.D.

The next result is obvious.

Lemma 1.5.2. *The triangulated category $\mathrm{D}(\mathscr{R}^{\mathrm{loc}})$ is equivalent to the full subcategory of $\mathrm{D}(\mathscr{R})$ consisting of objects $\mathscr{M}$ satisfying one of the following equivalent conditions:*

(i) $\mathrm{gr}_\hbar(\mathscr{M}) = 0$,

(ii) $\hbar\colon H^i(\mathscr{M}) \to H^i(\mathscr{M})$ *is an isomorphism for any integer* i,

(iii) $\mathscr{M} \to \mathscr{R}^{\mathrm{loc}} \overset{\mathrm{L}}{\otimes}_{\mathscr{R}} \mathscr{M}$ *is an isomorphism,*

(iv) $\mathrm{R}\mathscr{H}om_{\mathscr{R}}(\mathscr{R}^{\mathrm{loc}}, \mathscr{M}) \to \mathscr{M}$ *is an isomorphism,*

(v) $\mathrm{R}\mathscr{H}om_{\mathscr{R}}(\mathscr{R}^{\mathrm{loc}}/\mathscr{R}, \mathscr{M}) \simeq 0$.

Lemma 1.5.3. *Let K be a $\mathbb{Z}[\hbar]$-module with projective dimension ≤ 1. Then for any $\mathscr{M} \in \mathrm{D}(\mathscr{R})$, any open subset U and any integer i, we have an exact sequence*

$$\begin{aligned} 0 \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}[\hbar]}\big(K, H^{i-1}(U;\mathscr{M})\big) &\longrightarrow H^i\big(U;\mathrm{R}\mathscr{H}om_{\mathbb{Z}[\hbar]}(K,\mathscr{M})\big) \\ &\longrightarrow \mathrm{Hom}_{\mathbb{Z}[\hbar]}\big(K, H^i(U;\mathscr{M})\big) \longrightarrow 0. \end{aligned}$$

Proof. We have a distinguished triangle

$$\begin{aligned} \mathrm{RHom}_{\mathbb{Z}[\hbar]}\big(K, \tau^{<i}\mathrm{R}\Gamma(U;\mathscr{M})\big) &\to \mathrm{RHom}_{\mathbb{Z}[\hbar]}\big(K, \mathrm{R}\Gamma(U;\mathscr{M})\big) \\ &\to \mathrm{RHom}_{\mathbb{Z}[\hbar]}\big(K, \tau^{\geq i}\mathrm{R}\Gamma(U;\mathscr{M})\big) \xrightarrow{+1} . \end{aligned}$$

Since $H^k\mathrm{RHom}_{\mathbb{Z}[\hbar]}(K,N) = 0$ for any $k \neq 0,1$ and any $\mathbb{Z}[\hbar]$-module N, we have $H^{i+1}\mathrm{RHom}_{\mathbb{Z}[\hbar]}\big(K, \tau^{<i}\mathrm{R}\Gamma(U;\mathscr{M})\big) \simeq 0$. Hence we have an exact sequence

$$\begin{aligned} 0 \to H^i\mathrm{RHom}_{\mathbb{Z}[\hbar]}\big(K, \tau^{<i}\mathrm{R}\Gamma(U;\mathscr{M})\big) &\to H^i\mathrm{RHom}_{\mathbb{Z}[\hbar]}\big(K, \mathrm{R}\Gamma(U;\mathscr{M})\big) \\ &\to H^i\mathrm{RHom}_{\mathbb{Z}[\hbar]}\big(K, \tau^{\geq i}\mathrm{R}\Gamma(U;\mathscr{M})\big) \to 0. \end{aligned}$$

Then the result follows from

$$H^i\mathrm{RHom}_{\mathbb{Z}[\hbar]}\big(K, \tau^{<i}\mathrm{R}\Gamma(U;\mathscr{M})\big) \simeq \mathrm{Ext}^1_{\mathbb{Z}[\hbar]}\big(K, H^{i-1}(U;\mathscr{M})\big)$$

and $H^i\mathrm{RHom}_{\mathbb{Z}[\hbar]}\big(K, \tau^{\geq i}\mathrm{R}\Gamma(U;\mathscr{M})\big) \simeq \mathrm{Hom}_{\mathbb{Z}[\hbar]}\big(K, H^i(U;\mathscr{M})\big)$. Q.E.D.

Recall that we set

$$\widehat{\mathscr{M}} := \varprojlim_n \mathscr{M}/\hbar^n\mathscr{M}. \tag{1.5.2}$$

Lemma 1.5.4. *Let $\mathscr{M} \in \mathrm{Mod}(\mathscr{R})$ and assume that $\mathscr{M}$ has no $\hbar$-torsion.*

(i) $\mathscr{H}om_{\mathscr{R}}(\mathscr{R}^{\mathrm{loc}}/\mathscr{R}, \mathscr{M}^{\mathrm{loc}}/\mathscr{M}) \simeq \mathscr{E}xt^1_{\mathscr{R}}(\mathscr{R}^{\mathrm{loc}}/\mathscr{R}, \mathscr{M}) \simeq \widehat{\mathscr{M}}$.

(ii) $\mathrm{Ker}(\mathscr{M} \to \widehat{\mathscr{M}}) \simeq \mathscr{H}om_{\mathscr{R}}(\mathscr{R}^{\mathrm{loc}}, \mathscr{M})$. *In particular, $\mathscr{M}$ is $\hbar$-separated if and only if $\mathscr{H}om_{\mathscr{R}}(\mathscr{R}^{\mathrm{loc}}, \mathscr{M}) \simeq 0$.*

(iii) $\mathrm{Coker}(\mathscr{M} \to \widehat{\mathscr{M}}) \simeq \mathscr{E}xt^1_{\mathscr{R}}(\mathscr{R}^{\mathrm{loc}}, \mathscr{M})$. *In particular, $\mathscr{M}$ is $\hbar$-complete if and only if $\mathscr{E}xt^j_{\mathscr{R}}(\mathscr{R}^{\mathrm{loc}}, \mathscr{M}) \simeq 0$ for $j = 0, 1$.*

Proof. We have

$$\begin{aligned}\mathscr{H}om_{\mathscr{R}}(\mathscr{R}^{\mathrm{loc}}/\mathscr{R}, \mathscr{M}^{\mathrm{loc}}/\mathscr{M}) &\simeq \varprojlim_n \mathscr{H}om_{\mathscr{R}}(\hbar^{-n}\mathscr{R}/\mathscr{R}, \mathscr{M}^{\mathrm{loc}}/\mathscr{M}) \\ &\simeq \varprojlim_n \mathscr{H}om_{\mathscr{R}}(\hbar^{-n}\mathscr{R}/\mathscr{R}, \hbar^{-n}\mathscr{M}/\mathscr{M}) \\ &\simeq \varprojlim_n \mathscr{M}/\hbar^n\mathscr{M} \simeq \widehat{\mathscr{M}}.\end{aligned}$$

Since $\mathrm{R}\mathscr{H}om_{\mathscr{R}}(\mathscr{R}^{\mathrm{loc}}/\mathscr{R}, \mathscr{M}^{\mathrm{loc}}) \simeq 0$ by Lemma 1.5.2, applying the functor $\mathrm{R}\mathscr{H}om_{\mathscr{R}}(\mathscr{R}^{\mathrm{loc}}/\mathscr{R}, \bullet)$ to $0 \to \mathscr{M} \to \mathscr{M}^{\mathrm{loc}} \to \mathscr{M}^{\mathrm{loc}}/\mathscr{M} \to 0$, we obtain an isomorphism $\mathscr{H}om_{\mathscr{R}}(\mathscr{R}^{\mathrm{loc}}/\mathscr{R}, \mathscr{M}^{\mathrm{loc}}/\mathscr{M}) \xrightarrow{\sim} \mathscr{E}xt^1_{\mathscr{R}}(\mathscr{R}^{\mathrm{loc}}/\mathscr{R}, \mathscr{M})$. Hence we obtain (i).

By the long exact sequence associated with $0 \to \mathscr{R} \to \mathscr{R}^{\mathrm{loc}} \to \mathscr{R}^{\mathrm{loc}}/\mathscr{R} \to 0$, we obtain

$$\begin{aligned}&\mathscr{H}om_{\mathscr{R}}(\mathscr{R}^{\mathrm{loc}}/\mathscr{R}, \mathscr{M}) \to \mathscr{H}om_{\mathscr{R}}(\mathscr{R}^{\mathrm{loc}}, \mathscr{M}) \to \mathscr{H}om_{\mathscr{R}}(\mathscr{R}, \mathscr{M}) \\ &\qquad \to \mathscr{E}xt^1_{\mathscr{R}}(\mathscr{R}^{\mathrm{loc}}/\mathscr{R}, \mathscr{M}) \to \mathscr{E}xt^1_{\mathscr{R}}(\mathscr{R}^{\mathrm{loc}}, \mathscr{M}) \to 0,\end{aligned}$$

which reduces to

$$0 \to \mathscr{H}om_{\mathscr{R}}(\mathscr{R}^{\mathrm{loc}}, \mathscr{M}) \to \mathscr{M} \to \widehat{\mathscr{M}} \to \mathscr{E}xt^1_{\mathscr{R}}(\mathscr{R}^{\mathrm{loc}}, \mathscr{M}) \to 0.$$

Hence we obtain (ii) and (iii). Q.E.D.

Consider the right orthogonal category $\mathrm{D}(\mathscr{R}^{\mathrm{loc}})^{\perp r}$ to the full subcategory $\mathrm{D}(\mathscr{R}^{\mathrm{loc}})$ of $\mathrm{D}(\mathscr{R})$. By definition, this is the full triangulated subcategory consisting of objects $\mathscr{M} \in \mathrm{D}(\mathscr{R})$ satisfying $\mathrm{Hom}_{\mathrm{D}(\mathscr{R})}(\mathscr{N}, \mathscr{M}) \simeq 0$ for any $\mathscr{N} \in \mathrm{D}(\mathscr{R}^{\mathrm{loc}})$ (see [42, Exe. 10.15]).

Definition 1.5.5. One says that an object $\mathscr{M}$ of $\mathrm{D}(\mathscr{R})$ is cohomologically complete if it belongs to $\mathrm{D}(\mathscr{R}^{\mathrm{loc}})^{\perp r}$.

Proposition 1.5.6. (i) *For $\mathscr{M} \in \mathrm{D}(\mathscr{R})$, the following conditions are equivalent:*

(a) *$\mathscr{M}$ is cohomologically complete,*

(b) $\mathrm{R}\mathscr{H}om_{\mathscr{R}}(\mathscr{R}^{\mathrm{loc}}, \mathscr{M}) \simeq \mathrm{R}\mathscr{H}om_{\mathbb{Z}[\hbar]}(\mathbb{Z}[\hbar, \hbar^{-1}], \mathscr{M}) \simeq 0$,

(c) $\varinjlim_{U \ni x} \mathrm{Ext}^j_{\mathbb{Z}[\hbar]}\big(\mathbb{Z}[\hbar, \hbar^{-1}], H^i(U; \mathscr{M})\big) \simeq 0$ *for any* $x \in X$, $j = 0, 1$ *and any* $i \in \mathbb{Z}$. *Here,* U *ranges over an open neighborhood system of* x.

(ii) $\mathrm{R}\mathscr{H}om_{\mathscr{R}}(\mathscr{R}^{\mathrm{loc}}/\mathscr{R}, \mathscr{M})$ *is cohomologically complete for any* $\mathscr{M} \in \mathrm{D}(\mathscr{R})$.

(iii) *For any* $\mathscr{M} \in \mathrm{D}(\mathscr{R})$, *there exists a distinguished triangle*

$$\mathscr{M}' \to \mathscr{M} \to \mathscr{M}'' \xrightarrow{+1}$$

with $\mathscr{M}' \in \mathrm{D}(\mathscr{R}^{\mathrm{loc}})$ *and* $\mathscr{M}'' \in \mathrm{D}(\mathscr{R}^{\mathrm{loc}})^{\perp r}$.

(iv) *Conversely, if*

$$\mathscr{M}' \to \mathscr{M} \to \mathscr{M}'' \xrightarrow{+1}$$

is a distinguished triangle with $\mathscr{M}' \in \mathrm{D}(\mathscr{R}^{\mathrm{loc}})$ *and* $\mathscr{M}'' \in \mathrm{D}(\mathscr{R}^{\mathrm{loc}})^{\perp r}$, *then* $\mathscr{M}' \simeq \mathrm{R}\mathscr{H}om_{\mathscr{R}}(\mathscr{R}^{\mathrm{loc}}, \mathscr{M})$ *and* $\mathscr{M}'' \simeq \mathrm{R}\mathscr{H}om_{\mathscr{R}}(\mathscr{R}^{\mathrm{loc}}/\mathscr{R}[-1], \mathscr{M})$.

Proof. (i) (a)⇔(b) For any $\mathscr{N} \in \mathrm{D}(\mathscr{R}^{\mathrm{loc}})$, one has

$$\begin{aligned}\mathrm{Hom}_{\mathscr{R}}(\mathscr{N}, \mathscr{M}) &\simeq \mathrm{Hom}_{\mathscr{R}}(\mathscr{R}^{\mathrm{loc}} \overset{\mathrm{L}}{\otimes}_{\mathscr{R}} \mathscr{N}, \mathscr{M}) \\ &\simeq \mathrm{Hom}_{\mathscr{R}}(\mathscr{N}, \mathrm{R}\mathscr{H}om_{\mathscr{R}}(\mathscr{R}^{\mathrm{loc}}, \mathscr{M}))\end{aligned}$$

and it vanishes for all $\mathscr{N} \in \mathrm{D}(\mathscr{R}^{\mathrm{loc}})$ if and only if $\mathrm{R}\mathscr{H}om_{\mathscr{R}}(\mathscr{R}^{\mathrm{loc}}, \mathscr{M}) \simeq 0$.
(i) (b)⇔(c) follows from Lemma 1.5.3.
(ii) Since $\mathscr{R}^{\mathrm{loc}} \overset{\mathrm{L}}{\otimes}_{\mathscr{R}} (\mathscr{R}^{\mathrm{loc}}/\mathscr{R}) \simeq 0$, we have

$$\begin{aligned}&\mathrm{R}\mathscr{H}om_{\mathscr{R}}\big(\mathscr{R}^{\mathrm{loc}}, \mathrm{R}\mathscr{H}om_{\mathscr{R}}(\mathscr{R}^{\mathrm{loc}}/\mathscr{R}, \mathscr{M})\big) \\ &\qquad \simeq \mathrm{R}\mathscr{H}om_{\mathscr{R}}(\mathscr{R}^{\mathrm{loc}} \overset{\mathrm{L}}{\otimes}_{\mathscr{R}} (\mathscr{R}^{\mathrm{loc}}/\mathscr{R}), \mathscr{M}) \simeq 0,\end{aligned}$$

and hence $\mathrm{R}\mathscr{H}om_{\mathscr{R}}(\mathscr{R}^{\mathrm{loc}}/\mathscr{R}, \mathscr{M})$ is cohomologically complete.
(iii) We have obviously $\mathrm{R}\mathscr{H}om_{\mathscr{R}}(\mathscr{R}^{\mathrm{loc}}, \mathscr{M}) \in \mathrm{D}(\mathscr{R}^{\mathrm{loc}})$. Hence the distinguished triangle

$$\mathrm{R}\mathscr{H}om_{\mathscr{R}}(\mathscr{R}^{\mathrm{loc}}, \mathscr{M}) \to \mathrm{R}\mathscr{H}om_{\mathscr{R}}(\mathscr{R}, \mathscr{M}) \to \mathrm{R}\mathscr{H}om_{\mathscr{R}}(\mathscr{R}^{\mathrm{loc}}/\mathscr{R}[-1], \mathscr{M}) \xrightarrow{+1}$$

gives the result.
(iv) Since $\mathrm{R}\mathscr{H}om_{\mathscr{R}}(\mathscr{R}^{\mathrm{loc}}, \mathscr{M}'') \simeq 0$, we have

$$\mathscr{M}' \simeq \mathrm{R}\mathscr{H}om_{\mathscr{R}}(\mathscr{R}^{\mathrm{loc}}, \mathscr{M}') \xrightarrow{\sim} \mathrm{R}\mathscr{H}om_{\mathscr{R}}(\mathscr{R}^{\mathrm{loc}}, \mathscr{M}),$$

and hence $\mathscr{M}'' \simeq \mathrm{R}\mathscr{H}om_{\mathscr{R}}(\mathscr{R}^{\mathrm{loc}}/\mathscr{R}[-1], \mathscr{M})$. Q.E.D.

Note that $\mathscr{M} \mapsto \mathrm{R}\mathscr{H}om_{\mathscr{R}}(\mathscr{R}^{\mathrm{loc}}, \mathscr{M})$ is a right adjoint functor of the inclusion functor $\mathrm{D}(\mathscr{R}^{\mathrm{loc}}) \to \mathrm{D}(\mathscr{R})$, and the quotient category $\mathrm{D}(\mathscr{R})/\mathrm{D}(\mathscr{R}^{\mathrm{loc}})$ is equivalent to $\mathrm{D}(\mathscr{R}^{\mathrm{loc}})^{\perp r}$.

Remark that $\mathscr{M} \in \mathrm{D}(\mathscr{R})$ is cohomologically complete if and only if its image in $\mathrm{D}(\mathbb{Z}_X[\hbar])$ is cohomologically complete.

Corollary 1.5.7. *Let $\mathscr{M}$ be an $\mathscr{R}$-module. Assume the following conditions:*

(a) *$\mathscr{M}$ has no $\hbar$-torsion and is $\hbar$-complete,*

(b) *there is a base $\mathfrak{B}$ of open subsets such that $H^i(U;\mathscr{M}) = 0$ for any $i > 0$ and $U \in \mathfrak{B}$.*

Then $\mathscr{M}$ is cohomologically complete.

Proof. Let $U \in \mathfrak{B}$. We have $\mathrm{Ext}^j_{\mathbb{Z}[\hbar]}(\mathbb{Z}[\hbar, \hbar^{-1}], H^i(U;\mathscr{M})) = 0$ for $i > 0$ and $j = 0, 1$. Since $H^1(U;\mathscr{M}) = 0$, we have

$$\Gamma(U;\mathscr{M}/\hbar^n\mathscr{M}) \simeq \Gamma(U;\mathscr{M})/\hbar^n\Gamma(U;\mathscr{M}).$$

Hence we have

$$\begin{aligned}\Gamma(U;\mathscr{M})^{\wedge} &\simeq \varprojlim_n \Gamma(U;\mathscr{M})/\hbar^n\Gamma(U;\mathscr{M}) \simeq \varprojlim_n \Gamma(U;\mathscr{M}/\hbar^n\mathscr{M}) \\ &\simeq \Gamma(U;\varprojlim_n \mathscr{M}/\hbar^n\mathscr{M}) \simeq \Gamma(U;\mathscr{M}),\end{aligned}$$

which shows that $\Gamma(U;\mathscr{M})$ is $\hbar$-complete. Then Lemma 1.5.4 (iii) implies that $\mathrm{Ext}^j_{\mathbb{Z}[\hbar]}(\mathbb{Z}[\hbar, \hbar^{-1}], \Gamma(U;\mathscr{M})) = 0$ for $j = 0, 1$. Therefore $\mathscr{M}$ is cohomologically complete by Proposition 1.5.6. Q.E.D.

Proposition 1.5.8. *Let $\mathscr{M} \in \mathrm{D}(\mathscr{R})$ be a cohomologically complete object and $a \in \mathbb{Z}$. If $H^i(\mathrm{gr}_\hbar(\mathscr{M})) = 0$ for any $i < a$, then $H^i(\mathscr{M}) = 0$ for any $i < a$.*

Proof. The exact sequence $H^{i-1}(\mathrm{gr}_\hbar\mathscr{M}) \to H^i(\mathscr{M}) \xrightarrow{\hbar} H^i(\mathscr{M}) \to H^i(\mathrm{gr}_\hbar\mathscr{M})$ implies that $H^i(\mathscr{M}) \xrightarrow{\hbar} H^i(\mathscr{M})$ is an isomorphism for $i < a$. Hence $\tau^{<a}\mathscr{M} \in \mathrm{D}(\mathscr{R}^{\mathrm{loc}})$ and we have $\mathrm{R}\mathscr{H}om_{\mathscr{R}}(\mathscr{R}^{\mathrm{loc}}, \tau^{<a}\mathscr{M}) \simeq \tau^{<a}\mathscr{M}$. By the distinguished triangle,

$$\mathrm{R}\mathscr{H}om_{\mathscr{R}}(\mathscr{R}^{\mathrm{loc}}, \tau^{<a}\mathscr{M}) \to \mathrm{R}\mathscr{H}om_{\mathscr{R}}(\mathscr{R}^{\mathrm{loc}}, \mathscr{M}) \to \mathrm{R}\mathscr{H}om_{\mathscr{R}}(\mathscr{R}^{\mathrm{loc}}, \tau^{\geq a}\mathscr{M}) \xrightarrow{+1},$$

we have $\tau^{<a}\mathscr{M} \simeq \mathrm{R}\mathscr{H}om_{\mathscr{R}}(\mathscr{R}^{\mathrm{loc}}, \tau^{\geq a}\mathscr{M})[-1]$ and they belong to $\mathrm{D}^{<a}(\mathscr{R}) \cap \mathrm{D}^{\geq a+1}(\mathscr{R}) \simeq 0$. Q.E.D.

Corollary 1.5.9. *Let $\mathscr{M} \in \mathrm{D}(\mathscr{R})$ be a cohomologically complete object. If $\mathrm{gr}_\hbar(\mathscr{M}) \simeq 0$, then $\mathscr{M} \simeq 0$.*

Proposition 1.5.10. *Assume that $\mathscr{M} \in \mathrm{D}(\mathscr{R})$ is cohomologically complete. Then $\mathrm{R}\mathscr{H}om_{\mathscr{R}}(\mathscr{N}, \mathscr{M}) \in \mathrm{D}(\mathbb{Z}_X[\hbar])$ is cohomologically complete for any $\mathscr{N} \in \mathrm{D}(\mathscr{R})$.*

Proof. It follows from

$$\mathrm{R}\mathscr{H}om_{\mathbb{Z}[\hbar]}\big(\mathbb{Z}[\hbar, \hbar^{-1}], \mathrm{R}\mathscr{H}om_{\mathscr{R}}(\mathscr{N}, \mathscr{M})\big) \simeq \mathrm{R}\mathscr{H}om_{\mathscr{R}}\big(\mathscr{N}, \mathrm{R}\mathscr{H}om_{\mathbb{Z}[\hbar]}(\mathbb{Z}[\hbar, \hbar^{-1}], \mathscr{M})\big).$$

Q.E.D.

We can give an alternative definition of a cohomologically complete module.

Lemma 1.5.11. *Let $\mathscr{M} \in \mathrm{D}(\mathscr{R})$. Then we have*

(i) $\mathrm{R}\pi\big((\text{“}\varprojlim_n\text{”}\, \mathscr{R}\hbar^n)\overset{\mathrm{L}}{\otimes}_{\mathscr{R}}\mathscr{M}\big) \simeq \mathrm{R}\mathscr{H}om_{\mathscr{R}}(\mathscr{R}^{\mathrm{loc}}, \mathscr{M})$,

(ii) $\mathrm{R}\pi\big((\text{“}\varprojlim_n\text{”}\, \mathscr{R}/\mathscr{R}\hbar^n)\overset{\mathrm{L}}{\otimes}_{\mathscr{R}}\mathscr{M}\big) \simeq \mathrm{R}\mathscr{H}om_{\mathscr{R}}(\mathscr{R}^{\mathrm{loc}}/\mathscr{R}[-1], \mathscr{M})$.

Proof. It is enough to show (i). Set $L = \text{“}\varprojlim_n\text{”}(\mathscr{R}\hbar^n)$. Note that L is flat, *i.e.*, the functor $L \otimes_{\mathscr{R}} \bullet$ from $\mathrm{Mod}(\mathscr{R})$ to $\mathrm{Pro}(\mathrm{Mod}(\mathscr{R}))$ is exact.
One has the isomorphisms

$$\begin{aligned}
\mathscr{H}om_{\mathscr{R}}(\mathscr{R}^{\mathrm{loc}}, \mathscr{M}) &\simeq \mathscr{H}om_{\mathscr{R}}(\varinjlim_n \mathscr{R}\hbar^{-n}, \mathscr{M}) \\
&\simeq \varprojlim_n \mathscr{H}om_{\mathscr{R}}(\mathscr{R}\hbar^{-n}, \mathscr{M}) \\
&\simeq \varprojlim_n \mathscr{H}om_{\mathscr{R}}(\mathscr{R}\hbar^{-n}, \mathscr{R}) \otimes_{\mathscr{R}} \mathscr{M} \\
&\simeq \varprojlim_n \big(\mathscr{R}\hbar^n \otimes_{\mathscr{R}} \mathscr{M}\big).
\end{aligned}$$

It remains to show that $\mathrm{R}\pi(L\overset{\mathrm{L}}{\otimes}_{\mathscr{R}}\bullet)$ is the right derived functor of $\mathscr{M} \mapsto \varprojlim_n(\mathscr{R}\hbar^n \otimes_{\mathscr{R}} \mathscr{M})$. Hence, it is enough to check that if $\mathscr{M}$ is an injective $\mathscr{R}$-module, then $\mathrm{R}\pi(L\overset{\mathrm{L}}{\otimes}_{\mathscr{R}}\mathscr{M})$ is in degree zero. Applying Lemma 1.1.5 with $\mathscr{M}_n = \mathscr{R}\hbar^n \otimes_{\mathscr{R}} \mathscr{M}$, we find $H^i(U; \mathrm{R}\pi(L\overset{\mathrm{L}}{\otimes}_{\mathscr{R}}\mathscr{M})) \simeq 0$ for $i > 0$. Therefore, $\mathrm{R}^i\pi(L\overset{\mathrm{L}}{\otimes}_{\mathscr{R}}\mathscr{M}) \simeq 0$ for $i > 1$. On the other hand, since $\{\Gamma(U; \mathscr{M}_n)\}_n$ satisfies the Mittag-Leffler condition, we get that $\mathrm{R}^1\pi(L\overset{\mathrm{L}}{\otimes}_{\mathscr{R}}\mathscr{M}) \simeq 0$. Q.E.D.

Hence, $\mathscr{M}$ is cohomologically complete if and only if the morphism $\mathscr{M} \to \mathrm{R}\pi\big(\text{“}\varprojlim_n\text{”}(\mathscr{R}/\mathscr{R}\hbar^n)\overset{\mathrm{L}}{\otimes}_{\mathscr{R}}\mathscr{M}\big)$ is an isomorphism.

Proposition 1.5.12. *Let $f\colon X \to Y$ be a continuous map, and $\mathscr{M} \in \mathrm{D}(\mathbb{Z}_X[\hbar])$. If $\mathscr{M}$ is cohomologically complete, then so is $\mathrm{R}f_*\mathscr{M}$.*

Proof. It immediately follows from

$$\mathrm{R}\mathscr{H}om_{\mathbb{Z}_Y[\hbar]}(\mathbb{Z}_Y[\hbar,\hbar^{-1}],\mathrm{R}f_*\mathscr{M}) \simeq \mathrm{R}f_*\mathrm{R}\mathscr{H}om_{\mathbb{Z}_X[\hbar]}(\mathbb{Z}_X[\hbar,\hbar^{-1}],\mathscr{M}).$$

Q.E.D.

1.6 Cohomologically complete $\mathscr{A}$-modules

In this section, $\mathscr{A}$ is a $\mathbb{K}$-algebra satisfying hypotheses (1.2.2) and either (1.2.3) or (1.3.1).

Theorem 1.6.1. *Let $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A})$. Then $\mathscr{M}$ is cohomologically complete.*

Proof. Since any coherent module is an extension of a module without $\hbar$-torsion by an $\hbar$-torsion module, it is enough to treat each case.

Assume first that $\mathscr{M}$ is an $\hbar$-torsion coherent $\mathscr{A}$-module. Since the question is local, we may assume that there exists n such that $\hbar^n\mathscr{M} = 0$. Then the action of $\hbar$ on the cohomology groups of $\mathrm{R}\mathscr{H}om_{\mathscr{A}}(\mathscr{A}^{\mathrm{loc}},\mathscr{M})$ is nilpotent and invertible, and hence the cohomology groups vanishes.

Now assume that $\mathscr{M}$ is a coherent $\mathscr{A}$-module without $\hbar$-torsion. Then Corollary 1.5.7 shows that $\mathscr{M}$ is cohomologically complete. Q.E.D.

Corollary 1.6.2. *If $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A})$ and $\mathscr{N} \in \mathrm{D}(\mathscr{A})$, then $\mathrm{R}\mathscr{H}om_{\mathscr{A}}(\mathscr{N},\mathscr{M})$ is cohomologically complete.*

Proof. It is an immediate consequence of Proposition 1.5.10 and the theorem above. Q.E.D.

In the course of the proof of Theorem 1.6.4 below, we shall use the following elementary lemma that we state here without proof.

Lemma 1.6.3 (Cross Lemma)**.** *Let $\mathscr{C}$ be an abelian category and consider an exact diagram in $\mathscr{C}$*

$$\begin{tikzcd} & X_2 \arrow[d] & \\ X_1 \arrow[r] & Y \arrow[r] \arrow[d] & Z_1 \\ & Z_2. & \end{tikzcd}$$

Then the conditions below are equivalent:

(a) $\mathrm{Im}(X_2 \to Z_1) \xrightarrow{\sim} \mathrm{Im}(Y \to Z_1)$,

(b) $\mathrm{Im}(X_1 \to Z_2) \xrightarrow{\sim} \mathrm{Im}(Y \to Z_2)$,

(c) $X_1 \oplus X_2 \to Y$ *is an epimorphism.*

Theorem 1.6.4. *Let $\mathscr{M} \in \mathrm{D}^+(\mathscr{A})$ and assume:*

(a) *$\mathscr{M}$ is cohomologically complete,*

(b) $\mathrm{gr}_\hbar(\mathscr{M}) \in \mathrm{D}^+_{\mathrm{coh}}(\mathscr{A}_0)$.

Then, $\mathscr{M} \in \mathrm{D}^+_{\mathrm{coh}}(\mathscr{A})$, and we have the isomorphism

$$H^i(\mathscr{M}) \xrightarrow{\sim} \varprojlim_n H^i(\mathscr{A}_n \overset{\mathrm{L}}{\otimes}_{\mathscr{A}} \mathscr{M})$$

for all $i \in \mathbb{Z}$.

Proof. We shall assume (1.2.3). The case of hypothesis (1.3.1) could be treated with slight modifications.

Recall that $\mathscr{A}_n := \mathscr{A}/\hbar^{n+1}\mathscr{A}$ and set $\mathscr{M}_n = \mathscr{A}_n \overset{\mathrm{L}}{\otimes}_{\mathscr{A}} \mathscr{M}$, $\mathscr{N}_n^j := H^j(\mathscr{M}_n)$.

(1) For each $n \in \mathbb{N}$, the distinguished triangle $\mathscr{A}/\hbar^n\mathscr{A} \xrightarrow{\hbar} \mathscr{A}/\hbar^{n+1}\mathscr{A} \longrightarrow \mathscr{A}/\hbar\mathscr{A} \xrightarrow{+1}$ induces the distinguished triangle

$$\mathscr{M}_{n-1} \xrightarrow{\hbar} \mathscr{M}_n \longrightarrow \mathscr{M}_0 \xrightarrow{+1} . \tag{1.6.1}$$

This triangle gives rise to the long exact sequence

$$\mathscr{N}_0^{j-1} \longrightarrow \mathscr{N}_{n-1}^j \xrightarrow{\hbar} \mathscr{N}_n^j \longrightarrow \mathscr{N}_0^j \longrightarrow \mathscr{N}_{n-1}^{j+1} \tag{1.6.2}$$

from which we deduce by induction on n that $\mathscr{N}_n^j$ is a coherent $\mathscr{A}$-module for any j and $n \geq 0$ by using the hypothesis (b).

(2) Let us show that

(1.6.3) “$\varprojlim_n$” $\mathrm{Coker}(\mathscr{N}_n^j \xrightarrow{\hbar} \mathscr{N}_n^j)$ and “$\varprojlim_n$” $\mathrm{Ker}(\mathscr{N}_n^j \xrightarrow{\hbar} \mathscr{N}_n^j)$ are locally representable for all $j \in \mathbb{Z}$.

Consider the distinguished triangle:

$$\mathscr{M}_0 \xrightarrow{\hbar^{n+1}} \mathscr{M}_{n+1} \longrightarrow \mathscr{M}_n \xrightarrow{+1} . \tag{1.6.4}$$

It gives rise to the long exact sequence

$$(1.6.5) \qquad \cdots \longrightarrow \mathscr{N}_0^j \xrightarrow{\hbar^{n+1}} \mathscr{N}_{n+1}^j \longrightarrow \mathscr{N}_n^j \xrightarrow{\varphi_n^j} \mathscr{N}_0^{j+1} \longrightarrow \cdots .$$

Now consider the exact diagram, deduced from (1.6.2) and (1.6.5):

$$(1.6.6) \qquad \begin{tikzcd} & \mathscr{N}_{n+1}^j \arrow[d] & \\ \mathscr{N}_{n-1}^j \arrow[r, "\hbar\cdot"] \arrow[dr, "\varphi_{n-1}^j"'] & \mathscr{N}_n^j \arrow[r] \arrow[d, "\varphi_n^j"] & \mathscr{N}_0^j \\ & \mathscr{N}_0^{j+1}. & \end{tikzcd}$$

Here the commutativity of the triangle follows from the commutative diagram

$$\begin{tikzcd} \mathscr{M}_0 \arrow[r, "\hbar^n"] \arrow[d, "\mathrm{id}"] & \mathscr{M}_n \arrow[r] \arrow[d, "\hbar"] & \mathscr{M}_{n-1} \arrow[r, "+1"] \arrow[d, "\hbar"] & {} \\ \mathscr{M}_0 \arrow[r, "\hbar^{n+1}"] & \mathscr{M}_{n+1} \arrow[r] & \mathscr{M}_n \arrow[r, "+1"] & {} \end{tikzcd}$$

Hence $\mathrm{Im}(\varphi_{n-1}^j) \subset \mathrm{Im}(\varphi_n^j) \subset \mathscr{N}_0^{j+1}$. Therefore, the sequence $\{\mathrm{Im}\,\varphi_n^j\}_n$ of coherent $\mathscr{A}$-submodules of $\mathscr{N}_0^{j+1}$ is increasing and thus locally stationary. It follows from (1.6.6) and Lemma 1.6.3 that

(1.6.7) the decreasing sequence $\{\mathrm{Im}(\mathscr{N}_n^j \to \mathscr{N}_0^j)\}_n$ is locally stationary for any $j \in \mathbb{Z}$.

Since $\mathrm{Coker}(\mathscr{N}_{n-1}^j \xrightarrow{\hbar} \mathscr{N}_n^j) \simeq \mathrm{Im}(\mathscr{N}_n^j \to \mathscr{N}_0^j)$ by (1.6.2), we deduce that

$$\text{“}\varprojlim_n\text{”}\, \mathrm{Coker}(\mathscr{N}_n^j \xrightarrow{\hbar} \mathscr{N}_n^j) \simeq \text{“}\varprojlim_n\text{”}\, \mathrm{Coker}(\mathscr{N}_{n-1}^j \xrightarrow{\hbar} \mathscr{N}_n^j)$$

is locally representable.

Since $\mathrm{Ker}(\mathscr{N}_{n-1}^j \xrightarrow{\hbar} \mathscr{N}_n^j) \simeq \mathscr{N}_0^{j-1}/\,\mathrm{Im}(\mathscr{N}_n^{j-1} \to \mathscr{N}_0^{j-1})$ by (1.6.2), we get that $\text{“}\varprojlim_n\text{”}\, \mathrm{Ker}(\mathscr{N}_n^j \xrightarrow{\hbar} \mathscr{N}_n^j) \simeq \text{“}\varprojlim_n\text{”}\, \mathrm{Ker}(\mathscr{N}_{n-1}^j \xrightarrow{\hbar} \mathscr{N}_n^j)$ is locally representable.

Therefore, we have proved (1.6.3). Then by Proposition 1.2.18, $\varprojlim_n \mathscr{N}_n^j$ is a coherent $\mathscr{A}$-module and $\{\mathscr{N}_n^j\}_n$ satisfies the Mittag-Leffler condition.

(3) Hence it remains to prove that $H^j(\mathscr{M}) \xrightarrow{\sim} \varprojlim_n \mathscr{N}_n^j$ for any j. Set $\mathscr{M}' = (\text{“}\varprojlim_n\text{”}\, \mathscr{A}_n) \overset{\mathrm{L}}{\otimes}_{\mathscr{A}} \mathscr{M} \in \mathrm{D}^+(\mathrm{Pro}(\mathrm{Mod}(\mathscr{A})))$ and $\mathscr{N}^j = H^j(\mathscr{M}') \simeq \text{“}\varprojlim_n\text{”}\, \mathscr{N}_n^j \in \mathrm{Pro}(\mathrm{Mod}(\mathscr{A}))$. Lemma 1.5.11 implies that

$$\mathscr{M} \xrightarrow{\sim} \mathrm{R}\pi\mathscr{M}'.$$

Since the $\mathscr{N}_n^j$'s are coherent $\mathscr{A}$-modules, for any any $U \in \mathfrak{B}$, $H^i(U; \mathscr{N}_n^j) = 0$ $(i > 0)$ and $\{\mathscr{N}_n^j(U)\}_n$ satisfies the Mittag-Leffler condition. Hence in the exact sequence

$$0 \to \mathrm{R}^1\pi\big(\text{“}\varprojlim_n\text{”}\, H^{i-1}(U; \mathscr{N}_n^j)\big) \to H^i(U; \mathrm{R}\pi\mathscr{N}^j) \to \varprojlim_n H^i(U; \mathscr{N}_n^j) \to 0,$$

the first and the last term vanish, and we obtain $\mathrm{R}^i\pi\mathscr{N}^j = 0$ for any $i > 0$. Let us show that $H^j(\mathscr{M}) \xrightarrow{\sim} \varprojlim_n \mathscr{N}_n^j$ by induction on j. Assuming $H^j(\mathscr{M}) \xrightarrow{\sim} \varprojlim_n \mathscr{N}_n^j$ for $j < c$, let us show that $H^c(\mathscr{M}) \xrightarrow{\sim} \varprojlim_n \mathscr{N}_n^c$. By the assumption, $H^i(\mathscr{M}) \xrightarrow{\sim} \mathrm{R}\pi(\mathscr{N}^i)$ for any $i < c$. Hence $\tau^{<c}\mathscr{M} \xrightarrow{\sim} \mathrm{R}\pi(\tau^{<c}\mathscr{M}')$. Since $\mathscr{M} \xrightarrow{\sim} \mathrm{R}\pi\mathscr{M}'$, we obtain $\tau^{\geq c}\mathscr{M} \xrightarrow{\sim} \mathrm{R}\pi(\tau^{\geq c}\mathscr{M}')$. Hence taking the c-th cohomology, we obtain $H^c(\mathscr{M}) \xrightarrow{\sim} \mathrm{R}^0\pi H^c(\mathscr{M}') \simeq \varprojlim_n \mathscr{N}_n^c$.
Q.E.D.

The next result will be useful.

Proposition 1.6.5. *Assume that $\mathscr{A}^{\mathrm{op}}/\hbar\mathscr{A}^{\mathrm{op}}$ is a Noetherian ring and the flabby dimension of X is finite. If $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}(\mathscr{A})$ is cohomologically complete, then for any $\mathscr{N} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}^{\mathrm{op}})$, the object $\mathscr{N} \overset{\mathrm{L}}{\otimes}_{\mathscr{A}} \mathscr{M}$ of $\mathrm{D}^-(\mathbb{Z}[\hbar]_X)$ is cohomologically complete.*

Proof. By the assumption on the flabby dimension, there exists $a \in \mathbb{Z}$ such that $H^i\mathrm{R}\mathscr{H}om_{\mathbb{Z}[\hbar]}(\mathbb{Z}[\hbar, \hbar^{-1}], \mathscr{F}) = 0$ for any $\mathscr{F} \in \mathrm{D}^{\leq 0}(\mathbb{Z}_X[\hbar])$ and any $i > a$.

For any $n \in \mathbb{Z}$ we can locally find a finite complex L of free $\mathscr{A}^{\mathrm{op}}$-modules of finite rank such that there exists a distinguished triangle $L \overset{\mathrm{L}}{\otimes}_{\mathscr{A}} \mathscr{M} \to \mathscr{N} \overset{\mathrm{L}}{\otimes}_{\mathscr{A}} \mathscr{M} \to G$ where $G \in \mathrm{D}^{<n}(\mathbb{Z}_X[\hbar])$. Since $L \overset{\mathrm{L}}{\otimes}_{\mathscr{A}} \mathscr{M}$ is cohomologically complete, $H^i\mathrm{R}\mathscr{H}om_{\mathscr{A}}(\mathscr{A}^{\mathrm{loc}}, \mathscr{N} \overset{\mathrm{L}}{\otimes}_{\mathscr{A}} \mathscr{M}) \simeq H^i\mathrm{R}\mathscr{H}om_{\mathscr{A}}(\mathscr{A}^{\mathrm{loc}}, G) = 0$ for $i > n + a$. Hence $\mathscr{N} \overset{\mathrm{L}}{\otimes}_{\mathscr{A}} \mathscr{M}$ is cohomologically complete. Q.E.D.

Flatness

Theorem 1.6.6. *Assume that $\mathscr{A}^{\mathrm{op}}/\hbar\mathscr{A}^{\mathrm{op}}$ is a Noetherian ring and the flabby dimension of X is finite. Let $\mathscr{M}$ be an $\mathscr{A}$-module. Assume the following conditions:*

(a) *$\mathscr{M}$ has no $\hbar$-torsion,*

(b) *$\mathscr{M}$ is cohomologically complete,*

(c) *$\mathscr{M}/\hbar\mathscr{M}$ is a flat $\mathscr{A}_0$-module.*

Then $\mathscr{M}$ is a flat $\mathscr{A}$-module.

Proof. Let $\mathscr{N}$ be a coherent $\mathscr{A}^{\mathrm{op}}$-module. It is enough to show that we have $H^i(\mathscr{N}\overset{\mathrm{L}}{\otimes}_{\mathscr{A}}\mathscr{M}) = 0$ for any $i < 0$. We know by Proposition 1.6.5 that $\mathscr{N}\overset{\mathrm{L}}{\otimes}_{\mathscr{A}}\mathscr{M}$ is cohomologically complete. Since $\mathrm{gr}_\hbar(\mathscr{N}\overset{\mathrm{L}}{\otimes}_{\mathscr{A}}\mathscr{M}) \simeq (\mathrm{gr}_\hbar\mathscr{N})\overset{\mathrm{L}}{\otimes}_{\mathscr{A}_0}(\mathrm{gr}_\hbar\mathscr{M})$ belongs to $\mathrm{D}^{\geq 0}(\mathbb{Z}_X)$, we have $\mathscr{N}\overset{\mathrm{L}}{\otimes}_{\mathscr{A}}\mathscr{M} \in \mathrm{D}^{\geq 0}(\mathbb{Z}[\hbar]_X)$ by Proposition 1.5.8. Q.E.D.

Corollary 1.6.7. *In the situation of Theorem 1.6.6, assume moreover that $\mathscr{M}/\hbar\mathscr{M}$ is a faithfully flat $\mathscr{A}_0$-module. Then $\mathscr{M}$ is a faithfully flat $\mathscr{A}$-module.*

Proof. Let $\mathscr{N}$ be a coherent $\mathscr{A}^{\mathrm{op}}$-module such that $\mathscr{N}\otimes_{\mathscr{A}}\mathscr{M} \simeq 0$. We have to show that $\mathscr{N} \simeq 0$. By Theorem 1.6.6, we know that $\mathscr{M}$ is flat, so that $\mathscr{N}\otimes_{\mathscr{A}}\mathscr{M} \simeq \mathscr{N}\overset{\mathrm{L}}{\otimes}_{\mathscr{A}}\mathscr{M}$. Therefore

$$(\mathrm{gr}_\hbar\mathscr{N})\overset{\mathrm{L}}{\otimes}_{\mathscr{A}_0}(\mathrm{gr}_\hbar\mathscr{M}) \simeq \mathrm{gr}_\hbar(\mathscr{N}\otimes_{\mathscr{A}}\mathscr{M}) \simeq 0$$

and the hypothesis that $\mathscr{M}/\hbar\mathscr{M}$ is faithfully flat implies that $\mathrm{gr}_\hbar\mathscr{N} \simeq 0$. Since $\mathscr{N}$ is coherent, Corollary 1.4.6 implies that $\mathscr{N} \simeq 0$. Q.E.D.

Proposition 1.6.8. *Assume* (1.2.2) *and* (1.2.3)*. Let U be an open subset of X satisfying:*

$$U \cap V \in \mathfrak{B} \text{ for any } V \in \mathfrak{B}. \tag{1.6.8}$$

Then for any coherent $\mathscr{A}$-module $\mathscr{M}$, we have

(i) *$R^n\Gamma_U(\mathscr{M}) = 0$ for any $n \neq 0$,*

(ii) *$\Gamma_U(\mathscr{A})\otimes_{\mathscr{A}}\mathscr{M} \to \Gamma_U(\mathscr{M})$ is an isomorphism,*

(iii) *$\Gamma_U(\mathscr{A})$ is a flat $\mathscr{A}^{\mathrm{op}}$-module.*

Proof. (i) Since $R^n\Gamma_U(\mathscr{M})$ is the sheaf associated with the presheaf $V \mapsto H^n(U \cap V; \mathscr{M})$, (i) follows from Theorem 1.2.5 (v).

(ii) The question being local, we may assume that we have an exact sequence $0 \to \mathscr{N} \to \mathscr{L} \to \mathscr{M} \to 0$, where $\mathscr{L}$ is a free $\mathscr{A}$-module of finite rank. Then, we have a commutative diagram with exact rows by (i):

$$\begin{tikzcd}
& \Gamma_U(\mathscr{A}) \otimes_{\mathscr{A}} \mathscr{N} \arrow[r] \arrow[d] & \Gamma_U(\mathscr{A}) \otimes_{\mathscr{A}} \mathscr{L} \arrow[r] \arrow[d, "\wr"] & \Gamma_U(\mathscr{A}) \otimes_{\mathscr{A}} \mathscr{M} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Gamma_U(\mathscr{N}) \arrow[r] & \Gamma_U(\mathscr{L}) \arrow[r] & \Gamma_U(\mathscr{M}) \arrow[r] & 0.
\end{tikzcd}$$

Since the middle vertical arrow is an isomorphism, $\Gamma_U(\mathscr{A}) \otimes_{\mathscr{A}} \mathscr{M} \to \Gamma_U(\mathscr{M})$ is an epimorphism. Applying this to $\mathscr{N}$, $\Gamma_U(\mathscr{A}) \otimes_{\mathscr{A}} \mathscr{N} \to \Gamma_U(\mathscr{N})$ is an epimorphism. Hence, $\Gamma_U(\mathscr{A}) \otimes_{\mathscr{A}} \mathscr{M} \to \Gamma_U(\mathscr{M})$ is an isomorphism.

(iii) By (i) and (ii), $\mathscr{M} \mapsto \Gamma_U(\mathscr{A}) \otimes_{\mathscr{A}} \mathscr{M}$ is an exact functor on the category of coherent $\mathscr{A}$-modules. It follows that for all $x \in X$, the functor $\mathscr{M} \mapsto (\Gamma_U(\mathscr{A}))_x \otimes_{\mathscr{A}_x} \mathscr{M}_x$ is exact on the category $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{A})$. Therefore, $(\Gamma_U(\mathscr{A}))_x$ is a flat $\mathscr{A}_x^{\mathrm{op}}$-module. Q.E.D.

Remark 1.6.9. The results of this chapter can be generalized in the following situation. Let $\mathscr{A}$ be a sheaf of rings on a topological space X and let $\mathscr{I}$ be a both-sided sheaf of ideals of $\mathscr{A}$. We assume that:
there exists locally a section s of $\mathscr{I}$ such that $\mathscr{A} \ni a \mapsto as$ and $\mathscr{A} \ni a \mapsto sa$ give isomorphisms $\mathscr{A} \xrightarrow{\sim} \mathscr{I}$.

We set $\mathscr{A}_0 = \mathscr{A}/\mathscr{I}$, $\mathscr{A}(-n) = \mathscr{I}^n \subset \mathscr{A}$ and $\mathscr{A}(n) = \mathrm{R}\mathscr{H}om_{\mathscr{A}}(\mathscr{A}(-n), \mathscr{A})$ for $n \geq 0$.

Then we have $\mathscr{A}(n) \subset \mathscr{A}(n+1)$, and $\mathscr{A}(n) \otimes_{\mathscr{A}} \mathscr{A}(m) \simeq \mathscr{A}(n+m)$.

We set $\mathscr{A}^{\mathrm{loc}} = \varinjlim_n \mathscr{A}(n)$ and for an $\mathscr{A}$-module $\mathscr{M}$, we set $\mathscr{M}(n) = \mathscr{A}(n) \otimes_{\mathscr{A}} \mathscr{M}$.

We say that $\mathscr{M}$ is $\mathscr{I}$-torsion free if $\mathscr{M}(-1) \to \mathscr{M}$ is a monomorphism. Of course, $\mathscr{A}$ is $\mathscr{I}$-torsion free.

Finally, for an $\mathscr{A}$-module $\mathscr{M}$ we set $\widehat{\mathscr{M}} := \varprojlim_n \mathrm{Coker}(\mathscr{M}(-n) \to \mathscr{M})$.

Instead of (1.2.2), we assume

$$(1.6.9) \quad \begin{cases} \text{(i) } \mathscr{A} \xrightarrow{\sim} \widehat{\mathscr{A}}, \\ \text{(ii) } \mathscr{A}_0 \text{ is a left Noetherian ring.} \end{cases}$$

Under the assumptions (1.6.9) and (1.2.3), all the results of this chapter hold with suitable modifications.

In particular, our theory can be applied when $X = T^*M$ is the cotangent bundle to a complex manifold M and $\mathscr{A} = \widehat{\mathscr{E}_X}(0)$ is the ring of formal microdifferential operators of order 0 (see Section 6.1 for more details on the ring of formal microdifferential operators).

Chapter 2

DQ-algebroids

2.1 Algebroids

In this section, X denotes a topological space and recall that $\mathbb{K}$ is a commutative unital ring. A $\mathbb{K}$-linear category means a category $\mathscr{C}$ such that $\mathrm{Hom}_{\mathscr{C}}(X,Y)$ is endowed with a $\mathbb{K}$-module structure for any X, $Y \in \mathscr{C}$, and the composition map $\mathrm{Hom}_{\mathscr{C}}(X,Y) \times \mathrm{Hom}_{\mathscr{C}}(Y,Z) \to \mathrm{Hom}_{\mathscr{C}}(X,Z)$ is $\mathbb{K}$-bilinear for any X, Y, $Z \in \mathscr{C}$. One defines similarly the notion of a $\mathbb{K}$-linear stack.

The notion of an algebroid has been introduced in [46]. We refer to [19] for a more systematic study and to [42] for an introduction to stacks. Recall that a $\mathbb{K}$-algebroid $\mathscr{A}$ on X is a $\mathbb{K}$-linear stack locally non empty and such that for any open subset U of X, two objects of $\mathscr{A}(U)$ are locally isomorphic.

If A is a $\mathbb{K}$-algebra (an algebra, not a sheaf of algebras), we denote by A^+ the $\mathbb{K}$-linear category with one object and having A as the endomorphism ring of this object.

Let $\mathscr{A}$ be a sheaf of $\mathbb{K}$-algebras on X and consider the prestack $U \mapsto \mathscr{A}(U)^+$ (U open in X). We denote by $\mathscr{A}^+$ the associated stack. Then $\mathscr{A}^+$ is a $\mathbb{K}$-algebroid and is called the $\mathbb{K}$-algebroid associated with $\mathscr{A}$. The category $\mathscr{A}^+(X)$ is equivalent to the full subcategory of $\mathrm{Mod}(\mathscr{A}^{\mathrm{op}})$ consisting of objects locally isomorphic to $\mathscr{A}^{\mathrm{op}}$.

Conversely, if $\mathscr{A}$ is an algebroid on X and $\sigma \in \mathscr{A}(X)$, then $\mathscr{A}$ is equivalent to the algebroid $\mathscr{E}nd_{\mathscr{A}}(\sigma)^+$.

For an algebroid $\mathscr{A}$ and σ, $\tau \in \mathscr{A}(U)$, the $\mathbb{K}$-algebras $\mathscr{E}nd_{\mathscr{A}}(\sigma)$ and $\mathscr{E}nd_{\mathscr{A}}(\tau)$ are locally isomorphic. Hence, any definition of local nature concerning sheaves of $\mathbb{K}$-algebras, such as being coherent or Noetherian, extends to $\mathbb{K}$-algebroids.

Recall that for an algebroid $\mathscr{A}$, the algebroid $\mathscr{A}^{\mathrm{op}}$ is defined by $\mathscr{A}^{\mathrm{op}}(U) =$

$(\mathscr{A}(U))^{\mathrm{op}}$ (U open in X). Then, if $\mathscr{A}$ is a sheaf of $\mathbb{K}$-algebras, $(\mathscr{A}^{\mathrm{op}})^{+} \simeq (\mathscr{A}^{+})^{\mathrm{op}}$.

Convention 2.1.1. If $\mathscr{A}$ is a sheaf of algebras and if there is no risk of confusion, we shall keep the same notation $\mathscr{A}$ to denote the associated algebroid.

Note that two algebras may not be isomorphic even if the associated algebroids are equivalent.

Example 2.1.2. Let X be a complex manifold, $\mathscr{L}$ a line bundle on X and denote as usual by $\mathscr{D}_X$ the ring of differential operators on X. The ring of $\mathscr{L}$-twisted differential operators is given by

$$\mathscr{D}_X^{\mathscr{L}} := \mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{D}_X \otimes_{\mathscr{O}_X} \mathscr{L}^{\otimes -1}.$$

In general the two algebras $\mathscr{D}_X$ and $\mathscr{D}_X^{\mathscr{L}}$ are not isomorphic although the associated algebroids are equivalent. The equivalence is obtained by using the bi-invertible module $\mathscr{D}_X \otimes_{\mathscr{O}_X} \mathscr{L}^{\otimes -1}$ (see Definition 2.1.10 and Lemma 2.1.11 below).

Let $\mathscr{U} = \{U_i\}_{i\in I}$ be an open covering of X. In the sequel we set $U_{ij} := U_i \cap U_j$, $U_{ijk} := U_i \cap U_j \cap U_k$, etc.

Consider the data of

$$\text{(2.1.1)} \qquad \begin{cases} \text{a } \mathbb{K}\text{-algebroid } \mathscr{A} \text{ on } X, \\ \sigma_i \in \mathscr{A}(U_i) \text{ and isomorphisms } \varphi_{ij}\colon \sigma_j|_{U_{ij}} \xrightarrow{\sim} \sigma_i|_{U_{ij}}. \end{cases}$$

To these data, we associate:

- $\mathscr{A}_i = \mathscr{E}nd_{\mathscr{A}}(\sigma_i)$,
- $f_{ij}\colon \mathscr{A}_j|_{U_{ij}} \xrightarrow{\sim} \mathscr{A}_i|_{U_{ij}}$, the $\mathbb{K}$-algebra isomorphism $a \mapsto \varphi_{ij} \circ a \circ \varphi_{ij}^{-1}$,
- a_{ijk}, the invertible element of $\mathscr{A}_i(U_{ijk})$ given by $\varphi_{ij} \circ \varphi_{jk} \circ \varphi_{ik}^{-1}$.

Then:

$$\text{(2.1.2)} \qquad \begin{cases} f_{ij} \circ f_{jk} = \mathrm{Ad}(a_{ijk}) \circ f_{ik} \text{ on } U_{ijk}, \\ a_{ijk}a_{ikl} = f_{ij}(a_{jkl})a_{ijl} \text{ on } U_{ijkl}. \end{cases}$$

(Recall that $\mathrm{Ad}(a)(b) = aba^{-1}$.)

Conversely, let $\mathscr{A}_i$ be $\mathbb{K}$-algebras on U_i ($i \in I$), let $f_{ij}\colon \mathscr{A}_j|_{U_{ij}} \xrightarrow{\sim} \mathscr{A}_i|_{U_{ij}}$ ($i, j \in I$) be $\mathbb{K}$-algebra isomorphisms, and let a_{ijk} ($i, j, k \in I$) be invertible sections of $\mathscr{A}_i(U_{ijk})$ satisfying (2.1.2). One calls

$$\text{(2.1.3)} \qquad (\{\mathscr{A}_i\}_{i\in I}, \{f_{ij}\}_{i,j\in I}, \{a_{ijk}\}_{i,j,k\in I})$$

a gluing datum for $\mathbb{K}$-algebroids on $\mathscr{U}$. The following result, which easily follows from [28, Lem 3.8.1], is stated (in a different form) in [37] and goes back to [29].

Proposition 2.1.3. *Assume that X is paracompact. Consider a gluing datum* (2.1.3) *on $\mathscr{U}$. Then there exist an algebroid $\mathscr{A}$ on X and $\{\sigma_i, \varphi_{ij}\}_{i,j\in I}$ as in* (2.1.1) *to which this gluing datum is associated. Moreover, the data $(\mathscr{A}, \sigma_i, \varphi_{ij})$ are unique up to an equivalence of stacks, this equivalence being unique up to a unique isomorphism.*

We will give another construction in Proposition 2.1.13, which may be applied to non paracompact spaces such as algebraic varieties.

For an algebroid $\mathscr{A}$, one defines the $\mathbb{K}$-linear abelian category $\mathrm{Mod}(\mathscr{A})$, whose objects are called $\mathscr{A}$-modules, by setting

$$\mathrm{Mod}(\mathscr{A}) := \mathrm{Fct}_{\mathbb{K}}(\mathscr{A}, \mathfrak{Mod}(\mathbb{K}_X)). \tag{2.1.4}$$

Here $\mathfrak{Mod}(\mathbb{K}_X)$ is the $\mathbb{K}$-linear stack of sheaves of $\mathbb{K}$-modules on X and, for two $\mathbb{K}$-linear stacks $\mathscr{A}_1$ and $\mathscr{A}_2$, $\mathrm{Fct}_{\mathbb{K}}(\mathscr{A}_1, \mathscr{A}_2)$ is the category of $\mathbb{K}$-linear functors of stacks from $\mathscr{A}_1$ to $\mathscr{A}_2$. If $\mathscr{A}$ is the algebroid associated with a $\mathbb{K}$-algebra A on X, then $\mathrm{Mod}(\mathscr{A})$ is equivalent to $\mathrm{Mod}(A)$. The category $\mathrm{Mod}(\mathscr{A})$ is a Grothendieck category and we denote by $\mathrm{D}(\mathscr{A})$ its derived category and by $\mathrm{D}^{\mathrm{b}}(\mathscr{A})$ its bounded derived category.

For a $\mathbb{K}$-algebroid $\mathscr{A}$, the $\mathbb{K}$-linear prestack $U \mapsto \mathrm{Mod}(\mathscr{A}|_U)$ is a stack and we denote it by $\mathfrak{Mod}(\mathscr{A})$.

In the sequel, we shall write for short "$\sigma \in \mathscr{A}$" instead of "$\sigma \in \mathscr{A}(U)$ for some open set U".

Definition 2.1.4. An $\mathscr{A}$-module $\mathscr{L}$ is invertible if it is locally isomorphic to $\mathscr{A}$, namely for any $\sigma \in \mathscr{A}$, the $\mathscr{E}nd_{\mathscr{A}}(\sigma)$-module $\mathscr{L}(\sigma)$ is locally isomorphic to $\mathscr{E}nd_{\mathscr{A}}(\sigma)$.

This terminology is motivated by the fact that for an invertible module $\mathscr{L}$, if we set $\mathscr{B} := (\mathscr{E}nd_{\mathscr{A}}(\mathscr{L}))^{\mathrm{op}}$, then $\mathscr{H}om_{\mathscr{A}}(\mathscr{L}, \mathscr{A}) \otimes_{\mathscr{A}} \mathscr{L} \simeq \mathscr{B}$ and $\mathscr{L} \otimes_{\mathscr{B}} \mathscr{H}om_{\mathscr{A}}(\mathscr{L}, \mathscr{A}) \simeq \mathscr{A}$.

We denote by $\mathrm{Inv}(\mathscr{A})$ the full subcategory of $\mathrm{Mod}(\mathscr{A})$ consisting of invertible $\mathscr{A}$-modules and by $\mathfrak{Inv}(\mathscr{A})$ the corresponding full substack of $\mathfrak{Mod}(\mathscr{A})$. Then we have equivalences of $\mathbb{K}$-linear stacks $\mathscr{A} \xrightarrow{\sim} \mathfrak{Inv}(\mathscr{A}^{\mathrm{op}}) \xrightarrow{\sim} \mathfrak{Inv}(\mathscr{A})^{\mathrm{op}}$.

Recall that for two $\mathbb{K}$-linear categories $\mathscr{C}$ and $\mathscr{C}'$, one defines their tensor product $\mathscr{C} \otimes_{\mathbb{K}} \mathscr{C}'$ by setting $\mathrm{Ob}(\mathscr{C} \otimes_{\mathbb{K}} \mathscr{C}') = \mathrm{Ob}(\mathscr{C}) \times \mathrm{Ob}(\mathscr{C}')$ and

$$\mathrm{Hom}_{\mathscr{C}\otimes_{\mathbb{K}}\mathscr{C}'}((M, M'), (N, N')) = \mathrm{Hom}_{\mathscr{C}}(M, N) \otimes_{\mathbb{K}} \mathrm{Hom}_{\mathscr{C}'}(M', N')$$

for $M, N \in \mathscr{C}$ and $N, N' \in \mathscr{C}'$. Then $\mathscr{C} \otimes_{\mathbb{K}} \mathscr{C}'$ is a $\mathbb{K}$-linear category.

For a pair of $\mathbb{K}$-algebroids $\mathscr{A}$ and $\mathscr{A}'$, the $\mathbb{K}$-algebroid $\mathscr{A} \otimes_{\mathbb{K}} \mathscr{A}'$ is the $\mathbb{K}$-linear stack associated with the prestack $U \mapsto \mathscr{A}(U) \otimes_{\mathbb{K}} \mathscr{A}'(U)$ (U open in X). We have

$$\mathrm{Mod}(\mathscr{A} \otimes_{\mathbb{K}} \mathscr{A}') \simeq \mathrm{Fct}_{\mathbb{K}}(\mathscr{A}, \mathfrak{Mod}(\mathscr{A}')).$$

For a $\mathbb{K}$-algebroid $\mathscr{A}$, $\mathrm{Mod}(\mathscr{A} \otimes_{\mathbb{K}} \mathscr{A}^{\mathrm{op}})$ has a canonical object given by

$$\mathscr{A} \otimes_{\mathbb{K}} \mathscr{A}^{\mathrm{op}} \ni (\sigma, \sigma'^{\mathrm{op}}) \mapsto \mathscr{H}om_{\mathscr{A}}(\sigma', \sigma) \in \mathfrak{Mod}(\mathbb{K}_X).$$

We denote this object by the same letter $\mathscr{A}$. If $\mathscr{A}$ is associated with a $\mathbb{K}$-algebra A, this object corresponds to A, regarded as an $(A \otimes_{\mathbb{K}} A^{\mathrm{op}})$-module.

For $\mathbb{K}$-algebroids $\mathscr{A}_i$ ($i = 1, 2, 3$), we have the tensor product functor

$$\begin{aligned} \bullet \otimes_{\mathscr{A}_2} \bullet \colon \mathrm{Mod}(\mathscr{A}_1 \otimes_{\mathbb{K}} \mathscr{A}_2^{\mathrm{op}}) \times \mathrm{Mod}(\mathscr{A}_2 \otimes_{\mathbb{K}} \mathscr{A}_3^{\mathrm{op}}) \\ \to \mathrm{Mod}(\mathscr{A}_1 \otimes_{\mathbb{K}} \mathscr{A}_3^{\mathrm{op}}), \end{aligned} \tag{2.1.5}$$

and the $\mathscr{H}om$ functor

$$\begin{aligned} \mathscr{H}om_{\mathscr{A}_1}(\bullet, \bullet) \colon \mathrm{Mod}(\mathscr{A}_1 \otimes_{\mathbb{K}} \mathscr{A}_2^{\mathrm{op}})^{\mathrm{op}} \times \mathrm{Mod}(\mathscr{A}_1 \otimes_{\mathbb{K}} \mathscr{A}_3^{\mathrm{op}}) \\ \to \mathrm{Mod}(\mathscr{A}_2 \otimes_{\mathbb{K}} \mathscr{A}_3^{\mathrm{op}}). \end{aligned} \tag{2.1.6}$$

In particular, we have

$$\begin{array}{lcccl} \bullet \otimes_{\mathscr{A}} \bullet & : & \mathrm{Mod}(\mathscr{A}^{\mathrm{op}}) \times \mathrm{Mod}(\mathscr{A}) & \longrightarrow & \mathrm{Mod}(\mathbb{K}_X), \\ \mathscr{H}om_{\mathscr{A}}(\bullet, \bullet) & : & \mathrm{Mod}(\mathscr{A})^{\mathrm{op}} \times \mathrm{Mod}(\mathscr{A}) & \longrightarrow & \mathrm{Mod}(\mathbb{K}_X), \\ \mathscr{H}om_{\mathscr{A}}(\bullet, \mathscr{A}) & : & \mathrm{Mod}(\mathscr{A})^{\mathrm{op}} & \longrightarrow & \mathrm{Mod}(\mathscr{A}^{\mathrm{op}}). \end{array}$$

Since $\mathrm{Mod}(\mathscr{A})$ is a Grothendieck category, any left exact functor from $\mathrm{Mod}(\mathscr{A})$ to an abelian category admits a right derived functor.

Now consider the tensor product in (2.1.5). It admits a left derived functor as soon as $\mathscr{A}_3$ is $\mathbb{K}$-flat. Indeed, any $\mathscr{M} \in \mathrm{Mod}(\mathscr{A}_2 \otimes (\mathscr{A}_3)^{\mathrm{op}})$ is a quotient of an $\mathscr{A}_2$-flat module since there is an exact sequence

$$\bigoplus_{s \in \mathrm{Hom}\,(\mathscr{L}, \mathscr{M}|_U)} \mathscr{L} \to \mathscr{M} \to 0,$$

where $\mathscr{U}$ ranges over the family of open subsets of X and $\mathscr{L} \in (\mathscr{A}_2 \otimes (\mathscr{A}_3)^{\mathrm{op}})^{\mathrm{op}}(U)$. (Recall that for a $\mathbb{K}$-algebroid $\mathscr{A}$, $\mathscr{A}^{\mathrm{op}}(U)$ is equivalent to $\mathfrak{Inv}(\mathscr{A})(U)$.) Note that $\mathscr{L}$ is $\mathscr{A}_2$-flat since $(\mathscr{A}_3)^{\mathrm{op}}$ is $\mathbb{K}$-flat.

The following lemma is obvious.

Lemma 2.1.5. *Let $\mathscr{A}$ and $\mathscr{A}'$ be $\mathbb{K}$-algebroids. To give a functor of algebroids $\varphi\colon \mathscr{A}' \to \mathscr{A}$ is equivalent to giving an $(\mathscr{A}' \otimes \mathscr{A}^{\mathrm{op}})$-module $\mathscr{L}$ which is locally isomorphic to $\mathscr{A}$ (i.e. for $\sigma \in \mathscr{A}$ and $\sigma' \in \mathscr{A}'$, $\mathscr{L}(\sigma' \otimes \sigma^{\mathrm{op}})$ is locally isomorphic to $\mathscr{E}nd_{\mathscr{A}}(\sigma)$ as an $\mathscr{E}nd_{\mathscr{A}}(\sigma)^{\mathrm{op}}$-module).*

The $\mathscr{A}' \otimes \mathscr{A}^{\mathrm{op}}$-module $\mathscr{L}$ corresponding to φ is the module induced from the $\mathscr{A} \otimes \mathscr{A}^{\mathrm{op}}$-module $\mathscr{A}$ by $\varphi \otimes \mathscr{A}^{\mathrm{op}}\colon \mathscr{A}' \otimes \mathscr{A}^{\mathrm{op}} \to \mathscr{A} \otimes \mathscr{A}^{\mathrm{op}}$.

The forgetful functor

$$\mathrm{Mod}(\mathscr{A}) \to \mathrm{Mod}(\mathscr{A}')$$

is isomorphic to $\mathscr{M} \mapsto \mathscr{L} \otimes_{\mathscr{A}} \mathscr{M}$.

Let $f\colon X \to Y$ be a continuous map and let $\mathscr{A}$ be a $\mathbb{K}$-algebroid on Y. We denote by $f^{-1}\mathscr{A}$ the $\mathbb{K}$-linear stack associated with the prestack $\mathfrak{S}$ given by:

$$\begin{aligned}
\mathfrak{S}(U) = \{(\sigma, V)\,;\, & V \text{ is an open subset of } Y \text{ such that } f(U) \subset V \\
& \text{and } \sigma \in \mathscr{A}(V)\} \quad \text{for any open subset } U \text{ of } X, \\
\mathrm{Hom}_{\mathfrak{S}(U)}\big((\sigma, V), (\sigma', V')\big) &= \Gamma(U; f^{-1}\mathscr{H}om_{\mathscr{A}}(\sigma, \sigma')).
\end{aligned}$$

Then $f^{-1}\mathscr{A}$ is a $\mathbb{K}$-algebroid. We have functors

$$\begin{aligned}
f_*,\ f_!\colon \mathrm{Mod}(f^{-1}\mathscr{A}) &\longrightarrow \mathrm{Mod}(\mathscr{A}), \\
f^{-1}\colon \mathrm{Mod}(\mathscr{A}) &\longrightarrow \mathrm{Mod}(f^{-1}\mathscr{A}).
\end{aligned}$$

For two topological spaces X_1 and X_2, let $p_i\colon X_1 \times X_2 \to X_i$ be the projection. Let $\mathscr{A}_i$ be a $\mathbb{K}$-algebroid on X_i $(i = 1, 2)$. We define a $\mathbb{K}$-algebroid on $X_1 \times X_2$, called the external tensor product of $\mathscr{A}_1$ and $\mathscr{A}_2$, by setting:

$$\mathscr{A}_1 \boxtimes \mathscr{A}_2 := p_1^{-1}\mathscr{A}_1 \otimes p_2^{-1}\mathscr{A}_2.$$

We have a canonical bi-functor

$$\bullet \boxtimes \bullet\colon \mathrm{Mod}(\mathscr{A}_1) \times \mathrm{Mod}(\mathscr{A}_2) \longrightarrow \mathrm{Mod}(\mathscr{A}_1 \boxtimes \mathscr{A}_2).$$

Bi-invertible modules

The following notion of bi-invertible modules will appear all along these Notes since it describes equivalences of algebroids.

Definition 2.1.6. Let A and A' be two sheaves of $\mathbb{K}$-algebras. An $A \otimes A'$-module L is called *bi-invertible* if there exists locally a section w of L such that $A \ni a \mapsto (a \otimes 1)w \in L$ and $A' \ni a' \mapsto (1 \otimes a')w \in L$ give isomorphisms of A-modules and A'-modules, respectively.

Lemma 2.1.7. *Let L be a bi-invertible $A \otimes A'$-module and let u be a section of L. If $A \ni a \mapsto (a \otimes 1)u \in L$ is an isomorphism of A-modules, then $A' \ni a' \mapsto (1 \otimes a')u \in L$ is also an isomorphism of A'-modules.*

Proof. Let w be as above. There exist $a \in A$ and $b \in A$ such that $u = (a \otimes 1)w$ and $w = (b \otimes 1)u$. Then we have $u = (ab \otimes 1)u$ and hence $ab = 1$. Similarly $w = (ba \otimes 1)w$ implies $ba = 1$. Hence we have a commutative diagram

$$\begin{tikzcd}
A' \arrow[r, "\sim", "w"'] \arrow[dr, "u"'] & L \arrow[d, "a\otimes 1", "\wr"'] \\
& L
\end{tikzcd}$$

and we obtain the desired result. Q.E.D.

Remark 2.1.8. Let A and B be two $\mathbb{K}$-algebras and let L be an $(A \otimes B^{\mathrm{op}})$-module. Even if L is isomorphic to A as an A-module and isomorphic to B^{op} as a B^{op}-module, L is not necessarily bi-invertible, as shown by the following example.

Let I be an infinite set and take $\mathrm{o} \in I$. Set $I^* = I \setminus \{\mathrm{o}\}$. Then there exists a bijection $v \colon I^* \to I$. Set

$$\begin{aligned}
X &= \{a \in \mathrm{Hom}_{\mathbf{Set}}(I, I); a(\mathrm{o}) = \mathrm{o}\},\\
Y &= \{b \in \mathrm{Hom}_{\mathbf{Set}}(I, I); b(\mathrm{o}) = \mathrm{o} \text{ and } b(I^*) \subset I^*\}.
\end{aligned}$$

Set $Z = X$. Then X and Y are semi-groups and X acts on Z from the left and Y acts on Z from the right. Let $v' \in Z$ be the unique element extending v. Then $\mathrm{id}_I \in Z$ gives an isomorphism $X \xrightarrow{\sim} Z$ ($X \ni a \mapsto a \in Z$) and $v' \in Z$ induces an isomorphism $Y \xrightarrow{\sim} Z$ ($Y \ni b \mapsto v' \circ b \in Z$). Let $A = \mathbb{K}[X]$ and $B = \mathbb{K}[Y]$ be the semigroup algebras corresponding to X and Y. Set $L = \mathbb{K}[Z]$. Then L is an $(A \otimes B^{\mathrm{op}})$-module and L is isomorphic to A as an A-module and isomorphic to B^{op} as an B^{op}-module. Let u be the element of L corresponding to id_I. Then u gives an isomorphism $A \ni a \mapsto (a \otimes 1)u \in L$. Since the image of $B^{\mathrm{op}} \ni b \mapsto (1 \otimes b)u \in L$ is $\mathbb{K}[Y] \neq L$, L is not bi-invertible in view of Lemma 2.1.7.

However the following partial result holds.

Lemma 2.1.9. *Let A an A' be $\mathbb{K}$-algebras and let L be an $A \otimes A'$-module. Assume that L is isomorphic to A as an A-module and isomorphic to A' as an A'-module. If we assume moreover that A_x is a left noetherian ring for any $x \in X$, then L is bi-invertible.*

Proof. Assume that $A \ni a \mapsto (a \otimes 1)u \in L$ and $A' \ni a' \mapsto (1 \otimes a')v \in L$ are isomorphisms for some $u, v \in L$. Set $v = (a \otimes 1)u$ and $u = (1 \otimes a')v$. There exists $a'' \in A$ such that $(1 \otimes a')u = (a'' \otimes 1)u$. Then we have $u = (1 \otimes a')v = (1 \otimes a')(a \otimes 1)u = (a \otimes 1)(1 \otimes a')u = (aa'' \otimes 1)u$. Hence we obtain $aa'' = 1$. Therefore the A-linear endomorphism $f\colon A \ni z \mapsto za''$ is an epimorphism ($f(za) = z$). Since A_x is a left noetherian ring, f is an isomorphism. Hence, a'', as well as a, is an invertible element. Then the following commutative diagram implies the desired result:

$$\begin{tikzcd} A' \arrow[r, "\sim", "v"'] \arrow[dr, "u"'] & L \arrow[d, "a\otimes 1", "\wr"'] \\ & L. \end{tikzcd}$$

Q.E.D.

Definition 2.1.10. For two $\mathbb{K}$-algebroids $\mathscr{A}$ and $\mathscr{A}'$, we say that an $(\mathscr{A} \otimes \mathscr{A}')$-module $\mathscr{L}$ is bi-invertible if for any $\sigma \in \mathscr{A}$ and $\sigma' \in \mathscr{A}'$, $\mathscr{L}(\sigma \otimes \sigma')$ is a bi-invertible $\mathscr{E}nd_{\mathscr{A}}(\sigma) \otimes \mathscr{E}nd_{\mathscr{A}'}(\sigma')$-module.

Lemma 2.1.11. *To give an equivalence $\mathscr{A}' \xrightarrow{\sim} \mathscr{A}$ is equivalent to giving a bi-invertible $(\mathscr{A}' \otimes \mathscr{A}^{\mathrm{op}})$-module. More precisely, the forgetful functor $\mathfrak{Mod}(\mathscr{A}) \to \mathfrak{Mod}(\mathscr{A}')$ is given by $\mathscr{M} \mapsto \mathscr{L} \otimes_{\mathscr{A}} \mathscr{M}$ for a bi-invertible $(\mathscr{A}' \otimes \mathscr{A}^{\mathrm{op}})$-module $\mathscr{L}$.*

Let $\mathscr{M} \in \mathrm{Mod}(\mathscr{A})$. We shall denote by $\mathscr{E}nd_{\mathbb{K}}(\mathscr{M})$ the stack associated with the prestack $\mathfrak{S}$ whose objects are those of $\mathscr{A}$ and $\mathscr{H}om_{\mathfrak{S}}(\sigma, \sigma') = \mathscr{H}om_{\mathbb{K}}(\mathscr{M}(\sigma), \mathscr{M}(\sigma'))$ for σ, $\sigma' \in \mathscr{A}(U)$. Then $\mathscr{E}nd_{\mathbb{K}}(\mathscr{M})$ is a $\mathbb{K}$-algebroid and there exists a natural functor of $\mathbb{K}$-algebroids $\mathscr{A} \to \mathscr{E}nd_{\mathbb{K}}(\mathscr{M})$. Note that $\mathscr{M}$ may be regarded as an $\mathscr{E}nd_{\mathbb{K}}(\mathscr{M})$-module.

In particular, $\mathscr{E}nd_{\mathbb{K}}(\mathscr{A})$ is a $\mathbb{K}$-algebroid,there is a functor of $\mathbb{K}$-algebroids $\mathscr{A} \otimes \mathscr{A}^{\mathrm{op}} \to \mathscr{E}nd_{\mathbb{K}}(\mathscr{A})$, and $\mathscr{A}$ may be regarded as an $\mathscr{E}nd_{\mathbb{K}}(\mathscr{A})$-module.

Lemma 2.1.12. *Let $\mathscr{A}$ and $\mathscr{A}'$ be $\mathbb{K}$-algebroids, and let $\mathscr{M} \in \mathrm{Mod}(\mathscr{A})$, $\mathscr{M}' \in \mathrm{Mod}(\mathscr{A}')$. Assume that $\mathscr{M}$ and $\mathscr{M}'$ are locally isomorphic as $\mathbb{K}$-modules, that is, for any $\sigma \in \mathscr{A}$ and $\sigma' \in \mathscr{A}'$, $\mathscr{M}(\sigma)$ and $\mathscr{M}'(\sigma')$ are locally isomorphic as $\mathbb{K}_X$-modules. Then $\mathscr{E}nd_{\mathbb{K}}(\mathscr{M})$ and $\mathscr{E}nd_{\mathbb{K}}(\mathscr{M}')$ are equivalent as $\mathbb{K}$-algebroids.*

Proof. For $\sigma \in \mathscr{A}$ and $\sigma' \in \mathscr{A}'$, set $\mathscr{L}(\sigma' \otimes \sigma^{\mathrm{op}}) = \mathscr{H}om_{\mathbb{K}}(\mathscr{M}(\sigma), \mathscr{M}'(\sigma'))$. Then $\mathscr{L}$ is an $(\mathscr{E}nd_{\mathbb{K}}(\mathscr{M}') \otimes \mathscr{E}nd_{\mathbb{K}}(\mathscr{M})^{\mathrm{op}})$-module. By the assumption, $\mathscr{L}$ is a bi-invertible $\big(\mathscr{E}nd_{\mathbb{K}}(\mathscr{M}') \otimes \mathscr{E}nd_{\mathbb{K}}(\mathscr{M})^{\mathrm{op}}\big)$-module. Hence we obtain the desired result. Q.E.D.

Since Proposition 2.1.3 does not apply to algebraic varieties, we need an alternative local description of algebroids.

Let $\mathscr{U} = \{U_i\}_{i \in I}$ be an open covering of X. Consider the data of

$$\begin{cases} \text{a } \mathbb{K}\text{-algebroid } \mathscr{A} \text{ on } X, \\ \sigma_i \in \mathscr{A}(U_i). \end{cases} \tag{2.1.7}$$

To these data, we associate

- $\mathscr{A}_i := \mathscr{E}nd_{\mathscr{A}}(\sigma_i)$,
- $\mathscr{L}_{ij} := \mathscr{H}om_{\mathscr{A}_i|_{U_{ij}}}(\sigma_j|_{U_{ij}}, \sigma_i|_{U_{ij}})$, (hence $\mathscr{L}_{ij}$ is a bi-invertible $\mathscr{A}_i \otimes \mathscr{A}_j^{\mathrm{op}}$-module on U_{ij}),
- the natural isomorphisms

$$a_{ijk} \colon \mathscr{L}_{ij} \otimes_{\mathscr{A}_j} \mathscr{L}_{jk} \xrightarrow{\sim} \mathscr{L}_{ik} \quad \text{in } \mathrm{Mod}(\mathscr{A}_i \otimes \mathscr{A}_k^{\mathrm{op}}|_{U_{ijk}}). \tag{2.1.8}$$

Then the diagram below in $\mathrm{Mod}(\mathscr{A}_i \otimes \mathscr{A}_l^{\mathrm{op}}|_{U_{ijkl}})$ commutes:

$$\begin{tikzcd} \mathscr{L}_{ij} \otimes \mathscr{L}_{jk} \otimes \mathscr{L}_{kl} \arrow[r, "a_{ijk}"] \arrow[d, "a_{jkl}"] & \mathscr{L}_{ik} \otimes \mathscr{L}_{kl} \arrow[d, "a_{ikl}"] \\ \mathscr{L}_{ij} \otimes \mathscr{L}_{jl} \arrow[r, "a_{ijl}"] & \mathscr{L}_{il}\,. \end{tikzcd} \tag{2.1.9}$$

Conversely, let $\mathscr{A}_i$ be sheaves of $\mathbb{K}$-algebras on U_i ($i \in I$), let $\mathscr{L}_{ij}$ be a bi-invertible $\mathscr{A}_i \otimes \mathscr{A}_j^{\mathrm{op}}$-module on U_{ij}, and let a_{ijk} be isomorphisms as in (2.1.8) such that the diagram (2.1.9) commutes. One calls

$$(\{\mathscr{A}_i\}_{i \in I}, \{\mathscr{L}_{ij}\}_{i,j \in I}, \{a_{ijk}\}_{i,j,k \in I}) \tag{2.1.10}$$

an algebraic gluing datum for $\mathbb{K}$-algebroids on $\mathscr{U}$.

Proposition 2.1.13. *Consider an algebraic gluing datum* (2.1.10) *on* $\mathscr{U}$. *Then there exist an algebroid* $\mathscr{A}$ *on* X *and* $\{\sigma_i, \varphi_{ij}\}_{i,j \in I}$ *as in* (2.1.1) *to which this gluing datum is associated. Moreover, the data* $(\mathscr{A}, \sigma_i, \varphi_{ij})$ *are unique up to an equivalence of stacks, this equivalence being unique up to a unique isomorphism.*

Sketch of proof. We define a category $\mathrm{Mod}(\mathscr{A}_X)$ as follows. An object $\mathscr{M} \in \mathrm{Mod}(\mathscr{A}_X)$ is defined as a family $\{\mathscr{M}_i, q_{ij}\}_{i,j\in I}$ with $\mathscr{M}_i \in \mathrm{Mod}(\mathscr{A}_i)$ and the q_{ij}'s are isomorphisms

$$q_{ij} \colon \mathscr{L}_{ij} \otimes_{\mathscr{A}_j} \mathscr{M}_j \xrightarrow{\sim} \mathscr{M}_i$$

making the diagram below commutative:

$$\begin{tikzcd}
\mathscr{L}_{ij} \otimes \mathscr{L}_{jk} \otimes \mathscr{M}_k \arrow[r, "q_{jk}"] \arrow[d, "a_{ijk}"] & \mathscr{L}_{ij} \otimes \mathscr{M}_j \arrow[d, "q_{ij}"] \\
\mathscr{L}_{ik} \otimes \mathscr{M}_k \arrow[r, "q_{ik}"] & \mathscr{M}_i .
\end{tikzcd}$$

A morphism $\{\mathscr{M}_i, q_{ji}\}_{i,j\in I} \to \{\mathscr{M}'_i, q'_{ji}\}_{i,j\in I}$ in $\mathrm{Mod}(\mathscr{A}_X)$ is a family of morphisms $u_i \colon \mathscr{M}_i \to \mathscr{M}'_i$ satisfying the natural compatibility conditions. Replacing X with U open in X, we define a prestack $U \mapsto \mathrm{Mod}(\mathscr{A}_U)$ and one easily checks that this prestack is a stack and moreover that $\mathrm{Mod}(\mathscr{A}_{U_i})$ is equivalent to $\mathrm{Mod}(\mathscr{A}_i)$. We denote it by $\mathfrak{Mod}(\mathscr{A})$. Then we define the algebroid $\mathscr{A}_X$ as the substack of $(\mathfrak{Mod}(\mathscr{A}))^{\mathrm{op}}$ consisting of objects locally isomorphic to $\mathscr{A}_i$ on U_i. Q.E.D.

Invertible algebroids

In this subsection, $(X, \mathscr{R})$ denotes a topological space endowed with a sheaf of commutative $\mathbb{K}$-algebras. Recall (see [42, Chap.19 § 5]) that an $\mathscr{R}$-linear stack $\mathfrak{S}$ is a $\mathbb{K}$-linear stack $\mathfrak{S}$ together with a morphism of $\mathbb{K}$-algebras $\mathscr{R} \to \mathscr{E}nd(\mathrm{id}_{\mathfrak{S}})$. Here, $\mathscr{E}nd(\mathrm{id}_{\mathfrak{S}})$ is the sheaf of endomorphisms of the identity functor $\mathrm{id}_{\mathfrak{S}}$ from $\mathfrak{S}$ to itself.

Definition 2.1.14. (i) An $\mathscr{R}$-algebroid is a $\mathbb{K}$-algebroid $\mathscr{P}$ on X endowed with a morphism of $\mathbb{K}$-algebras $\mathscr{R} \to \mathscr{E}nd(\mathrm{id}_{\mathscr{P}})$.

(ii) An $\mathscr{R}$-algebroid $\mathscr{P}$ on X is called an invertible $\mathscr{R}$-algebroid if $\mathscr{R}_U \to \mathscr{E}nd_{\mathscr{P}}(\sigma)$ is an isomorphism for any open subset U of X and any $\sigma \in \mathscr{P}(U)$.

We shall state some properties of invertible $\mathscr{R}$-algebroids. Since the proofs are more or less obvious, we omit them.

For two $\mathscr{R}$-algebroids $\mathscr{P}_1$ and $\mathscr{P}_2$, the $\mathscr{R}$-algebroid $\mathscr{P}_1 \otimes_{\mathscr{R}} \mathscr{P}_2$ is defined as the $\mathscr{R}$-linear stack associated with the prestack $\mathfrak{S}$ given by

$$\begin{aligned}
&\mathfrak{S}(U) = \mathscr{P}_1(U) \times \mathscr{P}_2(U), \\
&\mathscr{H}om_{\mathfrak{S}}\big((\sigma_1, \sigma_2), (\sigma'_1, \sigma'_2)\big) = \mathscr{H}om_{\mathscr{P}_1}(\sigma_1, \sigma'_1) \otimes_{\mathscr{R}} \mathscr{H}om_{\mathscr{P}_2}(\sigma_2, \sigma'_2).
\end{aligned}$$

If $\mathscr{P}_1$ and $\mathscr{P}_2$ are invertible, then so is $\mathscr{P}_1 \otimes_{\mathscr{R}} \mathscr{P}_2$.

We have a functor of $\mathbb{K}$-linear stacks $\mathscr{P}_1 \otimes_{\mathbb{K}_X} \mathscr{P}_2 \to \mathscr{P}_1 \otimes_{\mathscr{R}} \mathscr{P}_2$.

Note that

(2.1.11) If $\mathscr{P}_1$ and $\mathscr{P}_2$ are two invertible $\mathscr{R}$-algebroids and $F\colon \mathscr{P}_1 \to \mathscr{P}_2$ is a functor of $\mathscr{R}$-linear stacks, then F is an equivalence.

(2.1.12) For any invertible $\mathscr{R}$-algebroid $\mathscr{P}$, $\mathscr{P} \otimes_{\mathscr{R}} \mathscr{P}^{\mathrm{op}}$ is equivalent to $\mathscr{R}$ as an $\mathscr{R}$-algebroid.

(2.1.13) The set of equivalence classes of invertible $\mathscr{R}$-algebroids has a structure of an additive group by the operation $\bullet \otimes_{\mathscr{R}} \bullet$ defined above, and this group is isomorphic to $H^2(X;\mathscr{R}^\times)$ (see [7, 42]). Here $\mathscr{R}^\times$ denotes the abelian sheaf of invertible sections of $\mathscr{R}$.

(2.1.14) For two invertible $\mathscr{R}$-algebroids $\mathscr{P}_1$ and $\mathscr{P}_2$, there is a natural functor

$$\bullet \otimes_{\mathscr{R}} \bullet\colon \mathrm{Mod}(\mathscr{P}_1) \times \mathrm{Mod}(\mathscr{P}_2) \to \mathrm{Mod}(\mathscr{P}_1 \otimes_{\mathscr{R}} \mathscr{P}_2),$$

and its derived version.

Invertible $\mathscr{O}_X$-algebroids

In this subsection, $(X, \mathscr{O}_X)$ denotes a complex manifold. As a particular case of Definition 2.1.14, taking $\mathbb{K} = \mathbb{C}$ and $\mathscr{R} = \mathscr{O}_X$, we get the notions of an $\mathscr{O}_X$-algebroid as well as that of an invertible $\mathscr{O}_X$-algebroid.

Lemma 2.1.15. *Any $\mathbb{C}$-algebra endomorphism of $\mathscr{O}_X$ is equal to the identity.*

Although this result is elementary and well-known, we give a proof.

Proof. Let φ be a $\mathbb{C}$-algebra endomorphism of $\mathscr{O}_X$. For $x \in X$, denote by φ_x the $\mathbb{C}$-algebra endomorphism of $\mathscr{O}_{X,x}$ induced by φ and by $\mathfrak{m}_x$ the unique maximal ideal of the ring $\mathscr{O}_{X,x}$. Then φ_x sends $\mathfrak{m}_x$ to $\mathfrak{m}_x$, φ_x induces an $\mathbb{C}$-algebra homomorphism $u_x\colon \mathscr{O}_{X,x}/\mathfrak{m}_x \to \mathscr{O}_{X,x}/\mathfrak{m}_x$. Since the composition $\mathbb{C} \xrightarrow{\sim} \mathscr{O}_{X,x}/\mathfrak{m}_x \xrightarrow{u_x} \mathscr{O}_{X,x}/\mathfrak{m}_x \xrightarrow{\sim} \mathbb{C}$ is the identity, we obtain that u_x is the identity. Hence, for any $f \in \mathscr{O}_X$, $\varphi(f)(x) = f(x)$. Therefore $\varphi(f) = f$. Q.E.D.

Lemma 2.1.16. *Let $\mathscr{P}$ be a $\mathbb{C}$-algebroid on a complex manifold X. Assume that, for any $\sigma \in \mathscr{P}$, $\mathscr{E}nd_{\mathscr{P}}(\sigma)$ is locally isomorphic to $\mathscr{O}_X$ as a $\mathbb{C}$-algebra. Then $\mathscr{P}$ is uniquely endowed with a structure of $\mathscr{O}_X$-algebroid, and $\mathscr{P}$ is invertible.*

Proof. By Lemma 2.1.15, for an open subset U and $\sigma \in \mathscr{P}(U)$, there exists a unique $\mathbb{C}$-algebra isomorphism $\mathscr{O}_X|_U \xrightarrow{\sim} \mathscr{E}nd_{\mathscr{P}}(\sigma)$. It gives a structure of $\mathscr{O}_X$-algebroid on $\mathscr{P}$. The remaining statements are obvious. Q.E.D.

Let $\mathscr{P}$ be an invertible $\mathscr{O}_X$-algebroid. For σ, $\sigma' \in \mathscr{P}(U)$, the two $\mathscr{O}_X$-module structures on $\mathscr{H}om_{\mathscr{P}}(\sigma, \sigma')$ induced by $\mathscr{E}nd_{\mathscr{P}}(\sigma) \simeq \mathscr{O}_X$ and by $\mathscr{E}nd_{\mathscr{P}}(\sigma') \simeq \mathscr{O}_X$ coincide, and $\mathscr{H}om_{\mathscr{P}}(\sigma, \sigma')$ is an invertible $\mathscr{O}_X$-module.

Let $f \colon X \to Y$ be a morphism of complex manifolds. For an invertible $\mathscr{O}_Y$-algebroid $\mathscr{P}_Y$, we set

$$f^*\mathscr{P}_Y := \mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\mathscr{P}_Y,$$

where the tensor product $\otimes_{f^{-1}\mathscr{O}_Y}$ is defined similarly as for $\mathbb{K}$-algebroids. Then $f^*\mathscr{P}_Y$ is an invertible $\mathscr{O}_X$-algebroid. We have functors

$$f^* : \mathrm{Mod}(\mathscr{P}_Y) \to \mathrm{Mod}(f^*\mathscr{P}_Y), \quad \mathrm{L}f^* : \mathrm{D^b}(\mathscr{P}_Y) \to \mathrm{D^b}(f^*\mathscr{P}_Y), \tag{2.1.15}$$

and

$$\begin{aligned} f_!,\ f_* &: \mathrm{Mod}(f^*\mathscr{P}_Y) \to \mathrm{Mod}(\mathscr{P}_Y), \\ \mathrm{R}f_!, \mathrm{R}f_* &: \mathrm{D^b}(f^*\mathscr{P}_Y) \to \mathrm{D^b}(\mathscr{P}_Y). \end{aligned} \tag{2.1.16}$$

Let $f \colon X \to Y$ be a morphism of complex manifolds, and let $\mathscr{P}_X$ (resp. $\mathscr{P}_Y$) be an invertible $\mathscr{O}_X$-algebroid (resp. an invertible $\mathscr{O}_Y$-algebroid). If $f^{-1}\mathscr{P}_Y \to \mathscr{P}_X$ is a functor of $\mathbb{C}$-linear stacks, then it defines a functor of $\mathbb{C}$-linear stacks $f^*\mathscr{P}_Y \to \mathscr{P}_X$ and this last functor is an equivalence by the preceding results.

Remark 2.1.17. Invertible $\mathscr{O}_X$-algebroids are trivial in the algebraic case. Indeed, for a smooth algebraic variety X, the group $H^2(X; \mathscr{O}_X^\times)$ is zero. Here the cohomology is calculated with respect to the Zariski topology. (With the étale topology, it does not vanish in general.) This result and its proof below have been communicated to us by Prof. Joseph Oesterlé, and we thank him here.

Let K be the field of rational functions on X, $K_X^\times$, the constant sheaf with the abelian group $K^\times$ as stalks, and denote by X_1 the set of closed irreducible hypersurfaces of X. One has an exact sequence

$$0 \to \mathscr{O}_X^\times \to K_X^\times \to \bigoplus_{S \in X_1} \mathbb{Z}_S \to 0.$$

Since $K_X^\times$ is constant, it is a flabby sheaf for the Zariski topology. On the other hand the sheaf $\bigoplus_{S \in X_1} \mathbb{Z}_S$ is also flabby. It follows that $H^j(X; \mathscr{O}_X^\times)$ is zero for $j > 1$.

2.2 DQ-algebras

From now on, X will be a complex manifold. We denote by $\delta_X\colon X \hookrightarrow X\times X$ the diagonal embedding and we set $\Delta_X = \delta_X(X)$. We denote by $\mathscr{O}_X$ the structure sheaf on X, by d_X the complex dimension, by Ω_X the sheaf of holomorphic forms of maximal degree and by Θ_X the sheaf of holomorphic vector fields. As usual, we denote by $\mathscr{D}_X$ the sheaf of rings of (finite order) differential operators on X and by $F_n(\mathscr{D}_X)$ the sheaf of differential operators of order $\leq n$. Recall that a bi-differential operator P on X is a $\mathbb{C}$-bilinear morphism $\mathscr{O}_X\times\mathscr{O}_X\to\mathscr{O}_X$ which is obtained as the composition $\delta_X^{-1}\circ\widetilde{P}$ where $\widetilde{P}$ is a differential operator on $X\times X$ defined on a neighborhood of the diagonal and δ^{-1} is the restriction to the diagonal:

$$P(f,g)(x) = (\widetilde{P}(x_1,x_2;\partial_{x_1},\partial_{x_2})(f(x_1)g(x_2))|_{x_1=x_2=x}. \tag{2.2.1}$$

Hence the sheaf of bi-differential operators is isomorphic to $\mathscr{D}_X\otimes_{\mathscr{O}_X}\mathscr{D}_X$, where both $\mathscr{D}_X$ are regarded as $\mathscr{O}_X$-modules by the left multiplications.

Star-products

Notation 2.2.1. We denote by $\mathbb{C}^\hbar$ the ring $\mathbb{C}[[\hbar]]$ of formal power series in an indeterminate $\hbar$ and by $\mathbb{C}^{\hbar,\mathrm{loc}}$ the field $\mathbb{C}((\hbar))$ of Laurent series in $\hbar$. Then $\mathbb{C}^{\hbar,\mathrm{loc}}$ is the fraction field of $\mathbb{C}^\hbar$.

We set

$$\mathscr{O}_X[[\hbar]] := \varprojlim_n \mathscr{O}_X\otimes(\mathbb{C}^\hbar/\hbar^n\mathbb{C}^\hbar) \simeq \prod_{n\geq 0}\mathscr{O}_X\hbar^n.$$

Let us recall a classical definition (see [1, 47]).

Definition 2.2.2. An associative multiplication law $\star$ on $\mathscr{O}_X[[\hbar]]$ is a star-product if it is $\mathbb{C}^\hbar$-bilinear and satisfies

$$f\star g = \sum_{i\geq 0} P_i(f,g)\hbar^i \text{ for } f,g\in\mathscr{O}_X, \tag{2.2.2}$$

where the P_i's are bi-differential operators such that $P_0(f,g) = fg$ and $P_i(f,1) = P_i(1,f) = 0$ for all $f\in\mathscr{O}_X$ and $i>0$. We call $(\mathscr{O}_X[[\hbar]],\star)$ a star-algebra.

Note that $1\in\mathscr{O}_X\subset\mathscr{O}_X[[\hbar]]$ is a unit with respect to $\star$. Note also that we have

$$(\sum_{i\geq 0} f_i\hbar^i)\star(\sum_{i\geq 0} g_i\hbar^i) = \sum_{n\geq 0}\Big(\sum_{i+j+k=n} P_k(f_i,g_j)\Big)\hbar^n.$$

Recall that a star-product defines a Poisson structure on $(X, \mathscr{O}_X)$ by setting for $f, g \in \mathscr{O}_X$:

$$\{f, g\} = P_1(f, g) - P_1(g, f) = \hbar^{-1}(f \star g - g \star f) \bmod \hbar\mathscr{O}_X[[\hbar]], \tag{2.2.3}$$

and that locally, (globally in the real case), any Poisson manifold $(X, \mathscr{O}_X)$ may be endowed with a star-product to which the Poisson structure is associated. This is a famous theorem of Kontsevich [47].

Proposition 2.2.3. *Let $\star$ and $\star'$ be star-products and let $\varphi\colon (\mathscr{O}_X[[\hbar]], \star) \to (\mathscr{O}_X[[\hbar]], \star')$ be a morphism of $\mathbb{C}^\hbar$-algebras. Then there exists a unique sequence of differential operators $\{R_i\}_{i \geq 0}$ on X such that $R_0 = 1$ and $\varphi(f) = \sum_{i \geq 0} R_i(f)\hbar^i$ for any $f \in \mathscr{O}_X$. In particular, φ is an isomorphism.*

First, we need a lemma. In this lemma, we set $F_\infty(\mathscr{D}_X) = \mathscr{D}_X$.

Lemma 2.2.4. *Let $l \in \mathbb{Z}_{\geq -1} \sqcup \{\infty\}$, and $\varphi \in \operatorname{End}_{\mathbb{C}_X}(\mathscr{O}_X)$. If $[\varphi, g] \in \mathrm{F}_l(\mathscr{D}_X)$ for all $g \in \mathscr{O}_X$, then $\varphi \in \mathrm{F}_{l+1}(\mathscr{D}_X)$.*

Proof. We may assume that X is an open subset of $\mathbb{C}^n$ and we denote by $(x_1, \ldots, x_n)$ the coordinates. Set $P_i = [\varphi, x_i]$. Then

$$[P_i, x_j] = [\,[\varphi, x_i], x_j] = [\,[\varphi, x_j], x_i] = [P_j, x_i].$$

This implies the existence of $P \in \mathrm{F}_{l+1}(\mathscr{D}_X)$ such that $[P, x_i] = P_i$ for all i. Setting $\psi := \varphi - P$, we have

$$[\psi, x_i] = 0 \text{ for all } i = 1, \ldots, n.$$

Let us show that $\psi \in \mathscr{O}_X$. Replacing ψ with $\theta := \psi - \psi(1)$, we get by induction on the order of the polynomials that $\theta(Q) = 0$ and $[\theta, Q] = 0$ for all $Q \in \mathbb{C}[x_1, \ldots, x_n]$. Let $f \in \mathscr{O}_X$. We shall prove that $\theta(f)(x) = 0$ for all $x \in X$. It is enough to prove it for $x = 0$. Then, writing $f = f(0) + \sum_i x_i f_i$, we get

$$\begin{aligned} \theta(f) &= \theta(f(0)) + \sum_i \theta(x_i f_i) = \theta(f(0)) + \sum_i \big(x_i\theta(f_i) + [\theta, x_i]f_i\big) \\ &= \sum_i x_i \theta(f_i), \end{aligned}$$

which vanishes at $x = 0$. Q.E.D.

Proof of Proposition 2.2.3. Let us write

$$\varphi(f) = \sum_{i \geq 0} \hbar^i \varphi_i(f), \quad f \in \mathscr{O}_X. \tag{2.2.4}$$

By Lemma 2.1.15, $\varphi_0 = 1$. We shall prove by induction that the φ_i's in (2.2.4) are differential operators and we assume that this is so for all $i < n$ for $n \in \mathbb{Z}_{>0}$.

Let $\{P_i\}$ and $\{P'_i\}$ be the sequence of bi-differential operators associated with the star-products $\star$ and $\star'$, respectively. We have

$$\begin{aligned}\varphi(f \star g) &= \varphi(\sum_{j\geq 0} \hbar^j P_j(f,g)) = \sum_{i,j\geq 0} \hbar^{i+j}\varphi_i(P_j(f,g)),\\ \varphi(f)\star'\varphi(g) &= \sum_{i\geq 0}\hbar^i\varphi_i(f)\star'\sum_{j\in\mathbb{N}}\hbar^j\varphi_j(g) = \sum_{i,j,k\geq 0}\hbar^{i+j+k}P'_k(\varphi_i(f),\varphi_j(g)).\end{aligned}$$

Since $\varphi(f \star g) = \varphi(f)\star'\varphi(g)$, we get:

$$\sum_{n=i+j}\varphi_i(P_j(f,g)) = \sum_{n=i+j+k} P'_k(\varphi_i(f),\varphi_j(g)). \tag{2.2.5}$$

By the induction hypothesis, the left hand side of (2.2.5) may be written as $\varphi_n(fg) + Q_n(f,g)$ where Q_n is a bi-differential operator. Similarly, the right hand side of (2.2.5) may be written as $\varphi_n(f)g + f\varphi_n(g) + R_n(f,g)$ where R_n is a bi-differential operator. For any $g \in \mathscr{O}_X$, considering g as an endomorphism of $\mathscr{O}_X$, we get

$$[\varphi_n, g](f) := \varphi_n(fg) - g\varphi_n(f) = f\varphi_n(g) + S_n(f),$$

where S_n is a differential operator. Then, the result follows from Lemma 2.2.4. Q.E.D.

DQ-algebras

Definition 2.2.5. A DQ-algebra $\mathscr{A}$ on X is a $\mathbb{C}^\hbar$-algebra locally isomorphic to a star-algebra $(\mathscr{O}_X[[\hbar]],\star)$ as a $\mathbb{C}^\hbar$-algebra.

Clearly a DQ-algebra $\mathscr{A}$ satisfies the conditions:

$$\begin{cases} \text{(i)}\ \ \hbar\colon \mathscr{A}\to\mathscr{A} \text{ is injective,}\\ \text{(ii)}\ \ \mathscr{A}\to\varprojlim_n \mathscr{A}/\hbar^n\mathscr{A} \text{ is an isomorphism,}\\ \text{(iii)}\ \ \mathscr{A}/\hbar\mathscr{A} \text{ is isomorphic to } \mathscr{O}_X \text{ as a } \mathbb{C}\text{-algebra.}\end{cases} \tag{2.2.6}$$

For a $\mathbb{C}^\hbar$-algebra $\mathscr{A}$ satisfying (2.2.6), the $\mathbb{C}$-algebra isomorphism $\mathscr{A}/\hbar\mathscr{A} \xrightarrow{\sim} \mathscr{O}_X$ in (2.2.6) (iii) is unique by Lemma 2.1.15. We denote by

$$\sigma_0\colon \mathscr{A}\to\mathscr{O}_X \tag{2.2.7}$$

the $\mathbb{C}^\hbar$-algebra morphism $\mathscr{A} \to \mathscr{A}/\hbar\mathscr{A} \xrightarrow{\sim} \mathscr{O}_X$. If φ is a $\mathbb{C}$-linear section of $\sigma_0 \colon \mathscr{A} \to \mathscr{O}_X$, then φ extends to an isomorphism of $\mathbb{C}^\hbar$-modules $\widetilde{\varphi} \colon \mathscr{O}_X[[\hbar]] \xrightarrow{\sim} \mathscr{A}$, given by $\widetilde{\varphi}(\sum_i f_i\hbar^i) = \sum_i \varphi(f_i)\hbar^i$.

Definition 2.2.6. We say that a $\mathbb{C}$-linear section $\varphi \colon \mathscr{O}_X \to \mathscr{A}$ of $\mathscr{A} \to \mathscr{O}_X$ is standard if there exists a sequence of bi-differential operators P_i such that

$$\varphi(f)\varphi(g) = \sum_{i\geq 0} \varphi(P_i(f,g))\hbar^i \text{ for any } f, g \in \mathscr{O}_X. \tag{2.2.8}$$

Consider a standard section $\varphi \colon \mathscr{O}_X \to \mathscr{A}$ of $\mathscr{A} \to \mathscr{O}_X$. Define a star-product $\star$ on $\mathscr{O}_X[[\hbar]]$ by setting

$$f \star g = \sum_{i\geq 0} P_i(f,g)\hbar^i \text{ for any } f, g \in \mathscr{O}_X.$$

Then we get an isomorphism of $\mathbb{C}^\hbar$-algebras

$$\widetilde{\varphi} \colon (\mathscr{O}_X[[\hbar]], \star) \xrightarrow{\sim} \mathscr{A}. \tag{2.2.9}$$

We call $\widetilde{\varphi}$ in (2.2.9) *a standard isomorphism.*

Hence, a DQ-algebra is nothing but a $\mathbb{C}^\hbar$-algebra satisfying (2.2.6) and admitting locally a standard section.

Remark 2.2.7. We conjecture that a $\mathbb{C}^\hbar$-algebra satisfying (2.2.6) locally admits a standard section.

Let $\mathscr{A}$ be a DQ-algebra. For f, $g \in \mathscr{O}_X$, taking a, $b \in \mathscr{A}$ such that $\sigma_0(a) = f$ and $\sigma_0(b) = g$, we set

$$\{f, g\} = \sigma_0(\hbar^{-1}(ab - ba)) \in \mathscr{O}_X. \tag{2.2.10}$$

Then this definition does not depend on the choice of a, b and it defines a Poisson structure on X. In particular, two DQ-algebras induce the same Poisson structure on X as soon as they are locally isomorphic.

By Proposition 2.2.3, if $\varphi, \varphi' \colon \mathscr{O}_X \to \mathscr{A}$ are two standard sections, then there exists a unique sequence of differential operators $\{R_i\}_{i\geq 0}$ such that $\varphi'(f) = \sum_{i\geq 0} \hbar^i \varphi(R_i(f))$ for any $f \in \mathscr{O}_X$.

Clearly, a DQ-algebra satisfies the hypotheses (1.2.2) and (1.2.3). Hence, a DQ-algebra is a right and left Noetherian ring (in particular, coherent).

Lemma 2.2.8. *Let $\mathscr{A}$ be a* DQ*-algebra. Then the opposite algebra $\mathscr{A}^{\mathrm{op}}$ is also a* DQ*-algebra.*

Proof. This follows from (2.2.2). Q.E.D.

Let X and Y be complex manifolds endowed with two star-products $\star_X$ and $\star_Y$. Denote by $\{P_i\}_i$ and $\{Q_j\}_j$ the bi-differential operators associated to these star-products as in (2.2.2). Let $P_i \boxtimes Q_j$ be a bi-differential operator on $X \times Y$ defined as follows. Let us take differential operators $\widetilde{P}_i(x_1, x_2, \partial_{x_1}, \partial_{x_2})$ and $\widetilde{Q}_j(y_1, y_2, \partial_{y_1}, \partial_{y_2})$ corresponding to P_i and Q_j as in (2.2.1). Then we set

$$\begin{aligned}&(P_i \boxtimes Q_j)(f,g)(x,y)\\ &\quad = \big(\widetilde{P}_i(x_1, x_2, \partial_{x_1}, \partial_{x_2})\widetilde{Q}_j(y_1, y_2, \partial_{y_1}, \partial_{y_2})(f(x_1,y_1)g(x_2,y_2))\big)|_{\substack{x_1=x_2=x\\ y_1=y_2=y}}.\end{aligned}$$

Hence, $P_i \boxtimes Q_j$ is the unique bi-differential operator on $X \times Y$ such that $(P_i \boxtimes Q_j)(f_1(x)g_1(y), f_2(x)g_2(y)) = P_i(f_1(x), f_2(x)) \cdot Q_j(g_1(y), g_2(y))$ for any $f_\nu(x) \in \mathscr{O}_X$ and $g_\nu(y) \in \mathscr{O}_Y$ ($\nu = 1, 2$).

One defines the external product of the star-products $\star_X$ and $\star_Y$ on $\mathscr{O}_{X\times Y}[[\hbar]]$ by setting

$$f \star g = \sum_{n\geq 0} \hbar^n \sum_{i+j=n} (P_i \boxtimes Q_j)(f,g).$$

Hence:

Lemma 2.2.9. *Let X and Y be complex manifolds, and let $\mathscr{A}_X$ be a* DQ-*algebra on X and $\mathscr{A}_Y$ a* DQ-*algebra on Y. Then there exists a* DQ-*algebra $\mathscr{A}$ on $X \times Y$ which contains $\mathscr{A}_X \boxtimes_{\mathbb{C}^\hbar} \mathscr{A}_Y$ as a $\mathbb{C}^\hbar$-subalgebra. Moreover such an $\mathscr{A}$ is unique up to a unique isomorphism.*

We call $\mathscr{A}$ the external product of the DQ-algebra $\mathscr{A}_X$ on X and the DQ-algebra $\mathscr{A}_Y$ on Y, and denote it by $\mathscr{A}_X \underline{\boxtimes} \mathscr{A}_Y$.

Remark 2.2.10. (i) Any commutative DQ-algebra is locally isomorphic to $(\mathscr{O}_X[[\hbar]], \star)$ where $\star$ is the trivial star-product $f \star g = fg$.

(ii) For the trivial DQ-algebra $\mathscr{O}_X[[\hbar]]$, we have

$$\mathscr{A}ut_{\,\mathbb{C}^\hbar\text{-alg}}(\mathscr{O}_X[[\hbar]]) \simeq \hbar\Theta_X[[\hbar]] := \prod_{n\geq 1} \hbar^n \Theta_X,$$

(recall that Θ_X is the sheaf of vector fields on X) and we associate to $v := \sum_{n\geq 1} \hbar^n v_n$ the automorphism $f \mapsto \exp(v)f$.

The ring $\mathscr{D}_X^{\mathscr{A}}$ and another construction for DQ-algebras

We define the $\mathbb{C}^\hbar$-algebra

$$\mathscr{D}_X[[\hbar]] := \varprojlim_n \mathscr{D}_X \otimes (\mathbb{C}^\hbar/\hbar^n\mathbb{C}^\hbar) \simeq \prod_{n\geq 0} \mathscr{D}_X \hbar^n.$$

Then $\mathscr{O}_X[[\hbar]]$ has a $\mathscr{D}_X[[\hbar]]$-module structure, and $\mathscr{D}_X[[\hbar]] \subset \mathscr{E}nd_{\mathbb{C}^\hbar}(\mathscr{O}_X[[\hbar]])$.

Let $\mathscr{A}_X$ be a DQ-algebra. Choose (locally) a standard section φ giving rise to a standard isomorphism of $\mathbb{C}^\hbar$-modules $\widetilde{\varphi}\colon \mathscr{O}_X[[\hbar]] \xrightarrow{\sim} \mathscr{A}_X$. This last isomorphism induces an isomorphism

$$\Phi\colon \mathscr{E}nd_{\mathbb{C}^\hbar}(\mathscr{O}_X[[\hbar]]) \xrightarrow{\sim} \mathscr{E}nd_{\mathbb{C}^\hbar}(\mathscr{A}_X). \tag{2.2.11}$$

Definition 2.2.11. Let $\mathscr{A}_X$ be a DQ-algebra and let φ be a standard section. The sheaf of rings $\mathscr{D}_X^{\mathscr{A}}$ is the $\mathbb{C}^\hbar$-subalgebra of $\mathscr{E}nd_{\mathbb{C}^\hbar}(\mathscr{A}_X)$, the image of $\mathscr{D}_X[[\hbar]] \subset \mathscr{E}nd_{\mathbb{C}^\hbar}(\mathscr{O}_X[[\hbar]])$ by the isomorphism Φ in (2.2.11).

It is easy to see that $\mathscr{D}_X^{\mathscr{A}} \subset \mathscr{E}nd_{\mathbb{C}^\hbar}(\mathscr{A}_X)$ does not depend on the choice of the standard section φ in virtue of Proposition 2.2.3. Hence $\mathscr{D}_X^{\mathscr{A}}$ is well-defined on X although standard sections only locally exist.

By its construction, we have $\mathscr{D}_X^{\mathscr{A}} \xrightarrow{\sim} \varprojlim_n \mathscr{D}_X^{\mathscr{A}}/\hbar^n\mathscr{D}_X^{\mathscr{A}}$. Moreover, the image of the algebra morphism $\mathscr{A}_X \otimes \mathscr{A}_X^{\mathrm{op}} \to \mathscr{E}nd_{\mathbb{C}^\hbar}(\mathscr{A}_X)$, as well as the one of $\delta_X^{-1}\mathscr{A}_{X\times X^a} \to \mathscr{E}nd_{\mathbb{C}^\hbar}(\mathscr{A}_X)$ is contained in $\mathscr{D}_X^{\mathscr{A}}$. Hence we have algebra morphisms

$$\mathscr{A}_X \otimes \mathscr{A}_{X^a} \to \delta_X^{-1}\mathscr{A}_{X\times X^a} \to \mathscr{D}_X^{\mathscr{A}}.$$

We shall show how to construct a star-algebra from the data of sections of $\mathscr{D}_X[[\hbar]]$ satisfying suitable commutation properties.

Let $\mathscr{A}_X := (\mathscr{O}_X[[\hbar]], \star)$ be a star-algebra. There are two $\mathbb{C}^\hbar$-linear morphisms from $\mathscr{O}_X[[\hbar]]$ to $\mathscr{D}_X[[\hbar]]$ given by

$$\Phi^l\colon f \mapsto f\,\star, \quad \Phi^r\colon f \mapsto \star\, f. \tag{2.2.12}$$

Hence, for $f \in \mathscr{O}_X$, we have:

$$\Phi^l(f) = \sum_{i\geq 0} P_i(f, \bullet)\hbar^i, \quad \Phi^r(f) = \sum_{i\geq 0} P_i(\bullet, f)\hbar^i.$$

Then $\Phi^l\colon \mathscr{A}_X \to \mathscr{D}_X[[\hbar]]$ and $\Phi^r\colon \mathscr{A}_X^{\mathrm{op}} \to \mathscr{D}_X[[\hbar]]$ are two $\mathbb{C}^\hbar$-algebra morphisms, and induce a $\mathbb{C}^\hbar$-algebra morphism $\mathscr{A}_X \otimes \mathscr{A}_X^{\mathrm{op}} \to \mathscr{D}_X[[\hbar]]$.

Assume to be given a local coordinate system $x = (x_1, \dots, x_n)$ on X and for $i = 1, \dots, n$, set $\Phi^l(x_i) = A_i$ and $\Phi^r(x_i) = B_i$. Then $\{A_i, B_j\}_{i,j=1,\dots,n}$ are sections of $\mathscr{D}_X[[\hbar]]$ which satisfy

$$\begin{cases} A_i(1) = B_i(1) = x_i, \\ A_i \equiv x_i \bmod \hbar\mathscr{D}_X[[\hbar]],\ B_i \equiv x_i \bmod \hbar\mathscr{D}_X[[\hbar]], \\ [A_i, B_j] = 0\ (i, j = 1, \dots, n). \end{cases} \tag{2.2.13}$$

Conversely, we have the following result.

Proposition 2.2.12. *Let $\{A_i, B_j\}_{i,j=1,\dots,n}$ be sections of $\mathscr{D}_X[[\hbar]]$ which satisfy* (2.2.13). *Define the subalgebra $\mathscr{A}_X \subset \mathscr{D}_X[[\hbar]]$ by*

$$\mathscr{A}_X = \{a \in \mathscr{D}_X[[\hbar]]\,;\, [a, B_i] = 0, i = 1, \dots, n\} \tag{2.2.14}$$

and define the $\mathbb{C}^\hbar$-linear map $\psi\colon \mathscr{A}_X \to \mathscr{O}_X[[\hbar]]$ by setting $\psi(a) = a(1)$. Then

(a) *ψ is a $\mathbb{C}^\hbar$-linear isomorphism,*

(b) *the product on $\mathscr{O}_X[[\hbar]]$ given by $\psi(a) \star \psi(b) := \psi(a \cdot b)$ is a star-product, $\mathscr{A}_X$ is a* DQ-*algebra and ψ^{-1} is a standard isomorphism,*

(c) *the algebra $\mathscr{A}_X^{\mathrm{op}}$ is obtained by replacing A_i with B_i $(i = 1, \dots, n)$ in the above construction.*

Proof. (a)-(i) $\mathscr{A}_X \cap \hbar\mathscr{D}_X[[\hbar]] = \hbar\mathscr{A}_X$, since $[\hbar a, B_i] = 0$ implies $[a, B_i] = 0$. Hence we have $\mathscr{A}_X/\hbar^j\mathscr{A}_X \subset \mathscr{D}_X[[\hbar]]/\hbar^j\mathscr{D}_X[[\hbar]]$ for any j.
(a)-(ii) $\mathscr{A}_X \xrightarrow{\sim} \varprojlim_j \mathscr{A}_X/\hbar^j\mathscr{A}_X$. Indeed, let $a = \sum_{i=0}^\infty \hbar^i a_i$ and assume that

$$[\sum_{i=0}^k \hbar^i a_i, B_l] = 0 \bmod \hbar^{k+1}\, (l = 1, \dots, n)$$

for all $k \in \mathbb{N}$. Then $[a, B_l] = 0$ for $l = 1, \dots, n$.
(a)-(iii) Let $\psi_j\colon \hbar^j\mathscr{A}_X/\hbar^{j+1}\mathscr{A}_X \to \hbar^j\mathscr{O}_X/\hbar^{j+1}\mathscr{O}_X$ be the morphisms induced by ψ. By (a)-(ii) it is enough to check that all ψ_j's are isomorphisms. Since all $\hbar^j\mathscr{A}_X/\hbar^{j+1}\mathscr{A}_X$ are isomorphic and all $\hbar^j\mathscr{O}_X/\hbar^{j+1}\mathscr{O}_X$ are isomorphic, we are reduced to prove that $\psi_0\colon \mathscr{A}_X/\hbar\mathscr{A}_X \to \mathscr{O}_X$ is an isomorphism.
(a)-(iv) ψ_0 is injective. Let $a_0 \in \mathscr{A}_X/\hbar\mathscr{A}_X \subset \mathscr{D}_X$. Since $[a_0, x_i] \in \hbar\mathscr{D}_X[[\hbar]]$ implies $[a_0, x_i] = 0$, we get $a_0 \in \mathscr{O}_X$. Therefore, $a_0(1) = 0$ implies $a_0 = 0$.
(a)-(v) ψ_0 is surjective. Let $y = (y_1, \dots, y_n)$ be a local coordinate system on a copy of X. Notice first that the sections $y_i - A_i$ of $\mathscr{D}_{X\times Y}[[\hbar]]$ are invertible on the open sets $\{y_i \neq x_i\}$. Let $f(x_1, \dots, x_n) \in \mathscr{O}_X$. Define the section $G(f)$ of $\mathscr{D}_X[[\hbar]]$ by

$$G(f) = \frac{1}{(2\pi i)^n} \oint f(y)(y_1 - A_1)^{-1} \cdots (y_n - A_n)^{-1}\, dy_1 \cdots dy_n\,. \tag{2.2.15}$$

Then $[G(f), B_i] = 0$ for all i. It is obvious that $G(f) - f \in \hbar\mathscr{D}_X[[\hbar]]$ and $\psi_0(G(f)) = f$.

(b) Clearly, the algebra $(\mathscr{O}_X[[\hbar]], \star)$ satisfies (2.2.6). Moreover, $f \mapsto G(f)$ is a standard section since there exist $P_i(f) \in \mathscr{D}_X[[\hbar]]$ $(i \in \mathbb{N})$ such that $G(f) = \sum_i P_i(f)\hbar^i$ and $P_i(f)$ is obtained as the action of a bidifferential operator P_i on f.

(c) follows from $\mathscr{A}^{\mathrm{op}} = \{b \in \mathscr{E}nd_{\mathbb{C}^\hbar}(\mathscr{A}_X); [b, \mathscr{A}_X] = 0\}$. Q.E.D.

Example 2.2.13. Let $M := \{a_{ij}\}_{i,j=1,\dots,n}$ be an $n \times n$ skew-symmetric matrix with entries in $\mathbb{C}$. Let $X = \mathbb{C}^n$ and consider the sections of $\mathscr{D}_X[[\hbar]]$:

$$A_i = x_i + \frac{\hbar}{2}\sum_j a_{ij}\partial_j, \quad B_i = x_i - \frac{\hbar}{2}\sum_j a_{ij}\partial_j.$$

Then $\{A_i, B_j\}_{i,j=1,\dots,n}$ satisfy (2.2.13), thus define a DQ-algebra $\mathscr{A}_X$. Note that the Poisson structure associated with the DQ-algebra $\mathscr{A}_X$ is symplectic if and only if the matrix M is non-degenerate.

2.3 DQ-algebroids

Let us introduce the notion of a deformation quantization algebroid, a DQ-algebroid for short.

Definition 2.3.1. A DQ-algebroid $\mathscr{A}$ on X is a $\mathbb{C}^\hbar$-algebroid such that for each open set $U \subset X$ and each $\sigma \in \mathscr{A}(U)$, the $\mathbb{C}^\hbar$-algebra $\mathscr{E}nd_{\mathscr{A}}(\sigma)$ is a DQ-algebra on U.

Note that a DQ-algebroid is called a twisted associative deformation of $\mathscr{O}_X$ in [65].

By (2.2.10), a DQ-algebroid $\mathscr{A}$ on the complex manifold X defines a Poisson structure on X. It is proved in [46] that, conversely, any complex Poisson manifold X may be endowed with a DQ-algebroid to which this Poisson structure is associated.

According to Convention 2.1.1, if $\mathscr{A}$ is a DQ-algebra, we shall often use the same notation $\mathscr{A}$ for the associated DQ-algebroid.

Note that any DQ-algebroid $\mathscr{A}$ on X may be obtained as the stack associated with a gluing datum as in (2.1.3), where the sheaves $\mathscr{A}_i$ are DQ-algebras.

Let $\mathscr{A}$ be a DQ-algebroid on X. For an $\mathscr{A}$-module $\mathscr{M}$, the local notions of being coherent or locally free, etc. make sense.

The category $\mathrm{Mod}(\mathscr{A})$ is a Grothendieck category. We denote by $\mathrm{D}(\mathscr{A})$ its derived category and by $\mathrm{D}^{\mathrm{b}}(\mathscr{A})$ its bounded derived category. We still call an object of this derived category an $\mathscr{A}$-module. We denote by $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A})$ the full triangulated subcategory of $\mathrm{D}^{\mathrm{b}}(\mathscr{A})$ consisting of objects with coherent cohomologies.

Opposite structure

If X is endowed with a DQ-algebroid $\mathscr{A}_X$, then we denote by X^a the manifold X endowed with the algebroid $\mathscr{A}_X^{\mathrm{op}}$, that is:

$$\mathscr{A}_{X^a} = \mathscr{A}_X^{\mathrm{op}}. \tag{2.3.1}$$

This is a DQ-algebroid by Lemma 2.2.8.

External product

Assume that complex manifolds X and Y are endowed with DQ-algebroids $\mathscr{A}_X$ and $\mathscr{A}_Y$ respectively. By Lemma 2.2.9, there is a canonical DQ-algebroid $\mathscr{A}_X \underline{\boxtimes} \mathscr{A}_Y$ on $X \times Y$ locally equivalent to the stack associated with the external product $\mathscr{A}_X \underline{\boxtimes} \mathscr{A}_Y$ of the DQ-algebras and there is a faithful functor of $\mathbb{C}^\hbar$-algebroids

$$\mathscr{A}_X \boxtimes \mathscr{A}_Y \to \mathscr{A}_X \underline{\boxtimes} \mathscr{A}_Y, \tag{2.3.2}$$

which induces a functor

$$\mathit{for}\colon \mathrm{Mod}(\mathscr{A}_X \underline{\boxtimes} \mathscr{A}_Y) \to \mathrm{Mod}(\mathscr{A}_X \boxtimes \mathscr{A}_Y). \tag{2.3.3}$$

When there is no risk of confusion, we set

$$\mathscr{A}_{X\times Y} := \mathscr{A}_X \underline{\boxtimes} \mathscr{A}_Y.$$

Then $\mathscr{A}_{X\times Y}$ belongs to $\mathrm{Mod}(\mathscr{A}_{X\times Y} \otimes (\mathscr{A}_{X^a} \boxtimes \mathscr{A}_{Y^a}))$ and the functor *for* admits a left adjoint functor $\mathscr{K} \mapsto \mathscr{A}_{X\times Y} \otimes_{\mathscr{A}_X \boxtimes \mathscr{A}_Y} \mathscr{K}$:

$$\mathrm{Mod}(\mathscr{A}_{X\times Y}) \underset{}{\overset{\mathit{for}}{\rightleftarrows}} \mathrm{Mod}(\mathscr{A}_X \boxtimes \mathscr{A}_Y). \tag{2.3.4}$$

We denote by $\bullet \underline{\boxtimes} \bullet$ the bi-functor $\mathscr{A}_{X\times Y} \underset{\mathscr{A}_X \boxtimes \mathscr{A}_Y}{\otimes} (\bullet \boxtimes \bullet)$:

$$\bullet \underline{\boxtimes} \bullet \quad : \quad \mathrm{Mod}(\mathscr{A}_X) \times \mathrm{Mod}(\mathscr{A}_Y) \longrightarrow \mathrm{Mod}(\mathscr{A}_{X\times Y}). \tag{2.3.5}$$

Lemma 2.3.2. *If $\mathscr{M}$ is an $\mathscr{A}_X$-module without $\hbar$-torsion, then the functor*

$$\mathscr{M} \underline{\boxtimes} \bullet : \mathrm{Mod}(\mathscr{A}_Y) \to \mathrm{Mod}(\mathscr{A}_{X\times Y})$$

is an exact functor.

Proof. We may assume that $\mathscr{A}_X$ and $\mathscr{A}_Y$ are DQ-algebras. Hence it is enough to show that for any $(x,y) \in X \times Y$, setting $\mathscr{N} := \mathscr{A}_{X\times Y} \otimes_{\mathscr{A}_X} \mathscr{M}$, $\mathscr{N}_{(x,y)}$ is a flat module over $\mathscr{A}^{\mathrm{op}}_{Y,y}$. We may assume further that $\mathscr{M}$ is a coherent $\mathscr{A}_X$-module without $\hbar$-torsion. For any Stein open subset U, let $p_U\colon U \times Y \to Y$ be the projection. Set $\mathscr{N}_U := (p_U)_*\big((\mathscr{A}_{X\times Y} \otimes_{\mathscr{A}_X} \mathscr{M})|_{U\times Y}\big)$. Then it is easy to check the conditions (a)–(c) in Theorem 1.6.6 are satisfied ((c) follows from the $\mathscr{O}$-module version of this lemma), and we conclude that $\mathscr{N}_U$ is a flat $\mathscr{A}_Y^{\mathrm{op}}$-module. Hence, $\mathscr{N}_{(x,y)} \simeq \varinjlim_{x\in U} (\mathscr{N}_U)_y$ is a flat $(\mathscr{A}_Y^{\mathrm{op}})_y$-module. Q.E.D.

Hence the left derived functor

$$\bullet\overset{\mathrm{L}}{\underline{\boxtimes}}\bullet\colon\ \mathrm{D}(\mathscr{A}_X)\times\mathrm{D}(\mathscr{A}_Y)\to\mathrm{D}(\mathscr{A}_{X\times Y})$$

satisfies $\mathscr{M}^\bullet\overset{\mathrm{L}}{\underline{\boxtimes}}\mathscr{N}^\bullet \xrightarrow{\sim} \mathscr{M}^\bullet\underline{\boxtimes}\mathscr{N}^\bullet$ as soon as $\mathscr{M}^\bullet$ or $\mathscr{N}^\bullet$ is a complex bounded from above of modules without $\hbar$-torsion.

Graded modules

For a $\mathbb{C}^\hbar$-algebroid $\mathscr{B}$ on X, one denotes by $\mathrm{gr}_\hbar(\mathscr{B})$ the $\mathbb{C}$-algebroid associated with the prestack $\mathfrak{S}$ given by

$$\begin{aligned}&\mathrm{Ob}(\mathfrak{S}(U))=\mathrm{Ob}(\mathscr{B}(U))\quad\text{for an open subset } U \text{ of } X,\\&\mathrm{Hom}_{\mathfrak{S}(U)}(\sigma,\sigma')=\mathrm{Hom}_{\mathscr{B}}(\sigma,\sigma')/\hbar\mathrm{Hom}_{\mathscr{B}}(\sigma,\sigma')\quad\text{for } \sigma,\ \sigma'\in\mathscr{B}(U).\end{aligned}$$

Let now $\mathscr{A}_X$ be a DQ-algebroid on X. Then it is easy to see that $\mathrm{gr}_\hbar(\mathscr{A}_X)$ is an invertible $\mathscr{O}_X$-algebroid and that we have a natural functor $\mathscr{A}_X\to\mathrm{gr}_\hbar(\mathscr{A}_X)$ of $\mathbb{C}$-algebroids. This functor induces a functor

$$\mathit{for}\colon \mathrm{Mod}(\mathrm{gr}_\hbar(\mathscr{A}_X))\to\mathrm{Mod}(\mathscr{A}_X). \tag{2.3.6}$$

The functor *for* above is fully faithful and $\mathrm{Mod}(\mathrm{gr}_\hbar(\mathscr{A}_X))$ is equivalent to the full subcategory of $\mathrm{Mod}(\mathscr{A}_X)$ consisting of objects M such that $\hbar\colon M\to M$ vanishes. The functor $\mathit{for}\colon \mathrm{Mod}(\mathrm{gr}_\hbar(\mathscr{A}_X))\to\mathrm{Mod}(\mathscr{A}_X)$ admits a left adjoint functor $M\mapsto M/\hbar M\simeq\mathbb{C}\otimes_{\mathbb{C}^\hbar}M$. The functor *for* is exact and it induces a functor

$$\mathit{for}\colon\ \mathrm{D}(\mathrm{gr}_\hbar(\mathscr{A}_X))\to\mathrm{D}(\mathscr{A}_X). \tag{2.3.7}$$

Remark 2.3.3. The functor in (2.3.7) is not full in general. Indeed, choose $X=\mathrm{pt}$, $\mathscr{A}_X=\mathbb{C}^\hbar$ and $L=\mathbb{C}^\hbar/\hbar\mathbb{C}^\hbar$ viewed as a $\mathrm{gr}_\hbar(A)$-module. Then

$$\begin{aligned}&\mathrm{Hom}_{\mathrm{D^b}(\mathbb{C}^\hbar)}\big(\mathit{for}(L),\mathit{for}(L[1])\big)\simeq\mathbb{C}^\hbar/\hbar\mathbb{C}^\hbar,\\&\mathrm{Hom}_{\mathrm{D^b}(\mathbb{C})}(L,L\,[1])\simeq 0.\end{aligned}$$

It could be also shown that this functor is not faithful in general.

One extends Definition 1.4.1 to the algebroid $\mathscr{A}_X$. As an $(\mathscr{A}_X\otimes\mathscr{A}_{X^a})$-module, $\mathrm{gr}_\hbar(\mathscr{A}_X)$ is isomorphic to $\mathbb{C}\otimes_{\mathbb{C}^\hbar}\mathscr{A}_X\simeq\mathscr{A}_X/\hbar\mathscr{A}_X$. We get the functor

$$\mathrm{gr}_\hbar\colon\ \mathrm{D}(\mathscr{A}_X)\to\mathrm{D}(\mathrm{gr}_\hbar(\mathscr{A}_X)),\ \mathscr{M}\mapsto\mathrm{gr}_\hbar(\mathscr{A}_X)\overset{\mathrm{L}}{\otimes}_{\mathscr{A}_X}\mathscr{M}\simeq\mathbb{C}\overset{\mathrm{L}}{\otimes}_{\mathbb{C}^\hbar}\mathscr{M}. \tag{2.3.8}$$

Note that Lemma 1.4.2, Propositions 1.4.3 and 1.4.5 as well as Corollary 1.4.6 still hold. Moreover

Corollary 2.3.4. *Let* $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X)$. *Then its support,* $\mathrm{Supp}(\mathscr{M})$, *is a closed complex analytic subset of* X.

Proof. By Corollary 1.4.6, $\mathrm{Supp}(\mathscr{M}) = \mathrm{Supp}(\mathrm{gr}_\hbar(\mathscr{M}))$. Since $\mathrm{gr}_\hbar(\mathscr{M}) \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathrm{gr}_\hbar(\mathscr{A}_X))$ and $\mathrm{gr}_\hbar(\mathscr{A}_X)$ is locally isomorphic to $\mathscr{O}_X$, the result follows. Q.E.D.

Let d_X denote the complex dimension of X. Applying Theorem 1.4.8, we get

Corollary 2.3.5. *Let* $\mathscr{A}_X$ *be a* DQ*-algebra and let* $\mathscr{M} \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_X)$. *Then, locally,* $\mathscr{M}$ *admits a resolution by free modules of finite rank of length* $\leq d_X + 1$.

Proposition 2.3.6. *The functors* $\mathrm{gr}_\hbar$ *in* (2.3.8) *and for in* (2.3.7) *define pairs of adjoint functors* $(\mathrm{gr}_\hbar, \mathit{for})$ *and* $(\mathit{for}, \mathrm{gr}_\hbar[-1])$.

Proof. Consider a pair (B, C) in which either $B = \mathscr{A}_X$ and $C = \mathrm{gr}_\hbar(\mathscr{A}_X)$ or $B = \mathrm{gr}_\hbar(\mathscr{A}_X)$ and $C = \mathscr{A}_X$, and let K be a (B, C)-bimodule. We have the adjunction formula, for $M \in \mathrm{D}^{\mathrm{b}}(B)$ and $N \in \mathrm{D}(C)$:

$$\mathrm{Hom}_{\mathrm{D}(B)}(K \overset{\mathrm{L}}{\otimes}_C N, M) \simeq \mathrm{Hom}_{\mathrm{D}(C)}(N, \mathrm{R}\mathscr{H}om_B(K, M)). \tag{2.3.9}$$

(i) Let us apply formula (2.3.9) with $B = \mathrm{gr}_\hbar(\mathscr{A}_X)$, $C = \mathscr{A}_X$ and $K = \mathrm{gr}_\hbar(\mathscr{A}_X)$ considered as a $(\mathrm{gr}_\hbar(\mathscr{A}_X), \mathscr{A}_X)$-bimodule. We get

$$\begin{aligned}&\mathrm{Hom}_{\mathrm{D}(\mathrm{gr}_\hbar(\mathscr{A}_X))}(\mathrm{gr}_\hbar(\mathscr{A}_X) \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_X} \mathscr{M}, \mathscr{N}) \\ &\qquad\simeq \mathrm{Hom}_{\mathrm{D}(\mathscr{A}_X)}(\mathscr{M}, \mathrm{R}\mathscr{H}om_{\mathrm{gr}_\hbar(\mathscr{A}_X)}(\mathrm{gr}_\hbar(\mathscr{A}_X), \mathscr{N})),\end{aligned}$$

and when remarking that $\mathrm{R}\mathscr{H}om_{\mathrm{gr}_\hbar(\mathscr{A}_X)}(\mathrm{gr}_\hbar(\mathscr{A}_X), \mathscr{N}) \simeq \mathit{for}(\mathscr{N})$, we get the first adjunction pairing.
(ii) Let us apply formula (2.3.9) with $C = \mathrm{gr}_\hbar(\mathscr{A}_X)$, $B = A$ and $K = \mathrm{gr}_\hbar(\mathscr{A}_X)$ considered as an $(\mathscr{A}_X, \mathrm{gr}_\hbar(\mathscr{A}_X))$-bimodule. We get

$$\begin{aligned}&\mathrm{Hom}_{\mathrm{D}(\mathscr{A}_X)}(\mathrm{gr}_\hbar(\mathscr{A}_X) \overset{\mathrm{L}}{\otimes}_{\mathrm{gr}_\hbar(\mathscr{A}_X)} \mathscr{N}, \mathscr{M}) \\ &\qquad\simeq \mathrm{Hom}_{\mathrm{D}(\mathrm{gr}_\hbar(\mathscr{A}_X))}(\mathscr{N}, \mathrm{R}\mathscr{H}om_{\mathscr{A}_X}(\mathrm{gr}_\hbar(\mathscr{A}_X), \mathscr{M})).\end{aligned}$$

We have $\mathrm{gr}_\hbar(\mathscr{A}_X) \overset{\mathrm{L}}{\otimes}_{\mathrm{gr}_\hbar(\mathscr{A}_X)} \mathscr{N} \simeq \mathit{for}(\mathscr{N})$ and to get the second adjunction pairing, notice that

$$\mathrm{R}\mathscr{H}om_{\mathscr{A}_X}(\mathrm{gr}_\hbar(\mathscr{A}_X), \mathscr{M}) \ \simeq \ \mathrm{R}\mathscr{H}om_{\mathscr{A}_X}(\mathrm{gr}_\hbar(\mathscr{A}_X), \mathscr{A}_X) \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_X} \mathscr{M},$$

and $\mathrm{R}\mathscr{H}om_{\mathscr{A}_X}(\mathrm{gr}_\hbar(\mathscr{A}_X), \mathscr{A}_X) \simeq \mathrm{gr}_\hbar(\mathscr{A}_X)\,[-1]$. Q.E.D.

Duality

Let $\mathscr{A}_X$ be a DQ-algebroid on X.

Definition 2.3.7. Let $\mathscr{M} \in \mathrm{D}(\mathscr{A}_X)$. Its dual $\mathrm{D}'_{\mathscr{A}_X}\mathscr{M} \in \mathrm{D}(\mathscr{A}_{X^a})$ is given by

$$\mathrm{D}'_{\mathscr{A}_X}\mathscr{M} := \mathrm{R}\mathscr{H}om_{\mathscr{A}_X}(\mathscr{M}, \mathscr{A}_X). \tag{2.3.10}$$

When there is no risk of confusion, we write $\mathrm{D}'_{\mathscr{A}}$ instead of $\mathrm{D}'_{\mathscr{A}_X}$.
By Corollary 2.3.5, $\mathrm{D}'_{\mathscr{A}}$ sends $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X)$ to $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_{X^a})$:

$$\mathrm{D}'_{\mathscr{A}} \colon \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X) \longrightarrow \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_{X^a}).$$

Assume that $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X)$. Then there is a canonical isomorphism:

$$\mathscr{M} \xrightarrow{\sim} \mathrm{D}'_{\mathscr{A}}\mathrm{D}'_{\mathscr{A}}\mathscr{M}. \tag{2.3.11}$$

For a $\mathrm{gr}_\hbar(\mathscr{A}_X)$-module $\mathscr{M}$, denote by $\mathrm{D}'_{\mathscr{O}}\mathscr{M}$ its dual,

$$\mathrm{D}'_{\mathscr{O}}\mathscr{M} := \mathrm{R}\mathscr{H}om_{\mathrm{gr}_\hbar(\mathscr{A}_X)}(\mathscr{M}, \mathrm{gr}_\hbar(\mathscr{A}_X)). \tag{2.3.12}$$

Proposition 2.3.8. *Let $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X)$. Then*

$$\mathrm{gr}_\hbar(\mathrm{D}'_{\mathscr{A}}\mathscr{M}) \simeq \mathrm{D}'_{\mathscr{O}}(\mathrm{gr}_\hbar(\mathscr{M})).$$

Proof. This follows from Proposition 1.4.3. Q.E.D.

Corollary 2.3.9. *Let $\mathscr{L} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X)$ and $j \in \mathbb{Z}$. Let us assume that $\mathscr{E}xt^j_{\mathrm{gr}_\hbar(\mathscr{A}_X)}(\mathrm{gr}_\hbar(\mathscr{L}), \mathrm{gr}_\hbar(\mathscr{A}_X)) \simeq 0$. Then $\mathscr{E}xt^j_{\mathscr{A}_X}(\mathscr{L}, \mathscr{A}_X) \simeq 0$.*

Proof. Applying the above proposition, we get

$$\begin{aligned} \mathscr{E}xt^j_{\mathrm{gr}_\hbar(\mathscr{A}_X)}(\mathrm{gr}_\hbar(\mathscr{L}), \mathrm{gr}_\hbar(\mathscr{A}_X)) &= H^j(\mathrm{D}'_{\mathscr{O}}(\mathrm{gr}_\hbar(\mathscr{L}))) \\ &\simeq H^j(\mathrm{gr}_\hbar(\mathrm{D}'_{\mathscr{A}}(\mathscr{L}))). \end{aligned}$$

Then the result follows from Proposition 1.4.5. Q.E.D.

Simple modules

Definition 2.3.10. Let Λ be a smooth submanifold of X and let $\mathscr{L}$ be a coherent $\mathscr{A}_X$-module supported by Λ. One says that $\mathscr{L}$ is simple along Λ if $\mathrm{gr}_\hbar(\mathscr{L})$ is concentrated in degree 0 and $H^0(\mathrm{gr}_\hbar(\mathscr{L}))$ is an invertible $\mathscr{O}_\Lambda \otimes_{\mathscr{O}_X} \mathrm{gr}_\hbar(\mathscr{A}_X)$-module. (In particular, $\mathscr{L}$ has no $\hbar$-torsion.)

Proposition 2.3.11. *Let Λ be a closed submanifold of X of codimension l and let $\mathscr{L}$ be a coherent $\mathscr{A}_X$-module simple along Λ. Then $H^j(\mathrm{D}'_{\mathscr{A}}(\mathscr{L})) = \mathscr{E}xt^j_{\mathscr{A}_X}(\mathscr{L}, \mathscr{A}_X)$ vanishes for $j \neq l$ and $H^l(\mathrm{D}'_{\mathscr{A}}(\mathscr{L}))$ is a coherent $\mathscr{A}_{X^a}$-module simple along Λ.*

Proof. The question being local, we may assume that $\mathscr{A}_X$ is a DQ-algebra so that $\mathrm{gr}_\hbar(\mathscr{A}_X) \simeq \mathscr{O}_X$ and $\mathrm{gr}_\hbar(\mathscr{L}) \simeq \mathscr{O}_\Lambda$. Then, we have $\mathscr{E}xt^j_{\mathscr{O}_X}(\mathrm{gr}_\hbar(\mathscr{L}), \mathscr{O}_X) \simeq 0$ for $j \neq l$. Therefore, $\mathscr{E}xt^j_{\mathscr{A}_X}(\mathscr{L}, \mathscr{A}_X) = 0$ for $j \neq l$ by Corollary 2.3.9 and

$$\begin{aligned}\mathrm{gr}_\hbar(\mathscr{E}xt^l_{\mathscr{A}_X}(\mathscr{L}, \mathscr{A}_X)) &\simeq \mathrm{D}'_{\mathscr{O}}(\mathrm{gr}_\hbar\mathscr{L})\,[l] \\ &\simeq \mathscr{E}xt^l_{\mathscr{O}_X}(\mathrm{gr}_\hbar(\mathscr{L}), \mathscr{O}_X)\end{aligned}$$

by Proposition 2.3.8. If $\mathrm{gr}_\hbar(\mathscr{L})$ is locally isomorphic to $\mathscr{O}_\Lambda$, then so is $\mathscr{E}xt^l_{\mathscr{O}_X}(\mathrm{gr}_\hbar(\mathscr{L}), \mathscr{O}_X)$. Q.E.D.

Homological dimension of $\mathscr{A}_X$-modules

The codimension of the support of a coherent $\mathscr{O}_X$-module $\mathscr{F}$ is related to the vanishing of the $\mathscr{E}xt^j_{\mathscr{O}_X}(\mathscr{F}, \mathscr{O}_X)$. Similar results hold for $\mathscr{A}_X$-modules.

Proposition 2.3.12. *Let $\mathscr{M}$ be a coherent $\mathscr{A}_X$-module. Then*

(a) $\mathscr{E}xt^j_{\mathscr{A}_X}(\mathscr{M}, \mathscr{A}_X) \simeq 0$ *for* $j < \operatorname{codim}\operatorname{Supp}\mathscr{M}$,

(b) $\operatorname{codim}\operatorname{Supp}\mathscr{E}xt^j_{\mathscr{A}_X}(\mathscr{M}, \mathscr{A}_X) \geq j$.

Proof. (a) First, note that $\mathrm{Supp}(\mathscr{M}) = \mathrm{Supp}(\mathrm{gr}_\hbar\mathscr{M})$. Therefore,

$$\mathscr{E}xt^j_{\mathrm{gr}_\hbar(\mathscr{A}_X)}(\mathrm{gr}_\hbar\mathscr{M}, \mathrm{gr}_\hbar(\mathscr{A}_X)) \simeq 0 \text{ for } j < \operatorname{codim}\operatorname{Supp}\mathscr{M}$$

and the result follows from Corollary 2.3.9.
(b) By Proposition 1.4.5, we know that

$$\operatorname{Supp}\mathscr{E}xt^j_{\mathscr{A}_X}(\mathscr{M}, \mathscr{A}_X) \subset \operatorname{Supp}\mathscr{E}xt^j_{\mathrm{gr}_\hbar(\mathscr{A}_X)}(\mathrm{gr}_\hbar\mathscr{M}, \mathrm{gr}_\hbar(\mathscr{A}_X)),$$

and $\operatorname{codim}\operatorname{Supp}\mathscr{E}xt^j_{\mathrm{gr}_\hbar(\mathscr{A}_X)}(\mathrm{gr}_\hbar\mathscr{M}, \mathrm{gr}_\hbar(\mathscr{A}_X)) \geq j$ by classical results for $\mathscr{O}_X$-modules. Q.E.D.

Extension of the base ring

Recall that $\mathbb{C}^{\hbar,\mathrm{loc}} := \mathbb{C}((\hbar))$ is the fraction field of $\mathbb{C}^\hbar$. To a DQ-algebroid $\mathscr{A}_X$ we associate the $\mathbb{C}^{\hbar,\mathrm{loc}}$-algebroid

$$\mathscr{A}_X^{\mathrm{loc}} = \mathbb{C}^{\hbar,\mathrm{loc}} \otimes_{\mathbb{C}^\hbar} \mathscr{A}_X \tag{2.3.13}$$

and we call $\mathscr{A}_X^{\mathrm{loc}}$ the *$\hbar$-localization* of $\mathscr{A}_X$. It follows from Lemma 1.4.10 that the algebroid $\mathscr{A}_X^{\mathrm{loc}}$ is Noetherian.

There naturally exists a faithful functor of $\mathbb{C}^\hbar$-algebroid

$$\mathscr{A}_X \to \mathscr{A}_X^{\mathrm{loc}}. \tag{2.3.14}$$

This functor gives rise to a pair of adjoint functors $(\mathrm{loc}, \mathit{for})$:

$$\mathrm{Mod}(\mathscr{A}_X^{\mathrm{loc}}) \underset{\mathrm{loc}}{\overset{\mathit{for}}{\rightleftarrows}} \mathrm{Mod}(\mathscr{A}_X). \tag{2.3.15}$$

Both functors are exact and we keep the same notations for their derived functors

$$\mathrm{D^b}(\mathscr{A}_X^{\mathrm{loc}}) \underset{\mathrm{loc}}{\overset{\mathit{for}}{\rightleftarrows}} \mathrm{D^b}(\mathscr{A}_X). \tag{2.3.16}$$

For $\mathscr{N} \in \mathrm{D^b}(\mathscr{A}_X)$, we have

$$\mathscr{N}^{\mathrm{loc}} := \mathbb{C}^{\hbar,\mathrm{loc}} \otimes_{\mathbb{C}^\hbar} \mathscr{N}. \tag{2.3.17}$$

We say that an $\mathscr{A}_X$-module $\mathscr{M}_0$ is a submodule of an $\mathscr{A}_X^{\mathrm{loc}}$-module $\mathscr{M}$ if there is a monomorphism $\mathscr{M}_0 \to \mathit{for}(\mathscr{M})$ in $\mathrm{Mod}(\mathscr{A}_X)$.

If $\mathscr{M}$ is an $\mathscr{A}_X^{\mathrm{loc}}$-module, $\mathscr{M}_0$ an $\mathscr{A}_X$-submodule and $\mathscr{M}_0 \otimes_{\mathbb{C}^\hbar} \mathbb{C}^{\hbar,\mathrm{loc}} \xrightarrow{\sim} \mathscr{M}$, then we shall say that $\mathscr{M}_0$ generates $\mathscr{M}$.

The following result is of constant use and follows from [38, Appendix A].

Lemma 2.3.13. *Any locally finitely generated $\mathscr{A}_X$-submodule of a coherent $\mathscr{A}_X^{\mathrm{loc}}$-module is coherent, i.e., any coherent $\mathscr{A}_X^{\mathrm{loc}}$-module is pseudo-coherent as an $\mathscr{A}_X$-module.*

Definition 2.3.14. A coherent $\mathscr{A}_X$-submodule $\mathscr{M}_0$ of a coherent $\mathscr{A}_X^{\mathrm{loc}}$-module $\mathscr{M}$ is called an $\mathscr{A}_X$-lattice of $\mathscr{M}$ if $\mathscr{M}_0$ generates $\mathscr{M}$.

We extend Definition 2.3.7 to $\mathscr{A}_X^{\mathrm{loc}}$-modules and, for $\mathscr{M} \in \mathrm{D^b}(\mathscr{A}_X^{\mathrm{loc}})$, we set

$$\mathrm{D}'_{\mathscr{A}^{\mathrm{loc}}}\mathscr{M} := \mathrm{R}\mathscr{H}om_{\mathscr{A}_X^{\mathrm{loc}}}(\mathscr{M}, \mathscr{A}_X^{\mathrm{loc}}). \tag{2.3.18}$$

Proposition 2.3.15. *Let $\mathscr{M}$ be a coherent $\mathscr{A}_X^{\mathrm{loc}}$-module. Then*

(a) $\mathscr{E}xt^j_{\mathscr{A}_X^{\mathrm{loc}}}(\mathscr{M}, \mathscr{A}_X^{\mathrm{loc}}) \simeq 0$ *for* $j < \operatorname{codim} \operatorname{Supp} \mathscr{M}$,

(b) $\operatorname{codim} \operatorname{Supp} \mathscr{E}xt^j_{\mathscr{A}_X^{\mathrm{loc}}}(\mathscr{M}, \mathscr{A}_X^{\mathrm{loc}}) \geq j$.

Proof. The result is local and we may choose an $\mathscr{A}_X$-lattice $\mathscr{M}_0$ of $\mathscr{M}$. Then the result follows from Proposition 2.3.12. Q.E.D.

Good modules

Definition 2.3.16. (i) A coherent $\mathscr{A}_X^{\mathrm{loc}}$-module $\mathscr{M}$ is good if, for any relatively compact open subset U of X, there exists an $(\mathscr{A}_X|_U)$-lattice of $\mathscr{M}|_U$.

(ii) One denotes by $\mathrm{Mod}_{\mathrm{gd}}(\mathscr{A}_X^{\mathrm{loc}})$ the full subcategory of $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_X^{\mathrm{loc}})$ consisting of good modules.

(iii) One denotes by $\mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{A}_X^{\mathrm{loc}})$ the full subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X^{\mathrm{loc}})$ consisting of objects $\mathscr{M}$ such that $H^j(\mathscr{M})$ is good for all $j \in \mathbb{Z}$.

Roughly speaking, a coherent $\mathscr{A}_X^{\mathrm{loc}}$-module $\mathscr{M}$ is good if it is endowed with a good filtration (see [38]) on each open relatively compact subset of X.

Proposition 2.3.17. (a) *The category* $\mathrm{Mod}_{\mathrm{gd}}(\mathscr{A}_X^{\mathrm{loc}})$ *is a thick subcategory of* $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_X^{\mathrm{loc}})$, (*i.e., stable by kernels, cokernels and extension*).

(b) *The full subcategory* $\mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{A}_X^{\mathrm{loc}})$ *of* $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X^{\mathrm{loc}})$ *is triangulated.*

(c) *An object* $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X^{\mathrm{loc}})$ *is good if and only if, for any open relatively compact subset U of X, there exists an $\mathscr{A}_X|_U$-module* $\mathscr{M}_0 \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X|_U)$ *such that* $\mathscr{M}_0^{\mathrm{loc}}$ *is isomorphic to* $\mathscr{M}|_U$.

Since the proof is similar to that of [38, Prop. 4.23], we shall not repeat it.

Proposition 2.3.18. *Let* $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X^{\mathrm{loc}})$. *Then* $\mathrm{Supp}(\mathscr{M})$ *is a closed complex analytic subset of X, involutive* (i.e., *co-isotropic*) *for the Poisson bracket on X.*

Proof. Since the problem is local, we may assume that $\mathscr{A}_X$ is a DQ-algebra. Then the proposition follows from Gabber's theorem [27]. Q.E.D.

Remark 2.3.19. One shall be aware that the support of a coherent $\mathscr{A}_X$-module is not involutive in general. Indeed, for a DQ-algebra $\mathscr{A}_X$, any coherent $\mathscr{O}_X$-module may be regarded as an $\mathscr{A}_X$-module. Hence any closed analytic subset can be the support of a coherent $\mathscr{A}_X$-module.

2.4 DQ-modules supported by the diagonal

Let X be a complex manifold endowed with a DQ-algebroid $\mathscr{A}_X$. We denote by $\mathscr{A}_{X\times X^a}$ the external product of $\mathscr{A}_X$ and $\mathscr{A}_{X^a}$ on $X \times X^a$. We still denote by $\delta_X\colon X \hookrightarrow X\times X^a$ the diagonal embedding and we denote by $\mathrm{Mod}_{\Delta_X}(\mathscr{A}_X \boxtimes \mathscr{A}_{X^a})$ the category of $(\mathscr{A}_X \boxtimes \mathscr{A}_{X^a})$-modules supported by the diagonal Δ_X. Then

$$\delta_{X*}\colon \mathrm{Mod}(\mathscr{A}_X \otimes \mathscr{A}_{X^a}) \to \mathrm{Mod}_{\Delta_X}(\mathscr{A}_X \boxtimes \mathscr{A}_{X^a})$$

gives an equivalence of categories, with quasi-inverse δ_X^{-1}. We shall often identify these two categories by this equivalence.

Recall that we have a canonical object $\mathscr{A}_X$ in $\mathrm{Mod}(\mathscr{A}_X \otimes \mathscr{A}_{X^a})$ (see § 2.1). We identify $\mathscr{A}_X$ with an $(\mathscr{A}_X \boxtimes \mathscr{A}_{X^a})$-module supported by the diagonal Δ_X of $X \times X^a$. In fact, it has a structure of $\mathscr{A}_{X\times X^a}$-module. More generally, we have:

Lemma 2.4.1. *Let $\mathscr{M}$ be an $(\mathscr{A}_X \otimes \mathscr{A}_{X^a})$-module.*

(a) *The following conditions are equivalent:*

  (i) *$\mathscr{M}$ is a bi-invertible $(\mathscr{A}_X \otimes \mathscr{A}_{X^a})$-module (see* Definition 2.1.10*),*

  (ii) *$\mathscr{M}$ is invertible as an $\mathscr{A}_X$-module (see* Definition 2.1.4*), that is, $\mathscr{M}$ is locally isomorphic to $\mathscr{A}_X$ as an $\mathscr{A}_X$-module,*

  (iii) *$\mathscr{M}$ is invertible as an $\mathscr{A}_{X^a}$-module.*

(b) *Under the equivalent conditions in* (a)*, $\delta_{X*}\mathscr{M} \to \mathscr{A}_{X\times X^a} \otimes_{\mathscr{A}_X\boxtimes\mathscr{A}_{X^a}} \delta_{X*}\mathscr{M}$ is an isomorphism and $\delta_{X*}\mathscr{M}$ has a structure of an $\mathscr{A}_{X\times X^a}$-module. Moreover, $\delta_{X*}\mathscr{M}$ is a simple $\mathscr{A}_{X\times X^a}$-module along the diagonal of $X \times X^a$.*

(c) *Conversely, if $\mathscr{N}$ is a simple $\mathscr{A}_{X\times X^a}$-module along the diagonal of $X \times X^a$, then $\delta_X^{-1}\mathscr{N}$ satisfies the equivalent conditions* (a) (i)–(iii)*.*

Proof. The statement is local and we may assume that $\mathscr{A}_X = (\mathscr{O}_X[[\hbar]], \star)$.

(a) Assume (ii) and take a generator $u \in \mathscr{M}$ as an $\mathscr{A}_X$-module. Then for any $a \in \mathscr{A}_X$, there exists a unique $\theta(a) \in \mathscr{A}_X$ such that $ua = \theta(a)u$. Then $\theta\colon \mathscr{A}_X \to \mathscr{A}_X$ gives a $\mathbb{C}^\hbar$-algebra endomorphism of $\mathscr{A}_X$. Hence θ is an isomorphism by Proposition 2.2.3. Thus we obtain (i). Similarly (iii) implies (i).

(b) Let us choose $u \in \mathscr{M}$ as in (a) and identify $\mathscr{M}$ with $\mathscr{O}_X[[\hbar]]$ that we regard as a sheaf supported by the diagonal. The action of $\mathscr{A}_X \otimes \mathscr{A}_X^{\mathrm{op}}$ on $\mathscr{M}$

can be expressed by differential operators. Namely, there exist differential operators $\{S_i(x, \partial_{x_1}, \partial_{x_2}, \partial_{x_3})\}_{i\in\mathbb{N}}$ such that

$$f \star a \star \theta(g) = \sum_i \big(S_i(x, \partial_{x_1}, \partial_{x_2}, \partial_{x_3}) f(x_1)g(x_2)a(x_3)\big)|_{x_1=x_2=x_3=x}\hbar^i$$
$$\text{for } f,\, g \in \mathscr{A}_X \text{ and } a \in \mathscr{O}_X[[\hbar]].$$

Then this action extends to an action of $\mathscr{A}_{X\times X^a}$ by setting

$$f(x,y) \star a(x) = \sum_i \big(S_i(x, \partial_{x_1}, \partial_{x_2}, \partial_{x_3}) f(x_1, x_2)a(x_3)\big)|_{x_1=x_2=x_3=x}\hbar^i$$
$$\text{for } f \in \mathscr{A}_{X\times X^a} \text{ and } a \in \mathscr{O}_X[[\hbar]].$$

We denote by $\widetilde{\mathscr{M}}$ the $\mathscr{A}_{X\times X^a}$-module thus obtained. Then, as an $(\mathscr{A}_X \otimes \mathscr{A}_{X^a})$-module, it is isomorphic to $\mathscr{M}$. Hence $\widetilde{\mathscr{M}}$ is a locally finitely generated $\mathscr{A}_{X\times X^a}$-module. Since $\hbar^n\widetilde{\mathscr{M}}/\hbar^{n+1}\widetilde{\mathscr{M}}$ is isomorphic to $\mathscr{O}_X$, $\widetilde{\mathscr{M}}$ is a coherent $\mathscr{A}_{X\times X^a}$-module by Theorem 1.2.5 (ii).

Let $\tilde{\mathscr{I}}$ be the annihilator of $u \in \mathscr{M} \simeq \widetilde{\mathscr{M}}$. Then $\tilde{\mathscr{I}}$ is a coherent left ideal of $\mathscr{A}_{X\times X^a}$. In the exact sequence

$$\mathscr{T}or_1^{\mathbb{C}^\hbar}(\widetilde{\mathscr{M}}, \mathbb{C}) \to \tilde{\mathscr{I}}/\hbar\tilde{\mathscr{I}} \to \mathscr{A}_{X\times X^a}/\hbar\mathscr{A}_{X\times X^a} \to \widetilde{\mathscr{M}}/\hbar\widetilde{\mathscr{M}} \to 0,$$

$\mathscr{T}or_1^{\mathbb{C}^\hbar}(\widetilde{\mathscr{M}}, \mathbb{C})$ vanishes. Therefore we obtain an exact sequence

$$0 \to \tilde{\mathscr{I}}/\hbar\tilde{\mathscr{I}} \to \mathscr{O}_{X\times X^a} \to \mathscr{O}_X \to 0,$$

and $\tilde{\mathscr{I}}/\hbar\tilde{\mathscr{I}}$ is isomorphic to the defining ideal $I_\Delta \subset \mathscr{O}_{X\times X^a}$ of the diagonal set $\Delta \subset X \times X^a$. This shows that $\widetilde{\mathscr{M}}$ is simple along the diagonal.

Denote by $\mathscr{I}'$ the left ideal of $\mathscr{A}_X \otimes \mathscr{A}_X^{\mathrm{op}}$ generated by the sections $\{a \otimes 1 - 1 \otimes \theta(a)\}$ where a ranges over the family of sections of $\mathscr{A}_X$ and by $\mathscr{I}$ the left ideal of $\mathscr{A}_{X\times X^a}$ generated by $\mathscr{I}'$. Set $\mathscr{M}' := \mathscr{A}_{X\times X^a} \otimes_{\mathscr{A}_X \boxtimes \mathscr{A}_{X^a}} \mathscr{M}$. We have:

$$\begin{aligned}
&\mathscr{M} \simeq (\mathscr{A}_X \otimes \mathscr{A}_{X^a})/\mathscr{I}',\\
&\mathscr{M}' \simeq \mathscr{A}_{X\times X^a}/\mathscr{I}.
\end{aligned}$$

There exists a surjective $\mathscr{A}_{X\times X^a}$-linear morphism $\mathscr{M}' \twoheadrightarrow \widetilde{\mathscr{M}}$, and hence $\mathscr{I} \subset \tilde{\mathscr{I}}$. Since $\mathscr{I}/\hbar\mathscr{I} \to \tilde{\mathscr{I}}/\hbar\tilde{\mathscr{I}} \simeq I_\Delta$ is surjective, we conclude that $\mathscr{I} = \tilde{\mathscr{I}}$. Hence we obtain $\mathscr{M}' \simeq \widetilde{\mathscr{M}}$.

(c) By the assumption, $p_{1*}\mathrm{gr}_\hbar(\mathscr{N}) \simeq \mathrm{gr}_\hbar(\delta_X^{-1}\mathscr{N})$ is an invertible $\mathscr{O}_X$-module. where $p_1\colon X \times X^a \to X$ is the projection. Hence Theorem 1.2.5 (iv) implies that $\delta_X^{-1}\mathscr{N}$ is a coherent $\mathscr{A}_X$-module. It is locally isomorphic to $\mathscr{A}_X$ by Lemma 1.2.17 because $\mathrm{gr}_\hbar(\delta_X^{-1}\mathscr{N})$ is locally isomorphic to $\mathscr{O}_X$. Q.E.D.

Thus we obtain:

Proposition 2.4.2. *The category of bi-invertible* $(\mathscr{A}_X \otimes \mathscr{A}_{X^a})$*-modules is equivalent to the category of coherent* $\mathscr{A}_{X\times X^a}$*-modules simple along the diagonal.*

Definition 2.4.3. We regard $\delta_{X*}\mathscr{A}_X$ as an $\mathscr{A}_{X\times X^a}$-module supported by the diagonal and denote it by $\mathscr{C}_X$. We call it the canonical module associated with the diagonal.

The next corollary immediately follows from Lemma 2.4.1.

Corollary 2.4.4. *The* $\mathscr{A}_{X\times X^a}$*-module* $\mathscr{C}_X$ *is coherent and simple along the diagonal. Moreover,* $\mathscr{A}_{X\times X^a} \otimes_{\mathscr{A}_X \boxtimes \mathscr{A}_{X^a}} \mathscr{C}_X \to \mathscr{C}_X$ *is an isomorphism in* $\mathrm{Mod}(\mathscr{A}_{X\times X^a})$*, and* $\mathscr{A}_X \to \delta_X^{-1}(\mathscr{C}_X)$ *is an isomorphism in* $\mathrm{Mod}(\mathscr{A}_X \otimes \mathscr{A}_{X^a})$.

The next result is obvious.

Lemma 2.4.5. *Let* Y *be another complex manifold endowed with a* DQ*-algebroid* $\mathscr{A}_Y$*. Then, there is a natural isomorphism* $\mathscr{C}_X \overset{\mathrm{L}}{\boxtimes} \mathscr{C}_Y \simeq \mathscr{C}_{X\times Y}$*. Here, we identify* $(X \times X^a) \times (Y \times Y^a)$ *with* $(X \times Y) \times (X \times Y)^a$*.*

Definition 2.4.6. We say that $\mathscr{P} \in \mathrm{D^b}(\mathscr{A}_X \otimes \mathscr{A}_{X^a})$ is bi-invertible if $\mathscr{P}$ is concentrated to some degree n and $H^n(\mathscr{P})$ is bi-invertible (see Definition 2.1.10).

We sometimes consider a bi-invertible $(\mathscr{A}_X \otimes \mathscr{A}_{X^a})$-module as an object of $\mathrm{D^b_{coh}}(\mathscr{A}_{X\times X^a})$ supported by the diagonal.

For a pair of bi-invertible $(\mathscr{A}_X \otimes \mathscr{A}_{X^a})$-modules $\mathscr{P}_1$ and $\mathscr{P}_2$, $\mathscr{P}_1 \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_X} \mathscr{P}_2$ is also a bi-invertible $(\mathscr{A}_X \otimes \mathscr{A}_{X^a})$-module. Hence the category of bi-invertible $(\mathscr{A}_X \otimes \mathscr{A}_{X^a})$-modules has a structure of a tensor category (see e.g. [42, § 4.2]). It is easy to see that $\mathscr{C}_X$ is a unit object. Namely, for any bi-invertible $(\mathscr{A}_X \otimes \mathscr{A}_{X^a})$-module $\mathscr{P}$, we have:

$$\mathscr{C}_X \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_X} \mathscr{P} \simeq \mathscr{P} \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_X} \mathscr{C}_X \simeq \mathscr{P}.$$

We have

$$\mathscr{P} \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_X} \mathrm{R}\mathscr{H}om_{\mathscr{A}_X}(\mathscr{P}, \mathscr{A}_X) \xrightarrow{\sim} \mathscr{C}_X,$$

$$\mathrm{R}\mathscr{H}om_{\mathscr{A}_{X^a}}(\mathscr{P}, \mathscr{A}_X) \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_X} \mathscr{P} \xrightarrow{\sim} \mathscr{C}_X.$$

Hence we have $\mathrm{R}\mathscr{H}om_{\mathscr{A}_X}(\mathscr{P}, \mathscr{A}_X) \simeq \mathrm{R}\mathscr{H}om_{\mathscr{A}_{X^a}}(\mathscr{P}, \mathscr{A}_X)$.

Definition 2.4.7. For a bi-invertible $(\mathscr{A}_X \otimes \mathscr{A}_{X^a})$-module $\mathscr{P}$, we set

$$\mathscr{P}^{\otimes -1} = \mathrm{R}\mathscr{H}om_{\mathscr{A}_X}(\mathscr{P}, \mathscr{A}_X) \simeq \mathrm{R}\mathscr{H}om_{\mathscr{A}_{X^a}}(\mathscr{P}, \mathscr{A}_X).$$

Hence we have

$$\mathscr{P}^{\otimes -1} \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_X} \mathscr{P} \simeq \mathscr{P} \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_X} \mathscr{P}^{\otimes -1} \simeq \mathscr{C}_X.$$

Note that, for two bi-invertible $(\mathscr{A}_X \otimes \mathscr{A}_{X^a})$-modules $\mathscr{P}_1$ and $\mathscr{P}_2$, we have

$$\begin{aligned} \mathrm{R}\mathscr{H}om_{\mathscr{A}_X}(\mathscr{P}_1, \mathscr{P}_2) &\simeq \mathscr{P}_1^{\otimes -1} \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_X} \mathscr{P}_2, \\ \mathrm{R}\mathscr{H}om_{\mathscr{A}_{X^a}}(\mathscr{P}_1, \mathscr{P}_2) &\simeq \mathscr{P}_2 \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_X} \mathscr{P}_1^{\otimes -1}. \end{aligned}$$

For a bi-invertible $(\mathscr{A}_X \otimes \mathscr{A}_{X^a})$-module $\mathscr{P}$ and $\mathscr{M}, \mathscr{N} \in \mathrm{D}(\mathscr{A}_{X\times Y\times Z})$, we have the isomorphism

$$\mathrm{R}\mathscr{H}om_{\mathscr{A}_{X\times Y}}(\mathscr{M}, \mathscr{N}) \simeq \mathrm{R}\mathscr{H}om_{\mathscr{A}_{X\times Y}}(\mathscr{P} \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_X} \mathscr{M}, \mathscr{P} \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_X} \mathscr{N}) \tag{2.4.1}$$

in $\mathrm{D}(\mathbb{C}^{\hbar}_{X\times Y} \boxtimes \mathscr{A}_Z)$.

Remark 2.4.8. Although it is sometimes convenient to identify $(X \times Y^a)^a$ with $Y \times X^a$, we do not take this point view in this Note. We identify $(X \times Y^a)^a$ with $X^a \times Y$. Hence, for example, we have functors

$$\begin{aligned} \mathrm{D}'_{\mathscr{A}_{X\times Y^a}} &: \mathrm{D}^{\mathrm{b}}(\mathscr{A}_{X\times Y^a}) \longrightarrow \mathrm{D}^{\mathrm{b}}(\mathscr{A}_{X^a\times Y}), \\ \mathrm{D}'_{\mathscr{A}_{X\times X^a}} &: \mathrm{D}^{\mathrm{b}}(\mathscr{A}_{X\times X^a}) \longrightarrow \mathrm{D}^{\mathrm{b}}(\mathscr{A}_{X^a\times X}). \end{aligned}$$

The next result may be useful.

Lemma 2.4.9. (i) *Let X and Y be manifolds endowed with* DQ*-algebroids $\mathscr{A}_X$ and $\mathscr{A}_Y$, let $\mathscr{M}$ be an $\mathscr{A}_{X\times Y^a}$-module and let $\mathscr{Q}$ be a bi-invertible $(\mathscr{A}_Y \otimes \mathscr{A}_{Y^a})$-module. Then*

$$\mathrm{D}'_{\mathscr{A}_{X\times Y^a}}(\mathscr{M} \otimes_{\mathscr{A}_Y} \mathscr{Q}) \simeq \mathscr{Q}^{\otimes -1} \otimes_{\mathscr{A}_Y} \mathrm{D}'_{\mathscr{A}_{X\times Y^a}}(\mathscr{M}).$$

(ii) *Let $\mathscr{P}$ and $\mathscr{Q}$ be bi-invertible $(\mathscr{A}_X \otimes \mathscr{A}_{X^a})$-modules. Then*

$$\begin{aligned} &\mathrm{D}'_{\mathscr{A}_{X\times X^a}}(\mathscr{P} \otimes_{\mathscr{A}_X} \mathscr{Q}) \simeq \mathscr{Q}^{\otimes -1} \otimes_{\mathscr{A}_X} \mathrm{D}'_{\mathscr{A}_{X\times X^a}}\mathscr{P} \simeq \mathrm{D}'_{\mathscr{A}_{X\times X^a}}\mathscr{Q} \otimes_{\mathscr{A}_X} \mathscr{P}^{\otimes -1}, \\ &\mathrm{D}'_{\mathscr{A}_{X\times X^a}}\mathscr{C}_X \otimes_{\mathscr{A}_X} \mathscr{P} \simeq \mathscr{P} \otimes_{\mathscr{A}_X} \mathrm{D}'_{\mathscr{A}_{X\times X^a}}\mathscr{C}_X \simeq \mathrm{D}'_{\mathscr{A}_{X\times X^a}}(\mathscr{P}^{\otimes -1}), \\ &(\mathrm{D}'_{\mathscr{A}_{X\times X^a}}\mathscr{C}_X)^{\otimes -1} \otimes_{\mathscr{A}_X} \mathscr{P} \simeq \mathscr{P} \otimes_{\mathscr{A}_X} (\mathrm{D}'_{\mathscr{A}_{X\times X^a}}\mathscr{C}_X)^{\otimes -1}. \end{aligned}$$

Proof. (i) We have the isomorphism

$$\begin{aligned}
\mathrm{D}'_{\mathscr{A}_{X\times Y^a}}(\mathscr{M}\otimes_{\mathscr{A}_Y}\mathscr{Q}) &= \mathscr{H}om_{\mathscr{A}_{X\times Y^a}}(\mathscr{M}\otimes_{\mathscr{A}_Y}\mathscr{Q},\mathscr{A}_{X\times Y^a})\\
&\simeq \mathscr{H}om_{\mathscr{A}_{X\times Y^a}}(\mathscr{M},\mathscr{A}_{X\times Y^a}\otimes_{\mathscr{A}_Y}\mathscr{Q}^{\otimes -1})\\
&\simeq \mathscr{H}om_{\mathscr{A}_{X\times Y^a}}(\mathscr{M},\mathscr{Q}^{\otimes -1}\otimes_{\mathscr{A}_Y}\mathscr{A}_{X\times Y^a})\\
&\simeq \mathscr{Q}^{\otimes -1}\otimes_{\mathscr{A}_Y}\mathrm{D}'_{\mathscr{A}_{X\times Y^a}}(\mathscr{M}).
\end{aligned}$$

(ii) The first isomorphism follows from (i) and the the second is proved similarly. The two last isomorphisms follow. Q.E.D.

The next result follows immediately from Corollary 2.4.4.

Lemma 2.4.10. *Let $\mathscr{M}\in\mathrm{D}^{\mathrm{b}}(\mathscr{A}_X)$, $\mathscr{L}\in\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X)$ and $\mathscr{N}\in\mathrm{D}^{\mathrm{b}}(\mathscr{A}_{X^a})$. Identifying Δ_X and X, there are natural isomorphisms*

$$\begin{aligned}
\mathscr{M}\simeq\mathscr{A}_X\overset{\mathrm{L}}{\otimes}_{\mathscr{A}_X}\mathscr{M} &\simeq \mathrm{R}\mathscr{H}om_{\mathscr{A}_X}(\mathscr{A}_X,\mathscr{M}) \quad \text{in } \mathrm{D}(\mathscr{A}_X),\\
\mathscr{N}\overset{\mathrm{L}}{\otimes}_{\mathscr{A}_X}\mathscr{M} &\simeq (\mathscr{N}\overset{\mathrm{L}}{\underline{\boxtimes}}\mathscr{M})\overset{\mathrm{L}}{\otimes}_{\mathscr{A}_{X\times X^a}}\mathscr{C}_X \quad \text{in } \mathrm{D}(\mathbb{C}^\hbar_X),\\
\mathrm{R}\mathscr{H}om_{\mathscr{A}_X}(\mathscr{L},\mathscr{M}) &\simeq \mathrm{D}'_{\mathscr{A}}\mathscr{L}\overset{\mathrm{L}}{\otimes}_{\mathscr{A}_X}\mathscr{M} \quad \text{in } \mathrm{D}(\mathbb{C}^\hbar_X),\\
\mathrm{R}\mathscr{H}om_{\mathscr{A}_X}(\mathscr{M},\mathscr{L}) &\simeq \mathrm{R}\mathscr{H}om_{\mathscr{A}_{X\times X^a}}(\mathscr{M}\overset{\mathrm{L}}{\underline{\boxtimes}}\mathrm{D}'_{\mathscr{A}}\mathscr{L},\mathscr{C}_X) \quad \text{in } \mathrm{D}(\mathbb{C}^\hbar_X).
\end{aligned}$$

2.5 Dualizing complex for DQ-algebroids

The algebroid $\mathscr{D}^{\mathscr{A}}_X$

We have seen that the $\mathbb{C}^\hbar$-algebra $\mathscr{D}^{\mathscr{A}}_X\subset\mathscr{E}nd_{\mathbb{C}^\hbar}(\mathscr{A}_X)$ is well-defined for a DQ-algebra $\mathscr{A}_X$ on X.

Now let $\mathscr{A}_X$ be a DQ-algebroid. Then we can regard $\mathscr{A}_X$ as an $(\mathscr{A}_X\otimes\mathscr{A}_X^{\mathrm{op}})$-module. In § 2.1, we have defined the $\mathbb{C}^\hbar$-algebroid $\mathscr{E}nd_{\mathbb{C}^\hbar}(\mathscr{A}_X)$ and introduced a functor of $\mathbb{C}^\hbar$-algebroids $\mathscr{A}_X\otimes\mathscr{A}_X^{\mathrm{op}}\to\mathscr{E}nd_{\mathbb{C}^\hbar}(\mathscr{A}_X)$.

Definition 2.5.1. The $\mathbb{C}^\hbar$-algebroid $\mathscr{D}^{\mathscr{A}}_X$ is the $\mathbb{C}^\hbar$-substack of $\mathscr{E}nd_{\mathbb{C}^\hbar}(\mathscr{A}_X)$ associated to the prestack $\mathfrak{S}$ defined as follows. The objects of $\mathfrak{S}$ are those of $\mathscr{A}_X\otimes\mathscr{A}_X^{\mathrm{op}}$. For $\sigma_1,\sigma_2\in\mathscr{A}_X\otimes\mathscr{A}_X^{\mathrm{op}}$, with $\sigma_1=\tau_1\otimes\lambda_1^{\mathrm{op}}$, $\sigma_2=\tau_2\otimes\lambda_2^{\mathrm{op}}$, we choose isomorphisms $\varphi_i\colon\tau_i\simeq\lambda_i$ $(i=1,2)$ and $\varphi_3\colon\tau_1\simeq\tau_2$. Set $\mathscr{B}=\mathscr{E}nd_{\mathscr{A}_X}(\lambda_1)$. It is a DQ-algebra. The isomorphisms φ_i $(i=1,2,3)$ induce an isomorphism

$$\begin{aligned}
\psi\colon\mathscr{H}om_{\mathbb{C}^\hbar}(\mathscr{B},\mathscr{B}) &\xrightarrow{\sim} \mathscr{H}om_{\mathbb{C}^\hbar}(\mathscr{H}om(\lambda_1,\tau_1),\mathscr{H}om(\lambda_2,\tau_2))\\
&\xrightarrow{\sim} \mathscr{H}om_{\mathbb{C}^\hbar}(\mathscr{A}_X(\sigma_1),\mathscr{A}_X(\sigma_2)).
\end{aligned}$$

We define $\mathscr{H}om_{\mathfrak{S}}(\sigma_1,\sigma_2) \subset \mathscr{H}om_{\mathbb{C}^\hbar}(\mathscr{A}_X(\sigma_1),\mathscr{A}_X(\sigma_2))$ as the image of $\mathscr{D}_X^{\mathscr{B}}$ by ψ. (This does not depend on the choice of the isomorphism φ_i ($i = 1, 2, 3$) in virtue of Proposition 2.2.3.)

Then there are functors of $\mathbb{C}^\hbar$-algebroids

$$\mathscr{A}_X \otimes \mathscr{A}_{X^a} \to \delta_X^{-1}\mathscr{A}_{X\times X^a} \to \mathscr{D}_X^{\mathscr{A}} \to \mathscr{E}nd_{\mathbb{C}^\hbar}(\mathscr{A}_X)$$

and $\mathscr{A}_X$ may be regarded as an object of $\mathrm{Mod}(\mathscr{D}_X^{\mathscr{A}})$.

Proposition 2.5.2. (i) *The $\mathbb{C}^\hbar$-algebroid $\mathscr{E}nd_{\mathbb{C}^\hbar}(\mathscr{A}_X)$ is equivalent to the $\mathbb{C}^\hbar$-algebroid $\mathscr{E}nd_{\mathbb{C}^\hbar}(\mathscr{O}_X[[\hbar]])$ (regarding the $\mathbb{C}^\hbar$-algebra $\mathscr{E}nd_{\mathbb{C}^\hbar}(\mathscr{O}_X[[\hbar]])$ as a $\mathbb{C}^\hbar$-algebroid).*

(ii) *The equivalence in* (i) *induces an equivalence of $\mathbb{C}^\hbar$-algebroids $\mathscr{D}_X^{\mathscr{A}} \simeq \mathscr{D}_X[[\hbar]]$.*

(iii) *The equivalence in* (ii) *induces an equivalence of $\mathbb{C}^\hbar$-linear stacks*

$$\mathfrak{Mod}(\mathscr{D}_X^{\mathscr{A}}) \simeq \mathfrak{Mod}(\mathscr{D}_X[[\hbar]]).$$

Moreover, the $\mathscr{D}_X^{\mathscr{A}}$-module $\mathscr{A}_X$ is sent to the $\mathscr{D}_X[[\hbar]]$-module $\mathscr{O}_X[[\hbar]]$ by this equivalence.

(iv) *The equivalence in* (ii) *also induces an equivalence of $\mathbb{C}$-algebroids*

$$\mathrm{gr}_\hbar(\mathscr{D}_X^{\mathscr{A}}) \simeq \mathscr{D}_X,$$

and an equivalence of $\mathbb{C}$-linear stacks $\mathfrak{Mod}(\mathrm{gr}_\hbar(\mathscr{D}_X^{\mathscr{A}})) \simeq \mathfrak{Mod}(\mathscr{D}_X)$. Moreover the $\mathrm{gr}_\hbar(\mathscr{D}_X^{\mathscr{A}})$-module $\mathrm{gr}_\hbar(\mathscr{A}_X)$ is sent to the $\mathscr{D}_X$-module $\mathscr{O}_X$ by this equivalence.

Proof. Recall first that for two $\mathbb{C}^\hbar$-algebroids $\mathscr{B}$ and $\mathscr{B}'$, to give an equivalence of $\mathbb{C}^\hbar$-algebroids $\mathscr{B} \simeq \mathscr{B}'$ is equivalent to giving a bi-invertible $\mathscr{B}^{\mathrm{op}}\otimes\mathscr{B}'$-module (Lemma 2.1.11).

(i) follows from Lemma 2.1.12. More precisely, we define an $\big(\mathscr{E}nd_{\mathbb{C}^\hbar}(\mathscr{A}_X) \otimes (\mathscr{E}nd_{\mathbb{C}^\hbar}(\mathscr{O}_X[[\hbar]]))^{\mathrm{op}}\big)$-module $\mathscr{L}'$ as follows. For $\sigma = (\sigma_1 \otimes \sigma_2^{\mathrm{op}}) \in \mathscr{A}_X \otimes \mathscr{A}_X^{\mathrm{op}}$, set

$$\mathscr{L}'(\sigma) := \mathscr{H}om_{\mathbb{C}^\hbar}(\mathscr{O}_X[[\hbar]], \mathscr{H}om_{\mathscr{A}_X}(\sigma_2,\sigma_1)).$$

Clearly, $\mathscr{L}'$ is bi-invertible.

(ii) For $\sigma = (\sigma_1\otimes\sigma_2^{\mathrm{op}}) \in \mathscr{A}_X\otimes\mathscr{A}_X^{\mathrm{op}}$, let us choose an isomorphism $\psi\colon \sigma_1 \xrightarrow{\sim} \sigma_2$ and a standard isomorphism $\widetilde{\varphi}\colon \mathscr{O}_X[[\hbar]] \xrightarrow{\sim} \mathscr{E}nd_{\mathscr{A}_X}(\sigma_1)$. Then they give an isomorphism

$$f\colon \mathscr{O}_X[[\hbar]] \xrightarrow{\sim} \mathscr{H}om_{\mathscr{A}_X}(\sigma_2,\sigma_1).$$

We define a $(\mathscr{D}_X^{\mathscr{A}} \otimes \mathscr{D}_X[[\hbar]]^{\mathrm{op}})$-submodule $\mathscr{L}$ of $\mathscr{L}'$ as follows: let $\mathscr{L}(\sigma)$ be the $\mathscr{D}_X[[\hbar]]^{\mathrm{op}}$-submodule of $\mathscr{L}'(\sigma)$ generated by f. Then $\mathscr{L}(\sigma)$ coincides with the submodule generated by f over the $\mathbb{C}^\hbar$-algebra $\mathscr{E}nd_{\mathscr{D}_X^{\mathscr{A}}}(\sigma) \subset \mathscr{E}nd_{\mathbb{C}^\hbar}(\mathscr{H}om_{\mathscr{A}_X}(\sigma_2, \sigma_1))$. Moreover, $\mathscr{L}(\sigma)$ does not depend on the choice of ψ and $\widetilde{\varphi}$. It is easy to see that $\mathscr{L}$ is a bi-invertible $(\mathscr{D}_X^{\mathscr{A}} \otimes \mathscr{D}_X[[\hbar]]^{\mathrm{op}})$-module.
(iii) The $(\mathscr{D}_X^{\mathscr{A}} \otimes \mathscr{D}_X[[\hbar]]^{\mathrm{op}})$-module $\mathscr{L}$ gives an equivalence of categories

$$\mathscr{L} \otimes_{\mathscr{D}_X[[\hbar]]} \bullet \colon \mathrm{Mod}(\mathscr{D}_X[[\hbar]]) \xrightarrow{\sim} \mathrm{Mod}(\mathscr{D}_X^{\mathscr{A}}), \tag{2.5.1}$$

which is isomorphic to the functor induced by the algebroid equivalence $\mathscr{D}_X^{\mathscr{A}} \xrightarrow{\sim} \mathscr{D}_X[[\hbar]]$. Consider the $(\mathscr{D}_X[[\hbar]] \otimes (\mathscr{D}_X^{\mathscr{A}})^{\mathrm{op}})$-module

$$\mathscr{L}^* := \mathscr{H}om_{\mathscr{D}_X^{\mathscr{A}}}(\mathscr{L}, \mathscr{D}_X^{\mathscr{A}}).$$

A quasi-inverse of the equivalence (2.5.1) is given by

$$\mathscr{L}^* \otimes_{\mathscr{D}_X^{\mathscr{A}}} \bullet \simeq \mathscr{H}om_{\mathscr{D}_X^{\mathscr{A}}}(\mathscr{L}, \bullet) \colon \mathrm{Mod}(\mathscr{D}_X^{\mathscr{A}}) \xrightarrow{\sim} \mathrm{Mod}(\mathscr{D}_X[[\hbar]]).$$

The results follow. Q.E.D.

Dualizing complex

Let $\mathscr{A}_X$ be a DQ-algebroid on X. We shall construct a deformation of the sheaf of differential forms of maximal degree and then the dualizing complex for $\mathscr{A}_X$.

Lemma 2.5.3. (i) *$\mathscr{A}_X$ has locally a resolution of length d_X by free $\mathscr{D}_X^{\mathscr{A}}$-modules of finite rank.*

(ii) $\mathrm{gr}_\hbar(\mathscr{E}xt^{d_X}_{\mathscr{D}_X^{\mathscr{A}}}(\mathscr{A}_X, \mathscr{D}_X^{\mathscr{A}})) \simeq \Omega_X$. *(Note that $\mathrm{gr}_\hbar(\mathscr{E}xt^{d_X}_{\mathscr{D}_X^{\mathscr{A}}}(\mathscr{A}_X, \mathscr{D}_X^{\mathscr{A}}))$ is a module over $\mathrm{gr}_\hbar(\mathscr{A}_X) \otimes_{\mathscr{O}_X} \mathrm{gr}_\hbar(\mathscr{A}_{X^a}) \simeq \mathscr{O}_X$ by (2.1.12)).*

(iii) $\mathscr{E}xt^i_{\mathscr{D}_X^{\mathscr{A}}}(\mathscr{A}_X, \mathscr{D}_X^{\mathscr{A}}) = 0$ *for $i \neq d_X$.*

Proof. We have $\mathscr{D}_X^{\mathscr{A}} \simeq \mathscr{D}_X[[\hbar]]$ and $\mathscr{A}_X \simeq \mathscr{O}_X[[\hbar]]$ as $\mathscr{D}_X^{\mathscr{A}}$-modules. Then the results follow from

$$\mathrm{R}\mathscr{H}om_{\mathscr{D}_X[[\hbar]]}(\mathscr{O}_X[[\hbar]], \mathscr{D}_X[[\hbar]]) \simeq \big(\Omega_X[[\hbar]]\big)[-d_X].$$

(ii) follows from

$$\begin{aligned} \mathrm{gr}_\hbar(\mathrm{R}\mathscr{H}om_{\mathscr{D}_X^{\mathscr{A}}}(\mathscr{A}_X, \mathscr{D}_X^{\mathscr{A}})) &\simeq \mathrm{R}\mathscr{H}om_{\mathrm{gr}_\hbar(\mathscr{D}_X^{\mathscr{A}})}(\mathrm{gr}_\hbar(\mathscr{A}_X), \mathrm{gr}_\hbar(\mathscr{D}_X^{\mathscr{A}})) \\ &\simeq \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{O}_X, \mathscr{D}_X) \simeq \Omega_X[-d_X]. \end{aligned}$$

Q.E.D.

We set

$$\Omega_X^{\mathscr{A}} := \mathscr{E}xt^{d_X}_{\mathscr{D}_X^{\mathscr{A}}}(\mathscr{A}_X, \mathscr{D}_X^{\mathscr{A}}) \in \mathrm{Mod}(\mathscr{A}_X \otimes \mathscr{A}_{X^a}). \tag{2.5.2}$$

Lemma 2.5.4. *The $(\mathscr{A}_X \otimes \mathscr{A}_X^{\mathrm{op}})$-module $\Omega_X^{\mathscr{A}}$ is bi-invertible.*

Proof. Under the equivalence $\mathscr{D}_X^{\mathscr{A}} \simeq \mathscr{D}_X[[\hbar]]$, we have $\Omega_X^{\mathscr{A}} \simeq \Omega_X[[\hbar]]$. Hence we have an isomorphism $\Omega_X^{\mathscr{A}} \xrightarrow{\sim} \varprojlim_n \Omega_X^{\mathscr{A}}/\hbar^n\Omega_X^{\mathscr{A}}$. Since $\mathrm{gr}_\hbar(\Omega_X^{\mathscr{A}}) \simeq \Omega_X$ is a coherent $\mathrm{gr}_\hbar(\mathscr{A}_X)$-module, $\Omega_X^{\mathscr{A}}$ is a coherent $\mathscr{A}_X$-module by Theorem 1.2.5 (iv). Since $\mathrm{gr}_\hbar(\Omega_X^{\mathscr{A}})$ is an invertible $\mathscr{O}_X$-module and $\Omega_X^{\mathscr{A}}$ has no $\hbar$-torsion, $\Omega_X^{\mathscr{A}}$ is locally isomorphic to $\mathscr{A}_X$ as an $\mathscr{A}_X$-module. Hence $\Omega_X^{\mathscr{A}}$ is a bi-invertible $(\mathscr{A}_X^{\mathrm{op}} \otimes \mathscr{A}_X)$-module by Lemma 2.4.1 (a). Q.E.D.

Lemma 2.5.5. *One has the isomorphisms*

$$\Omega_X^{\mathscr{A}} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X^{\mathscr{A}}} \mathscr{A}_X[-d_X] \simeq \mathrm{R}\mathscr{H}om_{\mathscr{D}_X^{\mathscr{A}}}(\mathscr{A}_X, \mathscr{A}_X) \simeq \mathbb{C}_X^\hbar. \tag{2.5.3}$$

Proof. The first isomorphism is obvious by Lemma 2.5.3. Hence, it is enough to prove that the natural morphism $\mathbb{C}_X^\hbar \to \mathrm{R}\mathscr{H}om_{\mathscr{D}_X^{\mathscr{A}}}(\mathscr{A}_X, \mathscr{A}_X)$ is an isomorphism. By the equivalence $\mathscr{D}_X^{\mathscr{A}} \simeq \mathscr{D}_X[[\hbar]]$, we may assume that $\mathscr{A}_X = \mathscr{O}_X[[\hbar]]$ and $\mathscr{D}_X^{\mathscr{A}} = \mathscr{D}_X[[\hbar]]$. Then $\mathrm{R}\mathscr{H}om_{\mathscr{D}_X^{\mathscr{A}}}(\mathscr{A}_X, \mathscr{A}_X)$ is represented by an infinite product of the de Rham complexes: $\prod_n \hbar^n \Omega_X^\bullet$. Then the assertion follows from a classical result: $\Omega_X^\bullet(U)$ is quasi-isomorphic to $\mathbb{C}$ when U is a contractible Stein open subset. Q.E.D.

Note that $\Omega_X^{\mathscr{A}}$ and $\Omega_{X^a}^{\mathscr{A}}$ are isomorphic as $\mathscr{A}_X \otimes \mathscr{A}_{X^a}$-modules.

Definition 2.5.6. We set

$$\omega_X^{\mathscr{A}} := \delta_{X*}\Omega_X^{\mathscr{A}}[d_X] \simeq \delta_{X*}\mathrm{R}\mathscr{H}om_{\mathscr{D}_X^{\mathscr{A}}}(\mathscr{A}_X, \mathscr{D}_X^{\mathscr{A}})[2d_X] \in \mathrm{D^b}(\mathscr{A}_{X\times X^a})$$

and call $\omega_X^{\mathscr{A}}$ the $\mathscr{A}_X$-dualizing sheaf.

Note that $\omega_X^{\mathscr{A}}$ is bi-invertible (see Definition 2.4.6). Using (2.5.3) and the morphism $\delta_{X*}\Omega_X^{\mathscr{A}} \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_{X\times X^a}} \mathscr{C}_X \to \Omega_X^{\mathscr{A}} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X^{\mathscr{A}}} \mathscr{A}_X$, we get the natural morphism

$$\omega_{X^a}^{\mathscr{A}} \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_{X\times X^a}} \mathscr{C}_X \to \delta_{X*}\mathbb{C}_X^\hbar\,[2d_X]. \tag{2.5.4}$$

Applying the functor $\mathrm{gr}_\hbar$ to the above morphisms, we get the morphism

$$\delta_{X*}(\mathrm{gr}_\hbar\omega_{X^a}^{\mathscr{A}}) \overset{\mathrm{L}}{\otimes}_{\mathrm{gr}_\hbar\mathscr{A}_{X\times X^a}} (\delta_{X*}\mathrm{gr}_\hbar\mathscr{C}_X) \;\to\; \delta_{X*}(\mathbb{C}_X\,[2d_X]), \tag{2.5.5}$$

which coincides with the morphism derived from

$$\delta_X^{-1}\big(\delta_{X*}(\mathrm{gr}_\hbar\omega_{X^a}^{\mathscr{A}})\overset{\mathrm{L}}{\otimes}_{\mathrm{gr}_\hbar\mathscr{A}_{X\times X^a}}(\delta_{X*}\mathrm{gr}_\hbar\mathscr{C}_X)\big)\to\Omega_X\,[d_X]\to\mathbb{C}_X\,[2d_X]. \tag{2.5.6}$$

Here we used the functor of algebroids $\delta_X^{-1}(\mathrm{gr}_\hbar\mathscr{A}_{X\times X^a})\to\mathscr{O}_X$.

Let Y be another manifold endowed with a DQ-algebroid $\mathscr{A}_Y$. We introduce the notation:

$$\omega_{X\times Y/Y}^{\mathscr{A}}=\omega_X^{\mathscr{A}}\overset{\mathrm{L}}{\underline{\boxtimes}}\mathscr{C}_Y\in\mathrm{D^b}(\mathscr{A}_{X\times X^a\times Y\times Y^a}).$$

Then $\omega_{X\times Y/Y}^{\mathscr{A}}$ also belongs to $\mathrm{D^b}\big((\mathscr{D}_X^{\mathscr{A}})^{\mathrm{op}}\boxtimes\mathscr{A}_{Y\times Y^a}\big)$, and we have an isomorphism $\omega_{X\times Y/Y}^{\mathscr{A}}\overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X^{\mathscr{A}}}\mathscr{A}_X\simeq\mathbb{C}_X^\hbar\boxtimes\mathscr{A}_Y$. Hence we have a canonical morphism

$$\omega_{X^a\times Y/Y}^{\mathscr{A}}\overset{\mathrm{L}}{\otimes}_{\mathscr{A}_{X\times X^a}}\mathscr{C}_X\to(\mathbb{C}_X^\hbar\boxtimes\mathscr{C}_Y)[2d_X] \tag{2.5.7}$$

in $\mathrm{D^b}(\mathbb{C}_X^\hbar\boxtimes\mathscr{A}_{Y\times Y^a})$.

The proof of the following fundamental result will be given later at the end of § 3.3.

Theorem 2.5.7. *We have the isomorphism*

$$\omega_X^{\mathscr{A}}\simeq(\mathrm{D}'_{\mathscr{A}_{X^a\times X}}\mathscr{C}_{X^a})^{\otimes-1}\quad \textit{in }\mathrm{D^b}(\mathscr{A}_{X\times X^a}). \tag{2.5.8}$$

Note that in formula (2.5.8), $\mathrm{D}'_{\mathscr{A}_{X^a\times X}}$ is the dual over $\mathscr{A}_{X^a\times X}$ and $(\bullet)^{\otimes-1}$ is the dual over $\mathscr{A}_X$.

Corollary 2.5.8. *For $\mathscr{M}\in\mathrm{D^b}(\mathscr{A}_{X\times X^a\times Y})$, we have*

$$\begin{aligned}\mathscr{C}_{X^a}\overset{\mathrm{L}}{\otimes}_{\mathscr{A}_{X\times X^a}}\mathscr{M}&\simeq\mathrm{R}\mathscr{H}om_{\mathscr{A}_{X\times X^a}}(\mathscr{C}_X,\omega_X^{\mathscr{A}}\overset{\mathrm{L}}{\otimes}_{\mathscr{A}_X}\mathscr{M})\\&\simeq\mathrm{R}\mathscr{H}om_{\mathscr{A}_{X\times X^a}}(\mathscr{C}_X,\mathscr{M}\overset{\mathrm{L}}{\otimes}_{\mathscr{A}_X}\omega_X^{\mathscr{A}}).\end{aligned}$$

Proof. We have

$$\begin{aligned}\mathscr{C}_{X^a}\overset{\mathrm{L}}{\otimes}_{\mathscr{A}_{X\times X^a}}\mathscr{M}&\simeq\mathrm{R}\mathscr{H}om_{\mathscr{A}_{X\times X^a}}(\mathrm{D}'_{\mathscr{A}_{X^a\times X}}\mathscr{C}_{X^a},\mathscr{M})\\&\simeq\mathrm{R}\mathscr{H}om_{\mathscr{A}_{X\times X^a}}(\omega_X^{\mathscr{A}}\overset{\mathrm{L}}{\otimes}_{\mathscr{A}_X}\mathrm{D}'_{\mathscr{A}_{X\times X^a}}\mathscr{C}_{X^a},\ \omega_X^{\mathscr{A}}\overset{\mathrm{L}}{\otimes}_{\mathscr{A}_X}\mathscr{M})\\&\simeq\mathrm{R}\mathscr{H}om_{\mathscr{A}_{X\times X^a}}(\mathscr{C}_X,\ \omega_X^{\mathscr{A}}\overset{\mathrm{L}}{\otimes}_{\mathscr{A}_X}\mathscr{M}).\end{aligned}$$

The other isomorphism is similarly proved. Q.E.D.

One shall be aware that, although $\Omega_X^{\mathscr{A}}$ is locally isomorphic to $\mathscr{A}_X$ as an $\mathscr{A}_X$-module, it is not always locally isomorphic to $\mathscr{A}_X$ as an $\mathscr{A}_X \otimes \mathscr{A}_{X^a}$-module.

Example 2.5.9. Let $X = \mathbb{C}^2$ with coordinates (x_1, x_2) and let $\mathscr{A}_X$ be the DQ-algebra given by the relation

$$[x_1, x_2] = \hbar\, x_1.$$

Let (y_1, y_2) denotes the coordinates on X^a. Hence

$$[y_1, y_2] = -\hbar\, y_1.$$

Then $\mathscr{C}_X$ is the $\mathscr{A}_{X\times X^a}$-module $\mathscr{A}_{X\times X^a} \cdot u$ where the generator u satisfies $(x_i - y_i) \cdot u = 0$ $(i = 1, 2)$. Therefore $\mathscr{C}_X$ is quasi-isomorphic to the complex

$$0 \to \mathscr{A}_{X\times X^a} \xrightarrow{\alpha} \mathscr{A}_{X\times X^a}^{\oplus 2} \xrightarrow{\beta} \mathscr{A}_{X\times X^a} \to 0, \tag{2.5.9}$$

where $\mathscr{A}_{X\times X^a}$ on the right is in degree 0, $\alpha(a) = (-a(x_2 - y_2 + \hbar), a(x_1 - y_1))$ and $\beta(b, c) = b(x_1 - y_1) + c(x_2 - y_2)$.

It follows that $\mathrm{D}'_{\mathscr{A}}(\mathscr{C}_X)\,[2]$ is isomorphic to $\mathscr{A}_{X\times X^a} \cdot w$ where the generator w satisfies $(x_1 - y_1) \cdot w = 0$, $(y_2 - x_2 + \hbar) \cdot w = 0$. The modules $\mathrm{D}'_{\mathscr{A}}(\mathscr{C}_X)\,[2]$ and $\mathscr{C}_X$ are isomorphic on $x_1 \neq 0$ by $u \leftrightarrow x_1 w$. However, $\mathrm{D}'_{\mathscr{A}}(\mathscr{C}_X)\,[2]$ and $\mathscr{C}_X$ are not isomorphic on a neighborhood of $x_1 = 0$. Indeed if they were isomorphic by $u \leftrightarrow aw$ for $a \in \mathscr{A}_X$, then $x_1 a = a x_1$ and $x_2 a = a(x_2 - \hbar)$. Then $\{x_2, \sigma_0(a)\} = -\sigma_0(a)$. Since $\{x_2, \bullet\} = -x_1 \partial_{x_1}$, we have $x_1 \partial_{x_1} \sigma_0(a) = \sigma_0(a)$, which contradicts the fact that $\sigma_0(a)$ is invertible.

Remark 2.5.10. The fact that $\mathrm{D}'_{\mathscr{A}}\mathscr{C}_X$ is concentrated in a single degree and plays the role of a dualizing complex in the sense of [61] was already proved (in a more restrictive situation) in [21, 22].

2.6 Almost free resolutions

We recall here and adapt to the framework of algebroids some results of [41].

In this section, $\mathbb{K}$ denotes a commutative unital ring, X a paracompact and locally compact space and $\mathscr{A}$ a $\mathbb{K}$-algebroid on X.

Let us take a family $\mathscr{S}$ of open subsets of X. We assume the following two conditions on $\mathscr{S}$:

(2.6.1)
- (i) for any $x \in X$, $\{U \in \mathscr{S}\,;\, x \in U\}$ is a neighborhood system of x,
- (ii) for U, $V \in \mathscr{S}$, $U \cap V$ is a finite union of open subsets belonging to $\mathscr{S}$.

Recall that invertible modules are defined in Definition 2.1.4.

Definition 2.6.1. (i) We define the additive category $\mathrm{Mod}^{\mathrm{af}}(\mathscr{A})$ of $\mathscr{S}$-almost free $\mathscr{A}$-modules as follows.

(a) An object of $\mathrm{Mod}^{\mathrm{af}}(\mathscr{A})$ is the data of $\{I, \{U_i, U'_i, L_i\}_{i\in I}\}$ where I is an index set, U_i and U'_i are open subsets of X, $U_i \in \mathscr{S}$, $\overline{U_i} \subset U'_i$, the family $\{U'_i\}_{i\in I}$ is locally finite and L_i is an invertible $\mathscr{A}|_{U'_i}$-module.

(b) Let $N = \{J, \{V_j, V'_j, K_j\}_{j\in J}\}$ and $M = \{I, \{U_i, U'_i, L_i\}_{i\in I}\}$ be two objects of $\mathrm{Mod}^{\mathrm{af}}(\mathscr{A})$. A morphism $u\colon N \to M$ is the data of $u_{ij} \in \Gamma(\overline{V_j}; \mathscr{H}om_{\mathscr{A}}(K_j, L_i))$ for all $(i,j) \in I \times J$ such that $V_j \subset U_i$.

(c) The composition of morphisms is the natural one.

(d) We denote by $\Phi\colon \mathrm{Mod}^{\mathrm{af}}(\mathscr{A}) \to \mathrm{Mod}(\mathscr{A})$ the functor which sends $\{I, \{U_i, U'_i, L_i\}_{i\in I}\}$ to $\bigoplus_{i\in I}(L_i)_{U_i}$ and which sends an element u_{ij} of $\Gamma(\overline{V_j}; \mathscr{H}om_{\mathscr{A}}(K_j, L_i))$ to its image in $\mathrm{Hom}_{\mathscr{A}}((K_j)_{V_j}, (L_i)_{U_i})$ if $V_j \subset U_i$ and 0 otherwise.

(ii) Similarly, we define the additive category $\mathrm{Mod}_{\mathrm{af}}(\mathscr{A})$ as follows.

(a) The set of objects of $\mathrm{Mod}_{\mathrm{af}}(\mathscr{A})$ is the same as the one of $\mathrm{Mod}^{\mathrm{af}}(\mathscr{A})$.

(b) Let $N = \{J, \{V_j, V'_j, K_j\}_{j\in J}\}$ and $M = \{I, \{U_i, U'_i, L_i\}_{i\in I}\}$ be two objects of $\mathrm{Mod}^{\mathrm{af}}(\mathscr{A})$. A morphism $u\colon N \to M$ is the data of $u_{ij} \in \Gamma(\overline{U_i}; \mathscr{H}om_{\mathscr{A}}(K_j, L_i))$ for all $(i,j) \in I \times J$ such that $U_i \subset V_j$.

(c) The composition of morphisms is the natural one.

(d) We denote by $\Psi\colon \mathrm{Mod}_{\mathrm{af}}(\mathscr{A}) \to \mathrm{Mod}(\mathscr{A})$ the functor which sends $\{I, \{U_i, U'_i, L_i\}_{i\in I}\}$ to $\bigoplus_{i\in I}\Gamma_{U_i}(L_i)$ and which sends an element u_{ij} of $\Gamma(\overline{U_i}; \mathscr{H}om_{\mathscr{A}}(K_j, L_i))$ to its image in $\mathrm{Hom}_{\mathscr{A}}(\Gamma_{V_j}(K_j), \Gamma_{U_i}(L_i))$ if $U_i \subset V_j$ and 0 otherwise.

Note that $\mathrm{Mod}_{\mathrm{af}}(\mathscr{A})$ is equivalent to $\mathrm{Mod}^{\mathrm{af}}(\mathscr{A}^{\mathrm{op}})^{\mathrm{op}}$ by the functor which sends $\{I, \{U_i, U'_i, L_i\}_{i\in I}\}$ to $\{I, \{U_i, U'_i, \mathscr{H}om_{\mathscr{A}}(L_i, \mathscr{A})\}_{i\in I}\}$.

Recall that for an additive category $\mathscr{C}$, we denote by $\mathrm{C}^-(\mathscr{C})$ (resp. $\mathrm{C}^+(\mathscr{C})$) the category of complexes of $\mathscr{C}$ bounded from above (resp. from below).

The following theorem is proved similarly as in [41, Appendix].

Theorem 2.6.2. *Let $\mathscr{A}$ be a left coherent algebroid and let $\mathscr{M} \in \mathrm{D}^-_{\mathrm{coh}}(\mathscr{A})$. Then there exist $L^\bullet \in \mathrm{C}^-(\mathrm{Mod}^{\mathrm{af}}(\mathscr{A}))$ and an isomorphism $\Phi(L^\bullet) \simeq \mathscr{M}$ in $\mathrm{D}^-(\mathscr{A})$.*

There is a dual version of Theorem 2.6.2.

Theorem 2.6.3. *Assume*

(a) $\mathscr{A}$ *being regarded as an object of* $\mathrm{Mod}(\mathscr{A}\otimes\mathscr{A}^{\mathrm{op}})$, $\mathrm{R}\Gamma_U(\mathscr{A})$ *is concentrated in degree* 0 *for all* $U \in \mathscr{S}$,

(b) $\mathscr{A}$ *is a right and left coherent algebroid,*

(c) *there exists an integer* d *such that, for any open subset* U, *any coherent* $\mathscr{A}|_U$*-module admits locally a finite free resolution of length* d.

Let $\mathscr{M} \in \mathrm{D}^+_{\mathrm{coh}}(\mathscr{A})$. *Then there exist* $L^\bullet \in \mathrm{C}^+(\mathrm{Mod}_{\mathrm{af}}(\mathscr{A}))$ *and an isomorphism* $\mathscr{M} \simeq \Psi(L^\bullet)$ *in* $\mathrm{D}^+(\mathscr{A})$.

Proof. Denote by D the duality functor $\mathrm{R}\mathscr{H}om_{\mathscr{A}}(\,\bullet\,,\mathscr{A})$ and keep the same notation with $\mathscr{A}^{\mathrm{op}}$ instead of $\mathscr{A}$. This functor sends $\mathrm{D}^+_{\mathrm{coh}}(\mathscr{A})$ to $\mathrm{D}^-_{\mathrm{coh}}(\mathscr{A}^{\mathrm{op}})$ by (c). It also sends $\mathrm{D}^-_{\mathrm{coh}}(\mathscr{A}^{\mathrm{op}})$ to $\mathrm{D}^+_{\mathrm{coh}}(\mathscr{A})$, and the composition

$$\mathrm{D}^+_{\mathrm{coh}}(\mathscr{A}) \xrightarrow{\mathrm{D}} \mathrm{D}^-_{\mathrm{coh}}(\mathscr{A}^{\mathrm{op}}) \xrightarrow{\mathrm{D}} \mathrm{D}^+_{\mathrm{coh}}(\mathscr{A})$$

is isomorphic to the identity functor.

On the other hand, if L is an invertible $\mathscr{A}^{\mathrm{op}}$-module, then $\mathrm{D}(L)$ is an invertible $\mathscr{A}$-module, and by the hypothesis (a), we have

$$\mathrm{D}(L_U) \simeq \Gamma_U(\mathrm{D}(L))$$

for any $U \in \mathscr{S}$.

Then we get the result by applying Theorem 2.6.2 to $\mathrm{D}(\mathscr{M}) \in \mathrm{D}^-_{\mathrm{coh}}(\mathscr{A}^{\mathrm{op}})$ and using $\mathscr{M} \xrightarrow{\sim} \mathrm{D}(\mathrm{D}(\mathscr{M}))$. Q.E.D.

2.7 DQ-algebroids in the algebraic case

In this section, X denotes a quasi-compact separated smooth algebraic variety over $\mathbb{C}$.

Clearly, the notions of a DQ-algebra and of a DQ-algebroid make sense in this settings and a detailed study of DQ-algebroids on algebraic variety is performed in [65].

Assume that X is endowed with a DQ-algebroid $\mathscr{A}_X$ for the Zariski topology. Then, in view of Remark 2.1.17, $\mathrm{gr}_\hbar(\mathscr{A}_X) \simeq \mathscr{O}_X$. However, this equivalence is not unique in general.

Let us denote by X_{an} the complex analytic manifold associated with X and by $\rho\colon X_{\mathrm{an}} \to X$ the natural morphism. Then we can naturally associate a DQ-algebroid $\mathscr{A}_{X_{\mathrm{an}}}$ to $\mathscr{A}_X$ and there is a natural functor $\rho^{-1}\mathscr{A}_X \to \mathscr{A}_{X_{\mathrm{an}}}$, whose construction is left to the reader. Then it induces functors

$$\mathrm{Mod}(\mathscr{A}_X) \to \mathrm{Mod}(\mathscr{A}_{X_{\mathrm{an}}}) \tag{2.7.1}$$

and

$$\mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_X) \to \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_{X_{\mathrm{an}}}). \tag{2.7.2}$$

When X is projective, the classical GAGA theorem of Serre extends to DQ-algebroids and it is proved in [17] that (2.7.2) is an equivalence.

Lemma 2.7.1. *Let $\mathscr{M} \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_X^{\mathrm{loc}})$. The two conditions below are equivalent.*

(a) *$\mathscr{M}$ is the inductive limit of its coherent sub-$\mathscr{A}_X$-modules,*

(b) *there exists an $\mathscr{A}_X$-lattice of $\mathscr{M}$ (see Definition 2.3.14).*

Proof. (a)$\Rightarrow$(b) Let $\mathscr{M} = \varinjlim \mathscr{N}$ where $\mathscr{N}$ ranges over the filtrant family of coherent $\mathscr{A}_X$-submodules of $\mathscr{M}$. Since $\mathscr{A}_X^{\mathrm{loc}}$ is Noetherian, the family $\{\mathbb{C}^{\hbar,\mathrm{loc}} \otimes_{\mathbb{C}^\hbar} \mathscr{N}\}$ is locally stationary. Since X is quasi-compact, this family is stationary.

(b)$\Rightarrow$(a) is obvious. Q.E.D.

Definition 2.7.2. Let $\mathscr{M} \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_X^{\mathrm{loc}})$. We say that $\mathscr{M}$ is algebraically good if it satisfies the equivalent conditions in Lemma 2.7.1.

We still denote by $\mathrm{Mod}_{\mathrm{gd}}(\mathscr{A}_X^{\mathrm{loc}})$ the full subcategory of $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_X^{\mathrm{loc}})$ consisting of algebraically good modules.

The proof of [38, Prop. 4.23] extends to this case and $\mathrm{Mod}_{\mathrm{gd}}(\mathscr{A}_X^{\mathrm{loc}})$ is a thick abelian subcategory of $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_X^{\mathrm{loc}})$. Hence, we still denote by $\mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{A}_X^{\mathrm{loc}})$ the full triangulated subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X^{\mathrm{loc}})$ consisting of objects $\mathscr{M}$ such that $H^j(\mathscr{M})$ is algebraically good for all $j \in \mathbb{Z}$.

Remark 2.7.3. We do not know if every coherent $\mathscr{A}_X^{\mathrm{loc}}$-module is algebraically good.

Almost free resolutions

Recall that X is endowed with a DQ-algebroid $\mathscr{A}_X$ for the Zariski topology.

We denote by $\mathfrak{B}$ the family of affine open subsets U of X on which the algebroid $\mathscr{A}_U$ is a sheaf of algebras. Note that this family is stable by intersection. Moreover, hypotheses (1.2.2) and (1.2.3) are satisfied.

Lemma 2.7.4. *Assume that X is affine and $\mathscr{A}_X$ is a* DQ*-algebra. Then, for any $\mathscr{M} \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_X)$, there exist a free $\mathscr{A}_X$-module $\mathscr{L}$ of finite rank and an epimorphism $u\colon \mathscr{L} \twoheadrightarrow \mathscr{M}$.*

Proof. Set $\mathscr{M}_0 = \mathscr{M}/\hbar\mathscr{M}$. Then $\mathscr{M}_0$ is a coherent $\mathscr{O}_X$-module and there exist finitely many sections $(v_1, \dots, v_N)$ of $\mathscr{M}_0$ on X which generate $\mathscr{M}_0$ over $\mathscr{O}_X$.

By Theorem 1.2.5, the morphism $\Gamma(X; \mathscr{M}) \to \Gamma(X; \mathscr{M}_0)$ is surjective. Let $(u_1, \dots, u_N)$ be sections of $\mathscr{M}$ whose image by this morphism are $(v_1, \dots, v_N)$. Let $\mathscr{L} = \mathscr{A}_X^N$ and denote by $(e_1, \dots, e_N)$ its canonical basis. It remains to define u by setting $u(e_i) = u_i$. Q.E.D.

Theorem 2.7.5. *Let $\mathscr{M} \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_X)$. Then there exists an isomorphism $\mathscr{M} \simeq \mathscr{L}^\bullet$ in $\mathrm{D^b}(\mathscr{A}_X)$ such that $\mathscr{L}^\bullet$ is a bounded complex of $\mathscr{A}_X$-modules and each $\mathscr{L}^i$ is a finite direct sum of modules of the form $i_{U*}\mathscr{L}_U$, where $i_U\colon U \hookrightarrow X$ is the embedding of an affine open set U such that $\mathscr{A}_U$ is isomorphic to a DQ-algebra and $\mathscr{L}_U$ is a locally free $\mathscr{A}_U$-module of finite rank.*

Before proving Theorem 2.7.5, we need some preliminary results.

Let $\mathscr{U} = \{U_i\}_{i\in I}$ be a finite covering of X by affine open sets such that $\mathscr{A}_X|_{U_i}$ is a DQ-algebra for all i.

We denote by Σ the category of non empty subsets of I (the morphisms are the inclusions maps). For $\sigma \in \Sigma$, we denote by $|\sigma|$ its cardinal. For $\sigma \in \Sigma$, we set

$$U_\sigma = \bigcap_{i\in\sigma} U_i, \quad \iota_\sigma\colon U_\sigma \hookrightarrow X \text{ the natural embedding.}$$

We introduce a category $\mathrm{Mod}(\mathscr{A}, \mathscr{U})$ as follows. An object M of $\mathrm{Mod}(\mathscr{A}, \mathscr{U})$ is the data of a family $(\{M_\sigma\}_{\sigma\in\Sigma}, \{q^M_{\sigma,\tau}\}_{\tau\subset\sigma\in\Sigma})$, where $M_\sigma \in \mathrm{Mod}(\mathscr{A}_{U_\sigma})$ and $q^M_{\sigma,\tau}\colon M_\tau|_{U_\sigma} \to M_\sigma$ are morphisms for $\emptyset \neq \tau \subset \sigma \in \Sigma$ satisfying $q^M_{\sigma,\sigma} = \mathrm{id}$ and for any $\sigma_1 \subset \sigma_2 \subset \sigma_3$, the diagram below commutes

$$\begin{tikzcd} M_{\sigma_1}|_{U_{\sigma_3}} \arrow[r, "q^M_{\sigma_2,\sigma_1}"] \arrow[dr, "q^M_{\sigma_3,\sigma_1}"'] & M_{\sigma_2}|_{U_{\sigma_3}} \arrow[d, "q^M_{\sigma_3,\sigma_2}"] \\ & M_{\sigma_3}. \end{tikzcd} \tag{2.7.3}$$

A morphism $M \to M'$ in $\mathrm{Mod}(\mathscr{A}, \mathscr{U})$ is a family of morphisms $M_\sigma \to M'_\sigma$ satisfying the natural compatibility conditions.

Clearly, $\mathrm{Mod}(\mathscr{A}, \mathscr{U})$ is an abelian category.

To an object $M \in \mathrm{Mod}(\mathscr{A}, \mathscr{U})$ we shall associate a Koszul complex $C^\bullet(M)$ using the construction of [42, § 12.4]. To M we associate a functor

$F : \Sigma \to \mathrm{Mod}(\mathscr{A}_X)$ as follows: $F(\sigma) = \iota_{\sigma *} M_\sigma$, and $F(\tau \subset \sigma) : F(\tau) \to F(\sigma)$ is given by the composition $\iota_{\tau *} M_\tau \to \iota_{\sigma *}(M_\tau|_{U_\sigma}) \xrightarrow{q^M_{\sigma,\tau}} \iota_{\sigma *} M_\sigma$.

According to loc. cit., we get a Koszul complex $C^\bullet(M)$

$$C^\bullet(M) := \cdots \longrightarrow 0 \longrightarrow C^1(M) \xrightarrow{d^1} C^2(M) \xrightarrow{d^2} \cdots \tag{2.7.4}$$

where

$$C^i(M) = \bigoplus_{|\sigma|=i} \iota_{\sigma *} M_\sigma$$

is in degree i. This construction being functorial, we get a functor

$$C^\bullet : \mathrm{Mod}(\mathscr{A}, \mathscr{U}) \to \mathrm{C}^{\mathrm{b}}(\mathrm{Mod}(\mathscr{A}_X)). \tag{2.7.5}$$

It is convenient to introduce some notations. We set

$$\begin{aligned} \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}, \mathscr{U}) &= \{M \in \mathrm{Mod}(\mathscr{A}, \mathscr{U})\,;\, M_\sigma \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_{U_\sigma}) \text{ for all } \sigma \in \Sigma\}, \\ \mathrm{Mod}_{\mathrm{ff}}(\mathscr{A}, \mathscr{U}) &= \{M \in \mathrm{Mod}(\mathscr{A}, \mathscr{U}); M_\sigma \text{ is a locally free } \mathscr{A}_{U_\sigma}\text{-module} \\ &\qquad\qquad \text{of finite rank for all } \sigma \in \Sigma\}. \end{aligned}$$

Clearly, $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}, \mathscr{U})$ is a full abelian subcategory of $\mathrm{Mod}(\mathscr{A}, \mathscr{U})$ and $\mathrm{Mod}_{\mathrm{ff}}(\mathscr{A}, \mathscr{U})$ is a full additive subcategory of $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}, \mathscr{U})$.

Lemma 2.7.6. *The functor* $C^\bullet : \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}, \mathscr{U}) \to \mathrm{C}^{\mathrm{b}}(\mathrm{Mod}(\mathscr{A}_X))$ *induced by* (2.7.5) *is exact.*

Proof. By Proposition 1.6.8 the functor $\iota_\sigma : \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_{U_\sigma}) \to \mathrm{Mod}(\mathscr{A}_X)$ is exact for each $\sigma \in \Sigma$. The result then easily follows. Q.E.D.

Let us denote by

$$\lambda : \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_X) \to \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}, \mathscr{U}) \tag{2.7.6}$$

the functor which, to $\mathscr{M} \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_X)$, associates the object M where $M_\sigma = \mathscr{M}|_{U_\sigma}$ and $q^M_{\sigma,\tau} : M_\tau|_{U_\sigma} \to M_\sigma$ is the restriction morphism.

Lemma 2.7.7. *The natural morphism* $\mathscr{M} \to C^\bullet(\lambda(\mathscr{M}))\,[1]$ *is a quasi-isomorphism.*

Proof. Apply [42, Th. 18.7.4 (ii)] with $A = \text{“}\bigsqcup_{i \in I}\text{”}\, U_i$, $u : A \to X$. By this result, the complex

$$F_u^\bullet := 0 \to \mathscr{M} \to C^1(\lambda(\mathscr{M})) \xrightarrow{d^1} C^2(\lambda(\mathscr{M})) \xrightarrow{d^2} \cdots$$

is exact. Q.E.D.

Lemma 2.7.8. *Let $M \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}, \mathscr{U})$. Then there exists an epimorphism $L \twoheadrightarrow M$ in $\mathrm{Mod}(\mathscr{A}, \mathscr{U})$ with $L \in \mathrm{Mod}_{\mathrm{ff}}(\mathscr{A}, \mathscr{U})$.*

Proof. Applying Lemma 2.7.4, we choose for each $\sigma \in \Sigma$ an epimorphism $L'_\sigma \twoheadrightarrow M_\sigma$ with a locally free $\mathscr{A}_{U_\sigma}$-module L'_σ of finite rank. Set

$$L_\sigma := \bigoplus_{\emptyset \neq \tau \subset \sigma} L'_\tau|_{U_\sigma}$$

and define the morphism $L_\sigma \to M_\sigma$ by the commutative diagrams in which $\tau \subset \sigma$:

$$\begin{tikzcd} L_\sigma \arrow[r] & M_\sigma \\ L'_\tau|_{U_\sigma} \arrow[u] \arrow[r] & M_\tau|_{U_\sigma}. \arrow[u] \end{tikzcd}$$

For $\tau \subset \sigma$, the morphism $q^L_{\sigma,\tau} \colon L_\tau|_{U_\sigma} \to L_\sigma$ is defined by the morphisms $(\lambda \subset \tau)$:

$$\begin{tikzcd} L_\tau|_{U_\sigma} \arrow[r, "q^L_{\sigma,\tau}"] & L_\sigma. \\ L'_\lambda|_{U_\sigma} \arrow[u] \arrow[ur] & \end{tikzcd}$$

Clearly, the family of morphisms $q^L_{\sigma,\tau}$ satisfies the compatibility conditions similar to those in diagram (2.7.3). We have thus constructed an object $L \in \mathrm{Mod}(\mathscr{A}, \mathscr{U})$, and the family of morphisms $L_\sigma \to M_\sigma$ defines the epimorphism $L \twoheadrightarrow M$ in $\mathrm{Mod}(\mathscr{A}, \mathscr{U})$. Q.E.D.

Proof of Theorem 2.7.5. By Lemma 2.7.8, there exists an exact sequence in $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}, \mathscr{U})$

$$0 \longrightarrow L_{d_X+1} \longrightarrow \cdots \to L_1 \to L_0 \to \lambda(\mathscr{M}) \to 0 \tag{2.7.7}$$

with the L_i's in $\mathrm{Mod}_{\mathrm{ff}}(\mathscr{A}, \mathscr{U})$ (see Corollary 2.3.5). Consider the complex

$$L^\bullet := \cdots \to L_1 \to L_0 \to 0. \tag{2.7.8}$$

Hence, we have a quasi-isomorphism $L^\bullet \xrightarrow{qis} \lambda(\mathscr{M})$. Using Lemma 2.7.6, we find a quasi-isomorphism

$$C^\bullet(L^\bullet) \xrightarrow{qis} C^\bullet(\lambda(\mathscr{M})). \tag{2.7.9}$$

Then, the result follows from Lemma 2.7.7. Q.E.D.

Chapter 3

Kernels

3.1 Convolution of kernels: definition

Integral transforms, also called "correspondences", are of constant use in algebraic and analytic geometry and we refer to the book [34] for an exposition. Here, we shall develop a similar formalism in the framework of DQ-modules (*i.e.,* modules over DQ-algebroids).

Consider complex manifolds X_i ($i = 1, 2$, etc.) endowed with DQ-algebroids $\mathscr{A}_{X_i}$.

Notation 3.1.1. (i) Consider a product of manifolds $X \times Y \times Z$. We denote by p_i the i-th projection and by p_{ij} the (i,j)-th projection (*e.g.,* p_{13} is the projection from $X_1 \times X_1^a \times X_2$ to $X_1 \times X_2$). We use similar notations for a product of four manifolds.

(ii) We write $\mathscr{A}_i$ and $\mathscr{A}_{ij^a}$ instead of $\mathscr{A}_{X_i}$ and $\mathscr{A}_{X_i \times X_j^a}$ and similarly with other products. We use the same notations for $\mathscr{C}_{X_i}$.

(iii) When there is no risk of confusion, we do note write the symbols p_i^{-1} and similarly with i replaced with ij, etc.

Let $\mathscr{K}_i \in \mathrm{D}^{\mathrm{b}}(\mathscr{A}_{X_i \times X_{i+1}^a})$ $(i = 1, 2)$. We set

$$
\begin{aligned}
\mathscr{K}_1 \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_2} \mathscr{K}_2 &:= p_{12}^{-1} \mathscr{K}_1 \overset{\mathrm{L}}{\otimes}_{p_2^{-1}\mathscr{A}_2} p_{23}^{-1} \mathscr{K}_2 \\
&\simeq (\mathscr{K}_1 \overset{\mathrm{L}}{\boxtimes} \mathscr{K}_2) \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_2 \boxtimes \mathscr{A}_{2^a}} \mathscr{C}_2 \in \mathrm{D}^{\mathrm{b}}(\mathscr{A}_1 \boxtimes \mathbb{C}^{\hbar}_{X_2} \boxtimes \mathscr{A}_{3^a}).
\end{aligned} \tag{3.1.1}
$$

Similarly, for $\mathscr{K}_i \in \mathrm{D}^{\mathrm{b}}(\mathscr{A}_{X_i \times X_{i+1}})$ $(i = 1, 2)$, we set

$$
\mathrm{R}\mathscr{H}om_{\mathscr{A}_2}(\mathscr{K}_1, \mathscr{K}_2) := \mathrm{R}\mathscr{H}om_{p_2^{-1}\mathscr{A}_2}(p_{12}^{-1}\mathscr{K}_1, p_{23}^{-1}\mathscr{K}_2). \tag{3.1.2}
$$

Here we identify $X_1 \times X_2 \times X_3^a$ with the diagonal set of $X_1 \times X_2^a \times X_2 \times X_3^a$.

This tensor product is not well suited to treat DQ-modules. For example, $\mathscr{A}_{X\times Y} \neq \mathscr{A}_X \boxtimes \mathscr{A}_Y$. This leads us to introduce a kind of completion of the tensor product as follows.

Definition 3.1.2. Let $\mathscr{K}_i \in \mathrm{D^b}(\mathscr{A}_{X_i\times X_{i+1}^a})$ $(i = 1, 2)$. We set

$$(3.1.3)\qquad \begin{aligned} \mathscr{K}_1 \underline{\overset{\mathrm{L}}{\otimes}}_{\mathscr{A}_2} \mathscr{K}_2 &= \delta_2^{-1}(\mathscr{K}_1 \underline{\overset{\mathrm{L}}{\boxtimes}} \mathscr{K}_2) \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_{22^a}} \mathscr{C}_2 \\ &= p_{12}^{-1}\mathscr{K}_1 \overset{\mathrm{L}}{\otimes}_{p_{12}^{-1}\mathscr{A}_{1^a2}} \mathscr{A}_{123} \overset{\mathrm{L}}{\otimes}_{p_{23}^{-1}\mathscr{A}_{23^a}} p_{23}^{-1}\mathscr{K}_2. \end{aligned}$$

It is an object of $\mathrm{D^b}(p_{13}^{-1}\mathscr{A}_{13^a})$ where $p_{13}\colon X_1 \times X_2 \times X_3 \to X_1 \times X_3$ is the projection.

We have a morphism in $\mathrm{D^b}(p_1^{-1}\mathscr{A}_{X_1} \otimes p_3^{-1}\mathscr{A}_{X_3^a})$:

$$(3.1.4)\qquad \mathscr{K}_1 \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_2} \mathscr{K}_2 \to \mathscr{K}_1 \underline{\overset{\mathrm{L}}{\otimes}}_{\mathscr{A}_2} \mathscr{K}_2.$$

Note that (3.1.4) is an isomorphism if $X_1 = \mathrm{pt}$ or $X_3 = \mathrm{pt}$.

Definition 3.1.3. Let $\mathscr{K}_i \in \mathrm{D^b}(\mathscr{A}_{X_i\times X_{i+1}^a})$ $(i = 1, 2)$. We set

$$(3.1.5)\qquad \mathscr{K}_1 \underset{X_2}{\circ} \mathscr{K}_2 = \mathrm{R}p_{13!}(\mathscr{K}_1 \underline{\overset{\mathrm{L}}{\otimes}}_{\mathscr{A}_2} \mathscr{K}_2) \in \mathrm{D^b}(\mathscr{A}_{X_1\times X_3^a}),$$

$$(3.1.6)\qquad \mathscr{K}_1 \underset{X_2}{*} \mathscr{K}_2 = \mathrm{R}p_{13*}(\mathscr{K}_1 \underline{\overset{\mathrm{L}}{\otimes}}_{\mathscr{A}_2} \mathscr{K}_2) \in \mathrm{D^b}(\mathscr{A}_{X_1\times X_3^a}).$$

We call $\underset{X_2}{\circ}$ the convolution of $\mathscr{K}_1$ and $\mathscr{K}_2$ (over X_2). If there is no risk of confusion, we write $\mathscr{K}_1 \circ \mathscr{K}_2$ for $\mathscr{K}_1 \underset{X_2}{\circ} \mathscr{K}_2$ and similarly with $*$.

Note that in case where $X_3 = \mathrm{pt}$ we get:

$$\mathscr{K}_1 \circ \mathscr{K}_2 \simeq \mathrm{R}p_{1!}(\mathscr{K}_1 \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_2} p_2^{-1}\mathscr{K}_2),$$

and in the general case, we have:

$$(3.1.7)\qquad \begin{aligned} \mathscr{K}_1 \underset{X_2}{\circ} \mathscr{K}_2 &\simeq (\mathscr{K}_1 \underline{\overset{\mathrm{L}}{\boxtimes}} \mathscr{K}_2) \underset{X_2\times X_2^a}{\circ} \mathscr{C}_{X_2} \\ &\simeq \mathrm{R}p_{14!}\big((\mathscr{K}_1 \underline{\overset{\mathrm{L}}{\boxtimes}} \mathscr{K}_2) \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_{22^a}} \mathscr{C}_2\big), \end{aligned}$$

where p_{14} is the projection $X_1 \times X_2 \times X_2^a \times X_3^a \to X_1 \times X_3^a$. There are canonical isomorphisms

$$(3.1.8)\qquad \mathscr{K}_1 \underset{X_2}{\circ} \mathscr{C}_{X_2} \simeq \mathscr{K}_1 \quad\text{and}\quad \mathscr{C}_{X_1} \underset{X_1}{\circ} \mathscr{K}_1 \simeq \mathscr{K}_1.$$

One shall be aware that $\circ$ and $*$ are not associative in general. (See Proposition 3.2.4 (ii).)

However, if $\mathscr{L}$ is a bi-invertible $\mathscr{A}_{X_2}\otimes\mathscr{A}_{X_2^a}$-module and the $\mathscr{K}_i$'s $(i=1,2)$ are as above, there are natural isomorphisms

$$\begin{aligned}
&\mathscr{K}_1 \underset{X_2}{\circ} \mathscr{L} \simeq \mathscr{K}_1 \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_{X_2}} \mathscr{L}, \quad \mathscr{L} \underset{X_2}{\circ} \mathscr{K}_2 \simeq \mathscr{L} \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_{X_2}} \mathscr{K}_2,\\
&(\mathscr{K}_1 \underset{X_2}{\circ} \mathscr{L}) \underset{X_2}{\circ} \mathscr{K}_2 \simeq \mathscr{K}_1 \underset{X_2}{\circ} (\mathscr{L} \underset{X_2}{\circ} \mathscr{K}_2).
\end{aligned}$$

For a closed subset Λ_i of $X_i \times X_{i+1}$ $(i=1,2)$, we set

$$\begin{aligned}
(3.1.9)\quad \Lambda_1 \circ \Lambda_2 \;&:=\; p_{13}(p_{12}^{-1}\Lambda_1 \cap p_{23}^{-1}\Lambda_2)\\
&=\; p_{14}((\Lambda_1 \times \Lambda_2) \cap (X_1 \times \Delta_2 \times X_3)) \subset X_1 \times X_3.
\end{aligned}$$

Note that if Λ_i is a closed complex analytic subvariety of $X_i \times X_{i+1}^a$ $(i=1,2)$ and p_{13} is proper on $p_{12}^{-1}\Lambda_1 \cap p_{23}^{-1}\Lambda_2$, then $\Lambda_1 \circ \Lambda_2$ is a closed complex analytic subvariety of $X_1 \times X_3^a$.

Let us still denote by $\circ$ the convolution of $\mathrm{gr}_\hbar(\mathscr{A})$-modules. More precisely for $\mathscr{L}_i \in \mathrm{D}^{\mathrm{b}}(\mathrm{gr}_\hbar(\mathscr{A}_{X_i \times X_{i+1}^a}))$ $(i=1,2)$, we set

$$\mathscr{L}_1 \circ \mathscr{L}_2 \;=\; \mathrm{R}p_{14!}\big((\mathscr{L}_1 \overset{\mathrm{L}}{\underline{\boxtimes}} \mathscr{L}_2) \overset{\mathrm{L}}{\otimes}_{\mathrm{gr}_\hbar(\mathscr{A}_{22^a})} \mathrm{gr}_\hbar(\mathscr{C}_2)\big).$$

Proposition 3.1.4. *For $\mathscr{K}_i \in \mathrm{D}^{\mathrm{b}}(\mathscr{A}_{X_i \times X_{i+1}^a})$ $(i=1,2)$, we have*

$$(3.1.10)\qquad \mathrm{gr}_\hbar(\mathscr{K}_1 \circ \mathscr{K}_2) \simeq \mathrm{gr}_\hbar(\mathscr{K}_1) \circ \mathrm{gr}_\hbar(\mathscr{K}_2).$$

Proof. Applying Proposition 1.4.3, it remains to remark that the functor $\mathrm{gr}_\hbar$ commutes with the functors of inverse images and proper direct images as well as with the functor $\underline{\boxtimes}$. Q.E.D.

3.2 Convolution of kernels: finiteness

In this section, we use Notation 3.1.1

Consider complex manifolds X_i $(i=1,2,3)$ endowed with DQ-algebroids $\mathscr{A}_{X_i}$ $(i=1,2,3,$ etc.$)$. We denote by d_X the complex dimension of X and we write for short d_i instead of d_{X_i}.

We shall prove the following coherency theorem for DQ-modules by reducing it to the corresponding result for $\mathscr{O}$-modules due to Grauert ([30]). In the sequel, for a closed subset Λ of X, we denote by $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh},\Lambda}(\mathscr{A}_X)$ the full triangulated subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X)$ consisting of objects supported by Λ. We define similarly $\mathrm{D}^{\mathrm{b}}_{\mathrm{gd},\Lambda}(\mathscr{A}_X^{\mathrm{loc}})$.

Theorem 3.2.1. *For $i = 1, 2$, let Λ_i be a closed subset of $X_i \times X_{i+1}$ and $\mathscr{K}_i \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh},\Lambda_i}(\mathscr{A}_{X_i \times X^a_{i+1}})$. Assume that $\Lambda_1 \times_{X_2} \Lambda_2$ is proper over $X_1 \times X_3$, and set $\Lambda = \Lambda_1 \circ \Lambda_2$. Then the object $\mathscr{K}_1 \circ \mathscr{K}_2$ belongs to $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh},\Lambda}(\mathscr{A}_{X_1 \times X^a_3})$.*

Proof. Since the question is local in X_1 and X_3, we may assume from the beginning that $\mathscr{A}_{X_1}$ and $\mathscr{A}_{X_3}$ are DQ-algebras.

We shall first show that

$$\mathscr{K}_1 \overset{\mathrm{L}}{\underline{\otimes}}_{\mathscr{A}_2} \mathscr{K}_2 \text{ is cohomologically complete.} \tag{3.2.1}$$

Since this statement is a local statement on $X_1 \times X_2 \times X_3$, we may assume that $\mathscr{A}_{X_2}$ is a DQ-algebra. Since $\mathscr{K}_1$ and $\mathscr{K}_2$ may be locally represented by finite complexes of free modules of finite rank, in order to see (3.2.1), we may assume $\mathscr{K}_i \simeq \mathscr{A}_{X_i \times X^a_{i+1}}$ $(i = 1, 2)$. Then $\mathscr{K}_1 \overset{\mathrm{L}}{\underline{\otimes}}_{\mathscr{A}_2} \mathscr{K}_2 \simeq \mathscr{A}_{X_1 \times X_2 \times X^a_3}$ is cohomologically complete by Theorem 1.6.1. Hence $\mathscr{K}_1 \circ \mathscr{K}_2 = \mathrm{R}p_{13*}(\mathscr{K}_1 \overset{\mathrm{L}}{\underline{\otimes}}_{\mathscr{A}_2} \mathscr{K}_2)$ is also cohomologically complete by Proposition 1.5.12.

On the other hand, $\mathrm{gr}_\hbar(\mathscr{K}_1 \circ \mathscr{K}_2) \simeq \mathrm{R}p_{13*}\big(p^*_{12}\mathrm{gr}_\hbar\mathscr{K}_1 \overset{\mathrm{L}}{\otimes}_{\mathscr{O}_{X_1 \times X_2 \times X_3}} p^*_{23}\mathrm{gr}_\hbar\mathscr{K}_2\big)$ belongs to $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_{X_1 \times X_3})$ by Grauert's direct image theorem ([30]). Hence Theorem 1.6.4 implies that $\mathscr{K}_1 \circ \mathscr{K}_2$ belongs to $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_{X_1 \times X^a_3})$. Q.E.D.

Remark 3.2.2. In [4], its authors use a variant of Theorem 3.2.1 in the symplectic case. They assert that the proof follows from Houzel's finiteness theorem on modules over sheaves of multiplicatively convex nuclear Fréchet algebras (see [33]). However, they do not give any proof, details being qualified of "routine".

Corollary 3.2.3. *Let $\mathscr{M}$ and $\mathscr{N}$ be two objects of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X)$ and assume that $\mathrm{Supp}(\mathscr{M}) \cap \mathrm{Supp}(\mathscr{N})$ is compact. Then the object $\mathrm{RHom}_{\mathscr{A}_X}(\mathscr{M}, \mathscr{N})$ belongs to $\mathrm{D}^{\mathrm{b}}_f(\mathbb{C}^\hbar)$.*

Proposition 3.2.4. *Let $\mathscr{K}_i \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_{X_i \times X^a_{i+1}})$ $(i = 1, 2, 3)$ and let $\mathscr{L} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_{X_4})$. Set $\Lambda_i = \mathrm{supp}(\mathscr{K}_i)$ and assume that $\Lambda_i \times_{X_{i+1}} \Lambda_{i+1}$ is proper over $X_i \times X_{i+2}$ $(i = 1, 2)$.*

(i) *There is a canonical isomorphism $(\mathscr{K}_1 \underset{X_2}{\circ} \mathscr{K}_2) \overset{\mathrm{L}}{\underline{\boxtimes}} \mathscr{L} \xrightarrow{\sim} \mathscr{K}_1 \underset{X_2}{\circ} (\mathscr{K}_2 \overset{\mathrm{L}}{\underline{\boxtimes}} \mathscr{L})$.*

(ii) *There is a canonical isomorphism $(\mathscr{K}_1 \underset{X_2}{\circ} \mathscr{K}_2) \underset{X_3}{\circ} \mathscr{K}_3 \simeq \mathscr{K}_1 \underset{X_2}{\circ} (\mathscr{K}_2 \underset{X_3}{\circ} \mathscr{K}_3)$.*

Proof. The morphism $(\mathscr{K}_1 \underset{X_2}{\circ} \mathscr{K}_2)\overset{\mathrm{L}}{\underline{\boxtimes}}\mathscr{L} \to \mathscr{K}_1 \underset{X_2}{\circ} (\mathscr{K}_2\overset{\mathrm{L}}{\underline{\boxtimes}}\mathscr{L})$ is deduced from the morphism (we do not write the functors p_i^{-1}, p_{ij}^{-1} for short):

$$\begin{aligned}
&\mathscr{A}_{13^a4}\otimes_{\mathscr{A}_{13^a}\boxtimes\mathscr{A}_4}\Big(\big((\mathscr{A}_{12^a23^a}\otimes_{\mathscr{A}_{12^a}\boxtimes\mathscr{A}_{23^a}}(\mathscr{K}_1\overset{\mathrm{L}}{\boxtimes}\mathscr{K}_2))\overset{\mathrm{L}}{\otimes}_{\mathscr{A}_{22^a}}\mathscr{C}_2\big)\overset{\mathrm{L}}{\boxtimes}\mathscr{L}\Big)\\
&\qquad\simeq\big((\mathscr{A}_{13^a4}\otimes_{\mathscr{A}_{13^a}\boxtimes\mathscr{A}_4}\mathscr{A}_{12^a23})\otimes_{\mathscr{A}_{12^a}\boxtimes\mathscr{A}_{23^a}\boxtimes\mathscr{A}_4}\mathscr{K}_1\overset{\mathrm{L}}{\boxtimes}\mathscr{K}_2\overset{\mathrm{L}}{\boxtimes}\mathscr{L}\big)\overset{\mathrm{L}}{\otimes}_{\mathscr{A}_{22^a}}\mathscr{C}_2\\
&\qquad\to\big(\mathscr{A}_{12^a23^a4}\otimes_{\mathscr{A}_{12^a}\boxtimes\mathscr{A}_{23^a}\boxtimes\mathscr{A}_4}(\mathscr{K}_1\overset{\mathrm{L}}{\boxtimes}\mathscr{K}_2\overset{\mathrm{L}}{\boxtimes}\mathscr{L})\big)\overset{\mathrm{L}}{\otimes}_{\mathscr{A}_{22^a}}\mathscr{C}_2.
\end{aligned}$$

Applying the functor $\mathrm{gr}_\hbar$ to this morphism in $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_{X_1\times X_3^a\times X^4})$, we get an isomorphism. This proves the result in view of Corollary 1.4.6.
(ii) By (i), we have

$$\begin{aligned}
(\mathscr{K}_1 \underset{X_2}{\circ} \mathscr{K}_2)\underset{X_3}{\circ}\mathscr{K}_3 &\simeq \big((\mathscr{K}_1 \underset{X_2}{\circ} \mathscr{K}_2)\overset{\mathrm{L}}{\underline{\boxtimes}}\mathscr{K}_3\big)\underset{X_3\times X_3^a}{\circ}\mathscr{C}_{X_3}\\
&\simeq \big(\mathscr{K}_1 \underset{X_2}{\circ}(\mathscr{K}_2\overset{\mathrm{L}}{\underline{\boxtimes}}\mathscr{K}_3)\big)\underset{X_3\times X_3^a}{\circ}\mathscr{C}_{X_3}\\
&\simeq \Big(\mathscr{C}_{X_2}\underset{X_2\times X_2^a}{\circ}(\mathscr{K}_1\overset{\mathrm{L}}{\underline{\boxtimes}}\mathscr{K}_2\overset{\mathrm{L}}{\underline{\boxtimes}}\mathscr{K}_3)\Big)\underset{X_3\times X_3^a}{\circ}\mathscr{C}_{X_3}.
\end{aligned}$$

Then this object is isomorphic to $(\mathscr{K}_1\overset{\mathrm{L}}{\underline{\boxtimes}}\mathscr{K}_2\overset{\mathrm{L}}{\underline{\boxtimes}}\mathscr{K}_3)\underset{X_2\times X_2^a\times X_3\times X_3^a}{\circ}(\mathscr{C}_{X_2}\overset{\mathrm{L}}{\underline{\boxtimes}}\mathscr{C}_{X_3})$.
Similarly, $\mathscr{K}_1 \underset{X_2}{\circ}(\mathscr{K}_2 \underset{X_3}{\circ}\mathscr{K}_3)$ is isomorphic to $(\mathscr{K}_1\overset{\mathrm{L}}{\underline{\boxtimes}}\mathscr{K}_2\overset{\mathrm{L}}{\underline{\boxtimes}}\mathscr{K}_3)\underset{X_2\times X_2^a\times X_3\times X_3^a}{\circ}(\mathscr{C}_{X_2}\overset{\mathrm{L}}{\underline{\boxtimes}}\mathscr{C}_{X_3})$.
Q.E.D.

3.3 Convolution of kernels: duality

The duality morphism for kernels

Denote as usual by $p_{13}\colon X_1\times X_2\times X_3^a\to X_1\times X_3^a$ the projection.

Lemma 3.3.1. *For $\mathscr{K}_i\in\mathrm{D}^{\mathrm{b}}(\mathscr{A}_{X_i\times X_{i+1}^a})$ $(i=1,2)$, we have a natural morphism in $\mathrm{D}^{\mathrm{b}}(\mathscr{A}_{X_1^a\times X_3})$:*

$$(\mathrm{D}'_{\mathscr{A}_{X_1\times X_2^a}}\mathscr{K}_1)\underset{X_2^a}{\circ}\omega^{\mathscr{A}}_{X_2^a}\underset{X_2^a}{\circ}(\mathrm{D}'_{\mathscr{A}_{X_2\times X_3^a}}\mathscr{K}_2)\to\mathrm{D}'_{\mathscr{A}_{X_1\times X_3^a}}(\mathscr{K}_1\underset{X_2}{\circ}\mathscr{K}_2).\tag{3.3.1}$$

Proof. We have

$$\begin{aligned}
\mathrm{D}'_{\mathscr{A}}\mathscr{K}_1\overset{\mathrm{L}}{\underline{\otimes}}_{\mathscr{A}_{2^a}}\omega^{\mathscr{A}}_{2^a}\overset{\mathrm{L}}{\underline{\otimes}}_{\mathscr{A}_{2^a}}\mathrm{D}'_{\mathscr{A}}\mathscr{K}_2 &\simeq (\mathrm{D}'_{\mathscr{A}}\mathscr{K}_1\overset{\mathrm{L}}{\underline{\boxtimes}}\mathrm{D}'_{\mathscr{A}}\mathscr{K}_2)\overset{\mathrm{L}}{\otimes}_{\mathscr{A}_{2^a2}}\omega^{\mathscr{A}}_{2^a}\\
&\simeq (\mathrm{D}'_{\mathscr{A}}\mathscr{K}_1\overset{\mathrm{L}}{\underline{\boxtimes}}\mathrm{D}'_{\mathscr{A}}\mathscr{K}_2)\overset{\mathrm{L}}{\otimes}_{\mathscr{A}_{12^a23^a}}\omega^{\mathscr{A}}_{12^a3^a/13^a}\\
&\simeq \mathrm{D}'_{\mathscr{A}}(\mathscr{K}_1\overset{\mathrm{L}}{\underline{\boxtimes}}\mathscr{K}_2)\overset{\mathrm{L}}{\otimes}_{\mathscr{A}_{12^a23^a}}\omega^{\mathscr{A}}_{12^a3^a/13^a}\\
&\simeq \mathrm{R}\mathscr{H}om_{\mathscr{A}_{12^a23^a}}(\mathscr{K}_1\overset{\mathrm{L}}{\underline{\boxtimes}}\mathscr{K}_2,\omega^{\mathscr{A}}_{12^a3^a/13^a}).
\end{aligned}$$

Hence we have morphisms

$$\begin{aligned}
\mathrm{D}'_{\mathscr{A}}\mathscr{K}_1\overset{\mathrm{L}}{\underline{\otimes}}_{\mathscr{A}_{2^a}}\omega^{\mathscr{A}}_{2^a}\overset{\mathrm{L}}{\underline{\otimes}}_{\mathscr{A}_{2^a}}\mathrm{D}'_{\mathscr{A}}\mathscr{K}_2 &\simeq \mathrm{R}\mathscr{H}om_{\mathscr{A}_{12^a23^a}}(\mathscr{K}_1\overset{\mathrm{L}}{\underline{\boxtimes}}\mathscr{K}_2,\omega^{\mathscr{A}}_{12^a3^a/13^a})\\
&\to \mathrm{R}\mathscr{H}om_{p_{13}^{-1}\mathscr{A}_{13^a}}\big((\mathscr{K}_1\overset{\mathrm{L}}{\underline{\boxtimes}}\mathscr{K}_2)\overset{\mathrm{L}}{\otimes}_{\mathscr{A}_{22^a}}\mathscr{C}_2,\omega^{\mathscr{A}}_{12^a3^a/13^a}\overset{\mathrm{L}}{\otimes}_{\mathscr{A}_{22^a}}\mathscr{C}_2\big)\\
&\to \mathrm{R}\mathscr{H}om_{p_{13}^{-1}\mathscr{A}_{13^a}}(\mathscr{K}_1\overset{\mathrm{L}}{\underline{\otimes}}_{\mathscr{A}_2}\mathscr{K}_2,p_{13}^{-1}\mathscr{A}_{13^a}[2d_2]).
\end{aligned}$$

The last arrow is induced by (2.5.7). Taking $\mathrm{R}p_{13!}$, we obtain

$$\begin{aligned}
(\mathrm{D}'_{\mathscr{A}}\mathscr{K}_1)\underset{X_2^a}{\circ}\omega^{\mathscr{A}}_{X_2^a}\underset{X_2^a}{\circ}(\mathrm{D}'_{\mathscr{A}}\mathscr{K}_2) &\simeq \mathrm{R}p_{13!}\big((\mathrm{D}'_{\mathscr{A}}\mathscr{K}_1)\overset{\mathrm{L}}{\underline{\otimes}}_{\mathscr{A}_{2^a}}\omega^{\mathscr{A}}_{2^a}\overset{\mathrm{L}}{\underline{\otimes}}_{\mathscr{A}_{2^a}}(\mathrm{D}'_{\mathscr{A}}\mathscr{K}_2)\big)\\
&\to \mathrm{R}p_{13*}\mathrm{R}\mathscr{H}om_{p_{13}^{-1}\mathscr{A}_{13^a}}(\mathscr{K}_1\overset{\mathrm{L}}{\underline{\otimes}}_{\mathscr{A}_2}\mathscr{K}_2,p_{13}^{-1}\mathscr{A}_{13^a}[2d_2])\\
&\xrightarrow{\sim} \mathrm{R}\mathscr{H}om_{\mathscr{A}_{13^a}}(\mathscr{K}_1\underset{X_2}{\circ}\mathscr{K}_2,\mathscr{A}_{13^a}).
\end{aligned}$$

Here the last isomorphism is given by the Poincaré duality. Q.E.D.

Serre duality

Let us recall the Serre duality for $\mathscr{O}$-modules. Let X and Y be complex manifolds. Denote by $f\colon X\times Y\to X$ the projection, by $\omega_Y=\Omega_Y^{d_Y}[d_Y]$ the dualizing complex on Y and by $\omega_{X\times Y/X}:=\mathscr{O}_X\overset{\mathrm{L}}{\underline{\boxtimes}}\omega_Y$ the relative dualizing complex. For $\mathscr{G}\in\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_X)$, we set

$$f^!\mathscr{G}=f^{-1}\mathscr{G}\overset{\mathrm{L}}{\otimes}_{f^{-1}\mathscr{O}_X}\omega_{X\times Y/X}.$$

Theorem 3.3.2. *For $\mathscr{F}\in\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_{X\times Y})$ and $\mathscr{G}\in\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_X)$, we have a morphism*

$$\mathrm{R}f_*\mathrm{R}\mathscr{H}om_{\mathscr{O}_{X\times Y}}(\mathscr{F},f^!\mathscr{G})\to\mathrm{R}\mathscr{H}om_{\mathscr{O}_X}(\mathrm{R}f_!\mathscr{F},\mathscr{G}). \tag{3.3.2}$$

If the support of $\mathscr{F}$ is proper over X, then this morphism is an isomorphism.

This result is classical and we shall only recall a construction of the morphism (3.3.2) adapted to our study. Since Ω_Y has a $\mathscr{D}_Y^{\mathrm{op}}$-module structure, we may regard $\omega_{X\times Y/X}$ as an object of $\mathrm{D}^{\mathrm{b}}(\mathscr{O}_X \boxtimes \mathscr{D}_Y^{\mathrm{op}})$. By the de Rham theorem, we have an isomorphism:

$$\omega_{X\times Y/X} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_Y} \mathscr{O}_Y \simeq f^{-1}\mathscr{O}_X[2d_Y].$$

By composing with the morphism $\omega_{X\times Y/X} \to \omega_{X\times Y/X} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_Y} \mathscr{O}_Y$, we get a morphism in $\mathrm{D}^{\mathrm{b}}(f^{-1}\mathscr{O}_X)$:

$$\omega_{X\times Y/X} \to f^{-1}\mathscr{O}_X[2d_Y].$$

Now we have a chain of morphisms in $\mathrm{D}^{\mathrm{b}}(f^{-1}\mathscr{O}_X)$

$$\begin{aligned}
\mathrm{R}\mathscr{H}om_{\mathscr{O}_{X\times Y}}(\mathscr{F}, f^!\mathscr{G}) &= \mathrm{R}\mathscr{H}om_{\mathscr{O}_{X\times Y}}(\mathscr{F}, f^{-1}\mathscr{G} \overset{\mathrm{L}}{\otimes}_{f^{-1}\mathscr{O}_X} \omega_{X\times Y/X}) \\
&\to \mathrm{R}\mathscr{H}om_{f^{-1}\mathscr{O}_X}(\mathscr{F}, f^{-1}\mathscr{G} \overset{\mathrm{L}}{\otimes}_{f^{-1}\mathscr{O}_X} f^{-1}\mathscr{O}_X[2d_Y]) \\
&\simeq \mathrm{R}\mathscr{H}om_{f^{-1}\mathscr{O}_X}(\mathscr{F}, f^{-1}\mathscr{G}[2d_Y]).
\end{aligned}$$

On the other hand, the Poincaré duality gives an isomorphism

$$\mathrm{R}f_*\mathrm{R}\mathscr{H}om_{f^{-1}\mathscr{O}_X}(\mathscr{F}, f^{-1}\mathscr{G}[2d_Y]) \simeq \mathrm{R}\mathscr{H}om_{\mathscr{O}_X}(\mathrm{R}f_!\mathscr{F}, \mathscr{G}).$$

Duality for kernels

Let X_i be complex manifolds of dimension d_i and let $\mathscr{A}_{X_i}$ be DQ-algebroids on X_i $(i = 1, 2, 3)$.

As in Notation 3.1.1, we often write for short X_{ij} instead of $X_i \times X_j$, X_{ij^a} instead of $X_i \times X_j^a$, etc. We also write $\mathscr{A}_{ij}$ instead of $\mathscr{A}_{X_{ij}}$, etc. and ij/i instead of X_{ij}/X_i etc.

Theorem 3.3.3. *Let* $\mathscr{K}_i \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_{X_i\times X_{i+1}^a})$ $(i = 1, 2)$. *We assume that* $\mathrm{Supp}(\mathscr{K}_1) \times_{X_2} \mathrm{Supp}(\mathscr{K}_2)$ *is proper over* $X_1 \times X_3^a$. *Then the natural morphism* (*see* (3.3.1))

$$(\mathrm{D}'_{\mathscr{A}}\mathscr{K}_1) \underset{X_2^a}{\circ} \omega_{X_2^a}^{\mathscr{A}} \underset{X_2^a}{\circ} (\mathrm{D}'_{\mathscr{A}}\mathscr{K}_2) \to \mathrm{D}'_{\mathscr{A}}(\mathscr{K}_1 \underset{X_2}{\circ} \mathscr{K}_2) \tag{3.3.3}$$

is an isomorphism in $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_{X_1^a\times X_3})$.

Proof. Since the question is local on $X_1 \times X_3^a$, we may assume that $\mathrm{gr}_\hbar(\mathscr{A}_{X_1})$ and $\mathrm{gr}_\hbar(\mathscr{A}_{X_3})$ are isomorphic to $\mathscr{O}_{X_1}$ and $\mathscr{O}_{X_3}$, respectively. Applying the functor $\mathrm{gr}_\hbar$, we get

$$
\begin{aligned}
&\mathrm{gr}_\hbar(\mathrm{D}'_{\mathscr{A}}(\mathscr{K}_2) \circ \omega_{X_2}^{\mathscr{A}} \circ \mathrm{D}'_{\mathscr{A}}(\mathscr{K}_1)) \\
&\qquad \simeq \mathrm{R}p_{13!}(\mathrm{R}\mathscr{H}om_{\mathscr{O}_{123}}(p_{12}^*\mathrm{gr}_\hbar(\mathscr{K}_1)\overset{\mathrm{L}}{\otimes}_{\mathscr{O}_{123}} p_{23}^*\mathrm{gr}_\hbar(\mathscr{K}_2), \omega_{X_{123}/X_{13}})) \\
&\qquad \simeq \mathrm{R}\mathscr{H}om_{\mathscr{O}_{X_{13}}}(\mathrm{R}p_{13!}\big(p_{12}^*\mathrm{gr}_\hbar(\mathscr{K}_1)\overset{\mathrm{L}}{\otimes}_{\mathscr{O}_{123}} p_{23}^*\mathrm{gr}_\hbar(\mathscr{K}_2)\big), \mathscr{O}_{13}) \\
&\qquad \simeq \mathrm{gr}_\hbar(\mathrm{D}'_{\mathscr{A}}(\mathscr{K}_1 \circ \mathscr{K}_2)).
\end{aligned}
$$

Here the second isomorphism follows from Theorem 3.3.2. Hence (3.3.3) is an isomorphism by Corollary 1.4.6. Q.E.D.

Recall that D'_X denotes the duality functor for $\mathbb{C}_X^\hbar$-modules: (see (1.1.1)) and $(\,\bullet\,)^\star$ the duality functor on $\mathrm{D}^{\mathrm{b}}_f(\mathbb{C}^\hbar)$ (see (1.1.2)).

Corollary 3.3.4. *Let $\mathscr{M}$ and $\mathscr{N}$ be two objects of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X)$.*

(i) *There is a natural morphism in $\mathrm{D}^{\mathrm{b}}(\mathbb{C}^\hbar)$*

$$
\mathrm{RHom}_{\mathscr{A}_X}(\mathscr{N}, \omega_X^{\mathscr{A}} \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_X} \mathscr{M}) \to (\mathrm{RHom}_{\mathscr{A}_X}(\mathscr{M}, \mathscr{N}))^\star. \tag{3.3.4}
$$

(ii) *If $\mathrm{Supp}(\mathscr{M}) \cap \mathrm{Supp}(\mathscr{N})$ is compact, then (3.3.4) is an isomorphism in $\mathrm{D}^{\mathrm{b}}_f(\mathbb{C}^\hbar)$.*

Proof. (i) In Lemma 3.3.1, take $X_1 = X_3 = \mathrm{pt}$, $X_2 = X$, $\mathscr{K}_1 = \mathscr{N}$ and $\mathscr{K}_2 = \mathrm{D}'_{\mathscr{A}}\mathscr{M}$.
(ii) follows from Theorem 3.3.3. Q.E.D.

In particular, if X is compact, then $\mathscr{M} \mapsto \omega_X^{\mathscr{A}} \otimes_{\mathscr{A}_X} \mathscr{M}$ is a Serre functor on the triangulated category $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X)$.

Remark 3.3.5. For $\mathscr{K}_i \in \mathrm{D}^{\mathrm{b}}(\mathscr{A}^{\mathrm{loc}}_{X_i \times X_{i+1}^a})$ $(i = 1, 2)$, one can define their product $\mathscr{K}_1 \underline{\overset{\mathrm{L}}{\otimes}}_{\mathscr{A}_2^{\mathrm{loc}}} \mathscr{K}_2$ similarly as in Definition 3.1.2 and their convolution similarly as in Definition 3.1.3. (Details are left to the reader.) One introduces

$$
\omega_X^{\mathscr{A}^{\mathrm{loc}}} := \mathbb{C}^{\hbar,\mathrm{loc}} \otimes_{\mathbb{C}^\hbar} \omega_X^{\mathscr{A}} \tag{3.3.5}
$$

and for $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}(\mathscr{A}_X^{\mathrm{loc}})$, one defines its dual by setting

$$
\mathrm{D}'_{\mathscr{A}}\mathscr{M} := \mathrm{R}\mathscr{H}om_{\mathscr{A}_X^{\mathrm{loc}}}(\mathscr{M}, \mathscr{A}_X^{\mathrm{loc}}) \in \mathrm{D}^{\mathrm{b}}(\mathscr{A}_{X^a}^{\mathrm{loc}}). \tag{3.3.6}
$$

Then Theorems 3.2.1 and 3.3.3 extend to good $\mathscr{A}^{\mathrm{loc}}$-modules.

Theorem 3.3.6. *Let Λ_i be a closed subset of $X_i \times X_{i+1}$ $(i = 1, 2)$ and assume that $\Lambda_1 \times_{X_2} \Lambda_2$ is proper over $X_1 \times X_3$. Set $\Lambda = \Lambda_1 \circ \Lambda_2$. Let $\mathscr{K}_i \in \mathrm{D}^{\mathrm{b}}_{\mathrm{gd},\Lambda_i}(\mathscr{A}^{\mathrm{loc}}_{X_i \times X^a_{i+1}})$ $(i = 1, 2)$. Then the object $\mathscr{K}_1 \circ \mathscr{K}_2$ belongs to $\mathrm{D}^{\mathrm{b}}_{\mathrm{gd},\Lambda}(\mathscr{A}^{\mathrm{loc}}_{X_1 \times X^a_3})$ and we have a natural isomorphism*

$$\mathrm{D}'_{\mathscr{A}}(\mathscr{K}_1) \underset{X_2^a}{\circ} \omega^{\mathscr{A}^{\mathrm{loc}}}_{X_2^a} \underset{X_2^a}{\circ} \mathrm{D}'_{\mathscr{A}}(\mathscr{K}_2) \xrightarrow{\sim} \mathrm{D}'_{\mathscr{A}}(\mathscr{K}_1 \underset{X_2}{\circ} \mathscr{K}_2).$$

Proof of Theorem 2.5.7

We are now ready to give a proof of Theorem 2.5.7. In Theorem 3.3.3, set $X_1 = X_2 = X_3 = X^a$ and $\mathscr{K}_1 = \mathscr{K}_2 = \mathscr{C}_{X^a}$. Then we obtain

$$\mathrm{D}'_{\mathscr{A}}\mathscr{C}_{X^a} \underset{X}{\circ} \omega^{\mathscr{A}}_X \underset{X}{\circ} \mathrm{D}'_{\mathscr{A}}\mathscr{C}_{X^a} \simeq \mathrm{D}'_{\mathscr{A}}(\mathscr{C}_{X^a} \underset{X^a}{\circ} \mathscr{C}_{X^a}) \simeq \mathrm{D}'_{\mathscr{A}}(\mathscr{C}_{X^a}).$$

By applying $\circ(\mathrm{D}'_{\mathscr{A}}\mathscr{C}_{X^a})^{\otimes -1}$, we obtain $\mathrm{D}'_{\mathscr{A}}\mathscr{C}_{X^a} \underset{X}{\circ} \omega^{\mathscr{A}}_X \simeq \mathscr{C}_X$.

3.4 Action of kernels on Grothendieck groups

Grothendieck group

For an abelian or a triangulated category $\mathscr{C}$, we denote as usual by $\mathrm{K}(\mathscr{C})$ its Grothendieck group. For an object M of $\mathscr{C}$, we denote by $[M]$ its image in $\mathrm{K}(\mathscr{C})$. Recall that if $\mathscr{C}$ is abelian, then $\mathrm{K}(\mathscr{C}) \simeq \mathrm{K}(\mathrm{D}^{\mathrm{b}}(\mathscr{C}))$.

If A is a ring, we write $\mathrm{K}(A)$ instead of $\mathrm{K}(\mathrm{Mod}(A))$ and write $\mathrm{K}_{\mathrm{coh}}(A)$ instead of $\mathrm{K}(\mathrm{Mod}_{\mathrm{coh}}(A))$.

In this subsection, we will adapt to DQ-modules well-known arguments concerning the Grothendieck group of filtered objects. References are made to [38, Ch. 2.2].

For a closed subset Λ of X, we shall write for short:

$$\begin{aligned}&\mathrm{K}_{\mathrm{coh},\Lambda}(\mathscr{A}_X) := \mathrm{K}(\mathrm{D}^{\mathrm{b}}_{\mathrm{coh},\Lambda}(\mathscr{A}_X)), \quad \mathrm{K}_{\mathrm{coh},\Lambda}(\mathrm{gr}_\hbar\mathscr{A}_X) := \mathrm{K}(\mathrm{D}^{\mathrm{b}}_{\mathrm{coh},\Lambda}(\mathrm{gr}_\hbar\mathscr{A}_X)),\\ &\mathrm{K}_{\mathrm{gd},\Lambda}(\mathscr{A}^{\mathrm{loc}}_X) := \mathrm{K}(\mathrm{D}^{\mathrm{b}}_{\mathrm{gd},\Lambda}(\mathscr{A}^{\mathrm{loc}}_X)).\end{aligned}$$

Recall that for an open subset U of X and $\mathscr{M} \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}^{\mathrm{loc}}_X)$, an $\mathscr{A}_U$-submodule $\mathscr{M}_0$ of $\mathscr{M}|_U$ is called a lattice of $\mathscr{M}$ on U if $\mathscr{M}_0$ is coherent over $\mathscr{A}_U$ and generates $\mathscr{M}|_U$.

Lemma 3.4.1. *Let $0 \to \mathscr{L} \to \mathscr{M} \to \mathscr{N} \to 0$ be an exact sequence in $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}^{\mathrm{loc}}_X)$. Then there locally exist lattices $\mathscr{L}_0$, $\mathscr{M}_0$ and $\mathscr{N}_0$ of $\mathscr{L}$, $\mathscr{M}$ and $\mathscr{N}$ respectively, such that this sequence induces an exact sequence of $\mathscr{A}_X$-modules: $0 \to \mathscr{L}_0 \to \mathscr{M}_0 \to \mathscr{N}_0 \to 0$.*

Proof. (i) Let $\mathscr{M}_0$ be a lattice of $\mathscr{M}$ and let $\mathscr{N}_0$ be its image in $\mathscr{N}$. We set $\mathscr{L}_0 := \mathscr{M}_0 \cap \mathscr{L}$. These $\mathscr{A}_X$-modules give rise to the exact sequence of the statement and it remains to check that $\mathscr{L}_0$ and $\mathscr{N}_0$ are lattices of $\mathscr{L}$ and $\mathscr{N}$, respectively.

(ii) Clearly, $\mathscr{N}_0$ generates $\mathscr{N}$, and being finitely generated, it is coherent over $\mathscr{A}_X$.

(iii) Let us show that $\mathscr{L}_0$ is a lattice of $\mathscr{L}$. Being the kernel of the morphism $\mathscr{M}_0 \to \mathscr{N}_0$, $\mathscr{L}_0$ is coherent. Since the functor $(\,\bullet\,)^{\mathrm{loc}}$ is exact, the sequence $0 \to \mathscr{L}_0^{\mathrm{loc}} \to \mathscr{M}_0^{\mathrm{loc}} \to \mathscr{N}_0^{\mathrm{loc}} \to 0$ is exact. Therefore, $\mathscr{L}_0^{\mathrm{loc}} \simeq \mathscr{L}$. Q.E.D.

Lemma 3.4.2. *Let $\mathscr{M} \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_X^{\mathrm{loc}})$, let U be a relatively compact open subset of X and assume that there exists a lattice $\mathscr{M}_0$ of $\mathscr{M}$ in a neighborhood of the closure $\overline{U}$ of U. Then the image of $\mathscr{M}_0$ in $\mathrm{K}_{\mathrm{coh}}(\mathrm{gr}_\hbar\mathscr{A}_U)$ depends only on $\mathscr{M}$.*

Proof. (i) Recall that $[\mathrm{gr}_\hbar\mathscr{M}_0]$ denotes the image of $\mathrm{gr}_\hbar\mathscr{M}_0$ in $\mathrm{K}_{\mathrm{coh}}(\mathrm{gr}_\hbar\mathscr{A}_U)$. First, remark that for $N \in \mathbb{N}$, the two $\mathrm{gr}_\hbar\mathscr{A}_X$-modules $\mathrm{gr}_\hbar\mathscr{M}_0$ and $\mathrm{gr}_\hbar\hbar^N\mathscr{M}_0$ are isomorphic, which implies

$$[\mathrm{gr}_\hbar\mathscr{M}_0] = [\mathrm{gr}_\hbar\hbar^N\mathscr{M}_0].$$

(ii) Now consider another lattice $\mathscr{M}_0'$ of $\mathscr{M}$ on $\overline{U}$. Since $\mathscr{M}$ is an $\mathscr{A}_X^{\mathrm{loc}}$-module of finite type and $\mathscr{M}_0'$ generates $\mathscr{M}$, there exists $n > 1$ such that $\mathscr{M}_0 \subset \hbar^{-n}\mathscr{M}_0'$. Similarly, there exists $m > 1$ with $\mathscr{M}_0' \subset \hbar^{-m}\mathscr{M}_0$, so that we have the inclusions

$$\hbar^{m+n}\mathscr{M}_0 \subset \hbar^m\mathscr{M}_0' \subset \mathscr{M}_0.$$

Using (i) we may replace $\mathscr{M}_0'$ with $\hbar^m\mathscr{M}_0'$. Hence, changing our notations, we may assume

$$\hbar^m\mathscr{M}_0 \subset \mathscr{M}_0' \subset \mathscr{M}_0. \tag{3.4.1}$$

(iii) Assume $m = 1$ in (3.4.1). Using $\hbar^m\mathscr{M}_0' \subset \hbar^m\mathscr{M}_0$, we get the exact sequences

$$\begin{gathered}0 \to \mathscr{M}_0'/\hbar\mathscr{M}_0 \to \mathscr{M}_0/\hbar\mathscr{M}_0 \to \mathscr{M}_0/\mathscr{M}_0' \to 0,\\ 0 \to \hbar\mathscr{M}_0/\hbar\mathscr{M}_0' \to \mathscr{M}_0'/\hbar\mathscr{M}_0' \to \mathscr{M}_0'/\hbar\mathscr{M}_0 \to 0,\end{gathered}$$

and the result follows in this case.
(iv) Now we argue by induction on m in (3.4.1) and we assume the result is true for $m-1$ with $m \geq 2$. Set

$$\mathscr{M}_0'' := \hbar^{m-1}\mathscr{M}_0 + \mathscr{M}_0'.$$

Then $\hbar\mathscr{M}_0'' \subset \mathscr{M}_0' \subset \mathscr{M}_0''$ and $\hbar^{m-1}\mathscr{M}_0 \subset \mathscr{M}_0'' \subset \mathscr{M}_0$. Then the result follows from (iii) and the induction hypothesis. Q.E.D.

We set

$$\widehat{\mathrm{K}}_{\mathrm{coh},\Lambda}(\mathrm{gr}_\hbar \mathscr{A}_X) := \varprojlim_U \mathrm{K}_{\mathrm{coh},\Lambda}(\mathrm{gr}_\hbar \mathscr{A}_U). \tag{3.4.2}$$

where U ranges over the family of relatively compact open subsets of X. Using Lemma 3.4.2, we get:

Proposition 3.4.3. *There is a natural morphism of groups*

$$\mathrm{gr}_\hbar \colon \mathrm{K}_{\mathrm{gd},\Lambda}(\mathscr{A}_X^{\mathrm{loc}}) \to \widehat{\mathrm{K}}_{\mathrm{coh},\Lambda}(\mathrm{gr}_\hbar \mathscr{A}_X).$$

Remark that when $X = \mathrm{pt}$, the morphism in Proposition 3.4.3 reduces to the isomorphism

$$\mathrm{K}_f(\mathbb{C}^{\hbar,\mathrm{loc}}) \xrightarrow{\sim} \mathrm{K}_f(\mathbb{C}), \tag{3.4.3}$$

and both are isomorphic to $\mathbb{Z}$ by $[M] \mapsto \dim M$.

Kernels

Consider the situation of Theorem 3.2.1. Let Λ_i be a closed subset of $X_i \times X_{i+1}$ $(i = 1, 2)$ and assume that $\Lambda_1 \times_{X_2} \Lambda_2$ is proper over $X_1 \times X_3$. Set $\Lambda = \Lambda_1 \circ \Lambda_2$. Since the convolution of kernels commutes with distinguished triangles, it factors through the Grothendieck groups. Moreover, one can define the convolution of $\mathrm{gr}_\hbar\mathscr{A}_X$-kernels and a variant of Theorem 3.2.1 with $\mathscr{A}_X$ replaced with $\mathrm{gr}_\hbar\mathscr{A}_X$ is well-known. Since the functor $\mathrm{gr}_\hbar$ commutes with the convolution of kernels, the diagram below commutes:

$$\begin{tikzcd}
\mathrm{Ob}\big(\mathrm{D}^{\mathrm{b}}_{\mathrm{coh},\Lambda_1}(\mathscr{A}_{12^a})\big) \times \mathrm{Ob}\big(\mathrm{D}^{\mathrm{b}}_{\mathrm{coh},\Lambda_2}(\mathscr{A}_{23^a})\big) \arrow[r, "\circ"] \arrow[d] & \mathrm{Ob}\big(\mathrm{D}^{\mathrm{b}}_{\mathrm{coh},\Lambda}(\mathscr{A}_{13^a})\big) \arrow[d] \\
\mathrm{K}_{\mathrm{coh},\Lambda_1}(\mathscr{A}_{12^a}) \times \mathrm{K}_{\mathrm{coh},\Lambda_2}(\mathscr{A}_{23^a}) \arrow[r, "\circ"] \arrow[d, "\mathrm{gr}_\hbar \times \mathrm{gr}_\hbar"] & \mathrm{K}_{\mathrm{coh},\Lambda}(\mathscr{A}_{13^a}) \arrow[d, "\mathrm{gr}_\hbar"] \\
\mathrm{K}_{\mathrm{coh},\Lambda_1}(\mathrm{gr}_\hbar\mathscr{A}_{12^a}) \times \mathrm{K}_{\mathrm{coh},\Lambda_2}(\mathrm{gr}_\hbar\mathscr{A}_{23^a}) \arrow[r, "\circ"] & \mathrm{K}_{\mathrm{coh},\Lambda}(\mathrm{gr}_\hbar\mathscr{A}_{13^a}).
\end{tikzcd} \tag{3.4.4}$$

Similarly to (3.4.4), the diagram below commutes:

$$\begin{tikzcd}
\mathrm{Ob}\big(\mathrm{D}^{\mathrm{b}}_{\mathrm{gd},\Lambda_1}(\mathscr{A}^{\mathrm{loc}}_{12^a})\big) \times \mathrm{Ob}\big(\mathrm{D}^{\mathrm{b}}_{\mathrm{gd},\Lambda_2}(\mathscr{A}^{\mathrm{loc}}_{23^a})\big) \arrow[r, "\circ"] \arrow[d] & \mathrm{Ob}\big(\mathrm{D}^{\mathrm{b}}_{\mathrm{gd},\Lambda}(\mathscr{A}^{\mathrm{loc}}_{13^a})\big) \arrow[d] \\
\mathrm{K}_{\mathrm{gd},\Lambda_1}(\mathscr{A}^{\mathrm{loc}}_{12^a}) \times \mathrm{K}_{\mathrm{gd},\Lambda_2}(\mathscr{A}^{\mathrm{loc}}_{23^a}) \arrow[r, "\circ"] \arrow[d, "\mathrm{gr}_\hbar \times \mathrm{gr}_\hbar"] & \mathrm{K}_{\mathrm{gd},\Lambda}(\mathscr{A}^{\mathrm{loc}}_{13^a}) \arrow[d, "\mathrm{gr}_\hbar"] \\
\widehat{\mathrm{K}}_{\mathrm{coh},\Lambda_1}(\mathrm{gr}_\hbar\mathscr{A}_{12^a}) \times \widehat{\mathrm{K}}_{\mathrm{coh},\Lambda_2}(\mathrm{gr}_\hbar\mathscr{A}_{23^a}) \arrow[r, "\circ"] & \widehat{\mathrm{K}}_{\mathrm{coh},\Lambda}(\mathrm{gr}_\hbar\mathscr{A}_{13^a}).
\end{tikzcd} \tag{3.4.5}$$

Chapter 4

Hochschild classes

4.1 Hochschild homology and Hochschild classes

Let X be a complex manifold and let $\mathscr{A}_X$ be a DQ-algebroid. Recall that $\delta_X\colon X \to X \times X^a$ is the diagonal embedding. We define the Hochschild homology $\mathcal{HH}(\mathscr{A}_X)$ of $\mathscr{A}_X$ by:

$$\mathcal{HH}(\mathscr{A}_X) := \delta_X^{-1}(\mathscr{C}_{X^a} \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_{X\times X^a}} \mathscr{C}_X), \text{ an object of } \mathrm{D^b}(\mathbb{C}^\hbar_X). \tag{4.1.1}$$

Note that by Theorem 2.5.7, we get the isomorphisms:

$$\begin{aligned}\mathcal{HH}(\mathscr{A}_X) &\simeq \delta_X^{-1}\mathrm{R}\mathscr{H}om_{\mathscr{A}_{X\times X^a}}(\mathrm{D}'_{\mathscr{A}_{X^a\times X}}\mathscr{C}_{X^a}, \mathscr{C}_X)\\ &\simeq \delta_X^{-1}\mathrm{R}\mathscr{H}om_{\mathscr{A}_{X\times X^a}}(\omega_X^{\mathscr{A}\otimes -1}, \mathscr{C}_X).\end{aligned}$$

We have also the isomorphisms

$$\begin{aligned}\mathrm{R}\mathscr{H}om_{\mathscr{A}_{X\times X^a}}(\omega_X^{\mathscr{A}\otimes -1}, \mathscr{C}_X) &\simeq \mathrm{R}\mathscr{H}om_{\mathscr{A}_{X\times X^a}}(\omega_X^{\mathscr{A}} \underset{X}{\circ} \omega_X^{\mathscr{A}\otimes -1}, \omega_X^{\mathscr{A}} \underset{X}{\circ} \mathscr{C}_X)\\ &\simeq \mathrm{R}\mathscr{H}om_{\mathscr{A}_{X\times X^a}}(\mathscr{C}_X, \omega_X^{\mathscr{A}}).\end{aligned}$$

One shall be aware that the composition of these isomorphisms does not coincide in general with the composition of

$$\begin{aligned}\mathrm{R}\mathscr{H}om_{\mathscr{A}_{X\times X^a}}(\omega_X^{\mathscr{A}\otimes -1}, \mathscr{C}_X) &\simeq \mathrm{R}\mathscr{H}om_{\mathscr{A}_{X\times X^a}}(\omega_X^{\mathscr{A}\otimes -1} \underset{X}{\circ} \omega_X^{\mathscr{A}}, \mathscr{C}_X \underset{X}{\circ} \omega_X^{\mathscr{A}})\\ &\simeq \mathrm{R}\mathscr{H}om_{\mathscr{A}_{X\times X^a}}(\mathscr{C}_X, \omega_X^{\mathscr{A}}).\end{aligned}$$

We shall see that they differ up to $\mathrm{hh}_X(\omega_X)\circ$ (see Lemma 4.3.4 below). For that reason, we shall not identify $\mathcal{HH}(\mathscr{A}_X)$ and $\mathrm{R}\mathscr{H}om_{\mathscr{A}_{X\times X^a}}(\mathscr{C}_X, \omega_X^{\mathscr{A}})$.

Lemma 4.1.1. *Let* $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X)$*. There are natural morphisms in* $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_{X\times X^a})$*:*

$$\omega_X^{\mathscr{A}\otimes -1} \longrightarrow \mathscr{M} \overset{\mathrm{L}}{\underline{\boxtimes}} \mathrm{D}'_{\mathscr{A}}\mathscr{M}, \tag{4.1.2}$$

$$\mathscr{M} \overset{\mathrm{L}}{\underline{\boxtimes}} \mathrm{D}'_{\mathscr{A}}\mathscr{M} \longrightarrow \mathscr{C}_X. \tag{4.1.3}$$

Proof. (i) We have

$$\begin{aligned} \mathrm{R}\mathscr{H}om_{\mathscr{A}_X}(\mathscr{M},\mathscr{M}) &\simeq (\mathrm{D}'_{\mathscr{A}}\mathscr{M}) \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_X} \mathscr{M} \\ &\simeq \mathscr{C}_{X^a} \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_{X\times X^a}} (\mathscr{M} \overset{\mathrm{L}}{\underline{\boxtimes}} \mathrm{D}'_{\mathscr{A}}\mathscr{M}) \\ &\simeq \mathrm{R}\mathscr{H}om_{\mathscr{A}_{X\times X^a}}(\omega_X^{\mathscr{A}\otimes -1}, \mathscr{M} \overset{\mathrm{L}}{\underline{\boxtimes}} \mathrm{D}'_{\mathscr{A}}\mathscr{M}). \end{aligned}$$

The identity of $\mathrm{Hom}_{\mathscr{A}_X}(\mathscr{M},\mathscr{M})$ defines the desired morphism.
(ii) Applying the duality functor $\mathrm{D}'_{\mathscr{A}_{X\times X^a}}$ to (4.1.2), we get (4.1.3). Q.E.D.

Let $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X)$. We have the chain of morphisms

$$\begin{aligned} \mathrm{R}\mathscr{H}om_{\mathscr{A}_X}(\mathscr{M},\mathscr{M}) &\xleftarrow{\sim} \mathrm{D}'_{\mathscr{A}}\mathscr{M} \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_X} \mathscr{M} \\ &\simeq \mathscr{C}_{X^a} \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_{X\times X^a}} (\mathscr{M} \overset{\mathrm{L}}{\underline{\boxtimes}} \mathrm{D}'_{\mathscr{A}}\mathscr{M}) \\ &\to \mathscr{C}_{X^a} \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_{X\times X^a}} \mathscr{C}_X = \mathcal{HH}(\mathscr{A}_X). \end{aligned} \tag{4.1.4}$$

We get a map

$$\mathrm{Hom}_{\mathscr{A}_X}(\mathscr{M},\mathscr{M}) \to H^0_{\mathrm{Supp}(\mathscr{M})}(X;\mathcal{HH}(\mathscr{A}_X)).$$

For $u \in \mathrm{End}(\mathscr{M})$, the image of u gives an element

$$\mathrm{hh}_X((\mathscr{M},u)) \in H^0_{\mathrm{Supp}(\mathscr{M})}(X;\mathcal{HH}(\mathscr{A}_X)). \tag{4.1.5}$$

Notation 4.1.2. For a closed subset Λ of X, we set

$$\mathrm{HH}_\Lambda(\mathscr{A}_X) := H^0\mathrm{R}\Gamma_\Lambda(X;\mathcal{HH}(\mathscr{A}_X)). \tag{4.1.6}$$

Definition 4.1.3. Let $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh},\Lambda}(\mathscr{A}_X)$. We set $\mathrm{hh}_X(\mathscr{M}) = \mathrm{hh}_X((\mathscr{M},\mathrm{id}_{\mathscr{M}})) \in \mathrm{HH}_\Lambda(\mathscr{A}_X)$ and call it the Hochschild class of $\mathscr{M}$.

Lemma 4.1.4. *Let* $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X)$*. Then, the composition of the two morphisms* (4.1.2) *and* (4.1.3)*:*

$$\omega_X^{\mathscr{A}\otimes -1} \longrightarrow \mathscr{M} \overset{\mathrm{L}}{\underline{\boxtimes}} \mathrm{D}'_{\mathscr{A}}\mathscr{M} \longrightarrow \mathscr{C}_X$$

coincides with the Hochschild class $\mathrm{hh}_X(\mathscr{M})$ *when identifying* $\mathcal{HH}(\mathscr{A}_X)$ *with* $\mathrm{R}\mathscr{H}om_{\mathscr{A}_{X\times X^a}}(\omega_X^{\mathscr{A}\otimes -1},\mathscr{C}_X)$.

Proof. The Hochschild class $\mathrm{hh}_X(\mathscr{M})$ is the image of $\mathrm{id}_{\mathscr{M}}$ by the composition

$$\begin{aligned}\mathrm{R}\mathscr{H}om_{\mathscr{A}_X}(\mathscr{M},\mathscr{M}) &\simeq \mathrm{R}\mathscr{H}om_{\mathscr{A}_{X\times X^a}}(\omega_X^{\mathscr{A}\otimes -1}, \mathscr{M}\overset{\mathrm{L}}{\underline{\boxtimes}}\mathrm{D}'_{\mathscr{A}}\mathscr{M})\\ &\to \mathrm{R}\mathscr{H}om_{\mathscr{A}_{X\times X^a}}(\omega_X^{\mathscr{A}\otimes -1}, \mathscr{C}_X) \simeq \mathcal{HH}(\mathscr{A}_X).\end{aligned}$$

Q.E.D.

Theorem 4.1.5. *The Hochschild class is additive with respect to distinguished triangles. In other words, for a distinguished triangle* $\mathscr{M}' \to \mathscr{M} \to \mathscr{M}'' \xrightarrow{+1}$ *in* $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X)$*, we have*

$$\mathrm{hh}_X(\mathscr{M}) = \mathrm{hh}_X(\mathscr{M}') + \mathrm{hh}_X(\mathscr{M}''). \tag{4.1.7}$$

Proof. Although the bifunctor $\bullet\overset{\mathrm{L}}{\otimes}_{\mathscr{A}_X}\bullet$ is not internal to our category, the theorem of May [50] is easily adapted to this situation. Q.E.D.

By this result, the Hochschild class factorizes through the Grothendieck group. Therefore, if Λ is a closed subset of X, we have the morphisms

$$\mathrm{D}^{\mathrm{b}}_{\mathrm{coh},\Lambda}(\mathscr{A}_X) \to \mathrm{K}_{\mathrm{coh},\Lambda}(\mathscr{A}_X) \to \mathrm{HH}_\Lambda(\mathscr{A}_X). \tag{4.1.8}$$

Duality

Denote by $s\colon X\times X^a \to X^a\times X$ the map $(x,y)\mapsto(y,x)$ and recall that δ_X is the diagonal embedding. Then $s\circ\delta_X = \delta_X$, $s^{-1}\mathscr{C}_X \simeq \mathscr{C}_{X^a}$, $s^{-1}\mathscr{A}_{X\times X^a} \simeq \mathscr{A}_{X^a\times X}$ and we obtain the isomorphisms

$$\begin{aligned}\mathcal{HH}(\mathscr{A}_X) &= \delta_X^{-1}(\mathscr{C}_{X^a}\overset{\mathrm{L}}{\otimes}_{\mathscr{A}_{X\times X^a}}\mathscr{C}_X)\\ &\simeq \delta_X^{-1}s^{-1}(\mathscr{C}_{X^a}\overset{\mathrm{L}}{\otimes}_{\mathscr{A}_{X\times X^a}}\mathscr{C}_X)\\ &\simeq \delta_X^{-1}(s^{-1}\mathscr{C}_{X^a}\overset{\mathrm{L}}{\otimes}_{s^{-1}\mathscr{A}_{X\times X^a}}s^{-1}\mathscr{C}_X)\\ &\simeq \delta_X^{-1}(\mathscr{C}_X\overset{\mathrm{L}}{\otimes}_{\mathscr{A}_{X^a\times X}}\mathscr{C}_{X^a}) = \mathcal{HH}(\mathscr{A}_{X^a}).\end{aligned}$$

After identifying $\mathcal{HH}(\mathscr{A}_X)$ and $\mathcal{HH}(\mathscr{A}_{X^a})$ by the isomorphism above, we have:

$$\mathrm{hh}_{X^a}(\mathrm{D}'_{\mathscr{A}}\mathscr{M}) = \mathrm{hh}_X(\mathscr{M}). \tag{4.1.9}$$

Remark 4.1.6. Let $\mathscr{A}$ be a DQ-algebroid and let $\mathscr{P}$ be an inverstible $\mathbb{C}^\hbar$-algebroid on X. Then

$$\mathscr{A}^{\mathscr{P}} := \mathscr{A}\otimes_{\mathbb{C}^\hbar_X}\mathscr{P} \tag{4.1.10}$$

is a DQ-algebroid on X. We have the natural equivalences

$$
\begin{aligned}
&(\mathscr{A}^{\mathrm{op}})^{\mathscr{P}^{\mathrm{op}}} \simeq (\mathscr{A}^{\mathscr{P}})^{\mathrm{op}},\\
&\delta_X^{-1}(\mathscr{A}^{\mathscr{P}} \underline{\boxtimes} (\mathscr{A}^{\mathscr{P}})^{\mathrm{op}}) \simeq \delta_X^{-1}(\mathscr{A} \underline{\boxtimes} (\mathscr{A}^{\mathscr{P}})).
\end{aligned}
$$

We deduce the isomorphism

$$
\mathcal{HH}(\mathscr{A}_X) \simeq \mathcal{HH}(\mathscr{A}_X^{\mathscr{P}}). \tag{4.1.11}
$$

4.2 Composition of Hochschild classes

Let X_i be complex manifolds endowed with DQ-algebroids $\mathscr{A}_{X_i}$ $(i = 1, 2, 3)$ and we denote as usual by p_{ij} the projection from $X_1 \times X_2 \times X_3$ to $X_i \times X_j$ $(1 \le i < j \le 3)$.

Proposition 4.2.1. *There is a natural morphism*

$$
\circ : \mathrm{R}p_{13!}(p_{12}^{-1}\mathcal{HH}(\mathscr{A}_{X_1\times X_2^a}) \overset{\mathrm{L}}{\otimes} p_{23}^{-1}\mathcal{HH}(\mathscr{A}_{X_2\times X_3^a})) \to \mathcal{HH}(\mathscr{A}_{X_1\times X_3^a}).
$$

Proof. (i) Set $Z_i = X_i \times X_i^a$. We shall denote by the same letter p_{ij} the projection from $Z_1 \times Z_2 \times Z_3$ to $Z_i \times Z_j$.

We have

$$
\begin{aligned}
&\mathcal{HH}(\mathscr{A}_{X_i\times X_j^a})\\
&\qquad \simeq (\mathscr{C}_{X_i^a} \overset{\mathrm{L}}{\underline{\boxtimes}} \mathscr{C}_{X_j}) \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_{Z_i\times Z_j^a}} (\mathscr{C}_{X_i} \overset{\mathrm{L}}{\underline{\boxtimes}} \mathscr{C}_{X_j^a})\\
&\qquad \simeq \mathrm{R}\mathscr{H}om_{\mathscr{A}_{Z_i\times Z_j^a}}(\omega_{X_i}^{\mathscr{A}\otimes -1} \overset{\mathrm{L}}{\underline{\boxtimes}} \omega_{X_j^a}^{\mathscr{A}\otimes -1}, \mathscr{C}_{X_i} \overset{\mathrm{L}}{\underline{\boxtimes}} \mathscr{C}_{X_j{}^a})\\
&\qquad \simeq \mathrm{R}\mathscr{H}om_{\mathscr{A}_{Z_i\times Z_j^a}}\big((\omega_{X_i}^{\mathscr{A}\otimes -1} \overset{\mathrm{L}}{\underline{\boxtimes}} \omega_{X_j^a}^{\mathscr{A}\otimes -1}) \overset{\mathrm{L}}{\underline{\otimes}}_{\mathscr{A}_{X_j^a}} \omega_{X_j^a}^{\mathscr{A}}, (\mathscr{C}_{X_i} \overset{\mathrm{L}}{\underline{\boxtimes}} \mathscr{C}_{X_j{}^a}) \overset{\mathrm{L}}{\underline{\otimes}}_{\mathscr{A}_{X_j^a}} \omega_{X_j^a}^{\mathscr{A}}\big)\\
&\qquad \simeq \mathrm{R}\mathscr{H}om_{\mathscr{A}_{Z_i\times Z_j^a}}(\omega_{X_i}^{\mathscr{A}\otimes -1} \overset{\mathrm{L}}{\underline{\boxtimes}} \mathscr{C}_{X_j^a}, \mathscr{C}_{X_i} \overset{\mathrm{L}}{\underline{\boxtimes}} \omega_{X_j{}^a}^{\mathscr{A}}).
\end{aligned}
$$

Set $S_{ij} := \omega_{X_i}^{\mathscr{A}\otimes -1} \overset{\mathrm{L}}{\underline{\boxtimes}} \mathscr{C}_{X_j^a} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_{Z_i\times Z_j^a})$ and $K_{ij} := \mathscr{C}_{X_i} \overset{\mathrm{L}}{\underline{\boxtimes}} \omega_{X_j^a}^{\mathscr{A}} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_{Z_i\times Z_j^a})$. Then we get

$$
\mathcal{HH}(\mathscr{A}_{X_i\times X_j^a}) \simeq \mathrm{R}\mathscr{H}om_{\mathscr{A}_{Z_i\times Z_j^a}}(S_{ij}, K_{ij}).
$$

Thus we obtain a morphism in $\mathrm{D}^{\mathrm{b}}(\mathbb{C}^{\hbar}_{Z_1\times Z_2\times Z_3})$

$$
\begin{aligned}
&p_{12}^{-1}\mathcal{HH}(\mathscr{A}_{X_1\times X_2^a}) \overset{\mathrm{L}}{\otimes} p_{23}^{-1}\mathcal{HH}(\mathscr{A}_{X_2\times X_3^a})\\
&\qquad\qquad \simeq p_{12}^{-1}\mathrm{R}\mathscr{H}om_{\mathscr{A}_{Z_1\times Z_2^a}}(S_{12}, K_{12}) \overset{\mathrm{L}}{\otimes} p_{23}^{-1}\mathrm{R}\mathscr{H}om_{\mathscr{A}_{Z_2\times Z_3^a}}(S_{23}, K_{23})\\
&\qquad\qquad \to p_{13}^{-1}\mathrm{R}\mathscr{H}om_{\mathscr{A}_{Z_1\times Z_3^a}}\big(S_{12} \overset{\mathrm{L}}{\underline{\otimes}}_{\mathscr{A}_{Z_2}} S_{23}, K_{12} \overset{\mathrm{L}}{\underline{\otimes}}_{\mathscr{A}_{Z_2}} K_{23}\big).
\end{aligned}
$$

We get a morphism

$$
\begin{aligned}
(4.2.1)\qquad & \mathrm{R}p_{13!}(p_{12}^{-1}\mathcal{HH}(\mathscr{A}_{X_1\times X_2^a})\overset{\mathrm{L}}{\otimes}p_{23}^{-1}\mathcal{HH}(\mathscr{A}_{X_2\times X_3^a}))\\
&\qquad \to \mathrm{R}p_{13!}\mathrm{R}\mathscr{H}om_{\mathscr{A}_{Z_1\times Z_3^a}}\big(S_{12}\underline{\overset{\mathrm{L}}{\otimes}}_{\mathscr{A}_{Z_2}}S_{23}, K_{12}\underline{\overset{\mathrm{L}}{\otimes}}_{\mathscr{A}_{Z_2}}K_{23}\big).
\end{aligned}
$$

(ii) We have a morphism

$$
\mathbb{C}^{\hbar}_{X_2}\to \mathrm{R}\mathscr{H}om_{\mathscr{A}_{Z_2^a}}(\mathscr{C}_{X_2^a},\mathscr{C}_{X_2^a})\simeq \mathscr{C}_{X_2^a}\underline{\overset{\mathrm{L}}{\otimes}}_{\mathscr{A}_{Z_2}}\omega_{X_2}^{\mathscr{A}\otimes -1},
$$

which induces the morphism:

$$
p_{13}^{-1}(\omega_{X_1}^{\mathscr{A}\otimes -1}\underline{\overset{\mathrm{L}}{\boxtimes}}\mathscr{C}_{X_3^a})\to(\omega_{X_1}^{\mathscr{A}\otimes -1}\underline{\overset{\mathrm{L}}{\boxtimes}}\mathscr{C}_{X_2^a})\underline{\overset{\mathrm{L}}{\otimes}}_{\mathscr{A}_{Z_2}}(\omega_{X_2}^{\mathscr{A}\otimes -1}\underline{\overset{\mathrm{L}}{\boxtimes}}\mathscr{C}_{X_3^a}),
$$

that is, the morphism in $\mathrm{D}^{\mathrm{b}}(\mathscr{A}_{Z_1\times Z_3^a})$:

$$
(4.2.2)\qquad S_{13}\to \mathrm{R}p_{13*}(S_{12}\underline{\overset{\mathrm{L}}{\otimes}}_{\mathscr{A}_{Z_2}}S_{23}).
$$

(iii) We have a morphism (see (2.5.7)):

$$
(\mathscr{C}_{X_1}\underline{\overset{\mathrm{L}}{\boxtimes}}\omega^{\mathscr{A}}_{X_2^a})\underline{\overset{\mathrm{L}}{\otimes}}_{\mathscr{A}_{Z_2}}(\mathscr{C}_{X_2}\underline{\overset{\mathrm{L}}{\boxtimes}}\omega^{\mathscr{A}}_{X_3^a})\to p_{13}^{-1}(\mathscr{C}_{X_1}\underline{\overset{\mathrm{L}}{\boxtimes}}\omega^{\mathscr{A}}_{X_3^a})[2d_2],
$$

which induces the morphism in $\mathrm{D}^{\mathrm{b}}(\mathscr{A}_{Z_1\times Z_3^a})$:

$$
(4.2.3)\qquad \mathrm{R}p_{13!}(K_{12}\underline{\overset{\mathrm{L}}{\otimes}}_{\mathscr{A}_{Z_2}}K_{23})\to K_{13}.
$$

(iv) Using (4.2.2) and (4.2.3) we obtain

$$
\begin{aligned}
& \mathrm{R}p_{13!}\mathrm{R}\mathscr{H}om_{\mathscr{A}_{Z_1\times Z_3^a}}\big(S_{12}\underline{\overset{\mathrm{L}}{\otimes}}_{\mathscr{A}_{Z_2}}S_{23}, K_{12}\underline{\overset{\mathrm{L}}{\otimes}}_{\mathscr{A}_{Z_2}}K_{23}\big)\\
(4.2.4)\qquad &\quad\to \mathrm{R}\mathscr{H}om_{\mathscr{A}_{Z_1\times Z_3^a}}\big(\mathrm{R}p_{13*}(S_{12}\underline{\overset{\mathrm{L}}{\otimes}}_{\mathscr{A}_{Z_2}}S_{23}), \mathrm{R}p_{13!}(K_{12}\underline{\overset{\mathrm{L}}{\otimes}}_{\mathscr{A}_{Z_2}}K_{23})\big)\\
&\quad\to \mathrm{R}\mathscr{H}om_{\mathscr{A}_{Z_1\times Z_3^a}}(S_{13},K_{13})\simeq \mathcal{HH}(\mathscr{A}_{X_1\times X_3^a}).
\end{aligned}
$$

Combining (4.2.1) and (4.2.4), we get the result. Q.E.D.

Let us denote by $X_{\mathbb{R}}$ the real underlying manifold to X and by $\omega^{\mathrm{top}}_{X_{\mathbb{R}}}$ the topological dualizing complex of the space $X_{\mathbb{R}}$ with coefficients in $\mathbb{C}^{\hbar}$. Note that X being smooth and oriented, $\omega^{\mathrm{top}}_{X_{\mathbb{R}}}$ is isomorphic to $\mathbb{C}^{\hbar}_X[2d_X]$.

Corollary 4.2.2. *There is a canonical morphism $\mathcal{HH}(\mathscr{A}_X)\otimes\mathcal{HH}(\mathscr{A}_X)\to \omega^{\mathrm{top}}_{X_{\mathbb{R}}}$.*

Proof. Let us apply Proposition 4.2.1 with $X_2=X$, $X_1=X_3=\mathrm{pt}$. Denoting by a_X the map $X\to\mathrm{pt}$, we get the morphism $\mathrm{R}a_{X!}(\mathcal{HH}(\mathscr{A}_X)\otimes\mathcal{HH}(\mathscr{A}_X))\to \mathbb{C}^{\hbar}_{\mathrm{pt}}$. By adjunction we get the desired morphism. Q.E.D.

4.3 Main theorem

Consider five manifolds X_i endowed with DQ-algebroids $\mathscr{A}_{X_i}$ $(i = 1, \dots, 5)$.

Notation 4.3.1. In the sequel and until the end of this section, when there is no risk of confusion, we use the following conventions.

(i) For $i, j \in \{1, 2, 3, 4, 5\}$, we set $X_{ij} := X_i \times X_j$, $X_{ij^a} := X_i \times X_j^a$ and similarly with X_{ijk}, etc.

(ii) We sometimes omit the symbols $p_{ij}, p_{ij*}, p_{ij}^{-1}$, etc.

(iii) We write $\mathscr{A}_i$ instead of $\mathscr{A}_{X_i}$, $\mathscr{A}_{ij^a}$ instead of $\mathscr{A}_{X_{ij^a}}$ and similarly with $\mathscr{C}_i, \omega_i^{\mathscr{A}}$, etc., and we write $\underset{i}{\circ}$ instead of $\underset{X_i}{\circ}$, $\underset{i}{*}$ instead of $\underset{X_i}{*}$, $\mathscr{H}om_i$ instead of $\mathscr{H}om_{\mathscr{A}_i}$ and $\otimes_i$ instead of $\otimes_{\mathscr{A}_i}$ and similarly with ij^a, ijk, etc.

(iv) We write D' instead of $\mathrm{D}'_{\mathscr{A}}$ and ω_X instead of $\omega_X^{\mathscr{A}}$.

(v) We often identify an invertible object of $\mathrm{D^b}(\mathscr{A}_X \otimes \mathscr{A}_{X^a})$ with an object of $\mathrm{D^b}(\mathscr{A}_{X\times X^a})$ supported by the diagonal.

(vi) We identify $(X_i \times X_j^a)^a$ with $X_i^a \times X_j$.

Let $\Lambda_{ij} \subset X_{ij}$ $(i = 1, 2,\ j = i+1)$ be a closed subset and assume that $\Lambda_{12} \times_{X_2} \Lambda_{23}$ is proper over $X_1 \times X_3$. Using Proposition 4.2.1, we get a map

$$\underset{2}{\circ} : \mathrm{HH}_{\Lambda_{12}}(\mathscr{A}_{X_{12^a}}) \times \mathrm{HH}_{\Lambda_{23}}(\mathscr{A}_{X_{23^a}}) \longrightarrow \mathrm{HH}_{\Lambda_{12}\circ\Lambda_{23}}(\mathscr{A}_{X_{13^a}}). \tag{4.3.1}$$

For $C_{ij} \in \mathrm{HH}_{\Lambda_{ij}}(\mathscr{A}_{X_{ij^a}})$ $(i = 1, 2,\ j = i+1)$, we obtain a class

$$C_{12} \underset{2}{\circ} C_{23} \in \mathrm{HH}_{\Lambda_{12}\circ\Lambda_{23}}(\mathscr{A}_{X_{13^a}}). \tag{4.3.2}$$

The morphism $(\mathscr{C}_{1^a} \overset{\mathrm{L}}{\underline{\otimes}}_{11^a} \mathscr{C}_1) \overset{\mathrm{L}}{\boxtimes} (\mathscr{C}_{2^a} \overset{\mathrm{L}}{\underline{\otimes}}_{22^a} \mathscr{C}_2) \to (\mathscr{C}_{1^a2^a} \overset{\mathrm{L}}{\underline{\otimes}}_{121^a2^a} \mathscr{C}_{12})$ induces the exterior product

$$\boxtimes : \mathrm{HH}_{\Lambda_1}(\mathscr{A}_{X_1}) \times \mathrm{HH}_{\Lambda_2}(\mathscr{A}_{X_2}) \to \mathrm{HH}_{\Lambda_1\times\Lambda_2}(\mathscr{A}_{X_1\times X_2}) \tag{4.3.3}$$

for $\Lambda_i \subset X_i$ $(i = 1, 2)$.

Lemma 4.3.2. *Let $\Lambda_{ij} \subset X_{ij}$ $(i = 1, 2, 3,\ j = i+1)$ and assume that $\Lambda_{ij} \times_{X_j} \Lambda_{jk}$ is proper over X_{ik} $(i = 1, 2,\ j = i+1,\ k = j+1)$. Let $C_{ij} \in \mathrm{HH}_{\Lambda_{ij}}(\mathscr{A}_{X_{ij^a}})$ $(i = 1, 2, 3,\ j = i+1)$.*

(a) *One has $(C_{12} \underset{2}{\circ} C_{23}) \underset{3}{\circ} C_{34} = C_{12} \underset{2}{\circ} (C_{23} \underset{3}{\circ} C_{34})$.*

(b) *For* $C_{245} \in \mathrm{HH}(\mathscr{A}_{X_{245^a}})$ *we have*

$$(C_{12} \boxtimes C_{34}) \underset{24}{\circ} C_{245} = C_{12} \underset{2}{\circ} (C_{34} \underset{4}{\circ} C_{245}).$$

(c) *Set* $\mathrm{C}_{\Delta_i} = \mathrm{hh}_{X_{ii^a}}(\mathscr{C}_{X_i})$. *Then* $C_{12} \underset{2}{\circ} \mathrm{C}_{\Delta_2} = \mathrm{C}_{\Delta_1} \underset{1}{\circ} C_{12} = C_{12}$.

(d) $(C_{12} \boxtimes \mathrm{C}_{\Delta_3}) \underset{23^a}{\circ} C_{23} = C_{12} \underset{2}{\circ} C_{23}$. *Here* $C_{12} \boxtimes \mathrm{C}_{\Delta_3} \in \mathrm{HH}_{\Lambda_{12} \times \Delta_3}(\mathscr{A}_{X_{12^a 33^a}})$ *is regarded as an element of* $\mathrm{HH}_{\Lambda_{12} \times \Delta_3}(\mathscr{A}_{X_{(13^a)(23^a)^a}})$.

Proof. The proof of (a) and (b) is left to the reader and (c) follows from Lemma 4.3.4 below. Indeed, $\Phi_{\mathscr{K}}$ in (4.3.7) is equal to the identity when $\mathscr{K} = \mathscr{C}_X$ since the functor $\mathscr{L} \mapsto \mathscr{K} \underset{2}{*} \mathscr{L} \underset{2}{\circ} \omega_2 \underset{2}{*} \mathrm{D}'\mathscr{K}$ is isomorphic to the identity functor.
(d) follows from (b) and (c). Q.E.D.

In order to prove Theorem 4.3.5 below, we need some lemmas.

Lemma 4.3.3. *Let* $\mathscr{K} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_{X_{12^a}})$. *Then, there are natural morphisms in* $\mathrm{D}^{\mathrm{b}}(\mathscr{A}_{X_{11^a}})$:

$$\omega_1^{\otimes -1} \to \mathscr{K} \underset{2}{*} \mathrm{D}'_{\mathscr{A}}\mathscr{K}, \tag{4.3.4}$$

$$\mathscr{K} \underset{2}{\circ} \omega_2 \underset{2}{\circ} \mathrm{D}'_{\mathscr{A}}\mathscr{K} \to \mathscr{C}_1. \tag{4.3.5}$$

Proof. (i) By (4.1.2), we have a morphism in $\mathrm{D}^{\mathrm{b}}(\mathscr{A}_{X_{12^a 21^a}})$

$$\omega_{12^a}^{\otimes -1} \to \mathscr{K} \underline{\boxtimes} \mathrm{D}'_{\mathscr{A}}\mathscr{K}.$$

Applying the functor $\bullet \overset{\mathrm{L}}{\otimes}_{22^a} \mathscr{C}_2$, we obtain

$$p_{11^a}^{-1}\omega_1^{\otimes -1} \to \omega_1^{\otimes -1} \boxtimes (\mathrm{D}'_{\mathscr{A}}\mathscr{C}_2 \overset{\mathrm{L}}{\otimes}_{22^a} \mathscr{C}_2) \to \omega_{12^a}^{\otimes -1} \overset{\mathrm{L}}{\otimes}_{22^a} \mathscr{C}_2 \to (\mathscr{K} \overset{\mathrm{L}}{\underline{\boxtimes}} \mathrm{D}'_{\mathscr{A}}\mathscr{K}) \otimes_{22^a} \mathscr{C}_2.$$

By adjunction, we get (4.3.4).
(ii) By (4.1.3), we have a morphism in $\mathrm{D}^{\mathrm{b}}(\mathscr{A}_{X_{12^a 21^a}})$

$$\mathscr{K} \underline{\boxtimes} \mathrm{D}'\mathscr{K} \to \mathscr{C}_{12^a}.$$

Applying the functor $\bullet \overset{\mathrm{L}}{\otimes}_{22^a} \omega_2$, we obtain

$$(\mathscr{K} \overset{\mathrm{L}}{\underline{\boxtimes}} \mathrm{D}'_{\mathscr{A}}\mathscr{K}) \otimes_{22^a} \omega_2 \to \mathscr{C}_{12^a} \overset{\mathrm{L}}{\otimes}_{22^a} \omega_2 \to \mathscr{C}_1 \overset{\mathrm{L}}{\underline{\otimes}} \mathbb{C}^{\hbar}_{X_2}[2d_2].$$

Here the last arrow is given by (2.5.7). By adjunction, we get (4.3.5). Q.E.D.

For the sake of brevity, we shall write $\Gamma_\Lambda\mathrm{Hom}$ instead of $H^0(\mathrm{R}\Gamma_\Lambda\mathrm{R}\mathscr{H}om)$.

Let Λ_{12} be a closed subset of $X_1 \times X_2^a$ and Λ_2 a closed subset of X_2. Let $\mathscr{K} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_{X_{12^a}})$ with support Λ_{12}. We assume

(4.3.6) $\Lambda_{12} \times_{X_2} \Lambda_2$ is proper over X_1.

We define the map

$$\Phi_{\mathscr{K}}\colon \mathrm{HH}_{\Lambda_2}(\mathscr{A}_{X_2}) \longrightarrow \mathrm{HH}_{\Lambda_{12}\circ\Lambda_2}(\mathscr{A}_{X_1}) \tag{4.3.7}$$

as the composition of the sequence of maps

$$\begin{aligned}
&\mathrm{HH}_{\Lambda_2}(\mathscr{A}_2) \simeq \Gamma_{\Lambda_2}\mathrm{Hom}_{22^a}(\omega_2^{\otimes-1}, \mathscr{C}_2)\\
&\to \Gamma_{\Lambda_{12}\times_{X_2}\Lambda_2}\mathrm{Hom}_{11^a}(\mathscr{K}\overset{\mathrm{L}}{\underline{\otimes}}_2\omega_2^{\otimes-1}\underset{2}{\circ}\omega_2\underset{2}{\circ}\mathrm{D}'_{\mathscr{A}}\mathscr{K}, \mathscr{K}\overset{\mathrm{L}}{\underline{\otimes}}_2\mathscr{C}_2\underset{2}{\circ}\omega_2\underset{2}{\circ}\mathrm{D}'_{\mathscr{A}}\mathscr{K})\\
&\to \Gamma_{\Lambda_{12}\circ\Lambda_2}\mathrm{Hom}_{11^a}\big(\mathrm{R}p_{1*}(\mathscr{K}\overset{\mathrm{L}}{\underline{\otimes}}_2(\omega_2^{\otimes-1}\underset{2}{\circ}\omega_2\underset{2}{\circ}\mathrm{D}'_{\mathscr{A}}\mathscr{K})), \mathrm{R}p_{1!}(\mathscr{K}\overset{\mathrm{L}}{\underline{\otimes}}_2(\mathscr{C}_2\underset{2}{\circ}\omega_2\underset{2}{\circ}\mathrm{D}'_{\mathscr{A}}\mathscr{K}))\big)\\
&\simeq \Gamma_{\Lambda_{12}\circ\Lambda_2}\mathrm{Hom}_{11^a}(\mathscr{K}\underset{2}{*}\mathrm{D}'_{\mathscr{A}}\mathscr{K}, \mathscr{K}\underset{2}{\circ}\omega_2\underset{2}{\circ}\mathrm{D}'_{\mathscr{A}}\mathscr{K})\\
&\to \Gamma_{\Lambda_{12}\circ\Lambda_2}\mathrm{Hom}_{11^a}(\omega_1^{\otimes-1}, \mathscr{C}_1) \simeq \mathrm{HH}_{\Lambda_{12}\circ\Lambda_2}(\mathscr{A}_1).
\end{aligned}$$

The first arrow is obtained by applying the functor $\mathscr{L} \mapsto \mathscr{K}\overset{\mathrm{L}}{\underline{\otimes}}_2(\mathscr{L}\underset{2}{\circ}\omega_2\underset{2}{\circ}\mathrm{D}'_{\mathscr{A}}\mathscr{K})$, The last arrow is associated with the morphisms in Lemma 4.3.3.

Lemma 4.3.4. *The map $\Phi_{\mathscr{K}}\colon \mathrm{HH}_{\Lambda_2}(\mathscr{A}_{X_2}) \longrightarrow \mathrm{HH}_{\Lambda_{12}\circ\Lambda_2}(\mathscr{A}_{X_1})$ in* (4.3.7) *is the map $\mathrm{hh}_{X_{12^a}}(\mathscr{K})\circ$ given in* (4.3.2).

Proof. In the proof, we do not write Λ_{12} and Λ_2. Let $\lambda = \mathrm{hh}_{12^a}(\mathscr{K}) \in \mathrm{HH}(\mathscr{A}_{12^a})$ and let $\lambda_2 \in \mathrm{HH}(\mathscr{A}_2)$. We regard λ as a morphism on $X_{12^a21^a}$:

$$\lambda\colon \omega_{12^a}^{\otimes-1} \to \mathscr{K}\underline{\boxtimes}\mathrm{D}'_{\mathscr{A}}\mathscr{K} \to \mathscr{C}_{12^a}.$$

We regard λ_2 as a morphism $\omega_2^{\otimes-1} \to \mathscr{C}_2$. Then $\Phi_{\mathscr{K}}(\lambda_2)$ is obtained as the composition

$$\begin{aligned}
\omega_1^{\otimes-1} &\to \mathscr{K}\underset{2}{*}\mathrm{D}'_{\mathscr{A}}\mathscr{K} \to \mathrm{R}p_{1*}\big(\mathscr{K}\overset{\mathrm{L}}{\underline{\otimes}}_2(\omega_2^{\otimes-1}\underset{2}{\circ}\omega_2\underset{2}{\circ}\mathrm{D}'_{\mathscr{A}}\mathscr{K})\big)\\
&\xrightarrow{\lambda_2} \mathrm{R}p_{1!}\big(\mathscr{K}\overset{\mathrm{L}}{\underline{\otimes}}_2(\mathscr{C}_2\underset{2}{\circ}\omega_2\underset{2}{\circ}\mathrm{D}'_{\mathscr{A}}\mathscr{K})\big) \to \mathscr{K}\underset{2}{\circ}\omega_2\underset{2}{\circ}\mathrm{D}'_{\mathscr{A}}\mathscr{K} \to \mathscr{C}_1.
\end{aligned}$$

In the following diagram in the category $\mathrm{D}^{\mathrm{b}}(\mathscr{A}_{11^a}\boxtimes\mathbb{C}^\hbar_{X_2\times X_2^a})$, we write D'

instead of $\mathrm{D}'_{\mathscr{A}}$ for sake of brevity:

$$\begin{tikzcd}[column sep=small]
p^{-1}_{11^a}\omega_1^{\otimes -1} \arrow[d] & & \\
(\omega_1^{\otimes -1}\boxtimes \mathscr{C}_{2^a})\overset{\mathrm{L}}{\underline{\otimes}}_{22^a}\omega_2^{\otimes -1} \arrow[r, "\lambda_2"] \arrow[d, "\wr"] & (\omega_1^{\otimes -1}\boxtimes \mathscr{C}_{2^a})\overset{\mathrm{L}}{\underline{\otimes}}_{22^a}\mathscr{C}_2 \arrow[d, "\wr"] & \\
((\omega_1^{\otimes -1}\boxtimes \omega_{2^a}^{\otimes -1})\underset{2^a}{\circ}\omega_{2^a})\overset{\mathrm{L}}{\underline{\otimes}}_{22^a}\omega_2^{\otimes -1} \arrow[r, "\lambda_2"] \arrow[d] & ((\omega_1^{\otimes -1}\boxtimes \omega_{2^a}^{\otimes -1})\underset{2}{\circ}\omega_{2^a})\overset{\mathrm{L}}{\underline{\otimes}}_{22^a}\mathscr{C}_2 \arrow[d] \arrow[dr] & \\
(\omega_2\underset{2}{\circ}\omega_{12^a}^{\otimes -1})\overset{\mathrm{L}}{\underline{\otimes}}_{22^a}\omega_2^{\otimes -1} \arrow[r, "\lambda_2"] \arrow[d] & (\omega_2\underset{2}{\circ}\omega_{12^a}^{\otimes -1})\overset{\mathrm{L}}{\underline{\otimes}}_{22^a}\mathscr{C}_2 \arrow[d] & ((\mathscr{K}\boxtimes \mathrm{D}'\mathscr{K})\underset{2^a}{\circ}\omega_{2^a})\overset{\mathrm{L}}{\underline{\otimes}}_{22^a}\mathscr{C}_2 \arrow[dl, "\sim"'] \arrow[d] \\
(\mathscr{K}\boxtimes\omega_2\underset{2}{\circ}\mathrm{D}'\mathscr{K})\overset{\mathrm{L}}{\underline{\otimes}}_{22^a}\omega_2^{\otimes -1} \arrow[r, "\lambda_2"] \arrow[d, "\wr"] & (\mathscr{K}\boxtimes\omega_2\underset{2}{\circ}\mathrm{D}'\mathscr{K})\overset{\mathrm{L}}{\underline{\otimes}}_{22^a}\mathscr{C}_2 \arrow[d, "\wr"] & (\mathscr{C}_{12^a}\underset{2^a}{\circ}\omega_{2^a})\overset{\mathrm{L}}{\underline{\otimes}}_{22^a}\mathscr{C}_2 \arrow[d, "\wr"] \\
\mathscr{K}\overset{\mathrm{L}}{\underline{\otimes}}_2(\omega_2^{\otimes -1}\underset{2}{\circ}\omega_2\underset{2}{\circ}\mathrm{D}'\mathscr{K}) \arrow[r, "\lambda_2"] & \mathscr{K}\overset{\mathrm{L}}{\underline{\otimes}}_2(\mathscr{C}_2\underset{2}{\circ}\omega_2\underset{2}{\circ}\mathrm{D}'\mathscr{K}) \arrow[d] & (\mathscr{C}_1\boxtimes\omega_{2^a})\overset{\mathrm{L}}{\underline{\otimes}}_{22^a}\mathscr{C}_2 \arrow[d] \\
 & \mathscr{K}\overset{\mathrm{L}}{\underline{\otimes}}_2\mathrm{D}'\mathscr{K} \arrow[r] & \mathscr{C}_1\boxtimes\mathbb{C}^{\hbar}_{X_2}[2d_2].
\end{tikzcd}$$

Here we used $\omega_2\underset{2}{\circ}L\simeq L\underset{2^a}{\circ}\omega_{2^a}$. This diagram commutes, and the rows on the top and the right columns $p^{-1}_{11^a}\omega_1^{\otimes -1}\to\big(\omega_2\underset{2}{\circ}(\mathscr{K}\boxtimes\mathrm{D}'_{\mathscr{A}}\mathscr{K})\big)\overset{\mathrm{L}}{\underline{\otimes}}_{22^a}\mathscr{C}_2\to p^{!}_{11^a}\mathscr{C}_1$ induce $\lambda\circ\lambda_2\colon\omega_1^{\otimes -1}\to\mathscr{C}_1$ by adjunction. Therefore, the diagram

$$\begin{tikzcd}
\omega_1^{\otimes -1} \arrow[r] \arrow[drr, "\lambda\circ\lambda_2"'] & \mathscr{K}\underset{2}{*}\mathrm{D}'_{\mathscr{A}}\mathscr{K} \arrow[r, "\sim"] & \mathscr{K}\underset{2}{*}(\omega_2^{\otimes -1}\underset{2}{\circ}\omega_2\underset{2}{\circ}\mathrm{D}'_{\mathscr{A}}\mathscr{K}) \arrow[d, "\lambda_2"] \\
 & & \mathscr{K}\underset{2}{\circ}\mathscr{C}_2\underset{2}{\circ}\omega_2\underset{2}{\circ}\mathrm{D}'_{\mathscr{A}}\mathscr{K} \arrow[d] \\
 & & \mathscr{C}_1
\end{tikzcd}$$

commutes, which gives the result since the composition of the rows on the top and the vertical arrows is $\Phi_{\mathscr{K}}(\lambda_2)$. Q.E.D.

Theorem 4.3.5. *Let Λ_i be a closed subset of $X_i\times X_{i+1}$ $(i=1,2)$ and assume that $\Lambda_1\times_{X_2}\Lambda_2$ is proper over $X_1\times X_3$. Set $\Lambda=\Lambda_1\circ\Lambda_2$. Let*

$\mathscr{K}_i \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh},\Lambda_i}(\mathscr{A}_{X_i \times X^a_{i+1}})$ $(i = 1,2)$. *Then*

$$\mathrm{hh}_{X_{13^a}}(\mathscr{K}_1 \circ \mathscr{K}_2) = \mathrm{hh}_{X_{12^a}}(\mathscr{K}_1) \circ \mathrm{hh}_{X_{23^a}}(\mathscr{K}_2) \tag{4.3.8}$$

as elements of $\mathrm{HH}_\Lambda(\mathscr{A}_{X_1 \times X^a_3})$.

Proof. For the sake of simplicity, we assume that $X_3 = \mathrm{pt}$. Consider the diagram in which we set $\lambda_2 = \mathrm{hh}_2(\mathscr{K}_2) \in \mathrm{HH}(\mathscr{A}_{X_2}) \simeq \mathrm{Hom}\,(\omega_2^{\otimes -1}, \mathscr{C}_2)$ and we write D' instead of $\mathrm{D}'_{\mathscr{A}}$:

$$\begin{tikzcd}
\omega_1^{\otimes -1} \arrow[r] \arrow[dddr, bend right] & \mathscr{K}_1 \underset{2}{\circ} \omega_2^{\otimes -1} \underset{2}{\circ} \omega_2 \underset{2}{\circ} \mathrm{D}'\mathscr{K}_1 \arrow[r, "\lambda_2"] \arrow[d] & \mathscr{K}_1 \underset{2}{\circ} \mathscr{C}_2 \underset{2}{\circ} \omega_2 \underset{2}{\circ} \mathrm{D}'\mathscr{K}_1 \arrow[r] & \mathscr{C}_1 \\
& \mathscr{K}_1 \underset{2}{\circ} (\mathscr{K}_2 \underline{\boxtimes} \mathrm{D}'\mathscr{K}_2) \underset{2}{\circ} \omega_2 \underset{2}{\circ} \mathrm{D}'\mathscr{K}_1 \arrow[ur] \arrow[d, "\wr"] & & \\
& (\mathscr{K}_1 \underset{2}{\circ} \mathscr{K}_2) \underline{\boxtimes} \mathrm{D}'\mathscr{K}_2 \underset{2}{\circ} \omega_2 \underset{2}{\circ} \mathrm{D}'\mathscr{K}_1 \arrow[d, "\wr"] & & \\
& (\mathscr{K}_1 \underset{2}{\circ} \mathscr{K}_2) \underline{\boxtimes} \mathrm{D}'(\mathscr{K}_1 \underset{2}{\circ} \mathscr{K}_2) \arrow[uuurr, bend right] & &
\end{tikzcd}$$

Here, the left horizontal arrow on the top is the composition of morphisms $\omega_1^{\otimes -1} \to \mathscr{K}_1 \underset{2}{\circ} \mathrm{D}'_{\mathscr{A}}\mathscr{K}_1 \to \mathscr{K}_1 \underset{2}{\circ} \omega_2^{\otimes -1} \underset{2}{\circ} \omega_2 \underset{2}{\circ} \mathrm{D}'_{\mathscr{A}}\mathscr{K}_1$. The composition of the arrows on the bottom is $\mathrm{hh}_1(\mathscr{K}_1 \circ \mathscr{K}_2)$ by Lemma 4.1.4 and the composition of the arrows on the top is $\Phi_{\mathscr{K}_1}(\mathrm{hh}_2(\mathscr{K}_2))$. Hence, the assertion follows from the commutativity of the diagram by Lemma 4.3.4. Q.E.D.

Recall Diagram 3.4.4. Using (4.1.8), we get the commutative diagram

$$\begin{tikzcd}
\mathrm{K}_{\mathrm{coh},\Lambda_1}(\mathscr{A}_{12^a}) \times \mathrm{K}_{\mathrm{coh},\Lambda_2}(\mathscr{A}_{23^a}) \arrow[r, "\circ"] \arrow[d, "\mathrm{hh}_{12^a} \times \mathrm{hh}_{23^a}"] & \mathrm{K}_{\mathrm{coh},\Lambda}(\mathscr{A}_{13^a}) \arrow[d, "\mathrm{hh}_{13^a}"] \\
\mathrm{HH}_{\Lambda_1}(\mathscr{A}_{12^a}) \times \mathrm{HH}_{\Lambda_2}(\mathscr{A}_{23^a}) \arrow[r, "\circ"] & \mathrm{HH}_\Lambda(\mathscr{A}_{13^a}).
\end{tikzcd} \tag{4.3.9}$$

Remark 4.3.6. (i) The fact that Hochschild homology of $\mathscr{O}$-modules is functorial seems to be well-known, although we do not know any paper in which it is explicitly stated (for closely related results, see *e.g.,* [59, 14, 34]).
(ii) In [16], its authors interpret Hochschild homology as a morphism of functors and the action of kernels as a 2-morphism in a suitable 2-category. Its authors claim that the the relation $\Phi_{\mathscr{K}_1} \circ \Phi_{\mathscr{K}_1} = \Phi_{\mathscr{K}_1 \circ \mathscr{K}_2}$ follows by general arguments on 2-categories. Their result applies in a general framework

including in particular $\mathscr{O}$-modules in the algebraic case and presumably DQ-modules but the precise axioms are not specified in loc. cit. See also [59] for related results. Note that, as far as we understand, these authors do not introduce the convolution of Hochschild homologies and they did not consider Lemma 4.3.4 nor Theorem 4.3.5.

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Let $\mathbb{K}$ be a field, let $M \in \mathrm{D}^{\mathrm{b}}_f(\mathbb{K})$ and let $u \in \mathrm{End}(M)$. One sets

$$\begin{aligned}\mathrm{tr}(u, M) &= \sum_{i\in\mathbb{Z}} (-1)^i \,\mathrm{tr}(H^i(u)\colon H^i(M) \to H^i(M)),\\ \chi(M) &= \sum_{i\in\mathbb{Z}} (-1)^i \dim_{\mathbb{K}}(H^i(M)).\end{aligned}$$

If $X = \mathrm{pt}$, then $\mathcal{HH}(\mathscr{A}_X)$ is isomorphic to $\mathbb{C}^\hbar$, and $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X) = \mathrm{D}^{\mathrm{b}}_f(\mathbb{C}^\hbar)$.

Recall that we have set $\mathscr{M}^{\mathrm{loc}} = \mathbb{C}^{\hbar,\mathrm{loc}} \otimes_{\mathbb{C}^\hbar} \mathscr{M}$. For $M \in \mathrm{D}^{\mathrm{b}}_f(\mathbb{C}^\hbar)$ and $u \in \mathrm{End}(M)$, we have

$$\mathrm{hh}_{\mathrm{pt}}((M, u)) = \mathrm{tr}(u^{\mathrm{loc}}, M^{\mathrm{loc}}). \tag{4.3.10}$$

In particular,

$$\mathrm{hh}_{\mathrm{pt}}(M) \;=\; \chi(M^{\mathrm{loc}}).$$

Moreover, we have

$$\begin{aligned}\chi(M^{\mathrm{loc}}) &= \chi(\mathrm{gr}_\hbar(M))\\ &= \sum_{i\in\mathbb{Z}} (-1)^i \big(\dim_{\mathbb{C}}(\mathbb{C} \otimes_{\mathbb{C}^\hbar} H^i(M)) - \dim_{\mathbb{C}} \mathrm{Tor}_1^{\mathbb{C}^\hbar}(\mathbb{C}, H^i(M))\big).\end{aligned}$$

In the sequel, we set

$$\chi(M) := \chi(M^{\mathrm{loc}}).$$

As a particular case of Theorem 4.3.5, consider two objects $\mathscr{M}$ and $\mathscr{N}$ in $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X)$ and assume that $\mathrm{Supp}(\mathscr{M}) \cap \mathrm{Supp}(\mathscr{N})$ is compact. Then $\mathrm{RHom}_{\mathscr{A}_X}(\mathscr{M}, \mathscr{N})$ belongs to $\mathrm{D}^{\mathrm{b}}_f(\mathbb{C}^\hbar)$ and

$$\begin{aligned}\chi(\mathrm{RHom}_{\mathscr{A}_X}(\mathscr{M}, \mathscr{N})) &= \mathrm{hh}_{\mathrm{pt}}(\mathrm{D}'_{\mathscr{A}}\mathscr{M} \underset{X}{\circ} \mathscr{N})\\ &= \mathrm{hh}_{X^a}(\mathrm{D}'_{\mathscr{A}}\mathscr{M}) \underset{X}{\circ} \mathrm{hh}_X(\mathscr{N})\\ &= \mathrm{hh}_X(\mathscr{M}) \underset{X}{\circ} \mathrm{hh}_X(\mathscr{N}).\end{aligned}$$

Note that we have

$$\begin{aligned}\chi\big(\mathrm{RHom}_{\mathscr{A}_X}(\mathscr{M}, \mathscr{N})\big) &= \chi\big(\mathrm{RHom}_{\mathscr{A}_X^{\mathrm{loc}}}(\mathscr{M}^{\mathrm{loc}}, \mathscr{N}^{\mathrm{loc}})\big)\\ &= \chi\big(\mathrm{RHom}_{\mathrm{gr}_\hbar(\mathscr{A}_X)}(\mathrm{gr}_\hbar(\mathscr{M}), \mathrm{gr}_\hbar(\mathscr{N}))\big).\end{aligned}$$

4.4 Graded and localized Hochschild classes

Graded Hochschild classes

Similarly to the case of $\mathscr{A}_X$, one defines

$$\mathcal{HH}(\mathrm{gr}_\hbar(\mathscr{A}_X)) \ := \ \mathrm{gr}_\hbar(\mathscr{C}_{X^a})\overset{\mathrm{L}}{\otimes}_{\mathrm{gr}_\hbar(\mathscr{A}_{X\times X^a})}\mathrm{gr}_\hbar(\mathscr{C}_X).$$

Note that $\mathcal{HH}(\mathrm{gr}_\hbar(\mathscr{A}_X)) \simeq \mathbb{C}\overset{\mathrm{L}}{\otimes}_{\mathbb{C}^\hbar}\mathcal{HH}(\mathscr{A}_X)$ and there is a natural morphism

$$\mathrm{gr}_\hbar\colon \mathcal{HH}(\mathscr{A}_X) \to \mathcal{HH}(\mathrm{gr}_\hbar(\mathscr{A}_X)).$$

Notation 4.4.1. For a closed subset Λ of X, we set

$$\mathrm{HH}_\Lambda(\mathrm{gr}_\hbar\mathscr{A}_X) := H^0\mathrm{R}\Gamma_\Lambda(X;\mathcal{HH}(\mathrm{gr}_\hbar(\mathscr{A}_X))). \tag{4.4.1}$$

We also need to introduce

$$\widehat{\mathrm{HH}}_\Lambda(\mathrm{gr}_\hbar\mathscr{A}_X) := \varprojlim_U \mathrm{HH}_\Lambda(\mathrm{gr}_\hbar\mathscr{A}_U), \tag{4.4.2}$$

where U ranges over the family of relatively compact open subsets of X.

For $\mathscr{F} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathrm{gr}_\hbar(\mathscr{A}_X))$, one defines its Hochschild class $\mathrm{hh}_X(\mathscr{F})$ by the same construction as for $\mathscr{A}_X$-modules. For $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X)$, we have:

$$\mathrm{gr}_\hbar(\mathrm{hh}_X(\mathscr{M})) = \mathrm{hh}_X(\mathrm{gr}_\hbar(\mathscr{M})).$$

Theorem 4.3.5 obviously also holds when replacing $\mathscr{A}_X$ with $\mathrm{gr}_\hbar(\mathscr{A}_X)$.

Corollary 4.4.2. *Let Λ_i be a closed subset of $X_i \times X_{i+1}$ $(i = 1,2)$ and assume that $\Lambda_1 \times_{X_2} \Lambda_2$ is proper over $X_1 \times X_3$. Set $\Lambda = \Lambda_1 \circ \Lambda_2$. Let $\mathscr{K}_i \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh},\Lambda_i}(\mathrm{gr}_\hbar(\mathscr{A}_{X_i\times X^a_{i+1}}))$ $(i = 1,2)$. Then*

$$\mathrm{hh}_{X_{13^a}}(\mathscr{K}_1 \circ \mathscr{K}_2) = \mathrm{hh}_{X_{12^a}}(\mathscr{K}_1) \circ \mathrm{hh}_{X_{23^a}}(\mathscr{K}_2) \tag{4.4.3}$$

as elements of $\mathrm{HH}_\Lambda(\mathrm{gr}_\hbar\mathscr{A}_{X_1\times X_3^a})$.

It follows that the diagram below commutes

$$\begin{CD} \mathrm{K}_{\mathrm{coh},\Lambda_1}(\mathrm{gr}_\hbar\mathscr{A}_{12^a}) \times \mathrm{K}_{\mathrm{coh},\Lambda_2}(\mathrm{gr}_\hbar\mathscr{A}_{23^a}) @>{\circ}>> \mathrm{K}_{\mathrm{coh},\Lambda}(\mathrm{gr}_\hbar\mathscr{A}_{13^a}) \\ @V{\mathrm{hh}}VV @V{\mathrm{hh}}VV \\ \mathrm{HH}_{\Lambda_1}(\mathrm{gr}_\hbar\mathscr{A}_{12^a}) \times \mathrm{HH}_{\Lambda_2}(\mathrm{gr}_\hbar\mathscr{A}_{23^a}) @>{\circ}>> \mathrm{HH}_\Lambda(\mathrm{gr}_\hbar\mathscr{A}_{23^a}). \end{CD} \tag{4.4.4}$$

We shall study the Hochschild class of $\mathscr{O}$-modules with some details in Chapter 5.

Hochschild classes for $\mathscr{A}_X^{\mathrm{loc}}$

One defines

$$\mathcal{HH}(\mathscr{A}_X^{\mathrm{loc}}) \ := \ \mathscr{C}_{X^a}^{\mathrm{loc}} \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_{X\times X^a}^{\mathrm{loc}}} \mathscr{C}_X^{\mathrm{loc}}.$$

We have $\mathcal{HH}(\mathscr{A}_X^{\mathrm{loc}}) \simeq \mathbb{C}^{\hbar,\mathrm{loc}} \otimes_{\mathbb{C}^\hbar} \mathcal{HH}(\mathscr{A}_X)$. and there is a natural morphism

$$(\bullet)^{\mathrm{loc}} \colon \mathcal{HH}(\mathscr{A}_X) \to \mathcal{HH}(\mathscr{A}_X^{\mathrm{loc}}).$$

For $\mathscr{F} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X^{\mathrm{loc}})$, one defines its Hochschild class $\mathrm{hh}_X(\mathscr{F})$ by the same construction as for $\mathscr{A}_X$-modules. For $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X)$, setting $\mathscr{M}^{\mathrm{loc}} = \mathbb{C}^{\hbar,\mathrm{loc}} \otimes_{\mathbb{C}^\hbar} \mathscr{M}$, we have

$$(\mathrm{hh}_X(\mathscr{M}))^{\mathrm{loc}} = \mathrm{hh}_{\mathrm{X}}(\mathscr{M}^{\mathrm{loc}}).$$

Recall that the notion of good modules and the category $\mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{A}_X^{\mathrm{loc}})$ have been given in Definition 2.3.16. One immediately deduces from Theorem 4.3.5 the following:

Corollary 4.4.3. *Let Λ_i be a closed subset of $X_i \times X_{i+1}$ $(i = 1, 2)$ and assume that $\Lambda_1 \times_{X_2} \Lambda_2$ is proper over $X_1 \times X_3$. Set $\Lambda = \Lambda_1 \circ \Lambda_2$. Let $\mathscr{K}_i \in \mathrm{D}^{\mathrm{b}}_{\mathrm{gd},\Lambda_i}(\mathscr{A}^{\mathrm{loc}}_{X_i \times X^a_{i+1}})$ $(i = 1, 2)$. Then*

$$\mathrm{hh}_{\mathrm{X}_{13^a}}(\mathscr{K}_1 \circ \mathscr{K}_2) = \mathrm{hh}_{\mathrm{X}_{12^a}}(\mathscr{K}_1) \circ \mathrm{hh}_{\mathrm{X}_{23^a}}(\mathscr{K}_2) \tag{4.4.5}$$

as elements of $\mathrm{HH}_\Lambda(\mathscr{A}^{\mathrm{loc}}_{X_1\times X_3^a})$.

Using Proposition 3.4.3 and the additivity of the Hochschild class in Theorem 4.1.5, we find that there is a natural map

$$\widehat{\mathrm{K}}_{\mathrm{coh},\Lambda}(\mathrm{gr}_\hbar \mathscr{A}_X) \to \widehat{\mathrm{HH}}_\Lambda(\mathrm{gr}_\hbar \mathscr{A}_X). \tag{4.4.6}$$

For $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{gd},\Lambda}(\mathscr{A}_X^{\mathrm{loc}})$, we denote by $\widehat{\mathrm{hh}}{}^{\mathrm{gr}}_X(\mathscr{M})$ the image of $\mathscr{M}$ by the sequence of maps

$$\mathrm{D}^{\mathrm{b}}_{\mathrm{gd},\Lambda}(\mathscr{A}_X^{\mathrm{loc}}) \to \widehat{\mathrm{K}}_{\mathrm{coh},\Lambda}(\mathrm{gr}_\hbar \mathscr{A}_X) \to \widehat{\mathrm{HH}}_\Lambda(\mathrm{gr}_\hbar \mathscr{A}_X).$$

Let Λ_i be a closed subset of $X_i \times X_{i+1}$ $(i = 1, 2)$ and assume that $\Lambda_1 \times_{X_2} \Lambda_2$ is proper over $X_1 \times X_3$. Set $\Lambda = \Lambda_1 \circ \Lambda_2$.

Using the commutativity of Diagrams 3.4.5, we get that the diagram below commutes

$$(4.4.7)\qquad \begin{tikzcd} \mathrm{Ob}\big(\mathrm{D}^{\mathrm{b}}_{\mathrm{gd},\Lambda_1}(\mathscr{A}^{\mathrm{loc}}_{12^a})\big) \times \mathrm{Ob}\big(\mathrm{D}^{\mathrm{b}}_{\mathrm{gd},\Lambda_2}(\mathscr{A}^{\mathrm{loc}}_{23^a})\big) \arrow[r, "\circ"] \arrow[d, "\mathrm{gr}_\hbar"'] & \mathrm{Ob}\big(\mathrm{D}^{\mathrm{b}}_{\mathrm{gd},\Lambda}(\mathscr{A}^{\mathrm{loc}}_{13^a})\big) \arrow[d, "\mathrm{gr}_\hbar"'] \\ \widehat{\mathrm{K}}_{\mathrm{coh},\Lambda_1}(\mathrm{gr}_\hbar\mathscr{A}_{12^a}) \times \widehat{\mathrm{K}}_{\mathrm{coh},\Lambda_2}(\mathrm{gr}_\hbar\mathscr{A}_{23^a}) \arrow[r, "\circ"] \arrow[d] & \widehat{\mathrm{K}}_{\mathrm{coh},\Lambda}(\mathrm{gr}_\hbar\mathscr{A}_{13^a}) \arrow[d] \\ \widehat{\mathrm{HH}}_{\Lambda_1}(\mathrm{gr}_\hbar\mathscr{A}_{12^a}) \times \widehat{\mathrm{HH}}_{\Lambda_2}(\mathrm{gr}_\hbar\mathscr{A}_{23^a}) \arrow[r, "\circ"] & \widehat{\mathrm{HH}}_{\Lambda}(\mathrm{gr}_\hbar\mathscr{A}_{23^a}). \end{tikzcd}$$

In other words,

$$(4.4.8)\qquad \widehat{\mathrm{hh}}^{\mathrm{gr}}_{13^a}(\mathscr{K}_1 \circ \mathscr{K}_2) = \widehat{\mathrm{hh}}^{\mathrm{gr}}_{12^a}(\mathscr{K}_1) \circ \widehat{\mathrm{hh}}^{\mathrm{gr}}_{23^a}(\mathscr{K}_2).$$

Corollary 4.4.4. *Let $\mathscr{M}, \mathscr{N} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{A}^{\mathrm{loc}}_X)$ and assume that $K := \mathrm{Supp}(\mathscr{M}) \cap \mathrm{Supp}(\mathscr{N})$ is compact. Then $R\mathscr{H}om_{\mathscr{A}_X}(\mathscr{M}, \mathscr{N})$ belongs to $\mathrm{D}^{\mathrm{b}}_f(\mathbb{C}^\hbar)$ and*

$$\begin{aligned} \chi(R\mathrm{Hom}_{\mathscr{A}^{\mathrm{loc}}_X}(\mathscr{M}, \mathscr{N})) &= \widehat{\mathrm{hh}}^{\mathrm{gr}}_{X^a}(\mathrm{D}'_{\mathscr{A}}\mathscr{M}) \circ \widehat{\mathrm{hh}}^{\mathrm{gr}}_X(\mathscr{N}) \\ &= \widehat{\mathrm{hh}}^{\mathrm{gr}}_X(\mathscr{M}) \circ \widehat{\mathrm{hh}}^{\mathrm{gr}}_U(\mathscr{N}). \end{aligned}$$

Proof. One has by (3.4.3)

$$\begin{aligned} \chi(R\mathrm{Hom}_{\mathscr{A}^{\mathrm{loc}}_X}(\mathscr{M}, \mathscr{N})) &= \mathrm{hh}_{\mathrm{pt}}(\mathrm{D}'_{\mathscr{A}}\mathscr{M} \circ \mathscr{N}) \\ &= \widehat{\mathrm{hh}}^{\mathrm{gr}}_{\mathrm{pt}}(\mathrm{D}'_{\mathscr{A}}\mathscr{M} \circ \mathscr{N}) = \widehat{\mathrm{hh}}^{\mathrm{gr}}_{X^a}(\mathrm{D}'_{\mathscr{A}}\mathscr{M}) \circ \widehat{\mathrm{hh}}^{\mathrm{gr}}_X(\mathscr{N}) \end{aligned}$$

and the last equality follows from (4.1.9). Q.E.D.

Remark 4.4.5. In the algebraic case, that is, in the situation of § 2.7, one should replace $\widehat{\mathrm{K}}_{\mathrm{coh},\Lambda}$ with $\mathrm{K}_{\mathrm{coh},\Lambda}$ and $\widehat{\mathrm{HH}}_\Lambda(\mathrm{gr}_\hbar\mathscr{A}_X)$ with $\mathrm{HH}_\Lambda(\mathrm{gr}_\hbar\mathscr{A}_X)$.

We shall explain how to calculate $\widehat{\mathrm{hh}}^{\mathrm{gr}}_X$ in Chapter 5.

Chapter 5

The commutative case

We shall make the link between the Hochschild class and the Chern and Euler classes of coherent $\mathscr{O}_X$-modules, following [36], an unpublished letter from the first named author (M.K) to the second (P.S), dated 18/11/1991.

5.1 Hochschild class of $\mathscr{O}$-modules

In this section, we shall study the Hochschild class in the particular case of a trivial deformation. In this case, the formal parameter $\hbar$ doesn't play any role, and we may work with $\mathscr{O}$-modules. We shall use the same notations for $\mathscr{O}_X$-modules as for $(\mathscr{O}_X[[\hbar]], \star)$-modules where $\star$ is the usual commutative product.

Note that the results of this section are well known from the specialists. Let us quote in particular [14, 15, 16, 34, 49, 54, 59, 64].

Let $(X, \mathscr{O}_X)$ be a complex manifold of complex dimension d_X. As usual, we denote by $\delta_X \colon X \hookrightarrow X \times X$ the diagonal embedding. We denote by Ω_X^i the sheaf of holomorphic i-forms and one sets $\Omega_X := \Omega_X^{d_X}$. We set

$$\omega_X := \Omega_X\,[d_X].$$

We denote by $\mathrm{D}'_{\mathscr{O}}$ and $\mathrm{D}_{\mathscr{O}}$ the duality functors

$$\mathrm{D}'_{\mathscr{O}}(\mathscr{F}) = \mathrm{R}\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{O}_X), \quad \mathrm{D}_{\mathscr{O}}(\mathscr{F}) = \mathrm{R}\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \omega_X).$$

When there is no risk of confusion, we write D' and D instead of $\mathrm{D}'_{\mathscr{O}}$ and $\mathrm{D}_{\mathscr{O}}$, respectively.

Let $f \colon X \to Y$ be a morphism of complex manifolds. For $\mathscr{G} \in \mathrm{D}^{\mathrm{b}}(\mathscr{O}_Y)$, we set

$$f^*\mathscr{G} := \mathscr{O}_X \overset{\mathrm{L}}{\otimes}_{f^{-1}\mathscr{O}_Y} f^{-1}\mathscr{G}.$$

We use the notation $H^0(f^*)\colon \mathrm{Mod}(\mathscr{O}_Y) \to \mathrm{Mod}(\mathscr{O}_X)$ for the (non derived) inverse image functor.

The Hochschild homology of $\mathscr{O}_X$ is given by:

$$\mathcal{HH}(\mathscr{O}_X) := \delta_X^*\delta_{X*}\mathscr{O}_X, \text{ an object of } \mathrm{D^b}(\mathscr{O}_X). \tag{5.1.1}$$

Note that $\delta_{X!} \simeq \delta_{X*} \simeq \mathrm{R}\delta_{X*}$, and moreover

$$\delta_{X*}\mathcal{HH}(\mathscr{O}_X) \simeq \delta_{X*}(\mathscr{O}_X \overset{\mathrm{L}}{\otimes} \delta_X^*(\delta_{X*}\mathscr{O}_X)) \simeq \delta_{X*}\mathscr{O}_X \overset{\mathrm{L}}{\otimes}_{\mathscr{O}_{X\times X}} \delta_{X*}\mathscr{O}_X. \tag{5.1.2}$$

By reformulating the construction of the Hochschild class for modules over DQ-algebroids, we get

Definition 5.1.1. For $\mathscr{F} \in \mathrm{D^b_{coh}}(\mathscr{O}_X)$, we define its Hochschild class $\mathrm{hh}_X(\mathscr{F}) \in H^0_{\mathrm{Supp}\,\mathscr{F}}(X; \delta_X^*\delta_{X*}\mathscr{O}_X)$ as the composition

$$\mathscr{O}_X \to \mathrm{R}\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F},\mathscr{F}) \xrightarrow{\sim} \delta_X^*(\mathscr{F} \underline{\boxtimes} \mathrm{D}'\mathscr{F}) \to \delta_X^*\delta_{X*}\mathscr{O}_X. \tag{5.1.3}$$

Here the morphism $\mathscr{F}\underline{\boxtimes}\mathrm{D}'\mathscr{F} \to \delta_{X*}\mathscr{O}_X$ is deduced from the morphism $\delta_X^*(\mathscr{F}\underline{\boxtimes}\mathrm{D}'\mathscr{F}) \xrightarrow{\sim} \mathscr{F} \overset{\mathrm{L}}{\otimes}_{\mathscr{O}_X} \mathrm{D}'\mathscr{F} \to \mathscr{O}_X$ by adjunction.

Applying Theorem 4.3.5, we get that for two complex manifolds X and Y and for $\mathscr{F} \in \mathrm{D^b_{coh}}(\mathscr{O}_X)$ and $\mathscr{G} \in \mathrm{D^b_{coh}}(\mathscr{O}_Y)$, we have

$$\mathrm{hh}_{X\times Y}(\mathscr{F}\underline{\boxtimes}\mathscr{G}) = \mathrm{hh}_X(\mathscr{F}) \boxtimes \mathrm{hh}_Y(\mathscr{G}).$$

Let $f\colon X \to Y$ be a morphism of complex manifolds and denote by $\Gamma_f \subset X \times Y$ its graph. We denote by $\mathrm{hh}_{X\times Y}(\mathscr{O}_{\Gamma_f})$ the Hochschild class of the coherent $\mathscr{O}_{X\times Y}$-module $\mathscr{O}_{\Gamma_f}$. Hence

$$\mathrm{hh}_{X\times Y}(\mathscr{O}_{\Gamma_f}) \in H^0(X\times Y; \mathcal{HH}(\mathscr{O}_{X\times Y})).$$

Applying Theorem 4.3.5, we get

Corollary 5.1.2. (i) *Let $\mathscr{G} \in \mathrm{D^b_{coh}}(\mathscr{O}_Y)$. Then*

$$\mathrm{hh}_X(f^*\mathscr{G}) = \mathrm{hh}_{X\times Y}(\mathscr{O}_{\Gamma_f}) \circ \mathrm{hh}_Y(\mathscr{G}).$$

(ii) *Let $\mathscr{F} \in \mathrm{D^b_{coh}}(\mathscr{O}_X)$ and assume that f is proper on* $\mathrm{Supp}(\mathscr{F})$. *Then*

$$\mathrm{hh}_Y(\mathrm{R}f_!\mathscr{F}) = \mathrm{hh}_X(\mathscr{F}) \circ \mathrm{hh}_{X\times Y}(\mathscr{O}_{\Gamma_f}).$$

In Proposition 5.1.3 and 5.2.3 below, we give a more direct description of the maps $\mathrm{hh}_{X\times Y}(\mathscr{O}_{\Gamma_f})\circ$ and $\circ\,\mathrm{hh}_{X\times Y}(\mathscr{O}_{\Gamma_f})$.

Proposition 5.1.3. *Let $f\colon X \to Y$ be a morphism of complex manifolds.*

(i) *There is a canonical morphism*

$$f^*\delta_Y^*\delta_{Y*}\mathscr{O}_Y \to \delta_X^*\delta_{X*}\mathscr{O}_X. \tag{5.1.4}$$

(ii) *This morphism together with the isomorphism $\mathscr{O}_X \xleftarrow{\sim} f^*\mathscr{O}_Y$ induces a morphism*

$$f^*\colon H^0(\mathrm{R}\Gamma(Y;\delta_Y^*\delta_{Y*}\mathscr{O}_Y)) \to H^0(\mathrm{R}\Gamma(X;\delta_X^*\delta_{X*}\mathscr{O}_X)) \tag{5.1.5}$$

and for $\mathscr{G} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_Y)$, we have

$$\mathrm{hh}_X(f^*\mathscr{G}) = f^*\mathrm{hh}_Y(\mathscr{G}). \tag{5.1.6}$$

Proof. (i) Consider the diagram

$$\begin{tikzcd} X \arrow[r, "\delta_X"] \arrow[d, "f"'] & X \times X \arrow[d, "\widetilde{f}"'] \\ Y \arrow[r, "\delta_Y"] & Y \times Y. \end{tikzcd} \tag{5.1.7}$$

Then we have morphisms

$$f^*\delta_Y^*\delta_{Y*}\mathscr{O}_Y \simeq \delta_X^*\widetilde{f}^*\delta_{Y*}\mathscr{O}_Y \to \delta_X^*\delta_{X*}f^*\mathscr{O}_Y \simeq \delta_X^*\delta_{X*}\mathscr{O}_X.$$

Here the arrow $\widetilde{f}^*\delta_{Y*} \to \delta_{X*}f^*$ is deduced by adjunction from

$$\delta_{Y*} \quad\to\quad \delta_{Y*}\mathrm{R}f_*f^* \simeq \mathrm{R}\widetilde{f}_*\delta_{X*}f^*.$$

(ii) The diagram

$$\begin{tikzcd} \widetilde{f}^*(\mathscr{G} \boxtimes \mathrm{D}'\mathscr{G}) \arrow[r] \arrow[d, "\sim"] & \widetilde{f}^*\delta_{Y*}\mathscr{O}_Y \arrow[d] \\ f^*\mathscr{G} \boxtimes f^*\mathrm{D}'\mathscr{G} \arrow[d, "\sim"] & \delta_{X*}f^*\mathscr{O}_Y \arrow[d, "\sim"] \\ f^*\mathscr{G} \boxtimes \mathrm{D}'f^*\mathscr{G} \arrow[r] & \delta_{X*}\mathscr{O}_X \end{tikzcd}$$

commutes. It follows that the diagram below commutes.

$$
\begin{tikzcd}
f^*\mathscr{O}_Y \arrow[r] \arrow[ddd, "\sim"] & f^*\delta_Y^*(\mathscr{G}\boxtimes \mathrm{D}'\mathscr{G}) \arrow[r] \arrow[d, "\sim"] & f^*\delta_Y^*\delta_{Y*}\mathscr{O}_Y \arrow[d, "\sim"] \\
 & \delta_X^*\widetilde{f}^*(\mathscr{G}\boxtimes \mathrm{D}'\mathscr{G}) \arrow[r] \arrow[d, "\sim"] & \delta_X^*\widetilde{f}^*\delta_{Y*}\mathscr{O}_Y \arrow[d] \\
 & \delta_X^*(f^*\mathscr{G}\boxtimes f^*\mathrm{D}'\mathscr{G}) \arrow[d, "\sim"] & \delta_X^*\delta_{X*}f^*\mathscr{O}_Y \arrow[d, "\sim"] \\
\mathscr{O}_X \arrow[r] & \delta_X^*(f^*\mathscr{G}\boxtimes \mathrm{D}'f^*\mathscr{G}) \arrow[r] & \delta_X^*\delta_{X*}\mathscr{O}_X .
\end{tikzcd}
$$

Therefore, the image of $\mathrm{hh}_Y(\mathscr{G}) \in \mathrm{Hom}_{\mathscr{O}_Y}(\mathscr{O}_Y, \delta_Y^*\delta_{Y*}\mathscr{O}_Y)$ by the maps

$$
\begin{aligned}
\mathrm{Hom}_{\mathscr{O}_Y}(\mathscr{O}_Y, \delta_Y^*\delta_{Y*}\mathscr{O}_Y) &\to \mathrm{Hom}_{\mathscr{O}_X}(f^*\mathscr{O}_Y, f^*\delta_Y^*\delta_{Y*}\mathscr{O}_Y) \\
&\to \mathrm{Hom}_{\mathscr{O}_X}(\mathscr{O}_X, \delta_X^*\delta_{X*}\mathscr{O}_X)
\end{aligned}
$$

is $\mathrm{hh}_X(f^*\mathscr{G})$. Q.E.D.

Remark 5.1.4. Although we omit the proof, the map in (5.1.5) coincides with $\mathrm{hh}_{X\times Y}(\mathscr{O}_{\Gamma_f})\circ$.

Ring structure

For an exposition on tensor categories, we refer to [42].

Proposition 5.1.5. (i) *The object $\delta_X^*\delta_{X*}\mathscr{O}_X$ is a ring in the tensor category $(\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_X), \overset{\mathrm{L}}{\otimes}_{\mathscr{O}_X})$. More precisely,*

(a) *the map μ obtained as the composition*

$$
\begin{aligned}
\delta_X^*\delta_{X*}\mathscr{O}_X \overset{\mathrm{L}}{\otimes}_{\mathscr{O}_X} \delta_X^*\delta_{X*}\mathscr{O}_X &\xrightarrow{\sim} \delta_X^*(\delta_{X*}\mathscr{O}_X \overset{\mathrm{L}}{\otimes}_{\mathscr{O}_{X\times X}} \delta_{X*}\mathscr{O}_X) \\
&\to \delta_X^*\delta_{X*}\mathscr{O}_X .
\end{aligned}
$$

is associative. Here the last arrow is induced by $\delta_{X*}\mathscr{O}_X \otimes \delta_{X*}\mathscr{O}_X \to \delta_{X*}\mathscr{O}_X$.

(b) *$\mathrm{hh}_X(\mathscr{O}_X)$ is a unit of this ring. More precisely, the natural morphism ε defined as the composition*

$$
\varepsilon\colon \mathscr{O}_X \xrightarrow{\sim} \delta_X^*\mathscr{O}_{X\times X} \to \delta_X^*\delta_{X*}\mathscr{O}_X
$$

has the property that the composition

$$\begin{aligned}\delta_X^*\delta_{X*}\mathscr{O}_X \simeq \delta_X^*\delta_{X*}\mathscr{O}_X\overset{\mathrm{L}}{\otimes}_{\mathscr{O}_X}\mathscr{O}_X &\xrightarrow{\varepsilon} \delta_X^*\delta_{X*}\mathscr{O}_X\overset{\mathrm{L}}{\otimes}_{\mathscr{O}_X}\delta_X^*\delta_{X*}\mathscr{O}_X \\ &\xrightarrow{\mu} \delta_X^*\delta_{X*}\mathscr{O}_X\end{aligned}$$

is the identity.

(ii) *The ring $(\delta_X^*\delta_{X*}\mathscr{O}_X,\mu)$ is commutative. More precisely, we have $\mu\circ\sigma=\mu$, where $\sigma\in\mathrm{Aut}_{\mathrm{D^b}(\mathscr{O}_X)}(\delta_X^*\delta_{X*}\mathscr{O}_X\overset{\mathrm{L}}{\otimes}_{\mathscr{O}_X}\delta_X^*\delta_{X*}\mathscr{O}_X)$ is the morphism associated with $x\otimes x'\mapsto x'\otimes x$.*

(iii) *The object $\delta_X^!\delta_{X!}\omega_X$ has a structure of a $\delta_X^*\delta_{X*}\mathscr{O}_X$-module. More precisely, the composition*

$$\begin{aligned}\delta_X^*\delta_{X*}\mathscr{O}_X\overset{\mathrm{L}}{\otimes}_{\mathscr{O}_X}\delta_X^!\delta_{X!}\omega_X &\to \delta_X^!(\delta_{X*}\mathscr{O}_X\overset{\mathrm{L}}{\otimes}_{\mathscr{O}_{X\times X}}\delta_{X!}\omega_X) \\ &\to \delta_X^!\delta_{X!}\omega_X.\end{aligned}$$

is associative and preserves the unit. Here, the last arrow is induced by $\delta_{X*}\mathscr{O}_X\overset{\mathrm{L}}{\otimes}_{\mathscr{O}_{X\times X}}\delta_{X!}\omega_X \simeq \delta_{X*}(\delta_X^*\delta_{X*}\mathscr{O}_X\otimes_{\mathscr{O}_X}\omega_X)\to\delta_{X*}(\mathscr{O}_X\otimes_{\mathscr{O}_X}\omega_X)\simeq \delta_{X*}\omega_X$ by adjunction.

Proof. The verification of these assertions is left to the reader. We only remark that the commutativity and associativity are consequences of the corresponding properties of $\delta_{X*}\mathscr{O}_X$. For example, the commutativity is the consequence of the commutativity of $\delta_{X*}\mathscr{O}_X\overset{\mathrm{L}}{\otimes}_{\mathscr{O}_{X\times X}}\delta_{X*}\mathscr{O}_X\to\delta_{X*}\mathscr{O}_X$. Q.E.D.

Notation 5.1.6. For $\lambda_i\in H^0_{\Lambda_i}(X;\delta_X^*\delta_{X*}\mathscr{O}_X)$ $(i=1,2)$, we define their product $\lambda_1\bullet\lambda_2$ as the composition

$$\mathscr{O}_X\xrightarrow{\sim}\mathscr{O}_X\overset{\mathrm{L}}{\otimes}_{\mathscr{O}_X}\mathscr{O}_X\xrightarrow{\lambda_1\otimes\lambda_2}\delta_X^*\delta_{X*}\mathscr{O}_X\overset{\mathrm{L}}{\otimes}_{\mathscr{O}_X}\delta_X^*\delta_{X*}\mathscr{O}_X\xrightarrow{\mu}\delta_X^*\delta_{X*}\mathscr{O}_X.$$

Proposition 5.1.7. *Let $\mathscr{F}_i\in\mathrm{D^b_{coh}}(\mathscr{O}_X)$ $(i=1,2)$. Then*

$$\mathrm{hh}_X(\mathscr{F}_1\overset{\mathrm{L}}{\otimes}_{\mathscr{O}_X}\mathscr{F}_2)=\mathrm{hh}_X(\mathscr{F}_1)\bullet\mathrm{hh}_X(\mathscr{F}_2). \tag{5.1.8}$$

Proof. Consider the commutative diagram below (in which $\otimes$ stands for $\otimes_{\mathscr{O}}$):

$$
\begin{tikzcd}
\mathscr{O}_X \arrow[r] \arrow[dddr, rounded corners] & \mathscr{O}_X \otimes \mathscr{O}_X \arrow[d] & \\
 & \delta_X^*(\mathscr{F}_1 \boxtimes \mathrm{D}'\mathscr{F}_1) \otimes \delta_X^*(\mathscr{F}_2 \boxtimes \mathrm{D}'\mathscr{F}_2) \arrow[r] \arrow[d, "\sim"] & \delta_X^*\delta_{X*}\mathscr{O}_X \otimes \delta_X^*\delta_{X*}\mathscr{O}_X \arrow[d, "\wr"] \\
 & \delta_X^*\big((\mathscr{F}_1 \boxtimes \mathrm{D}'\mathscr{F}_1) \otimes (\mathscr{F}_2 \boxtimes \mathrm{D}'\mathscr{F}_2)\big) \arrow[r] \arrow[d] & \delta_X^*(\delta_{X*}\mathscr{O}_X \otimes \delta_{X*}\mathscr{O}_X) \arrow[d] \\
 & \delta_X^*\big((\mathscr{F}_1 \otimes \mathscr{F}_2) \boxtimes \mathrm{D}'(\mathscr{F}_1 \otimes \mathscr{F}_2)\big) \arrow[r] & \delta_X^*(\delta_{X*}\mathscr{O}_X).
\end{tikzcd}
$$

The composition of the arrows on the top and the right gives $\mathrm{hh}_X(\mathscr{F}_1) \bullet \mathrm{hh}_X(\mathscr{F}_2)$ and the composition of the arrows on the left and the bottom gives $\mathrm{hh}_X(\mathscr{F}_1 \overset{\mathrm{L}}{\otimes}_{\mathscr{O}_X} \mathscr{F}_2)$. Q.E.D.

Note that

$$\mathrm{hh}_X(\mathscr{F}_1 \overset{\mathrm{L}}{\otimes}_{\mathscr{O}_X} \mathscr{F}_2) = \delta_X^*(\mathrm{hh}_X(\mathscr{F}_1) \boxtimes \mathrm{hh}_X(\mathscr{F}_2)).$$

5.2 co-Hochschild class

Definition 5.2.1. For $\mathscr{F} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_X)$, we define its co-Hochschild class $\mathrm{thh}_X(\mathscr{F}) \in H^0_{\mathrm{Supp}\,\mathscr{F}}(X; \delta_X^! \delta_{X!}\omega_X)$ as the composition

$$\mathscr{O}_X \to \mathrm{R}\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{F}) \simeq \delta_X^!(\mathscr{F} \underline{\boxtimes} \mathrm{D}_{\mathscr{O}}\mathscr{F}) \to \delta_X^! \delta_{X!}\omega_X. \tag{5.2.1}$$

Here, the morphism $(\mathscr{F} \underline{\boxtimes} \mathrm{D}_{\mathscr{O}}\mathscr{F}) \to \delta_{X!}\omega_X$ is induced from $\delta_X^*(\mathscr{F} \underline{\boxtimes} \mathrm{D}_{\mathscr{O}}\mathscr{F}) \simeq \mathscr{F} \overset{\mathrm{L}}{\otimes}_{\mathscr{O}_X} \mathrm{D}_{\mathscr{O}}\mathscr{F} \to \omega_X$ by adjunction.

Consider the sequence of isomorphisms

$$
\begin{aligned}
\delta_X^*\delta_{X*}\mathscr{O}_X &\xrightarrow{\sim} \mathscr{O}_X \overset{\mathrm{L}}{\otimes}_{\mathscr{O}_X} \delta_X^*\delta_{X*}\mathscr{O}_X \xrightarrow{\sim} \delta_X^!(\mathscr{O}_X \underline{\boxtimes} \omega_X) \overset{\mathrm{L}}{\otimes}_{\mathscr{O}_X} \delta_X^*\delta_{X*}\mathscr{O}_X \\
&\xrightarrow{\sim} \delta_X^!((\mathscr{O}_X \underline{\boxtimes} \omega_X) \overset{\mathrm{L}}{\otimes}_{\mathscr{O}_X} \delta_{X*}\mathscr{O}_X) \xrightarrow{\sim} \delta_X^!\delta_{X*}(\delta_X^*(\mathscr{O}_X \underline{\boxtimes} \omega_X) \overset{\mathrm{L}}{\otimes}_{\mathscr{O}_X} \mathscr{O}_X) \\
&\xrightarrow{\sim} \delta_X^!\delta_{X!}\omega_X.
\end{aligned}
$$

We denote by td the isomorphism

$$\mathrm{td}\colon \delta_X^*\delta_{X*}\mathscr{O}_X \xrightarrow{\sim} \delta_X^!\delta_{X!}\omega_X \tag{5.2.2}$$

constructed above. For a closed subset $S \subset X$, we keep the same notation td to denote the isomorphism

$$\mathrm{td}\colon H^0_S(X; \delta_X^* \delta_{X*} \mathscr{O}_X) \xrightarrow{\sim} H^0_S(X; \delta_X^! \delta_{X!} \omega_X). \tag{5.2.3}$$

Proposition 5.2.2. *For $\mathscr{F} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_X)$, we have*

$$\mathrm{thh}_X(\mathscr{F}) = \mathrm{td} \circ \mathrm{hh}_X(\mathscr{F}). \tag{5.2.4}$$

Proof. The proof follows from the commutativity of the diagram below in which we use the natural morphism $\mathscr{O}_X \to \delta_X^!(\mathscr{O}_X \boxtimes \omega_X)$

$$\begin{tikzcd}
\mathscr{O}_X \arrow[r] \arrow[ddddr, bend right] & \delta_X^*(\mathscr{F} \boxtimes \mathrm{D}'\mathscr{F}) \arrow[r] \arrow[d] & \delta_X^* \delta_{X*} \mathscr{O}_X \arrow[d, "\wr"] \\
 & \delta_X^!(\mathscr{O}_X \boxtimes \omega_X) \otimes \delta_X^*(\mathscr{F} \boxtimes \mathrm{D}'\mathscr{F}) \arrow[r] \arrow[d] & \delta_X^!(\mathscr{O}_X \boxtimes \omega_X) \otimes \delta_X^* \delta_{X*} \mathscr{O}_X \arrow[d, "\wr"] \\
 & \delta_X^!((\mathscr{O}_X \boxtimes \omega_X) \otimes (\mathscr{F} \boxtimes \mathrm{D}'\mathscr{F})) \arrow[r] \arrow[dd] & \delta_X^!((\mathscr{O}_X \boxtimes \omega_X) \otimes \delta_{X*} \mathscr{O}_X) \arrow[d, "\wr"] \\
 & & \delta_X^! \delta_{X*}\big(\delta_X^!(\mathscr{O}_X \boxtimes \omega_X) \otimes \mathscr{O}_X\big) \arrow[d, "\wr"] \\
 & \delta_X^!(\mathscr{F} \boxtimes \mathrm{D}\mathscr{F}) \arrow[r] & \delta_X^! \delta_{X!} \omega_X .
\end{tikzcd}$$

Q.E.D.

For a morphism $f\colon X \to Y$ of complex manifolds, we denote by $\Gamma_{\mathrm{f-pr}}(X; \bullet)$ the functor of global sections with f-proper supports.

Proposition 5.2.3. *Let $f\colon X \to Y$ be a morphism of complex manifolds.*

(i) *There is a canonical morphism*

$$\mathrm{R}f_! \delta_X^! \delta_{X!} \omega_X \to \delta_Y^! \delta_{Y!} \omega_Y. \tag{5.2.5}$$

(ii) *This morphism together with the morphism $\mathscr{O}_Y \to \mathrm{R}f_* \mathscr{O}_X$ induces a morphism*

$$f_!\colon H^0(\mathrm{R}\Gamma_{\mathrm{f-pr}}(X; \delta_X^! \delta_{X!} \omega_X)) \to H^0(\mathrm{R}\Gamma(Y; \delta_Y^! \delta_{Y!} \omega_Y)) \tag{5.2.6}$$

and for $\mathscr{F} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_X)$ such that f is proper on $\mathrm{Supp}(\mathscr{F})$*, we have*

$$\mathrm{thh}_Y(\mathrm{R}f_! \mathscr{F}) = f_! \,\mathrm{thh}_X(\mathscr{F}). \tag{5.2.7}$$

Proof. (i) Consider the diagram (5.1.7). Then we have morphisms

$$\mathrm{R}f_!\delta_X^!\delta_{X!}\omega_X \to \delta_Y^!\mathrm{R}\widetilde{f}_!\delta_{X!}\omega_X \simeq \delta_Y^!\delta_{Y!}\mathrm{R}f_!\omega_X \to \delta_Y^!\delta_{Y!}\omega_Y.$$

Here, the first morphism is deduced by adjunction from

$$\delta_X^! \to \delta_X^!\widetilde{f}^!\mathrm{R}\widetilde{f}_! \simeq f^!\delta_Y^!\mathrm{R}\widetilde{f}_!.$$

(ii) The proof is similar to that of Proposition 5.1.3 and follows from the commutativity of the diagram below in which we write for short $f_!$ and f_* instead of $\mathrm{R}f_!$ and $\mathrm{R}f_*$ and similarly with $\widetilde{f}$.

$$\begin{tikzcd}
f_*\mathscr{O}_X \arrow[r] & f_*\delta_X^!(\mathscr{F}\boxtimes\mathrm{D}\mathscr{F}) \arrow[r] \arrow[d, "\sim"] & f_!\delta_X^!\delta_{X!}\omega_X \arrow[d] \\
& \delta_Y^!\widetilde{f}_!(\mathscr{F}\boxtimes\mathrm{D}'\mathscr{F}) \arrow[r] \arrow[d, "\sim"] & \delta_Y^!\widetilde{f}_!\delta_{X!}\omega_X \arrow[d, "\sim"] \\
& \delta_Y^!(f_!\mathscr{F}\boxtimes f_!\mathrm{D}\mathscr{F}) \arrow[d, "\sim"] & \delta_Y^!\delta_{Y!}f_!\omega_X \arrow[d] \\
\mathscr{O}_Y \arrow[r] \arrow[uuu] & \delta_Y^!(f_!\mathscr{F}\boxtimes\mathrm{D}f_!\mathscr{F}) \arrow[r] & \delta_Y^!\delta_{Y!}\omega_Y.
\end{tikzcd}$$

Therefore, the image of $\mathrm{thh}_X(\mathscr{F}) \in \mathrm{Hom}_{\mathscr{O}_X}(\mathscr{O}_X, \delta_X^!\delta_{X!}\omega_X)$ by the maps

$$\begin{aligned}
\Gamma_{\mathrm{f-pr}}(X; \mathscr{H}om_{\mathscr{O}_X}(\mathscr{O}_X, \delta_X^!\delta_{X!}\omega_X)) &\to \mathrm{Hom}_{\mathscr{O}_X}(\mathrm{R}f_*\mathscr{O}_X, \mathrm{R}f_!\delta_X^!\delta_{X!}\omega_X) \\
&\to \mathrm{Hom}_{\mathscr{O}_Y}(\mathscr{O}_Y, \delta_Y^!\delta_{Y!}\omega_Y)
\end{aligned}$$

is $\mathrm{thh}_Y(f_!\mathscr{F})$. Q.E.D.

Remark 5.2.4. Although we omit the proof, the map in (5.2.6) coincides with $\circ\ \mathrm{hh}_{X\times Y}(\mathscr{O}_{\Gamma_f})$.

5.3 Chern and Euler classes of $\mathscr{O}$-modules

The Hodge cohomology of $\mathscr{O}_X$ is given by:

(5.3.1) $$\mathcal{HD}(\mathscr{O}_X) := \bigoplus_{i=0}^{d_X} \Omega_X^i\,[i], \text{ an object of } \mathrm{D^b}(\mathscr{O}_X).$$

Lemma 5.3.1. *Let $f\colon X \to Y$ be a morphism of complex manifolds. There are canonical morphisms*

$$\underline{\boxtimes} \;:\; \mathcal{HD}(\mathscr{O}_X) \boxtimes \mathcal{HD}(\mathscr{O}_Y) \to \mathcal{HD}(\mathscr{O}_{X\times Y}), \tag{5.3.2}$$

$$f^* \;:\; f^*\mathcal{HD}(\mathscr{O}_Y) \to \mathcal{HD}(\mathscr{O}_X), \tag{5.3.3}$$

$$f_! \;:\; \mathrm{R}f_!\mathcal{HD}(\mathscr{O}_X) \to \mathcal{HD}(\mathscr{O}_Y). \tag{5.3.4}$$

Proof. The morphisms (5.3.2), (5.3.3) and (5.3.4) are respectively associated with the morphisms

$$\begin{aligned}
&\Omega^i_X\,[i] \boxtimes \Omega^j_Y\,[j] \longrightarrow \Omega^{i+j}_{X\times Y}\,[i+j],\\
&f^*\Omega^i_Y\,[i] \longrightarrow \Omega^i_X\,[i],\\
&\mathrm{R}f_!\Omega^{i+d_X}_X\,[i+d_X] \longrightarrow \Omega^{i+d_Y}_Y\,[i+d_Y].
\end{aligned}$$

Q.E.D.

Theorem 5.3.2. (a) *There is an isomorphism*

$$\alpha_X\colon \delta_X^*\delta_{X*}\mathscr{O}_X \xrightarrow{\sim} \mathcal{HD}(\mathscr{O}_X)$$

which commutes with the functors $\boxtimes$ and f^*.

(b) *There is an isomorphism*

$$\beta_X\colon \mathcal{HD}(\mathscr{O}_X) \xrightarrow{\sim} \delta_X^!\delta_{X!}\omega_X$$

which commutes with the functors $\boxtimes$ and $f_!$.

Setting $\tau := \beta_X^{-1} \circ \mathrm{td} \circ \alpha_X^{-1}$, we get a commutative diagram in $\mathrm{D^b}(\mathscr{O}_X)$:

$$\begin{tikzcd}
\delta_X^*\delta_{X*}\mathscr{O}_X \arrow[r, "\sim", "\mathrm{td}"'] \arrow[d, "\alpha_X"', "\sim"] & \delta_X^!\delta_{X!}\omega_X \\
\mathcal{HD}(\mathscr{O}_X) \arrow[r, "\sim", "\tau"'] & \mathcal{HD}(\mathscr{O}_X). \arrow[u, "\sim", "\beta_X"']
\end{tikzcd} \tag{5.3.5}$$

The construction of α_X and β_X and the proof are given in the next section[1].

Definition 5.3.3. For $\mathscr{F} \in \mathrm{D^b_{coh}}(\mathscr{O}_X)$, we set

$$\mathrm{ch}(\mathscr{F}) = \alpha_X \circ \mathrm{hh}_X(\mathscr{F}) \in \bigoplus_{i=0}^{d_X} H^i_{\mathrm{Supp}(\mathscr{F})}(X;\Omega^i_X), \tag{5.3.6}$$

$$\mathrm{eu}(\mathscr{F}) = \beta_X^{-1} \circ \mathrm{thh}_X(\mathscr{F}) \in \bigoplus_{i=0}^{d_X} H^i_{\mathrm{Supp}(\mathscr{F})}(X;\Omega^i_X). \tag{5.3.7}$$

We call $\mathrm{ch}(\mathscr{F})$ the Chern class of $\mathscr{F}$ and $\mathrm{eu}(\mathscr{F})$ the Euler class of $\mathscr{F}$.

[1]Theorem 5.3.2 was first proved in the (unpublished) letter [36]. The first published proof seems to be in [64].

Of course, $\mathrm{ch}(\mathscr{F})$ coincides with the classical Chern character and the morphism α_X is the so-called Hochschild-Kostant-Rosenberg map.

The following conjecture was stated in [36].

Conjecture 5.3.4. One has $\mathrm{eu}(\mathscr{O}_X) = \mathrm{td}_X(TX)$, where $\mathrm{td}_X(TX)$ is the Todd class of the tangent bundle TX. This implies that $\mathrm{eu}(\mathscr{F}) = \mathrm{ch}(\mathscr{F}) \cup \mathrm{td}_X(TX)$. Indeed, for $a, b \in H^*(X; \delta_X^* \delta_{X*} \mathscr{O}_X)$, we have $\mathrm{td}(a \circ b) = a \circ \mathrm{td}(b)$ by Proposition 5.1.5 (iii) and Lemma 5.4.7 below.

This conjecture has recently been proved by A. Ramadoss [54] in the algebraic case and by J. Grivaux [31] in the analytic case.

An index theorem

Consider the particular case of two coherent $\mathscr{O}_X$-modules $\mathscr{L}_i$ $(i = 1, 2)$ such that $\mathrm{Supp}(\mathscr{L}_1) \cap \mathrm{Supp}(\mathscr{L}_2)$ is compact. In this case we have (see [34, 54]):

$$\begin{aligned} \mathrm{hh}_{\mathrm{pt}}(\mathscr{L}_1 \circ \mathscr{L}_2) &= \chi(\mathrm{R}\Gamma(X; \mathscr{L}_1 \overset{\mathrm{L}}{\otimes}_{\mathscr{O}_X} \mathscr{L}_2)) \\ &= \int_X (\mathrm{ch}(\mathscr{L}_1) \cup \mathrm{ch}(\mathscr{L}_2) \cup \mathrm{td}_X(TX)). \end{aligned} \tag{5.3.8}$$

We consider the situation of Corollary 4.4.4. Hence, $\mathscr{A}_X$ is a DQ-algebroid on X.

Corollary 5.3.5. *Let $\mathscr{M}, \mathscr{N} \in \mathrm{D}^{\mathrm{b}}_{gd}(\mathscr{A}_X^{\mathrm{loc}})$ and assume that $K := \mathrm{Supp}(\mathscr{M}) \cap \mathrm{Supp}(\mathscr{N})$ is compact. Let U be a relatively compact open subset of X containing K. Then $\mathrm{RHom}_{\mathscr{A}_X^{\mathrm{loc}}}(\mathscr{M}, \mathscr{N})$ belongs to $\mathrm{D}^{\mathrm{b}}_f(\mathbb{C}^{\hbar,\mathrm{loc}})$ and its Euler-Poincaré index is given by the formula*

$$\chi(\mathrm{RHom}_{\mathscr{A}_X^{\mathrm{loc}}}(\mathscr{M}, \mathscr{N})) = \int_U \mathrm{ch}_U((\mathrm{gr}_\hbar^U \mathrm{D}'_{\mathscr{A}} \mathscr{M})) \cup \mathrm{ch}_U(\mathrm{gr}_\hbar^U(\mathscr{N})) \cup \mathrm{td}_U(TU).$$

Proof. Applying Corollary 4.4.4, we have

$$\begin{aligned} \chi(\mathrm{RHom}_{\mathscr{A}_X^{\mathrm{loc}}}(\mathscr{M}, \mathscr{N})) &= \mathrm{hh}_{\mathrm{pt}}(\mathrm{D}'_{\mathscr{A}} \mathscr{M} \circ \mathscr{N}) \\ &= \mathrm{hh}_{\mathrm{pt}}(\mathrm{gr}_\hbar \mathrm{D}'_{\mathscr{A}} \mathscr{M}_0 \circ \mathrm{gr}_\hbar \mathscr{N}_0), \end{aligned}$$

where $\mathscr{M}_0$ (resp. $\mathscr{N}_0$) is an object of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_U)$ which generates $\mathscr{M}$ (resp. $\mathscr{N}$) on U. Then, the result follows from (5.3.8). Q.E.D.

5.4 Proof of Theorem 5.3.2

As usual, we denote by $p_i \colon X \times X \to X$ the i-th projection ($i = 1, 2$). The following lemma is well-known.

Lemma 5.4.1. *Let $\mathscr{F}$ be an $(\mathscr{O}_X \boxtimes \mathscr{O}_X)$-module supported by the diagonal. Then the following conditions are equivalent:*

(i) *$p_{1*}\mathscr{F}$ is a coherent $\mathscr{O}_X$-module,*

(ii) *$p_{2*}\mathscr{F}$ is a coherent $\mathscr{O}_X$-module.*

If these conditions are satisfied, then the map $\mathscr{F} \to \mathscr{O}_{X\times X} \otimes_{\mathscr{O}_X \boxtimes \mathscr{O}_X} \mathscr{F}$ is an isomorphism. In particular, the $(\mathscr{O}_X \boxtimes \mathscr{O}_X)$-module structure on $\mathscr{F}$ extends uniquely to an $\mathscr{O}_{X\times X}$-module structure.

We define the $p_1^{-1}\mathscr{O}_X$-module

$$P_k := \delta_{X*}\Omega_X^k \oplus \delta_{X*}\Omega_X^{k+1} \text{ for } k \geq 0, \quad P_k = 0 \text{ for } k < 0.$$

We endow the P_k's with a structure of $p_2^{-1}\mathscr{O}_X$-module by setting

$$p_2^*(a)(\omega_k \oplus \theta_{k+1}) \quad = \quad a\omega_k \oplus (a\theta_{k+1} - da \wedge \omega_k)$$

for $a \in \mathscr{O}_X$, $\omega_k \in \Omega_X^k$, $\theta_{k+1} \in \Omega_X^{k+1}$. This defines an action of $p_2^{-1}\mathscr{O}_X$ since

$$\begin{aligned}
p_2^*(a_1)p_2^*(a_2)(\omega_k \oplus \theta_{k+1}) &= p_2^*(a_1)(a_2\omega_k \oplus (a_2\theta_{k+1} - da_2 \wedge \omega_k)) \\
&= a_1a_2\omega_k \oplus (a_1a_2\theta_{k+1} - a_1 da_2 \wedge \omega_k - da_1 \wedge a_2\,\omega_k) \\
&= a_1a_2\omega_k \oplus (a_1a_2\theta_{k+1} - d(a_1a_2) \wedge \omega_k) \\
&= p_2^*(a_1a_2)(\omega_k \oplus \theta_{k+1}).
\end{aligned}$$

By Lemma 5.4.1, we get that P_k has a structure of $\mathscr{O}_{X\times X}$-module and we have an exact sequence:

$$0 \to \delta_{X*}\Omega_X^{k+1} \xrightarrow{\alpha_k} P_k \xrightarrow{\beta_k} \delta_{X*}\Omega_X^k \to 0. \tag{5.4.1}$$

Hence $\delta_{X*}\Omega^k[k] \xleftarrow{\sim} (\delta_{X*}\Omega_X^{k+1} \to P_k) \to \delta_{X*}\Omega_X^{k+1}[k+1]$ defines the morphism

$$\xi_k \colon \delta_{X*}\Omega^k[k] \to \delta_{X*}\Omega_X^{k+1}[k+1].$$

It induces a morphism

$$\xi \colon \bigoplus_k \delta_{X*}\Omega_X^k[k] \longrightarrow \bigoplus_k \delta_{X*}\Omega_X^k[k]. \tag{5.4.2}$$

Let $d_k^{\mathrm{stan}}\colon P_k \to P_{k-1}$ be the composition

$$d_k^{\mathrm{stan}}\colon P_k \xrightarrow{\beta_k} \delta_{X*}\Omega_X^k \xrightarrow{\alpha_{k-1}} P_{k-1}. \tag{5.4.3}$$

We define the complex $P_\bullet$ whose differential $d_P^{-k}\colon P_k \to P_{k-1}$ is given by kd_k^{stan}. Then $\operatorname{Im} d_k^{\mathrm{stan}} \simeq \operatorname{Im}\beta_k \simeq \delta_{X*}\Omega_X^k$ and $\operatorname{Ker} d_k^{\mathrm{stan}} \simeq \operatorname{Ker}\beta_k \simeq \delta_{X*}\Omega_X^{k+1}$. Therefore we have a quasi-isomorphism $P_\bullet \to \delta_{X*}\mathscr{O}_X$.

Lemma 5.4.2. *The morphism*

$$\alpha_X\colon \delta_X^*\delta_{X*}\mathscr{O}_X \to H^0(\delta_X^*)(P_\bullet) \simeq \bigoplus_k \Omega_X^k[k] \tag{5.4.4}$$

is an isomorphism in $\mathrm{D^b}(\mathscr{O}_X)$.

Proof of Lemma 5.4.2. Since the question is local, we may assume that X is a vector space V. Then we have a Koszul complex

$$\mathscr{O}_{X\times X}\otimes \bigwedge^{\bullet} V^* \simeq \Big(\cdots \to \mathscr{O}_{X\times X}\otimes \bigwedge^{2} V^* \to \mathscr{O}_{X\times X}\otimes V^* \to \mathscr{O}_{X\times X}\Big)$$

and an isomorphism $\mathscr{O}_{X\times X}\otimes\bigwedge^\bullet V^* \to \delta_{X*}\mathscr{O}_X$ in $\mathrm{D^b}(\mathscr{O}_{X\times X})$. Then applying $H^0(\delta_X^*)$, we obtain an isomorphism in $\mathrm{D^b}(\mathscr{O}_X)$:

$$\delta_X^*\delta_{X*}\mathscr{O}_X \xrightarrow{\sim} H^0(\delta_X^*)(\mathscr{O}_{X\times X}\otimes \bigwedge^{\bullet} V^*).$$

The $\mathbb{C}$-linear maps $\bigwedge^k V^* \to \Omega_X^k(V) \to P_k(X\times X)$ induce a morphism of complexes $\mathscr{O}_{X\times X}\otimes\bigwedge^\bullet V^* \to P_\bullet$ such that the diagram below commutes:

$$\begin{tikzcd}
\mathscr{O}_{X\times X}\otimes \bigwedge^\bullet V^* \arrow[d] \arrow[dr] & \\
P_\bullet \arrow[r] & \delta_{X*}\mathscr{O}_X
\end{tikzcd}$$

Since $H^0(\delta_X^*)(\mathscr{O}_{X\times X}\otimes\bigwedge^\bullet V^*)[d_X] \to H^0(\delta_X^*)(P_\bullet)$ is an isomorphism, we obtain the desired result. Q.E.D.

Remark 5.4.3. (i) Let $I \subset \mathscr{O}_{X\times X}$ be the defining ideal of the diagonal set $\delta_X(X)$. Then the morphism $\xi_0\colon \delta_{X*}\mathscr{O}_X \to \delta_{X*}\Omega_X^1[1]$ is given by the exact sequence $0 \to \delta_{X*}\Omega_X^1 \to \mathscr{O}_{X\times X}/I^2 \to \delta_{X*}\mathscr{O}_X \to 0$. Indeed, we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & I/I^2 \arrow[r] \arrow[d, "\wr"] & \mathscr{O}_{X\times X}/I^2 \arrow[r] \arrow[d, "\wr"] & \delta_{X*}\mathscr{O}_X \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & \delta_{X*}\Omega_X^1 \arrow[r, "\beta_0"] & P_0 \arrow[r, "\alpha_0"] & \delta_{X*}\mathscr{O}_X \arrow[r] & 0.
\end{tikzcd}$$

Here, the left vertical isomorphism is given by

$$I/I^2 \ni p_1^*(a) - p_2^*(a) \longleftrightarrow da \in \delta_{X*}\Omega_X^1 \quad (a \in \mathscr{O}_X).$$

(ii) Moreover the morphism $\xi_k \colon \delta_{X*}\Omega_X^k[k] \to \delta_{X*}\Omega_X^{k+1}[k+1]$ coincides with the composition

$$\begin{aligned}\delta_{X*}\Omega_X^k[k] &\simeq \delta_{X*}\Omega_X^k[k]\overset{\mathrm{L}}{\otimes}_{\mathscr{O}_{X\times X}}\mathscr{O}_{X\times X} \to \delta_{X*}\Omega_X^k[k]\overset{\mathrm{L}}{\otimes}_{\mathscr{O}_{X\times X}}\delta_{X*}\mathscr{O}_X \\ &\xrightarrow{\xi_0} \delta_{X*}\Omega_X^k[k]\overset{\mathrm{L}}{\otimes}_{\mathscr{O}_{X\times X}}\delta_{X*}\Omega_X^1[1] \to \delta_{X*}(\Omega_X^k[k]\overset{\mathrm{L}}{\otimes}_{\mathscr{O}_X}\Omega_X^1[1]) \\ &\to \delta_{X*}\Omega_X^{k+1}[k+1].\end{aligned}$$

(iii) Note that the morphism $\alpha_X \colon \delta_X^*\delta_{X*}\mathscr{O}_X \xrightarrow{\sim} \bigoplus_k \Omega_X^k[k]$ coincides with the morphism obtained from $\delta_{X*}\mathscr{O}_X \to \bigoplus_k \delta_{X*}\Omega_X^k[k] \xrightarrow{\exp(\xi)} \bigoplus_k \delta_{X*}\Omega_X^k[k]$ by adjunction.

Lemma 5.4.4. *The morphism α_X in* (5.4.4) *interchanges the composition of the ring $\delta_X^*\delta_{X*}\mathscr{O}_X$ given in* Proposition 5.1.5 (a) *with the composition*

$$\Omega_X^i[i]\overset{\mathrm{L}}{\otimes}_{\mathscr{O}_X}\Omega_X^j[j] \simeq (\Omega_X^i\overset{\mathrm{L}}{\otimes}_{\mathscr{O}_X}\Omega_X^j)[i+j] \xrightarrow{\wedge} \Omega_X^{i+j}[i+j].$$

Note that the unit $\mathscr{O}_X \to \delta_X^*\delta_{X*}\mathscr{O}_X$ is given by $\mathscr{O}_X \simeq \delta_X^*\mathscr{O}_{X\times X} \to \delta_X^*\delta_{X*}\mathscr{O}_X$, where the last arrow is induced by $\mathscr{O}_{X\times X} \to \delta_{X*}\mathscr{O}_X$.

Proof. We define

$$\mu_{ij} \colon P_i \otimes_{\mathscr{O}_{X\times X}} P_j \to P_{i+j}$$

by

$$\begin{aligned}\mu_{ij}&\big(((\omega_i \oplus \theta_{i+1}) \otimes (\omega_j \oplus \theta_{j+1}))\big) \\ &= (\omega_i \wedge \omega_j) \oplus (\theta_{i+1} \wedge \omega_j + (-1)^i\omega_i \wedge \theta_{j+1}).\end{aligned} \tag{5.4.5}$$

This map is $p_2^{-1}(\mathscr{O}_X)$-bilinear since:

$$\begin{aligned}\mu_{ij}&\Big(\big(p_2^*(a)(\omega_i \oplus \theta_{i+1})\big) \otimes (\omega_j \oplus \theta_{j+1})\Big) \\ &= \mu_{ij}\Big(\big(a\omega_i \oplus (a\theta_{i+1} - da \wedge \omega_i)\big) \otimes (\omega_j \oplus \theta_{j+1})\Big) \\ &= (a\omega_i \wedge \omega_j) \oplus \big((a\theta_{i+1} - da \wedge \omega_i) \wedge \omega_j + (-1)^i a\omega_i \wedge \theta_{j+1}\big) \\ &= p_2^*(a)\big((\omega_i \wedge \omega_j) \oplus (\theta_{i+1} \wedge \omega_j + (-1)^i\omega_i \wedge \theta_{j+1})\big) \\ &= p_2^*(a)\mu_{ij}\big((\omega_i \oplus \theta_{i+1}) \otimes (\omega_j \oplus \theta_{j+1})\big),\end{aligned}$$

and

$$\begin{aligned}
&\mu_{ij}\big((\omega_i \oplus \theta_{i+1}) \otimes p_2^*(a)(\omega_j \oplus \theta_{j+1})\big)\\
&\quad = \mu_{ij}\Big(\big(\omega_i \oplus \theta_{i+1}\big) \otimes \big(a\omega_j \oplus (a\theta_{j+1} - da \wedge \omega_j)\big)\Big)\\
&\quad = \big(a\omega_i \wedge \omega_j\big) \oplus \big(\theta_{i+1} \wedge a\omega_j + (-1)^i \omega_i \wedge (a\theta_{j+1} - da \wedge \omega_j)\big)\\
&\quad = (a\omega_i \wedge \omega_j) \oplus (a\theta_{i+1} \wedge \omega_j + (-1)^i a\omega_i \wedge \theta_{j+1} - da \wedge \omega_i \wedge \omega_j)\\
&\quad = p_2^*(a)\big(\omega_i \wedge \omega_j \oplus (\theta_{i+1} \wedge \omega_j + (-1)^i \omega_i \wedge \theta_{j+1})\big)\\
&\quad = p_2^*(a)\mu_{ij}\big((\omega_i \oplus \theta_{i+1}) \otimes (\omega_j \oplus \theta_{j+1})\big).
\end{aligned}$$

The morphism μ commutes with the differentials since:

$$\begin{aligned}
&\mu d\big((\omega_i \oplus \theta_{i+1}) \otimes (\omega_j \oplus \theta_{j+1})\big)\\
&\quad = \mu_{i-1,j}\big((0 \oplus i\omega_i) \otimes (\omega_j \oplus \theta_{j+1})\big) + (-1)^i \mu_{i,j-1}\big((\omega_i \oplus \theta_{i+1}) \otimes (0 \oplus j\omega_j)\big)\\
&\quad = 0 \oplus \big(i\omega_i \wedge \omega_j + (-1)^i(-1)^i j\omega_i \wedge \omega_j\big) = 0 \oplus (i+j)\omega_i \wedge \omega_j\\
&\quad = d\mu\big((\omega_i \oplus \theta_{i+1}) \otimes (\omega_j \oplus \theta_{j+1})\big).
\end{aligned}$$

Hence we have a commutative diagram in $\mathrm{D^b}(\mathscr{O}_{X\times X})$

$$\begin{tikzcd}
\delta_{X*}\mathscr{O}_X \overset{\mathrm{L}}{\otimes}_{\mathscr{O}_{X\times X}} \delta_{X*}\mathscr{O}_X \arrow[r] \arrow[d] & \delta_{X*}\mathscr{O}_X \\
P_\bullet \otimes_{\mathscr{O}_{X\times X}} P_\bullet \arrow[r, "\mu"] & P_\bullet \, . \arrow[u]
\end{tikzcd}$$

Therefore, applying δ_X^*, the morphism $\delta_X^*\delta_{X*}\mathscr{O}_X \overset{\mathrm{L}}{\otimes} \delta_X^*\delta_{X*}\mathscr{O}_X \to \delta_X^*\delta_{X*}\mathscr{O}_X$ is represented by

$$H^0(\delta_X^*)P_\bullet \otimes_{\mathscr{O}_X} H^0(\delta_X^*)P_\bullet \to H^0(\delta_X^*)P_\bullet .$$

Thus we obtain the desired result. Q.E.D.

Lemma 5.4.5. *Consider a morphism $f \colon X \to Y$. Then the diagram below commutes:*

$$\begin{tikzcd}
f^*\delta_Y^*\delta_{Y*}\mathscr{O}_Y \arrow[r] \arrow[d, "\alpha_Y"] & \delta_X^*\delta_{X*}\mathscr{O}_X \arrow[d, "\alpha_X"] \\
f^*(\bigoplus_k \Omega_Y^k[k]) \arrow[r] & \bigoplus_k \Omega_X^k[k] \, .
\end{tikzcd}$$

Proof. Let $\tilde{f}\colon X \times X \to Y \times Y$ be the morphism associated with f. Let us denote by $P^X_\bullet$ the complex on X constructed above. Then we easily construct a commutative diagram

$$\begin{tikzcd} H^0(\tilde{f}^*)P^Y_\bullet \arrow[r] \arrow[d, "\varphi"'] & H^0(\tilde{f}^*)\delta_{Y*}\mathscr{O}_Y \arrow[d] \\ P^X_\bullet \arrow[r] & \delta_{X*}\mathscr{O}_X \end{tikzcd}$$

such that

$$\begin{tikzcd} H^0(\delta_X^*\tilde{f}^*)P^Y_\bullet \arrow[r] \arrow[d, "\delta_X^*\varphi"'] & H^0(f^*\delta_Y^*)P^Y_\bullet \arrow[r, "\sim"] & f^*(\bigoplus_k \Omega^k_Y[k]) \arrow[d, "\psi"] \\ H^0(\delta_X^*)P^X_\bullet \arrow[rr] & & \bigoplus_k \Omega^k_X[k] \end{tikzcd}$$

commutes where ψ is given in (5.3.3). Q.E.D.

Now we set

$$Q_k = \begin{cases} P_{k-1} & \text{for } 1 \le k \le d_X, \\ \delta_{X*}\mathscr{O}_X & \text{for } k = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{5.4.6}$$

and define the differential d^Q with $d^Q_k = (k-1-d_X)\ d^{\mathrm{stan}}_{k-1}$, where d^{stan}_{k-1} is given by (5.4.3) and $d^{\mathrm{stan}}_0\colon \mathscr{O}_X \oplus \Omega^1_X \to \mathscr{O}_X$ is the canonical morphism. Then $Q_\bullet$ is a complex of $\mathscr{O}_{X\times X}$-modules and the canonical homomorphism $\Omega^{d_X}_X \to \Omega^{d_X-1}_X \oplus \Omega^{d_X}_X$ induces a morphism of complexes $\delta_{X*}\omega_X \to Q_\bullet$, which is an isomorphism in $\mathrm{D}^{\mathrm{b}}(\mathscr{O}_{X\times X})$.

Let us denote by $H^0(\delta^!_X)$ the functor $\delta_X^{-1}\mathscr{H}om_{\mathscr{O}_{X\times X}}(\delta_*\mathscr{O}_X, \bullet)$.

Lemma 5.4.6. *The morphism*

$$\beta_X\colon \bigoplus_k \Omega^k_X \simeq H^0(\delta^!_X)Q_\bullet \to \delta^!_X\delta_{X*}\omega_X$$

is an isomorphism in $\mathrm{D}^{\mathrm{b}}(\mathscr{O}_X)$.

Since the proof is similar to that of Lemma 5.4.2, we omit it.

Note that the morphism β_X coincides with the morphism obtained by adjunction from

$$\bigoplus_k \delta_{X!}\Omega^k_X \xrightarrow{\exp(-\xi)} \bigoplus_k \delta_{X!}\Omega^k_X \to \delta_{X!}\Omega^n_X[n] \simeq \delta_{X!}\omega_X.$$

Lemma 5.4.7. *The morphism* $\delta_X^*\delta_{X*}\mathscr{O}_X\overset{\mathrm{L}}{\otimes}_{\mathscr{O}_X}\delta_X^!\delta_{X!}\omega_X \to \delta_X^!\delta_{X!}\omega_X$ *in* Proposition 5.1.5 (d) *coincides with* $\Omega_X^i[i]\otimes_{\mathscr{O}_X}\Omega_X^j[j]\xrightarrow{\wedge}\Omega_X^{i+j}[i+j]$.

Proof. We define the morphism $\mu_{ij}\colon P_i\otimes_{\mathscr{O}_{X\times X}}Q_j\to Q_{i+j}$ by the same formula as in (5.4.5). Then it commutes with the differential. Indeed the proof is similar to that of Lemma 5.4.4 except when $i+j=d_X+1$. In this case,

$$\mu d\big((\omega_i\oplus\theta_{i+1})\otimes(\omega_{j-1}\oplus\theta_j)\big)=0\oplus(i+j-d_X-1)\omega_i\wedge\omega_{j-1}=0.$$

With this morphism $\mu\colon P_\bullet\otimes_{\mathscr{O}_{X\times X}}Q_\bullet\to Q_\bullet$, the following diagram in the category of complexes is commutative:

$$\begin{tikzcd}
P_\bullet\otimes_{\mathscr{O}_{X\times X}}Q_\bullet \arrow[r, "\mu"] \arrow[d] & Q_\bullet \arrow[d] \\
P_\bullet\otimes_{\mathscr{O}_{X\times X}}\delta_{X!}\omega_X \arrow[r] & \delta_{X!}\omega_X\,.
\end{tikzcd}$$

Thus we have a commutative diagram in $\mathrm{D^b}(\mathscr{O}_X)$:

$$\begin{tikzcd}
H^0(\delta_X^*)P_\bullet\otimes_{\mathscr{O}_X}H^0(\delta_X^!)Q_\bullet \arrow[r] \arrow[d, "\wr"] & H^0(\delta_X^!)(P_\bullet\otimes_{\mathscr{O}_{X\times X}}Q_\bullet) \arrow[r] \arrow[d] & H^0(\delta_X^!)(Q_\bullet) \arrow[d] \\
\delta_X^*\delta_{X*}\mathscr{O}_X\overset{\mathrm{L}}{\otimes}_{\mathscr{O}_X}\delta_X^!\delta_{X!}\omega_X \arrow[r] & \delta_X^!(\delta_{X*}\mathscr{O}_X\overset{\mathrm{L}}{\otimes}_{\mathscr{O}_{X\times X}}\delta_{X!}\omega_X) \arrow[r] & \delta_X^!\delta_{X!}\omega_X\,.
\end{tikzcd}$$

Q.E.D.

Recall that in Corollary 4.2.2, we have constructed a morphism $\mathcal{HH}(\mathscr{A}_X)\otimes\mathcal{HH}(\mathscr{A}_X)\to\omega_{X_\mathbb{R}}^{\mathrm{top}}$. Let us describe its image via the isomorphisms α_X and β_X. Consider the diagram

$$\begin{tikzcd}
\mathcal{HH}(\mathscr{O}_X)\otimes\mathcal{HH}(\mathscr{O}_X) \arrow[d, "\lambda"'] \arrow[dr, "u"] & \\
\mathcal{HD}(\mathscr{O}_X)\otimes\mathcal{HD}(\mathscr{O}_X) \arrow[r, "v"'] & \omega_{X_\mathbb{R}}^{\mathrm{top}}\,.
\end{tikzcd} \tag{5.4.7}$$

Here, u is the map given by Corollary 4.2.2, λ is the isomorphism $\alpha_X\otimes\beta_X^{-1}$ and v is the composition

$$\bigoplus_k\Omega_X^k[k]\otimes\bigoplus_k\Omega_Y^k[k]\longrightarrow\bigoplus_k\Omega_Y^k[k]\to\omega_{X_\mathbb{R}}^{\mathrm{top}},$$

where the first morphism is given by the wedge product and the last one by the map $\Omega_X^{d_X}[d_X]\to\omega_{X_\mathbb{R}}^{\mathrm{top}}$. Then diagram (5.4.7) commutes.

Chapter 6

Symplectic case and $\mathscr{D}$-modules

6.1 Deformation quantization on cotangent bundles

Consider the case where X is an open subset of the cotangent bundle T^*M of a complex manifold M. We denote by $\pi\colon T^*M \to M$ the projection. As usual, we denote by $\mathscr{D}_M$ the $\mathbb{C}$-algebra of differential operators on M. This is a right and left Noetherian sheaf of rings.

The space T^*M is endowed with the filtered sheaf of $\mathbb{C}$-algebras $\widehat{\mathscr{E}}_{T^*M}$ of formal microdifferential operators of [55], and its subsheaf $\widehat{\mathscr{E}}_{T^*M}(0)$ of operators of order ≤ 0.

On T^*M, there is also a DQ-algebra, denoted by $\widehat{\mathscr{W}}_{T^*M}(0)$ and constructed in [52] as follows. Consider the complex line $\mathbb{C}$ endowed with the coordinate t and denote by $(t;\tau)$ the associated symplectic coordinates on $T^*\mathbb{C}$. Let $T^*_{\tau\neq 0}(M\times\mathbb{C})$ be the open subset of $T^*(M\times\mathbb{C})$ defined by $\tau \neq 0$ and consider the map

$$\rho\colon T^*_{\tau\neq 0}(M\times\mathbb{C}) \to T^*M, \quad (x,t;\xi,\tau)\mapsto (x;\tau^{-1}\xi).$$

Denote by $\widehat{\mathscr{E}}_{T^*(M\times\mathbb{C}),\widehat{t}}(0)$ the subalgebra of $\widehat{\mathscr{E}}_{T^*(M\times\mathbb{C})}(0)$ consisting of operators not depending on t, that is, commuting with ∂_t. Setting $\hbar = \partial_t^{-1}$, the DQ-algebra $\widehat{\mathscr{W}}_X(0)$ is defined as

$$\widehat{\mathscr{W}}_X(0) = \rho_*\widehat{\mathscr{E}}_{T^*(M\times\mathbb{C}),\widehat{t}}(0).$$

One denotes by $\widehat{\mathscr{W}}_{T^*M}$ the localization of $\widehat{\mathscr{W}}_{T^*M}(0)$, that is, $\widehat{\mathscr{W}}_{T^*M} = \mathbb{C}^{\hbar,\mathrm{loc}}\otimes_{\mathbb{C}^\hbar} \widehat{\mathscr{W}}_{T^*M}(0)$.

Remark 6.1.1. One shall be aware that $\widehat{\mathscr{E}}_{T^*M}$ and $\widehat{\mathscr{E}}_{T^*M}(0)$ are denoted by $\widehat{\mathscr{E}}_M$ and $\widehat{\mathscr{E}}_M(0)$, respectively, in [55]. Similarly, $\widehat{\mathscr{W}}_{T^*M}$ and $\widehat{\mathscr{W}}_{T^*M}(0)$ are denoted by $\widehat{\mathscr{W}}_M$ and $\widehat{\mathscr{W}}_M(0)$, respectively, in [52].

There are natural morphisms of algebras

$$\pi_M^{-1}\mathscr{D}_M \hookrightarrow \widehat{\mathscr{E}}_{T^*M} \hookrightarrow \widehat{\mathscr{W}}_{T^*M}. \tag{6.1.1}$$

Lemma 6.1.2. (a) *The algebra $\widehat{\mathscr{W}}_{T^*M}(0)$ is faithfully flat over $\widehat{\mathscr{E}}_{T^*M}(0)$.*

(b) *The algebra $\widehat{\mathscr{W}}_{T^*M}$ is faithfully flat over $\widehat{\mathscr{E}}_{T^*M}$.*

(c) *$\widehat{\mathscr{E}}_{T^*M}$ is flat over $\pi_M^{-1}\mathscr{D}_M$.*

Proof. In the sequel, we set $X = T^*M$. For an $\widehat{\mathscr{E}}_X(0)$-module $\mathscr{M}$, we set

$$\begin{aligned} \mathscr{M}^{\mathrm{W}} &:= \widehat{\mathscr{W}}_X(0) \otimes_{\widehat{\mathscr{E}}_X(0)} \mathscr{M}, \\ \mathrm{gr}_{\mathscr{E}}(\mathscr{M}) &= \big(\widehat{\mathscr{E}}_X(0)/\widehat{\mathscr{E}}_X(-1)\big) \overset{\mathrm{L}}{\otimes}_{\widehat{\mathscr{E}}_X(0)} \mathscr{M}. \end{aligned}$$

Note that the analogue of Corollary 1.4.6 holds for $\widehat{\mathscr{E}}_X(0)$-modules, that is, the functor $\mathrm{gr}_{\mathscr{E}}$ above is conservative on $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\widehat{\mathscr{E}}_X(0))$. We have

$$\mathrm{gr}_\hbar(\mathscr{M}^{\mathrm{W}}) \simeq \mathscr{O}_X \otimes_{\mathscr{O}_X(0)} \mathrm{gr}_{\mathscr{E}}(\mathscr{M}), \tag{6.1.2}$$

where $\mathscr{O}_X(0)$ denotes the subsheaf of $\mathscr{O}_X$ of sections homogeneous of degree 0 in the fiber variable of the vector bundle T^*M, and $\mathscr{O}_X$ is faithfully flat over $\mathscr{O}_X(0)$.

(a) (i) Let us first prove the result outside of the zero-section, that is, on $T^*M \setminus T^*_M M$. Let us show that

$$H^j(\widehat{\mathscr{W}}_X(0) \overset{\mathrm{L}}{\otimes}_{\widehat{\mathscr{E}}_X(0)} \mathscr{M}) = 0 \quad \text{for any } j < 0 \tag{6.1.3}$$

holds for any coherent $\widehat{\mathscr{E}}_X(0)$-module $\mathscr{M}$. First assume that $\mathscr{M}$ is torsion-free, i.e., $\widehat{\mathscr{E}}_X(-1) \otimes_{\widehat{\mathscr{E}}_X(0)} \mathscr{M} \to \mathscr{M}$ is a monomorphism. Since $\mathscr{O}_X$ is flat over $\mathscr{O}_X(0)$,

$$\mathrm{gr}_\hbar(\widehat{\mathscr{W}}_X(0) \overset{\mathrm{L}}{\otimes}_{\widehat{\mathscr{E}}_X(0)} \mathscr{M}) \simeq \mathscr{O}_X \overset{\mathrm{L}}{\otimes}_{\mathscr{O}_X(0)} \mathrm{gr}_{\mathscr{E}}(\mathscr{M})$$

has zero cohomologies in degree < 0. Hence Proposition 1.4.5 implies (6.1.3).

Now assume that $\widehat{\mathscr{E}}_X(-1)\mathscr{M} = 0$. Then we have

$$\begin{aligned}\widehat{\mathscr{W}_X}(0)\overset{\mathrm{L}}{\otimes}_{\widehat{\mathscr{E}}_X(0)}\mathscr{M} &\simeq \widehat{\mathscr{W}_X}(0)\overset{\mathrm{L}}{\otimes}_{\widehat{\mathscr{E}}_X(0)}\widehat{\mathscr{E}}_X(0)\overset{\mathrm{L}}{\otimes}_{\widehat{\mathscr{E}}_X(0)}\mathscr{O}_X(0)\overset{\mathrm{L}}{\otimes}_{\mathscr{O}_X(0)}\mathscr{M}\\ &\simeq \widehat{\mathscr{W}_X}(0)\overset{\mathrm{L}}{\otimes}_{\widehat{\mathscr{E}}_X(0)}\mathscr{O}_X(0)\overset{\mathrm{L}}{\otimes}_{\mathscr{O}_X(0)}\mathscr{M}\\ &\simeq \mathscr{O}_X\overset{\mathrm{L}}{\otimes}_{\mathscr{O}_X(0)}\mathscr{M},\end{aligned}$$

which implies (6.1.3).

Since any coherent $\widehat{\mathscr{E}}_X(0)$-module is successive extension of torsion-free $\widehat{\mathscr{E}}_X(0)$-modules and $(\widehat{\mathscr{E}}_X(0)/\widehat{\mathscr{E}}_X(-1))$-modules, we obtain (6.1.3) for any coherent $\widehat{\mathscr{E}}_X(0)$-module.

Consider a coherent $\widehat{\mathscr{E}}_X(0)$-module $\mathscr{M}$ and assume that $\mathscr{M}^{\mathrm{W}} \simeq 0$. Then $\mathrm{gr}_\hbar(\mathscr{M}^{\mathrm{W}}) \simeq 0$ and this implies that $\mathrm{gr}_{\mathscr{E}}(\mathscr{M}) \simeq 0$ in view of (6.1.2) since $\mathscr{O}_X$ is faithfully flat over $\mathscr{O}_X(0)$. Since $\mathrm{gr}_{\mathscr{E}}$ is conservative, the result follows.

(a) (ii) To prove the result in a neighborhood of the zero section, we use the classical trick of the dummy variable. Let $(t;\tau)$ denote a homogeneous symplectic coordinate system on $T^*\mathbb{C}$. Consider the functors

$$\begin{aligned}\alpha\colon \mathrm{Mod}_{\mathrm{coh}}(\mathscr{O}_M) &\to \mathrm{Mod}_{\mathrm{coh}}(\widehat{\mathscr{E}}_{X\times T^*\mathbb{C}}(0)|_{\tau\neq 0}),\\ \mathscr{M} &\mapsto \mathscr{M}\underline{\boxtimes}(\widehat{\mathscr{E}}_{\mathbb{C}}(0)/\widehat{\mathscr{E}}_{\mathbb{C}}(0)\cdot t),\\ \beta\colon \mathrm{Mod}_{\mathrm{coh}}(\widehat{\mathscr{W}_{X|_M}}(0)) &\to \mathrm{Mod}_{\mathrm{coh}}(\widehat{\mathscr{W}}_{X\times T^*\mathbb{C}}(0)|_{\tau\neq 0}),\\ \mathscr{M} &\mapsto \mathscr{M}\underline{\boxtimes}(\widehat{\mathscr{W}_{T^*\mathbb{C}}}(0)/\widehat{\mathscr{W}_{T^*\mathbb{C}}}(0)\cdot t).\end{aligned}$$

These two functors α and β are exact and faithful. Then the result follows from (a) (i).

(b) (i) Here again, we prove the result first on $T^*M \setminus T^*_M M$. In this case, it follows from the isomorphism

$$\widehat{\mathscr{W}_X} \simeq \widehat{\mathscr{W}_X}(0)\otimes_{\widehat{\mathscr{E}}_{T^*\mathbb{C}}(0)}\widehat{\mathscr{E}}_{T^*\mathbb{C}}.$$

(b) (ii) The case of the zero-section is deduced from (b) (i) similarly as for (a).

(c) is proved for example in [38, Th. 7.25]. Q.E.D.

Recall that for a coherent $\mathscr{D}_M$-module $\mathscr{M}$, the support of $\widehat{\mathscr{E}}_{T^*M}\otimes_{\pi_M^{-1}\mathscr{D}_M}\pi_M^{-1}\mathscr{M}$ is called its characteristic variety and denoted by $\mathrm{char}(\mathscr{M})$. It is a closed $\mathbb{C}^\times$-conic complex analytic involutive subset of T^*M.

Now assume that M is open in some finite-dimensional $\mathbb{C}$-vector space. Denote by (x) a linear coordinate system on M and by $(x;u)$ the associated symplectic coordinate system on T^*M. Let $f,g \in \mathscr{O}_X[[\hbar]]$. In this case, the DQ-algebra $\widehat{\mathscr{W}}_X(0)$ is isomorphic to the star algebra $(\mathscr{O}_X[[\hbar]],\star)$ where:

$$f \star g \;=\; \sum_{\alpha\in\mathbb{N}^n} \frac{\hbar^{|\alpha|}}{\alpha!}(\partial_u^\alpha f)(\partial_x^\alpha g). \tag{6.1.4}$$

This product is similar to the product of the total symbols of differential operators on M and indeed, the morphism of $\mathbb{C}$-algebras $\pi_M^{-1}\mathscr{D}_M \longrightarrow \widehat{\mathscr{W}}_X$ is given by

$$f(x) \mapsto f(x), \qquad \partial_{x_i} \mapsto \hbar^{-1}u_i.$$

Note that there also exists an analytic version of $\widehat{\mathscr{E}}_{T^*M}$ and $\widehat{\mathscr{W}}_{T^*M}$, obtained by using the $\mathbb{C}$-subalgebra of $(\mathscr{O}_X[[\hbar]],\star)$ consisting of sections $f = \sum_{j\geq 0} f_j\hbar^j$ of $\mathscr{O}_X[[\hbar]](U)$ (U open in T^*M) satisfying:

$$\begin{cases}\text{for any compact subset } K \text{ of } U \text{ there exists a positive con-}\\ \text{stant } C_K \text{ such that } \sup\limits_K |f_j| \leq C_K^j j! \text{ for all } j>0.\end{cases} \tag{6.1.5}$$

They are the total symbols of the analytic (no more formal) microdifferential operators of [55].

Remark 6.1.3. (i) Let X be a complex symplectic manifold. Then X is locally isomorphic to an open subset of a cotangent bundle T^*M, for a complex manifold M (Darboux's theorem), and it is a well-known fact that if $\mathscr{A}_X$ is a DQ-algebra and the associated Poisson structure is the symplectic structure of X, then $\mathscr{A}_X$ is locally isomorphic to $\widehat{\mathscr{W}}_{T^*M}(0)$.
(ii) On X, there is a canonical DQ-algebroid, still denoted by $\widehat{\mathscr{W}}_X(0)$. It has been constructed in [52], after [37] had first treated the contact case. Clearly, any DQ-algebroid $\mathscr{A}$ is equivalent to $\widehat{\mathscr{W}}_X(0)\otimes_{\mathbb{C}_X^\hbar}\mathscr{P}$, where $\mathscr{P}$ is an invertible $\mathbb{C}_X^\hbar$-algebroid. It follows that the DQ-algebroids on X are classified by $H^2(X;(\mathbb{C}_X^\hbar)^\times)$. See [51] for a detailed study.
(iii) Using (4.1.11), we get the isomorphism

$$\mathcal{HH}(\mathscr{A}_X) \simeq \mathcal{HH}(\widehat{\mathscr{W}}_X(0)). \tag{6.1.6}$$

6.2 Hochschild homology of $\mathscr{A}$

Throughout this section, X denotes a complex manifold endowed with a DQ-algebroid $\mathscr{A}_X$ such that the associated Poisson structure is symplectic. Hence, X is symplectic and we denote by α_X the symplectic 2-form on X.

We set $2n = d_X$, $Z = X \times X^a$ and we denote by dv the volume form on X given by $dv = \alpha_X^n/n!$.

Lemma 6.2.1. *Let Λ be a smooth Lagrangian submanifold of X and let $\mathscr{L}_i$ $(i = 0, 1)$ be simple $\mathscr{A}_X$-modules along Λ. Then:*

(i) *$\mathscr{L}_0$ and $\mathscr{L}_1$ are locally isomorphic,*

(ii) *the natural morphism $\mathbb{C}^\hbar \to \mathscr{H}om_{\mathscr{A}_X}(\mathscr{L}_0, \mathscr{L}_0)$ is an isomorphism.*

Note that the lemma above does not hold if one removes the hypothesis that X is symplectic (see Example 2.5.9).

Proof. (i) We may assume that $X = T^*M$ for a complex manifold M, $\mathscr{A}_X = \widehat{\mathscr{W}}_{T^*M}(0)$. Choose a local coordinate system $(x_1, \ldots, x_n)$ on M, and denote by $(x; u)$ the associated coordinates on X. We shall identify the section u_i of $\mathscr{A}_X$ with the differential operator $\hbar\partial_i$.

We may assume that Λ is the zero-section $T^*_M M$ and $\mathscr{L}_0 = \mathscr{O}_M[[\hbar]] \simeq \mathscr{A}_X/\mathscr{I}_0$, where $\mathscr{I}_0$ is the left ideal generated by $(\hbar\partial_1, \ldots, \hbar\partial_n)$. Since $\mathscr{L}_1$ is simple, it locally admits a generator, say u. Denote by $\mathscr{I}_1$ the annihilator ideal of u in $\mathscr{A}_X$. Since $\mathscr{I}_1/\hbar\mathscr{I}_1$ is reduced, there exist sections $(P_1, \cdots, P_n)$ of $\mathscr{A}_X$ such that

$$\{\hbar\partial_1 + \hbar P_1, \ldots, \hbar\partial_n + \hbar P_n\} \subset \mathscr{I}_1.$$

By identifying $\widehat{\mathscr{W}}_{T^*M}(0)$ with the sheaf of microdifferential operators of order ≤ 0 in the variable $(x_1, \ldots, x_n, t)$ not depending on t and $\hbar$ with ∂_t^{-1}, a classical result of [55] (see also [57, Th 6.2.1] for an exposition) shows that there exists an invertible section $P \in \mathscr{A}_X$ such that $\mathscr{I}_0 = \mathscr{I}_1 P$. Hence, $\mathscr{L}_1 \simeq \mathscr{L}_0$.

(ii) We may assume $\mathscr{L}_0 = \mathscr{O}_M[[\hbar]]$. Then $\mathscr{H}om_{\mathscr{A}_X}(\mathscr{O}_M[[\hbar]], \mathscr{O}_M[[\hbar]])$ is isomorphic to the kernel of the map

$$u \colon \mathscr{O}_M[[\hbar]] \to (\mathscr{O}_M[[\hbar]])^n, \quad u = (\hbar\partial_1, \ldots, \hbar\partial_n).$$

Q.E.D.

Recall that the objects $\Omega_X^{\mathscr{A}}$ and $\omega_X^{\mathscr{A}}$ are defined in § 2.5.

Lemma 6.2.2. *There exists a local system L of rank one over $\mathbb{C}_X^\hbar$ such that $\Omega_X^{\mathscr{A}} \simeq L \otimes_{\mathbb{C}_X^\hbar} \mathscr{C}_X$ in $\mathrm{Mod}(\mathscr{A}_{X \times X^a})$.*

Proof. Both $\Omega_X^{\mathscr{A}}$ and $\mathscr{C}_X$ are simple $\mathscr{A}_{X \times X^a}$-modules along the diagonal Δ. By Lemma 6.2.1, $L := \mathscr{H}om_{\mathscr{A}_Z}(\mathscr{C}_X, \Omega_X^{\mathscr{A}})$ is a local system of rank one over $\mathbb{C}^\hbar$ and we have $\Omega_X^{\mathscr{A}} \simeq L \otimes_{\mathbb{C}_X^\hbar} \mathscr{C}_X$. Q.E.D.

Note that this implies the isomorphisms

$$\mathrm{D}'_{\mathscr{A}_{X\times X^a}}\mathscr{C}_X \simeq L^{\otimes -1}\otimes\mathscr{C}_X\,[-d_X]. \tag{6.2.1}$$

Hence we obtain the chain of morphisms

$$\begin{aligned}
L \;&\to\; L\otimes \mathrm{R}\mathscr{H}om_{\mathscr{A}_Z}(\mathscr{C}_X,\mathscr{C}_X)\simeq L\otimes\mathrm{D}'_{\mathscr{A}_{X\times X^a}}\mathscr{C}_X\overset{\mathrm{L}}{\otimes}_{\mathscr{A}_Z}\mathscr{C}_X\\
&\simeq\; \mathscr{C}_X\overset{\mathrm{L}}{\otimes}_{\mathscr{A}_Z}\mathscr{C}_X\,[-d_X]=\mathcal{HH}(\mathscr{A}_X)\,[-d_X]\simeq L^{\otimes -1}\otimes\Omega_X^{\mathscr{A}}\overset{\mathrm{L}}{\otimes}_{\mathscr{A}_Z}\mathscr{C}_X\,[-d_X]\\
&\to\; L^{\otimes -1}\otimes\Omega_X^{\mathscr{A}}\overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X^{\mathscr{A}}}\mathscr{C}_X\,[-d_X]\simeq L^{\otimes -1}.
\end{aligned}$$

Therefore, we get the morphism:

$$L \xrightarrow{\sim} H^{-d_X}(\mathcal{HH}(\mathscr{A}_X))\to L^{\otimes -1}. \tag{6.2.2}$$

Lemma 6.2.3. (i) $\mathrm{gr}_\hbar(L)\to\mathscr{H}om_{\mathrm{gr}_\hbar(\mathscr{A}_Z)}(\mathrm{gr}_\hbar(\mathscr{C}_X),\mathrm{gr}_\hbar(\Omega_X^{\mathscr{A}}))\simeq\Omega_X$ *gives an isomorphism* $\mathrm{gr}_\hbar(L)\xrightarrow{\sim}\mathbb{C}_X\cdot dv$.

(ii) *The morphism* $L^{\otimes 2}\to\mathbb{C}_X^\hbar$ *induced by* (6.2.2) *decomposes as* $L^{\otimes 2}\xrightarrow{\varphi}\hbar^{2n}\mathbb{C}_X^\hbar\hookrightarrow\mathbb{C}_X^\hbar$ *and* φ *is an isomorphism.*

(iii) *The diagram below commutes:*

$$\begin{tikzcd}
\mathrm{gr}_\hbar(L^{\otimes 2}) \arrow[r, "\sim"] \arrow[d, "\wr"] & \mathrm{gr}_\hbar(\hbar^{2n}\mathbb{C}_X^\hbar) & \mathrm{gr}_\hbar(\mathbb{C}_X^\hbar) \arrow[l, "\sim", "\hbar^{2n}"'] \arrow[d, "\wr"] \\
(\mathrm{gr}_\hbar(L))^{\otimes 2} \arrow[r, "\sim"] & \mathbb{C}_X^{\otimes 2} \arrow[r, dash, "\sim"] & \mathbb{C}_X
\end{tikzcd}$$

Proof. The question being local, we may assume to be given a local coordinate system $x=(x_1,\dots,x_{2n})$ on X and a scalar-valued non-degenerate skew-symmetric matrix $B=(b_{ij})_{1\le i,j\le 2n}$ such that the symplectic form α_X is given by

$$\alpha_X=\sum_{i,j}b_{ij}\,dx_i\wedge dx_j.$$

We set

$$A=(a_{ij})_{1\le i,j\le 2n}=B^{-1}.$$

We may assume that $\mathscr{A}_X=(\mathscr{O}_X[[\hbar]],\star)$ is a star-algebra with a star product

$$f\star g=\Big(\exp\big(\sum_{ij}\frac{\hbar a_{ij}}{2}\frac{\partial^2}{\partial x_i\partial x'_j}\big)f(x)g(x')\Big)\Big|_{x'=x}.$$

Set

$$\delta_i = \sum_{j=1}^{2n} a_{ij}\partial_{x_j}\ (i = 1, \dots, 2n).$$

Then, the $\mathbb{C}^\hbar$-linear morphisms from $\mathscr{O}_X[[\hbar]]$ to $\mathscr{D}_X[[\hbar]]$

$$\Phi^l \colon f \mapsto f\,\star, \quad \Phi^r \colon f \mapsto \star\, f \tag{6.2.3}$$

are given by

$$\Phi^l(x_i) = x_i + \frac{\hbar}{2}\delta_i, \quad \Phi^r(x_i) = x_i - \frac{\hbar}{2}\delta_i.$$

These morphisms define the morphism

$$\begin{aligned} \Phi \colon \mathscr{A}_X \otimes \mathscr{A}_{X^a} &\to \mathscr{D}_X[[\hbar]] \\ x_i \mapsto x_i + \frac{\hbar}{2}\delta_i, &\quad y_i \mapsto x_i - \frac{\hbar}{2}\delta_i. \end{aligned} \tag{6.2.4}$$

where we denote by $y = (y_1, \dots, y_{2n})$ a copy of the local coordinate system on X^a.

We identify $\Omega_X^{\mathscr{A}}$ with the $(\mathscr{D}_X[[\hbar]])^{\mathrm{op}}$-module $\Omega_X[[\hbar]]$. Then, regarding $\Omega_X[[\hbar]]$ as an $\mathscr{A}_Z$-module through $\mathscr{A}_Z|_X \to \mathscr{A}_Z^{\mathrm{op}}|_X \to (\mathscr{D}_X[[\hbar]])^{\mathrm{op}}$, we have

$$\begin{aligned} x_i(a\,dv) &= (a\,dv)\Phi^r(x_i) = (a\,dv)(x_i - \frac{\hbar}{2}\delta_i) \\ &= ((x_i + \frac{\hbar}{2}\delta_i)a)dv \end{aligned}$$

and similarly

$$y_i(adv) = ((x_i - \frac{\hbar}{2}\delta_i)a)dv.$$

Hence, $a \mapsto a\,dv$ gives an $\mathscr{A}_Z$-linear isomorphism

$$\mathscr{C}_X \simeq \mathscr{O}_X[[\hbar]] \xrightarrow{\sim} \Omega_X[[\hbar]] \simeq \Omega_X^{\mathscr{A}}.$$

Hence it gives an isomorphism $L := \mathscr{H}om_{\mathscr{A}_Z}(\mathscr{C}_X, \Omega_X^{\mathscr{A}}) \simeq \mathscr{H}om_{\mathscr{A}_Z}(\mathscr{C}_X, \mathscr{C}_X) \simeq \mathbb{C}_X^\hbar$, and the induced morphism $\mathrm{gr}_\hbar(L) \to \mathscr{H}om_{\mathrm{gr}_\hbar(\mathscr{A}_Z)}(\mathrm{gr}_\hbar(\mathscr{C}_X), \mathrm{gr}_\hbar(\Omega_X^{\mathscr{A}})) \simeq \Omega_X$ gives an isomorphism $\mathrm{gr}_\hbar(L) \xrightarrow{\sim} \mathbb{C}_X\,dv$. Hence we obtain (i).

For a sheaf of $\mathbb{C}^\hbar$-modules $\mathscr{F}$, we set

$$\mathscr{F}^{(p)} = \big(\bigwedge^p (\mathbb{C}_X^\hbar)^{2n}\big) \otimes_{\mathbb{C}_X^\hbar} \mathscr{F}.$$

Let $(e_1,\dots,e_{2n})$ be the basis of $(\mathbb{C}^\hbar)^{2n}$. Consider the Koszul complex $K^\bullet(\mathscr{A}_Z;b)$ where $b=(b_1,\dots,b_{2n})$, $b_i=(x_i-y_i)$ is the right multiplication by (x_i-y_i) on $\mathscr{A}_Z$:

$$\begin{aligned} K^\bullet(\mathscr{A}_Z;b) &:= 0\to\mathscr{A}_Z^{(0)}\xrightarrow{b}\cdots\xrightarrow{b}\mathscr{A}_Z^{(2n)}\to 0,\\ b &= \sum_i \bullet\wedge b_ie_i\colon K^p(\mathscr{A}_Z;b)\to K^{p+1}(\mathscr{A}_Z;b).\end{aligned}$$

On the other hand, consider the Koszul complex $K^\bullet(\mathscr{D}_X[[\hbar]];\delta)$ where $\delta=(\delta_1,\dots,\delta_{2n})$:

$$\begin{aligned} K^\bullet(\mathscr{D}_X[[\hbar]];\delta) &:= 0\to(\mathscr{D}_X[[\hbar]])^{(0)}\xrightarrow{\delta}\cdots\xrightarrow{\delta}(\mathscr{D}_X[[\hbar]])^{(2n)}\to 0,\\ \delta &= (\delta_1,\dots,\delta_{2n}).\end{aligned}$$

There is a quasi-isomorphism $K^\bullet(\mathscr{A}_Z;b)\xrightarrow{qis}\mathscr{C}_X[-2n]$ in the category of complexes in $\mathrm{Mod}(\mathscr{A}_Z)$.

Set $\delta=(\delta_1^l-\delta_1^r,\dots,\delta_{2n}^l-\delta_{2n}^r)$. Then the morphism Φ in (6.2.4) sends (x_i-y_i) to $\hbar\delta_i$. Consider the Koszul complex $K^\bullet(\mathscr{D}_X[[\hbar]];\delta)$:

$$\begin{aligned} K^\bullet(\mathscr{D}_X[[\hbar]];\delta) &:= 0\to(\mathscr{D}_X[[\hbar]])^{(0)}\xrightarrow{\delta}\cdots\xrightarrow{\delta}(\mathscr{D}_X[[\hbar]])^{(2n)}\to 0,\\ \delta &= (\delta_1,\dots,\delta_{2n}).\end{aligned}$$

There is a quasi-isomorphism $K^\bullet(\mathscr{D}_X[[\hbar]];\delta)\xrightarrow{qis}\mathscr{O}_X[[\hbar]][-2n]$. Therefore we get a commutative diagram in $\mathrm{Mod}(\mathscr{A}_Z)$:

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{A}_Z^{(0)} \arrow[r, "b"] \arrow[d, "\hbar^{2n}\Phi"] & \cdots \arrow[r] & \mathscr{A}_Z^{(2n-1)} \arrow[r, "b"] \arrow[d, "\hbar\Phi"] & \mathscr{A}_Z^{(2n)} \arrow[r] \arrow[d, "\hbar^0\Phi"] & 0 \\ 0 \arrow[r] & (\mathscr{D}_X[[\hbar]])^{(0)} \arrow[r, "\delta"] & \cdots \arrow[r] & (\mathscr{D}_X[[\hbar]])^{(2n-1)} \arrow[r, "\delta"] & (\mathscr{D}_X[[\hbar]])^{(2n)} \arrow[r] & 0. \end{tikzcd}$$

The object $\Omega_X^{\mathscr{A}}\overset{\mathrm{L}}{\otimes}_{\mathscr{A}_Z}\mathscr{C}_X$ is obtained by applying the functor $\Omega_X^{\mathscr{A}}\otimes_{\mathscr{A}_Z}\bullet$ to the row on the top and the object $\Omega_X^{\mathscr{A}}\overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X^{\mathscr{A}}}\mathscr{C}_X$ is obtained by applying the functor $\Omega_X^{\mathscr{A}}\otimes_{\mathscr{D}_X^{\mathscr{A}}}\bullet$ to the row on the bottom. By identifying $\Omega_X^{\mathscr{A}}$ with $\Omega_X[[\hbar]]$, the morphism $\Omega_X^{\mathscr{A}}\overset{\mathrm{L}}{\otimes}_{\mathscr{A}_Z}\mathscr{C}_X[-d_X]\to\Omega_X^{\mathscr{A}}\overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X^{\mathscr{A}}}\mathscr{C}_X[-d_X]$ is described by the morphism of complexes:

$$\begin{tikzcd} 0 \arrow[r] & \Omega_X^0[[\hbar]] \arrow[r, "\hbar d"] \arrow[d, "\hbar^{2n}"] & \cdots \arrow[r] & \Omega_X^{2n-1}[[\hbar]] \arrow[r, "\hbar d"] \arrow[d, "\hbar"] & \Omega_X^{2n}[[\hbar]] \arrow[r] \arrow[d, "\hbar^0"] & 0 \\ 0 \arrow[r] & \Omega_X^0[[\hbar]] \arrow[r, "d"] & \cdots \arrow[r] & \Omega_X^{2n-1}[[\hbar]] \arrow[r, "d"] & \Omega_X^{2n}[[\hbar]] \arrow[r] & 0. \end{tikzcd} \tag{6.2.5}$$

Here d denotes the usual exterior derivative.

Therefore, we find the commutative diagram with exact rows:

$$\begin{tikzcd}
 & L^{\otimes 2} \arrow[d, "\wr"] & & \\
0 \arrow[r] & \mathbb{C}^{\hbar}_X \arrow[r] \arrow[d, "\hbar^{2n}"] & \Omega^0_X[[\hbar]] \arrow[r, "\hbar d"] \arrow[d, "\hbar^{2n}"] & \Omega^1_X[[\hbar]] \arrow[d, "\hbar^{2n-1}"] \\
0 \arrow[r] & \mathbb{C}^{\hbar}_X \arrow[r] & \Omega^0_X[[\hbar]] \arrow[r, "d"] & \Omega^1_X[[\hbar]]
\end{tikzcd}$$

in which the morphism $L^{\otimes 2} \to \mathbb{C}^{\hbar}_X$ corresponds to the morphism $L[d_X] \to L^{\otimes -1} \otimes \Omega^{\mathscr{A}}_X \otimes_{\mathscr{A}_Z} \mathscr{C}_X$.

This completes the proof. Q.E.D.

Theorem 6.2.4. *Assume that X is symplectic.*

(i) *Let L be the local system given by Lemma 6.2.2. Then there is a canonical $\mathbb{C}^{\hbar}$-linear isomorphism $L \xrightarrow{\sim} \hbar^{d_X/2}\mathbb{C}^{\hbar}_X$, hence, a canonical $\mathscr{A}_Z$-linear isomorphism*

$$\Omega^{\mathscr{A}}_X \xrightarrow{\sim} \hbar^{d_X/2}\mathbb{C}^{\hbar}_X \otimes_{\mathbb{C}^{\hbar}_X} \mathscr{C}_X. \tag{6.2.6}$$

(ii) *The isomorphism* (6.2.6) *together with* (6.2.2) *induce canonical morphisms*

$$\hbar^{d_X/2}\mathbb{C}^{\hbar}_X[d_X] \xrightarrow{\iota_X} \mathcal{HH}(\mathscr{A}_X) \xrightarrow{\tau_X} \hbar^{-d_X/2}\mathbb{C}^{\hbar}_X[d_X] \tag{6.2.7}$$

and the composition $\tau_X \circ \iota_X$ is the canonical morphism $\hbar^{d_X/2}\mathbb{C}^{\hbar}_X[d_X] \to \hbar^{-d_X/2}\mathbb{C}^{\hbar}_X[d_X]$.

(iii) *$H^j(\mathcal{HH}(\mathscr{A}_X)) \simeq 0$ unless $-d_X \le j \le 0$ and the morphism ι_X induces an isomorphism*

$$\iota_X \colon \hbar^{d_X/2}\mathbb{C}^{\hbar}_X \xrightarrow{\sim} H^{-d_X}(\mathcal{HH}(\mathscr{A}_X)). \tag{6.2.8}$$

In particular, there is a canonical non-zero section in $H^{-d_X}(X;\mathcal{HH}(\mathscr{A}_X))$.

Proof. (i) By Lemma 6.2.3, we have an isomorphism $(\hbar^{-d_X/2}L)^{\otimes 2} \simeq \mathbb{C}^{\hbar}_X$ together with a compatible isomorphism $\mathrm{gr}_{\hbar}(\hbar^{-d_X/2}L) \simeq \mathbb{C}_X$. This implies $\hbar^{-d_X/2}L \simeq \mathbb{C}^{\hbar}_X$ since the only invertible element $a \in \mathbb{C}^{\hbar}$ satisfying $a^2 = 1$, $\sigma_0(a) = 1$ is $a = 1$.

(ii)–(iii) Denote by $(\Omega_X^\bullet[[\hbar]], \hbar d)$ and $(\Omega_X^\bullet[[\hbar]], d)$ the complexes given by the top row and the bottom row of (6.2.5), respectively. The morphism ι_X is represented by

$$L[d_X] \to L^{\otimes -1} \otimes (\Omega_X^\bullet[[\hbar]], \hbar d)[d_X]$$

and the morphism τ_X is the composition

$$L^{\otimes -1} \otimes (\Omega_X^\bullet[[\hbar]], \hbar d)[d_X] \to L^{\otimes -1} \otimes (\Omega_X^\bullet[[\hbar]], d)[d_X] \xrightarrow{\sim} L^{\otimes -1}[d_X].$$

Q.E.D.

Applying Theorem 6.2.4 together with Corollary 3.3.4, we obtain:

Corollary 6.2.5. *Let X be a compact complex symplectic manifold. Then $\mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{A}_X^{\mathrm{loc}})$ is a Calabi-Yau triangulated category of dimension d_X over $\mathbb{C}^{\hbar,\mathrm{loc}}$.*

Remark 6.2.6. The statement in Theorem 9.2 (ii) of [43] is not correct. If Y is a compact complex contact manifold of dimension d_Y, then the dimension of the Calabi-Yau category associated to it in loc. cit. is d_Y, not $d_Y - 1$.

6.3 Euler classes of $\mathscr{A}^{\mathrm{loc}}$-modules

Theorem 6.3.1. *The complex $\mathcal{HH}(\mathscr{A}_X^{\mathrm{loc}})$ is concentrated in degree $-d_X$ and the morphisms ι_X and τ_X in Theorem 6.2.4 induce isomorphisms*

$$\mathbb{C}_X^{\hbar,\mathrm{loc}}[d_X] \xrightarrow[\iota_X]{\sim} \mathcal{HH}(\mathscr{A}_X^{\mathrm{loc}}) \xrightarrow[\tau_X]{\sim} \mathbb{C}_X^{\hbar,\mathrm{loc}}[d_X]. \tag{6.3.1}$$

Proof. This follows from the fact that $(\Omega_X^\bullet[[\hbar]], \hbar d) \to (\Omega_X^\bullet[[\hbar]], d)$ becomes a quasi-isomorphism after applying the functor $(\,\bullet\,)^{\mathrm{loc}} = \mathbb{C}^{\hbar,\mathrm{loc}} \otimes_{\mathbb{C}^\hbar} (\,\bullet\,)$. Q.E.D.

Definition 6.3.2. Let $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X^{\mathrm{loc}})$. We set

$$\mathrm{eu}_X(\mathscr{M}) = \tau_X(\mathrm{hh}_X(\mathscr{M})) \in H^{d_X}_{\mathrm{Supp}(\mathscr{M})}(X; \mathbb{C}_X^{\hbar,\mathrm{loc}}) \tag{6.3.2}$$

and call $\mathrm{eu}_X(\mathscr{M})$ the Euler class of $\mathscr{M}$.

Remark 6.3.3. (i) The existence of a canonical section in $H^{-d_X}(X; \mathcal{HH}(\mathscr{A}_X^{\mathrm{loc}}))$ is well known when $X = T^*M$ is a cotangent bundle, see in particular [12, 26, 63]. It is intensively used in [11] where these authors call it the “trace density map”.
(ii) The Hochschild and cyclic homology of an algebroid stack have been defined in [9] where the Chern character of a perfect complex is constructed in the negative cyclic homology. It gives in particular an alternative construction of the Hochschild class of a coherent DQ-module, but it is not clear whether the two constructions give the same class.

Consider the diagram

$$\begin{tikzcd} p_{13!}(p_{12}^{-1}\mathcal{HH}(\mathscr{A}^{\mathrm{loc}}_{X_1\times X_2^a})\otimes p_{23}^{-1}\mathcal{HH}(\mathscr{A}^{\mathrm{loc}}_{X_2\times X_3^a})) \arrow[r, "\star"] \arrow[d, "\tau_{12^a}\otimes\tau_{23^a}"] & \mathcal{HH}(\mathscr{A}^{\mathrm{loc}}_{X_1\times X_3^a}) \arrow[d, "\tau_{13^a}"] \\ p_{13!}(p_{12}^{-1}\mathbb{C}^{\hbar,\mathrm{loc}}_{X_{12}}[d_{12}]\otimes p_{23}^{-1}\mathbb{C}^{\hbar,\mathrm{loc}}_{X_{23}}[d_{23}]) \arrow[r, "\int_2(\cdot\cup\cdot)"] & \mathbb{C}^{\hbar,\mathrm{loc}}_{X_{13}}[d_{13}]. \end{tikzcd} \tag{6.3.3}$$

Here, the horizontal arrow in the bottom denoted by $\int_2(\bullet\cup\bullet)$ is obtained by taking the cup product and integrating on X_2 (Poincaré duality), using the fact that the manifold X_2 has real dimension $2\,d_2$ and is oriented. The arrow in the top denoted by $\star$ is obtained by Proposition 4.2.1.

Proposition 6.3.4. Diagram 6.3.3 *commutes.*

Proof. Since X_1 and X_3 play the role of parameter spaces, we may assume that $X_1 = X_3 = \{\mathrm{pt}\}$. We set $X_2 = X$ and denote by a_X the projection $X\to\{\mathrm{pt}\}$. We are reduce to prove the commutativity of the diagram below:

$$\begin{tikzcd} a_{X!}(\mathcal{HH}(\mathscr{A}^{\mathrm{loc}}_X)\otimes\mathcal{HH}(\mathscr{A}^{\mathrm{loc}}_X)) \arrow[d, "\tau\otimes\tau"'] \arrow[dr, "\star"] & \\ a_{X!}(\mathbb{C}^{\hbar,\mathrm{loc}}_X[d_X]\otimes\mathbb{C}^{\hbar,\mathrm{loc}}_X[d_X]) \arrow[r, "\int_X(\cdot\cup\cdot)"'] & \mathbb{C}^{\hbar,\mathrm{loc}}. \end{tikzcd} \tag{6.3.4}$$

This will follow by applying the functor $a_{X!}$ to Diagram 6.3.5 below. Q.E.D.

Lemma 6.3.5. *The diagram below commutes.*

$$\begin{tikzcd} \mathcal{HH}(\mathscr{A}^{\mathrm{loc}}_X)\otimes\mathcal{HH}(\mathscr{A}^{\mathrm{loc}}_X) \arrow[d, "\tau\otimes\tau"'] \arrow[dr, "\star"] & \\ \mathbb{C}^{\hbar,\mathrm{loc}}_X[d_X]\otimes\mathbb{C}^{\hbar,\mathrm{loc}}_X[d_X] \arrow[r] & \mathbb{C}^{\hbar,\mathrm{loc}}_X[2d_X]. \end{tikzcd} \tag{6.3.5}$$

Proof. The morphism $L\otimes L[2d_X]\simeq\mathbb{C}^{\hbar,\mathrm{loc}}_X[d_X]\otimes\mathbb{C}^{\hbar,\mathrm{loc}}_X[d_X]\to\mathbb{C}^{\hbar,\mathrm{loc}}_X[2d_X]$ is given by

$$\begin{aligned} L\otimes L[2d_X] &\to L[d_X]\otimes\mathrm{R}\mathscr{H}om_{\mathscr{A}_Z}(\mathscr{C}_X,\Omega^{\mathscr{A}}_X)[d_X] \\ &\simeq L\otimes\mathrm{D}'_{\mathscr{A}}\mathscr{C}_X[d_X]\otimes_{\mathscr{A}_Z}\omega^{\mathscr{A}}_X\simeq\mathscr{C}_{X^a}\otimes_{\mathscr{A}_Z}\omega^{\mathscr{A}}_X\to\mathbb{C}^{\hbar}_X[2d_X]. \end{aligned}$$

On the other hand, $L\otimes L[2d_X]\to\mathcal{HH}(\mathscr{A}_X)\otimes\mathcal{HH}(\mathscr{A}_X)\to\mathbb{C}^{\hbar}_X[2d_X]$ is given by

$$\begin{aligned} L\otimes L[2d_X] &\to \mathrm{R}\mathscr{H}om_{\mathscr{A}_{Z^a}}(\mathrm{D}'_{\mathscr{A}}\mathscr{C}_X,\mathscr{C}_{X^a})\otimes\mathrm{R}\mathscr{H}om_{\mathscr{A}_Z}(\mathscr{C}_X,\omega^{\mathscr{A}}_X) \\ &\simeq \mathrm{R}\mathscr{H}om_{\mathscr{A}_{Z^a}}(\mathrm{D}'_{\mathscr{A}}\mathscr{C}_X,\mathscr{C}_{X^a})\otimes\big(\mathrm{D}'_{\mathscr{A}}(\mathscr{C}_X)\otimes_{\mathscr{A}_Z}\omega^{\mathscr{A}}_X\big) \\ &\to \mathscr{C}_{X^a}\otimes_{\mathscr{A}_Z}\omega^{\mathscr{A}}_X\to\mathbb{C}^{\hbar}_X[2d_X]. \end{aligned}$$

These two morphisms give the same morphism from $L \otimes L[2d_X]$ to $\mathbb{C}_X^{\hbar}[2d_X]$.
Q.E.D.

Corollary 6.3.6. *Let $\mathscr{K}_i \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}^{\mathrm{loc}}_{X_i \times X^a_{i+1}})$ $(i = 1, 2)$. Assume that the projection p_{13} defined on $X_1 \times X_2 \times X_3$ is proper on $p_{12}^{-1} \operatorname{Supp}(\mathscr{K}_1) \cap p_{23}^{-1} \operatorname{Supp}(\mathscr{K}_2)$. Then*

$$\mathrm{eu}_{X_{13^a}}(\mathscr{K}_1 \underset{2}{\circ} \mathscr{K}_2) = \int_{X_2} \mathrm{eu}_{X_{12^a}}(\mathscr{K}_1) \cup \mathrm{eu}_{X_{23^a}}(\mathscr{K}_2). \tag{6.3.6}$$

Remark 6.3.7. Consider an object $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}^{\mathrm{loc}}_X)$. Then, according to Definition 6.3.2, its Euler class is well-defined in the de Rham cohomology of X with values in $\mathbb{C}^{\hbar,\mathrm{loc}}$. Now assume that $\mathscr{M}$ is generated by $\mathscr{M}_0 \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X)$ and consider $\mathrm{gr}_\hbar(\mathscr{M}_0)$. Assume for simplicity that $\mathrm{gr}_\hbar(\mathscr{A}_X) = \mathscr{O}_X$ (the general case can be treated with suitable modifications). Then $\mathrm{gr}_\hbar(\mathscr{M}_0) \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_X)$ and we may consider its Chern class in de Rham cohomology. A natural question is to compare these two classes. A precise conjecture had been made in the case of $\mathscr{D}$-modules by one of the authors (PS) and J-P. Schneiders in [58] and proved by P. Bressler, R. Nest and B. Tsygan in [11]. These authors, together with A. Gorokhovsky, recently treated the general case of DQ-algebroids in the symplectic setting in [10]. The formula they obtain makes use of a cohomology class naturally associated to the deformation $\mathscr{A}_X$.

6.4 Hochschild classes of $\mathscr{D}$-modules

We shall apply the preceding result to the study of the Euler class of $\mathscr{D}$-modules.

Recall after [38] that a coherent $\mathscr{D}_M$-module $\mathscr{M}$ is *good* if, for any open relatively compact set $U \subset M$, there exists a coherent sub-$\mathscr{O}_U$-module $\mathscr{F}$ of $\mathscr{M}|_U$ which generates it on U as a $\mathscr{D}_M$-module. One denotes by $\mathrm{D}^b_{\mathrm{gd}}(\mathscr{D}_M)$ the full sub-triangulated category of $\mathrm{D}^b_{\mathrm{coh}}(\mathscr{D}_M)$ consisting of objects with good cohomology.

From now on, we set

$$X = T^*M.$$

We introduce the functor

$$\begin{aligned} (\bullet)^{\mathrm{W}} \colon \mathrm{Mod}(\mathscr{D}_M) &\to \mathrm{Mod}(\widehat{\mathscr{W}}_X) \\ \mathscr{M} &\mapsto \widehat{\mathscr{W}}_X \otimes_{\pi_M^{-1}\mathscr{D}_M} \pi_M^{-1}\mathscr{M}. \end{aligned} \tag{6.4.1}$$

The next result shows that one can, in some sense, reduce the study of $\mathscr{D}$-modules to that of $\widehat{\mathscr{W}}_X$-modules.

Proposition 6.4.1. *The functor $\mathscr{M} \mapsto \mathscr{M}^{\mathrm{W}}|_{T^*_M M}$ is exact and faithful.*

Proof. The morphism

$$\mathscr{M} \quad \to \quad (\widehat{\mathscr{E}}_{T^*M} \otimes_{\pi_M^{-1}\mathscr{D}_M} \pi_M^{-1}\mathscr{M})|_{T^*_M M}.$$

is an isomorphism, and hence the result is a particular case of Lemma 6.1.2.
Q.E.D.

It follows that $(\bullet)^{\mathrm{W}}$ sends $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_M)$ to $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\widehat{\mathscr{W}_X})$ and $\mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{D}_M)$ to $\mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\widehat{\mathscr{W}_X})$.

Definition 6.4.2. Let $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{D}_M)$. We set

$$\mathrm{hh}^{\mathrm{gr}}_{\mathrm{X}}(\mathscr{M}) = \mathrm{hh}^{\mathrm{gr}}_{\mathrm{X}}(\mathscr{M}^{\mathrm{W}}) \in \mathrm{HH}_{\mathrm{char}(\mathscr{M})}(\mathscr{O}_{\mathrm{X}}). \tag{6.4.2}$$

For Λ a closed subset of T^*M, we denote by $\mathrm{K}_{\mathrm{gd},\Lambda}(\mathscr{D}_M)$ the Grothendieck group of the full abelian subcategory of $\mathrm{Mod}_{\mathrm{gd}}(\mathscr{D}_M)$ consisting of $\mathscr{D}$-modules whose characteristic is contained in Λ.

Let V be an open relatively compact subset of M. By slightly modifying the proof of Proposition 3.4.3, we get morphisms of groups

$$\mathrm{K}_{\mathrm{gd},\Lambda}(\mathscr{D}_M) \longrightarrow \mathrm{K}_{\mathrm{coh},\Lambda}(\mathscr{O}_{\pi^{-1}V}). \tag{6.4.3}$$

Let M_i $(i = 1, 2, 3)$ be three complex manifolds and set $X_i = T^*M_i$. Denote by q_{ij} the ij-th projection defined on $M_1 \times M_2 \times M_3$ and by p_{ij} the ij-th projection defined on $X_1 \times X_2 \times X_3$ $(1 \le i < j \le 3)$. We set, as for DQ-algebra, $\mathscr{D}_{M^a} := (\mathscr{D}_M)^{\mathrm{op}}$ and we write for short M_{ij} or M_{ij^a} instead of $M_i \times M_j$ or $M_i \times M_j^a$ and similarly with X_{ij}. We also write $\mathscr{D}_{ij}$ instead of $\mathscr{D}_{M_{ij}}$ and similarly with ij^a, etc. For example,

$$\mathscr{D}_{12^a} = \mathscr{O}_{M_{12}} \otimes_{(\mathscr{O}_{M_1} \boxtimes \mathscr{O}_{M_2})} (\mathscr{D}_{M_1} \boxtimes (\mathscr{D}_{M_2})^{\mathrm{op}}).$$

Then $\mathscr{D}_1$ may be regarded as a $\mathscr{D}_{11^a}$-module supported on the diagonal of $X_1 \times X_{1^a}$. Let $\mathscr{K}_i \in \mathrm{D}^{\mathrm{b}}(\mathscr{D}_{ij^a})$ $(i = 1, 2,\ j = i + 1)$. Set

$$\mathscr{K}_1 \underset{M_2}{\circ} \mathscr{K}_2 := \mathrm{R}q_{13^a!}\Big(\mathscr{D}_2 \otimes_{\mathscr{D}_{2^a 2}} \mathscr{D}_{12^a 23^a} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_{12^a} \boxtimes \mathscr{D}_{23^a}} (\mathscr{K}_1 \boxtimes \mathscr{K}_2)\Big).$$

Theorem 6.4.3. *Let Λ_i be a closed subset of $X_i \times X_{i+1}$ $(i = 1, 2)$ and assume that the projection p_{13} defined on $X_1 \times X_2 \times X_3$ is proper on $p_{12}^{-1}\Lambda_1 \cap p_{23}^{-1}\Lambda_2$. Set $\Lambda = \Lambda_1 \circ \Lambda_2$. Let $\mathscr{K}_i \in \mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{D}_{ij^a})$ $(i = 1, 2,\ j = i+1)$ with $\mathrm{char}(\mathscr{K}_i) \subset \Lambda_i$ $(i = 1, 2)$. Then $\mathscr{K}_1 \underset{M_2}{\circ} \mathscr{K}_2 \in \mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{D}_{13^a})$, $\mathrm{char}(\mathscr{K}_1 \underset{M_2}{\circ} \mathscr{K}_2) \subset \Lambda$ and*

$$(\mathscr{K}_1 \underset{M_2}{\circ} \mathscr{K}_2)^{\mathrm{W}} \xrightarrow{\sim} \mathscr{K}_1^{\mathrm{W}} \underset{X_2}{\circ} \mathscr{K}_2^{\mathrm{W}}. \tag{6.4.4}$$

The proof is straightforward and is left to the reader. By using Diagram 4.4.7, we get:

Theorem 6.4.4. *In the situation of Theorem 6.4.3, let V_{ij} be a relatively compact open subset of $M_i \times M_j$ $(i = 1, 2,\ j = i+1)$ and assume that $\pi^{-1}V_{12^a} \times_{M_2} \pi^{-1}V_{23^a}$ contains $(\Lambda_1 \times_{X_2} \Lambda_2) \cap q_{13^a}^{-1}\pi^{-1}V_{13^a}$. Then the diagram below commutes*

$$\begin{tikzcd}
\mathrm{D}^{\mathrm{b}}_{\mathrm{gd},\Lambda_1}(\mathscr{D}_{12^a}) \times \mathrm{D}^{\mathrm{b}}_{\mathrm{gd},\Lambda_2}(\mathscr{D}_{23^a}) \arrow[r, "\circ"] \arrow[d, "\mathrm{gr}_\hbar"'] & \mathrm{D}^{\mathrm{b}}_{\mathrm{gd},\Lambda}(\mathscr{D}_{13^a}) \arrow[d, "\mathrm{gr}_\hbar"'] \\
\mathrm{K}_{\mathrm{coh},\Lambda_1}(\mathscr{O}_{\pi^{-1}V_{12^a}}) \times \mathrm{K}_{\mathrm{coh},\Lambda_2}(\mathscr{O}_{\pi^{-1}V_{23^a}}) \arrow[r, "\circ"] \arrow[d, "\mathrm{hh}\times\mathrm{hh}"'] & \mathrm{K}_{\mathrm{coh},\Lambda}(\mathscr{O}_{\pi^{-1}V_{13^a}}) \arrow[d, "\mathrm{hh}"'] \\
\mathrm{HH}_{\Lambda_1}(\mathscr{O}_{\pi^{-1}V_{12^a}}) \times \mathrm{HH}_{\Lambda_2}(\mathscr{O}_{\pi^{-1}V_{23^a}}) \arrow[r, "\circ"] & \mathrm{HH}_{\Lambda}(\mathscr{O}_{\pi^{-1}V_{13^a}}).
\end{tikzcd}$$

In particular

$$\mathrm{hh}^{\mathrm{gr}}_{\pi^{-1}\mathrm{V}_{13^{\mathrm{a}}}}(\mathscr{K}_1 \underset{2}{\circ} \mathscr{K}_2) = \mathrm{hh}^{\mathrm{gr}}_{\pi^{-1}\mathrm{V}_{12^{\mathrm{a}}}}(\mathscr{K}_1) \circ \mathrm{hh}^{\mathrm{gr}}_{\pi^{-1}\mathrm{V}_{23^{\mathrm{a}}}}(\mathscr{K}_2) \tag{6.4.5}$$

in $\mathrm{HH}_\Lambda(\mathscr{O}_{\pi^{-1}V_{13^a}})$.

As a particular case, and using Corollary 5.3.5, we recover a theorem of Laumon [48] in the analytic framework.

6.5 Euler classes of $\mathscr{D}$-modules

We keep the notations of § 6.4 and we set $X = T^*M$. One defines the Hochschild homology $\mathcal{HH}(\widehat{\mathscr{E}}_X)$ of $\widehat{\mathscr{E}}_X$ and the Hochschild class $\mathrm{hh}_X(\mathscr{M})$ of a coherent $\widehat{\mathscr{E}}_X$-module $\mathscr{M}$ similarly as for $\mathcal{HH}(\mathscr{A}_X)$.

In the sequel, we identify a coherent $\mathscr{D}_M$-module $\mathscr{M}$ with $\widehat{\mathscr{E}}_X \otimes_{\pi^{-1}\mathscr{D}_M} \pi^{-1}\mathscr{M}$. In particular, we define by this way the Hochschild class $\mathrm{hh}_X(\mathscr{M})$ of a coherent $\mathscr{D}$-module $\mathscr{M}$. Hence

$$\mathrm{hh}_X(\mathscr{M}) \in H^{d_X}_{\mathrm{char}(\mathscr{M})}(X; \mathcal{HH}(\widehat{\mathscr{E}}_X)). \tag{6.5.1}$$

Lemma 6.5.1. *There is a natural isomorphism*

$$\mathcal{HH}(\widehat{\mathscr{E}}_X) \xrightarrow{\sim} \mathbb{C}_X\,[d_X] \tag{6.5.2}$$

which makes the diagram below commutative:

$$\begin{tikzcd}
\mathcal{HH}(\widehat{\mathscr{E}}_X) \arrow[r, "\sim"] \arrow[d] & \mathbb{C}_X[d_X] \arrow[d] \\
\mathcal{HH}(\widehat{\mathscr{W}}_X) \arrow[r, "\sim", "\tau"'] & \mathbb{C}^{\hbar,\mathrm{loc}}_X\,[d_X].
\end{tikzcd}$$

Sketch of proof. We take coordinates $(x_1, \dots, x_n, u_1, \dots u_n)$, and set $\widetilde{\mathscr{O}_X} := \prod_{k \le 0} \hbar^{-k} \mathscr{O}_X(k)$, where $\mathscr{O}_X(k)$ is the sheaf of holomorphic functions on X homogeneous of degree k with respect to the variables $(u_1, \dots, u_n)$. Then $\widetilde{\mathscr{O}_X}$ is isomorphic to $\widehat{\mathscr{E}}_X(0)$ as a sheaf. Moreover, $\mathcal{HH}(\widehat{\mathscr{E}}_X)$ is represented by the Koszul complex of $\partial/\partial x_i$, $\hbar\partial/\partial u_i \in \mathscr{E}nd(\widetilde{\mathscr{O}_X})$ $(i = 1, \dots, n)$. On the other hand, as we have seen, $\mathcal{HH}(\widehat{\mathscr{W}_X})$ is represented by the Koszul complex of $\hbar\partial/\partial x_i$, $\hbar\partial/\partial u_i \in \mathscr{E}nd(\mathscr{O}_X((\hbar)))$ $(i = 1, \dots, n)$. Hence we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \widetilde{\mathscr{O}_X} \arrow[r] \arrow[d, "\hbar^{-n}"] & \cdots \arrow[r] & \widetilde{\mathscr{O}_X}^{2n} \arrow[r] \arrow[d] & \widetilde{\mathscr{O}_X} \arrow[r] \arrow[d, "\hbar^{0}"] & 0 \\
0 \arrow[r] & \mathscr{O}_X((\hbar)) \arrow[r] & \cdots \arrow[r] & \mathscr{O}_X((\hbar))^{2n} \arrow[r] & \mathscr{O}_X((\hbar)) \arrow[r] & 0,
\end{tikzcd}$$

in which the top row represents $\mathcal{HH}(\widehat{\mathscr{E}}_X)$ and the bottom row represents $\mathcal{HH}(\widehat{\mathscr{W}_X})$. Q.E.D.

Definition 6.5.2. Let $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\widehat{\mathscr{E}}_X)$. We denote by $\mathrm{eu}_X(\mathscr{M})$ the image of $\mathrm{hh}_X(\mathscr{M})$ in $H^{d_X}_{\mathrm{char}(\mathscr{M})}(X; \mathbb{C}_X)$ by the morphism in (6.5.2) and call it the Euler class of $\mathscr{M}$.

The next result immediately follows from Lemma 6.5.1.

Proposition 6.5.3. *For $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_M)$, $\mathrm{eu}_X(\mathscr{M}^W)$ is the image of $\mathrm{eu}_X(\mathscr{M})$ by the natural map $H^{d_X}_{\mathrm{char}(\mathscr{M})}(X; \mathbb{C}_X) \to H^{d_X}_{\mathrm{char}(\mathscr{M})}(X; \mathbb{C}^{\hbar,\mathrm{loc}}_X)$.*

Applying Theorem 4.3.5, we get:

Theorem 6.5.4. *In the situation of Theorem 6.4.3, one has:*

$$\mathrm{eu}_{13^a}(\mathscr{K}_1 \underset{2}{\circ} \mathscr{K}_2) = \mathrm{eu}_{12^a}(\mathscr{K}_1) \circ \mathrm{eu}_{23^a}(\mathscr{K}_2) \tag{6.5.3}$$

in $H^{d_1+d_3}_{\Lambda_1 \circ \Lambda_2}(X_{13}; \mathbb{C}_{X_{13}})$.

This formula is equivalent to the results of [58] on the functoriality of the Euler class of $\mathscr{D}$-modules. Note that the results of loc. cit. also deal with constructible sheaves.

Chapter 7

Holonomic DQ-modules

The aim of this chapter is to study holonomic DQ-modules on symplectic manifolds. More precisely, we will prove that, if $\mathscr{L}$ and $\mathscr{M}$ are two holonomic $\mathscr{A}_X^{\mathrm{loc}}$-modules on a symplectic manifold X, then the complex $\mathrm{R}\mathscr{H}om_{\mathscr{A}_X^{\mathrm{loc}}}(\mathscr{M},\mathscr{L})$ is perverse (hence, in particular, $\mathbb{C}$-constructible) over the field $\mathbb{C}^{\hbar,\mathrm{loc}}$. It follows from the preceding results in Chapter 6 that if the intersection of the supports of $\mathscr{M}$ and $\mathscr{L}$ is compact, then the Euler-Poincaré index of this complex is given by the integral $\int_X \mathrm{eu}_X(\mathscr{M})\cdot\mathrm{eu}_X(\mathscr{L})$. We show here that the Euler class of a holonomic module is a Lagrangian cycle, which makes its calculation easy.

If moreover $\mathscr{L}$ and $\mathscr{M}$ are simple holonomic modules supported on smooth Lagrangian submanifolds Λ_0 and Λ_1, then the microsupport of the complex $\mathrm{R}\mathscr{H}om_{\mathscr{A}_X^{\mathrm{loc}}}(\mathscr{M},\mathscr{L})$ is contained in the normal cone $\mathrm{C}(\Lambda_0,\Lambda_1)$. This last result was first obtained in [43] in the analytic framework, that is, using $\mathscr{W}_X$-modules, not $\widehat{\mathscr{W}}_X$-modules, which made the proofs much more intricate.

Finally we prove that, in some sense, the complex $\mathrm{R}\mathscr{H}om_{\mathscr{A}_X^{\mathrm{loc}}}(\mathscr{M},\mathscr{L})$ is invariant by Hamiltonian symplectomorphism.

7.1 $\mathscr{A}$-modules along a Lagrangian submanifold

Let X be a complex symplectic manifold endowed with a DQ-algebroid $\mathscr{A}_X$.

The algebra $\mathscr{A}_{\Lambda/X}$

Let Λ be a smooth Lagrangian submanifold of X and let $\mathscr{L}$ be a coherent $\mathscr{A}_X$-module simple along Λ.

Locally, X is isomorphic as a symplectic manifold to $T^*\Lambda$, the cotangent bundle to Λ. We set for short

$$\mathscr{O}_\Lambda^\hbar := \mathscr{O}_\Lambda[[\hbar]], \quad \mathscr{O}_\Lambda^{\hbar,\mathrm{loc}} := \mathscr{O}_\Lambda((\hbar)).$$

There are local isomorphisms

$$\mathscr{A}_X \simeq \widehat{\mathscr{W}_X}(0), \quad \mathscr{L} \simeq \mathscr{O}_\Lambda^\hbar.$$

Then $\mathscr{E}nd_{\mathbb{C}^\hbar}(\mathscr{L}) \simeq \mathscr{E}nd_{\mathbb{C}^\hbar}(\mathscr{O}_\Lambda^\hbar)$ (see Lemma 2.1.12) and the subalgebroid of $\mathscr{E}nd_{\mathbb{C}^\hbar}(\mathscr{L})$ corresponding to the subring $\mathscr{D}_\Lambda[[\hbar]]$ of $\mathscr{E}nd_{\mathbb{C}^\hbar}(\mathscr{O}_\Lambda^\hbar)$ is well-defined. We denote it by $\mathscr{D}_\mathscr{L}$.

Lemma 7.1.1. (i) *$\mathscr{D}_\mathscr{L}$ is equivalent to $\mathscr{D}_\Lambda[[\hbar]]$ as a $\mathbb{C}^\hbar$-algebroid.*

(ii) *The $\mathbb{C}^\hbar$-algebra $\mathscr{D}_\mathscr{L}$ satisfies* (1.2.2) *and* (1.3.1). *In particular, it is right and left Noetherian.*

Proof. (i) follows by similar arguments as in Proposition 2.5.2 (ii).
(ii) follows from Example 1.3.1. Q.E.D.

The functor $\mathscr{A}_X|_\Lambda \to \mathscr{E}nd_{\mathbb{C}^\hbar}(\mathscr{L})$ factorizes as

$$\mathscr{A}_X|_\Lambda \to \mathscr{D}_\mathscr{L}, \tag{7.1.1}$$

and setting $\mathscr{D}_\mathscr{L}^{\mathrm{loc}} := (\mathscr{D}_\mathscr{L})^{\mathrm{loc}}$, this functor induces a functor

$$\mathscr{A}_X^{\mathrm{loc}}|_\Lambda \to \mathscr{D}_\mathscr{L}^{\mathrm{loc}}. \tag{7.1.2}$$

We denote by $I_\Lambda \subset \mathscr{O}_X$ the defining ideal of Λ. Let $\mathscr{I}$ be the kernel of the composition

$$\hbar^{-1}\mathscr{A}_X \xrightarrow{\hbar} \mathscr{A}_X \xrightarrow{\sigma} \mathscr{O}_X \to \mathscr{O}_\Lambda.$$

Then we have $\mathscr{I}/\mathscr{A}_X \simeq I_\Lambda$.

Definition 7.1.2. We denote by $\mathscr{A}_{\Lambda/X}$ the $\mathbb{C}^\hbar$-subalgebroid of $\mathscr{A}_X^{\mathrm{loc}}$ generated by $\mathscr{I}$.

Note that the algebra $\mathscr{A}_{\Lambda/X}$ is the analogue in the framework of DQ-algebras of the algebra $\mathscr{E}_\Lambda$ constructed in [39].

The ideal $\hbar\mathscr{I}$ is contained in $\mathscr{A}_X$, hence acts on $\mathscr{L}$ and one sees easily that $\hbar\mathscr{I}$ sends $\mathscr{L}$ to $\hbar\mathscr{L}$. Hence, $\mathscr{I}$ acts on $\mathscr{L}$ and defines a functor $\mathscr{A}_{\Lambda/X} \to \mathscr{D}_\mathscr{L}$. We thus have the functors of algebroids

$$\begin{tikzcd}
\mathscr{A}_X|_\Lambda \arrow[r] \arrow[dr] & \mathscr{A}_{\Lambda/X}|_\Lambda \arrow[r] \arrow[d] & \mathscr{A}_X^{\mathrm{loc}}|_\Lambda \arrow[d] \\
& \mathscr{D}_\mathscr{L} \arrow[r] & \mathscr{D}_\mathscr{L}^{\mathrm{loc}}.
\end{tikzcd}$$

In particular, $\mathscr{L}$ is naturally an $\mathscr{A}_{\Lambda/X}$-module.

Lemma 7.1.3. (i) $\mathscr{I}^k = \mathscr{A}_{\Lambda/X} \cap \hbar^{-k}\mathscr{A}_X$ *for any* $k \geq 0$,

(ii) $\mathscr{I}^k/\mathscr{I}^{k-1} \simeq I_\Lambda^k$ *for* $k > 0$,

(iii) $\mathscr{A}_{\Lambda/X}$ *is a right and left Noetherian algebroid,*

(iv) $\mathrm{gr}_\hbar(\mathscr{A}_{\Lambda/X})|_\Lambda \xrightarrow{\sim} \mathrm{gr}_\hbar \mathscr{D}_{\mathscr{L}} \simeq \mathscr{D}_\Lambda$,

(v) $(\mathscr{A}_{\Lambda/X})^{\mathrm{loc}} \simeq \mathscr{A}_X^{\mathrm{loc}}$ *and* $\mathscr{A}_X^{\mathrm{loc}}$ *is flat over* $\mathscr{A}_{\Lambda/X}$.

Proof. Since the question is local, we may assume that $X = T^*\mathbb{C}^n$ with coordinates (x,u), $\Lambda = \{u = 0\}$, and $\mathscr{A}_X$ is the star-algebra as in (6.1.4). Set

$$\mathscr{A}' := \{\sum_k f_k(x,u)\hbar^k \in \mathscr{A}_X^{\mathrm{loc}};\ f_k(x,u) \in I_\Lambda^{-k} \text{ for } k < 0\}.$$

Then we can check that $\mathscr{A}'$ is a subalgebra of $\mathscr{A}_X^{\mathrm{loc}}$ and it contains $\mathscr{I}$. Hence it contains $\mathscr{A}_{\Lambda/X}$. It is easy to see that the image of $\mathscr{I}^k \to \hbar^{-k}\mathscr{A}_X/\hbar^{-k+1}\mathscr{A}_X$ contains $\hbar^{-k}I_\Lambda^k$. On the other hand, the image of $\mathscr{A}'\cap\hbar^{-k}\mathscr{A}_X \to \hbar^{-k}\mathscr{A}_X/\hbar^{-k+1}\mathscr{A}_X$ coincides with $\hbar^{-k}I_\Lambda^k$. Hence, $\mathscr{A}_{\Lambda/X} \cap \hbar^{-k}\mathscr{A}_X$ and $\mathscr{A}' \cap \hbar^{-k}\mathscr{A}_X$ have the same image $\hbar^{-k}I_\Lambda^k$ in $\hbar^{-k}\mathscr{A}_X/\hbar^{-k+1}\mathscr{A}_X$. We conclude that $\mathscr{A}_{\Lambda/X} = \mathscr{A}'$ and $\mathscr{A}_{\Lambda/X} \cap \hbar^{-k}\mathscr{A}_X \subset \mathscr{I}^k + \hbar^{-k+1}\mathscr{A}_X$. Hence, an induction on k shows (i).

(ii) is now obvious.

(iii) Considering the filtration $\{\mathscr{A}_{\Lambda/X}\cap\hbar^{-k}\mathscr{A}_X\}_{k\geq 0}$ of $\mathscr{A}_{\Lambda/X}$, the result follows by [38, Theorem A.32].

(iv) is obvious.

(v) follows from $\mathscr{A}_X \subset \mathscr{A}_{\Lambda/X} \subset \mathscr{A}_X^{\mathrm{loc}}$. Q.E.D.

By this lemma, for a coherent $\mathscr{A}_{\Lambda/X}$-module $\mathscr{N}$, we may regard $\mathrm{gr}_\hbar(\mathscr{N})$ as an object of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_\Lambda)$. Recall that $\mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_\Lambda)$ denotes the full triangulated category of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_\Lambda)$ consisting of objects with holonomic cohomology.

Lemma 7.1.4. *The algebroid* $\mathscr{D}_{\mathscr{L}}$ *is flat over* $\mathscr{A}_{\Lambda/X}$ *and* $\mathscr{D}_{\mathscr{L}}^{\mathrm{loc}}$ *is flat over* $\mathscr{A}_X^{\mathrm{loc}}$.

Proof. It is enough to prove the first statement.

Let us show that $H^j(\mathscr{D}_{\mathscr{L}} \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_{\Lambda/X}} \mathscr{M}) \simeq 0$ for any coherent $\mathscr{A}_{\Lambda/X}$-module $\mathscr{M}$ and any $j < 0$.

(i) Assume that $\mathscr{M}$ has no $\hbar$-torsion. Using Lemma 7.1.3 (iv), we have for $j < 0$, $H^j\mathrm{gr}_\hbar(\mathscr{D}_{\mathscr{L}} \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_{\Lambda/X}} \mathscr{M}) \simeq H^j\mathrm{gr}_\hbar\mathscr{M} \simeq 0$, and hence $H^j(\mathscr{D}_{\mathscr{L}} \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_{\Lambda/X}} \mathscr{M}) \simeq 0$ by Proposition 1.4.5.

(ii) Assume that $\hbar\mathscr{M} = 0$. Then

$$\mathscr{D}_{\mathscr{L}} \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_{\Lambda/X}} \mathscr{M} \simeq \mathscr{D}_{\mathscr{L}} \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_{\Lambda/X}} \mathrm{gr}_\hbar \mathscr{A}_{\Lambda/X} \overset{\mathrm{L}}{\otimes}_{\mathrm{gr}_\hbar \mathscr{A}_{\Lambda/X}} \mathscr{M} \simeq \mathrm{gr}_\hbar \mathscr{D}_{\mathscr{L}} \overset{\mathrm{L}}{\otimes}_{\mathrm{gr}_\hbar \mathscr{A}_{\Lambda/X}} \mathscr{M} \simeq \mathscr{M}.$$

(iii) In the general case, set ${}_n\mathscr{N} := \mathrm{Ker}(\hbar^n \colon \mathscr{M} \to \mathscr{M})$ and $\mathscr{M}_{tor} := \bigcup_n {}_n\mathscr{N}$. Note that this union is locally stationary. Defining $\mathscr{M}_{tf}$ by the exact sequence,

$$0 \to \mathscr{M}_{tor} \to \mathscr{M} \to \mathscr{M}_{tf} \to 0,$$

this module has no $\hbar$-torsion. It is thus enough to prove the result for the ${}_n\mathscr{N}$'s and this follows from (ii) by induction on n, using the exact sequence

$$0 \to {}_n\mathscr{N} \to {}_{n+1}\mathscr{N} \to {}_{n+1}\mathscr{N}/{}_n\mathscr{N} \to 0.$$

Q.E.D.

Definition 7.1.5. An object $\mathscr{N}$ of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_{\Lambda/X})$ is holonomic if $\mathrm{gr}_\hbar(\mathscr{N})$ is Lagrangian in $T^*\Lambda$, that is, if $\mathrm{gr}_\hbar(\mathscr{N})$ belongs to $\mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_\Lambda)$.

Note that this condition is equivalent to saying that $H^i(\mathscr{N})/\hbar H^i(\mathscr{N})$ and $\mathrm{Ker}(\hbar\colon H^i(\mathscr{N}) \to H^i(\mathscr{N}))$ are holonomic $\mathscr{D}_\Lambda$-modules for any i (see Lemma 1.4.2).

Microsupport and constructible sheaves

Let us recall some notions and results of [40].

Let M be a *real analytic* manifold and $\mathbb{K}$ a Noetherian commutative ring of finite global dimension. For $F \in \mathrm{D}^{\mathrm{b}}(\mathbb{K}_M)$, we denote by $\mathrm{SS}(F)$ its microsupport, a closed $\mathbb{R}^+$-conic (*i.e.,* invariant by the $\mathbb{R}^+$-action on T^*M) subset of T^*M. Recall that this set is involutive (one also says *co-isotropic*), see [40, Def. 6.5.1].

An object F of $\mathrm{D}^{\mathrm{b}}(\mathbb{K}_M)$ is *weakly* $\mathbb{R}$*-constructible* if there exists a subanalytic stratification $M = \bigsqcup_{\alpha\in A} M_\alpha$ such that $H^j(F)|_{M_\alpha}$ is locally constant for all $j \in \mathbb{Z}$ and all $\alpha \in A$. The object F is $\mathbb{R}$*-constructible* if moreover $H^j(F)_x$ is finitely generated for all $x \in M$ and all $j \in \mathbb{Z}$. One denotes by $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbb{K}_M)$ the full subcategory of $\mathrm{D}^{\mathrm{b}}(\mathbb{K}_M)$ consisting of $\mathbb{R}$-constructible objects. Recall that the duality functor $\mathrm{D}'_X(\bullet)$ (see (1.1.1)) is an anti-auto-equivalence of the category $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbb{K}_M)$.

If M is complex analytic, one defines similarly the notions of (weakly) $\mathbb{C}$-constructible sheaf, replacing "subanalytic" with "complex analytic". We denote by $\mathrm{D}^{\mathrm{b}}_{\mathrm{w}\mathbb{C}\mathrm{c}}(\mathbb{K}_M)$ the full subcategory of $\mathrm{D}^{\mathrm{b}}(\mathbb{K}_M)$ consisting of weakly-$\mathbb{C}$-constructible objects and by $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\mathrm{c}}(\mathbb{K}_M)$ the full subcategory consisting of

$\mathbb{C}$-constructible objects. Also recall ([40]) that $F \in \mathrm{D}^{\mathrm{b}}(\mathbb{K}_M)$ is weakly-$\mathbb{C}$-constructible if and only if its microsupport is a closed $\mathbb{C}^\times$-conic (*i.e.,* invariant by the $\mathbb{C}^\times$-action on T^*M)) complex analytic Lagrangian subset of T^*M or, equivalently, if it is contained in a closed $\mathbb{C}^\times$-conic complex analytic isotropic subset of T^*M.

Proposition 7.1.6. *Let $F \in \mathrm{D}^{\mathrm{b}}(\mathbb{Z}_M[\hbar])$ and assume that F is cohomologically complete. Then*

$$\mathrm{SS}(F) = \mathrm{SS}(\mathrm{gr}_\hbar(F)). \tag{7.1.3}$$

Proof. The inclusion

$$\mathrm{SS}(\mathrm{gr}_\hbar(F)) \subset \mathrm{SS}(F)$$

follows from the distinguished triangle $F \xrightarrow{\hbar} F \to \mathrm{gr}_\hbar(F) \xrightarrow{+1}$. Let us prove the converse inclusion.

Using the definition of the microsupport, it is enough to prove that given two open subsets $U \subset V$ of M, $\mathrm{R}\Gamma(V;F) \to \mathrm{R}\Gamma(U;F)$ is an isomorphism as soon as $\mathrm{R}\Gamma(V;\mathrm{gr}_\hbar(F)) \to \mathrm{R}\Gamma(U;\mathrm{gr}_\hbar(F))$ is an isomorphism. Consider a distinguished triangle $\mathrm{R}\Gamma(V;F) \to \mathrm{R}\Gamma(U;F) \to G \xrightarrow{+1}$. Then we get a distinguished triangle $\mathrm{R}\Gamma(V;\mathrm{gr}_\hbar(F)) \to \mathrm{R}\Gamma(U;\mathrm{gr}_\hbar(F)) \to \mathrm{gr}_\hbar(G) \xrightarrow{+1}$. Therefore, $\mathrm{gr}_\hbar(G) \simeq 0$. On the other hand, G is cohomologically complete, thanks to Proposition 1.5.12 and $G \simeq 0$ by Corollary 1.5.9. Q.E.D.

Proposition 7.1.7. *Let $F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbb{C}^\hbar_X)$. Then F is cohomologically complete.*

Proof. One has

$$\begin{aligned}
\text{“}\varinjlim_{U \ni x}\text{”}\, \mathrm{Ext}^j_{\mathbb{Z}[\hbar]}\big(\mathbb{Z}[\hbar,\hbar^{-1}], H^i(U;F)\big) &\simeq \mathrm{Ext}^j_{\mathbb{Z}[\hbar]}\big(\mathbb{Z}[\hbar,\hbar^{-1}], \text{“}\varinjlim_{U \ni x}\text{”}\, H^i(U;F)\big) \\
&\simeq \mathrm{Ext}^j_{\mathbb{Z}[\hbar]}\big(\mathbb{Z}[\hbar,\hbar^{-1}], F_x\big) \simeq 0
\end{aligned}$$

where the last isomorphism follows from the fact that F_x is cohomologically complete when taking $X = \mathrm{pt}$.

Hence, the hypothesis (i) (c) of Proposition 1.5.6 is satisfied. Q.E.D.

Propagation for solutions of $\mathscr{A}_{\Lambda/X}$-modules

Proposition 7.1.8. *Let $\mathscr{N}$ be a coherent $\mathscr{A}_{\Lambda/X}$-module. Then*

$$\mathrm{SS}(\mathrm{R}\mathscr{H}om_{\mathscr{A}_{\Lambda/X}}(\mathscr{N},\mathscr{L})) \subset \mathrm{char}(\mathrm{gr}_\hbar\mathscr{N}). \tag{7.1.4}$$

Proof. By Lemma 7.1.4, we have

$$\mathrm{R}\mathscr{H}om_{\mathscr{A}_{\Lambda/X}}(\mathscr{N},\mathscr{L}) \simeq \mathrm{R}\mathscr{H}om_{\mathscr{D}_{\mathscr{L}}}(\mathscr{D}_{\mathscr{L}} \otimes_{\mathscr{A}_{\Lambda/X}} \mathscr{N},\mathscr{L}).$$

Since $\mathrm{gr}_\hbar(\mathscr{D}_{\mathscr{L}} \otimes_{\mathscr{A}_{\Lambda/X}} \mathscr{N}) = \mathrm{gr}_\hbar(\mathscr{N})$, Proposition 7.1.8 will follow from Proposition 7.1.9 below, already obtained in [18]. Q.E.D.

Proposition 7.1.9. *Let $\mathscr{N}$ be a coherent $\mathscr{D}_{\mathscr{L}}$-module. Then*

$$\mathrm{SS}(\mathrm{R}\mathscr{H}om_{\mathscr{D}_{\mathscr{L}}}(\mathscr{N},\mathscr{L})) = \mathrm{char}(\mathrm{gr}_\hbar\mathscr{N}). \tag{7.1.5}$$

Proof. Set $F = \mathrm{R}\mathscr{H}om_{\mathscr{D}_{\mathscr{L}}}(\mathscr{N},\mathscr{L})$. Then F is cohomologically complete by Corollary 1.6.2 and $\mathrm{SS}(F) = \mathrm{SS}(\mathrm{gr}_\hbar(F))$ by Proposition 7.1.6. On the other hand, $\mathrm{gr}_\hbar(F) \simeq \mathrm{R}\mathscr{H}om_{\mathscr{D}_\Lambda}(\mathrm{gr}_\hbar\mathscr{N},\mathscr{O}_\Lambda)$ by Proposition 1.4.3 and the microsupport of this complex is equal to $\mathrm{char}(\mathrm{gr}_\hbar\mathscr{N})$ by [40, Th 11.3.3]. Q.E.D.

Constructibility of solutions

Theorem 7.1.10 below has already been obtained in [18] in the framework of $\mathscr{D}_M[[\hbar]]$-modules.

Recall that $\mathscr{L}$ is a coherent $\mathscr{A}_X$-module, simple along Λ.

Theorem 7.1.10. *Let $\mathscr{N}$ be a holonomic $\mathscr{A}_{\Lambda/X}$-module.*

(a) *The objects $\mathrm{R}\mathscr{H}om_{\mathscr{A}_{\Lambda/X}}(\mathscr{N},\mathscr{L})$ and $\mathrm{R}\mathscr{H}om_{\mathscr{A}_{\Lambda/X}}(\mathscr{L},\mathscr{N})$ belong to $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\mathrm{c}}(\mathbb{C}^\hbar_\Lambda)$ and their microsupports are contained in $\mathrm{char}(\mathrm{gr}_\hbar\mathscr{N})$.*

(b) *There is a natural isomorphism in $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\mathrm{c}}(\mathbb{C}^\hbar_\Lambda)$*

$$\mathrm{R}\mathscr{H}om_{\mathscr{A}_{\Lambda/X}}(\mathscr{N},\mathscr{L}) \xrightarrow{\sim} \mathrm{D}'_X\big(\mathrm{R}\mathscr{H}om_{\mathscr{A}_{\Lambda/X}}(\mathscr{L},\mathscr{N})\big)\,[d_X]. \tag{7.1.6}$$

The morphism in (b) is similar to the morphism in Lemma 3.3.1 and is associated with

$$\begin{aligned}&\mathrm{R}\mathscr{H}om_{\mathscr{A}_{\Lambda/X}}(\mathscr{N},\mathscr{L}) \otimes \mathrm{R}\mathscr{H}om_{\mathscr{A}_{\Lambda/X}}(\mathscr{L},\mathscr{N})\\ &\qquad \to \mathrm{R}\mathscr{H}om_{\mathscr{A}_{\Lambda/X}}(\mathscr{L},\mathscr{L}) \to \mathrm{R}\mathscr{H}om_{\mathscr{D}_{\mathscr{L}}}(\mathscr{L},\mathscr{L}) \simeq \mathbb{C}^\hbar_\Lambda \to \mathbb{C}^\hbar_X\,[d_X].\end{aligned}$$

Proof. (a) It is enough to treat $F := \mathrm{R}\mathscr{H}om_{\mathscr{A}_{\Lambda/X}}(\mathscr{N},\mathscr{L})$. In view of Proposition 7.1.8, F is weakly $\mathbb{C}$-constructible and it remains to show that for each $x \in \Lambda$, F_x belongs to $\mathrm{D}^{\mathrm{b}}_f(\mathbb{C}^\hbar)$.

If U is a sufficiently small open ball centered at x, then $\mathrm{R}\Gamma(U;F) \to F_x$ is an isomorphism ([40]). The finiteness of the complex $\mathrm{gr}_\hbar(F_x)$ follows from the classical finiteness theorem for holonomic $\mathscr{D}$-modules of [35]. Since F is cohomologically complete, Proposition 1.5.12 implies that $\mathrm{R}\Gamma(U;F)$ is cohomologically complete. Hence the result follows from Theorem 1.6.4.

(b) follows from Corollary 1.4.6, since we know by [35] that (7.1.6) is an isomorphism after applying the functor $\mathrm{gr}_\hbar$. Q.E.D.

$\mathscr{A}_{\Lambda/X}$ modules and $\mathscr{A}_X^{\mathrm{loc}}$-modules

Definition 7.1.11. A coherent $\mathscr{A}_{\Lambda/X}$-submodule $\mathscr{N}$ of a coherent $\mathscr{A}_X^{\mathrm{loc}}$-module $\mathscr{M}$ is called an $\mathscr{A}_{\Lambda/X}$-lattice of $\mathscr{M}$ if $\mathscr{N}$ generates $\mathscr{M}$ as an $\mathscr{A}_X^{\mathrm{loc}}$-module.

Lemma 7.1.12. *Let $\mathscr{M}$ be a coherent $\mathscr{A}_X^{\mathrm{loc}}$-module and let $\mathscr{N}\subset\mathscr{M}$ be an $\mathscr{A}_{\Lambda/X}$-lattice of $\mathscr{M}$. Then $\mathrm{char}\big(\mathrm{gr}_\hbar(\mathscr{N})\big)\subset T^*\Lambda$ does not depend on the choice of $\mathscr{N}$.*

The proof is similar to the one of Lemma 3.4.2, and we shall not repeat it.

Definition 7.1.13. Let $\mathscr{M}$ be a coherent $\mathscr{A}_X^{\mathrm{loc}}$-module and let $\mathscr{N}\subset\mathscr{M}$ be an $\mathscr{A}_{\Lambda/X}$-lattice of $\mathscr{M}$. We set

$$\mathrm{char}_\Lambda(\mathscr{M}):=\mathrm{char}(\mathrm{gr}_\hbar\mathscr{N}).$$

Example 7.1.14. Let $X=\mathbb{C}^2$ endowed with the symplectic coordinates $(x;u)$ and let Λ be the Lagrangian manifold given by the equation $\{u=0\}$. In this case, $\mathscr{A}_{\Lambda/X}=\mathscr{A}_X[u\hbar^{-1}]$.

Now let $\alpha\in\mathbb{C}$ and consider the module $\mathscr{M}=\mathscr{A}_X^{\mathrm{loc}}/\mathscr{A}_X^{\mathrm{loc}}(xu-\alpha\hbar)$ and $\mathscr{N}=\mathscr{A}_{\Lambda/X}/\mathscr{A}_{\Lambda/X}(xu\hbar^{-1}-\alpha)$. Then $\mathscr{N}$ is an $\mathscr{A}_{\Lambda/X}$-lattice of $\mathscr{M}$ and $\mathrm{gr}_\hbar\mathscr{N}\simeq\mathscr{D}_\Lambda/\mathscr{D}_\Lambda(x\partial_x-\alpha)$.

Lemma 7.1.15. *Let $\mathscr{M}$ be a coherent $\mathscr{A}_X^{\mathrm{loc}}$-module.*

(i) $\mathrm{char}_\Lambda(\mathscr{M})$ *is a closed conic complex analytic subset of $T^*\Lambda$ and this set is involutive.*

(ii) *Let $0\to\mathscr{M}'\to\mathscr{M}\to\mathscr{M}''\to 0$ be an exact sequence of $\mathscr{A}_X^{\mathrm{loc}}$-modules. Then $\mathrm{char}_\Lambda(\mathscr{M})=\mathrm{char}_\Lambda(\mathscr{M}')\cup\mathrm{char}_\Lambda(\mathscr{M}'')$.*

Proof. (i) is a well-known result of $\mathscr{D}$-module theory, see [38].
(ii) Let $\mathscr{N}$ be an $\mathscr{A}_{\Lambda/X}$-lattice of $\mathscr{M}$. Set $\mathscr{N}'=\mathscr{M}'\cap\mathscr{N}$ and $\mathscr{N}''\subset\mathscr{M}''$ be the image of $\mathscr{N}$. Then $\mathscr{N}'$ and $\mathscr{N}''$ are $\mathscr{A}_{\Lambda/X}$-lattices of $\mathscr{M}'$ and $\mathscr{M}''$, respectively. Since we have an exact sequence

$$0\to\mathscr{N}'/\hbar\mathscr{N}'\to\mathscr{N}/\hbar\mathscr{N}\to\mathscr{N}''/\hbar\mathscr{N}''\to 0,$$

we have $\mathrm{char}_\Lambda(\mathscr{M}) = \mathrm{char}(\mathscr{N}/\hbar\mathscr{N}) = \mathrm{char}(\mathscr{N}'/\hbar\mathscr{N}') \cup \mathrm{char}(\mathscr{N}''/\hbar\mathscr{N}'') = \mathrm{char}_\Lambda(\mathscr{M}') \cup \mathrm{char}_\Lambda(\mathscr{M}'')$. Q.E.D.

Proposition 7.1.16. *For a coherent $\mathscr{A}_X^{\mathrm{loc}}$-module $\mathscr{M}$, we have*

$$\operatorname{codim} \mathrm{char}_\Lambda(\mathscr{M}) \geq \operatorname{codim} \mathrm{Supp}(\mathscr{M}).$$

Proof. In the course of the proof, we shall have to consider the analogue of the algebra $\mathscr{A}_{\Lambda/X}$ but with $\mathscr{A}_{X^a}$ instead of $\mathscr{A}_X$. We shall denote by $\mathscr{A}_{\Lambda^a}$ this algebra. We shall show that $\operatorname{codim} \mathrm{Supp}(\mathscr{M}) \geq r$ implies $\operatorname{codim} \mathrm{char}_\Lambda(\mathscr{M}) \geq r$ by descending induction on r. Applying Proposition 2.3.15 (a), we have $\mathrm{R}\mathscr{H}om_{\mathscr{A}_X^{\mathrm{loc}}}(\mathscr{M}, \mathscr{A}_X^{\mathrm{loc}}) \simeq \tau^{\geq r}\mathrm{R}\mathscr{H}om_{\mathscr{A}_X^{\mathrm{loc}}}(\mathscr{M}, \mathscr{A}_X^{\mathrm{loc}})$, where $\tau^{\geq r}$ is the truncation functor. Hence we have a distinguished triangle in $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_{X^a}^{\mathrm{loc}})$:

$$\mathscr{E}xt^r_{\mathscr{A}_X^{\mathrm{loc}}}(\mathscr{M}, \mathscr{A}_X^{\mathrm{loc}})[-r] \to \mathrm{R}\mathscr{H}om_{\mathscr{A}_X^{\mathrm{loc}}}(\mathscr{M}, \mathscr{A}_X^{\mathrm{loc}}) \to \mathscr{K} \xrightarrow{+1}, \tag{7.1.7}$$

where $\mathscr{K} = \tau^{>r}\mathrm{R}\mathscr{H}om_{\mathscr{A}_X^{\mathrm{loc}}}(\mathscr{M}, \mathscr{A}_X^{\mathrm{loc}})$. Note that $\operatorname{codim}(\mathrm{Supp}(\mathscr{K})) > r$ by Proposition 2.3.15 (b). Setting $\mathscr{M}' = \mathscr{E}xt^r_{\mathscr{A}_X^{\mathrm{loc}}}(\mathscr{M}, \mathscr{A}_X^{\mathrm{loc}})$, the distinguished triangle (7.1.7) induces a distinguished triangle in $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X^{\mathrm{loc}})$:

$$\mathrm{R}\mathscr{H}om_{\mathscr{A}_{X^a}^{\mathrm{loc}}}(\mathscr{K}, \mathscr{A}_{X^a}^{\mathrm{loc}}) \to \mathscr{M} \to \mathrm{R}\mathscr{H}om_{\mathscr{A}_{X^a}^{\mathrm{loc}}}(\mathscr{M}', \mathscr{A}_{X^a}^{\mathrm{loc}})[r] \xrightarrow{+1} .$$

Setting $\mathscr{M}_1 = \mathscr{E}xt^r_{\mathscr{A}_{X^a}^{\mathrm{loc}}}(\mathscr{M}', \mathscr{A}_{X^a}^{\mathrm{loc}})$, we obtain a morphism $\varphi\colon \mathscr{M} \to \mathscr{M}_1$ and $\mathrm{Ker}(\varphi)$ has codimension greater than r. Hence, $\operatorname{codim} \mathrm{char}_\Lambda(\mathrm{Ker}(\varphi)) > r$ by the induction hypothesis. Since $\mathrm{char}_\Lambda(\mathscr{M}) \subset \mathrm{char}_\Lambda(\mathscr{M}_1) \cup \mathrm{char}_\Lambda(\mathrm{Ker}(\varphi))$, it is enough to show that $\operatorname{codim} \mathrm{char}_\Lambda(\mathscr{M}_1) \geq r$.

Hence we may assume from the beginning that $\mathscr{M} = \mathscr{E}xt^r_{\mathscr{A}_{X^a}^{\mathrm{loc}}}(\mathscr{M}', \mathscr{A}_{X^a}^{\mathrm{loc}})$ for a coherent $\mathscr{A}_{X^a}^{\mathrm{loc}}$-module $\mathscr{M}'$. Let us take an $\mathscr{A}_{\Lambda^a}$-lattice $\mathscr{N}'$ of $\mathscr{M}'$. Set $\mathscr{N}_0 = \mathscr{E}xt^r_{\mathscr{A}_{\Lambda^a}}(\mathscr{N}', \mathscr{A}_{\Lambda^a})$. Then we have $\mathscr{N}_0^{\mathrm{loc}} \simeq \mathscr{M}$, and it induces a morphism $\mathscr{N}_0 \to \mathscr{M}$. Let $\mathscr{N}$ be the image of the morphism $\mathscr{N}_0 \to \mathscr{M}$. Then $\mathscr{N}$ is an $\mathscr{A}_{\Lambda/X}$-lattice of $\mathscr{M}$. Hence we have $\mathrm{char}_\Lambda(\mathscr{M}) = \mathrm{char}(\mathscr{N}/\hbar\mathscr{N})$, which implies

$$\mathrm{char}_\Lambda(\mathscr{M}) \subset \mathrm{char}(\mathscr{N}_0/\hbar\mathscr{N}_0). \tag{7.1.8}$$

On the other hand, we have an exact sequence

$$\mathscr{E}xt^r_{\mathscr{A}_{\Lambda^a}}(\mathscr{N}', \mathscr{A}_{\Lambda^a}) \xrightarrow{\hbar} \mathscr{E}xt^r_{\mathscr{A}_{\Lambda^a}}(\mathscr{N}', \mathscr{A}_{\Lambda^a}) \to \mathscr{E}xt^r_{\mathscr{A}_{\Lambda^a}}(\mathscr{N}', \mathrm{gr}_\hbar(\mathscr{A}_{\Lambda^a})).$$

Since we have $\mathscr{E}xt^r_{\mathscr{A}_{\Lambda^a}}(\mathscr{N}', \mathrm{gr}_\hbar(\mathscr{A}_{\Lambda^a})) \simeq \mathscr{E}xt^r_{\mathrm{gr}_\hbar(\mathscr{A}_{\Lambda^a})}(\mathrm{gr}_\hbar\mathscr{N}', \mathrm{gr}_\hbar(\mathscr{A}_{\Lambda^a}))$, we have a monomorphism

$$\mathscr{N}_0/\hbar\mathscr{N}_0 \rightarrowtail \mathscr{E}xt^r_{\mathrm{gr}_\hbar(\mathscr{A}_{\Lambda^a})}(\mathrm{gr}_\hbar\mathscr{N}', \mathrm{gr}_\hbar(\mathscr{A}_{\Lambda^a})).$$

Hence we obtain $\mathrm{char}(\mathscr{N}_0/\hbar\mathscr{N}_0) \subset \mathrm{char}\Big(\mathscr{E}xt^r_{\mathrm{gr}_\hbar(\mathscr{A}_{\Lambda^a})}(\mathrm{gr}_\hbar\mathscr{N}', \mathrm{gr}_\hbar(\mathscr{A}_{\Lambda^a}))\Big)$. Since $\mathrm{char}\Big(\mathscr{E}xt^r_{\mathrm{gr}_\hbar(\mathscr{A}_{\Lambda^a})}(\mathrm{gr}_\hbar\mathscr{N}', \mathrm{gr}_\hbar(\mathscr{A}_{\Lambda^a}))\Big)$ has codimension $\geq r$ by *e.g.,* [38, Theorem 2.19], we conclude that $\mathrm{codim}\,\mathrm{char}(\mathscr{N}_0/\hbar\mathscr{N}_0) \geq r$. By (7.1.8), we obtain $\mathrm{codim}\,\mathrm{char}_\Lambda(\mathscr{M}) \geq r$. Q.E.D.

7.2 Holonomic DQ-modules

In a complex symplectic manifold X, an isotropic subvariety Λ is a locally closed complex analytic subvariety such that Λ_{reg} is isotropic, *i.e.,* the 2-form defining the symplectic structure vanishes on Λ_{reg}. Here, Λ_{reg} denotes the smooth part of Λ.

A Lagrangian subvariety Λ is an isotropic subvariety of pure dimension $d_X/2$. Equivalently, Λ is a subvariety of pure dimension $d_X/2$ such that Λ_{reg} is involutive.

Definition 7.2.1. (a) An $\mathscr{A}_X^{\mathrm{loc}}$-module $\mathscr{M}$ is holonomic if it is coherent and its support is a Lagrangian subvariety of X.

(b) An $\mathscr{A}_X$-module $\mathscr{N}$ is holonomic if it is coherent, without $\hbar$-torsion and $\mathscr{N}^{\mathrm{loc}}$ is a holonomic $\mathscr{A}_X^{\mathrm{loc}}$-module.

(c) Let Λ be a smooth Lagrangian submanifold of X. We say that an $\mathscr{A}_X^{\mathrm{loc}}$-module $\mathscr{M}$ is simple holonomic along Λ if there exists locally an $\mathscr{A}_X$-module $\mathscr{M}_0$ simple along Λ such that $\mathscr{M} \simeq \mathscr{M}_0^{\mathrm{loc}}$.

Lemma 7.2.2. *Let $\mathscr{M}$ be a holonomic $\mathscr{A}_X^{\mathrm{loc}}$-module. Then $\mathrm{D}'_{\mathscr{A}^{\mathrm{loc}}}\mathscr{M}\,[d_X/2]$ is concentrated in degree 0 and is holonomic.*

Proof. This follows from Proposition 2.3.15 and the involutivity theorem (Proposition 2.3.18). Q.E.D.

Let X be a complex symplectic manifold and let $\mathscr{M}$ and $\mathscr{L}$ be two holonomic $\mathscr{A}_X^{\mathrm{loc}}$-modules. Using Lemma 2.4.10 (more precisely, an $\mathscr{A}_X^{\mathrm{loc}}$-variant of this lemma) and Theorem 6.2.4, we have

$$
\begin{aligned}
\mathrm{R}\mathscr{H}om_{\mathscr{A}_X^{\mathrm{loc}}}(\mathscr{M},\mathscr{L}) &\simeq \mathrm{R}\mathscr{H}om_{\mathscr{A}^{\mathrm{loc}}_{X\times X^a}}(\mathscr{M}\overset{\mathrm{L}}{\boxtimes}\mathrm{D}'_{\mathscr{A}}\mathscr{L}, \mathscr{C}_X^{\mathrm{loc}}),\\
\mathrm{R}\mathscr{H}om_{\mathscr{A}_X^{\mathrm{loc}}}(\mathscr{L},\mathscr{M}) &\simeq \mathrm{R}\mathscr{H}om_{\mathscr{A}^{\mathrm{loc}}_{X\times X^a}}(\mathscr{L}\overset{\mathrm{L}}{\boxtimes}\mathrm{D}'_{\mathscr{A}}\mathscr{M}, \mathscr{C}_X^{\mathrm{loc}})\\
&\simeq \mathrm{R}\mathscr{H}om_{\mathscr{A}^{\mathrm{loc}}_{X\times X^a}}(\mathrm{D}'_{\mathscr{A}}(\mathscr{C}_X^{\mathrm{loc}}), \mathscr{M}\overset{\mathrm{L}}{\boxtimes}\mathrm{D}'_{\mathscr{A}}\mathscr{L})\\
&\simeq \mathrm{R}\mathscr{H}om_{\mathscr{A}^{\mathrm{loc}}_{X\times X^a}}(\mathscr{C}_X^{\mathrm{loc}}, \mathscr{M}\overset{\mathrm{L}}{\boxtimes}\mathrm{D}'_{\mathscr{A}}\mathscr{L})[d_X].
\end{aligned}
\tag{7.2.1}
$$

Theorem 7.2.3. *Let X be a complex symplectic manifold and let $\mathscr{M}$ and $\mathscr{L}$ be two holonomic $\mathscr{A}_X^{\mathrm{loc}}$-modules. Then*

(i) *the object $\mathrm{R}\mathscr{H}om_{\mathscr{A}_X^{\mathrm{loc}}}(\mathscr{M},\mathscr{L})$ belongs to $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\mathrm{c}}(\mathbb{C}_X^{\hbar,\mathrm{loc}})$,*

(ii) *there is a canonical isomorphism:*

$$\mathrm{R}\mathscr{H}om_{\mathscr{A}_X^{\mathrm{loc}}}(\mathscr{M},\mathscr{L}) \xrightarrow{\sim} \big(\mathrm{D}'_X\mathrm{R}\mathscr{H}om_{\mathscr{A}_X^{\mathrm{loc}}}(\mathscr{L},\mathscr{M})\big)\,[d_X]. \tag{7.2.2}$$

(iii) *the object $\mathrm{R}\mathscr{H}om_{\mathscr{A}_X^{\mathrm{loc}}}(\mathscr{M},\mathscr{L})[d_X/2]$ is perverse.*

Proof. Using (7.2.1), we may assume from the beginning that $\mathscr{L}$ is a simple holonomic $\mathscr{A}_X^{\mathrm{loc}}$-module supported on a smooth Lagrangian submanifold Λ of X. Let $\mathscr{L}_0$ be an $\mathscr{A}_X$-module simple along Λ such that $\mathscr{L}\simeq\mathscr{L}_0^{\mathrm{loc}}$.
(i)-(ii) Let $\mathscr{N}$ be an $\mathscr{A}_{\Lambda/X}$-lattice of $\mathscr{M}$. By Lemma 7.1.3 (v), we have

$$\mathrm{R}\mathscr{H}om_{\mathscr{A}_X^{\mathrm{loc}}}(\mathscr{M},\mathscr{L}) \;\simeq\; \mathrm{R}\mathscr{H}om_{\mathscr{A}_{\Lambda/X}}(\mathscr{N},\mathscr{L}_0)^{\mathrm{loc}}.$$

Then the results follow from Proposition 7.1.16 and Theorem 7.1.10.
(iii) Since the problem is local, we may assume that $X=T^*M$, $\mathscr{A}_X^{\mathrm{loc}}=\widehat{\mathscr{W}}_X$ and $\mathscr{L}_0=\mathscr{O}_M^{\hbar}$.

By (ii), it is enough to check the statement:

$$H^j\Big(\mathrm{R}\Gamma_N\big(\mathrm{R}\mathscr{H}om_{\mathscr{A}_{\Lambda/X}}(\mathscr{N},\mathscr{L}_0)\big)\Big) \text{ vanishes for } j<l \text{ and for any closed smooth submanifold } N \text{ of } M \text{ of codimension } l. \tag{7.2.3}$$

Since $F:=\mathrm{R}\Gamma_N\big(\mathrm{R}\mathscr{H}om_{\mathscr{A}_{\Lambda/X}}(\mathscr{N},\mathscr{L}_0)\big)$ is $\mathbb{C}$-constructible, it is enough to show that $H^j(\mathrm{gr}_\hbar(F))=0$ for $j<l$. This follows from the well-known fact that $H^j(\mathrm{R}\Gamma_N(\mathscr{O}_M))=0$ for $j<l$. Q.E.D.

Assume for simplicity that X is open in some cotangent bundle T^*M. We shall compare the sheaf of solutions of holonomic $\widehat{\mathscr{E}}_X$-modules and $\widehat{\mathscr{W}}_X$-modules. Recall that $\widehat{\mathscr{W}}_X$ is faithfully flat over $\widehat{\mathscr{E}}_X$ by Lemma 6.1.2.

Corollary 7.2.4. *Let $\mathscr{M}$ and $\mathscr{L}$ be two holonomic $\widehat{\mathscr{E}}_X$-modules. Then the object $\mathrm{R}\mathscr{H}om_{\widehat{\mathscr{E}}_X}(\mathscr{M},\mathscr{L})$ belongs to $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\mathrm{c}}(\mathbb{C}_X)$.*

Proof. Let t denote the coordinate on the complex line $\mathbb{C}$, let E denote the ring $\widehat{\mathscr{E}}_{T^*\mathbb{C}}|_{t=0,\tau=1}$ and let L be the E-module $E/E\cdot t$. Then we have the embedding

$$\mathbb{C}^{\hbar,\mathrm{loc}} \hookrightarrow E,\quad \hbar\mapsto\partial_t^{-1}.$$

Set for short $F := \mathrm{R}\mathscr{H}om_{\widehat{\mathscr{E}}_X}(\mathscr{M}, \mathscr{L})$. Then

$$\begin{aligned} F &\simeq \mathrm{R}\mathscr{H}om_E(L, \mathrm{R}\mathscr{H}om_{\widehat{\mathscr{E}}_X}(\mathscr{M}, (\widehat{\mathscr{E}}_{X\times T^*\mathbb{C}}/\widehat{\mathscr{E}}_{X\times T^*\mathbb{C}}\cdot t)|_{t=0,\tau=1} \overset{\mathrm{L}}{\otimes}_{\widehat{\mathscr{E}}_X} \mathscr{L})) \\ &\simeq \mathrm{R}\mathscr{H}om_E(L, \mathrm{R}\mathscr{H}om_{\widehat{\mathscr{W}}_X}(\widehat{\mathscr{W}}_X \otimes_{\widehat{\mathscr{E}}_X} \mathscr{M}, \widehat{\mathscr{W}}_X \otimes_{\widehat{\mathscr{E}}_X} \mathscr{L})). \end{aligned}$$

Set $G := \mathrm{R}\mathscr{H}om_{\widehat{\mathscr{W}}_X}(\widehat{\mathscr{W}}_X \otimes_{\widehat{\mathscr{E}}_X} \mathscr{M}, \widehat{\mathscr{W}}_X \otimes_{\widehat{\mathscr{E}}_X} \mathscr{L})$. Applying Theorem 7.2.3, we find that $G \in \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\mathrm{c}}(\mathbb{C}_X^{\hbar,\mathrm{loc}})$ and it follows that $F \in \mathrm{D}^{\mathrm{b}}_{\mathrm{w}\mathbb{C}\mathrm{c}}(\mathbb{C}_X)$.

Moreover, for each $x \in X$, G_x is of finite type over $\mathbb{C}^{\hbar,\mathrm{loc}}$ and is an E-module. One easily deduces that $F_x \simeq \mathrm{RHom}_E(L, G_x)$ is a $\mathbb{C}$-vector space of finite dimension. Q.E.D.

7.3 Lagrangian cycles

Given two holonomic $\mathscr{A}_X^{\mathrm{loc}}$ modules $\mathscr{M}$ and $\mathscr{L}$ such that $\mathrm{Supp}(\mathscr{M}) \cap \mathrm{Supp}(\mathscr{L})$ is compact, the Euler-Poincaré index is given by

$$\begin{aligned} \chi(X;\mathscr{M},\mathscr{L}) &= \chi(\mathrm{RHom}_{\mathscr{A}_X^{\mathrm{loc}}}(\mathscr{M},\mathscr{L})) \\ &= \textstyle\sum_i (-)^i \dim \mathrm{Ext}^i_{\mathscr{A}_X^{\mathrm{loc}}}(\mathscr{M},\mathscr{L}). \end{aligned} \tag{7.3.1}$$

Applying (6.3.6), we get

$$\chi(X;\mathscr{M},\mathscr{L}) = \int_X (\mathrm{eu}_X(\mathscr{M}) \cdot \mathrm{eu}_X(\mathscr{L})). \tag{7.3.2}$$

Recall that $\mathrm{eu}_X(\mathscr{M}) = (-1)^{d_X/2}\mathrm{eu}_X(\mathrm{D}'_{\mathscr{A}^{\mathrm{loc}}}\mathscr{M})$, and also recall that d_X being even, $\mathrm{eu}_X(\mathscr{M}) \cdot \mathrm{eu}_X(\mathscr{L}) = \mathrm{eu}_X(\mathscr{L}) \cdot \mathrm{eu}_X(\mathscr{M})$.

We shall explain how to calculate the Euler classes by using the theory of Lagrangian cycles. We refer to [40, Ch. 9 § 3] for a detailed study of these cycles.

Recall that $\mathbb{K}$ denotes a commutative Noetherian unital ring of finite global dimension.

Consider a closed Lagrangian subvariety Λ of X. We define the sheaf:

$$\mathrm{L}_\Lambda^{\mathbb{K}} := H^{d_X}_\Lambda(\mathbb{K}_X), \tag{7.3.3}$$

and we simply write L_Λ instead of $\mathrm{L}_\Lambda^{\mathbb{Z}}$. The next results are obvious and well-known (see loc. cit.).

Lemma 7.3.1. (i) *$U \mapsto H^{d_X}_{\Lambda\cap U}(U;\mathbb{K}_X)$ (U open in X) is a sheaf and this sheaf coincides with $\mathrm{L}_\Lambda^{\mathbb{K}}$,*

(ii) $H^i_{\Lambda\setminus\Lambda_{\mathrm{reg}}}(\mathrm{L}^{\mathbb{K}}_\Lambda) \simeq 0$ *for* $i = 0, 1$,

(iii) *if* s *is a section of* $\mathrm{L}^{\mathbb{K}}_\Lambda$, *then its support is open and closed in* Λ,

(iv) *there is a canonical section in* $\Gamma(\Lambda; \mathrm{L}_\Lambda)$ *which gives an isomorphism* $\mathrm{L}_\Lambda|_{\Lambda_{\mathrm{reg}}} \xrightarrow{\sim} \mathbb{Z}_{\Lambda_{\mathrm{reg}}}$.

We denote by $[\Lambda]$ the section given in (iv) above.

Definition 7.3.2. We call a section of $\mathrm{L}^{\mathbb{K}}_\Lambda$ on an open set U of Λ a Lagrangian cycle on U.

Recall that $\mathrm{K}_{\mathrm{coh},\Lambda}(\mathscr{O}_X)$ denotes the Grothendieck group of the category $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh},\Lambda}(\mathscr{O}_X)$. We denote by $\mathscr{K}_{\mathrm{coh},\Lambda}(\mathscr{O}_X)$ the sheaf associated with the presheaf $U \mapsto \mathrm{K}_{\mathrm{coh},\Lambda\cap U}(\mathscr{O}_U)$. Then, there is a well defined $\mathbb{Z}$-linear map

$$\kappa \quad : \quad \mathscr{K}_{\mathrm{coh},\Lambda}(\mathscr{O}_X) \to \mathrm{L}_\Lambda. \tag{7.3.4}$$

This map is characterized by the property that

$$\kappa(\mathscr{O}_\Lambda) = [\Lambda] \in \Gamma(\Lambda; \mathrm{L}_\Lambda). \tag{7.3.5}$$

Let $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{A}^{\mathrm{loc}}_X)$ and let Λ be a closed Lagrangian subvariety of X which contains $\mathrm{Supp}(\mathscr{M})$.

Let $\mathscr{M}_0$ be an $\mathscr{A}_X$-lattice of $\mathscr{M}$ on an open set U of X. Then $\mathrm{gr}_\hbar(\mathscr{M}_0)$ defines an element $[\mathrm{gr}_\hbar(\mathscr{M}_0)] \in \mathrm{K}_{\mathrm{coh},\Lambda}(\mathscr{O}_X|_U)$, hence an element of $\Gamma(U; \mathscr{K}_{\mathrm{coh},\Lambda}(\mathscr{O}_X))$. This element depends only on $\mathscr{M}$, and we thus have a morphism

$$\mathrm{K}_{\mathrm{coh},\Lambda}(\mathscr{A}^{\mathrm{loc}}_X) \to \Gamma(\Lambda; \mathscr{K}_{\mathrm{coh},\Lambda}(\mathscr{O}_X)).$$

Composing with the map κ, we obtain a map

$$\mathrm{K}_{\mathrm{coh},\Lambda}(\mathscr{A}^{\mathrm{loc}}_X) \to \Gamma(\Lambda; \mathrm{L}_\Lambda). \tag{7.3.6}$$

Definition 7.3.3. We denote by $\mathrm{lc}_X(\mathscr{M})$ the image of $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh},\Lambda}(\mathscr{A}^{\mathrm{loc}}_X)$ by the morphism in (7.3.6) and call it the Lagrangian cycle of $\mathscr{M}$.

On the other-hand, recall (see Definition 6.3.2) that the Euler class $\mathrm{eu}_X(\mathscr{M})$ of $\mathscr{M}$ belongs to $H^{d_X}_\Lambda(X; \mathbb{C}^{\hbar,\mathrm{loc}}_X)$. Hence, the Euler class of $\mathscr{M}$ is a Lagrangian cycle supported by Λ:

$$\mathrm{eu}_X(\mathscr{M}) \in \Gamma(\Lambda; \mathrm{L}^{\mathbb{C}^{\hbar,\mathrm{loc}}}_\Lambda). \tag{7.3.7}$$

The map $\mathbb{Z} \to \mathbb{C}^{\hbar,\mathrm{loc}}$ induces the morphism

$$\iota_X \colon \mathrm{L}_\Lambda \to \mathrm{L}^{\mathbb{C}^{\hbar,\mathrm{loc}}}_\Lambda. \tag{7.3.8}$$

The next lemma is easily checked.

Lemma 7.3.4. *Let Λ be a smooth Lagrangian submanifold of X and let $\mathscr{L}$ be a coherent $\mathscr{A}_X^{\mathrm{loc}}$-module, simple along Λ. Then* $\mathrm{eu}_X(\mathscr{L}) = \iota_X([\Lambda])$.

Theorem 7.3.5. *One has* $\mathrm{eu}_X(\mathscr{M}) = \iota_X \circ \mathrm{lc}_X(\mathscr{M})$.

Proof. By Lemma 7.3.1, it is enough to prove the result at the generic point of Λ. Hence, we may assume that Λ is smooth. Let $x \in \Lambda$ and let us choose a smooth Lagrangian submanifold S_x of X which intersects Λ transversally at the single point x. Let us also choose a simple $\mathscr{A}_X^{\mathrm{loc}}$-module $\mathscr{L}$ simple along S_x. Using (7.3.2), we find

$$\chi(\mathrm{R}\mathscr{H}om_{\mathscr{A}_X^{\mathrm{loc}}}(\mathscr{L}, \mathscr{M})_x) = \int_X (\mathrm{eu}_X(\mathscr{L}) \cdot \mathrm{eu}_X(\mathscr{M})).$$

Let $\mathscr{L}_0$ and $\mathscr{M}_0$ be $\mathscr{A}_X$-lattices of $\mathscr{L}$ and $\mathscr{M}$, respectively. We also have

$$\begin{aligned} \chi(\mathrm{R}\mathscr{H}om_{\mathscr{A}_X^{\mathrm{loc}}}(\mathscr{L}, \mathscr{M})_x) &= \chi(\mathrm{R}\mathscr{H}om_{\mathrm{gr}_\hbar(\mathscr{A}_X)}(\mathrm{gr}_\hbar(\mathscr{L}_0), \mathrm{gr}_\hbar(\mathscr{M}_0))_x) \\ &= \int_X (\kappa([\mathrm{gr}_\hbar(\mathscr{L}_0)]) \cdot \kappa([\mathrm{gr}_\hbar(\mathscr{M}_0)])). \end{aligned}$$

Clearly, we have

$$\kappa([\mathrm{gr}_\hbar(\mathscr{L}_0)]) = [S_x]. \tag{7.3.9}$$

By Lemma 7.3.4, $\mathrm{eu}(\mathscr{L}_0) = [S_x]$. Therefore,

$$\int_X ([S_x] \cdot \mathrm{eu}_X(\mathscr{M})) = \int_X ([S_x] \cdot \mathrm{lc}_X(\mathscr{M})) \tag{7.3.10}$$

for any smooth Lagrangian submanifold S_x which intersects Λ transversally at x. This completes the proof. Q.E.D.

Remark 7.3.6. The Euler class of a holonomic $\mathscr{A}_X^{\mathrm{loc}}$-module supported by a Lagrangian variety Λ is easy to calculate, since it is enough to calculate it at generic points of Λ. Moreover, the integral in (7.3.2) is invariant by smooth (real) homotopy of the Lagrangian cycles $\mathrm{lc}_X(\mathscr{M})$ and $\mathrm{lc}_X(\mathscr{L})$ and one may deform them in order that they intersect transversally at the smooth part of their support. See [40, Ch. 9,§ 3] for a detailed study.

7.4 Simple holonomic modules

When $\mathscr{L}_0$ and $\mathscr{L}_1$ are simple along smooth Lagrangian manifolds, one can give an estimate on the microsupport of $\mathrm{R}\mathscr{H}om_{\mathscr{A}_X^{\mathrm{loc}}}(\mathscr{L}_1, \mathscr{L}_0)$. It follows from Lemma 6.2.1 that two simple holonomic modules along Λ are locally isomorphic.

Example 7.4.1. Assume $X = T^*M$ for a complex manifold M and $\mathscr{A}_X = \widehat{\mathscr{W}}_X(0)$. Then $\mathscr{O}_M^{\hbar,\mathrm{loc}}$ is a simple holonomic $\mathscr{A}_X^{\mathrm{loc}}$-module along M.

Recall that on a complex symplectic manifold X, the symplectic form gives the Hamiltonian isomorphism from the cotangent bundle to the tangent bundle:

$$H\colon T^*X \xrightarrow{\sim} TX, \quad \langle \theta, v\rangle = \omega(v, H(\theta)), \quad v \in TX,\, \theta \in T^*X. \tag{7.4.1}$$

For a smooth Lagrangian submanifold Λ of X the isomorphism (7.4.1) induces an isomorphism between the normal bundle to Λ in X and its cotangent bundle $T^*\Lambda$.

For the notion of normal cone, see *e.g.,* [40, Def. 4.1.1]. The next result is proved in [43, Prop. 7.1].

Proposition 7.4.2. *Let X be a complex symplectic manifold and let Λ_0 and Λ_1 be two closed complex analytic isotropic subvarieties of X. Then, after identifying TX and T^*X by* (7.4.1)*, the normal cone* $\mathrm{C}(\Lambda_0, \Lambda_1)$ *is a complex analytic $\mathbb{C}^\times$-conic isotropic subvariety of T^*X.*

Theorem 7.4.3. *Let $\mathscr{L}_i$ be a simple holonomic $\mathscr{A}_X^{\mathrm{loc}}$-module along a smooth Lagrangian manifold Λ_i $(i = 0, 1)$. Then*

$$\mathrm{SS}\bigl(\mathrm{R}\mathscr{H}om_{\mathscr{A}_X^{\mathrm{loc}}}(\mathscr{L}_1, \mathscr{L}_0)\bigr) \subset \mathrm{C}(\Lambda_0, \Lambda_1). \tag{7.4.2}$$

Idea of the proof of Theorem 7.4.3. (i) By identifying $\mathrm{R}\mathscr{H}om_{\mathscr{A}_X^{\mathrm{loc}}}(\mathscr{L}_1, \mathscr{L}_0)$ with a sheaf supported by Λ_0, the estimate (7.4.2) is equivalent to the estimate

$$\mathrm{SS}(\mathrm{R}\mathscr{H}om_{\mathscr{A}_X^{\mathrm{loc}}}(\mathscr{L}_1, \mathscr{L}_0)) \subset \mathrm{C}_{\Lambda_0}(\Lambda_1). \tag{7.4.3}$$

(ii) The problem being local, we may assume $X = T^*M$, $\mathscr{A}_X = \widehat{\mathscr{W}}_X(0)$, $\Lambda_0 = M$, $\mathscr{L}_0 = \mathscr{O}_M^{\hbar,\mathrm{loc}}$. If $\Lambda_1 = \Lambda_0$, Theorem 7.4.3 is immediate. Hence, we assume $\Lambda_0 \neq \Lambda_1$.

Then there exists a non constant holomorphic function $\varphi\colon M \to \mathbb{C}$ such that

$$\Lambda_1 = \{(x; u) \in X\,;\, u = \operatorname{grad} \varphi(x)\}$$

Consider the ideal

$$\mathscr{I}_W = \sum_{i=1}^{n} \widehat{\mathscr{W}}_X \cdot (\hbar\partial_{x_i} - \varphi'_i). \tag{7.4.4}$$

We may assume that $\mathscr{L}_1 = \widehat{\mathscr{W}_X}/\mathscr{I}_W$. Let $u \in \mathscr{L}_1$ be the image of $1 \in \widehat{\mathscr{W}_X}$ and denote by $\mathscr{N}$ the $\mathscr{A}_{\Lambda_0/X}$-submodule of $\mathscr{L}_1$ generated by u.

To conclude, it remains to prove the inclusion

$$\mathrm{char}(\mathrm{gr}_\hbar(\mathscr{N})) \subset \mathrm{C}(\Lambda_1, \mathrm{T}^*_{\mathrm{M}}\mathrm{M}). \tag{7.4.5}$$

We shall not give the proof of (7.4.5) here and refer to [43]. Let us simply mention that the proof uses [38, Th. 6.8]. Q.E.D.

Remark 7.4.4. Consider a smooth Lagrangian submanifold Λ of X and denote by $\mathrm{ch}(\Omega_\Lambda) \in H^1(\Lambda; \mathscr{O}_\Lambda^\times)$ the class corresponding to the line bundle Ω_Λ. To the exact sequence

$$1 \to \mathbb{C}_\Lambda^\times \to \mathscr{O}_\Lambda^\times \xrightarrow{d\log} d\mathscr{O}_\Lambda \to 0$$

one associates the maps β and γ:

$$H^1(\Lambda; \mathscr{O}_\Lambda^\times) \xrightarrow{\beta} H^1(\Lambda; d\mathscr{O}_\Lambda) \xrightarrow{\gamma} H^2(\Lambda; \mathbb{C}_\Lambda^\times).$$

We shall denote by $\mathbb{C}_\Lambda^{1/2}$ the invertible $\mathbb{C}_\Lambda$-algebroid associated with the cohomology class $\gamma(\frac{1}{2}\beta(\mathrm{ch}(\Omega_\Lambda)) \in H^2(\Lambda; \mathbb{C}_\Lambda^\times)$ (see (2.1.13)).

Consider an invertible $\mathbb{C}_\Lambda^\hbar$-algebroid $\mathfrak{A}$ on Λ and denote by $\mathrm{Inv}(\mathfrak{A})$ the category of invertible $\mathfrak{A}$-modules (see Definition 2.1.4). On the other hand, denote by $\mathrm{Simple}(\Lambda)$ the category of simple $\mathscr{A}_X$-modules along Λ. It can be easily deduced from Lemma 6.2.1 that, given a DQ-algebroid $\mathscr{A}_X$, there exist an invertible $\mathbb{C}_\Lambda^\hbar$-algebroid $\mathfrak{A}$ and an equivalence of categories

$$\mathrm{Simple}(\Lambda) \simeq \mathrm{Inv}(\mathfrak{A}). \tag{7.4.6}$$

When $\mathscr{A}_X$ is the canonical algebroid $\widehat{\mathscr{W}_X}(0)$ (see remark 6.1.3), it is proved in [20] that one has an equivalence $\mathfrak{A} \simeq \mathbb{C}_\Lambda^\hbar \otimes_{\mathbb{C}_\Lambda} \mathbb{C}_\Lambda^{1/2}$.

7.5 Invariance by deformation

We shall show that in the situation of Theorem 7.2.3, $\mathrm{R}\mathscr{H}om_{\mathscr{A}_X^{\mathrm{loc}}}(\mathscr{M}, \mathscr{L})$ is, in some sense, invariant by Hamiltonian symplectomorphism.

First, we need a lemma.

Lemma 7.5.1. *Let M be a complex manifold, $X = T^*M$ and let $\mathscr{M}$ be a holonomic $\widehat{\mathscr{W}_X}$-module. Assume that the projection $\pi_M \colon X \to M$ is proper (hence, finite) on* $\mathrm{Supp}(\mathscr{M})$. *Then $\pi_{M*}\mathscr{M}$ is a locally free $\mathscr{O}_M^{\hbar,\mathrm{loc}}$-module of finite rank.*

Proof. (i) In the sequel, we write $\mathscr{A}_X$ and $\mathscr{A}_X^{\mathrm{loc}}$ instead of $\widehat{\mathscr{W}}_X(0)$ and $\widehat{\mathscr{W}}_X$, respectively. Since π_M is finite on $\mathrm{Supp}(\mathscr{M})$, $\mathrm{R}\pi_{M*}\mathscr{M}$ is concentrated in degree 0. Let us prove that this sheaf is $\mathscr{O}_M^{\hbar,\mathrm{loc}}$-coherent. Denote by Γ_π the graph of the projection π_M and consider the diagram

$$\begin{tikzcd} & M \times X \arrow[dl] \arrow[dr] & \Gamma_\pi \arrow[l, hook', "s"'] \arrow[d, "p"] \\ M & & X. \end{tikzcd}$$

Using the morphism of $\mathbb{C}^\hbar$-algebras $\pi_M^{-1}\mathscr{O}_M^\hbar \hookrightarrow \mathscr{A}_X$, we may regard $\mathscr{L} := s_* p^{-1}\mathscr{A}_{X^a}$ as a coherent $\mathscr{A}_{M\times X^a}$-module simple along Γ_π. Then

$$\mathrm{R}\pi_{M*}\mathscr{M} \simeq \mathscr{L}^{\mathrm{loc}} \underset{X}{\circ} \mathscr{M}.$$

We may apply Theorem 3.3.6 and we get that $\mathrm{R}\pi_{M*}\mathscr{M}$ is $\mathscr{O}_M^{\hbar,\mathrm{loc}}$-coherent.

(ii) Let $n = d_M = \frac{1}{2}d_X$. By Lemma 7.2.2, $\mathrm{D}'_{\mathscr{A}^{\mathrm{loc}}}(\mathscr{M})\,[n]$ is concentrated in degree 0 and it follows from a similar argument as in (i) that $\mathrm{D}'_{\mathscr{A}}(\mathscr{M}) \circ \mathscr{L}'\,[n]$ is $\mathscr{O}_M^{\hbar,\mathrm{loc}}$-coherent and concentrated in degree 0 for any coherent $\mathscr{A}^{\mathrm{loc}}_{X\times M}$-module $\mathscr{L}'$ simple along Γ_π. Denote by $\mathrm{D}'_{\mathscr{O}^{\hbar,\mathrm{loc}}}$ the duality functor over $\mathscr{O}_M^{\hbar,\mathrm{loc}}$. Applying again Theorem 3.3.6, we get

$$\begin{aligned} \mathrm{D}'_{\mathscr{O}^{\hbar,\mathrm{loc}}}(\mathscr{M} \circ \mathscr{L}^{\mathrm{loc}}) &\simeq \mathrm{D}'_{\mathscr{A}^{\mathrm{loc}}}(\mathscr{L}^{\mathrm{loc}}) \circ \omega_X^{\mathscr{A}^{\mathrm{loc}}} \circ \mathrm{D}'_{\mathscr{A}^{\mathrm{loc}}}(\mathscr{M}) \\ &\simeq \mathrm{R}\pi_{M*}\Big(\mathrm{R}p_*(\mathrm{D}'_{\mathscr{A}}(\mathscr{L}) \circ \omega_X^{\mathscr{A}}) \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_X} \mathrm{D}'_{\mathscr{A}^{\mathrm{loc}}}(\mathscr{M})\Big). \end{aligned}$$

Since $\omega_X^{\mathscr{A}} \circ \mathrm{D}'_{\mathscr{A}}(\mathscr{L}) \simeq \mathscr{L}'\,[n]$ for an $\mathscr{A}_{M\times X}$-module $\mathscr{L}'$ simple along Γ_π and $\mathrm{D}'_{\mathscr{A}^{\mathrm{loc}}}(\mathscr{M})$ is concentrated in degree n, $\mathrm{D}'_{\mathscr{O}^{\hbar,\mathrm{loc}}}(\pi_{M*}\mathscr{M})$ is concentrated in degree zero. Therefore, $\pi_{M*}\mathscr{M}$ is a locally projective $\mathscr{O}_M^{\hbar,\mathrm{loc}}$-module of finite rank. To conclude, note that, for $x \in M$, any finitely generated projective $\mathscr{O}_{M,x}^{\hbar,\mathrm{loc}}$-module is free, by a result of [53] (see [60]). Q.E.D.

Recall the situation of (3.1.9): we have three symplectic manifolds X_i ($i = 1,2,3$) and closed subsets Λ_i of $X_i \times X_{i+1}$ ($i = 1,2$). Assume that the Λ_i ($i = 1,2$) are closed subvarieties and the projection p_{13} is proper on $p_{12}^{-1}\Lambda_1 \cap p_{23}^{-1}\Lambda_2$. Then $\Lambda_1 \circ \Lambda_2$ is a closed subvariety of $X_1 \times X_3$. Now assume that Λ_i ($i = 1,2$) is isotropic in $X_i \times X_{i+1}^a$. Then $\Lambda_1 \circ \Lambda_2$ is isotropic in $X_1 \times X_3^a$ by classical results (see *e.g.,* [40, Prop. 8.3.11]).

In the sequel, we denote by $\mathbb{D}$ the open unit disc in the complex line $\mathbb{C}$, endowed with the coordinate t. We set for short

$$Y := T^*\mathbb{D},$$

and we consider the projections

$$\begin{tikzcd} & X\times Y \arrow[r, "p_2"] \arrow[dl, "p_1"'] \arrow[d, "p"] \arrow[dr, "q"] & Y \arrow[d, "\pi"] \\ X & X\times\mathbb{D} \arrow[r, "s"] & \mathbb{D}. \end{tikzcd}$$

Assume to be given a Lagrangian subvariety $\Lambda \subset X \times Y$ satisfying

(7.5.1) the restriction $p|_\Lambda \colon \Lambda \to X \times \mathbb{D}$ is finite.

For $a \in \mathbb{D}$, writing for short $T^*_a\mathbb{D}$ instead of $T^*_{\{a\}}\mathbb{D}$, we set

$$\Lambda_a := \Lambda \circ T^*_a\mathbb{D} = p_1(\Lambda \cap q^{-1}(a)),$$

and this set is a Lagrangian subvariety of X.

We introduce the "skyscraper" $\mathscr{A}_Y^{\mathrm{loc}}$-module

$$\mathscr{C}_a := \mathscr{A}_Y^{\mathrm{loc}}/\mathscr{A}_Y^{\mathrm{loc}}\cdot(t-a). \tag{7.5.2}$$

Theorem 7.5.2. *Let X be a complex symplectic manifold, let Λ be a Lagrangian subvariety of $X \times Y$ satisfying* (7.5.1)*, and let V be a Lagrangian subvariety of X. Let $\mathscr{L}$ be a holonomic $\mathscr{A}_{X\times Y}^{\mathrm{loc}}$-module such that* $\mathrm{Supp}(\mathscr{L}) \subset \Lambda$ *and let $\mathscr{N}$ be a holonomic $\mathscr{A}_X^{\mathrm{loc}}$-module such that* $\mathrm{Supp}(\mathscr{N}) \subset V$*. Assume that the map $q\colon \Lambda \cap (p_1^{-1}V) \to \mathbb{D}$ is proper. For $a \in \mathbb{D}$, we set $\mathscr{L}_a := \mathscr{L} \underset{Y}{\circ} \mathscr{C}_a$ and $\mathscr{M} := \mathrm{R}p_{2*}\mathrm{R}\mathscr{H}om_{p_1^{-1}\mathscr{A}_X^{\mathrm{loc}}}(p_1^{-1}\mathscr{N}, \mathscr{L})$. Then*

(i) *$\mathscr{L}_a$ is concentrated in degree 0 and is a holonomic $\mathscr{A}_X^{\mathrm{loc}}$-module supported by Λ_a,*

(ii) *$\mathscr{M}$ is a coherent $\mathscr{A}_Y^{\mathrm{loc}}$-module supported by $V \underset{X}{\circ} \Lambda$,*

(iii) *$F_a := \mathrm{RHom}_{\mathscr{A}_X^{\mathrm{loc}}}(\mathscr{N}, \mathscr{L}_a) \simeq \mathrm{R}\Gamma(Y; (\mathscr{O}_\mathbb{D}/\mathscr{O}_\mathbb{D}(t-a)) \overset{\mathrm{L}}{\otimes}_{\mathscr{O}_\mathbb{D}} \mathscr{M})$ is an object of $\mathrm{D}^{\mathrm{b}}_f(\mathbb{C}^{\hbar,\mathrm{loc}})$, and F_a and F_b are isomorphic for any $a, b \in \mathbb{D}$.*

Proof. (i) First note that $t-a\colon \mathscr{L} \to \mathscr{L}$ is a monomorphism. Indeed for any $s \in \mathrm{Ker}(t-a\colon \mathscr{L} \to \mathscr{L})$, $\mathscr{A}_{X\times Y}^{\mathrm{loc}} s \subset \mathscr{L}$ is a coherent $\mathscr{A}_{X\times Y}^{\mathrm{loc}}$-module whose support is involutive and of codimension $> d_{X\times Y}/2$, hence empty. Therefore $\mathscr{L}_a = \mathscr{L} \underset{Y}{\circ} (\mathscr{A}_Y^{\mathrm{loc}}/\mathscr{A}_Y^{\mathrm{loc}}\cdot(t-a)) \simeq \mathrm{R}p_{1*}\big((\mathscr{O}_\mathbb{D}/\mathscr{O}_\mathbb{D}(t-a)) \otimes_{\mathscr{O}_\mathbb{D}} \mathscr{L}\big)$, and (i) follows immediately from the hypothesis (7.5.1).

(ii) We have

$$\mathrm{R}p_{2*}\mathrm{R}\mathscr{H}om_{p_1^{-1}\mathscr{A}_X^{\mathrm{loc}}}(p_1^{-1}\mathscr{N}, \mathscr{L}) \simeq \mathrm{D}'_{\mathscr{A}}(\mathscr{N}) \circ \mathscr{L}.$$

By the hypothesis, the projection $\Lambda \cap (V \times Y) \to Y$ is proper. It follows from Theorem 3.2.1 that $\mathscr{M}$ belongs to $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_Y^{\mathrm{loc}})$ and is supported by the isotropic variety $\Lambda_Y := V \underset{X}{\circ} \Lambda$.

(iii) By the hypothesis, the projection $\pi \colon \Lambda_Y \to \mathbb{D}$ is proper, hence finite. It follows easily that $H^i(\mathscr{M})$ is a holonomic $\mathscr{A}_Y^{\mathrm{loc}}$-module and $H^i(\mathrm{R}\pi_*\mathscr{M}) \simeq \pi_* H^i(\mathscr{M})$ is a locally free $\mathscr{O}_{\mathbb{D}}^{\hbar,\mathrm{loc}}$-module of finite rank by Lemma 7.5.1. Hence

$$H^i\Big(\mathrm{R}\Gamma(Y; (\mathscr{O}_{\mathbb{D}}/\mathscr{O}_{\mathbb{D}}(t-a)) \overset{\mathrm{L}}{\otimes}_{\mathscr{O}_{\mathbb{D}}} \mathscr{M})\Big) \simeq \Gamma(Y; H^i(\mathscr{M})/(t-a)H^i(\mathscr{M}))$$

is a finite-dimensional $\mathbb{C}^{\hbar,\mathrm{loc}}$-vector space whose dimension does not depend on $a \in \mathbb{D}$. Q.E.D.

We shall make a link between the hypotheses in Theorem 7.5.2 and the Hamiltonian deformations of a Lagrangian variety Λ_0.

Assume to be given a holomorphic map

$$\Phi(x,t) \colon X \times \mathbb{D} \to X$$

such that $\Phi(\cdot, a) \colon X \to X$ is a symplectomorphism for each $a \in \mathbb{D}$ and is the identity for $a = 0$. Set

$$\Gamma := \{(x, t, \Phi(x,t))\}, \text{ the graph of } \Phi \text{ in } X \times X^a \times \mathbb{D}.$$

Consider the differential

$$\frac{\partial \Phi}{\partial t} \colon X \times \mathbb{D} \to TX \simeq T^*X.$$

We make the hypothesis:

$$\text{there exists } f \colon X \times \mathbb{D} \to \mathbb{C} \text{ such that } \frac{\partial \Phi}{\partial t} = H_f, \tag{7.5.3}$$

where H_f denotes as usual the Hamiltonian vector field. In this case, we can define (identifying $T^*\mathbb{D}$ with $\mathbb{D} \times \mathbb{C}$)

$$\widetilde{\Gamma} := \{((x, \Phi(x,t)); (t, f(x,t)))\} \subset X \times X^a \times T^*\mathbb{D}$$

and $\widetilde{\Gamma}$ is Lagrangian. Let Λ_0 be a Lagrangian subvariety of X. We set:

$$\Lambda := \Lambda_0 \circ \widetilde{\Gamma}.$$

Then Λ will satisfy hypotheses (7.5.1) and $\Lambda_a = \Phi(x,a)(\Lambda_0)$.

Example 7.5.3. Let $X = T^*M$, $V = T^*_M M$ and let $\varphi\colon M \times \mathbb{D} \to \mathbb{C}$ be a holomorphic function. Set $Y = T^*\mathbb{D}$ and let

$$\begin{aligned}\Lambda &= \{(x,t;u,\tau) \in X \times Y; (u,\tau) = \operatorname{grad}_{x,t}\varphi(x,t)\},\\ \Lambda_a &= \{(x;u) \in X; u = \operatorname{grad}_x \varphi(x,a)\}.\end{aligned}$$

Consider the family of symplectomorphisms

$$\Phi(x,u,t) = (x, u + \varphi'_x(x,t) - \varphi'_x(x,0)).$$

Then

$$\frac{\partial \Phi}{\partial t} = -H_{\partial_t \varphi} \text{ and } \Lambda_a = \Phi(x,u,a)\Lambda_0.$$

Set $Z = \{(x,t) \in M \times \mathbb{D}; \operatorname{grad}_x \varphi(x,t) = 0\}$ and assume that

$$\text{the projection } Z \to \mathbb{D} \text{ is proper.}$$

Consider the ideals

$$\begin{aligned}\mathscr{I} &= \sum_{i=1}^n \mathscr{A}^{\mathrm{loc}}_{X\times Y} \cdot (\hbar\partial_{x_i} - \varphi'_{x_i}) + \mathscr{A}^{\mathrm{loc}}_{X\times Y} \cdot (\hbar\partial_t - \varphi'_t),\\ \mathscr{I}_a &= \sum_{i=1}^n \mathscr{A}^{\mathrm{loc}}_{X} \cdot (\hbar\partial_{x_i} - \varphi'_{x_i}(\cdot,a)).\end{aligned}$$

Set $\mathscr{N} = \mathscr{A}^{\mathrm{loc}}_X \otimes_{\mathscr{D}_M} \mathscr{O}_M$ and $\mathscr{L} = \mathscr{A}_{X\times Y}/\mathscr{I}$. Hence we have $\mathscr{L}_a = \mathscr{A}^{\mathrm{loc}}_X/\mathscr{I}_a$ and $H^i\big(\mathrm{RHom}_{\mathscr{A}^{\mathrm{loc}}_X}(\mathscr{L}_a,\mathscr{N})\big)$ does not depend on $a \in \mathbb{D}$.

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