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IND-SHEAVES

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Abstract. — Sheaf theory is not well suited to the study of various objects in Analysis which are not defined by local properties. The aim of this paper is to show that it is possible to overcome this difficulty by enlarging the category of sheaves to that of ind-sheaves, and by extending to ind-sheaves the machinery of sheaves.

Let X be a locally compact topological space and let k be a commutative ring. We define the category $\mathrm{I}(k_X)$ of ind-sheaves of k-modules on X as the category of ind-objects of the category $\mathrm{Mod}^c(k_X)$ of sheaves of k-modules on X with compact support, and we construct "Grothendieck's six operations" in the derived categories of ind-sheaves, as well as new functors which naturally arise.

A method for constructing ind-sheaves is the use of Grothendieck topologies associated with families $\mathcal{T}$ of open subsets satisfying suitable properties. Sheaves on the site $X_{\mathcal{T}}$ naturally define ind-sheaves.

When X is a real analytic manifold, we consider the subanalytic site X_{sa} associated with the family of open subanalytic subsets, and construct various ind-sheaves by this way. We obtain in particular the ind-sheaf $\mathcal{C}_X^{\infty,t}$ of tempered C^∞-functions, the ind-sheaf $\mathcal{C}_X^{\infty,w}$ of Whitney C^∞-functions and the ind-sheaf $\mathcal{D}b_X^t$ of tempered distributions. On a complex manifold X, we concentrate on the study of the ind-sheaf $\mathcal{O}_X^t$ of "tempered holomorphic functions" and prove an adjunction formula for integral transforms in this framework.

Résumé. — La théorie des faisceaux n'est pas bien adaptée à l'étude de divers objets de l'Analyse qui ne sont pas définis par des propriétés locales. Le but de cet article est de montrer que l'on peut surmonter cette difficulté en élargissant la catégorie des faisceaux à celle des ind-faisceaux, et étendre à ceux-ci le formalisme des faisceaux.

Soit X un espace localement compact et soit k un anneau commutatif. Nous définissons la catégorie $\mathrm{I}(k_X)$ des ind-faisceaux de k-modules sur X comme la catégorie des ind-objets de la catégorie $\mathrm{Mod}^c(k_X)$ des faisceaux de k-modules sur X à support compact, et nous construisons les "six opérations de Grothendieck" dans la catégorie dérivée des ind-faisceaux, ainsi que de nouveaux foncteurs qui apparaissent naturellement.

Une méthode pour construire des ind-faisceaux est l'utilisation de topologies de Grothendieck associées à des familles $\mathcal{T}$ d'ouverts de X satisfaisant certaines propriétés. Les faisceaux sur le site $X_{\mathcal{T}}$ définissent alors naturellement des ind-faisceaux.

Quand X est une variété analytique, nous considérons le site sous-analytique X_{sa} associé à la famille des ouverts sous-analytiques et nous construisons ainsi divers ind-faisceaux. Nous obtenons en particulier le ind-faisceau $\mathcal{C}_X^{\infty,t}$ des fonctions C^∞ tempérées, le ind-faisceau $\mathcal{C}_X^{\infty,w}$ des fonctions C^∞ de type Whitney, et le ind-faisceau $\mathcal{D}b_X^t$ des distributions tempérées.

Sur une variété complexe X, nous concentrons notre étude sur le ind-faisceau $\mathcal{O}_X^t$ des "fonctions holomorphes tempérées" et prouvons une formule d'adjonction dans ce cadre.

CONTENTS

INTRODUCTION

Sheaf theory is not well suited to the study of various objects in Analysis which are not defined by local properties, such as for example holomorphic functions with tempered growth. The aim of this paper is to show that it is possible to overcome this difficulty by enlarging the category of sheaves to that of ind-sheaves, and by extending to ind-sheaves the machinery of sheaves.

Recall that if $\mathcal{C}$ is an abelian category, the category $\mathrm{Ind}(\mathcal{C})$ of ind-objects of $\mathcal{C}$ has many remarkable properties: it contains $\mathcal{C}$ and admits small inductive limits, it is abelian and the natural functor $\mathcal{C} \to \mathrm{Ind}(\mathcal{C})$ is exact and fully faithful. Moreover $\mathrm{Ind}(\mathcal{C})$ is, in a certain sense, "dual" to $\mathcal{C}$.

For a locally compact topological space X and a commutative ring k, we introduce the category $\mathrm{I}(k_X)$ of ind-sheaves of k-modules on X as the category of ind-objects of the category $\mathrm{Mod}^c(k_X)$ of sheaves of k-modules with compact support in X. This construction has some analogy with that of distributions: the space of distributions is bigger than that of functions, and is dual to the space of functions with compact support. This last condition implies the local nature of distributions, and similarly, we prove that ind-sheaves form a stack (a "sheaf of categories").

We construct "Grothendieck's six operations" in the derived categories of ind-sheaves, as well as new functors which naturally arise.

There is a method for constructing ind-sheaves using Grothendieck topologies. We consider on X a family $\mathcal{T}$ of open subsets satisfying suitable properties and associate to it a site. In particular, when X is a real analytic manifold and $\mathcal{T}$ is the family of subanalytic open subsets, we obtain the "subanalytic site X_{sa}". We prove that the category of ind-objects of $\mathcal{T}$-coherent sheaves is equivalent to the category of sheaves on the site $X_{\mathcal{T}}$. Therefore, such sheaves naturally define ind-sheaves.

As already mentioned, ind-sheaves allow us to treat functions with growth conditions in the formalism of sheaves. On a complex manifold X, we can define the ind-sheaf of "tempered holomorphic functions" $\mathcal{O}^t_X$, or the ind-sheaf of "Whitney holomorphic functions" $\mathcal{O}_X^{\mathrm{w}}$, and obtain for example the sheaves of distributions or of C^∞-functions using Sato's construction of hyperfunctions, simply replacing $\mathcal{O}_X$ with $\mathcal{O}_X^t$ or $\mathcal{O}_X^{\mathrm{w}}$. We also prove an adjunction formula for integral transforms in this framework.

The contents of these Notes is as follows.

Chapters I and II are a short review, without proofs, of the theory of ind-objects with some applications to derived categories, and the theory of sheaves on Grothendieck topologies. Of course all these theories (invented by Grothendieck) are now classical. However, we shall also recall some technical statements extracted from [**13**] which are new.

Chapter III is devoted to stacks on a locally compact space X. We introduce the notion of a proper stack, show that this notion is stronger than the usual one of a stack, although its axioms are quite easy to check, and prove that the indization of a proper stack is a proper stack. There are new functors: ι from a stack to the associated ind-stack, and its left inverse α. Under reasonable conditions which will be satisfied by sheaves, α also admits a left adjoint β.

Ind-sheaves are introduced in Chapter IV, in which we first construct the internal operations: tensor product denoted by $\otimes$, and internal hom denoted by $\mathcal{I}hom$. We then construct the external operations: inverse image f^{-1}, direct image f_* and proper direct image $f_{!!}$. Finally, we study the various relations among all these functors. Note that the proper direct image of a sheaf is not the same in general whether we calculate it in sheaf theory or ind-sheaf theory.

In Chapter V, we derive all the functors we have constructed, and give relations among the derived functors. Moreover, as in the classical case, the functor $Rf_{!!}$ admits a right adjoint $f^!$, and we study its main properties. One of the difficulties of this study is that the category of ind-sheaves does not have enough injective objects. In this chapter, we also introduce the notions of ind-sheaves of rings and modules. This will be necessary for applications. For example, the ind-sheaf $\mathcal{O}_X^t$ of "tempered holomorphic functions" cannot be defined in the derived category of ind-sheaves of $\mathcal{D}_X$-modules, and one has to replace $\mathcal{D}_X$ with the ind-sheaf of rings $\beta_X(\mathcal{D}_X)$. As we shall see, this does not cause much trouble.

Chapter VI is devoted to the construction of ind-sheaves using Grothendieck topologies. We consider a family $\mathcal{T}$ of open subsets of X satisfying suitable properties, and its subfamily $\mathcal{T}_c$ of relatively compact open sets. We define the category $\mathrm{Coh}(\mathcal{T}_c)$ as the full subcategory of $\mathrm{Mod}(k_X)$ consisting of cokernels of morphisms $F \to G$ with F and G finite sums of sheaves of the type k_{XU} with $U \in \mathcal{T}_c$ and prove that this category is abelian. Then we define the site $X_\mathcal{T}$ whose family of objects is $\mathcal{T}$, a covering of $U \in \mathcal{T}$ being a *locally finite covering in* X. We study the category $\mathrm{Mod}(k_\mathcal{T})$ of sheaves on this site and prove that it is equivalent to the category of ind-objects of $\mathrm{Coh}(\mathcal{T}_c)$. Hence, there is a natural fully faithful exact functor from $\mathrm{Mod}(k_\mathcal{T})$ to

$\mathrm{I}(k_X)$, and this is a useful tool for constructing ind-sheaves. The category $\mathrm{I}(k_X)$ of ind-sheaves on X is much bigger than the category $\mathrm{Mod}(k_{\mathcal{T}})$. The first one does not have enough injectives, which make the theory rather difficult, unlike the second one. On the other hand the natural functor from $\mathrm{Mod}(k_X)$ to $\mathrm{I}(k_X)$ is exact, which fails when we replace $\mathrm{I}(k_X)$ with $\mathrm{Mod}(k_{\mathcal{T}})$.

We apply these results in Chapter VII and obtain the "subanalytic site" X_{sa} on a real analytic manifold X by taking the family of open subanalytic subsets as $\mathcal{T}$. We construct various ind-sheaves on this site, and when X is a complex manifold this allows us to define in particular the ind-sheaves $\mathcal{O}_X^t$ and $\mathcal{O}_X^{\mathrm{w}}$ of "tempered holomorphic functions" and "Whitney holomorphic functions", respectively. We prove formulas for direct images, inverse images and composition with a regular holonomic kernel for the ind-sheaf $\mathcal{O}_X^t$ (in the framework of $\mathcal{D}$-modules), from which we deduce a general adjunction formula for integral transforms.

CHAPTER 1

INDIZATION OF CATEGORIES: A REVIEW

In all these Notes, a ring means an associative unitary ring, and the action of a ring on a module is unitary. For a commutative ring k, a k-algebra is a ring A endowed with a ring morphism $k \to A$ whose image is contained in the center of A.

In this chapter, we recall some particular results on indization and localization of categories that we shall need in the sequel. References are made to [**16**] for the theory of universes and ind-objects and to [**10**], [**13**] or [**5**] for an exposition on categories and homological algebra, in particular ind-objects, localization, and derived categories. Some complementary results to the classical ones may be found in [**13**].

1.1. Ind-objects

Let $\mathcal{U}$ be a universe. A set is called $\mathcal{U}$-small if it is isomorphic to a set belonging to $\mathcal{U}$. Recall that a $\mathcal{U}$-category $\mathcal{C}$ is a category such that for any $X, Y \in \mathcal{C}$, the set $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ is $\mathcal{U}$-small. If moreover the family of objects (a set in a bigger universe) of $\mathcal{C}$ is $\mathcal{U}$-small, then one says that the category is $\mathcal{U}$-small. One says that a category is essentially $\mathcal{U}$-small if it is equivalent to a small category.

In these notes, we fix a universe $\mathcal{U}$ and we shall not refer to $\mathcal{U}$. We shall often abusively refer to a $\mathcal{U}$-category as a category. A category without the $\mathcal{U}$-small condition is called a big category. We shall often simply denote by $\mathbf{Set}$ the category of $\mathcal{U}$-small sets.

For a category $\mathcal{C}$, we denote by $\mathcal{C}^{\mathrm{op}}$ the opposite category of $\mathcal{C}$, i. e. $\mathrm{Ob}(\mathcal{C}^{\mathrm{op}}) = \mathrm{Ob}(\mathcal{C})$ and $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(Y,X)$.

Definition 1.1.1. — Let $\mathcal{C}$ be a category. One sets

$$\mathcal{C}^{\wedge} \quad : \quad \text{the big category of functors from } \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}\,.$$

One shall be aware that $\mathcal{C}^{\wedge}$ is not a $\mathcal{U}$-category in general (unless $\mathcal{C}$ is essentially small).

One defines the functor

$$\begin{array}{rclcl} h^\wedge & : & \mathcal{C} & \to & \mathcal{C}^\wedge \\ & & X & \mapsto & \mathrm{Hom}_{\mathcal{C}}(\,\cdot\,, X)\,. \end{array}$$

Then for $G \in \mathcal{C}^\wedge$ and $X \in \mathcal{C}$,

$$\mathrm{Hom}_{\mathcal{C}^\wedge}(h^\wedge(X), G) \simeq G(X). \tag{1.1.1}$$

In particular,

$$\mathrm{Hom}_{\mathcal{C}^\wedge}(h^\wedge(X), h^\wedge(Y)) \simeq \mathrm{Hom}_{\mathcal{C}}(X, Y),$$

and $h^\wedge$ is fully faithful. We shall identify $\mathcal{C}$ with a full subcategory of $\mathcal{C}^\wedge$ by $h^\wedge$. We denote by $\mathcal{C}^\vee$ the big category $((\mathcal{C}^{\mathrm{op}})^\wedge)^{\mathrm{op}}$. Then $\mathcal{C}$ is also embedded into $\mathcal{C}^\vee$.

The big category $\mathcal{C}^\wedge$ admits small inductive limits, but in general, even if $\mathcal{C}$ also admits small inductive limits, the functor $h^\wedge$ does not commute with $\varinjlim$. In order to avoid confusion, we denote by "$\varinjlim$" the inductive limit in $\mathcal{C}^\wedge$ and by $\varinjlim$ the inductive limit in $\mathcal{C}$. If I is small and $\alpha\colon I \to \mathcal{C}$ is a functor, we set "$\varinjlim$" $\alpha = \varinjlim(h^\wedge \circ \alpha)$. In other words, "$\varinjlim$" α is the object of $\mathcal{C}^\wedge$ defined by:

$$\text{“}\varinjlim\text{”}\,\alpha \,:\, \mathcal{C} \ni X \mapsto \varinjlim_i \mathrm{Hom}_{\mathcal{C}}(X, \alpha(i))\,.$$

With this convention

$$\varinjlim_i \mathrm{Hom}_{\mathcal{C}}(X, \alpha(i)) \quad = \quad \mathrm{Hom}_{\mathcal{C}^\wedge}(h^\wedge(X), \text{“}\varinjlim\text{”}\,\alpha).$$

Recall that a category I is filtrant if it satisfies the conditions (i)–(iii) below.

(i) I is non empty,
(ii) for any i and j in I, there exists $k \in I$ and morphisms $i \to k, j \to k$,
(iii) for any parallel morphisms $f, g\colon i \rightrightarrows j$, there exists a morphism $h\colon j \to k$ such that $h \circ f = h \circ g$.

Recall also that I is called cofinally small if there is a small subset S of $\mathrm{Ob}(I)$ such that any $i \in I$ admits an arrow $i \to j$ with $j \in S$.

Definition 1.1.2. — Let $\mathcal{C}$ be a category. An ind-object in $\mathcal{C}$ is an object $A \in \mathcal{C}^\wedge$ which is isomorphic to "$\varinjlim$" α for some functor $\alpha\colon I \to \mathcal{C}$ with I filtrant and small. One denotes by $\mathrm{Ind}(\mathcal{C})$ the full subcategory of $\mathcal{C}^\wedge$ consisting of ind-objects, and calls it the indization of $\mathcal{C}$.

Note that $\mathrm{Ind}(\mathcal{C})$ is a $\mathcal{U}$-category.

For $A \in \mathcal{C}^\wedge$, we define the category $\mathcal{C}_A$ and the functor $\alpha_A \colon \mathcal{C}_A \to \mathcal{C}$ by:

$$\begin{aligned}
\mathrm{Ob}(\mathcal{C}_A) &= \{(X,a); X \in \mathcal{C}, a \in A(X)\},\\
\mathrm{Hom}_{\mathcal{C}_A}((X,a),(Y,b)) &= \{f : X \to Y; a = b \circ f\},\\
\alpha_A &: (X,a) \mapsto X.
\end{aligned}$$

One proves easily that $A \in \mathrm{Ind}(\mathcal{C})$ if and only if $\mathcal{C}_A$ is filtrant and cofinally small, and $A \simeq$ "$\varinjlim$" α_A in this case.

One extends a functor $F \colon \mathcal{C} \to \mathcal{C}'$ to a functor $IF \colon \mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{C}')$ as follows. For $A \in \mathrm{Ind}(\mathcal{C})$ one defines $IF(A) \in \mathrm{Ind}(\mathcal{C}')$ by

$$IF(A) = \text{“}\varinjlim_{X \in \mathcal{C}_A}\text{”}\, F(X).$$

For $B \in \mathrm{Ind}(\mathcal{C})$ and a morphism $f \colon A \to B$ in $\mathrm{Ind}(\mathcal{C})$, f defines a functor $\mathcal{C}_A \to \mathcal{C}_B$ $(A(X) \ni a \mapsto f \circ a \in B(X))$. Hence we get a morphism

$$IF(f) : \text{“}\varinjlim_{X \in \mathcal{C}_A}\text{”}\, F(X) \to \text{“}\varinjlim_{Y \in \mathcal{C}_B}\text{”}\, F(Y),$$

and one checks that IF is a functor.

When $A \simeq \text{“}\varinjlim_i\text{”}\, \alpha(i)$, $B \simeq \text{“}\varinjlim_j\text{”}\, \beta(j)$, one has

$$\mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(A,B) \simeq \varprojlim_i \varinjlim_j \mathrm{Hom}_{\mathcal{C}}(\alpha(i),\beta(j))\,,$$

and the map $IF \colon \mathrm{Hom}\,(A,B) \to \mathrm{Hom}\,(IF(A), IF(B))$ is given by

$$\varprojlim_i \varinjlim_j \mathrm{Hom}_{\mathcal{C}}(\alpha(i),\beta(j)) \to \varprojlim_i \varinjlim_j \mathrm{Hom}_{\mathcal{C}'}(F(\alpha(i)), F(\beta(j))).$$

Proposition 1.1.3. — *Let $F \colon \mathcal{C} \to \mathcal{C}'$. Then*

(i) *the diagram below commutes*

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d] & \mathcal{C}' \arrow[d] \\
\mathrm{Ind}(\mathcal{C}) \arrow[r, "IF"] & \mathrm{Ind}(\mathcal{C}'),
\end{tikzcd}$$

(ii) *the functor $IF \colon \mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{C}')$ commutes with filtrant inductive limits,*

(iii) *if F is faithful (resp. fully faithful), so is IF.*

1.2. Indization and localization

Let $\mathcal{C}$ be a category and let $\mathcal{S}$ be a family of morphisms in $\mathcal{C}$.

Definition 1.2.1. — A localization of $\mathcal{C}$ by $\mathcal{S}$ is the data of a big category $\mathcal{C}_{\mathcal{S}}$ and a functor $Q \colon \mathcal{C} \to \mathcal{C}_{\mathcal{S}}$ satisfying:

(i) for every $s \in \mathcal{S}$, $Q(s)$ is an isomorphism,

(ii) for any functor $F\colon \mathcal{C} \to \mathcal{C}'$ such that $F(s)$ is an isomorphism for all $s \in \mathcal{S}$, there exist a functor $F_{\mathcal{S}}\colon \mathcal{C}_{\mathcal{S}} \to \mathcal{C}'$ and an isomorphism $F \simeq F_{\mathcal{S}} \circ Q$,

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "Q"'] & \mathcal{C}' \\ \mathcal{C}_{\mathcal{S}} \arrow[ur, dotted, "F_{\mathcal{S}}"'] & \end{tikzcd}$$

(iii) if G_1 and G_2 are two functors from $\mathcal{C}_{\mathcal{S}}$ to a big category $\mathcal{C}'$, then the natural map $\mathrm{Hom}\,(G_1, G_2) \to \mathrm{Hom}\,(G_1 \circ Q, G_2 \circ Q)$ is bijective.

Note that $\mathcal{C}_{\mathcal{S}}$ is unique up to equivalence and $F_{\mathcal{S}}$ in (ii) is unique up to unique isomorphism. It is well-known that if $\mathcal{S}$ is "a multiplicative system", then the localization exists. Using $\mathrm{Ind}(\mathcal{C})$, the localization is constructed as follows.

For any object $X \in \mathcal{C}$ let us define the categories $\mathcal{S}^l_X, \mathcal{S}^r_X$ and the functors α_X, β_X as follows.

$$\begin{aligned} \mathrm{Ob}(\mathcal{S}^r_X) &= \{s : X \to X'; s \in \mathcal{S}\} \\ \mathrm{Hom}_{\mathcal{S}^r_X}((s : X \to X'), (s' : X \to X'')) &= \{h : X' \to X''; h \circ s = s'\} \\ \alpha_X : \mathcal{S}^r_X \to \mathcal{C} \quad &\text{is} \quad \alpha_X(X \underset{s}{\to} X') = X', \\ \mathrm{Ob}(\mathcal{S}^l_X) &= \{s : X' \to X; s \in \mathcal{S}\} \\ \mathrm{Hom}_{\mathcal{S}^l_X}((s : X' \to X), (s' : X'' \to X)) &= \{h : X'' \to X'; s \circ h = s'\} \\ \beta_X : (\mathcal{S}^l_X)^{\mathrm{op}} \to \mathcal{C} \quad &\text{is} \quad \beta_X(X' \underset{s}{\to} X) = X'. \end{aligned}$$

By the definition, $\mathcal{S}$ is a multiplicative system if and only if the categories $(\mathcal{S}^l_X)^{\mathrm{op}}$ and $\mathcal{S}^r_X$ are filtrant and contain $X \xrightarrow{\mathrm{id}_X} X$.

In the sequel, we shall have to consider the inductive limit $\varinjlim \mathrm{Hom}\,(Y, \alpha_X)$ with $X, Y \in \mathcal{C}$. We shall often denote it by

$$\varinjlim_{X \underset{s}{\to} X',\, s \in \mathcal{S}} \mathrm{Hom}\,(Y, X').$$

One has:

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}_{\mathcal{S}}}(X, Y) &\simeq \varinjlim \mathrm{Hom}_{\mathcal{C}}(X, \alpha_Y) = \varinjlim_{Y \underset{t}{\to} Y',\, t \in \mathcal{S}} \mathrm{Hom}_{\mathcal{C}}(X, Y') \\ &\simeq \varinjlim \mathrm{Hom}_{\mathcal{C}}(\beta_X, \alpha_Y) = \varinjlim_{X' \underset{s}{\to} X,\, Y \underset{t}{\to} Y',\, s,t \in \mathcal{S}} \mathrm{Hom}_{\mathcal{C}}(X', Y') \\ &\simeq \varinjlim \mathrm{Hom}_{\mathcal{C}}(\beta_X, Y) = \varinjlim_{X' \underset{s}{\to} X,\, s \in \mathcal{S}} \mathrm{Hom}_{\mathcal{C}}(X', Y). \end{aligned}$$

One defines the functor

$$\alpha : \mathcal{C} \to \mathrm{Ind}(\mathcal{C}) \subset \mathcal{C}^{\wedge}$$

by setting

$$\alpha(X) = \text{“}\varinjlim\text{”}\, \alpha_X.$$

If $f\colon X \to Y$ is a morphism in $\mathcal{C}$, one constructs $\alpha(f)\colon \alpha(X) \to \alpha(Y)$ using the axioms of multiplicative systems, and one obtains a functor $\alpha\colon \mathcal{C} \to \mathrm{Ind}(\mathcal{C})$.

Proposition 1.2.2. — (i) *The functor α factorizes through $\mathcal{C}_{\mathcal{S}}$, hence defines a functor $\alpha_{\mathcal{S}}\colon \mathcal{C}_{\mathcal{S}} \to \mathrm{Ind}(\mathcal{C})$.*

(ii) *The functor $\alpha_{\mathcal{S}}$ is fully faithful.*

One shall be aware that the diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "Q"] \arrow[dr, "h^\wedge"'] & \mathcal{C}_{\mathcal{S}} \arrow[d, "\alpha_{\mathcal{S}}"] \\ & \mathrm{Ind}(\mathcal{C}) \end{tikzcd}$$

is *not* commutative in general. However, there is a natural morphism:

$$h^\wedge \to \alpha = \alpha_{\mathcal{S}} \circ Q,\ X \to \text{“}\varinjlim\text{”}\, \alpha_X. \tag{1.2.1}$$

Localization of functors. — Let $\mathcal{C}$ be a category, $\mathcal{S}$ a multiplicative system and $F\colon \mathcal{C} \to \mathcal{C}'$ a functor. In general, F does not send morphisms in $\mathcal{S}$ to isomorphisms in $\mathcal{C}'$, hence, does not factorizes through $\mathcal{C}_{\mathcal{S}}$. It is however possible in some cases to define a localization of F as follows.

Definition 1.2.3. — (i) A right localization of F (if it exists) is a functor $F_{\mathcal{S}}$: $\mathcal{C}_{\mathcal{S}} \to \mathcal{C}'$ and a morphism of functors $\tau\colon F \to F_{\mathcal{S}} \circ Q$ such that, for any morphism $G\colon \mathcal{C}_{\mathcal{S}} \to \mathcal{C}'$, the map $\mathrm{Hom}\,(F_{\mathcal{S}}, G) \to \mathrm{Hom}\,(F, G \circ Q)$ is bijective.
We say that F is right localizable if it admits a right localization.

(ii) We say that F is universally right localizable, if for any functor $K\colon \mathcal{C}' \to \mathcal{C}''$, the functor $K \circ F$ is right localizable and moreover $(K \circ F)_{\mathcal{S}} \xrightarrow{\sim} K \circ F_{\mathcal{S}}$.

Proposition 1.2.4. — *Let $\mathcal{C}$ be a category, $\mathcal{I}$ a full subcategory, $\mathcal{S}$ a multiplicative system in $\mathcal{C}$, and $\mathcal{T}$ the family of morphisms in $\mathcal{I}$ which belong to $\mathcal{S}$. Let $F\colon \mathcal{C} \to \mathcal{C}'$ be a functor. Assume that*

(i) *$\mathcal{T}$ is a multiplicative system in $\mathcal{I}$,*

(ii) *for any $X \in \mathcal{C}$ there exists $s\colon X \to W$ with $W \in \mathcal{I}$ and $s \in \mathcal{S}$,*

(iii) *for any $t \in \mathcal{T}$, $F(t)$ is an isomorphism.*

Then F is universally right localizable.

Indeed, the restriction of F to $\mathcal{I}$ is localizable, and the natural functor $\mathcal{I}_{\mathcal{T}} \to \mathcal{C}_{\mathcal{S}}$ is an equivalence. This is visualized by the diagram

$$
\begin{tikzcd}
\mathcal{C} \arrow[rr, "Q"] & & \mathcal{C}_{\mathcal{S}} \arrow[dd, dotted, "F_{\mathcal{S}}"] \\
\mathcal{I} \arrow[u, "\iota"] \arrow[r, "Q"] \arrow[rrd, "F\circ\iota"'] & \mathcal{I}_{\mathcal{T}} \arrow[ru, "\sim", "\iota_Q"'] \arrow[rd, "F_{\mathcal{T}}"] & \\
& & \mathcal{C}'
\end{tikzcd}
$$

Definition 1.2.5. — Let $X \in \mathcal{C}$. One says that F is right localizable at X if "$\varinjlim$"$(F \circ \alpha_X)$ is representable in $\mathcal{C}'$.

Recall that

$$\text{“}\varinjlim\text{”}(F \circ \alpha_X) = \text{“}\varinjlim_{X \to X'}\text{”}\, F(X'),\ (X \to X' \in \mathcal{S}).$$

Proposition 1.2.6. — *Let $F\colon \mathcal{C} \to \mathcal{C}'$ be a functor and $\mathcal{S}$ a multiplicative system in $\mathcal{C}$. The two conditions below are equivalent:*

(i) *F is right localizable at each $X \in \mathcal{C}$,*

(ii) *F is universally right localizable.*

1.3. Indization of abelian categories

From now on, $\mathcal{C}$ is an abelian category. One denotes by $\mathcal{C}^{\wedge,add}$ the big category of additive functors from $\mathcal{C}^{\mathrm{op}}$ to $\mathrm{Mod}(\mathbb{Z})$. This big category is clearly abelian. One denotes by $\mathcal{C}^{\wedge,add,l}$ the full big subcategory consisting of left exact functors. The functor $h^{\wedge}\colon \mathcal{C} \to \mathcal{C}^{\wedge}$ makes $\mathcal{C}$ a full abelian subcategory of $\mathcal{C}^{\wedge,add}$ and this functor is left exact, but not exact.

As seen in §1.1, an ind-object in $\mathcal{C}$ is an object $A \in \mathcal{C}^{\wedge}$ which is isomorphic to "$\varinjlim$" α for some functor $\alpha\colon I \to \mathcal{C}$ with I filtrant and small. Hence, $\mathrm{Ind}(\mathcal{C})$ is a full additive subcategory of $\mathcal{C}^{\wedge,add,l}$. Recall that it is a $\mathcal{U}$-category. If $\mathcal{C}$ is small, then $\mathrm{Ind}(\mathcal{C}) \simeq \mathcal{C}^{\wedge,add,l}$.

The category $\mathrm{Ind}(\mathcal{C})$ admits kernels and cokernels. Indeed, if $f\colon A \to B$ is a morphism in $\mathrm{Ind}(\mathcal{C})$, one may construct a small filtrant category I, two functors $\alpha, \beta\colon I \to \mathcal{C}$ and a morphism $\varphi\colon \alpha \to \beta$ such that $A \simeq$ "$\varinjlim$" α, $B \simeq$ "$\varinjlim$" β and $f =$ "$\varinjlim$" φ. Then "$\varinjlim$" $\ker\varphi$ and "$\varinjlim$" $\operatorname{coker}\varphi$ will be a kernel and a cokernel of f, respectively.

Theorem 1.3.1 (*see* [**13**]). — (i) *The category* $\mathrm{Ind}(\mathcal{C})$ *is abelian.*

(ii) *The natural functor* $\mathcal{C} \to \mathrm{Ind}(\mathcal{C})$ *is fully faithful and exact and the natural functor* $\mathrm{Ind}(\mathcal{C}) \to \mathcal{C}^{\wedge,add}$ *is fully faithful and left exact.*
(iii) *The category* $\mathrm{Ind}(\mathcal{C})$ *admits exact small filtrant inductive limits.*
(iv) *Assume that for any family* $\{X_i\}_{i\in I}$ *of objects of* $\mathcal{C}$ *indexed by a small set* I*, the product* $\prod_i X_i$ *(which is well-defined in* $\mathcal{C}^{\wedge}$*) belongs to* $\mathrm{Ind}(\mathcal{C})$*. Then the category* $\mathrm{Ind}(\mathcal{C})$ *admits small projective limits, and the functor* $\varprojlim$ *is left exact.*

In particular, $\mathrm{Ind}(\mathcal{C})$ admits small direct sums, which are denoted by "$\oplus$".

As a consequence of the preceding results, one gets that if $0 \to A' \to A \to A'' \to 0$ is an exact sequence in $\mathrm{Ind}(\mathcal{C})$, then one may construct a filtrant and small category I and an exact sequence of functors from I to $\mathcal{C}$, $0 \to \alpha' \to \alpha \to \alpha'' \to 0$ such that $0 \to \text{“}\varinjlim\text{”}\,\alpha' \to \text{“}\varinjlim\text{”}\,\alpha \to \text{“}\varinjlim\text{”}\,\alpha'' \to 0$ is isomorphic to $0 \to A' \to A \to A'' \to 0$.

This immediately implies that if $F\colon \mathcal{C} \to \mathcal{C}'$ is an additive functor of abelian categories and $IF\colon \mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{C}')$ is the associated functor, then IF is left (resp. right) exact as soon as F is left (resp. right) exact.

Proposition 1.3.2. — *A sequence of morphisms* $A \underset{f}{\to} B \underset{g}{\to} C$ *in* $\mathrm{Ind}(\mathcal{C})$ *with* $g\circ f = 0$ *is exact if and only if for any commutative diagram in* $\mathrm{Ind}(\mathcal{C})$ *with* $Y \in \mathcal{C}$

$$\begin{tikzcd} X \arrow[r, dotted, "h"] \arrow[d, dotted] & Y \arrow[d] \arrow[dr, "0"] & \\ A \arrow[r, "f"'] & B \arrow[r, "g"'] & C \end{tikzcd}$$

the dotted arrows may be completed to a commutative diagram, with $X \in \mathcal{C}$ *and* h *an epimorphism.*

Proposition 1.3.3. — *Let* $\mathcal{C}$ *be an abelian category.*
(i) $\mathcal{C}$ *is stable by extension in* $\mathrm{Ind}(\mathcal{C})$.
(ii) *Let* $\mathcal{C}_0 \subset \mathcal{C}$ *be an abelian subcategory stable by extension in* $\mathcal{C}$*. Then* $\mathrm{Ind}(\mathcal{C}_0)$ *is stable by extension in* $\mathrm{Ind}(\mathcal{C})$.

Let $\mathcal{C}$ be an abelian category and $\mathcal{J}$ a full additive subcategory.

Definition 1.3.4. — We say that $\mathcal{J}$ is generating (resp. cogenerating) in $\mathcal{C}$ if for any $X \in \mathcal{C}$ there exists an epimorphism $Y \twoheadrightarrow X$ (resp. a monomorphism $X \rightarrowtail Y$) with $Y \in \mathcal{J}$.

1.4. Derived categories

In this subsection, $\mathcal{C}$, $\mathcal{C}'$, etc. are abelian categories.

One denotes by $C(\mathcal{C})$ the abelian category of complexes in $\mathcal{C}$. By regarding morphisms in $C(\mathcal{C})$ which are homotopic to 0 as the zero morphism, one obtains the triangulated category $K(\mathcal{C})$, whose distinguished triangles are those isomorphic to $X \underset{f}{\to} Y \to M(f) \underset{+1}{\longrightarrow}$, where $M(f)$ denotes the mapping cone of the morphism f in $C(\mathcal{C})$.

A morphism $f\colon X \to Y$ in $K(\mathcal{C})$ (or in $C(\mathcal{C})$) is called a quasi-isomorphism (a qis for short) if it induces an isomorphism $H^j(f)\colon H^j(X) \to H^j(Y)$ for all $j \in \mathbb{Z}$.

The derived category $D(\mathcal{C})$ is the localization of $K(\mathcal{C})$ by the multiplicative system of quasi-isomorphisms. It is naturally a triangulated category. It is a big category in general.

One denotes as usual by $D^*(\mathcal{C})$ ($* = +, -$ or b) the full triangulated subcategory of $D(\mathcal{C})$ consisting of complexes bounded from below, from above, or bounded.

Now consider a left exact functor $F\colon \mathcal{C} \to \mathcal{C}'$. It defines a functor $K(F)\colon K(\mathcal{C}) \to K(\mathcal{C}')$. We shall often write F instead of $K(F)$ for short.

Definition 1.4.1. — Let $\mathcal{J}$ be a full additive subcategory of $\mathcal{C}$. One says that $\mathcal{J}$ is F-injective if:

(i) the category $\mathcal{J}$ is cogenerating in $\mathcal{C}$,
(ii) for any $X \in K^+(\mathcal{J})$ such that X is qis to 0, $F(X)$ is qis to 0.

By considering $\mathcal{C}^{\mathrm{op}}$, one obtains the notion of an F-projective subcategory, when F is right exact.

Proposition 1.4.2. — *Let $\mathcal{J}$ be an F-injective subcategory of $\mathcal{C}$. Then F is right derivable, i.e. $F\colon K^+(\mathcal{C}) \to D^+(\mathcal{C}')$ is universally right localizable with respect to the multiplicative system of quasi-isomorphisms. In particular F admits a right derived functor $RF\colon D^+(\mathcal{C}) \to D^+(\mathcal{C}')$. Moreover we have $H^k(RF(X)) = 0$ for any $X \in \mathcal{J}$ and $k \neq 0$.*

Recall that if $X \in K^+(\mathcal{C})$, then

$$RF(X) = \underset{X \xrightarrow[qis]{} X'}{\text{“}\varinjlim\text{”}}\ F(X')$$

in $\mathrm{Ind}(D^+(\mathcal{C}'))$.

There is a useful tool to check that a subcategory is F-injective.

Theorem 1.4.3 (**[13]**). — *Let $\mathcal{J}$ be a full additive subcategory of $\mathcal{C}$. Assume:*

(i) *the category $\mathcal{J}$ is cogenerating in $\mathcal{C}$,*
(ii) *for any monomorphism $Y' \rightarrowtail X$ with $Y' \in \mathcal{J}$ there exists an exact sequence $0 \to Y' \to Y \to Y'' \to 0$ with Y, Y'' in $\mathcal{J}$ such that $Y' \to Y$ factorizes through $Y' \to X$ and such that the sequence $0 \to F(Y') \to F(Y) \to F(Y'') \to 0$ is exact.*

Then the category $\mathcal{J}$ is F-injective.

As a corollary, we recover a classical result:

Corollary 1.4.4. — *Let $\mathcal{J}$ be a full additive subcategory of $\mathcal{C}$. Assume:*

(i) *the category $\mathcal{J}$ is cogenerating in $\mathcal{C}$,*
(ii) *for any exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathcal{C}$, if $X', X \in \mathcal{J}$, then $X'' \in \mathcal{J}$,*
(iii) *for any exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathcal{C}$ with $X', X \in \mathcal{J}$, the sequence $0 \to F(X') \to F(X) \to F(X'') \to 0$ is exact.*

Then the category $\mathcal{J}$ is F-injective.

We shall also have to derive bifunctors. Consider three abelian categories $\mathcal{C}$, $\mathcal{C}'$, $\mathcal{C}''$ and an additive bifunctor $F: \mathcal{C} \times \mathcal{C}' \to \mathcal{C}''$. We shall assume that F is left exact with respect to each of its arguments.

Definition 1.4.5. — Let $\mathcal{J}$ and $\mathcal{J}'$ be full additive subcategories of $\mathcal{C}$ and $\mathcal{C}'$ respectively. One says that $\mathcal{J} \times \mathcal{J}'$ is F-injective if

(i) $\mathcal{J}$ and $\mathcal{J}'$ are cogenerating,
(ii) for any $Y \in \mathcal{J}$, $\mathcal{J}'$ is $F(Y, \cdot\,)$-injective,
(iii) for any $Y' \in \mathcal{J}'$, $\mathcal{J}$ is $F(\,\cdot\,, Y')$-injective.

Proposition 1.4.6. — *Assume that there exist $\mathcal{J}$ and $\mathcal{J}'$ such that $\mathcal{J} \times \mathcal{J}'$ is F-injective. Then F is right derivable and defines $RF : D^+(\mathcal{C}) \times D^+(\mathcal{C}') \to D^+(\mathcal{C}'')$. Moreover, for $(X, X') \in K^+(\mathcal{C}) \times K^+(\mathcal{C}')$ one has:*

$$RF(X, X') \simeq \underset{X \xrightarrow[qis]{} Y,\, X' \xrightarrow[qis]{} Y'}{\text{“}\varinjlim\text{”}} F(Y, Y').$$

1.5. Indization and derivation

We shall study the derived category $D(\mathrm{Ind}(\mathcal{C}))$ of the category $\mathrm{Ind}(\mathcal{C})$ associated with an abelian category $\mathcal{C}$. In such a study, the universe $\mathcal{U}$ plays an important role. In fact, even if $\mathcal{C}$ has enough injectives, the category $\mathrm{Ind}(\mathcal{C})$ does not have enough injectives in general (see [**13**]). Instead, we shall use the following notion of quasi-injectives. Recall that unless otherwise specified, when we consider a category $\mathcal{C}$, it is a $\mathcal{U}$-category.

Let $\mathcal{C}$ be an abelian category, $\mathcal{C}' \subset \mathcal{C}$ a full abelian subcategory (hence, the natural functor $\mathcal{C}' \to \mathcal{C}$ is exact). One denotes by $D^b_{\mathcal{C}'}(\mathcal{C})$ the full subcategory of $D^b(\mathcal{C})$ consisting of objects with cohomology in $\mathcal{C}'$. If $\mathcal{C}'$ is stable by extension in $\mathcal{C}$, then $D^b_{\mathcal{C}'}(\mathcal{C})$ is triangulated.

The next result is easily deduced from Propositions 1.3.3 and 1.3.2.

Proposition 1.5.1. — *Let $\mathcal{C}$ be an abelian category. The natural functor $D^b(\mathcal{C}) \to D^b_{\mathcal{C}}(\mathrm{Ind}(\mathcal{C}))$ is an equivalence of triangulated categories.*

Definition 1.5.2. — Let $A \in \mathrm{Ind}(\mathcal{C})$. We say that A is quasi-injective if the functor

$$\begin{array}{rcl} \mathcal{C}^{\mathrm{op}} & \to & \mathrm{Mod}(\mathbb{Z}), \\ X & \mapsto & A(X)\ (= \mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(X, A)) \end{array}$$

is exact.

Assuming that $\mathcal{C}$ has enough injectives, one proves easily that A is quasi-injective if and only if there exist a small filtrant category I and $\alpha\colon I \to \mathcal{C}$ such that $A \simeq$ "$\varinjlim$" α and $\alpha(i)$ is injective in $\mathcal{C}$ for all $i \in I$.

Definition 1.5.3. — Let $\mathcal{C}$ be an abelian category. A system of strict $\mathcal{U}$-generators in $\mathcal{C}$ is a family $\{G_a; a \in A\}$ of objects of $\mathcal{C}$ such that A is $\mathcal{U}$-small and:

(i) for all $X \in \mathcal{C}$ and all $a \in A$, the object $G_a^{\oplus \mathrm{Hom}\,(G_a, X)}$ exists,

(ii) for all $X \in \mathcal{C}$, there exists $a \in A$ such that the morphism $G_a^{\oplus \mathrm{Hom}\,(G_a, X)} \to X$ is an epimorphism.

A system of strict $\mathcal{U}$-generators is a system of $\mathcal{U}$-generators in the sense of Grothendieck ([**7**]).

Theorem 1.5.4 **([13])**. — *Let $\mathcal{C}$ be an abelian category with a system of strict generators and let $S \subset \mathrm{Ob}(\mathrm{Ind}(\mathcal{C}))$ be a small subset.*

(a) *There exists an essentially small full abelian subcategory $\mathcal{C}_0$ of $\mathcal{C}$ such that $S \subset \mathrm{Ob}(\mathrm{Ind}(\mathcal{C}_0))$ with the properties:*
   (i) *$\mathcal{C}_0$ is stable under sub-object, quotient and extension in $\mathcal{C}$,*
   (ii) *for any epimorphism $X \twoheadrightarrow Y''$ with $Y'' \in \mathcal{C}_0$ and $X \in \mathcal{C}$, there exists a morphism $Y' \to X$ with $Y' \in \mathcal{C}_0$ such that the composition $Y' \to Y''$ is an epimorphism,*
   (iii) *$\mathrm{Ind}(\mathcal{C}_0)$ is stable by sub-object, quotient and extension in $\mathrm{Ind}(\mathcal{C})$,*
   (iv) *for any epimorphism $X \twoheadrightarrow Y''$ with $Y'' \in \mathrm{Ind}(\mathcal{C}_0)$ and $X \in \mathrm{Ind}(\mathcal{C})$, there exists a morphism $Y' \to X$ with $Y' \in \mathrm{Ind}(\mathcal{C}_0)$ such that the composition $Y' \to Y''$ is an epimorphism,*
   (v) *$\mathrm{Ind}(\mathcal{C}_0)$ has enough injectives.*

(b) *Assume moreover that $\mathcal{C}$ has enough injectives. Then we may choose $\mathcal{C}_0$ having the above properties and such that the injective objects of $\mathrm{Ind}(\mathcal{C}_0)$ are quasi-injective in $\mathrm{Ind}(\mathcal{C})$.*

Corollary 1.5.5. — *Assume that $\mathcal{C}$ has enough injectives and a system of strict generators. Then $\mathrm{Ind}(\mathcal{C})$ admits enough quasi-injectives.*

We denote by $\mathcal{I}_q$ the category of quasi-injective objects in $\mathrm{Ind}(\mathcal{C})$.

As above, denote by $F\colon \mathcal{C} \to \mathcal{C}'$ a left exact functor, and by $IF\colon \mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{C}')$ the associated left exact functor.

Theorem 1.5.6. — *Assume that $\mathcal{C}$ has enough injectives and a system of strict generators.*

(i) *The category $\mathcal{I}_q$ is IF-injective.*

(ii) *The diagram below commutes :*

$$\begin{tikzcd} D^+(\mathcal{C}) \arrow[r, "RF"] \arrow[d] & D^+(\mathcal{C}') \arrow[d] \\ D^+(\mathrm{Ind}(\mathcal{C})) \arrow[r, "RIF"] & D^+(\mathrm{Ind}(\mathcal{C}')). \end{tikzcd}$$

(iii) *The functor $R^kIF\colon \mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{C}')$ commutes with "$\varinjlim$". In other words, if I is small and filtrant and $\alpha\colon I \to \mathrm{Ind}(\mathcal{C})$, then*

$$R^kIF(\text{“}\varinjlim\text{”}\,\alpha) \simeq \text{“}\varinjlim\text{”}(R^kIF \circ \alpha).$$

Corollary 1.5.7. — *We keep the notations and the hypotheses of Theorem 1.5.6. Let $X \in D^b(\mathcal{C})$. Then there is a natural isomorphism*

$$\varinjlim \mathrm{Hom}_{D^+(\mathcal{C})}(X, \alpha) \simeq \mathrm{Hom}_{D^+(\mathrm{Ind}(\mathcal{C}))}(X, \text{“}\varinjlim\text{”}\,\alpha).$$

Remark 1.5.8. — One can prove that if $\mathcal{C}$ denotes the category $\mathrm{Mod}(\mathbb{C})$ of vector spaces over the field $\mathbb{C}$, then $\mathrm{Ind}(\mathcal{C})$ does not have enough injectives.

Exercises to Chapter 1

Exercise 1.1. — Let k be a field and let $\mathcal{C} = \mathrm{Mod}(k)$. Define $\beta\colon \mathcal{C} \to \mathrm{Ind}(\mathcal{C})$ by setting $\beta(V) = \text{“}\varinjlim_{W \subset V}\text{”}\, W$, where W ranges over the family of finite-dimensional vector subspaces of V.

(i) Prove that k is projective in $\mathrm{Ind}(\mathcal{C})$.

(ii) Prove that the natural morphism $\beta(V) \to V$ is a monomorphism.

(iii) Prove that if V is infinite-dimensional, then $\beta(V)$ is not representable in $\mathrm{Mod}(k)$. In particular, $\beta(V) \to V$ is not an isomorphism.

(iv) Prove that $\mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(k, V/\beta(V)) = 0$ for all $V \in \mathcal{C}$.

CHAPTER 2

GROTHENDIECK TOPOLOGIES: A REVIEW

In this chapter we briefly recall without proofs some classical constructions. References are made to [**16**].

2.1. Sites, presheaves and sheaves

The Grothendieck topology was introduced by A. Grothendieck in order to have a cohomology theory on algebraic varieties. The idea underlying this construction is that the notion of sheaves on a topological space X essentially relies on the category $\mathrm{Op}(X)$ of open subsets of X and on the notion of open coverings, and nothing else. Hence to construct the category of sheaves, we may start with an arbitrary category and axiomatize the notion of a covering.

However, we shall not treat the most general case, and for simplicity we shall only consider $\mathcal{U}$-small categories admitting finite products and fiber products. If $\mathcal{C}$ is such a category and $U \in \mathcal{C}$, one denotes by $\mathcal{C}_U$ the category of arrows $V \to U$. Clearly, $\mathcal{C}_U$ admits finite products and fiber products.

Note that if $\mathcal{C}$ admits a terminal object X, then $\mathcal{C}$ admits finite products and fiber products if and only if $\mathcal{C}$ admits finite projective limits. Moreover the product is the fiber product over X.

If $V \to U$ is a morphism and $S \subset \mathrm{Ob}(\mathcal{C}_U)$, we denote by $V \times_U S$ the set $\{V \times_U W \to V; W \in S\} \subset \mathrm{Ob}(\mathcal{C}_V)$.

Definition 2.1.1. — If $S_1, S_2 \subset \mathrm{Ob}(\mathcal{C}_U)$, one says that S_1 is a refinement of S_2 if any $V \to U$ in S_1 factorizes as $V \to V' \to U$ with $V' \to U \in S_2$. In such a situation one writes $S_1 \preceq S_2$.

Definition 2.1.2. — A Grothendieck topology on $\mathcal{C}$ is the data associating to any $U \in \mathcal{C}$ a family $\mathrm{Cov}(U)$ of subsets of $\mathrm{Ob}(\mathcal{C}_U)$ satisfying the axioms:

GT1 $\{U \underset{\mathrm{id}}{\to} U\}$ belongs to $\mathrm{Cov}(U)$,

GT2 if $S_1 \in \mathrm{Cov}(U)$ is a refinement of $S_2 \subset \mathrm{Ob}(\mathcal{C}_U)$, then $S_2 \in \mathrm{Cov}(U)$,
GT3 if $S \in \mathrm{Cov}(U)$, then $V \times_U S \in \mathrm{Cov}(V)$ for any $V \to U$,
GT4 if $S_1, S_2 \subset \mathrm{Ob}(\mathcal{C}_U)$, $S_1 \in \mathrm{Cov}(U)$ and $V \times_U S_2 \in \mathrm{Cov}(V)$ for any $V \in S_1$, then $S_2 \in \mathrm{Cov}(U)$.

A subset $S \in \mathrm{Cov}(U)$ is called a covering of U.
A site X is a category $\mathcal{C}_X$ which admits finite products and fiber products and endowed with a Grothendieck topology.

In the case when $\mathcal{C}_X$ admits a terminal object, we denote it by the same letter X.
Let X and Y be two sites.

Definition 2.1.3. — (i) A functor $f^t\colon \mathcal{C}_Y \to \mathcal{C}_X$ is said to be continuous if it commutes with fiber products and if for any $V \in \mathcal{C}_Y$ and any $S \in \mathrm{Cov}(V)$, $f^t(S) \in \mathrm{Cov}(f^t(V))$.
(ii) A morphism of sites $f\colon X \to Y$ is a continuous functor $f^t\colon \mathcal{C}_Y \to \mathcal{C}_X$.

Examples 2.1.4. — (i) Let X be a topological space. The set $\mathrm{Op}(X)$ of open subsets of X ordered by inclusion defines a category, still denoted by $\mathrm{Op}(X)$. Note that if $U \in \mathrm{Op}(X)$, then $\mathrm{Op}(X)_U = \mathrm{Op}(U)$. We keep the same symbol X to denote the site obtained by endowing $\mathrm{Op}(X)$ with the following topology: a subset $S \subset \mathrm{Op}(U)$ is a covering of U if $\bigcup_{V \in S} V = U$.
(ii) If $f\colon X \to Y$ is a continuous map of topological spaces, we shall denote by $f^t\colon \mathrm{Op}(Y) \to \mathrm{Op}(X)$ the functor $V \mapsto f^{-1}(V)$ and by $f\colon X \to Y$ the associated functor of sites. Hence, identifying a topological space with a site, one identifies a continuous map with a functor of sites.
(iii) Let X be a topological space. We can also endow $\mathrm{Op}(X)$ with the following topology: a subset $S \subset \mathrm{Op}(U)$ is a covering of U if there exists a finite subset $S_0 \subset S$ such that $\bigcup_{V \in S_0} V = U$. We denote by X_f the site so-obtained.
(iv) Assume that X is a locally compact topological space. We denote by X_{lf} the category $\mathrm{Op}(X)$ endowed with the following topology. A subset $S \subset \mathrm{Op}(U)$ is a covering of U in X_{lf} if for any compact K of X, there exists a finite subset $S_0 \subset S$ such that $K \cap (\cup_{V \in S_0} V) = K \cap U$.
If $U \in \mathrm{Op}(X)$, we denote by $U_{X_{lf}}$ the category $\mathrm{Op}(U)$ endowed with the topology induced by X_{lf}: a covering of $V \subset U$ for the topology $U_{X_{lf}}$ is a covering of V in X_{lf}. There is a natural morphism of sites $U_{lf} \to U_{X_{lf}}$ which is not an isomorphism in general.

In these Notes, we shall restrict ourselves to the study of sheaves of k-modules. (Recall that k denotes a commutative ring.)

Definition 2.1.5. — Let X be a site.
(i) A presheaf F of k-modules on X is a functor $\mathcal{C}_X^{\mathrm{op}} \to \mathrm{Mod}(k)$ and a morphism of presheaves is a morphism of such functors.

(ii) One denotes by $\mathrm{Psh}(k_X)$ the abelian category of presheaves of k-modules on X.

(iii) If F is a presheaf of k-modules on X and $S \subset \mathcal{C}_U$, one sets

$$F(S) = \ker(\prod_{V \in S} F(V) \rightrightarrows \prod_{V', V'' \in S} F(V' \times_U V'')).$$

(Recall that the kernel of the double arrow is the kernel of the difference of the two arrows. Here the two arrows are associated to $F(V') \to F(V' \times_U V'')$ and $F(V'') \to F(V' \times_U V'')$.)

(iv) We say that a presheaf F of k-modules on X is separated (resp. is a sheaf) if for any $U \in \mathcal{C}_X$ and any covering $S \in \mathrm{Cov}(U)$, the natural morphism $F(U) \to F(S)$ is a monomorphism (resp. an isomorphism).

(v) One denotes by $\mathrm{Mod}(k_X)$ the full additive subcategory of $\mathrm{Psh}(k_X)$ consisting of sheaves of k-modules on X. We shall often write Hom_{k_X} instead of $\mathrm{Hom}_{\mathrm{Mod}(k_X)}$.

Notation 2.1.6. — Let F be a presheaf on $\mathcal{C}_X$, let $U \in \mathcal{C}_X$, let $V \to U \in \mathcal{C}_U$, and let $s \in F(U)$. One sometimes writes $s|_U$ to denote the image of s in $F(V)$ by the morphism $F(U) \to F(V)$.

Let X be a site. In order to construct the sheaf associated with a presheaf, we need some preparation. For $U \in \mathcal{C}_X$, notice first that the relation $S_1 \preceq S_2$ is a pre-order on $\mathrm{Cov}(U)$. Hence, $\mathrm{Cov}(U)$ inherits a structure of a category: $\mathrm{Hom}_{\mathrm{Cov}(U)}(S_1, S_2) = \{\mathrm{pt}\}$ or $\varnothing$ according whether S_1 is a refinement of S_2 or not, and $\mathrm{Cov}(U)^{\mathrm{op}}$ is filtrant. Note that for $S_1, S_2 \in \mathrm{Cov}(U)$, $\{V_1 \times_U V_2; V_i \in S_i, i = 1, 2\}$ again belongs to $\mathrm{Cov}(U)$.

Let $F \in \mathrm{Psh}(k_X)$, and let $S_1 \preceq S_2$. For $V_1 \in S_1$, define first $\prod_{V \in S_2} F(V) \to F(V_1)$ by choosing $V_2 \in S_2$ such that $V_1 \to U$ factorizes through $V_2 \to U$. The composition $F(S_2) \to \prod_{V \in S_2} F(V) \to F(V_1)$ does not depend on the choice of $V_2 \in S_2$, and defines $F(S_2) \to F(S_1)$. Hence, F gives a functor $\mathrm{Cov}(U)^{\mathrm{op}} \to \mathrm{Mod}(k)$. One defines the presheaf F^+ by setting for all $U \in \mathcal{C}_X$:

$$F^+(U) = \varinjlim_{S \in \mathrm{Cov}(U)} F(S). \tag{2.1.1}$$

Theorem 2.1.7. — (i) *The functor* $^+ : \mathrm{Psh}(k_X) \to \mathrm{Psh}(k_X)$ *is left exact.*

(ii) *For any* $F \in \mathrm{Psh}(k_X)$, F^+ *is a separated presheaf.*

(iii) *For any separated presheaf* F, F^+ *is a sheaf.*

(iv) *The functor* $^{++}\colon \mathrm{Psh}(k_X) \to \mathrm{Mod}(k_X)$ *is a left adjoint to the embedding functor* $\iota\colon \mathrm{Mod}(k_X) \to \mathrm{Psh}(k_X)$.

In the sequel, we shall often omit to write the symbol ι. Hence, (iv) may be written as

$$\mathrm{Hom}_{\mathrm{Psh}(k_X)}(F, G) \simeq \mathrm{Hom}_{\mathrm{Mod}(k_X)}(F^{++}, G), \tag{2.1.2}$$

with $F \in \mathrm{Psh}(k_X)$ and $G \in \mathrm{Mod}(k_X)$. If F is a presheaf on X, the sheaf F^{++} is called the sheaf associated with F.

Definition 2.1.8. — (i) Let $M \in \mathrm{Mod}(k)$. One denotes by M_X the sheaf associated with the presheaf $U \mapsto M$ and calls M_X the constant sheaf with stalk M.

(ii) For $U \in \mathcal{C}_X$, one defines $k_{XU} \in \mathrm{Mod}(k_X)$ as the sheaf associated with the presheaf $V \mapsto k^{\oplus \mathrm{Hom}\,(V,U)}$.

Proposition 2.1.9. — *Let $F \in \mathrm{Mod}(k_X)$. There is a natural isomorphism*

$$F(U) \simeq \mathrm{Hom}_{k_X}(k_{XU}, F).$$

Theorem 2.1.10. — (i) *The category $\mathrm{Mod}(k_X)$ admits projective limits. More precisely, if $\{F_i\}_{i\in I}$ is a projective system of sheaves, its projective limit in $\mathrm{Psh}(k_X)$ is a sheaf and is a projective limit in $\mathrm{Mod}(k_X)$.*

(ii) *The category $\mathrm{Mod}(k_X)$ admits inductive limits. More precisely, if $\{F_i\}_{i\in I}$ is an inductive system of sheaves, its inductive limit in $\mathrm{Mod}(k_X)$ is the sheaf associated with its inductive limit in $\mathrm{Psh}(k_X)$.*

(iii) *The category $\mathrm{Mod}(k_X)$ is abelian.*

(iv) *The functor $\iota : \mathrm{Mod}(k_X) \to \mathrm{Psh}(k_X)$ is fully faithful and left exact. The functor ${}^{++} : \mathrm{Psh}(k_X) \to \mathrm{Mod}(k_X)$ is exact.*

(v) *Filtrant inductive limits in $\mathrm{Mod}(k_X)$ are exact.*

(vi) *The $\mathcal{U}$-category $\mathrm{Mod}(k_X)$ admits enough injectives.*

Notation 2.1.11. — Let $U \in \mathcal{C}_X$ and let $F \in \mathrm{Mod}(k_X)$. One sets

$$\Gamma(U; F) = F(U).$$

Proposition 2.1.12. — *A morphism $\varphi : F \to G$ in $\mathrm{Mod}(k_X)$ is an epimorphism if and only if for any $U \in \mathcal{C}_X$ and any $t \in G(U)$, there exists a covering $S \in \mathrm{Cov}(U)$ such that for each $V \in S$ there exists $s_V \in F(V)$ with $\varphi(s_V) = t|_V$.*

2.2. Inverse and direct images

Consider a morphism of sites $f\colon X \to Y$ associated with $f^t\colon \mathcal{C}_Y \to \mathcal{C}_X$. One defines a functor

$$f_*\colon\ \mathrm{Psh}(k_X) \to \mathrm{Psh}(k_Y)$$

by setting for $V \in \mathcal{C}_Y$ and $F \in \mathrm{Psh}(k_X)$, $(f_*F)(V) = F(f^t(V))$. This functor induces a functor that one denotes by the same symbol:

$$f_*\colon \mathrm{Mod}(k_X) \to \mathrm{Mod}(k_Y).$$

One defines a functor

$$If\colon\ \mathrm{Psh}(k_Y) \to \mathrm{Psh}(k_X)$$

by setting for $G \in \mathrm{Psh}(k_Y)$, $If(G)(U) = \varinjlim_{U \to f^t(V)} G(V)$.

One defines

$$f^{-1}\colon \mathrm{Mod}(k_Y) \to \mathrm{Mod}(k_X)$$

by setting $f^{-1}G = (If(G))^{++}$.

Theorem 2.2.1. — (i) *The functor* $f^{-1}\colon \mathrm{Mod}(k_Y) \to \mathrm{Mod}(k_X)$ *is left adjoint to* $f_*\colon \mathrm{Mod}(k_X) \to \mathrm{Mod}(k_Y)$. *In other words, we have for* $F \in \mathrm{Mod}(k_X)$ *and* $G \in \mathrm{Mod}(k_Y)$*:*

$$\mathrm{Hom}_{k_X}(f^{-1}G, F) \simeq \mathrm{Hom}_{k_Y}(G, f_*F).$$

(ii) *The functor* f_* *is left exact and commutes with projective limits.*
(iii) *The functor* f^{-1} *is exact and commutes with inductive limits.*

Notation 2.2.2. — (i) We denote by $\{\mathrm{pt}\}$ the category with one object $\{\mathrm{pt}\}$ and one morphism, endowed with its natural topology for which $\mathrm{Cov}(\{\mathrm{pt}\})$ consists of $\{\mathrm{id}_{\{\mathrm{pt}\}}\}$.
(ii) Let X be a site with a terminal object X. We denote by a_X the morphism of sites $X \to \{\mathrm{pt}\}$ defined by $a_X^t(\{\mathrm{pt}\}) = X$.

2.3. The functor i_U^{-1}

Definition 2.3.1. — Let $U \in \mathcal{C}_X$ and let $V \in \mathcal{C}_U$. A subset of $\mathcal{C}_V$ is called a covering of V in $\mathcal{C}_U$ if it is a covering in $\mathcal{C}_X$.

Clearly, the conditions of Definition 2.1.2 are satisfied and we get a site (with a terminal object U) that we denote by U. We define the functor

$$\begin{aligned} i_U^t\colon \mathcal{C}_X &\to \mathcal{C}_U \\ i_U^t(V) &= U \times V. \end{aligned}$$

Since i_U^t commutes with fiber products, it defines a functor of sites

$$i_U\colon U \to X.$$

Note that for $F \in \mathrm{Mod}(k_X)$ and for a morphism $W \to U$, we have

$$(i_U^{-1}F)(W) \simeq F(W). \tag{2.3.1}$$

Notation 2.3.2. — (i) For $F \in \mathrm{Mod}(k_X)$ and $U \in \mathcal{C}_X$, one sets

$$\Gamma_U(F) = i_{U*}i_U^{-1}F \in \mathrm{Mod}(k_X).$$

(ii) We sometimes write for short

$$F|_U := i_U^{-1}F.$$

If $F \in \mathrm{Mod}(k_V)$ with $U \to V$, we keep the same notation $F|_U$.

Clearly, the functor $\Gamma_U : \mathrm{Mod}(k_X) \to \mathrm{Mod}(k_X)$ is left exact, there is a natural morphism $F \to \Gamma_U(F)$, and $\Gamma(X;\cdot) \circ \Gamma_U(\cdot) \simeq \Gamma(U;\cdot)$. Moreover we have for $F, G \in \mathrm{Mod}(k_X)$:

$$\mathrm{Hom}_{k_X}(F, \Gamma_U(G)) \simeq \mathrm{Hom}_{k_U}(F|_U, G|_U).$$

Consider the functor

$$\begin{aligned} j_U^t\colon \mathcal{C}_U &\to \mathcal{C}_X,\\ V &\mapsto V. \end{aligned}$$

This functor defines a functor of sites

$$j_U\colon X \to U.$$

We get functors:

$$\mathrm{Mod}(k_U) \underset{j_{U*}}{\overset{j_U{}^{-1}}{\rightleftarrows}} \mathrm{Mod}(k_X) \underset{i_{U*}}{\overset{i_U{}^{-1}}{\rightleftarrows}} \mathrm{Mod}(k_U).$$

Definition 2.3.3. — One sets $i_{U!} = j_U{}^{-1}$.

Proposition 2.3.4. — (i) *One has* $j_{U*} \simeq i_U^{-1}$.

(ii) i_U^{-1} *is exact and commutes with projective limits.*

(iii) $i_{U!} := j_U{}^{-1}$ *is a left adjoint to* $i_U{}^{-1}$. *In other words, for* $F \in \mathrm{Mod}(k_U)$ *and* $G \in \mathrm{Mod}(k_X)$ *one has*

$$\mathrm{Hom}_{k_X}(i_{U!}F, G) \simeq \mathrm{Hom}_{k_U}(F, i_U{}^{-1}G).$$

(iv) $i_{U!}$ *is exact and commutes with inductive limits.*

(v) *For* $F \in \mathrm{Mod}(k_U)$, *the sheaf* $i_{U!}F$ *is the sheaf associated with the presheaf* $\mathcal{C}_X \ni V \mapsto \bigoplus_{V\to U} F(V \to U)$.

Notation 2.3.5. — For $F \in \mathrm{Mod}(k_X)$ and $U \in \mathcal{C}_X$, one sets

$$F_U = i_{U!}i_U{}^{-1}F.$$

Note that the functor $F \mapsto F_U$ is exact and there is a natural morphism $F_U \to F$. Moreover, if $F = k_X$, this definition agrees with Definition 2.1.8 (ii).

Consider the hypothesis for $U \in \mathcal{C}_X$:

(2.3.2) For any $V \in \mathcal{C}_X$, $\mathrm{Hom}_{\mathcal{C}_X}(V, U)$ has at most one element.

Proposition 2.3.6. — *Assume* (2.3.2).

(i) *One has* $i_U^{-1} \circ i_{U*} \simeq \mathrm{id}$ *and* $i_U^{-1} \circ i_{U!} \simeq \mathrm{id}$.

(ii) i_{U*} *and* $i_{U!}$ *are fully faithful.*

(iii) *The natural morphism* $F_U \to F$ *is a monomorphism.*

Consider a morphism of sites $f\colon X \to Y$ and assume that $\mathcal{C}_X$ and $\mathcal{C}_Y$ have terminal objects X and Y and $f^t(Y) = X$. Let $V \in \mathcal{C}_Y$, $U = f^t(V)$. The morphism f defines a morphism $f|_U\colon U \to V$. Consider the commutative diagrams of sites

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ U \arrow[u, "i_U"] \arrow[r, "f|_U"] & V \arrow[u, "i_V"] \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "f"] \arrow[d, "j_U"] & Y \arrow[d, "j_V"] \\ U \arrow[r, "f|_U"] & V. \end{tikzcd}$$

We deduce

$$i_{U!} \circ (f|_U)^{-1} \simeq f^{-1} \circ i_{V!}, \tag{2.3.3}$$

$$(f|_U)_* \circ i_U^{-1} \simeq i_V^{-1} \circ f_*. \tag{2.3.4}$$

Using (2.3.3), we get the isomorphism

$$(f|_U)^{-1} k_{YV} \simeq k_{XU}. \tag{2.3.5}$$

2.4. Internal hom and tensor product

Definition 2.4.1. — Let X be a site and let $F, G \in \mathrm{Mod}(k_X)$.

(i) We denote by $\mathcal{H}om_{k_X}(F, G)$ the presheaf on X, $U \mapsto \mathrm{Hom}_{k_U}(F|_U, G|_U)$ and call it the "internal hom" of F and G. If there is no risk of confusion, we write $\mathcal{H}om(F, G)$ instead of $\mathcal{H}om_{k_X}(F, G)$.

(ii) We denote by $F \otimes_{k_X} G$ the sheaf associated with the presheaf on X, $U \mapsto F(U) \otimes_k G(U)$ and call it the tensor product of F and G. If there is no risk of confusion, we write $F \otimes G$ instead of $F \otimes_{k_X} G$.

Proposition 2.4.2. — *Let $F, G, K \in \mathrm{Mod}(k_X)$.*

(i) *The presheaf $\mathcal{H}om(F, G)$ is a sheaf on X,*

(ii) *for any $U \in \mathcal{C}_X$, $i_U^{-1}\mathcal{H}om(F, G) \simeq \mathcal{H}om(i_U^{-1}F, i_U^{-1}G)$,*

(iii) $\mathcal{H}om(k_X, F) \simeq F$,

(iv) $k_X \otimes F \simeq F$,

(v) $\mathcal{H}om(F \otimes G, K) \simeq \mathcal{H}om(F, \mathcal{H}om(G, K))$.

Now consider a morphism of sites $f : X \to Y$.

Proposition 2.4.3. — (a) *Let $F \in \mathrm{Mod}(k_X)$ and let $G \in \mathrm{Mod}(k_Y)$. There is a natural isomorphism in $\mathrm{Mod}(k_Y)$*

$$\mathcal{H}om_{k_Y}(G, f_*F) \xrightarrow{\sim} f_*\mathcal{H}om_{k_X}(f^{-1}G, F). \tag{2.4.1}$$

(b) *Let $G_1, G_2 \in \mathrm{Mod}(k_Y)$. There is a natural isomorphism in $\mathrm{Mod}(k_X)$*

$$f^{-1}(G_1 \otimes G_2) \xrightarrow{\sim} f^{-1}G_1 \otimes f^{-1}G_2. \tag{2.4.2}$$

As a corollary, we find that the functor $i_{U!}$ commutes with tensor product.

Proposition 2.4.4. — *Let $U \in \mathcal{C}_X$, let $F \in \mathrm{Mod}(k_U)$ and let $G \in \mathrm{Mod}(k_X)$.*

(a) *There is a natural isomorphism*

$$i_{U!}(F \otimes i_U^{-1}G) \xrightarrow{\sim} i_{U!}F \otimes G. \tag{2.4.3}$$

(b) *There is a natural isomorphism*

$$\mathcal{H}om(i_{U!}F, G) \simeq i_{U*}\mathcal{H}om(F, i_U^{-1}G). \tag{2.4.4}$$

Proposition 2.4.5. — *Let $U \in \mathcal{C}_X$ and let $F, G \in \mathrm{Mod}(k_X)$. Then*

(i) $F_U \otimes G_U \simeq (F \otimes G)_U$,
(ii) $\mathrm{Hom}_{k_X}(F_U, G) \simeq \mathrm{Hom}_{k_U}(F|_U, G|_U)$,
(iii) $F_U \simeq F \otimes k_{XU}$,
(iv) $\mathcal{H}om\,(k_{XU}, F) \simeq \Gamma_U(F)$.

Exercises to Chapter 2

Exercise 2.1. — Let X be a site.
(i) Prove that hypothesis (2.3.2) on $U \in \mathcal{C}_X$ is equivalent to

for any morphisms $V \to U$ and $W \to U$, one has $V \times_U W \xrightarrow{\sim} V \times W$.

Assume hypothesis (2.3.2) is satisfied for every $U \in \mathcal{C}_X$.
(ii) Let $U, V \in \mathcal{C}_X$ and set $W = U \times V$. Denote by $i^U_W : \mathcal{C}_W \to \mathcal{C}_U$ the natural morphism of sites, and similarly for i^V_W. Prove the isomorphism $i_U^{-1} \circ i_{V!} \simeq i^U_{W!} \circ {i^V_W}^{-1}$.
(iii) Let $F \in \mathrm{Mod}(k_X)$. Under the notations in (ii), prove the isomorphisms $i_U^{-1} F_W \xrightarrow{\sim} i_U^{-1} F_V$ and $(F_V)_U \simeq F_W$.

CHAPTER 3

STACKS

In this chapter, we work in a fixed universe $\mathcal{U}$. Hence, otherwise stated, a category means a $\mathcal{U}$-category and all sets are $\mathcal{U}$-small. We shall denote by X a topological space and by k a commutative ring.

3.1. Definition of stacks

References are made to [**6**] (see also [**9**]).

Definition 3.1.1. — A prestack $\mathcal{C}$ on a topological space X is the data of:

(i) for each open subset U of X, a category $\mathcal{C}(U)$,

(ii) for each open inclusion $V \subset U$, a functor (sometimes called the restriction functor) $\rho_{VU} \colon \mathcal{C}(U) \to \mathcal{C}(V)$,

(iii) for each open inclusions $W \subset V \subset U$, an isomorphism of functors $\lambda_{WVU} \colon \rho_{WV} \circ \rho_{VU} \xrightarrow{\sim} \rho_{WU}$,

these data satisfying:

(i) $\rho_{UU} = \mathrm{id}_{\mathcal{C}(U)}$,

(ii) for each open inclusions $U_1 \subset U_2 \subset U_3 \subset U_4$, the diagram below is commutative (in this diagram, we shall write ρ_{ij} instead of $\rho_{U_iU_j}$ and λ_{ijk} instead of $\lambda_{U_iU_jU_k}$):

$$\begin{tikzcd}
\rho_{12} \circ \rho_{23} \circ \rho_{34} \arrow[r, "\lambda_{234}"] \arrow[d, "\lambda_{123}"] & \rho_{12} \circ \rho_{24} \arrow[d, "\lambda_{124}"] \\
\rho_{13} \circ \rho_{34} \arrow[r, "\lambda_{134}"] & \rho_{14}.
\end{tikzcd}$$

It follows from the axioms that $\lambda_{VVU} = \mathrm{id}$ and $\lambda_{VUU} = \mathrm{id}$ for $V \subset U$.

A prestack of k-additive (resp. k-abelian) categories is a prestack such that for each U, $\mathcal{C}(U)$ is k-additive (resp. k-abelian) and the functors ρ_{VU} are k-linear.

Notation 3.1.2. — (i) For $F, G \in \mathcal{C}(X)$ one denotes by $\mathcal{H}om_{\mathcal{C}}(F,G)$ the presheaf of sets on X: $U \mapsto \mathrm{Hom}_{\mathcal{C}(U)}(\rho_{UX}F, \rho_{UX}G)$ (U open). One sets $\mathcal{E}nd_{\mathcal{C}}(F) = \mathcal{H}om_{\mathcal{C}}(F,F)$.

(ii) For $F \in \mathcal{C}(U)$ one often writes $F|_V$ instead of $\rho_{VU}(F)$ for short, and one calls it the restriction of F to V. Hence, for $W \subset V \subset U$, λ_{WVU} defines an isomorphism $\lambda_{WVU}\colon (F|_V)|_W \xrightarrow{\sim} F|_W$.

(iii) If $\mathcal{C}$ is a prestack on X, one defines in an obvious way the prestack $\mathcal{C}|_U$ on U, its restriction to U.

(iv) If $\{U_i; i \in I\}$ is a family of open subsets, one writes $U_{ij} = U_i \cap U_j, U_{ijk} = U_i \cap U_j \cap U_k$, etc.

Definition 3.1.3. — We say that a prestack $\mathcal{C}$ satisfies the axiom ST1, if for any open subset U of X and any $F, G \in \mathcal{C}(U)$, the presheaf $\mathcal{H}om_{\mathcal{C}|_U}(F,G)$ is a sheaf on U.

If $\mathcal{C}$ is a prestack of additive categories satisfying ST1 and $F \in \mathcal{C}(U)$, one defines the support of F, denoted by $\operatorname{supp} F$, as the complementary set in U of the union of all open subsets $V \subset U$ such that $F|_V = 0$. This coincides with the support of $\mathrm{id}_F \in \Gamma(X; \mathcal{E}nd_{\mathcal{C}}(F))$. Note that if $V = \bigcup_{i \in I} V_i$ and $F|_{V_i} = 0$ for all i, then $F|_V = 0$.

Definition 3.1.4. — We say that a prestack $\mathcal{C}$ satisfies the axiom ST2, if for any open subset $U \subset X$, any open covering $U = \bigcup_{i \in I} U_i$, any family $F_i \in \mathcal{C}(U_i)$, any family of isomorphisms $\theta_{ji}\colon F_i|_{U_{ji}} \xrightarrow{\sim} F_j|_{U_{ji}}$ such that:

$$\theta_{ij}|_{U_{ijk}} \circ \theta_{jk}|_{U_{ijk}} = \theta_{ik}|_{U_{ijk}}, \tag{3.1.1}$$

there exist $F \in \mathcal{C}(U)$ and isomorphisms $\theta_i\colon F|_{U_i} \xrightarrow{\sim} F_i$ such that

$$\theta_{ij} \circ (\theta_j|_{U_{ij}}) = \theta_i|_{U_{ij}}. \tag{3.1.2}$$

More precisely, (3.1.1) means that the diagram below (in which we do not write explicitly the morphisms λ_{ijk}) commutes:

$$\begin{tikzcd}
F_k|_{U_{jk}}|_{U_{ijk}} \arrow[r, "\theta_{jk}"] \arrow[d, "\sim"] & F_j|_{U_{jk}}|_{U_{ijk}} \arrow[r, "\sim"] & F_j|_{U_{ijk}} \\
F_k|_{U_{ijk}} & & F_j|_{U_{ij}}|_{U_{ijk}} \arrow[u, "\sim"'] \arrow[d, "\theta_{ij}"] \\
F_k|_{U_{ik}}|_{U_{ijk}} \arrow[u, "\sim"'] \arrow[d, "\theta_{ik}"] & & F_i|_{U_{ij}}|_{U_{ijk}} \arrow[d, "\sim"'] \\
F_i|_{U_{ik}}|_{U_{ijk}} \arrow[rr, "\sim"] & & F_i|_{U_{ijk}}
\end{tikzcd}$$

and the equation (3.1.2) means that the diagram below commutes

$$\begin{tikzcd} F|_{U_j}|_{U_{ij}} \arrow[r, "\sim"] \arrow[d, "\theta_j"] & F|_{U_{ij}} & F|_{U_i}|_{U_{ij}} \arrow[l, "\sim"'] \arrow[d, "\theta_i"] \\ F_j|_{U_{ij}} \arrow[rr, "\theta_{ij}"] & & F_i|_{U_{ij}}. \end{tikzcd}$$

Definition 3.1.5. — (i) A separated prestack is a prestack which satisfies the axiom ST1.

(ii) A stack is a prestack which satisfies both the axioms ST1 and ST2.

Note that if $\mathcal{C}$ is a stack and if F is defined as in ST2, then F is unique up to isomorphism. Indeed, if (F', θ'_i) is another candidate, the isomorphisms $\alpha_i : \theta'^{-1}_i \circ \theta_i : F|_{U_i} \xrightarrow{\sim} F'|_{U_i}$ will glue as an isomorphism $\alpha\colon F \xrightarrow{\sim} F'$ by ST1.

Note also that for a stack of additive categories, $\mathcal{C}(\varnothing)$ is equivalent to 0, the category consisting of the zero object.

Definition 3.1.6. — Let $\mathcal{C}$ and $\mathcal{C}'$ be two prestacks on X. We shall denote by ρ_{VU}, λ_{WVU} (resp. ρ'_{VU}, λ'_{WVU}) the associated functors and morphisms of functors on $\mathcal{C}$ (resp. $\mathcal{C}'$). A functor of prestacks $\varphi\colon \mathcal{C} \to \mathcal{C}'$ is the data of:

(i) for each open subset U, a functor $\varphi_U\colon \mathcal{C}(U) \to \mathcal{C}'(U)$,

(ii) for each open inclusion $V \subset U$, an isomorphism of functors $\theta_{VU}\colon \varphi_V \circ \rho_{VU} \xrightarrow{\sim} \rho'_{VU} \circ \varphi_U$,

such that for each open inclusions $W \subset V \subset U$, the diagram below commutes:

$$\begin{tikzcd} \varphi_W \circ \rho_{WV} \circ \rho_{VU} \arrow[r, "\lambda_{WVU}"] \arrow[d, "\theta_{WV}"'] & \varphi_W \circ \rho_{WU} \arrow[dd, "\theta_{WU}"'] \\ \rho'_{WV} \circ \varphi_V \circ \rho_{VU} \arrow[d, "\theta_{VU}"'] & \\ \rho'_{WV} \circ \rho'_{VU} \circ \varphi_U \arrow[r, "\lambda'_{WVU}"] & \rho'_{WU} \circ \varphi_U\,. \end{tikzcd}$$

A functor of stacks is a functor of the underlying prestacks. One defines naturally the notion of a functor of k-additive or k-abelian stacks.

Definition 3.1.7. — Let $\mathcal{C}$ and $\mathcal{C}'$ be two prestacks on X. We shall use the same notations ρ_{VU} and ρ'_{VU} as in the preceding Definition 3.1.6. Let $\varphi\colon \mathcal{C} \to \mathcal{C}'$ and $\varphi'\colon \mathcal{C} \to \mathcal{C}'$ be two functors of prestacks. We denote by θ_{VU} (resp. θ'_{VU}) the associated morphism to φ (resp. φ'). A morphism $f\colon \varphi \to \varphi'$ of functors of prestacks is the data which associate, to each open subset U, a morphism $f_U\colon \varphi_U \to \varphi'_U$ of functors from $\mathcal{C}(U)$ to $\mathcal{C}'(U)$, such that for each open inclusion $V \subset U$ and $F \in \mathcal{C}(U)$, the following diagram commutes:

$$\begin{tikzcd}
\varphi_V(\rho_{VU}F) \arrow[r, "f_V(\rho_{VU}F)"] \arrow[d, "\theta_{VU}(F)"'] & \varphi'_V(\rho_{VU}F) \arrow[d, "\theta'_{VU}(F)"] \\
\rho'_{VU}(\varphi_U F) \arrow[r, "\rho'_{VU}(f_U F)"] & \rho'_{VU}(\varphi'_U F)\,.
\end{tikzcd}$$

Hence, one has the notion of equivalence of stacks.

Examples 3.1.8. — (i) If $\mathcal{C}$ is a stack on X, then $\mathcal{C}|_U$ is a stack on U.

(ii) Let $\mathcal{A}$ be a sheaf of k-algebras on X. Then $U \mapsto \mathrm{Mod}(\mathcal{A}|_U)$ is a stack of k-abelian categories on X.

3.2. Proper stacks

From now on, we assume that the topological space X is Hausdorff and locally compact. Recall that, for open subsets U and V of X, $V \subset\subset U$ means that the closure $\overline{V}$ of V is compact and contained in U. In this section, $\mathcal{C}$ is a prestack of abelian categories on X.

Notation 3.2.1. — For an open subset U of X, we denote by i_U the open embedding $U \hookrightarrow X$. We often write i_U^{-1} instead of ρ_{UX} to denote the restriction functor $\mathcal{C}(X) \to \mathcal{C}(U)$. Hence, for $F \in \mathcal{C}(X)$, we have three notations:

$$\rho_{UX}F = F|_U = i_U^{-1}F.$$

Definition 3.2.2. — A proper stack $\mathcal{C}$ on X is a prestack $\mathcal{C}$ of abelian categories satisfying the following axioms:

(i) $\mathcal{C}$ is a separated prestack,
(ii) for all open subsets $V \subset U \subset X$, the restriction functor ρ_{VU} is exact,
(iii) for all open subset $U \subset X$, $\mathcal{C}(U)$ admits small filtrant inductive limits, and the functor $\varinjlim$ is exact over such limits and commutes with ρ_{VU},
(iv) for all open subset $U \subset X$, $\mathcal{C}(U)$ admits small filtrant projective limits, and the functor $\varprojlim$ commutes with ρ_{VU},
(v) for all open subset $U \subset X$, the functor i_U^{-1} admits a left adjoint, and denoting this functor by $i_{U!}$, it satisfies $\mathrm{id}_{\mathcal{C}(U)} \xrightarrow{\sim} i_U^{-1} \circ i_{U!}$.

We shall prove later that a proper stack is actually a stack.

In the sequel let $\mathcal{C}$ be a proper stack.

Lemma 3.2.3. — *For any small filtrant inductive system $\{F_i\}_i$ in $\mathcal{C}(X)$ and for any small filtrant projective system $\{G_j\}_j$ in $\mathcal{C}(X)$, we have*

$$\mathcal{H}om_{\mathcal{C}}(\varinjlim_i F_i, \varprojlim_j G_j) \simeq \varprojlim_{i,j} \mathcal{H}om_{\mathcal{C}}(F_i, G_j).$$

Proof. — This immediately follows from the fact that inductive and projective limits commute with the restriction functors. q.e.d.

Lemma 3.2.4. — *For open subsets $V \subset V' \subset U$ and $F \in \mathcal{C}(U)$, there exists a canonical morphism*

$$i_{V!}(F|_V) \to i_{V'!}(F|_{V'}).$$

Moreover, if $V' \cap \overline{\operatorname{supp}(F)} \subset V$, this morphism is an isomorphism.

Proof. — For any $G \in \mathcal{C}(X)$, we have a chain of morphisms

$$\begin{aligned}\operatorname{Hom}_{\mathcal{C}(X)}(i_{V'!}(F|_{V'}), G) &\simeq \operatorname{Hom}_{\mathcal{C}(V')}(F|_{V'}, G|_{V'})\\ &\xrightarrow{\beta} \operatorname{Hom}_{\mathcal{C}(V)}(F|_V, G|_V) \simeq \operatorname{Hom}_{\mathcal{C}(X)}(i_{V!}(F|_V), G)\end{aligned}$$

Hence we have the desired morphism.

Assume $V' \cap \overline{\operatorname{supp}(F)} \subset V$. Then $V' \cap \operatorname{supp}(\mathcal{H}om_{\mathcal{C}|_U}(F, G|_U)) \subset V$ and hence $\Gamma(V'; \mathcal{H}om_{\mathcal{C}|_U}(F, G|_U)) \to \Gamma(V; \mathcal{H}om_{\mathcal{C}|_U}(F, G|_U))$ is an isomorphism. It means that β is an isomorphism. q.e.d.

Lemma 3.2.5. — *For any $F \in \mathcal{C}(U)$, one has $\operatorname{supp}(i_{U!}F) \subset \overline{\operatorname{supp} F}$.*

Proof. — It is enough to show

(3.2.1) $\quad (i_{U!}F)|_V = 0$ for any open subset V such that $\operatorname{supp} F \cap V = \varnothing$.

Set $W = U \cup V$, $\tilde{F} = i_{U!}F$ and $H = \mathcal{H}om_{\mathcal{C}}(\tilde{F}, \tilde{F})$. Then $H|_{U\cap V} = 0$. Define $\varphi \in \operatorname{End}_{\mathcal{C}(W)}(\tilde{F}|_W) = \Gamma(W; H)$ by $\varphi|_U = 0$ and $\varphi|_V = \mathrm{id}_{\tilde{F}|_V}$. Let G be the cokernel of $\varphi : \tilde{F}|_W \to \tilde{F}|_W$, and let $\tilde{G} = i_{W!}G \in \mathcal{C}(X)$. Then we have

$$\tilde{G}|_V \simeq G|_V \simeq \operatorname{coker}(\varphi|_V) = 0,$$

and similarly $G|_U \simeq F$.

Hence it is enough to show that $\tilde{G}$ is isomorphic to $i_{U!}F$. For any $K \in \mathcal{C}(X)$, we have

$$\begin{aligned}\operatorname{Hom}_{\mathcal{C}(X)}(\tilde{G}, K) &\simeq \operatorname{Hom}_{\mathcal{C}(W)}(G, K|_W)\\ &\simeq \Gamma(W; \mathcal{H}om_{\mathcal{C}|_W}(G, K|_W)) \simeq \Gamma(U; \mathcal{H}om_{\mathcal{C}|_W}(G, K|_W))\\ &\simeq \operatorname{Hom}_{\mathcal{C}(U)}(G|_U, K|_U) \simeq \operatorname{Hom}_{\mathcal{C}(U)}(F, K|_U).\end{aligned}$$

Here the third isomorphism follows from $\mathcal{H}om_{\mathcal{C}|_W}(G, K|_W)|_V = 0$. q.e.d.

Lemma 3.2.6. — *For $F \in \mathcal{C}(U)$, we have $i_{U!}F \simeq \varinjlim_{V \subset\subset U} i_{V!}(F|_V)$.*

Proof. — Let $G \in \mathcal{C}(X)$. Then

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}(X)}(\varinjlim_{V\subset\subset U} i_{V!}(F|_V), G) &\simeq \varprojlim_{V\subset\subset U} \mathrm{Hom}_{\mathcal{C}(X)}(i_{V!}(F|_V), G)\\ \simeq \varprojlim_{V\subset\subset U} \mathrm{Hom}_{\mathcal{C}(V)}(F|_V, i_V^{-1}G) &\simeq \varprojlim_{V\subset\subset U} \Gamma(V\,;\mathcal{H}om_{\mathcal{C}|_U}(F, i_U^{-1}G))\\ \simeq \Gamma(U\,;\mathcal{H}om_{\mathcal{C}|_U}(F, i_U^{-1}G)) &\simeq \mathrm{Hom}_{\mathcal{C}(U)}(F, i_U^{-1}G) \simeq \mathrm{Hom}_{\mathcal{C}(X)}(i_{U!}F, G).\end{aligned}$$

q.e.d.

Lemma 3.2.7. — *Let $F \in \mathcal{C}(X)$ with* $\operatorname{supp} F \subset U$. *Then $i_{U!}i_U^{-1}F \xrightarrow{\sim} F$.*

Proof. — Apply Lemma 3.2.4 with V, V', U replaced by U, X, X. q.e.d.

Lemma 3.2.8. — *Let $G \in \mathcal{C}(X)$. If there is a monomorphism $G \rightarrowtail i_{U!}F$ with $F \in \mathcal{C}(U)$, then we have*

$$i_{U!}i_U^{-1}G \xrightarrow{\sim} G.$$

Proof. — Recall the isomorphism $i_{U!}F \simeq \varinjlim_{V\subset\subset U} i_{V!}(F|_V)$. Define $G_V \in \mathcal{C}(X)$ by the Cartesian square

$$\begin{tikzcd} G_V \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & G \arrow[d] \\ i_{V!}(F|_V) \arrow[r] & i_{U!}F. \end{tikzcd}$$

Since the functor $\varinjlim_{V\subset\subset U}$ is exact, we get the isomorphism $\varinjlim_{V\subset\subset U} G_V \xrightarrow{\sim} G$. Hence, we get the isomorphisms

$$i_{U!}i_U^{-1}G \simeq i_{U!}i_U^{-1} \varinjlim_{V\subset\subset U} G_V \simeq \varinjlim_{V\subset\subset U} i_{U!}i_U^{-1}G_V.$$

Indeed, i_U^{-1} commutes with $\varinjlim$ by the axioms, and $i_{U!}$ commutes with $\varinjlim$ since it has a right adjoint. On the other hand, the monomorphism $G_V \rightarrowtail i_{V!}(F|_V)$ implies that $\operatorname{supp} G_V$ is contained in $\overline{V} \subset U$. Applying Lemma 3.2.7, we get the isomorphism $i_{U!}i_U^{-1}G_V \xrightarrow{\sim} G_V$, and then $i_{U!}i_U^{-1}G \xrightarrow{\sim} \varinjlim_{V\subset\subset U} G_V \xrightarrow{\sim} G$. q.e.d.

Lemma 3.2.9. — *For $F \in \mathcal{C}(X)$, the natural morphism $i_{U!}i_U^{-1}F \to F$ is a monomorphism.*

Proof. — Define N by the exact sequence $0 \to N \to i_{U!}i_U^{-1}F \to F$. Applying the exact functor i_U^{-1} and using the isomorphism $i_U^{-1}i_{U!}i_U^{-1} \simeq i_U^{-1}$, we get $i_U^{-1}N = 0$. Since N is a subobject of $i_{U!}i_U^{-1}F$, we find $N = 0$ by Lemma 3.2.8. q.e.d.

Proposition 3.2.10. — *The functor $i_{U!}$ is exact.*

Proof. — This functor is right exact since it admits a right adjoint. Consider a monomorphism $G \rightarrowtail F$ in $\mathcal{C}(U)$, and define N by the exact sequence $0 \to N \to i_{U!}G \to i_{U!}F$. Applying the exact functor i_U^{-1}, we find $i_U^{-1}N = 0$. Since $N \rightarrowtail i_{U!}G$, we have $N \simeq i_{U!}i_U^{-1}N$ by Lemma 3.2.8. Hence $N = 0$. q.e.d.

Proposition 3.2.11. — *Let U and V be open subsets, and let $F \in \mathcal{C}(U)$. Then*

$$i_{V!}i_V^{-1}i_{U!}F \simeq i_{U\cap V!}(F|_{U\cap V}).$$

Proof. — For $U' \subset\subset U$ and $V' \subset\subset V$, set

$$\begin{aligned} G_{U'V'} &= i_{V'!}i_{V'}^{-1}i_{U'!}(F|_{U'}), \\ G &= \varinjlim_{U',V'} G_{U'V'}. \end{aligned}$$

Then we have

$$G \simeq i_{V!}i_V^{-1}i_{U!}F. \tag{3.2.2}$$

Since $\operatorname{supp} G_{U'V'} \subset \overline{U'} \cap \overline{V'} \subset U \cap V$,

$$i_{U\cap V!}i_{U\cap V}^{-1}G_{U'V'} \xrightarrow{\sim} G_{U'V'}.$$

Taking the inductive limit with respect to (U', V'), we obtain

$$i_{U\cap V!}i_{U\cap V}^{-1}G \xrightarrow{\sim} G. \tag{3.2.3}$$

On the other hand, (3.2.2) implies $i_{U\cap V}^{-1}G \simeq F|_{U\cap V}$, and we get

$$i_{U\cap V!}i_{U\cap V}^{-1}G \simeq i_{U\cap V!}(F|_{U\cap V}). \tag{3.2.4}$$

Then the assertion follows from (3.2.2), (3.2.3) and (3.2.4). q.e.d.

Proposition 3.2.12. — *Let $V \subset U \subset X$ be open inclusions. Then $i_U^{-1}i_{V!}$ is a left adjoint to ρ_{VU}:*

$$\operatorname{Hom}_{\mathcal{C}(U)}(i_U^{-1}i_{V!}G, F) \simeq \operatorname{Hom}_{\mathcal{C}(V)}(G, \rho_{VU}F) \quad \textit{for } G \in \mathcal{C}(V) \textit{ and } F \in \mathcal{C}(U).$$

Proof. — Applying the preceding proposition, we have

$$i_{U!}i_U^{-1}i_{V!}G \simeq i_{V\cap U!}(G|_{V\cap U}) \simeq i_{V!}G.$$

Hence we have

$$\begin{aligned} \operatorname{Hom}_{\mathcal{C}(U)}(i_U^{-1}i_{V!}G, F) &\simeq \operatorname{Hom}_{\mathcal{C}(U)}(i_U^{-1}i_{V!}G, i_U^{-1}i_{U!}F) \\ &\simeq \operatorname{Hom}_{\mathcal{C}(X)}(i_{U!}i_U^{-1}i_{V!}G, i_{U!}F) \\ &\simeq \operatorname{Hom}_{\mathcal{C}(X)}(i_{V!}G, i_{U!}F) \\ &\simeq \operatorname{Hom}_{\mathcal{C}(V)}(G, i_V^{-1}i_{U!}F) \simeq \operatorname{Hom}_{\mathcal{C}(V)}(G, \rho_{VU}F). \end{aligned}$$

q.e.d.

Proposition 3.2.13. — *Let $U \subset X$. Then $\mathcal{C}|_U$ is a proper stack.*

Proof. — By the preceding proposition, ρ_{VU} admits a left adjoint $i_U^{-1} i_{V!}$. The axioms are now easily checked. q.e.d.

Proposition 3.2.14. — *Let $F \in \mathcal{C}(U), G \in \mathcal{C}(X)$. Then*

$$\mathcal{H}om_{\mathcal{C}}(i_{U!}F, G) \simeq i_{U*}\mathcal{H}om_{\mathcal{C}|_U}(F, i_U^{-1}G).$$

Proof. — Let $V \subset X$. Then

$$\begin{aligned}\Gamma(V; \mathcal{H}om_{\mathcal{C}}(i_{U!}F, G)) &\simeq \mathrm{Hom}_{\mathcal{C}(V)}(i_V^{-1} i_{U!}F, i_V^{-1}G) \simeq \mathrm{Hom}_{\mathcal{C}(X)}(i_{V!} i_V^{-1} i_{U!}F, G)\\ &\simeq \mathrm{Hom}_{\mathcal{C}(X)}(i_{U\cap V!}(F|_{U\cap V}), G) \simeq \mathrm{Hom}_{\mathcal{C}(U\cap V)}(F|_{U\cap V}, i_{U\cap V}^{-1}G)\\ &\simeq \Gamma(U\cap V; \mathcal{H}om_{\mathcal{C}|_U}(F, i_U^{-1}G)) \simeq \Gamma(V; i_{U*}\mathcal{H}om_{\mathcal{C}|_U}(F, i_U^{-1}G)).\end{aligned}$$

Here the third isomorphism follows from Proposition 3.2.11. q.e.d.

Definition 3.2.15. — For $F \in \mathcal{C}(X)$, we set $F_U := i_{U!} i_U^{-1} F$.

The functor $\mathcal{C}(X) \ni G \mapsto \Gamma(U; \mathcal{H}om_{\mathcal{C}}(F, G))$ is representable by F_U.

Let $V \subset U$. The morphism $\Gamma(U; \mathcal{H}om_{\mathcal{C}}(F, G)) \to \Gamma(V; \mathcal{H}om_{\mathcal{C}}(F, G))$ defines a morphism

$$F_V \to F_U.$$

Now, consider an open covering $U = \bigcup_{i\in I} U_i$. The families of morphisms $F_{U_{ij}} \to F_{U_i}$ and $F_{U_{ij}} \to F_{U_j}$ define the two morphisms α and β:

$$\bigoplus\nolimits_{i,j\in I} F_{U_{ij}} \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \bigoplus\nolimits_{k\in I} F_{U_k}. \tag{3.2.5}$$

Here we have used the convention in Notation 3.1.2 (iv).

Lemma 3.2.16. — *Let $F \in \mathcal{C}(X)$. There is a natural isomorphism* $\mathrm{coker}(\alpha - \beta) \xrightarrow{\sim} F_U$.

Proof. — Set for short $F_0 := \bigoplus_{k\in I} F_{U_k}$ and $F_1 := \bigoplus_{ij} F_{U_{ij}}$. Let $G \in \mathcal{C}(X)$ and set $\mathcal{H} := \mathcal{H}om_{\mathcal{C}}(F, G)$. The two sequences

$$\begin{gathered}0 \to \Gamma(U; \mathcal{H}) \to \prod_k \Gamma(U_k; \mathcal{H}) \to \prod_{i,j\in I} \Gamma(U_{ij}; \mathcal{H}) \quad \text{and}\\ 0 \to \mathrm{Hom}_{\mathcal{C}(X)}(F_U, G) \to \mathrm{Hom}_{\mathcal{C}(X)}(F_0, G) \to \mathrm{Hom}_{\mathcal{C}(X)}(F_1, G)\end{gathered}$$

are isomorphic. Since the first one is exact, the result follows. q.e.d.

Theorem 3.2.17. — *Let $\mathcal{C}$ be a proper stack. Then $\mathcal{C}$ is a stack.*

Proof. — Let $U = \bigcup_{i\in I} U_i$ be an open covering of $U \subset X$. Let $F_i \in \mathcal{C}(U_i)$ and let $\theta_{ji}\colon F_i|_{U_{ij}} \xrightarrow{\sim} F_j|_{U_{ij}}$, and assume that these isomorphisms satisfy the condition (3.1.1).

Let us introduce the following notations:

$$\begin{array}{lll} f_i : U_i \hookrightarrow X, & f_{i!} = i_{U_i!}, & f_i^{-1} = i_{U_i}^{-1}, \\ f_{ij} : U_{ij} \hookrightarrow X, & f_{ij!} = i_{U_{ij}!}, & f_{ij}^{-1} = i_{U_{ij}}^{-1}, \\ g_{ij} : U_{ij} \hookrightarrow U_i, & g_{ij!} = i_{U_i}^{-1} \circ i_{U_{ij}!}, & g_{ij}^{-1} = \rho_{U_{ij}U_i}. \end{array}$$

and define

$$F_{ij} := g_{ij}^{-1} F_i = F_i|_{U_{ij}}.$$

Using Lemma 3.2.4, we have a morphism $f_{ij!}g_{ij}^{-1} \to f_{i!}$ from which we deduce the morphism

$$\alpha_{ij} : f_{ij!}F_{ij} \to f_{i!}F_i. \tag{3.2.6}$$

Denote by β_{ij} the composition of the morphisms

$$f_{ij!}F_{ij} \xrightarrow[\theta_{ji}]{} f_{ij!}F_{ji} \xrightarrow[\alpha_{ji}]{} f_{j!}F_j.$$

Then:

$$\beta_{ij} : f_{ij!}F_{ij} \to f_{j!}F_j. \tag{3.2.7}$$

We thus get two morphisms in $\mathcal{C}(U)$:

$$\oplus_{i,j\in I} f_{ij!}F_{ij} \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \oplus_k f_{k!}F_k.$$

Set $F := \operatorname{coker}(\alpha - \beta)$ and define $\theta_i \colon F_i \to f_i^{-1}F$ by the natural morphism $f_{i!}F_i \to \oplus_k f_{k!}F_k \to F$. It remains to show that θ_{i_0} is an isomorphism for any $i_0 \in I$.

We may assume from the beginning that $U = U_{i_0}$. Set $F^0 = F_{i_0}$, $F_i^0 = F^0|_{U_i}$, $F_{ij}^0 = F^0|_{U_{ij}}$. The isomorphisms $\theta_{i_0 i} \colon F_i \xrightarrow{\sim} F_i^0$ define isomorphisms

$$\begin{array}{lcl} \Theta^0 = \oplus_k \theta_{i_0 k} & : & \oplus_k f_{k!}F_k \xrightarrow{\sim} \oplus_k f_{k!}F_k^0 \\ \Theta^1 = \oplus_{ij} \theta_{i_0 i} & : & \oplus_{ij} f_{ij!}F_{ij} \xrightarrow{\sim} \oplus_{ij} f_{ij!}F_{ij}^0. \end{array}$$

Consider the diagram below where α^0 and β^0 are defined as in (3.2.5) for F_0, so that $F^0 \simeq \operatorname{coker}(\alpha^0 - \beta^0)$:

$$\begin{tikzcd} \oplus_{ij} f_{ij!}F_{ij} \arrow[r, "\alpha-\beta"] \arrow[d, "\Theta^1", "\sim"'] & \oplus_k f_{k!}F_k \arrow[d, "\Theta^0", "\sim"'] \\ \oplus_{ij} f_{ij!}F_{ij}^0 \arrow[r, "\alpha^0-\beta^0"] & \oplus_k f_{k!}F_k^0. \end{tikzcd}$$

This diagram commutes, thanks to the commutativity of the diagrams

$$\begin{tikzcd} f_{ij!}F_{ij} \arrow[r] \arrow[d, "\theta_{i_0 i}"] & f_{i!}F_i \arrow[d, "\theta_{i_0 i}"] & f_{ij!}F_{ij} \arrow[r, "\theta_{ji}"] \arrow[d, "\theta_{i_0 i}"] & f_{j!}F_j \arrow[d, "\theta_{i_0 j}"] \\ f_{ij!}F_{ij}^0 \arrow[r] & f_{i!}F_i^0 & f_{ij!}F_{ij}^0 \arrow[r] & f_{j!}F_j^0. \end{tikzcd}$$

Hence, F is isomorphic to F^0. q.e.d.

We shall extend classical constructions in sheaf theory to proper stacks.

Let $F \in \mathcal{C}(X)$. For an open subset U, we have already defined the object $F_U \in \mathcal{C}(X)$. For a closed subset S, we define the object F_S by the exact sequence

$$0 \to F_{X \setminus S} \to F \to F_S \to 0. \tag{3.2.8}$$

For any locally closed subset Z, one can find an open subset U and a closed subset S such that $Z = U \cap S$. We shall see below that $(F_U)_S$ depends only on Z, and we shall denote this object by F_Z.

Proposition 3.2.18. — *Let $F \in \mathcal{C}(X)$, and let Z be a locally closed subset of X. Then*

(i) *the functor $\mathcal{C}(X) \ni G \mapsto \Gamma_Z(X; \mathcal{H}om_{\mathcal{C}}(F, G))$ is representable by F_Z,*

(ii) *one has $\mathcal{H}om_{\mathcal{C}}(F_Z, G) \simeq \Gamma_Z \mathcal{H}om_{\mathcal{C}}(F, G)$,*

(iii) *the functor $F \mapsto F_Z$ is exact and commutes with inductive limits,*

(iv) *if Z_1 and Z_2 are two locally closed subsets, then $(F_{Z_1})_{Z_2} \simeq F_{Z_1 \cap Z_2}$,*

(v) *if $Z' \subset Z$ is closed in Z, the sequence $0 \to F_{Z \setminus Z'} \to F_Z \to F_{Z'} \to 0$ is exact,*

(vi) *if U is open, then we have $i_U^{-1}(F_Z) \simeq (i_U^{-1} F)_{Z \cap U}$ i.e. $\mathcal{C}(U) \ni F \mapsto F_{Z \cap U} \in \mathcal{C}(U)$ is a functor of stacks,*

(vii) $\operatorname{supp}(F_Z) \subset \overline{Z} \cap \operatorname{supp} F$.

Proof. — (i) The formula $\Gamma_Z(X; \mathcal{H}om_{\mathcal{C}}(F, G)) \simeq \operatorname{Hom}_{\mathcal{C}(X)}(F_Z, G)$ is true when Z is open. Applying the left exact functor $\operatorname{Hom}_{\mathcal{C}(X)}(\,\cdot\,, G)$ to the exact sequence (3.2.8), we find that this formula remains true when Z is closed. Now let $Z = U \cap S$. Applying Proposition 3.2.14, we get

$$\begin{aligned}
\Gamma_Z(X; \mathcal{H}om_{\mathcal{C}}(F, G)) &\simeq \Gamma_S(X; i_{U*} i_U^{-1} \mathcal{H}om_{\mathcal{C}}(F, G)) \\
&\simeq \Gamma_S(X; \mathcal{H}om_{\mathcal{C}}(F_U, G)) \simeq \operatorname{Hom}_{\mathcal{C}(X)}((F_U)_S, G).
\end{aligned}$$

(ii) The formula is true when Z is open. Since $\mathcal{H}om_{\mathcal{C}}$ is left exact, this formula remains true when Z is closed. Assume $Z = U \cap S$. Then the result follows from $\Gamma_Z \simeq \Gamma_S \circ \Gamma_U$.

(iii) Let us first show that $F \mapsto F_Z$ is exact. The functor $F \mapsto F_U \simeq i_{U!} i_U^{-1} F$ is exact. Hence, we may assume that $Z = S$ is closed. By the definition of F_S in (3.2.8), this functor is exact.

The functor $F \mapsto \Gamma_Z(X; \mathcal{H}om_{\mathcal{C}}(F, G))$ sends inductive limits to projective limits. Therefore, the functor $F \mapsto F_Z$ commutes with inductive limits.

(iv) For any $G \in \mathcal{C}(X)$, one has

$$\begin{aligned}
\operatorname{Hom}_{\mathcal{C}(X)}(F_{Z_1 \cap Z_2}, G) &\simeq \Gamma_{Z_1 \cap Z_2}(X; \mathcal{H}om_{\mathcal{C}}(F, G)) \simeq \Gamma_{Z_1}(X; \Gamma_{Z_2} \mathcal{H}om_{\mathcal{C}}(F, G)) \\
&\simeq \Gamma_{Z_1}(X; \mathcal{H}om_{\mathcal{C}}(F_{Z_2}, G)) \simeq \operatorname{Hom}_{\mathcal{C}(X)}((F_{Z_2}))_{Z_1}, G)).
\end{aligned}$$

(v) We may assume $Z' = S \cap Z$, with S closed in X. Then the result follows from the exact sequence (3.2.8) applied to F_Z and (iv).

(vi) For any $G \in \mathcal{C}(U)$, one has

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}(U)}(i_U^{-1}(F_Z), G) &\simeq \Gamma(U; \mathcal{H}om_{\mathcal{C}}(F_Z, i_{U!}G)) \simeq \Gamma_{Z\cap U}(U; \mathcal{H}om_{\mathcal{C}}(F, i_{U!}G)) \\ &\simeq \Gamma_{Z\cap U}(U; \mathcal{H}om_{\mathcal{C}|_U}(i_U^{-1}F, G)) \simeq \mathrm{Hom}_{\mathcal{C}(U)}((i_U^{-1}F)_{Z\cap U}, G).\end{aligned}$$

Hence we have $i_U^{-1}(F_Z) \simeq (i_U^{-1}F)_{Z\cap U}$.

(vii) For any $G \in \mathcal{C}(X)$, we have $\mathrm{supp}\,\mathcal{H}om_{\mathcal{C}}(F, G) \subset \mathrm{supp}\,F$, and hence

$$\mathrm{supp}\,\mathcal{H}om_{\mathcal{C}}(F_Z, G) = \mathrm{supp}\,\Gamma_Z \mathcal{H}om_{\mathcal{C}}(F, G) \subset \overline{Z} \cap \mathrm{supp}\,F.$$

Hence setting $G = F_Z$, one has $\mathrm{supp}\,F = \mathrm{supp}(\mathcal{E}nd_{\mathcal{C}}(F)) \subset \overline{Z} \cap \mathrm{supp}\,F$. q.e.d.

Proposition 3.2.19. — *Let $F \in \mathcal{C}(X)$. Then*

$$\varinjlim_{U\subset\subset X} F_U \xrightarrow{\sim} F \xrightarrow{\sim} \varprojlim_K F_K.$$

As usual U is open, and K ranges through the family of compact subsets of X.

Proof. — Since $\mathcal{C}$ is a stack it is enough to show that they are isomorphisms on any relatively compact open subset of X. This is obvious because inductive limits and projective limits commute with the restriction functors. q.e.d.

Definition 3.2.20. — Let U be an open subset of X and let $G \in \mathcal{C}(U)$. One sets

$$i_{U*}G = \varprojlim_{K\subset U} i_{U!}(G_K), \ K \text{ compact}.$$

Proposition 3.2.21. — *The functor i_{U*} is a right adjoint to the functor i_U^{-1}.*

Proof. — For $F \in \mathcal{C}(X)$ and $G \in \mathcal{C}(U)$, we have

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}(X)}(F, \varprojlim_{K\subset U} i_{U!}G_K) &\simeq \varprojlim_{K\subset U} \mathrm{Hom}_{\mathcal{C}(X)}(F, i_{U!}G_K) \\ &\simeq \varprojlim_{K\subset U} \Gamma(X; \mathcal{H}om_{\mathcal{C}}(F, i_{U!}G_K)).\end{aligned}$$

Since $\mathrm{supp}(\mathcal{H}om_{\mathcal{C}}(F, i_{U!}G_K)) \subset K \subset U$ we have

$$\begin{aligned}\Gamma(X; \mathcal{H}om_{\mathcal{C}}(F, i_{U!}G_K)) &\simeq \Gamma(U; \mathcal{H}om_{\mathcal{C}}(F, i_{U!}G_K)) \\ &\simeq \mathrm{Hom}_{\mathcal{C}(U)}(i_U^{-1}F, G_K).\end{aligned}$$

Hence we have

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}(X)}(F, \varprojlim_{K\subset U} i_{U!}G_K) &\simeq \varprojlim_{K\subset U} \mathrm{Hom}_{\mathcal{C}(U)}(i_U^{-1}F, G_K) \\ &\simeq \mathrm{Hom}_{\mathcal{C}(U)}(i_U^{-1}F, \varprojlim_{K\subset U} G_K) \simeq \mathrm{Hom}_{\mathcal{C}(U)}(i_U^{-1}F, G).\end{aligned}$$

q.e.d.

Note that the morphisms $G \to G_K$ define the morphism

$$i_{U!}G \to i_{U*}G.$$

If G has compact support in U, this morphism is an isomorphism.

Proposition 3.2.22. — *Let Z be a locally closed subset of X. The functor $F \mapsto F_Z$ admits a right adjoint.*

We shall denote this adjoint functor by $\Gamma_Z(\,\cdot\,)$. Hence, for F and G in $\mathcal{C}(X)$, one has:

$$\mathrm{Hom}_{\mathcal{C}(X)}(G_Z, F) \simeq \mathrm{Hom}_{\mathcal{C}(X)}(G, \Gamma_Z(F)).$$

Proof. — (i) If Z is open, $i_{U*}i_U^{-1}$ is a right adjoint to $i_{U!}i_U^{-1}$.

(ii) If $Z = S$ is closed, set $U = X \setminus S$ and define $\Gamma_S(F)$ by the exact sequence $0 \to \Gamma_S(F) \to F \to \Gamma_U(F)$. Since $\mathrm{Hom}_{\mathcal{C}(X)}(G, \,\cdot\,)$ is left exact, the result follows in this case.

(iii) Now assume that $Z = S \cap U$ with S closed and U open. Set $\Gamma_Z = \Gamma_U \circ \Gamma_S$. q.e.d.

As an immediate consequence of the properties of the functor $F \mapsto F_Z$, one gets:

Corollary 3.2.23. — (i) *The functor Γ_Z is left exact and commutes with projective limits,*

(ii) *one has $\Gamma_{Z_1} \circ \Gamma_{Z_2} \simeq \Gamma_{Z_1 \cap Z_2}$,*

(iii) *if Z' is closed in Z, there is an exact sequence of functors $0 \to \Gamma_{Z'} \to \Gamma_Z \to \Gamma_{Z \setminus Z'}$,*

(iv) *$\Gamma_Z(F)$ represents the contravariant functor $G \to \Gamma_Z(X; \mathcal{H}om_{\mathcal{C}}(G, F))$.*

3.3. Indization of proper stacks

We assume that X is a Hausdorff locally compact space with a countable base of open sets. Let $\mathcal{C}$ be a proper stack of abelian categories on X. We define the full abelian subcategory $\mathcal{C}_c(X)$ of $\mathcal{C}(X)$ by

$$\mathrm{Ob}(\mathcal{C}_c(X)) = \{F \in \mathcal{C}(X)\,;\, \mathrm{supp}\, F \text{ is compact}\}.$$

We denote for short

$$\mathrm{I}\mathcal{C}(X) = \mathrm{Ind}(\mathcal{C}_c(X)).$$

The reason why we consider the ind-objects of $\mathcal{C}_c(X)$ is that the correspondence $U \mapsto \mathrm{I}\mathcal{C}(U)$ is a stack as we shall see later, contrarily to the correspondence $U \mapsto \mathrm{Ind}(\mathcal{C}(U))$ as we shall see now.

Example 3.3.1. — Let $X = \mathbb{R}$, and consider the stack $\mathrm{Mod}(k_X)$ of sheaves of k-modules on X. Let $F = k_X$, $G_n = k_{[n,+\infty[}$, $G = \text{“}\varinjlim_n\text{”}\, G_n$. Then $G|_U = 0$ in $\mathrm{Ind}(\mathrm{Mod}(k_U))$ for any relatively compact open subset U of X. On the other hand, $\mathrm{Hom}_{\mathrm{Ind}(\mathrm{Mod}(k_X))}(k_X, G) \simeq \varinjlim_n \mathrm{Hom}_{k_X}(k_X, G_n) \simeq k$.

Lemma 3.3.2. — *$\mathrm{I}\mathcal{C}(X)$ admits small projective limits.*

Proof. — By the general result in Theorem 1.3.1, it is enough to show that for a small family $\{F_i \in \mathcal{C}_c(X)\}$, its product (which is well-defined in the category $\mathcal{C}_c(X)^\wedge$ of contravariant functors on $\mathcal{C}_c(X)$) belongs to $\mathrm{I}\mathcal{C}(X)$. For $G \in \mathcal{C}_c(X)$ we have

$$\begin{aligned}\prod_i \mathrm{Hom}_{\mathcal{C}(X)}(G, F_i) &\xleftarrow{\sim} \varinjlim_{U \subset\subset X} \mathrm{Hom}_{\mathcal{C}(X)}(G, \prod_i (F_{iU}))\\ &\simeq \mathrm{Hom}_{\mathrm{I}\mathcal{C}(X)}(G, \text{“}\varinjlim_U\text{”} \prod_i (F_{iU})).\end{aligned}$$

Here the first arrow is an isomorphism because $\mathrm{Hom}_{\mathcal{C}(X)}(G, F_{iU}) \to \mathrm{Hom}_{\mathcal{C}(X)}(G, F_i)$ is an isomorphism whenever $\mathrm{supp}(G) \subset U$. Thus the product of the F_i's is represented by $\text{“}\varinjlim_U\text{”} \prod_i (F_{iU})$. q.e.d.

We introduce the functor:

$$\begin{aligned}\iota_X : \mathcal{C}(X) &\to \mathrm{I}\mathcal{C}(X)\\ F &\mapsto (\mathcal{C}_c(X) \ni G \mapsto \mathrm{Hom}_{\mathcal{C}(X)}(G, F)).\end{aligned} \tag{3.3.1}$$

This functor is well-defined by the lemma below.

Lemma 3.3.3. — *For $F \in \mathcal{C}(X)$ one has the isomorphisms:*

$$\iota_X F \simeq \text{“}\varinjlim_{U \subset\subset X}\text{”} F_U \simeq \varprojlim_{K \subset X} F_K\,,$$

where U ranges through the family of relatively compact open subsets of X and K through that of compact subsets.

Proof. — Let $G \in \mathcal{C}_c(X)$. Then:

$$\mathrm{Hom}_{\mathcal{C}(X)}(G, F) \simeq \varinjlim_U \mathrm{Hom}_{\mathcal{C}(X)}(G, F_U) \simeq \varprojlim_K \mathrm{Hom}_{\mathcal{C}(X)}(G, F_K).$$

q.e.d.

Proposition 3.3.4. — *The functor ι_X of* (3.3.1) *is fully faithful, exact, and commutes with $\varprojlim$.*

Proof. — Let F, G belong to $\mathcal{C}(X)$. One has the chain of isomorphisms:

$$\begin{aligned}\mathrm{Hom}_{\mathrm{I}\mathcal{C}(X)}(\iota_X F, \iota_X G) &\simeq \varprojlim_{U \subset\subset X} \mathrm{Hom}_{\mathrm{I}\mathcal{C}(X)}(F_U, \iota_X G)\\ &\simeq \varprojlim_{U \subset\subset X} \mathrm{Hom}_{\mathcal{C}(X)}(F_U, G) \simeq \mathrm{Hom}_{\mathcal{C}(X)}(F, G).\end{aligned}$$

Hence, ι_X is fully faithful. It is exact since the functor $F \mapsto F_U$ is exact as well as "$\varinjlim_U$". Finally let us prove that ι_X commutes with projective limits. Let $\{F_i\}_i$ be a small projective system in $\mathcal{C}(X)$. Then we have for $G \in \mathcal{C}_c(X)$

$$\begin{aligned}\mathrm{Hom}_{\mathrm{I}\mathcal{C}(X)}(G, \iota_X(\varprojlim F_i)) &\simeq \mathrm{Hom}_{\mathcal{C}(X)}(G, \varprojlim F_i) \simeq \varprojlim \mathrm{Hom}_{\mathcal{C}(X)}(G, F_i)\\ &\simeq \varprojlim \mathrm{Hom}_{\mathrm{I}\mathcal{C}(X)}(G, \iota_X(F_i)) \simeq \mathrm{Hom}_{\mathrm{I}\mathcal{C}(X)}(G, \varprojlim(\iota_X F_i)).\end{aligned}$$

q.e.d.

Notation 3.3.5. — In the sequel, we shall identify $\mathcal{C}(X)$ with a full subcategory of $\mathrm{I}\mathcal{C}(X)$ by ι_X and often write F instead of $\iota_X F$.

Definition 3.3.6. — Let U be an open subset of X. We introduce the functor $\rho_{UX} \colon \mathrm{I}\mathcal{C}(X) \to \mathrm{I}\mathcal{C}(U)$, also denoted by $F \mapsto F|_U$, by the formula:

$$\mathrm{Hom}_{\mathrm{I}\mathcal{C}(U)}(G, F|_U) = \mathrm{Hom}_{\mathrm{I}\mathcal{C}(X)}(i_{U!}G, F) \quad \text{for } G \in \mathcal{C}_c(U).$$

This functor is well-defined by the following lemma.

Lemma 3.3.7. — *Let $F =$ "$\varinjlim_i$" F_i, $F_i \in \mathcal{C}_c(X)$. Then:*

$$F|_U \simeq \text{“}\varinjlim_{i, V \subset\subset U}\text{”}\,(F_{iV}|_U) \simeq \varprojlim_{K \subset U} \text{“}\varinjlim_i\text{”}(F_{iK}|_U) \quad V \text{ open, } K \text{ compact.}$$

Proof. — Let $G \in \mathcal{C}_c(U)$. Then $i_{U!}G \in \mathcal{C}_c(X)$ and one has

$$\mathrm{Hom}_{\mathrm{I}\mathcal{C}(X)}(i_{U!}G, F) \simeq \varinjlim_i \mathrm{Hom}_{\mathcal{C}(X)}(i_{U!}G, F_i) \simeq \varinjlim_i \mathrm{Hom}_{\mathcal{C}(U)}(G, F_i|_U).$$

Furthermore we have

$$\varinjlim_i \mathrm{Hom}_{\mathcal{C}(U)}(G, F_i|_U) \simeq \varinjlim_i \varinjlim_{V \subset\subset U} \mathrm{Hom}_{\mathcal{C}(U)}(G, F_{iV}|_U)$$

and

$$\begin{aligned}\varinjlim_i \mathrm{Hom}_{\mathcal{C}(U)}(G, F_i|_U) &\simeq \varprojlim_{K \subset U} \varinjlim_i \mathrm{Hom}_{\mathcal{C}(U)}(G, F_{iK}|_U)\\ &\simeq \mathrm{Hom}_{\mathrm{I}\mathcal{C}(U)}(G, \varprojlim_{K \subset U} \text{“}\varinjlim_i\text{”}\, F_{iK}|_U).\end{aligned}$$

q.e.d.

Lemma 3.3.8. — *For $F \in \mathcal{C}(X)$, we have $\iota_U(F|_U) \simeq (\iota_X F)|_U$.*

Proof. — For any $G \in \mathcal{C}_c(U)$ we have

$$\begin{aligned}\mathrm{Hom}_{\mathrm{I}\mathcal{C}(U)}(G, \iota_U(F|_U)) &\simeq \mathrm{Hom}_{\mathcal{C}(U)}(G, F|_U) \simeq \mathrm{Hom}_{\mathcal{C}(X)}(i_{U!}G, F)\\ &\simeq \mathrm{Hom}_{\mathrm{I}\mathcal{C}(X)}(i_{U!}G, \iota_X F) \simeq \mathrm{Hom}_{\mathrm{I}\mathcal{C}(U)}(G, (\iota_X F)|_U).\end{aligned}$$

q.e.d.

Lemma 3.3.9. — *The correspondence* $U \mapsto \mathrm{I}\mathcal{C}(U)$ *is a prestack of abelian categories and* $\iota_U : \mathcal{C}(U) \to \mathrm{I}\mathcal{C}(U)$ *is a functor of prestacks.*

The proof is obvious.

Proposition 3.3.10. — *The functor* $\rho_{UX} : \mathrm{I}\mathcal{C}(X) \to \mathrm{I}\mathcal{C}(U)$ *is exact and commutes with* "$\varinjlim$" *and* $\varprojlim$.

Proof. — The functor ρ_{UX} commutes with $\varprojlim$ since the functor $\mathrm{Hom}_{\mathrm{I}\mathcal{C}(X)}(i_{U!}G, \cdot\,)$ does. The exactness is obvious, and the commutativity with "$\varinjlim$" follows from Lemma 3.3.7. q.e.d.

Recall that if F and G belong to $\mathrm{I}\mathcal{C}(X)$, the presheaf $U \mapsto \mathrm{Hom}_{\mathrm{I}\mathcal{C}(U)}(F|_U, G|_U)$ is denoted by $\mathcal{H}om_{\mathrm{I}\mathcal{C}}(F, G)$.

Proposition 3.3.11. — *Let* F, G *belong to* $\mathrm{I}\mathcal{C}(X)$.

(i) *The presheaf* $\mathcal{H}om_{\mathrm{I}\mathcal{C}}(F, G)$ *is a sheaf.*
(ii) *For* $F \simeq$ "$\varinjlim\limits_i$" F_i *and* $G \simeq$ "$\varinjlim\limits_j$" G_j *with* F_i, $G_j \in \mathcal{C}_c(X)$, *one has*

$$\mathcal{H}om_{\mathrm{I}\mathcal{C}}(F, G) \simeq \varprojlim_i \varinjlim_j \mathcal{H}om_{\mathcal{C}}(F_i, G_j).$$

Proof. — One has the chain of isomorphisms:

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{I}\mathcal{C}(U)}(F|_U, G|_U) &\simeq \mathrm{Hom}_{\mathrm{I}\mathcal{C}(U)}\big(\text{“}\varinjlim_{i,\, V\subset\subset U}\text{”}\ F_{iV}|_U,\ \text{“}\varinjlim_{j,\, W\subset\subset U}\text{”}\ G_{jW}|_U\big) \\
&\simeq \varprojlim_{i,\, V\subset\subset U} \big(\varinjlim_{j,\, W\subset\subset U} \mathrm{Hom}_{\mathcal{C}(U)}(F_{iV}|_U, G_{jW}|_U)\big) \\
&\simeq \varprojlim_{i,\, V\subset\subset U} \big(\varinjlim_j \mathrm{Hom}_{\mathcal{C}(U)}(F_{iV}|_U, G_j|_U)\big) \\
&\simeq \varprojlim_{i,\, V\subset\subset U} \varinjlim_j \Gamma(V; \mathcal{H}om_{\mathcal{C}}(F_i, G_j)).
\end{aligned}$$

On the other hand we have

$$\begin{aligned}
\varprojlim_{V\subset\subset U} \varinjlim_j \Gamma(V; \mathcal{H}om_{\mathcal{C}}(F_i, G_j)) &\simeq \varprojlim_{V, V'\subset\subset U} \varinjlim_j \Gamma(V; \mathcal{H}om_{\mathcal{C}}(F_i, G_j)_{\overline{V'}}) \\
&\simeq \varprojlim_{V'\subset\subset U} \varinjlim_j \Gamma(U; \mathcal{H}om_{\mathcal{C}}(F_i, G_j)_{\overline{V'}}).
\end{aligned}$$

Finally we obtain

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{I}\mathcal{C}(U)}(F|_U, G|_U) &\simeq \varprojlim_{i,\, V\subset\subset U} \varinjlim_j \Gamma(\overline{V}; \mathcal{H}om_{\mathcal{C}}(F_i, G_j)) \\
&\simeq \varprojlim_{i,\, V\subset\subset U} \Gamma(\overline{V}; \varinjlim_j \mathcal{H}om_{\mathcal{C}}(F_i, G_j)) \\
&\simeq \varprojlim_i \Gamma(U; \varinjlim_j \mathcal{H}om_{\mathcal{C}}(F_i, G_j)) \\
&\simeq \Gamma(U; \varprojlim_i \varinjlim_j \mathcal{H}om_{\mathcal{C}}(F_i, G_j)).
\end{aligned}$$

q.e.d.

Corollary 3.3.12. — *Let $\{F_i\}$ be a small filtrant inductive system in $\mathrm{I}\mathcal{C}(X)$.*

(i) *For any $G \in \mathrm{I}\mathcal{C}(X)$, we have*

$$\mathcal{H}om_{\mathrm{I}\mathcal{C}}(\text{“}\varinjlim_i\text{”}\, F_i, G) \simeq \varprojlim_i \mathcal{H}om_{\mathrm{I}\mathcal{C}}(F_i, G).$$

(ii) *For any $G \in \mathcal{C}(X)$, we have*

$$\mathcal{H}om_{\mathrm{I}\mathcal{C}}(G, \text{“}\varinjlim_i\text{”}\, F_i) \simeq \varinjlim_i \mathcal{H}om_{\mathrm{I}\mathcal{C}}(G, F_i).$$

Note that (ii) does not hold in general for $G \in \mathrm{I}\mathcal{C}(X)$ (see Exercise 3.5).

Lemma 3.3.13. — *The functor i_U^{-1} admits a left adjoint. In other words, denoting this adjoint by $i_{U!!}$, we have for $F \in \mathrm{I}\mathcal{C}(U)$ and $G \in \mathrm{I}\mathcal{C}(X)$:*

$$\mathrm{Hom}_{\mathrm{I}\mathcal{C}(X)}(i_{U!!}F, G) \simeq \mathrm{Hom}_{\mathrm{I}\mathcal{C}(U)}(F, i_U^{-1}G).$$

Moreover, if $F = \text{“}\varinjlim_i\text{”}\, F_i$ with $F_i \in \mathcal{C}_c(U)$, then

$$i_{U!!}\, \text{“}\varinjlim_i\text{”}\, F_i \simeq \text{“}\varinjlim_i\text{”}\, i_{U!}F_i.$$

Proof. — We may assume that $F = \text{“}\varinjlim_i\text{”}\, F_i$ with $F_i \in \mathcal{C}_c(U)$ and $G = \text{“}\varinjlim_j\text{”}\, G_j$ with $G_j \in \mathcal{C}_c(X)$. One defines $i_{U!!}F \in \mathrm{I}\mathcal{C}(X)$ as in the statement. Then

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{I}\mathcal{C}(X)}(i_{U!!}F, G) &\simeq \varprojlim_i \varinjlim_j \mathrm{Hom}_{\mathcal{C}(X)}(i_{U!}F_i, G_j) \simeq \varprojlim_i \varinjlim_j \mathrm{Hom}_{\mathcal{C}(U)}(F_i, i_U^{-1}G_j) \\
&\simeq \varprojlim_i \varinjlim_{j, V\subset\subset U} \mathrm{Hom}_{\mathcal{C}(U)}(F_i, (i_U^{-1}G_j)_V) \simeq \mathrm{Hom}_{\mathrm{I}\mathcal{C}(U)}(F, i_U^{-1}G).
\end{aligned}$$

q.e.d.

One shall be aware that the natural morphism $i_{U!!}\iota_U F \to \iota_X i_{U!}F$ for $F \in \mathcal{C}(U)$ is not an isomorphism in general. Here this morphism is defined by the morphisms

$$i_{U!!}F \simeq \text{“}\varinjlim_{V\subset\subset U}\text{”}\, i_{U!}(F_V) \to \text{“}\varinjlim_{W\subset\subset X}\text{”}\, (i_{U!}F)_W$$

(we have not written the functors ι_U, ι_X for short). This is the reason why we employ the different notation $i_{U!!}$. Of course, if $\operatorname{supp} F$ is compact, then $i_{U!!}F \xrightarrow{\sim} i_{U!}F$.

Theorem 3.3.14. — *The prestack* $\mathrm{I}\mathcal{C}$ *is a proper stack.*

Proof. — By the preceding results, it remains to check properties (v) of Definition 3.2.2 for the functor $i_{U!!}$. Let $F = \text{“}\varinjlim_i\text{”}\, F_i \in \mathrm{I}\mathcal{C}(U)$ with $F_i \in \mathcal{C}_c(U)$. Since i_U^{-1} commutes with inductive limits, one has

$$i_U^{-1} i_{U!!} F \simeq i_U^{-1} \text{“}\varinjlim_i\text{”}\, i_{U!} F_i \simeq \text{“}\varinjlim_i\text{”}\, i_U^{-1} i_{U!} F_i \simeq \text{“}\varinjlim_i\text{”}\, F_i.$$

q.e.d.

Let Z be a locally closed subset of X and let $F \in \mathrm{I}\mathcal{C}(X)$. Since $\mathrm{I}\mathcal{C}$ is a proper stack, the objects F_Z and $\Gamma_Z(F)$ are well-defined in $\mathrm{I}\mathcal{C}(X)$. For $F \in \mathcal{C}(X)$, we have

$$\iota_X(F_Z) \not\simeq (\iota_X(F))_Z \text{ in general, and } \iota_X(\Gamma_Z(F)) \simeq \Gamma_Z(\iota_X F), \tag{3.3.2}$$

as we shall see later. Therefore we shall use another notation ${}_ZF$ instead of F_Z.

Definition 3.3.15. — For a locally closed subset Z and $F \in \mathrm{I}\mathcal{C}(X)$, let ${}_ZF$ be an object of $\mathrm{I}\mathcal{C}(X)$ that represents the functor $G \mapsto \Gamma_Z(X; \mathcal{H}om_{\mathrm{I}\mathcal{C}}(F, G))$.

This functor is representable by Proposition 3.2.18, and the functor $F \mapsto {}_ZF$ shares the properties in the same proposition. In particular we have

$$\begin{aligned} \mathcal{H}om_{\mathrm{I}\mathcal{C}}({}_ZF, G) &\simeq \Gamma_Z \mathcal{H}om_{\mathrm{I}\mathcal{C}}(F, G)), \\ \mathrm{Hom}_{\mathrm{I}\mathcal{C}(X)}({}_ZF, G) &\simeq \Gamma_Z(X; \mathcal{H}om_{\mathrm{I}\mathcal{C}}(F, G)). \end{aligned}$$

Proposition 3.3.16. — *Let Z be a locally closed subset and $F \in \mathrm{I}\mathcal{C}(X)$.*

(i) *If $F = \text{“}\varinjlim_i\text{”}\, F_i$ with $F_i \in \mathcal{C}(X)$ and if $Z = U \cap S$ for a closed subset S and an open subset U, then we have*

$$ {}_ZF \simeq \text{“}\varinjlim_{V \subset\subset U, W \supset S}\text{”}\, (F_i)_{V \cap \overline{W}} \qquad V,\ W \text{ open.}$$

(ii) *For $G \in \mathcal{C}(X)$ and $F \in \mathrm{I}\mathcal{C}(X)$, we have the isomorphism*

$$\mathcal{H}om_{\mathrm{I}\mathcal{C}}(G, F)_Z \simeq \mathcal{H}om_{\mathrm{I}\mathcal{C}}(G, {}_ZF).$$

(iii) *The functor $\mathcal{C}_c(X) \ni G \mapsto \Gamma(X; \mathcal{H}om_{\mathrm{I}\mathcal{C}}(G, F)_Z)$ is representable by ${}_ZF$.*

Proof. — (i) Since $F \simeq \text{“}\varinjlim_{i, W \subset\subset X}\text{”}\, F_{iW}$, we may assume from the beginning $F_i \in \mathcal{C}_c(X)$. Assume first $Z = U$. Then

$$\begin{aligned} {}_UF &\simeq i_{U!!} i_U^{-1} F \simeq i_{U!!} \text{“}\varinjlim_{V \subset\subset U}\text{”}(F_{iV}|_U) \\ &\simeq \text{“}\varinjlim_{V \subset\subset U}\text{”}\, i_{U!}(F_{iV}|_U) \simeq \text{“}\varinjlim_{V \subset\subset U}\text{”}\, F_{iV}. \end{aligned}$$

Hence we have the desired result. Now assume that $Z = S$ is closed. For any $G = \text{“}\varinjlim\text{”}\, G_j \in \mathrm{I}\mathcal{C}(X)$ with $G_j \in \mathcal{C}_c(X)$, we have a chain of isomorphisms

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{I}\mathcal{C}(X)}(\text{“}\varinjlim_{i,\, W\supset S}\text{”}\, F_{i\overline{W}}, G) &\simeq \varprojlim_{i,\, W\supset S} \varinjlim_{j} \mathrm{Hom}_{\mathcal{C}(X)}(F_{i\overline{W}}, G_j)\\
&\simeq \varprojlim_{i,\, W\supset S} \varinjlim_{j} \Gamma_{\overline{W}}(X; \mathcal{H}om_{\mathcal{C}}(F_i, G_j)) \simeq \varprojlim_{i,\, W\supset S} \varinjlim_{j} \Gamma(X; (\mathcal{H}om_{\mathcal{C}}(F_i, G_j))_W)\\
&\simeq \varprojlim_{i,\, W\supset S} \Gamma(X; \varinjlim_{j}(\mathcal{H}om_{\mathcal{C}}(F_i, G_j))_W) \simeq \varprojlim_{i,\, W\supset S} \Gamma(X; (\mathcal{H}om_{\mathrm{I}\mathcal{C}}(F_i, G))_W)\\
&\simeq \varprojlim_{i} \Gamma_S(X; \mathcal{H}om_{\mathrm{I}\mathcal{C}}(F_i, G)) \simeq \Gamma_S(X; \mathcal{H}om_{\mathrm{I}\mathcal{C}}(F, G)).
\end{aligned}$$

The general case follows from ${}_U({}_S F) \simeq {}_{S\cap U}F$.

(ii) First assume that $Z = U$ is open. Then the formula follows from the isomorphisms

$$\begin{aligned}
\mathcal{H}om_{\mathrm{I}\mathcal{C}}(G, F)_U &\simeq \varinjlim_{V'\subset\subset U,\, i} \mathcal{H}om_{\mathrm{I}\mathcal{C}}(G, F_i)_{V'} \simeq \varinjlim_{V, V'\subset\subset U,\, i} \mathcal{H}om_{\mathrm{I}\mathcal{C}}(G, F_{iV})_{V'}\\
&\simeq \varinjlim_{V\subset\subset U,\, i} \mathcal{H}om_{\mathrm{I}\mathcal{C}}(G, F_{iV}) \simeq \mathcal{H}om_{\mathrm{I}\mathcal{C}}(G, \text{“}\varinjlim_{V\subset\subset U,\, i}\text{”}\, F_{iV}).
\end{aligned}$$

Here the first and last isomorphisms follow from Corollary 3.3.12 (ii).

Next, assume that $Z = S$ is closed. In this case, the formula follows from the isomorphisms

$$\begin{aligned}
\mathcal{H}om_{\mathrm{I}\mathcal{C}}(G, {}_S F) &\simeq \varinjlim_{W\supset S,\, i} \mathcal{H}om_{\mathrm{I}\mathcal{C}}(G, F_{i\overline{W}}) \simeq \varinjlim_{W, W'\supset S,\, i} \mathcal{H}om_{\mathrm{I}\mathcal{C}}(G, F_{i\overline{W}})_{\overline{W'}}\\
&\simeq \varinjlim_{W'\supset S,\, i} \mathcal{H}om_{\mathrm{I}\mathcal{C}}(G, F_i)_{\overline{W'}} \simeq \mathcal{H}om_{\mathrm{I}\mathcal{C}}(G, F)_S.
\end{aligned}$$

The general case follows, since ${}_{U\cap S}F \simeq {}_S({}_U F)$.

(iii) follows from (ii) by applying the functor $\Gamma(X; \cdot\,)$. q.e.d.

Lemma 3.3.17. — *Let Z be a locally closed subset and $F \in \mathcal{C}(X)$. Then we have $\iota_X(\Gamma_Z(F)) \simeq \Gamma_Z(\iota_X F)$.*

Proof. — For any $G \in \mathcal{C}_c(X)$, we have

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{I}\mathcal{C}(X)}(G, \iota_X(\Gamma_Z F)) &\simeq \mathrm{Hom}_{\mathcal{C}(X)}(G, \Gamma_Z F) \simeq \Gamma_Z(X; \mathcal{H}om_{\mathcal{C}}(G, F))\\
&\simeq \Gamma_Z(X; \mathcal{H}om_{\mathrm{I}\mathcal{C}}(G, \iota_X F)) \simeq \mathrm{Hom}_{\mathrm{I}\mathcal{C}(X)}(G, \Gamma_Z(\iota_X F)).
\end{aligned}$$

q.e.d.

We shall introduce new functors between the categories $\mathcal{C}(X)$ and $\mathrm{I}\mathcal{C}(X)$.

Definition 3.3.18. — One defines the functor $\alpha_X \colon \mathrm{I}\mathcal{C}(X) \to \mathcal{C}(X)$ by

$$\alpha_X(\text{“}\varinjlim_{i}\text{”}\, F_i) = \varinjlim_{i} F_i.$$

Hence, we have the chain of functors

$$\mathcal{C}(X) \xrightarrow[\iota_X]{} \mathrm{I}\mathcal{C}(X) \xrightarrow[\alpha_X]{} \mathcal{C}(X)$$

and

$$\alpha_X \circ \iota_X \simeq \mathrm{id}_{\mathcal{C}(X)},$$

since for $F \in \mathcal{C}(X)$, one has $F \simeq \varinjlim_{U \subset\subset X} F_U$.

Proposition 3.3.19. — *The functor ι_X is a right adjoint to α_X, that is, for $F \in \mathcal{C}(X)$ and $G \in \mathrm{I}\mathcal{C}(X)$ one has the isomorphism:*

$$\mathrm{Hom}_{\mathcal{C}(X)}(\alpha_X G, F) \simeq \mathrm{Hom}_{\mathrm{I}\mathcal{C}(X)}(G, \iota_X F).$$

Proof. — Let $G \simeq \text{“}\varinjlim_j\text{”}\, G_j$. then

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}(X)}(\alpha_X G, F) &\simeq \mathrm{Hom}_{\mathcal{C}(X)}(\varinjlim_j G_j, F) \simeq \varprojlim_j \mathrm{Hom}_{\mathcal{C}(X)}(G_j, F) \\ &\simeq \varprojlim_j \mathrm{Hom}_{\mathrm{I}\mathcal{C}(X)}(G_j, \iota_X F) \simeq \mathrm{Hom}_{\mathrm{I}\mathcal{C}(X)}(\text{“}\varinjlim_j\text{”}\, G_j, \iota_X F). \end{aligned}$$

q.e.d.

Corollary 3.3.20. — *For any open subset U, we have*

$$\alpha_U \circ i_U^{-1} \simeq i_U^{-1} \circ \alpha_X,$$

or equivalently, $\alpha \colon \mathrm{I}\mathcal{C} \to \mathcal{C}$ is a functor of stacks.

Proof. — If $F = \text{“}\varinjlim_i\text{”}\, F_i$ with $F_i \in \mathcal{C}_c(X)$ then

$$\begin{aligned} \alpha_U i_U^{-1} F &\simeq \alpha_U\Big(\text{“}\varinjlim_{i,\, V \subset\subset U}\text{”}\, (F_{iV})|_U\Big) \simeq \varinjlim_{i,\, V \subset\subset U} (F_{iV})|_U \\ &\simeq \varinjlim_{V \subset\subset U} ((\alpha_X F)_V)|_U \simeq i_U^{-1}(\alpha_X F). \end{aligned}$$

q.e.d.

Corollary 3.3.21. — *For $F \in \mathcal{C}(X)$ and $G \in \mathrm{I}\mathcal{C}(X)$, one has the isomorphism $\mathcal{H}om_{\mathcal{C}}(\alpha_X G, F) \simeq \mathcal{H}om_{\mathrm{I}\mathcal{C}}(G, \iota_X F)$.*

It follows from Proposition 3.3.19 that there is a natural morphism of functors $\mathrm{id} \to \iota_X \circ \alpha_X$. Recall that the natural morphism $\alpha_X \circ \iota_X \to \mathrm{id}$ is an isomorphism.

Proposition 3.3.22. — *The functor α_X is exact and commutes with small $\varinjlim$ and small $\varprojlim$.*

Proof. — (i) Let $0 \to F' \to F \to F'' \to 0$ be an exact sequence in $\mathrm{I}\mathcal{C}(X)$. There exists an inductive system of exact sequences $0 \to F'_i \to F_i \to F''_i \to 0$, whose "$\varinjlim$" is the above exact sequence. Then the sequence $0 \to \varinjlim_i F'_i \to \varinjlim_i F_i \to \varinjlim_i F''_i \to 0$ is exact.

(ii) Since α_X admits a right adjoint, it commutes with $\varinjlim$.

(iii) Let us show that α_X commutes with $\varprojlim$. Since this functor is exact, it is enough to show that it commutes with products. Let A be a small set and let $F_a \in \mathrm{I}\mathcal{C}(X), a \in A$. For each $a \in A$, there exists a small filtrant category I_a such that $F_a \simeq$ "$\varinjlim_{i \in I_a}$" $F_{a,i}$ with $F_{a,i} \in \mathcal{C}(X)$. Define the small set

$$B := \{\varphi \colon A \to \bigsqcup I_a; \varphi(a) \in I_a\}.$$

Then $\prod_{a\in A}(\text{"}\varinjlim_{i\in I_a}\text{"} F_{a,i}) \simeq \text{"}\varinjlim_{\varphi\in B}\text{"} \prod_{a\in A} \iota_X(F_{a,i}) \simeq \text{"}\varinjlim_{U\subset\subset X, \varphi\in B}\text{"} (\prod_{a\in A} F_{a,\varphi(a)})_U$ holds, and we obtain the chain of isomorphisms

$$\alpha_X(\prod_{a\in A} F_a) \simeq \varinjlim_{U\subset\subset X, \varphi\in B} (\prod_{a\in A} F_{a,\varphi(a)})_U \simeq \prod_{a\in A} \varinjlim_{i\in I_a} F_{a,i} \simeq \prod_{a\in A} \alpha_X(F_a).$$

q.e.d.

Corollary 3.3.23. — *Assume that $F \in \mathcal{C}(X)$ is injective. Then $\iota_X F \in \mathrm{I}\mathcal{C}(X)$ is injective.*

Proof. — By Proposition 3.3.22, the functor

$$G \mapsto \mathrm{Hom}_{\mathrm{I}\mathcal{C}(X)}(G, \iota_X F) = \mathrm{Hom}_{\mathcal{C}(X)}(\alpha_X(G), F)$$

is exact. q.e.d.

In order to construct a left adjoint to the functor α_X, we need a hypothesis.

Definition 3.3.24. — (i) Let $F \in \mathcal{C}(X)$. We say that F is light on $U \subset X$ if, for any small filtrant inductive system $i \mapsto G_i$ in $\mathcal{C}(X)$, the natural morphism $\varinjlim_i \mathcal{H}om_{\mathcal{C}}(F, G_i)|_U \to \mathcal{H}om_{\mathcal{C}}(F, \varinjlim_i G_i)|_U$ is an isomorphism. This condition is equivalent to saying that $\mathcal{H}om_{\mathrm{I}\mathcal{C}}(F, G)|_U \to \mathcal{H}om_{\mathcal{C}}(F, \alpha_X G)|_U$ is an isomorphism for any $G \in \mathrm{I}\mathcal{C}(X)$.

(ii) We denote by $\mathcal{L}(X)$ the full additive subcategory of $\mathcal{C}(X)$ whose objects are the direct sums $\bigoplus_{i\in I}(F_i)_{U_i}$ with I small and F_i light on U_i, U_i open in X.

(iii) If $\mathcal{L}(X)$ is generating in $\mathcal{C}(X)$ (i.e. for any $F \in \mathcal{C}(X)$, there exists an epimorphism $G \twoheadrightarrow F$ with $G \in \mathcal{L}(X)$), we say that $\mathcal{C}(X)$ has enough light objects.

Example 3.3.25. — If $\mathcal{A}$ is a sheaf of rings on X and $\mathcal{C}(X) = \mathrm{Mod}(\mathcal{A})$, the sheaf $\mathcal{A}_U$ is light on U, and the category $\mathcal{C}(X)$ has enough light objects.

Theorem 3.3.26. — *Assume that $\mathcal{C}(X)$ has enough light objects. Then the functor α_X admits a left adjoint β_X:*

$$\mathrm{Hom}_{\mathrm{I}\mathcal{C}(X)}(\beta_X F, G) \simeq \mathrm{Hom}_{\mathcal{C}(X)}(F, \alpha_X G).$$

Moreover, if F is light on U, then $\beta_X(F_U) \simeq {}_U F \simeq$ "$\varinjlim_{V \subset\subset U}$" F_V.

Proof. — It is equivalent to saying that the object $\beta_X F \in \mathrm{I}\mathcal{C}(X)^\vee$ defined by

$$(\beta_X F)(G) = \mathrm{Hom}_{\mathcal{C}(X)}(F, \alpha_X G),\ G \in \mathrm{I}\mathcal{C}(X)$$

is representable.

Now remark that if $F_1 \to F_0 \to F \to 0$ is an exact sequence and $\beta_X F_1$, $\beta_X F_0$ are representable, then $\beta_X F$ is representable by $\mathrm{coker}(\beta_X F_1 \to \beta_X F_0)$. Similarly, if $\{F_i\}$ is a small family of objects of $\mathcal{C}(X)$ such that each $\beta_X F_i$ is representable, then $\beta_X(\oplus_i F_i)$ is representable by "$\bigoplus_i$" $\beta_X F_i$. Hence, it is enough to prove the result for F_U when F is light on U. For any $G \in \mathrm{I}\mathcal{C}(X)$, one has

$$\begin{aligned} \mathrm{Hom}_{\mathrm{I}\mathcal{C}(X)}({}_U F, G) &\simeq \Gamma(U; \mathcal{H}om_{\mathrm{I}\mathcal{C}}(F, G)) \\ &\xrightarrow{\sim} \Gamma(U; \mathcal{H}om_{\mathcal{C}}(F, \alpha_X G)) \simeq \mathrm{Hom}_{\mathrm{I}\mathcal{C}(X)}(F_U, \alpha_X G)\,. \end{aligned}$$

Here the first isomorphism follows from by Definition 3.3.15 and the second isomorphism from the fact that F is light on U. q.e.d.

Until the end of this section, we assume that $\mathcal{C}(X)$ has enough light objects. The functor β_X has the following properties.

Proposition 3.3.27. — (i) *β_X commutes with $\varinjlim$ (i.e. $\beta_X \circ \varinjlim \simeq$ "$\varinjlim$" $\circ \beta_X$),*

(ii) *β_X is right exact,*
(iii) *$\alpha_X \circ \beta_X \simeq \mathrm{id}$,*
(iv) *β_X is fully faithful.*

Proof. — (i) and (ii) follows from the fact that β_X has a right adjoint.

(iii) For F and G in $\mathcal{C}(X)$, we have

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}(X)}(\alpha_X \beta_X F, G) &\simeq \mathrm{Hom}_{\mathrm{I}\mathcal{C}(X)}(\beta_X F, \iota_X G) \\ &\simeq \mathrm{Hom}_{\mathcal{C}(X)}(F, \alpha_X \iota_X G) \simeq \mathrm{Hom}_{\mathcal{C}(X)}(F, G). \end{aligned}$$

(iv) For F and G in $\mathcal{C}(X)$, we have

$$\mathrm{Hom}_{\mathcal{C}(X)}(F, G) \simeq \mathrm{Hom}_{\mathcal{C}(X)}(F, \alpha_X \circ \beta_X G) \simeq \mathrm{Hom}_{\mathrm{I}\mathcal{C}(X)}(\beta_X F, \beta_X G).$$

q.e.d.

Lemma 3.3.28. — *For any open subset U, one has $i_U^{-1}\beta_X \simeq \beta_U i_U^{-1}$, that is, $\beta \colon \mathcal{C} \to \mathrm{I}\mathcal{C}$ is a functor of stacks.*

Proof. — By Corollary 3.3.20 we have $i_U^{-1} \simeq i_U^{-1}\alpha_X\beta_X \simeq \alpha_U i_U^{-1}\beta_X$. By adjunction we obtain $\beta_U i_U^{-1} \to i_U^{-1}\beta_X$.

Let us show that $\beta_U i_U^{-1}F \to i_U^{-1}\beta_X F$ is an isomorphism for any $F \in \mathcal{C}(X)$. Since both sides are right exact and commute with inductive limits, it is enough to prove the result when $F = L_V$ for L light on V. In this case the assertion follows from Proposition 3.2.18 (vi). q.e.d.

This lemma along with Corollary 3.3.20 implies the following result.

Lemma 3.3.29. — *For $F \in \mathcal{C}(X)$ and $G \in \mathrm{I}\mathcal{C}(X)$, we have*

$$\mathcal{H}om_{\mathrm{I}\mathcal{C}}(\beta_X F, G) \simeq \mathcal{H}om_{\mathcal{C}}(F, \alpha_X G).$$

Remark 3.3.30. — The morphism of functors $\beta_X \circ \alpha_X \to \mathrm{id}$ defines

$$\beta_X \simeq \beta_X \circ \alpha_X \circ \iota_X \to \iota_X.$$

This morphism is not an isomorphism in general, even when $X = \{pt\}$ and $\mathcal{C} = \mathrm{Mod}(k_X)$. Indeed, if F is a k-vector space, $\beta_X F = \text{“}\varinjlim_i\text{”}\, F_i$, where F_i ranges through the family of finite-dimensional subspaces of F.

Exercises to Chapter 3

Exercise 3.1. — Let $X = \bigcup_{i\in I} U_i$ be an open covering and let $\mathcal{C}_i$ be a stack on U_i. Assume to be given equivalences of stacks $\varphi_{ij}\colon \mathcal{C}_i|_{U_{ij}} \to \mathcal{C}_j|_{U_{ij}}$ and isomorphisms of functors $\lambda_{ijk}\colon \varphi_{ij} \circ \varphi_{jk} \xrightarrow{\sim} \varphi_{ik}$ from $\mathcal{C}_i|_{U_{ijk}}$ to $\mathcal{C}_k|_{U_{ijk}}$. Assume that for any i, j, k, l, the diagram below commutes:

$$\begin{tikzcd}
\varphi_{ij} \circ \varphi_{jk} \circ \varphi_{kl} \arrow[r, "\lambda_{jkl}"] \arrow[d, "\lambda_{ijk}"] & \varphi_{ij} \circ \varphi_{jl} \arrow[d, "\lambda_{ijl}"] \\
\varphi_{ik} \circ \varphi_{kl} \arrow[r, "\lambda_{ikl}"] & \varphi_{il}.
\end{tikzcd}$$

Prove that there exists a stack $\mathcal{C}$ on X and equivalence of stacks $\varphi_i\colon \mathcal{C}|_{U_i} \to \mathcal{C}_i$ satisfying the natural conditions (i.e. $\varphi_{ij} \circ \varphi_j \simeq \varphi_i$, etc.).
Prove that $\mathcal{C}$ is unique up to equivalence.

Exercise 3.2. — Give an example of $F \in \mathcal{C}(X)$ such that $\beta_X F \to \iota_X F$ is not a monomorphism (resp. epimorphism).

Exercise 3.3. — Let $\mathcal{C}$ be a proper stack on X, and let $K' \underset{\varphi}{\to} K \underset{\psi}{\to} K''$ be a complex in $\mathrm{I}\mathcal{C}(X)$. Prove that this complex is exact as soon as the sequence of sheaves

$$\mathcal{H}om_{\mathrm{I}\mathcal{C}}(F, K') \to \mathcal{H}om_{\mathrm{I}\mathcal{C}}(F, K) \to \mathcal{H}om_{\mathrm{I}\mathcal{C}}(F, K'')$$

is exact for any $F \in \mathcal{C}_c(X)$.

(Hint: prove that $\mathcal{H}om_{\mathrm{I}\mathcal{C}}(F, \mathrm{im}\,\varphi) \to \mathcal{H}om_{\mathrm{I}\mathcal{C}}(F, \ker\psi)$ is an isomorphism.)

Exercise 3.4. — Let $\mathcal{C}$ denote a proper stack. By definition, $\mathrm{End}\,(\mathrm{id}_{\mathcal{C}})$ is the ring of endomorphisms of the functor $\mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$. Clearly, the presheaf $U \mapsto \mathrm{End}\,(\mathrm{id}_{\mathcal{C}|_U})$ is a sheaf of commutative rings. One denotes this sheaf by $\mathcal{E}nd(\mathrm{id}_{\mathcal{C}})$.

Let $\mathcal{B}$ be a sheaf of commutative rings on X. Let us say that $\mathcal{C}$ is a $\mathcal{B}$-stack if one is given a morphism of rings $\mathcal{B} \to \mathcal{E}nd(\mathrm{id}_{\mathcal{C}})$.

Let $\mathcal{A}$ be a sheaf of (not necessarily commutative) rings. One defines the stack $\mathrm{Mod}(\mathcal{A}, \mathcal{C})$ by

$$\begin{aligned}
\mathrm{Ob}(\mathrm{Mod}(\mathcal{A},\mathcal{C})(U)) = \{(F, \xi_F)\,;\, & F \in \mathcal{C}(U), \\
& \xi_F : \mathcal{A}|_U \to \mathcal{E}nd_{\mathcal{C}|_U}(F) \text{ is a ring morphism}\}, \\
\mathrm{Hom}_{\mathrm{Mod}(\mathcal{A},\mathcal{C})(U)}((F,\xi_F),(G,\xi_G)) = \{ & f \in \mathrm{Hom}_{\mathcal{C}(U)}(F,G)\,;\, \xi_G(a) \circ f = f \circ \xi_F(a) \\
& \text{for all } V \subset U, a \in \mathcal{A}(V)\}.
\end{aligned}$$

(i) Prove that $\mathrm{Mod}(\mathcal{A}, \mathcal{C})$ is a proper stack of abelian categories, and if $\mathcal{A}$ is commutative, it is an $\mathcal{A}$-stack.

(ii) Prove that the natural functor of abelian categories $\mathrm{Mod}(\mathcal{A}, \mathcal{C})(X) \to \mathcal{C}(X)$ is exact and faithful.

(iii) Let $\mathcal{A}$ be a sheaf of rings on X. Show that the stack $\mathrm{Mod}(\mathcal{A}, \mathrm{Mod}(\mathbb{Z}_X))$ is equivalent to $\mathrm{Mod}(\mathcal{A})$.

Exercise 3.5. — Let $F \in \mathrm{I}\mathcal{C}(X)$.

(i) Prove that the functor $\mathrm{I}\mathcal{C}(X) \ni G \mapsto \mathcal{H}om_{\mathrm{I}\mathcal{C}}(F,G) \in \mathrm{Mod}(\mathbb{Z}_X)$ commutes with filtrant inductive limits if and only if $F \in \mathcal{C}(X)$.

(ii) Prove that the functor $\mathrm{I}\mathcal{C}(X) \ni G \mapsto \mathrm{Hom}_{\mathrm{I}\mathcal{C}}(F,G) \in \mathrm{Mod}(\mathbb{Z})$ commutes with filtrant inductive limits if and only if $F \in \mathcal{C}_c(X)$.

Exercise 3.6. — Let $\mathcal{C}$ and $\mathcal{C}'$ be two stacks on X. For an open subset U of X, denote by $\mathcal{S}(U)$ the category of functors of stacks on U from $\mathcal{C}|_U$ to $\mathcal{C}'|_U$. Prove that $U \mapsto \mathcal{S}(U)$ is a stack on X.

CHAPTER 4

IND-SHEAVES

As in the preceding chapter, we work in a given universe $\mathcal{U}$. Hence, otherwise specified, a category means a $\mathcal{U}$-category and all sets are $\mathcal{U}$-small. Moreover, all topological spaces are assumed to be Hausdorff, locally compact, and with a countable base of open sets. We denote by k a commutative ring.

4.1. The stack of ind-sheaves

In this chapter, $\mathcal{A}$ will be a sheaf of k-algebras on a topological space X (with the image of k_X contained in the center of $\mathcal{A}$).

We denote by $\mathrm{Mod}(\mathcal{A})$ the abelian category of $\mathcal{A}$-modules, and by $\mathrm{Mod}^c(\mathcal{A})$ its full abelian subcategory consisting of sheaves with compact support. Recall that $\mathrm{Mod}(\mathcal{A})$ and $\mathrm{Mod}^c(\mathcal{A})$ have enough injectives and also have systems of strict generators. Moreover, the injective objects of $\mathrm{Mod}^c(\mathcal{A})$ are injective in $\mathrm{Mod}(\mathcal{A})$. Also recall that $U \mapsto \mathrm{Mod}(\mathcal{A}|_U)$ is a proper stack.

Notation 4.1.1. — We shall use the functors $\mathcal{H}om_{\mathcal{A}}$ and $\otimes_{\mathcal{A}}$ on $\mathrm{Mod}(\mathcal{A})$. When $\mathcal{A} = k_X$ we shall simply denote these functors by $\mathcal{H}om$ and $\otimes$. Moreover, we write as usual $\mathrm{Hom}_{\mathcal{A}}$ instead of $\mathrm{Hom}_{\mathrm{Mod}(\mathcal{A})}$.

Definition 4.1.2. — We call an object of $\mathrm{Ind}(\mathrm{Mod}^c(\mathcal{A}))$ an ind-sheaf of $\mathcal{A}$-modules on X.

We set for short:

$$\mathrm{I}(\mathcal{A}) := \mathrm{Ind}(\mathrm{Mod}^c(\mathcal{A})).$$

Hence, denoting by $\mathcal{C}$ the proper stack $U \mapsto \mathrm{Mod}(\mathcal{A}|_U)$, $\mathrm{I}(\mathcal{A})$ is nothing but the category $\mathrm{I}\mathcal{C}(X)$, and we may apply the preceding results. In particular,

$$U \mapsto \mathrm{I}(\mathcal{A}|_U) \text{ is a proper stack.}$$

Let us denote by $\mathcal{F}(\mathcal{A})$ the full additive subcategory of $\mathrm{Mod}(\mathcal{A})$ consisting of objects isomorphic to $\bigoplus_{i\in I}\mathcal{A}_{U_i}$ with I small and U_i open in X. Then $\mathcal{F}(\mathcal{A})$ is generating. Since $\mathcal{A}$ is light on X, $\mathcal{F}(\mathcal{A})$ is contained in $\mathcal{L}(X)$ (see Definition 3.3.24). Applying the constructions in §3.3, we get the functors

$$\begin{aligned}\iota_X &: \mathrm{Mod}(\mathcal{A}) \to \mathrm{I}(\mathcal{A}),\\ \alpha_X &: \mathrm{I}(\mathcal{A}) \to \mathrm{Mod}(\mathcal{A}),\\ \beta_{\mathcal{A}} &: \mathrm{Mod}(\mathcal{A}) \to \mathrm{I}(\mathcal{A}).\end{aligned}$$

See Remark 4.1.4 for the reason why we write $\beta_{\mathcal{A}}$ instead of β_X as in Chapter 3.

These functors satisfy the following properties:

(i) ι_X is exact, fully faithful, and commutes with $\varprojlim$,

(ii) α_X is exact and commutes with $\varprojlim$ and $\varinjlim$,

(iii) $\beta_{\mathcal{A}}$ is right exact, fully faithful and commutes with $\varinjlim$,

(iv) α_X is left adjoint to ι_X,

(v) α_X is right adjoint to $\beta_{\mathcal{A}}$,

(vi) $\alpha_X\circ\iota_X \simeq \mathrm{id}_{\mathrm{Mod}(\mathcal{A})}$ and $\alpha_X\circ\beta_{\mathcal{A}} \simeq \mathrm{id}_{\mathrm{Mod}(\mathcal{A})}$.

We have also defined the functor $F \mapsto F|_U$ from $\mathrm{I}(\mathcal{A})$ to $\mathrm{I}(\mathcal{A}|_U)$, as well as the functor $(F,G) \mapsto \mathcal{H}om_{\mathrm{I}(\mathcal{A})}(F,G)$ from $\mathrm{I}(\mathcal{A})^{\mathrm{op}} \times \mathrm{I}(\mathcal{A})$ to $\mathrm{Mod}(k_X)$. Recall that if $i \mapsto F_i$ is a small filtrant inductive system with $F_i \in \mathrm{Mod}^c(\mathcal{A})$ and $F \simeq \text{“}\varinjlim_i\text{”}\, F_i$, then

$$F|_U \simeq \text{“}\varinjlim_{i,V\subset\subset U}\text{”}\,(F_{iV}|_U),$$

and if $j \mapsto G_j$ is a small filtrant inductive system with $G_j \in \mathrm{Mod}^c(\mathcal{A})$, then

$$\mathcal{H}om_{\mathrm{I}(\mathcal{A})}(\text{“}\varinjlim_i\text{”}\,F_i, \text{“}\varinjlim_j\text{”}\,G_j) \simeq \varprojlim_i \varinjlim_j \mathcal{H}om_{\mathcal{A}}(F_i, G_j).$$

Note that we have

$$\beta_{\mathcal{A}}(\mathcal{A}_U) \simeq {}_U\mathcal{A} \simeq \text{“}\varinjlim_{V\subset\subset U}\text{”}\,\mathcal{A}_V\,.$$

Notation 4.1.3. — (i) If $\mathcal{A} = k_X$, we shall often simply denote by $\mathcal{H}om$ the functor $\mathcal{H}om_{\mathrm{I}(k_X)}$.
(ii) We shall often identify a sheaf $F \in \mathrm{Mod}(\mathcal{A})$ and its image $\iota_X F \in \mathrm{I}(\mathcal{A})$, and we shall not write ι_X.
(iii) When $\mathcal{A} = k_X$, we write β_X instead of β_{k_X}.

Remark 4.1.4. — Denote for a while by *for* one of the natural functors $\mathrm{Mod}(\mathcal{A}) \to \mathrm{Mod}(k_X)$ or $\mathrm{I}(\mathcal{A}) \to \mathrm{I}(k_X)$. Then, clearly, α_X and ι_X commute with *for*. One shall be aware that *for* $\circ\,\beta_{\mathcal{A}} \neq \beta_X \circ$ *for*.

The functor β_X. — In Chapter 3, we have introduced the functor β_X. This functor has no counterpart in classical sheaf theory, and we shall study here some of its properties.

Proposition 4.1.5. — *Let F and G be in* $\mathrm{Mod}(\mathcal{A})$.

(i) *There is a canonical morphism*

$$\mathcal{H}om_{\mathcal{A}}(G,\mathcal{A})\otimes_{\mathcal{A}}F\to\mathcal{H}om_{\mathrm{I}(\mathcal{A})}(G,\beta_{\mathcal{A}}F). \tag{4.1.1}$$

(ii) *The canonical morphism $\mathcal{H}om_{\mathcal{A}}(G,\mathcal{A})\otimes_{\mathcal{A}}F\to\mathcal{H}om_{\mathcal{A}}(G,F)$ factors as*

$$\mathcal{H}om_{\mathcal{A}}(G,\mathcal{A})\otimes_{\mathcal{A}}F\to\mathcal{H}om_{\mathrm{I}(\mathcal{A})}(G,\beta_{\mathcal{A}}F)\to\mathcal{H}om_{\mathcal{A}}(G,F), \tag{4.1.2}$$

where the first arrow is given by (4.1.1) *and the second arrow is induced by $\beta_{\mathcal{A}}F\to F$.*

(iii) *For $F\in\mathcal{F}(\mathcal{A})$, the morphism* (4.1.1) *is an isomorphism.*

Proof. — (i) (ii) Let U be an open subset of X. For $\varphi\in\mathrm{Hom}_{\mathcal{A}|_U}(G|_U,\mathcal{A}|_U)$ and $s\in F(U)\simeq\mathrm{Hom}_{\mathcal{A}|_U}(\mathcal{A}|_U,F|_U)$, we define $\psi\in\mathrm{Hom}_{\mathrm{I}(\mathcal{A}|_U)}(G|_U,\beta_{\mathcal{A}}(F)|_U)$ by

$$G|_U\xrightarrow{\varphi}\mathcal{A}|_U\xleftarrow{\sim}\beta_{\mathcal{A}}\mathcal{A}|_U\xrightarrow{\beta_{\mathcal{A}}(s)}\beta_{\mathcal{A}}(F)|_U.$$

(Here, we write $\beta_{\mathcal{A}}$ instead of $\beta_{\mathcal{A}|_U}$ for short in view of Lemma 3.3.28.)

Then $\varphi\otimes s\mapsto\psi$ defines $\mathrm{Hom}_{\mathcal{A}|_U}(G|_U,\mathcal{A}|_U)\otimes_{\mathcal{A}(U)}F(U)\to\mathrm{Hom}_{\mathrm{I}(\mathcal{A}|_U)}(G|_U,\beta_{\mathcal{A}}F|_U)$.

(iii) Since both sides of (4.1.1) commute with direct sums with respect to F, we may assume that $F=\mathcal{A}_U$ for some open subset U. In this case the assertion follows from Proposition 3.3.16. q.e.d.

Lemma 4.1.6. — *Let $F\in\mathcal{F}(\mathcal{A})$, $G\in\mathrm{Mod}(\mathcal{A})$.*

(i) *Let $\psi\in\mathrm{Hom}_{\mathrm{I}(\mathcal{A})}(G,\beta_{\mathcal{A}}F)$. For each $x\in X$, there exist an open neighborhood U of x, $g\colon G|_U\to\mathcal{A}^n|_U$ and $f\colon\mathcal{A}^n|_U\to F|_U$ such that $\psi|_U$ factorizes as*

$$G|_U\underset{g}{\to}\mathcal{A}^n|_U\xleftarrow{\sim}\beta_{\mathcal{A}}(\mathcal{A}^n|_U)\xrightarrow[\beta_{\mathcal{A}}(f)]{}\beta_{\mathcal{A}}(F|_U).$$

(ii) *Assume that $g\colon G\to\mathcal{A}^n$ and $f\colon\mathcal{A}^n\to F$ satisfy $\beta_{\mathcal{A}}(f)\circ g=0$ in $\mathrm{I}(\mathcal{A})$. Then for any $x\in X$, there exist an open neighborhood V of x and a commutative diagram*

$$\begin{tikzcd} & \mathcal{A}_V^k \arrow[r, "b"] \arrow[d, "a"] & \mathcal{A}_V^m \arrow[d, "h"] \\ G_V \arrow[ur, "\varphi"] \arrow[r, "g"'] & \mathcal{A}_V^n \arrow[r, "f"'] & F_V \end{tikzcd}$$

such that $b\circ\varphi=0$.

Proof. — (i) follows from Proposition 4.1.5 (iii).

(ii) Let $\sum_i g_i \otimes f_i \in \mathcal{H}om_{\mathcal{A}}(G, \mathcal{A})_x \otimes_{\mathcal{A}_x} F_x$, and assume this section vanishes. There exist $\varphi_j \in \mathcal{H}om_{\mathcal{A}}(G, \mathcal{A})_x$, $h_k \in F_x$ and $a_{ij}, b_{jk} \in \mathcal{A}_x$ with

$$\begin{aligned} g_i &= \sum_j \varphi_j a_{ji} \ \text{ for all } i, \\ \sum_k b_{jk} h_k &= \sum_i a_{ji} f_i \ \text{ for all } j, \\ \sum_j \varphi_j b_{jk} &= 0 \ \text{ for all } k. \end{aligned}$$

q.e.d.

Proposition 4.1.7. — *Let F be an $\mathcal{A}$-module.*

(i) *$\beta_{\mathcal{A}} F \to F$ is an epimorphism if and only if F is locally of finite type.*
(ii) *$\beta_{\mathcal{A}} F \to F$ is an isomorphism if and only if F is locally of finite presentation.*

Proof. — (i) (a) Assume that $\beta_{\mathcal{A}} F \to F$ is an epimorphism. Let us choose an epimorphism $L = \oplus_{i \in I} \mathcal{A}_{U_i} \twoheadrightarrow F$. Let $s_i \in F(U_i)$ be the image of $1_{U_i} \in \Gamma(U_i; \mathcal{A}_{U_i})$. Since $\beta_{\mathcal{A}} L \to \beta_{\mathcal{A}} F$ and $\beta_{\mathcal{A}} F \to F$ are epimorphisms, $\beta_{\mathcal{A}} L \simeq \mathop{\text{“}\bigoplus\text{”}}_{i \in I} {}_{U_i}\mathcal{A} \to F$ is an epimorphism. Hence there locally exist a finite subset J of I and $V_i \subset\subset U_i$ such that $\oplus_{i \in J} \mathcal{A}_{V_i} \to F$ is an epimorphism. Hence $F = \sum_{i \in J} \mathcal{A}_{V_i} s_i$. For any $x \in X$, set $J(x) = \{i \in J\,;\, x \in U_i\}$. Then $W = \bigcap_{i \in J(x)} U_i \setminus (\bigcup_{i \in J \setminus J(x)} \overline{V_i})$ is an open neighborhood of x. Since $V_i \cap W = \varnothing$ for any $i \in J \setminus J(x)$, we have $F|_W = \sum_{i \in J(x)} (\mathcal{A}|_W) s_i$.

(i) (b) Conversely, assume that F is locally of finite type. There locally exists an epimorphism $g\colon \mathcal{A}^n \twoheadrightarrow F$. This morphism factors through $\mathcal{A}^n \xleftarrow{\sim} \beta_{\mathcal{A}}(\mathcal{A}^n) \xrightarrow[\beta_{\mathcal{A}}(g)]{} \beta_{\mathcal{A}} F \to F$. Hence $\beta_{\mathcal{A}} F \to F$ is an epimorphism.

(ii) (a) Assume that $\beta_{\mathcal{A}} F \to F$ is an isomorphism. By (i), F is locally of finite type. Hence there locally exists an exact sequence $0 \to N \to \mathcal{A}^n \to F \to 0$. Consider the commutative exact diagram

$$\begin{tikzcd} & \beta_{\mathcal{A}} N \arrow[r] \arrow[d] & \beta_{\mathcal{A}} \mathcal{A}^n \arrow[r] \arrow[d, "\sim"] & \beta_{\mathcal{A}} F \arrow[r] \arrow[d, "\sim"] & 0 \\ 0 \arrow[r] & N \arrow[r] & \mathcal{A}^n \arrow[r] & F \arrow[r] & 0. \end{tikzcd}$$

It shows that $\beta_{\mathcal{A}} N \to N$ is an epimorphism, and hence N is locally of finite type. This implies that F is locally of finite presentation.

(ii) (b) Conversely, assume that F is locally of finite presentation and let us show that $\beta_{\mathcal{A}} F \to F$ is an isomorphism. There locally exists an exact sequence $\mathcal{A}^m \to \mathcal{A}^n \to F \to 0$. Then the assertion is obvious by the following diagram with exact rows

$$\begin{tikzcd}
\beta_{\mathcal{A}}\mathcal{A}^m \arrow[r] \arrow[d, "\sim"] & \beta_{\mathcal{A}}\mathcal{A}^n \arrow[r] \arrow[d, "\sim"] & \beta_{\mathcal{A}}F \arrow[r] \arrow[d] & 0 \\
\mathcal{A}^m \arrow[r] & \mathcal{A}^n \arrow[r] & F \arrow[r] & 0.
\end{tikzcd}$$

q.e.d.

Corollary 4.1.8. — *$F \in \mathrm{Mod}(\mathcal{A})$ is light if and only if F is locally of finite presentation.*

Proposition 4.1.9. — *The following two conditions are equivalent:*

(i) *the functor $\beta_{\mathcal{A}} \colon \mathrm{Mod}(\mathcal{A}) \to \mathrm{I}(\mathcal{A})$ is exact,*
(ii) *the sheaf of rings $\mathcal{A}$ is left coherent.*

Proof. — (i) $\Rightarrow$ (ii). Consider an exact sequence $0 \to F \to \mathcal{A}^m \to \mathcal{A}$. Applying the exact functor $\beta_{\mathcal{A}}$, we find the isomorphism $\beta_{\mathcal{A}}F \xrightarrow{\sim} F$. Then Proposition 4.1.7 implies that F is locally of finite type.

In order to prove (ii) $\Rightarrow$ (i), we need a lemma.

Lemma 4.1.10. — *Assume that $\mathcal{A}$ is coherent and consider an exact sequence $0 \to F \underset{f}{\to} L_1 \underset{g}{\to} L_0$ in $\mathrm{Mod}(\mathcal{A})$ with L_1 and L_0 in $\mathcal{F}(\mathcal{A})$. Then the sequence $\beta_{\mathcal{A}}F \xrightarrow[\beta_{\mathcal{A}}f]{} \beta_{\mathcal{A}}L_1 \xrightarrow[\beta_{\mathcal{A}}g]{} \beta_{\mathcal{A}}L_0$ is exact.*

Proof. — By the result of Exercise 3.3, it is enough to show that for any $G \in \mathrm{Mod}(\mathcal{A})$, the sequence below is exact:

$$\mathcal{H}om_{\mathrm{I}(\mathcal{A})}(G, \beta_{\mathcal{A}}F) \to \mathcal{H}om_{\mathrm{I}(\mathcal{A})}(G, \beta_{\mathcal{A}}L_1) \to \mathcal{H}om_{\mathrm{I}(\mathcal{A})}(G, \beta_{\mathcal{A}}L_0). \tag{4.1.3}$$

Let $x \in X$. In a neighborhood of x, take $u \colon G \to \beta_{\mathcal{A}}L_1$ such that the composition $G \to \beta_{\mathcal{A}}L_1 \to \beta_{\mathcal{A}}L_0$ vanishes. Applying Lemma 4.1.6, there is a commutative diagram

$$\begin{tikzcd}
 & \mathcal{A}^k \arrow[rr, "b"] \arrow[d] & & \mathcal{A}^m \arrow[d] \\
G \arrow[ur, "\varphi"] \arrow[r, "v"] & \mathcal{A}^n \arrow[r, "w"] & L_1 \arrow[r, "g"] & L_0
\end{tikzcd}$$

such that $b \circ \varphi = 0$ and u is the composition

$$G \underset{v}{\to} \mathcal{A}^n \simeq \beta_{\mathcal{A}}\mathcal{A}^n \xrightarrow[\beta_{\mathcal{A}}(w)]{} \beta_{\mathcal{A}}L_1.$$

Set $K = \ker(b)$ and consider the diagram:

$$\begin{tikzcd}
 & & G \arrow[dl] \arrow[d] & \\
0 \arrow[r] & K \arrow[r] \arrow[d] & A^k \arrow[r] \arrow[d] & A^m \arrow[d] \\
0 \arrow[r] & F \arrow[r, "f"] & L_1 \arrow[r, "g"] & L_0.
\end{tikzcd}$$

Hence, we get the commutative diagram

$$\begin{tikzcd}
 & G \arrow[dl] \arrow[d] \\
K \arrow[r] & A^k \\
\beta_{\mathcal{A}}K \arrow[r] \arrow[u, "\sim"] \arrow[d] & \beta_{\mathcal{A}}A^k \arrow[u, "\sim"] \arrow[d] \\
\beta_{\mathcal{A}}F \arrow[r, "\beta_{\mathcal{A}}f"] & \beta_{\mathcal{A}}L_1.
\end{tikzcd}$$

Therefore, the morphism $u : G \to \beta_{\mathcal{A}}L_1$ factorizes through $\beta_{\mathcal{A}}F$. q.e.d.

We can now complete the proof of Proposition 4.1.9.

Consider an exact sequence $0 \to F' \to F \to F'' \to 0$ in $\mathrm{Mod}(\mathcal{A})$. We can find an exact commutative diagram as below such that the sheaves L'_j, L_j, L''_j $(j = 0, 1)$ are in $\mathcal{F}(\mathcal{A})$, and moreover such that the second and third rows split:

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
& N' \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & N'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L_1' \arrow[r] \arrow[d] & L_1 \arrow[r] \arrow[d] & L_1'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L_0' \arrow[r] \arrow[d] & L_0 \arrow[r] \arrow[d] & L_0'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F' \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & F'' \arrow[r] \arrow[d] & 0 \\
& 0 & 0 & 0 &
\end{tikzcd}$$

Let us apply the right exact functor $\beta_{\mathcal{A}}$ to the diagram. The second and third rows being split exact, they will remain exact. The columns remain exact by Lemma 4.1.10. Hence, the bottom row remains exact after applying $\beta_{\mathcal{A}}$. q.e.d.

4.2. Internal operations

Definition 4.2.1. — We define the internal tensor product, denoted by $\otimes_{\mathcal{A}}$, and the internal hom, denoted by $\mathcal{I}hom_{\mathcal{A}}$:

$$\otimes_{\mathcal{A}} : \mathrm{I}(\mathcal{A}^{\mathrm{op}}) \times \mathrm{I}(\mathcal{A}) \to \mathrm{I}(k_X)$$
$$\mathcal{I}hom_{\mathcal{A}} : \mathrm{I}(\mathcal{A})^{\mathrm{op}} \times \mathrm{I}(\mathcal{A}) \to \mathrm{I}(k_X)$$

by the formulas:

$$\begin{aligned}
(\text{“}\varinjlim_i\text{”}\, F_i) \otimes_{\mathcal{A}} (\text{“}\varinjlim_j\text{”}\, G_j) &= \text{“}\varinjlim_{i,j}\text{”}(F_i \otimes_{\mathcal{A}} G_j),\\
\mathcal{I}hom_{\mathcal{A}}(\text{“}\varinjlim_i\text{”}\, F_i, \text{“}\varinjlim_j\text{”}\, G_j) &= \varprojlim_i \text{“}\varinjlim_j\text{”}\, \mathcal{H}om_{\mathcal{A}}(F_i, G_j).
\end{aligned}$$

Similarly we define $\otimes$ and $\mathcal{I}hom$ using $\otimes_{k_X}$ and $\mathcal{H}om_{k_X}$. Then those functors $\otimes$ and $\mathcal{I}hom$ induce

$$\begin{aligned}
\otimes &: \mathrm{I}(k_X) \times \mathrm{I}(\mathcal{A}) \to \mathrm{I}(\mathcal{A}),\\
\otimes &: \mathrm{I}(\mathcal{A}) \times \mathrm{I}(k_X) \to \mathrm{I}(\mathcal{A}),\\
\mathcal{I}hom &: \mathrm{I}(k_X)^{\mathrm{op}} \times \mathrm{I}(\mathcal{A}) \to \mathrm{I}(\mathcal{A}),\\
\mathcal{I}hom &: \mathrm{I}(\mathcal{A}^{\mathrm{op}})^{\mathrm{op}} \times \mathrm{I}(k_X) \to \mathrm{I}(\mathcal{A}).
\end{aligned}$$

Proposition 4.2.2. — (i) *The functor* $\otimes_{\mathcal{A}}$ *commutes with* "$\varinjlim$" *and is right exact. Moreover,* $\mathcal{A} \otimes_{\mathcal{A}} F \simeq F$.

(ii) *The functor* $\mathcal{I}hom_{\mathcal{A}}$ *is left exact. Moreover,* $\mathcal{I}hom_{\mathcal{A}}(\mathcal{A}, F) \simeq F$.

The proof is obvious.

Proposition 4.2.3. — *The diagram below commutes:*

$$\begin{tikzcd}
\mathrm{Mod}(\mathcal{A}^{\mathrm{op}}) \times \mathrm{Mod}(\mathcal{A}) \arrow[r, "\otimes_{\mathcal{A}}"] \arrow[d, "\iota_X \times \iota_X"] & \mathrm{Mod}(k_X) \arrow[d, "\iota_X"] \\
\mathrm{I}(\mathcal{A}^{\mathrm{op}}) \times \mathrm{I}(\mathcal{A}) \arrow[r, "\otimes_{\mathcal{A}}"] \arrow[d, "\alpha_X \times \alpha_X"] & \mathrm{I}(k_X) \arrow[d, "\alpha_X"] \\
\mathrm{Mod}(\mathcal{A}^{\mathrm{op}}) \times \mathrm{Mod}(\mathcal{A}) \arrow[r, "\otimes_{\mathcal{A}}"] & \mathrm{Mod}(k_X).
\end{tikzcd}$$

Proof. — Let $F \in \mathrm{Mod}(\mathcal{A}^{\mathrm{op}})$ and $G \in \mathrm{Mod}(\mathcal{A})$. Then:

$$\begin{aligned}
\iota_X(F \otimes_{\mathcal{A}} G) &= \text{“}\varinjlim_{U}\text{”}(F \otimes_{\mathcal{A}} G)_U \simeq \text{“}\varinjlim_{U}\text{”}(F_U \otimes_{\mathcal{A}} G_U) \\
&\simeq \text{“}\varinjlim_{U,U'}\text{”}(F_U \otimes_{\mathcal{A}} G_{U'}) \simeq (\text{“}\varinjlim_{U}\text{”}\, F_U) \otimes_{\mathcal{A}} (\text{“}\varinjlim_{U'}\text{”}\, G_{U'})\,.
\end{aligned}$$

Let $F = \text{“}\varinjlim_{i}\text{”}\, F_i$ and $G = \text{“}\varinjlim_{j}\text{”}\, G_j$. Then

$$\begin{aligned}
\alpha_X((\text{“}\varinjlim_{i}\text{”}\, F_i) \otimes_{\mathcal{A}} (\text{“}\varinjlim_{j}\text{”}\, G_j)) &\simeq \alpha_X(\text{“}\varinjlim_{i,j}\text{”}(F_i \otimes_{\mathcal{A}} G_j)) \\
&\simeq \varinjlim_{i,j}(F_i \otimes_{\mathcal{A}} G_j) \simeq (\varinjlim_{i} F_i) \otimes_{\mathcal{A}} (\varinjlim_{j} G_j)\,.
\end{aligned}$$

q.e.d.

Proposition 4.2.4. — *The diagram below commutes:*

$$\begin{tikzcd}
\mathrm{Mod}(\mathcal{A})^{\mathrm{op}} \times \mathrm{Mod}(\mathcal{A}) \arrow[r, "\mathcal{H}om_{\mathcal{A}}"'] \arrow[d, "\iota_X \times \iota_X"] & \mathrm{Mod}(k_X) \arrow[d, "\iota_X"] \\
\mathrm{I}(\mathcal{A})^{\mathrm{op}} \times \mathrm{I}(\mathcal{A}) \arrow[r, "\mathcal{I}hom_{\mathcal{A}}"'] \arrow[dr, "\mathcal{H}om_{I(\mathcal{A})}"'] & \mathrm{I}(k_X) \arrow[d, "\alpha_X"] \\
& \mathrm{Mod}(k_X).
\end{tikzcd}$$

Proof. — (i) Let $F, G \in \mathrm{Mod}(\mathcal{A})$. Then

$$\begin{aligned}
\mathcal{I}hom_{\mathcal{A}}(\iota_X F, \iota_X G) &\simeq \varprojlim_{U} \text{“}\varinjlim_{V}\text{”}\, \mathcal{H}om_{\mathcal{A}}(F_U, G_V) \\
&\simeq \varprojlim_{U} \mathcal{H}om_{\mathcal{A}}(F_U, G) \simeq \iota_X(\mathcal{H}om_{\mathcal{A}}(F, G)).
\end{aligned}$$

(ii) Let $F = \text{“}\varinjlim_i\text{”}\, F_i, G = \text{“}\varinjlim_j\text{”}\, G_j$. Since α_X commutes with $\varprojlim$, we get the chain of isomorphisms

$$\begin{aligned}\alpha_X(\mathcal{I}hom_{\mathcal{A}}(F,G)) &\simeq \alpha_X(\varprojlim_i \text{“}\varinjlim_j\text{”}\, \mathcal{H}om_{\mathcal{A}}(F_i,G_j))\\ &\simeq \varprojlim_i \varinjlim_j \mathcal{H}om_{\mathcal{A}}(F_i,G_j) \simeq \mathcal{H}om_{\mathrm{I}(\mathcal{A})}(F,G).\end{aligned}$$

q.e.d.

Proposition 4.2.5. — *Let $K \in \mathrm{I}(k_X)$ and $F,\ G \in \mathrm{I}(\mathcal{A})$. Then:*

$$\begin{aligned}\mathrm{Hom}_{\mathrm{I}(\mathcal{A})}(K \otimes F, G) &\simeq \mathrm{Hom}_{\mathrm{I}(k_X)}(K, \mathcal{I}hom_{\mathcal{A}}(F,G))\\ &\simeq \mathrm{Hom}_{\mathrm{I}(\mathcal{A})}(F, \mathcal{I}hom\,(K,G)).\end{aligned}$$

In particular, the functors $K \otimes \cdot$ is a left adjoint of $\mathcal{I}hom\,(K,\cdot)$.

Proof. — One has the chain of isomorphisms:

$$\begin{aligned}&\mathrm{Hom}_{\mathrm{I}(\mathcal{A})}(\text{“}\varinjlim_k\text{”}\, K_k \otimes \text{“}\varinjlim_i\text{”}\, F_i\,,\ \text{“}\varinjlim_j\text{”}\, G_j)\\ &\quad\simeq \mathrm{Hom}_{\mathrm{I}(\mathcal{A})}(\text{“}\varinjlim_{i,k}\text{”}(K_k \otimes F_i), \text{“}\varinjlim_j\text{”}\, G_j)\\ &\quad\simeq \varprojlim_{i,k} \varinjlim_j \mathrm{Hom}_{\mathcal{A}}(K_k \otimes F_i, G_j)\\ &\quad\simeq \varprojlim_{i,k} \varinjlim_j \mathrm{Hom}_{k_X}(K_k, \mathcal{H}om_{\mathcal{A}}(F_i,G_j))\\ &\quad\simeq \varprojlim_i \mathrm{Hom}_{\mathrm{I}(k_X)}(\text{“}\varinjlim_k\text{”}\, K_k, \text{“}\varinjlim_j\text{”}\, \mathcal{H}om_{\mathcal{A}}(F_i,G_j))\\ &\quad\simeq \mathrm{Hom}_{\mathrm{I}(k_X)}(\text{“}\varinjlim_k\text{”}\, K_k, \varprojlim_i \text{“}\varinjlim_k\text{”}\, \mathcal{H}om_{\mathcal{A}}(F_i,G_j))\\ &\quad\simeq \mathrm{Hom}_{\mathrm{I}(k_X)}\Big(\text{“}\varinjlim_k\text{”}\, K_k, \mathcal{I}hom_{\mathcal{A}}(\text{“}\varinjlim_i\text{”}\, F_i,\ \text{“}\varinjlim_j\text{”}\, G_j)\Big).\end{aligned}$$

The second isomorphism is proved similarly. q.e.d.

Corollary 4.2.6. — *For $F, G \in \mathrm{I}(\mathcal{A})$ there is a canonical morphism*

$$F \otimes \mathcal{I}hom_{\mathcal{A}}(F,G) \to G.$$

Corollary 4.2.7. — *For $F,\ G \in \mathrm{I}(\mathcal{A})$ and $K \in \mathrm{I}(k_X)$, there is a canonical morphism*

$$\mathcal{I}hom_{\mathcal{A}}(F,G) \otimes K \to \mathcal{I}hom_{\mathcal{A}}(F, G \otimes K).$$

Proof. — By the preceding corollary there is a morphism $F \otimes \mathcal{I}hom_{\mathcal{A}}(F,G) \otimes K \to G \otimes K$, and we obtain the desired morphism by adjunction (Proposition 4.2.5). q.e.d.

Corollary 4.2.8. — *Let $K \in \mathrm{I}(\mathcal{A})$ and let $i \mapsto F_i \in \mathrm{I}(\mathcal{A})$ be a small filtrant inductive system of ind-sheaves, and $j \mapsto G_j \in \mathrm{I}(\mathcal{A})$ a small projective system of ind-sheaves. Then we have the isomorphisms*

(i) $\mathcal{I}hom_{\mathcal{A}}(K, \varprojlim_j G_j) \simeq \varprojlim_j \mathcal{I}hom_{\mathcal{A}}(K, G_j)$,

(ii) $\mathcal{I}hom_{\mathcal{A}}(\text{“}\varinjlim_i\text{”}\, F_i, K) \simeq \varprojlim_i \mathcal{I}hom_{\mathcal{A}}(F_i, K)$.

(iii) *If* $K \in \mathrm{Mod}(\mathcal{A})$*, then one has* $\mathcal{I}hom_{\mathcal{A}}(K, \text{“}\varinjlim_i\text{”}\, F_i) \simeq \text{“}\varinjlim_i\text{”}\, \mathcal{I}hom_{\mathcal{A}}(K, F_i)$,

Proof. — (i) Let $S \in \mathrm{Mod}^c(k_X)$. Then

$$\begin{aligned} \mathrm{Hom}_{\mathrm{I}(k_X)}(S, \mathcal{I}hom_{\mathcal{A}}(K, \varprojlim_j G_j)) &\simeq \mathrm{Hom}_{\mathrm{I}(\mathcal{A})}(S \otimes K, \varprojlim_j G_j) \\ &\simeq \varprojlim_j \mathrm{Hom}_{\mathrm{I}(\mathcal{A})}(S \otimes K, G_j) \\ &\simeq \varprojlim_j \mathrm{Hom}_{\mathrm{I}(k_X)}(S, \mathcal{I}hom_{\mathcal{A}}(K, G_j)) \\ &\simeq \mathrm{Hom}_{\mathrm{I}(k_X)}(S, \varprojlim_j \mathcal{I}hom_{\mathcal{A}}(K, G_j)). \end{aligned}$$

(ii) Let $S \in \mathrm{Mod}^c(k_X)$. Then

$$\begin{aligned} &\mathrm{Hom}_{\mathrm{I}(k_X)}(S, \mathcal{I}hom_{\mathcal{A}}(\text{“}\varinjlim_i\text{”}\, F_i, K)) \simeq \mathrm{Hom}_{\mathrm{I}(\mathcal{A})}(S \otimes \text{“}\varinjlim_i\text{”}\, F_i, K) \\ &\quad \simeq \mathrm{Hom}_{\mathrm{I}(\mathcal{A})}(\text{“}\varinjlim_i\text{”}(S \otimes F_i), K) \simeq \varprojlim_i \mathrm{Hom}_{\mathrm{I}(\mathcal{A})}(S \otimes F_i, K) \\ &\quad \simeq \varprojlim_i \mathrm{Hom}_{\mathrm{I}(k_X)}(S, \mathcal{I}hom_{\mathcal{A}}(F_i, K)) \simeq \mathrm{Hom}_{\mathrm{I}(k_X)}(S, \varprojlim_i \mathcal{I}hom_{\mathcal{A}}(F_i, K)). \end{aligned}$$

(iii) Let $S \in \mathrm{Mod}^c(k_X)$. Then

$$\begin{aligned} &\mathrm{Hom}_{\mathrm{I}(k_X)}(S, \mathcal{I}hom_{\mathcal{A}}(K, \text{“}\varinjlim_i\text{”}\, F_i)) \simeq \mathrm{Hom}_{\mathrm{I}(\mathcal{A})}(S \otimes K, \text{“}\varinjlim_i\text{”}\, F_i) \\ &\quad \simeq \varinjlim_i \mathrm{Hom}_{\mathrm{I}(\mathcal{A})}(S \otimes K, F_i) \simeq \varinjlim_i \mathrm{Hom}_{\mathrm{I}(k_X)}(S, \mathcal{I}hom_{\mathcal{A}}(K, F_i)) \\ &\quad \simeq \mathrm{Hom}_{\mathrm{I}(k_X)}(S, \text{“}\varinjlim_i\text{”}\, \mathcal{I}hom_{\mathcal{A}}(K, F_i)). \end{aligned}$$

q.e.d.

Corollary 4.2.9. — *Let* $K \in \mathrm{I}(k_X)$ *and* $F,\ G \in \mathrm{I}(\mathcal{A})$*. Then:*

$$\begin{aligned} \mathcal{I}hom_{\mathcal{A}}(K \otimes F, G) &\simeq \mathcal{I}hom\,(K, \mathcal{I}hom_{\mathcal{A}}(F, G)) \\ &\simeq \mathcal{I}hom_{\mathcal{A}}(F, \mathcal{I}hom\,(K, G)). \end{aligned}$$

Proof. — For $S \in \mathrm{Mod}^c(k_X)$ we have

$$\begin{aligned} \mathrm{Hom}_{\mathrm{I}(k_X)}(S, \mathcal{I}hom_{\mathcal{A}}(K \otimes F, G)) &\simeq \mathrm{Hom}_{\mathrm{I}(\mathcal{A})}(S \otimes K \otimes F, G) \\ &\simeq \mathrm{Hom}_{\mathrm{I}(k_X)}(S \otimes K, \mathcal{I}hom_{\mathcal{A}}(F, G)) \\ &\simeq \mathrm{Hom}_{\mathrm{I}(k_X)}(S, \mathcal{I}hom\,(K, \mathcal{I}hom_{\mathcal{A}}(F, G)))\,. \end{aligned}$$

This shows the first isomorphism. The proof of the second isomorphism is similar.

q.e.d.

Remark 4.2.10. — It follows from Corollary 4.2.8 that, for $F \in \mathrm{I}(\mathcal{A})$ and $K \in \mathrm{Mod}(\mathcal{A})$, the natural morphism $\mathcal{I}hom_{\mathcal{A}}(F, K) \to \mathcal{H}om_{\mathrm{I}(\mathcal{A})}(F, K)$ is an isomorphism.

We shall now study the relations among the functors $\otimes$, $\mathcal{I}hom$ and β_X. Here β_X is the functor $\mathrm{Mod}(k_X) \to \mathrm{I}(k_X)$.

Proposition 4.2.11. — *Let* $K \in \mathrm{Mod}(k_X)$ *and* F, $G \in \mathrm{I}(\mathcal{A})$. *Then:*

$$\begin{aligned}\mathrm{Hom}_{k_X}(K, \mathcal{H}om_{\mathcal{A}}(F, G)) &\simeq \mathrm{Hom}_{\mathrm{I}(\mathcal{A})}(\beta_X K \otimes F, G),\\ \mathcal{H}om(K, \mathcal{H}om_{\mathcal{A}}(F, G)) &\simeq \mathcal{H}om_{\mathcal{A}}(\beta_X K \otimes F, G).\end{aligned}$$

Proof. — Consider the chain of isomorphisms:

$$\begin{aligned}\mathrm{Hom}_{k_X}(K, \mathcal{H}om_{\mathcal{A}}(F, G)) &\simeq \mathrm{Hom}_{\mathrm{I}(k_X)}(K, \alpha_X(\mathcal{I}hom_{\mathcal{A}}(F, G)))\\ &\simeq \mathrm{Hom}_{\mathrm{I}(k_X)}(\beta_X K, \mathcal{I}hom_{\mathcal{A}}(F, G))\\ &\simeq \mathrm{Hom}_{\mathrm{I}(k_X)}(\beta_X K \otimes F, G).\end{aligned}$$

The second formula follows. q.e.d.

Proposition 4.2.12. — *Let* F *and* G *be in* $\mathrm{Mod}(k_X)$. *Then:*

$$\beta_X F \otimes \beta_X G \simeq \beta_X(F \otimes G).$$

Proof. — Let $K \in \mathrm{I}(k_X)$. Consider the chain of isomorphisms:

$$\begin{aligned}&\mathrm{Hom}_{\mathrm{I}(k_X)}(\beta_X F \otimes \beta_X G, K) \simeq \mathrm{Hom}_{\mathrm{I}(k_X)}(\beta_X F, \mathcal{I}hom(\beta_X G, K))\\ &\quad\simeq \mathrm{Hom}_{k_X}(F, \mathcal{H}om(\beta_X G, K)) \simeq \mathrm{Hom}_{k_X}(F, \mathcal{H}om(G, \alpha_X K))\\ &\quad\simeq \mathrm{Hom}_{k_X}(F \otimes G, \alpha_X K) \simeq \mathrm{Hom}_{\mathrm{I}(k_X)}(\beta_X(F \otimes G), K).\end{aligned}$$

Here the third isomorphism follows from Lemma 3.3.29. q.e.d.

Remark 4.2.13. — For $F \in \mathrm{I}(\mathcal{A})$, $\mathcal{H}om_{\mathcal{A}}(F, \cdot\,)$ and $\mathcal{I}hom_{\mathcal{A}}(F, \cdot\,)$ commute with filtrant inductive limits if and only if $F \in \mathrm{Mod}(\mathcal{A})$. Moreover $\mathrm{Hom}_{\mathrm{I}(\mathcal{A})}(F, \cdot\,)$ commutes with filtrant inductive limits if and only if $F \in \mathrm{Mod}^c(\mathcal{A})$. See Exercise 3.5.

Sheaves associated with locally closed subsets. — In this subsection, the base ring is the sheaf k_X. Recall that if Z is a locally closed subset and $F \in \mathrm{I}(k_X)$, we have constructed the ind-sheaf ${}_Z F$ in Definition 3.3.15, by setting, for $F = \text{“}\varinjlim_i\text{”}\, F_i$, $Z = U \cap S$, U open, S closed:

$${}_Z F = \text{“}\varinjlim_{i,\, V \subset\subset U,\, W \supset S}\text{”}\, (F_i)_{V \cap \overline{W}}.$$

When there is no risk of confusion, we often write k_Z and ${}_Z k$ instead of $(k_X)_Z$ and ${}_Z(k_X)$, respectively.

If $Z = \{x\}$ with $x \in X$, we shall write ${}_x k$ and k_x instead of ${}_{\{x\}}k$ and $k_{\{x\}}$, respectively.

Proposition 4.2.14. — (i) *One has the isomorphisms*

$$_Zk \simeq \beta_X k_Z,\ {}_ZF \simeq F \otimes {}_Zk \quad \text{for } F \in \mathrm{I}(k_X).$$

(ii) *Let U be open and let $F, G \in \mathrm{I}(k_X)$. Then*

$$\mathrm{Hom}_{\mathrm{I}(k_U)}(F|_U, G|_U) \simeq \mathrm{Hom}_{\mathrm{I}(k_X)}(F \otimes {}_Uk, G).$$

(iii) *Let Z_1 and Z_2 be two locally closed subsets. Then*

$$_{Z_1 \cap Z_2}k \simeq {}_{Z_1}k \otimes {}_{Z_2}k.$$

(iv) *Assume that Z' is closed in Z. Then there is an exact sequence*

$$0 \to {}_{Z \setminus Z'}k \to {}_Zk \to {}_{Z'}k \to 0.$$

(v) *Let Z be a locally closed subset. Then for $G \in \mathrm{Mod}(k_X)$ and $F \in \mathrm{I}(k_X)$, one has*

$$\mathcal{H}om\,(G, F \otimes {}_Zk) \simeq \mathcal{H}om\,(G, F) \otimes k_Z.$$

Proof. —
(ii) By Proposition 4.2.11, one has $\mathrm{Hom}_{\mathrm{I}(k_X)}(F \otimes {}_Uk, G) \simeq \mathrm{Hom}_{k_X}(k_U, \mathcal{H}om\,(F, G))$.
(v) is a particular case of Proposition 3.3.16.
The other results are obvious. q.e.d.

Corollary 4.2.15. — (i) *If U is open, the morphism $_Uk \to k_U$ is a monomorphism.*

(ii) *If S is closed, the morphism $_Sk \to k_S$ is an epimorphism.*

Proposition 4.2.16. — *For any locally closed subset Z and $F \in \mathrm{I}(k_X)$, one has $\Gamma_Z F \simeq \mathcal{I}hom\,({}_Zk, F)$.*

Proof. — For any $G \in \mathrm{I}(k_X)$, one has

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{I}(k_X)}(G, \Gamma_Z F) &\simeq \Gamma_Z(X\,; \mathcal{H}om\,(G, F)) \simeq \mathrm{Hom}_{\mathrm{I}(k_X)}(G_Z, F) \\
&\simeq \mathrm{Hom}_{\mathrm{I}(k_X)}({}_Zk \otimes G, F) \simeq \mathrm{Hom}_{\mathrm{I}(k_X)}(G, \mathcal{I}hom\,({}_Zk, F)).
\end{aligned}$$

q.e.d.

Example 4.2.17. — Let X be a real manifold of dimension $n \geq 1$ and let $x \in X$. Define $N \in \mathrm{I}(k_X)$ by the exact sequence

$$0 \to N \to {}_xk \to k_x \to 0. \tag{4.2.1}$$

Since $\mathrm{Hom}\,(k_x, {}_xk) \simeq \varinjlim \mathrm{Hom}\,(k_x, k_{\overline{V}})$, where V ranges through a neighborhood system of x, we find $\mathrm{Hom}\,(k_x, {}_xk) = 0$, hence $N \neq 0$. On the other hand,

$$\mathrm{Hom}\,(k_U, N) = 0$$

for any open neighborhood U of x.

Example 4.2.18. — By Exercise 1.1, we get a non-zero ind-sheaf F over $X = \{\mathrm{pt}\}$ such that

$$\mathrm{Hom}_{\mathrm{I}(k_X)}(k_{XU}, F) = 0 \text{ for all open set } U \subset X. \tag{4.2.2}$$

In fact, one can construct ind-sheaves with such a property on every non-empty space X (see Exercise 4.8).

Quasi-injective ind-sheaves. — In this subsection, the base ring is the sheaf k_X.

Recall that an ind-sheaf $K \in \mathrm{I}(k_X)$ is called quasi-injective if the functor $G \mapsto \mathrm{Hom}_{\mathrm{I}(k_X)}(G, K)$ is exact on $\mathrm{Mod}^c(k_X)$, or equivalently, if $K \simeq$ "$\varinjlim\limits_i$" K_i with $K_i \in \mathrm{Mod}^c(k_X)$ and K_i injective in $\mathrm{Mod}^c(k_X)$.

Proposition 4.2.19. — *Let $K \in \mathrm{I}(k_X)$ and consider the following properties:*

(i) *K is quasi-injective,*
(ii) *$K \simeq$ "$\varinjlim\limits_i$" K_i with $K_i \in \mathrm{Mod}(k_X)$ and K_i injective,*
(iii) *the functor $\mathrm{Hom}_{\mathrm{I}(k_X)}(\,\cdot\,, K)$ is exact on $\mathrm{Mod}(k_X)$,*
(iv) *the functor $\mathcal{H}om\,(\,\cdot\,, K)$ is exact on $\mathrm{Mod}(k_X)$, and the sheaf $\mathcal{H}om\,(F, K)$ is soft for any $F \in \mathrm{Mod}(k_X)$,*
(v) *the functor $\mathcal{I}hom\,(\,\cdot\,, K)$ is exact on $\mathrm{Mod}(k_X)$.*

Then (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) $\Rightarrow$ (v).

Proof. — (i) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (i). One has the isomorphism $K \simeq$ "$\varinjlim\limits_{Z,i}$" $\Gamma_Z(K_i)$, where Z ranges over the family of compact subsets of X. Since $\Gamma_Z(K_i)$ is injective in $\mathrm{Mod}^c(k_X)$, the result follows.

(ii) $\Rightarrow$ (v) follows from the isomorphism $\mathcal{I}hom\,(F, K) \simeq$ "$\varinjlim\limits_i$" $\mathcal{H}om\,(F, K_i)$, and the fact that $\mathcal{H}om\,(\,\cdot\,, K_i)$ is exact on $\mathrm{Mod}(k_X)$.

(ii)+(v) $\Rightarrow$ (iv). By applying the exact functor α_X to $\mathcal{I}hom\,(\,\cdot\,, K)$, the functor $\mathcal{H}om_{\mathrm{I}(k_X)}(\,\cdot\,, K)$ is exact. Since $\mathcal{H}om_{\,k_X}(F, K_i)$ is a flabby sheaf, $\mathcal{H}om_{\mathrm{I}(k_X)}(F, K) \simeq \varinjlim\limits_i \mathcal{H}om\,(F, K_i)$ is a soft sheaf.

(iv) $\Rightarrow$ (iii) follows from $\mathrm{Hom}_{\mathrm{I}(k_X)}(\,\cdot\,, K) = \Gamma(X; \mathcal{H}om\,(\,\cdot\,, K))$.

(iii) $\Rightarrow$ (i) is obvious. q.e.d.

Corollary 4.2.20. — *Let $0 \to K' \to K \to K'' \to 0$ be an exact sequence in $\mathrm{I}(k_X)$, and assume that K' is quasi-injective. Then for any $F \in \mathrm{Mod}(k_X)$ the following sequences are exact:*

$$\begin{aligned}
&0 \to \mathcal{I}hom\,(F, K') \to \mathcal{I}hom\,(F, K) \to \mathcal{I}hom\,(F, K'') \to 0,\\
&0 \to \mathcal{H}om\,(F, K') \to \mathcal{H}om\,(F, K) \to \mathcal{H}om\,(F, K'') \to 0,\\
&0 \to \mathrm{Hom}_{\mathrm{I}(k_X)}(F, K') \to \mathrm{Hom}_{\mathrm{I}(k_X)}(F, K) \to \mathrm{Hom}_{\mathrm{I}(k_X)}(F, K'') \to 0\,.
\end{aligned}$$

Proof. — There exists a filtrant inductive system of exact sequences $0 \to K'_i \to K_i \to K''_i \to 0$ in $\mathrm{Mod}(k_X)$ such that every K'_i is injective and its inductive limit gives $0 \to K' \to K \to K'' \to 0$. Then $0 \to \mathcal{H}om\,(F, K'_i) \to \mathcal{H}om\,(F, K_i) \to \mathcal{H}om\,(F, K''_i) \to 0$ is exact. Hence we obtain the exactness of the first and second sequences. Since $\mathcal{H}om\,(F, K')$ is soft by the preceding proposition, the last sequence is exact by applying $\Gamma(X;\,\cdot\,)$ to the second one. q.e.d.

Proposition 4.2.21. — *Assume that k is a field. Let $K \in \mathrm{I}(k_X)$ be quasi-injective. Let $F \in \mathrm{Mod}(k_X)$. Then $\beta_X F \otimes K$ is quasi-injective.*

Proof. — (i) Let us first prove the result when $F = k_Z$ for a locally closed set Z. By the result of Proposition 4.2.14 (v), we have for any $G \in \mathrm{Mod}^c(k_X)$:

$$\mathrm{Hom}_{\mathrm{I}(k_X)}(G, \beta_X k_Z \otimes K) \simeq \Gamma(X; k_Z \otimes \mathcal{H}om\,(G, K))\,,$$

and this functor is exact with respect to G.

(ii) Let $F \in \mathrm{Mod}(k_X)$. There exists an epimorphism $\bigoplus_{i\in I} k_{U_i} \twoheadrightarrow F$. For a finite subset J of I, denote by F_J the image of $\bigoplus_{i\in J} k_{U_i}$. Then $\beta_X F \simeq \text{“}\varinjlim_{J}\text{”}(\beta_X F_J)$, and it is enough to prove the result with F replaced by F_J. If $|J| = 1$, then F is isomorphic to k_Z for some locally closed subset Z, and the result follows by (i). Then the proof goes by induction on the cardinal of J. Indeed, if one has an exact sequence $0 \to F' \to F \to F'' \to 0$ then $0 \to \beta_X F' \otimes K \to \beta_X F \otimes K \to \beta_X F'' \otimes K \to 0$ is exact, and if $\beta_X F' \otimes K$ and $\beta_X F'' \otimes K$ are quasi-injective then $\beta_X F \otimes K$ will be quasi-injective. q.e.d.

Proposition 4.2.22. — *Assume that k is a field. Let F, $G \in \mathrm{Mod}(k_X)$ and let $K \in \mathrm{I}(k_X)$. Then there are natural isomorphisms*

$$\begin{aligned}
\mathcal{I}hom\,(G, K) \otimes \beta_X F &\xrightarrow{\sim} \mathcal{I}hom\,(G, K \otimes \beta_X F)\,,\\
\mathcal{H}om\,(G, K) \otimes F &\xrightarrow{\sim} \mathcal{H}om\,(G, K \otimes \beta_X F)\,,\\
\Gamma(X; F \otimes \mathcal{H}om\,(G, k_X)) &\xrightarrow{\sim} \mathrm{Hom}_{\mathrm{I}(k_X)}(G, \beta_X F)\,.
\end{aligned}$$

Note that the third formula gives an alternative definition of the ind-object $\beta_X F$ when $\mathcal{A} = k_X$.

Proof. — The first morphism is obtained by Corollary 4.2.7, the second one by applying α to the first one, and the last one by applying $\Gamma(X;\,\cdot\,)$ to the second one.

First let us prove that the second morphism is an isomorphism. Remark that both sides are left exact with respect to K. Hence we may assume that K is quasi-injective.

Since the functor $\mathcal{H}om\,(G,\cdot\,)$ sends exact sequence of quasi-injective objects to exact sequences, we get by Proposition 4.2.21 that both sides are exact with respect to F. Since both sides commute with $\varinjlim$ with respect to F, we may assume $F = k_U$ for an open subset U. Then the second formula follows from Proposition 4.2.14.

The third formula follows from the second one by applying the functor $\Gamma(X;\,\cdot\,)$ with $K = k_X$.

Finally let us prove the first formula. For any $S \in \mathrm{Mod}^c(k_X)$, one has

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{I}(k_X)}(S, \mathcal{I}hom\,(G,K)\otimes\beta_X F) &\simeq \Gamma\Big(X\,;\,\mathcal{H}om\,\big(S, \mathcal{I}hom\,(G,K)\otimes\beta_X F\big)\Big)\\
&\simeq \Gamma\Big(X\,;\,\mathcal{H}om\,\big(S, \mathcal{I}hom\,(G,K)\big)\otimes F\Big)\\
&\simeq \Gamma\Big(X\,;\,\mathcal{H}om\,(S\otimes G, K)\otimes F\Big)\\
&\simeq \Gamma\Big(X\,;\,\mathcal{H}om\,(S\otimes G, K\otimes\beta_X F)\Big)\\
&\simeq \Gamma\Big(X\,;\,\mathcal{H}om\,\big(S, \mathcal{I}hom\,(G, K\otimes\beta_X F)\big)\Big)\\
&\simeq \mathrm{Hom}_{\mathrm{I}(k_X)}(S, \mathcal{I}hom\,(G, K\otimes\beta_X F))\,.
\end{aligned}$$

q.e.d.

4.3. External operations

Recall that all spaces are Hausdorff, locally compact and with a countable base of open sets. Let $f\colon X\to Y$ be a continuous map.

From now on, for the sake of simplicity, *we assume that k is a field,* and we consider only the base rings k_X, k_Y, etc. The case of sheaves of rings will be treated in the next chapter.

Inverse image. — Let $G\in\mathrm{I}(k_Y)$. We define $f^{-1}G\in\mathrm{I}(k_X)$ as follows. If $G = \text{“}\varinjlim_i\text{”}\, G_i$ for $G_i\in\mathrm{Mod}^c(k_Y)$, we set:

$$f^{-1}G = \text{“}\varinjlim_{i,U\subset\subset X}\text{”}\,(f^{-1}G_i)_U.$$

Note that one also has $f^{-1}G = \text{“}\varinjlim_i\text{”}\, f^{-1}G_i$, where $f^{-1}G_i\in\mathrm{Mod}(k_X)$.

If $i_U\colon U\hookrightarrow X$ is an open inclusion, and $F\in\mathrm{I}(k_X)$, then $i_U^{-1}F\simeq F|_U$, by Lemma 3.3.7.

Proposition 4.3.1. — *The diagram below commutes:*

$$\begin{tikzcd}
\mathrm{Mod}(k_Y) \arrow[r, "f^{-1}"] \arrow[d, "\iota_Y"'] & \mathrm{Mod}(k_X) \arrow[d, "\iota_X"'] \\
\mathrm{I}(k_Y) \arrow[r, "f^{-1}"] \arrow[d, "\alpha_Y"'] & \mathrm{I}(k_X) \arrow[d, "\alpha_X"'] \\
\mathrm{Mod}(k_Y) \arrow[r, "f^{-1}"] & \mathrm{Mod}(k_X).
\end{tikzcd}$$

Proof. — (i) Let $G \in \mathrm{Mod}(k_Y)$. Then $\iota_Y(G) = \text{“}\varinjlim_{V\subset\subset Y}\text{”}\, G_V$ and

$$\begin{aligned}
f^{-1}\iota_Y(G) &\simeq \text{“}\varinjlim_{V\subset\subset Y, U\subset\subset X}\text{”}\, (f^{-1}G_V)_U \\
&\simeq \text{“}\varinjlim_{V\subset\subset Y, U\subset\subset X}\text{”}\, (f^{-1}G)_{f^{-1}(V)\cap U} \simeq \text{“}\varinjlim_{U\subset\subset X}\text{”}(f^{-1}G)_U.
\end{aligned}$$

(ii) Let $G = \text{“}\varinjlim_i\text{”}\, G_i \in \mathrm{I}(k_Y)$. Then $\alpha_Y(G) = \varinjlim_i G_i$ and

$$f^{-1}\alpha_Y(G) \simeq f^{-1}\varinjlim_i G_i \simeq \varinjlim_i f^{-1}G_i \simeq \alpha_X(\text{“}\varinjlim_i\text{”}\, f^{-1}G_i)\,.$$

q.e.d.

Proposition 4.3.2. — *The functor $f^{-1}\colon \mathrm{I}(k_Y) \to \mathrm{I}(k_X)$ is exact and commutes with “$\varinjlim$” and $\otimes$.*

Proof. — Recall that $f^{-1}G = \text{“}\varinjlim_{i, U\subset\subset X}\text{”}\, (f^{-1}G_i)_U$, and the functors “$\varinjlim$”, f^{-1} and $(\,\cdot\,)_U$ are exact and commute with “$\varinjlim$” and $\otimes$. q.e.d.

Proposition 4.3.3. — *For $f\colon X \to Y$ and $g\colon Y \to Z$, one has $(g\circ f)^{-1} \simeq f^{-1}\circ g^{-1}$.*

The proof is straightforward.

We shall prove later (Corollary 4.3.7) that the functors f^{-1} and β commute.

Direct image. — Let $F = \text{“}\varinjlim_i\text{”}\, F_i \in \mathrm{I}(k_X)$ for $F_i \in \mathrm{Mod}^c(k_X)$. One defines $f_*F \in \mathrm{I}(k_Y)$ by the formula:

$$f_*(\text{“}\varinjlim_i\text{”}\, F_i) = \varprojlim_K \text{“}\varinjlim_i\text{”}\, f_*(F_{iK}) \simeq \varprojlim_{U\subset\subset X} \text{“}\varinjlim_i\text{”}\, f_*(\Gamma_U(F_i)).$$

Here, K (resp. U) ranges through the family of compact (resp. relatively compact open) subsets of X. The isomorphism above is described by the morphisms

$$\varprojlim_K \text{“}\varinjlim_i\text{”}\, f_*(F_{iK}) \xrightarrow{\sim} \varprojlim_{K,U} \text{“}\varinjlim_i\text{”}\, f_*(\Gamma_U F_i)_K \xleftarrow{\sim} \varprojlim_U \text{“}\varinjlim_i\text{”}\, f_*(\Gamma_U F_i).$$

Proposition 4.3.4. — *The two functors f^{-1} and f_* are adjoint. More precisely, let $F \in \mathrm{I}(k_X)$ and let $G \in \mathrm{I}(k_Y)$. Then*

$$\mathrm{Hom}_{\mathrm{I}(k_X)}(f^{-1}G, F) \simeq \mathrm{Hom}_{\mathrm{I}(k_Y)}(G, f_*F).$$

Proof. — Let $F = \text{“}\varinjlim_i\text{”}\, F_i$ and $G = \text{“}\varinjlim_j\text{”}\, G_j$. One has the chain of isomorphisms

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{I}(k_X)}(f^{-1}G, F) &\simeq \mathrm{Hom}_{\mathrm{I}(k_X)}(\text{“}\varinjlim_{j, U\subset\subset X}\text{”}\, (f^{-1}G_j)_U, \text{“}\varinjlim_i\text{”}\, F_i) \\
&\simeq \varprojlim_{j, U\subset\subset X} \varinjlim_i \mathrm{Hom}_{\mathrm{Mod}(k_X)}((f^{-1}G_j)_U, F_i) \\
&\simeq \varprojlim_{j, U\subset\subset X} \varinjlim_i \mathrm{Hom}_{\mathrm{Mod}(k_X)}(f^{-1}G_j, \Gamma_U F_i) \\
&\simeq \varprojlim_{j, U\subset\subset X} \varinjlim_i \mathrm{Hom}_{\mathrm{Mod}(k_Y)}(G_j, f_*\Gamma_U F_i) \\
&\simeq \varprojlim_j \mathrm{Hom}_{\mathrm{I}(k_Y)}(G_j, \varprojlim_{U\subset\subset X} \text{“}\varinjlim_i\text{”}\, f_*\Gamma_U F_i) \\
&\simeq \mathrm{Hom}_{\mathrm{I}(k_Y)}(G, f_*F).
\end{aligned}$$

q.e.d.

Corollary 4.3.5. — *The functor $f_*\colon \mathrm{I}(k_X) \to \mathrm{I}(k_Y)$ is left exact and commutes with $\varprojlim$.*

Proposition 4.3.6. — *The diagram below commutes:*

$$\begin{tikzcd}
\mathrm{Mod}(k_X) \arrow[r, "f_*"] \arrow[d, "\iota_X"'] & \mathrm{Mod}(k_Y) \arrow[d, "\iota_Y"] \\
\mathrm{I}(k_X) \arrow[r, "f_*"] \arrow[d, "\alpha_X"'] & \mathrm{I}(k_Y) \arrow[d, "\alpha_Y"] \\
\mathrm{Mod}(k_X) \arrow[r, "f_*"] & \mathrm{Mod}(k_Y).
\end{tikzcd}$$

Proof. — (i) $f_*\iota_X \simeq \iota_Y f_*$ follows from $\alpha_X f^{-1} \simeq f^{-1}\alpha_Y$ and the fact that f_* and ι are right adjoint of f^{-1} and α.

(ii) Let $F \simeq \text{“}\varinjlim_i\text{”}\, F_i$ with $F_i \in \mathrm{Mod}^c(k_X)$. Then

$$\begin{aligned}
f_*(\alpha_X F) &\simeq \varprojlim_K f_*((\alpha_X F)_K) \simeq \varprojlim_K f_*(\varinjlim_i F_{iK}) \\
&\simeq \varprojlim_K f_!(\varinjlim_i F_{iK}) \simeq \varprojlim_K \varinjlim_i f_!(F_{iK}) \\
&\simeq \varprojlim_K \alpha_Y(\text{“}\varinjlim_i\text{”}\, f_! F_{iK}) \simeq \alpha_Y(\varprojlim_K \text{“}\varinjlim_i\text{”}\, f_* F_{iK}) \simeq \alpha_Y(f_*F).
\end{aligned}$$

q.e.d.

Corollary 4.3.7. — *The diagram below commutes:*

$$\begin{tikzcd} \mathrm{Mod}(k_Y) \arrow[r, "f^{-1}"] \arrow[d, "\beta_Y"'] & \mathrm{Mod}(k_X) \arrow[d, "\beta_X"'] \\ \mathrm{I}(k_Y) \arrow[r, "f^{-1}"] & \mathrm{I}(k_X). \end{tikzcd}$$

Proof. — By adjunction, using Proposition 4.3.6. q.e.d.

Proposition 4.3.8. — *Let $F \in \mathrm{I}(k_X)$ and $G \in \mathrm{I}(k_Y)$. Then*

$$\begin{aligned} \mathcal{I}hom\,(G, f_*F) &\simeq f_*\mathcal{I}hom\,(f^{-1}G, F), \\ \mathcal{H}om\,(G, f_*F) &\simeq f_*\mathcal{H}om\,(f^{-1}G, F). \end{aligned}$$

Proof. — The second formula follows from the first one by applying the functor α_Y. To prove the first formula, consider $K \in \mathrm{I}(k_Y)$. Then

$$\begin{aligned} &\mathrm{Hom}_{\mathrm{I}(k_Y)}(K, \mathcal{I}hom\,(G, f_*F)) \simeq \mathrm{Hom}_{\mathrm{I}(k_Y)}(K \otimes G, f_*F) \\ &\quad \simeq \mathrm{Hom}_{\mathrm{I}(k_X)}(f^{-1}(K \otimes G), F) \simeq \mathrm{Hom}_{\mathrm{I}(k_X)}(f^{-1}K \otimes f^{-1}G, F) \\ &\quad \simeq \mathrm{Hom}_{\mathrm{I}(k_X)}(f^{-1}K, \mathcal{I}hom\,(f^{-1}G, F)) \simeq \mathrm{Hom}_{\mathrm{I}(k_Y)}(K, f_*\mathcal{I}hom\,(f^{-1}G, F)). \end{aligned}$$

q.e.d.

Proposition 4.3.9. — *For $f\colon X \to Y$ and $g\colon Y \to Z$, one has $(g \circ f)_* \simeq g_* \circ f_*$.*

By adjunction, this follows from Proposition 4.3.3.

Note that f_* does not commute with β in general.

Proposition 4.3.10. — *Assume that $f\colon X \to Y$ is the embedding of a locally closed subset. Then $f^{-1}f_* \xrightarrow{\sim} \mathrm{id}$.*

Proof. — First, that assume f is an open embedding. Let $F \in \mathrm{I}(k_X)$ and let $G \in \mathrm{Mod}^c(k_X)$. One has the isomorphisms:

$$\begin{aligned} \mathrm{Hom}_{\mathrm{I}(k_X)}(G, f^{-1}f_*F) &\simeq \mathrm{Hom}_{\mathrm{I}(k_Y)}(f_{!!}G, f_*F) \\ &\simeq \mathrm{Hom}_{\mathrm{I}(k_X)}(f^{-1}f_{!!}G, F) \\ &\simeq \mathrm{Hom}_{\mathrm{I}(k_X)}(G, F). \end{aligned}$$

Here, the first and second isomorphisms follow from Theorem 3.3.14.

If f is a closed embedding the result follows from the classical one for sheaves since both f^{-1} and f_* commute with inductive limits in this case. q.e.d.

Proper direct image. — Let $F = \text{“}\varinjlim_i\text{”}\, F_i \in \mathrm{I}(k_X)$ for $F_i \in \mathrm{Mod}^c(k_X)$. One defines $f_{!!}F \in \mathrm{I}(k_Y)$ by the formula

$$f_{!!}\,\text{“}\varinjlim_i\text{”}\, F_i = \text{“}\varinjlim_i\text{”}\, f_! F_i.$$

If $i_U : U \hookrightarrow X$ is an open embedding, $i_{U!!}$ coincides with the previous construction of section 3.3.

Note that the natural morphism $f_{!!}\iota_X F \to \iota_Y f_! F$ is not an isomorphism for $F \in \mathrm{Mod}(k_X)$, in general. Here, this morphism is described by the morphisms (we do not write the functors ι_X, ι_Y for short)

$$f_{!!}F \simeq \text{“}\varinjlim_{U\subset\subset X}\text{”}\, f_!(F_U) \simeq \text{“}\varinjlim_{U\subset\subset X,\, V\subset\subset Y}\text{”}\, (f_!(F_U))_V \to \text{“}\varinjlim_{V\subset\subset Y}\text{”}(f_!F)_V. \tag{4.3.1}$$

To avoid any confusion, we use the different notation $f_{!!}$.

Lemma 4.3.11. — *If the support of $F \in \mathrm{Mod}(k_X)$ is proper over Y, then $f_{!!}\iota_X F \xrightarrow{\sim} \iota_Y f_! F$.*

Proof. — In this case it is obvious that the last morphism in (4.3.1) is an isomorphism. q.e.d.

Proposition 4.3.12. — *The functor $f_{!!}$ is left exact and commutes with “$\varinjlim$”.*

The proof is evident.

Proposition 4.3.13. — *The diagram below commutes:*

$$\begin{array}{ccc} \mathrm{I}(k_X) & \xrightarrow{f_{!!}} & \mathrm{I}(k_Y) \\ {\scriptstyle \alpha_X}\downarrow & & {\scriptstyle \alpha_Y}\downarrow \\ \mathrm{Mod}(k_X) & \xrightarrow{f_!} & \mathrm{Mod}(k_Y). \end{array}$$

Proof. — One has the chain of isomorphisms

$$\begin{aligned} \alpha_Y f_{!!}(\text{“}\varinjlim_i\text{”}\, F_i) &\simeq \alpha_Y(\text{“}\varinjlim_i\text{”}\, f_! F_i) \simeq \varinjlim_i f_! F_i \\ &\simeq f_!(\varinjlim_i F_i) \simeq f_! \alpha_X(\text{“}\varinjlim_i\text{”}\, F_i). \end{aligned}$$

q.e.d.

Theorem 4.3.14. — *For $G \in \mathrm{I}(k_Y)$ and let $F \in \mathrm{I}(k_X)$, one has*

$$G \otimes f_{!!}F \simeq f_{!!}(f^{-1}G \otimes F).$$

Proof. — Let $G = \text{“}\varinjlim_j\text{”}\, G_j$ for $G_j \in \mathrm{Mod}^c(k_Y)$ and $F = \text{“}\varinjlim_i\text{”}\, F_i$ for $F_i \in \mathrm{Mod}^c(k_X)$. Then we have

$$f^{-1}G \otimes F \simeq \text{“}\varinjlim_{i,j}\text{”}(f^{-1}G_j) \otimes F_i\,.$$

Since the F_i's have compact support, $(f^{-1}G_j) \otimes F_i \in \mathrm{Mod}^c(k_X)$, and

$$\begin{aligned} f_{!!}(f^{-1}G \otimes F) &\simeq \text{“}\varinjlim_{i,j}\text{”}\, f_!(f^{-1}G_j \otimes F_i) \simeq \text{“}\varinjlim_{i,j}\text{”}\, G_j \otimes f_!F_i \\ &\simeq (\text{“}\varinjlim_j\text{”}\, G_j) \otimes (\text{“}\varinjlim_i\text{”}\, f_!F_i) \simeq G \otimes f_{!!}F. \end{aligned}$$

q.e.d.

Corollary 4.3.15. — *Let $G \in \mathrm{I}(k_Y)$ and let $F \in \mathrm{I}(k_X)$.*

(i) *There are natural morphisms*

$$\begin{aligned} f_{!!}\mathcal{I}hom\,(f^{-1}G, F) &\to \mathcal{I}hom\,(G, f_{!!}F)\,, \\ f_!\mathcal{H}om\,(f^{-1}G, F) &\to \mathcal{H}om\,(G, f_{!!}F)\,. \end{aligned}$$

(ii) *If $G \in \mathrm{Mod}(k_Y)$, these morphisms are isomorphisms.*

Proof. — Since the assertions for the second morphism follows from the first one by applying α, we shall prove the assertion for the first morphism.

(i) Applying Corollary 4.2.6, we have a natural morphism

$$f^{-1}G \otimes \mathcal{I}hom\,(f^{-1}G, F) \to F\,.$$

Hence, by Theorem 4.3.14, we get the morphism

$$G \otimes f_{!!}\mathcal{I}hom\,(f^{-1}G, F) \simeq f_{!!}(f^{-1}G \otimes \mathcal{I}hom\,(f^{-1}G, F)) \to f_{!!}F.$$

Applying Proposition 4.2.5, we get the desired morphism.

(ii) Let $F \simeq \text{“}\varinjlim_i\text{”}\, F_i$ with the F_i's in $\mathrm{Mod}^c(k_X)$. One has the chain of isomorphisms

$$\begin{aligned} \mathcal{I}hom\,(G, f_{!!}F) &\simeq \mathcal{I}hom\,(G, f_{!!}\,\text{“}\varinjlim_i\text{”}\, F_i) \\ &\simeq \mathcal{I}hom\,(G, \text{“}\varinjlim_i\text{”}\, f_!F_i) \simeq \text{“}\varinjlim_i\text{”}\, \mathcal{H}om\,(G, f_*F_i) \\ &\simeq \text{“}\varinjlim_i\text{”}\, f_*\mathcal{H}om\,(f^{-1}G, F_i) \simeq \text{“}\varinjlim_i\text{”}\, f_!\mathcal{H}om\,(f^{-1}G, F_i) \\ &\simeq f_{!!}\,\text{“}\varinjlim_i\text{”}\, \mathcal{H}om\,(f^{-1}G, F_i) \simeq f_{!!}\mathcal{I}hom\,(f^{-1}G, F). \end{aligned}$$

Note that we have used the fact that $\mathcal{I}hom\,(K, \cdot\,)$ commutes with “$\varinjlim$” when K is a sheaf, and this does not hold when K is an ind-sheaf. q.e.d.

Proposition 4.3.16. — *There is a natural morphism of functors from* $\mathrm{I}(k_X)$ *to* $\mathrm{I}(k_Y)$*:*

$$f_{!!} \to f_*. \tag{4.3.2}$$

If $F \in \mathrm{I}(k_X)$ *has proper support over* Y*, this morphism induces an isomorphism* $f_{!!}F \xrightarrow{\sim} f_*F$*.*

Proof. — Let $F = \text{“}\varinjlim_i\text{”}\, F_i$ with $F_i \in \mathrm{Mod}^c(k_X)$. We have the morphisms

$$f_{!!}F \simeq \text{“}\varinjlim_i\text{”}\, f_! F_i \to \varprojlim_{K \subset\subset X} \text{“}\varinjlim_i\text{”}\, f_!(F_i)_K \simeq f_*F. \tag{4.3.3}$$

Here, K ranges over the family of compact subsets of X.

Assume that $\operatorname{supp} F$ is proper over Y. Let S be a closed subset of X proper over Y such that $\operatorname{supp} F$ is contained in the interior of S. We may replace F_i with $(F_i)_S$. Then the arrow in (4.3.3) is an isomorphism. q.e.d.

In general $f_{!!}$ and β do not commute. However:

Proposition 4.3.17. — (i) *There is a natural morphism of functors*

$$\beta_Y \circ f_{!!} \to f_{!!} \circ \beta_X. \tag{4.3.4}$$

(ii) *If f is an open embedding,* (4.3.4) *is an isomorphism.*

Proof. — (i) There is a chain of morphisms

$$f_{!!} \to f_{!!} \circ \alpha_X \circ \beta_X \simeq \alpha_Y \circ f_{!!} \circ \beta_X.$$

The result follows by adjunction.
(ii) The functor $f_{!!} \circ \beta_X$ is left adjoint to $\alpha_X \circ f^{-1} \simeq f^{-1} \circ \alpha_Y$ which is right adjoint to $\beta_Y \circ f_!$. q.e.d.

Consider now a Cartesian square of topological spaces

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\square"] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y. \end{tikzcd}$$

Theorem 4.3.18. — *There is a natural isomorphism of functors from* $\mathrm{I}(k_X)$ *to* $\mathrm{I}(k_{Y'})$*:*

$$f'_{!!}g'^{-1} \simeq g^{-1}f_{!!}.$$

Proof. — Let $F = \text{“}\varinjlim_i\text{”}\, F_i$ with $F_i \in \mathrm{Mod}^c(k_X)$. Since $\operatorname{supp} F_i$ is compact, the map $f'\colon \operatorname{supp}(g'^{-1}F_i) \to Y'$ is proper. Hence $f'_{!!}g'^{-1}F_i \simeq f'_! g'^{-1}F_i$, and we get

$$\begin{aligned} f'_{!!}g'^{-1}F &\simeq f'_{!!}(\text{“}\varinjlim_i\text{”}\, g'^{-1}F_i) \simeq \text{“}\varinjlim_i\text{”}\, f'_! g'^{-1}F_i \\ &\simeq \text{“}\varinjlim_i\text{”}\, g^{-1}f_! F_i \simeq g^{-1}\text{“}\varinjlim_i\text{”}\, f_! F_i \simeq g^{-1}f_{!!}F. \end{aligned}$$

q.e.d.

Corollary 4.3.19. — *Let V be an open subset of Y. Set $U = f^{-1}(V)$, $f_V := f|_V : U \to V$, and recall that i_V (resp. i_U) denotes the open inclusion $V \hookrightarrow Y$, (resp. $U \hookrightarrow X$). Let $F \in \mathrm{I}(k_X)$. Then*

$$i_V^{-1} f_* F \simeq f_{V*} i_U^{-1} F.$$

Proof. — Let $G \in \mathrm{Mod}^c(k_V)$. Then

$$\begin{aligned} \mathrm{Hom}_{\mathrm{I}(k_V)}(G, i_V^{-1}f_*F) &\simeq \mathrm{Hom}_{\mathrm{I}(k_Y)}(i_{V!!}G, f_*F) \simeq \mathrm{Hom}_{\mathrm{I}(k_X)}(f^{-1}i_{V!!}G, F) \\ &\simeq \mathrm{Hom}_{\mathrm{I}(k_X)}(i_{U!!}f_V^{-1}G, F) \simeq \mathrm{Hom}_{\mathrm{I}(k_V)}(G, f_{V*}i_U^{-1}F). \end{aligned}$$

q.e.d.

Ind-stalk. — Let $x \in X$, and denote by j_x the embedding $\{x\} \hookrightarrow X$. For $F \in \mathrm{I}(k_X)$, it is natural to define its stalk at x by setting $F_x = j_x^{-1}F$. However, there exist non-zero ind-sheaves on X whose stalk at every $x \in X$ vanishes.

Example 4.3.20. — We keep the same notation as in Example 4.2.17, that is, we define the ind-sheaf N by the exact sequence $0 \to N \to {}_xk \to k_x \to 0$. Since j_x^{-1} commutes with inductive limits, we get that $j_y^{-1}N = 0$ for all $y \in X$.

Hence, we define the ind-stalk of F at x as ${}_xF$. This is an ind-sheaf whose support is contained in $\{x\}$. By Proposition 3.3.16, for $F \in \mathrm{Mod}(k_X)$ and $K \in \mathrm{I}(k_X)$ one has the isomorphism:

$$\mathcal{H}om\,(F, K)_x \simeq \mathrm{Hom}_{\mathrm{I}(k_X)}(F, {}_xK).$$

The functor which associates with F the family of its ind-stalks is faithful:

Proposition 4.3.21. — *Let $f\colon F \to G$ be a morphism in $\mathrm{I}(k_X)$, and assume that for each $x \in X$ the induced morphism ${}_xf\colon {}_xF \to {}_xG$ is the zero morphism. Then f is the zero morphism.*

Proof. — One has the chain of equivalences

$f = 0 \Leftrightarrow$ for all $K \in \mathrm{Mod}^c(k_X)$, the induced morphism $\mathrm{Hom}_{\mathrm{I}(k_X)}(K, F) \to \mathrm{Hom}_{\mathrm{I}(k_X)}(K, G)$ is zero $\Leftrightarrow$ for all $x \in X$ and all K, $\mathcal{H}om\,(K, F)_x \to \mathcal{H}om\,(K, G)_x$ is zero $\Leftrightarrow$ for all $x \in X$ and all K, $\mathrm{Hom}_{\mathrm{I}(k_X)}(K, {}_xF) \to \mathrm{Hom}_{\mathrm{I}(k_X)}(K, {}_xG)$ is zero $\Leftrightarrow$ for all $x \in X$, ${}_xF \to {}_xG$ is zero. q.e.d.

Exercises to Chapter 4

Exercise 4.1. — Let $Y = \mathbb{R}, X = \{0\}$ and denote by $f\colon X \to Y$ the embedding. Let $G_n = k_{]-\frac{1}{n+1},\frac{1}{n+1}[} \in \mathrm{Mod}(k_Y)$. Prove that $f^{-1}(\prod_{n\in\mathbb{N}} G_n) \simeq k_X^{\oplus\mathbb{N}}$ and $\prod_{n\in\mathbb{N}} f^{-1}(G_n) \simeq k_X^{\mathbb{N}}$. Hence, f^{-1} does not commute with $\varprojlim$.

Exercise 4.2. — Let $f\colon X \to Y$ be as in Exercise 4.1. By choosing $F = k_X \in \mathrm{Mod}(k_X)$, prove that both f_* and $f_{!!}$ do not commute with β.

Exercise 4.3. — Let $X =]-1,1[$, $Y = \mathbb{R}$ and $f\colon X \hookrightarrow Y$ the embedding. Let $F_n = k_{]-1+\frac{1}{n},1-\frac{1}{n}[} \in \mathrm{Mod}(k_X)$. Show that "$\varinjlim_n$" $F_n \simeq k_X$, "$\varinjlim_n$" $f_*F_n \simeq {}_X(k_Y)$ and $f_*($"$\varinjlim_n$" $F_n) \simeq (k_Y)_{\overline{X}}$, Hence, f_* does not commute with "$\varinjlim$".

Exercise 4.4. — Let $f\colon X \to Y$ be as in Exercise 4.3. Prove that $f_{!!}k_X \simeq {}_X(k_Y)$ and $f_!k_X \simeq (k_Y)_X$. Hence, $f_{!!}\iota_X \not\simeq \iota_Y f_!$ in general.

Exercise 4.5. — Let $X = \{\mathrm{pt}\}$, $F_n = k$. Prove that $\beta_X(\prod_{n\in\mathbb{Z}} F_n)$ is not isomorphic to $\prod_{n\in\mathbb{Z}} \beta_X(F_n)$. Hence, β_X does not commute with $\varprojlim$.

Exercise 4.6. — Let $X = \mathbb{R}$, $Y = \{0\}$, and denote by $f\colon X \to Y$ the projection. Let $F_n = k_{\{n\}} \in \mathrm{Mod}(k_X)$. Prove that $\prod_{n\in\mathbb{N}} F_n \simeq \bigoplus_{n\in\mathbb{N}} F_n \simeq$ "$\bigoplus_{n\in\mathbb{N}}$" F_n. Deduce that $f_{!!}(\prod_{n\in\mathbb{N}} F_n) \simeq k^{\oplus\,\mathbb{N}}$ and $\prod_{n\in\mathbb{N}} f_{!!}F_n \simeq k^{\mathbb{N}}$. Hence, $f_{!!}$ does not commute with $\varprojlim$.

Exercise 4.7. — Assume that k is a field. Let $K \in \mathrm{Mod}(k_X)$ be a soft sheaf. Prove that $F \otimes K$ is soft for any sheaf $F \in \mathrm{Mod}(k_X)$.
(Hint: adapt the proof of Proposition 4.2.21.)

Exercise 4.8. — Assume that k is a field. Let $W = k^{\mathbb{N}}$, let Σ denote the the family of subspaces of W with countable dimension and let $G =$ "$\varinjlim_{V\in\Sigma}$" V.

(i) Let U be a relatively compact open subset of X. Prove that

$$\mathrm{Hom}_{\mathrm{I}(k_X)}(k_{XU}, W_X) \simeq \varinjlim_{V\in\Sigma} \mathrm{Hom}_{\mathrm{I}(k_X)}(k_{XU}, V_X)$$

(ii) Define $F = (W/G)_X \simeq$ "$\varinjlim_V$" W_X/V_X. Prove that for any open subset U of X one has

$$\mathrm{Hom}_{\mathrm{I}(k_X)}(k_{XU}, F) = 0.$$

(Hint: in (i) remark that any open covering of U admits a countable subcovering.)

CHAPTER 5

DERIVED CATEGORIES OF IND-SHEAVES

In this chapter as in the preceding one, we work in a given universe $\mathcal{U}$. All topological spaces (denoted by X, Y, etc.) are assumed to be Hausdorff, locally compact, and with a countable base of open subsets. Moreover k is a field.

Notation 5.0.1. — If $\varphi\colon \mathcal{C} \to \mathcal{C}'$ is an additive functor of abelian categories, we shall usually still denote by φ instead of $K^+(\varphi)$ the associated functor from $K^+(\mathcal{C})$ to $K^+(\mathcal{C}')$, and similarly with bifunctors. For example, we still denote by $\mathcal{I}hom$ the functor $K^-(\mathrm{I}(k_X))^{\mathrm{op}} \times K^+(\mathrm{I}(k_X)) \to K^+(\mathrm{I}(k_X))$ associated with $\mathcal{I}hom$.

5.1. Internal operations

Recall that the categories $\mathrm{Mod}(k_X)$ and $\mathrm{Mod}^c(k_X)$ have enough injectives and systems of strict generators.

As usual, we denote by $D(k_X)$ the derived category of the category $\mathrm{Mod}(k_X)$ of sheaves of k-vector spaces on X. More generally, if $\mathcal{A}$ is a sheaf of rings on X we write $D(\mathcal{A})$ instead of $D(\mathrm{Mod}(\mathcal{A}))$.

We denote by $D(\mathrm{I}(k_X))$ the derived category of the category $\mathrm{I}(k_X)$ of ind-objects of $\mathrm{Mod}^c(k_X)$.

Proposition 5.1.1. — *The natural functor $D(k_X) \to D(\mathrm{I}(k_X))$ induced by ι_X is fully faithful. In particular, $D(k_X)$ is equivalent to the full triangulated subcategory of $D(\mathrm{I}(k_X))$ consisting of objects F such that $H^j(F) \in \mathrm{Mod}(k_X)$ for all j.*

Proof. — Let $F, G \in K(\mathrm{Mod}(k_X))$. We have

$$\begin{aligned}
\mathrm{Hom}_{D(\mathrm{I}(k_X))}(\iota_X G, \iota_X F) &\simeq \varinjlim_{G' \xrightarrow[qis]{} G} \mathrm{Hom}_{K(\mathrm{I}(k_X))}(G', \iota_X F)\,, \quad G' \in K(\mathrm{I}(k_X)) \\
&\simeq \varinjlim_{G' \xrightarrow[qis]{} G} \mathrm{Hom}_{K(\mathrm{Mod}(k_X))}(\alpha_X G', F)\,, \quad G' \in K(\mathrm{I}(k_X)) \\
&\simeq \varinjlim_{G'' \xrightarrow[qis]{} G} \mathrm{Hom}_{K(\mathrm{Mod}(k_X))}(G'', F)\,, \quad G'' \in K(\mathrm{Mod}(k_X)) \\
&\simeq \mathrm{Hom}_{D(k_X)}(G, F).
\end{aligned}$$

Here, we have used the fact that the category $\{\alpha_X G' \to G\}$ is cofinal in the category $\{G'' \to G\}$.

Let us prove that if $F \in D(\mathrm{I}(k_X))$ satisfies $H^j(F) \in \mathrm{Mod}(k_X)$ for any j, then $F \in D(k_X)$. There is a natural morphism $F \to \iota_X \alpha_X F$, and it is an isomorphism because it induces isomorphisms in cohomologies. q.e.d.

We introduce the categories

$$\begin{aligned}
\mathcal{P}(k_X) &:= \{\text{“}\bigoplus_j\text{”}\, G_j\,;\, G_j \in \mathrm{Mod}(k_X)\}, \\
\mathcal{P}_c(k_X) &:= \{\text{“}\bigoplus_j\text{”}\, G_j\,;\, G_j \in \mathrm{Mod}^c(k_X)\}, \qquad (5.1.1)\\
\mathcal{I}_q(k_X) &:= \{F \in \mathrm{I}(k_X)\,;\, F \text{ is quasi-injective}\}.
\end{aligned}$$

If there is no risk of confusion, we shall write $\mathcal{P}$, $\mathcal{P}_c$ and $\mathcal{I}_q$ instead of $\mathcal{P}(k_X)$, $\mathcal{P}_c(k_X)$ and $\mathcal{I}_q(k_X)$, respectively. Note that $\mathcal{P}$ and $\mathcal{P}_c$ are generating and $\mathcal{I}_q$ is cogenerating.

Lemma 5.1.2. — (i) *The category $\mathcal{P}$ is stable by $\otimes$ and the category $\mathcal{I}_q$ is stable by $\prod$,*
(ii) *if $G \in \mathcal{P}$ and $F \in \mathcal{I}_q$, then $\mathcal{I}hom\,(G, F) \in \mathcal{I}_q$,*
(iii) *if $G \in \mathcal{P}$ and $F \in \mathcal{I}_q$, then $\mathcal{H}om\,(G, F)$ is soft.*

Proof. — (i) is obvious.

(ii) Since $\mathcal{I}hom\,(\,\cdot\,, F)$ sends direct sums to direct products, we may assume by (i) that $G \in \mathrm{Mod}(k_X)$. In this case, $\mathrm{Hom}_{\mathrm{I}(k_X)}(\,\cdot\,, \mathcal{I}hom\,(G, F)) \simeq \mathrm{Hom}_{\mathrm{I}(k_X)}(\,\cdot \otimes G, F)$ is an exact functor on $\mathrm{Mod}^c(k_X)$.

(iii) follows from (ii) and Proposition 4.2.19. Note that

$$\mathcal{H}om\,(G, F) \simeq \mathcal{H}om\,(k_X, \mathcal{I}hom\,(F, G)).$$

q.e.d.

Theorem 5.1.3. — (a) *The category $\mathcal{P}^{\mathrm{op}} \times \mathcal{I}_q$ is injective with respect to the functors $\mathrm{Hom}_{\mathrm{I}(k_X)}$, $\mathcal{I}hom$, $\mathcal{H}om$.*

(b) *The functors below are well-defined:*

$$\begin{aligned}\mathrm{RHom}_{\mathrm{I}(k_X)} &: D^-(\mathrm{I}(k_X))^{\mathrm{op}} \times D^+(\mathrm{I}(k_X)) \to D^+(\mathrm{Mod}(k))\\ R\mathcal{I}hom &: D^-(\mathrm{I}(k_X))^{\mathrm{op}} \times D^+(\mathrm{I}(k_X)) \to D^+(\mathrm{I}(k_X))\\ R\mathcal{H}om &: D^-(\mathrm{I}(k_X))^{\mathrm{op}} \times D^+(\mathrm{I}(k_X)) \to D^+(k_X).\end{aligned}$$

Proof. — (b) follows from (a) by Proposition 1.4.6.

(a1) We shall show first that $\mathcal{P}_c^{\mathrm{op}} \times \mathcal{I}_q$ is injective with respect to $\mathrm{Hom}_{\mathrm{I}(k_X)}$.

(i) Let $G = \text{“}\bigoplus_j\text{”}\, G_j \in \mathcal{P}_c$. In order to see that $\mathcal{I}_q$ is injective with respect to the functor $\mathrm{Hom}_{\mathrm{I}(k_X)}(G,\,\cdot\,)$, we shall apply Corollary 1.4.4. Consider an exact sequence $0 \to F' \underset{f}{\to} F \underset{g}{\to} F'' \to 0$ in $\mathrm{I}(k_X)$ with F', F, F'' in $\mathcal{I}_q$. Let us prove that the functor $\mathrm{Hom}_{\mathrm{I}(k_X)}(G,\,\cdot\,)$ applied to this exact sequence gives an exact sequence. We have

$$\mathrm{Hom}_{\mathrm{I}(k_X)}(G,F) \simeq \prod_j \mathrm{Hom}_{\mathrm{I}(k_X)}(G_j,F) \tag{5.1.2}$$

and similarly with F replaced by F' or F''. Since the functor $\prod_j \mathrm{Hom}\,(G_j,\,\cdot\,)$ is exact on quasi-injective sheaves by Corollary 4.2.20, the result follows.

(ii) Let $F \in \mathcal{I}_q$. In order to see that $\mathcal{P}_c$ is injective with respect to the functor $\mathrm{Hom}_{\mathrm{I}(k_X)}(\,\cdot\,,F)$, we shall apply Theorem 1.4.3. Consider an epimorphism $H \twoheadrightarrow G''$ in $\mathrm{I}(k_X)$ with $G'' = \text{“}\bigoplus_j\text{”}\, G''_j \in \mathcal{P}_c$. By Proposition 1.3.2, for each j, there exist $G_j \in \mathrm{Mod}^c(k_X)$ and an epimorphism $G_j \twoheadrightarrow G''_j$ such that the composition $G_j \to G''_j \to G''$ factors through H. Let $G'_j = \ker(G_j \to G''_j)$. We get an exact sequence

$$0 \to \text{“}\bigoplus_j\text{”}\, G'_j \to \text{“}\bigoplus_j\text{”}\, G_j \to \text{“}\bigoplus_j\text{”}\, G''_j \to 0 \tag{5.1.3}$$

such that the morphism $\text{“}\bigoplus_j\text{”}\, G_j \to G''$ factors through H. By (5.1.2), the sequence (5.1.3) will remain exact after applying the functor $\mathrm{Hom}_{\mathrm{I}(k_X)}(\,\cdot\,,F)$.

(a2) Next let us show that $\mathcal{P}^{\mathrm{op}} \times \mathcal{I}_q$ is injective with respect to the functor $\mathcal{I}hom$. For $G \in \mathcal{P}$, let us prove that $\mathcal{I}hom\,(G, I^\bullet)$ is an exact sequence if $I^\bullet$ is an exact sequence in $\mathcal{I}_q$ bounded from below. It is enough to prove that for any $S \in \mathrm{Mod}^c(k_X)$, $H^\bullet := \mathrm{Hom}_{\mathrm{I}(k_X)}(S, \mathcal{I}hom\,(G, I^\bullet))$ is exact. We have the isomorphism

$$H^\bullet \simeq \mathrm{Hom}_{\mathrm{I}(k_X)}(S \otimes G, I^\bullet).$$

Since $S \otimes G \in \mathcal{P}_c$, (a1) implies the exactness of $H^\bullet$.

The exactness in G is similarly proved.

(a3) The case of the functor $\mathcal{H}om$ follows from (a2) by applying the exact functor α_X.

(a4) Finally let us prove that $\mathcal{P}^{\mathrm{op}} \times \mathcal{I}_q$ is injective with respect to $\mathrm{Hom}_{\mathrm{I}(k_X)}$. Note that $\mathcal{H}om\,(F,G)$ is soft for $G \in \mathcal{P}$ and $F \in \mathcal{I}_q$ by Lemma 5.1.2 (iii). Hence the assertion follows from (a3) and the isomorphism $\mathrm{Hom}_{\mathrm{I}(k_X)}(G,F) \simeq \Gamma(X\,;\,\mathcal{H}om\,(G,F))$. q.e.d.

Proposition 5.1.4. — *Let $G, K \in D^-(\mathrm{I}(k_X))$ and let $F \in D^+(\mathrm{I}(k_X))$. Then*
(i) $\alpha_X R\mathcal{I}hom\,(G,F) \simeq R\mathcal{H}om\,(G,F)$,
(ii) $R\mathcal{I}hom\,(G\otimes K,F) \simeq R\mathcal{I}hom\,(G,R\mathcal{I}hom\,(K,F))$.

Proof. — (i) follows from the isomorphism $\alpha_X \circ \mathcal{I}hom \simeq \mathcal{H}om$ and the exactness of α_X.

(ii) We may assume $G, K \in K^-(\mathcal{P})$ and $F \in K^+(\mathcal{I}_q)$. Since $G\otimes K \in K^-(\mathcal{P})$, and $\mathcal{I}hom\,(K,F) \in K^+(\mathcal{I}_q)$, we get the isomorphisms

$$\begin{aligned} R\mathcal{I}hom\,(G\otimes K,F) &\simeq \mathcal{I}hom\,(G\otimes K,F) \\ &\simeq \mathcal{I}hom\,(G,\mathcal{I}hom\,(K,F)) \simeq R\mathcal{I}hom\,(G,R\mathcal{I}hom\,(K,F)). \end{aligned}$$

q.e.d.

Proposition 5.1.5. — *Let $G \in D^-(\mathrm{I}(k_X))$ and $F \in D^+(\mathrm{I}(k_X))$. Then*

$$\mathrm{RHom}_{\mathrm{I}(k_X)}(G,F) \simeq \mathrm{R}\Gamma(X;R\mathcal{H}om\,(G,F)).$$

Proof. — We may assume that $G \in K^-(\mathcal{P})$ and $F \in K^+(\mathcal{I}_q)$. Then

$$\begin{aligned} \mathrm{RHom}_{\mathrm{I}(k_X)}(G,F) &\simeq \mathrm{Hom}_{\mathrm{I}(k_X)}(G,F) \\ &\simeq \Gamma(X;\mathcal{H}om\,(G,F)) \simeq \mathrm{R}\Gamma(X;R\mathcal{H}om\,(G,F)). \end{aligned}$$

Note that the last isomorphism follows since $\mathcal{H}om\,(G,F)$ is soft. q.e.d.

One difficulty of the theory of ind-sheaves is that the category $\mathrm{I}(k_X)$ does not have enough injectives. This difficulty is partly overcome by the use of Theorem 1.5.4.

Theorem 5.1.6. — *Let S be a small set contained in $\mathrm{Ob}(\mathrm{I}(k_X))$. Then there exists a small full abelian subcategory $\mathcal{C}_0$ of $\mathrm{Mod}^c(k_X)$ with the properties below.*
(i) $S \subset \mathrm{Ind}(\mathcal{C}_0)$.
(ii) *$\mathcal{C}_0$ satisfies the properties in (a) of* Theorem 1.5.4. *In particular, $\mathcal{C}_0$ as well as $\mathrm{Ind}(\mathcal{C}_0)$ are stable by sub-objects and quotients,*
(iii) *$\mathrm{Ind}(\mathcal{C}_0)$ has enough injectives and such objects are quasi-injective in $\mathrm{I}(k_X)$.*
(iv) *$\mathcal{C}_0$ is stable by $\otimes$, $\mathcal{H}om$. Moreover, $k_U \in \mathcal{C}_0$ for all open $U \subset\subset X$.*
(v) *If $F \in S$, then $\mathcal{I}hom\,(F,\cdot\,)$ sends $\mathrm{Ind}(\mathcal{C}_0)$ to $\mathrm{Ind}(\mathcal{C}_0)$.*
(vi) *Let $K \in \mathrm{Ind}(\mathcal{C}_0)$ and assume K is injective in this category.*
  (a) *If $F \in S$, then $\mathcal{I}hom\,(F,K)$ is quasi-injective in $\mathrm{I}(k_X)$.*
  (b) *If $F \in S$, then $H^kR\mathcal{I}hom\,(F,K) = 0$ for $k \neq 0$.*
  (c) *If $F \in \mathrm{Ind}(\mathcal{C}_0)$, then $\mathcal{H}om\,(F,K)$ is injective in $\mathrm{Mod}(k_X)$.*
  (d) *If $F \in \mathrm{Ind}(\mathcal{C}_0)$, then $H^kR\mathcal{H}om\,(F,K) = 0$ and $H^k\mathrm{RHom}_{\mathrm{I}(k_X)}(F,K) = 0$ for $k \neq 0$.*

Proof. — (i)–(iv) There exists a small set B contained in $\mathrm{Ob}(\mathrm{Mod}^c(k_X))$ such that S consists of F_s's and $F_s \simeq$ "$\varinjlim_{i\in I_s}$" $F_{s,i}$ with I_s small and filtrant and $F_{s,i} \in B$. Set

$I = \bigsqcup_s I_s$. We may assume from the beginning that S contains the sheaves k_U, for all open subsets $U \subset\subset X$.

With slight modifications of the proof of Theorem 1.5.4 (see [**13**]), we may construct a small full subcategory $\mathcal{C}_0$ satisfying conditions (a) and (b) of this theorem and such that $\mathcal{C}_0$ contains B and is stable by $\otimes$, $\mathcal{H}om$. Moreover, we may assume that $\mathcal{C}_0$ is stable by products indexed by I of sheaves in $\mathcal{C}_0$ with supports contained in a fixed compact subset. Then (i)–(iv) follow from this statement.

(v) Let $F = F_s \simeq$ "$\varinjlim_{i\in I_s}$" F_i and $G \simeq$ "$\varinjlim_{j\in J}$" G_j with $G_j, F_i \in \mathcal{C}_0$. Then $\mathcal{I}hom\,(F,G)$ is a sub-object of $\prod_{i\in I_s}$ "$\varinjlim_{j\in J}$" $\mathcal{H}om\,(F_i, G_j)$. One has

$$\prod_{i\in I_s} \text{“}\varinjlim_{j\in J}\text{”}\, \mathcal{H}om\,(F_i,G_j) \simeq \text{“}\varinjlim_{\varphi,U}\text{”}\Big(\prod_{i\in I_s} \mathcal{H}om\,(F_i, G_{\varphi(i)})_U\Big),$$

where φ ranges over the set of maps $J \to I_s$ and U over the family of open subsets $U \subset\subset X$.
Then (v) follows since $\mathcal{C}_0$ is stable by $\mathcal{H}om$ and by product indexed by I_s.

(vi) (a) Let $G \in \mathrm{Ind}(\mathcal{C}_0)$. Then

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Ind}(\mathcal{C}_0)}(G, \mathcal{I}hom\,(F,K)) &\simeq \mathrm{Hom}_{\mathrm{I}(k_X)}(G, \mathcal{I}hom\,(F,K)) \\ &\simeq \mathrm{Hom}_{\mathrm{I}(k_X)}(G\otimes F, K) \simeq \mathrm{Hom}_{\mathrm{Ind}(\mathcal{C}_0)}(G\otimes F, K).\end{aligned}$$

and this functor is exact with respect to G since $G \otimes F \in \mathrm{Ind}(\mathcal{C}_0)$. Therefore, $\mathcal{I}hom\,(F,K)$ is injective in $\mathrm{Ind}(\mathcal{C}_0)$, hence quasi-injective in $\mathrm{I}(k_X)$.

(vi) (b) Let $F_\bullet \to F$ be a resolution of F with the components of $F_\bullet$ in $\mathcal{P}$. We may assume from the beginning that the F_j's belong to S. We may also assume that S is stable by kernels and cokernels.
Since K is quasi-injective, $R\mathcal{I}hom\,(F,K)$ is represented by $\mathcal{I}hom\,(F_\bullet, K)$. Hence, it is enough to prove that this complex is qis to $\mathcal{I}hom\,(F,K)$. By standard arguments, it is enough to show that $\mathcal{I}hom\,(\,\cdot\,,K)$ is exact on S. This follows from the fact that $\mathrm{Hom}_{\mathrm{Ind}(\mathcal{C}_0)}(G, \mathcal{I}hom\,(H,K)) \simeq \mathrm{Hom}_{\mathrm{Ind}(\mathcal{C}_0)}(G\otimes H, K)$ is exact with respect to $H \in S$ for any $G \in \mathcal{C}_0$.

(vi) (c) It is enough to prove that $\mathcal{H}om\,(F,K)$ is flabby. Let U be an open subset of X. The monomorphism $F \otimes {}_U k \rightarrowtail F$ gives rise to the epimorphism

$$\begin{aligned}\Gamma(X; \mathcal{H}om\,(F,K)) &\simeq \mathrm{Hom}_{\mathrm{Ind}(\mathcal{C}_0)}(F,K) \\ &\twoheadrightarrow \mathrm{Hom}_{\mathrm{Ind}(\mathcal{C}_0)}(F\otimes {}_U k, K) \simeq \Gamma(U; \mathcal{H}om\,(F,K)).\end{aligned}$$

(vi) (d) Consider a resolution $F_\bullet \to F$ with the components of $F_\bullet$ in $\mathcal{P}_c \cap \mathrm{Ind}(\mathcal{C}_0)$. Then $R\mathcal{H}om\,(F,K)$ is represented by the complex $\mathcal{H}om\,(F_\bullet, K)$. Hence, it is enough to show that $\mathcal{H}om\,(\,\cdot\,,K)$ is exact on $\mathrm{Ind}(\mathcal{C}_0)$. This follows from the formula

$$\Gamma(U; \mathcal{H}om\,(F,K)) \simeq \mathrm{Hom}_{\mathrm{Ind}(\mathcal{C}_0)}({}_U k\otimes F, K). \tag{5.1.4}$$

The case of $\mathrm{RHom}_{\mathrm{I}(k_X)}(F, K)$ is similar. q.e.d.

Definition 5.1.7. — Let S be a small subset of $\mathrm{Ob}(\mathrm{I}(k_X))$. We denote by $\mathcal{J}(S)$ the subcategory of $\mathrm{I}(k_X)$ consisting of objects F with the following properties:

(i) for any $G \in S$, $\mathcal{I}hom\,(G, F)$ is in $\mathcal{I}_q(k_X)$,

(ii) $H^k R\mathcal{I}hom\,(G, F) = 0$, $H^k R\mathcal{H}om\,(G, F) = 0$ and $H^k R\mathrm{Hom}_{\mathrm{I}(k_X)}(G, F) = 0$ for any $G \in S$ and any $k \neq 0$,

(iii) $\mathcal{H}om\,(G, F)$ is an injective sheaf for any $G \in S$.

Applying Theorem 5.1.6, we get the following result.

Corollary 5.1.8. — *For any small subset S of $\mathrm{Ob}(\mathrm{I}(k_X))$, the category $\mathcal{J}(S)$ is cogenerating in $\mathrm{I}(k_X)$.*

Proof. — Let $F \in \mathrm{I}(k_X)$. Let us take $\mathcal{C}_0$ as in Theorem 5.1.6 replacing S with $S \cup \{F\}$. Since $\mathrm{Ind}(\mathcal{C}_0)$ has enough injectives, F is embedded into an injective object in $\mathrm{Ind}(\mathcal{C}_0)$. The assertion follows from the fact that any injective objects in $\mathrm{Ind}(\mathcal{C}_0)$ is contained in $\mathcal{J}(S)$. q.e.d.

Corollary 5.1.9. — *Let $F \in D^+(\mathrm{I}(k_X))$ and $G \in D^-(\mathrm{I}(k_X))$. Then*

(i) $R\mathcal{I}hom\,(G, F) \simeq \text{“}\varinjlim_{F \xrightarrow[qis]{} F'}\text{”}\, \mathcal{I}hom\,(G, F')$ *in* $\mathrm{Ind}(D^+(\mathrm{I}(k_X)))$,

(ii) $R\mathcal{H}om\,(G, F) \simeq \text{“}\varinjlim_{F \xrightarrow[qis]{} F'}\text{”}\, \mathcal{H}om\,(G, F')$ *in* $\mathrm{Ind}(D^+(k_X))$,

(iii) $R\mathrm{Hom}_{\mathrm{I}(k_X)}(G, F) \simeq \text{“}\varinjlim_{F \xrightarrow[qis]{} F'}\text{”}\, \mathrm{Hom}_{\mathrm{I}(k_X)}(G, F')$ *in* $\mathrm{Ind}(D^+(\mathrm{Mod}(k)))$.

Here $F \xrightarrow[qis]{} F'$ ranges over the family of quasi-isomorphisms in $K^+(\mathrm{I}(k_X))$.

Proof. — (i) Choose a qis $G' \to G$ with $G' \in K^-(\mathcal{P})$ and a qis $F \to F'$ with $F' \in K^+(\mathcal{I}_q)$. By Theorem 5.1.3, $R\mathcal{I}hom\,(G, F) \simeq \mathcal{I}hom\,(G', F')$. Hence it is enough to prove that, given a qis $G' \to G$ as above, there exists a qis $F \to F'$ with $F' \in K^+(\mathcal{I}_q)$ such that the morphism $\mathcal{I}hom\,(G, F') \to \mathcal{I}hom\,(G', F')$ is a qis. Let G'' denote the mapping cone of $G' \to G$. We choose a small set S in $\mathrm{Ob}(\mathrm{I}(k_X))$ such that S contains all objects of the complexes F, G'' and is stable by kernels and cokernels. Let us apply Corollary 5.1.8. There exists a qis $F \to F'$ with $F' \in K^+(\mathcal{J}(S))$. By Corollary 5.1.8 (ii), $\mathcal{I}hom\,(G'', F')$ is qis to 0.

(ii) (iii) The proof is similar. q.e.d.

This corollary means that, for a fixed $G \in K^-(\mathrm{I}(k_X))$, the functor $\mathcal{I}hom\,(G, \cdot\,)$ is right derivable, and its right derived functor coincides with the right derived functor $R\mathcal{I}hom\,(\cdot\,,\cdot\,)$ of the bifunctor $\mathcal{I}hom\,(\cdot\,,\cdot\,)$.

Proposition 5.1.10. — *Let $K \in D^-(k_X)$, $K' \in D^+(k_X)$, $G \in D^-(\mathrm{I}(k_X))$ and $F \in D^+(\mathrm{I}(k_X))$. Then*

(i) $R\mathcal{H}om\,(\beta_X K, F) \simeq R\mathcal{H}om\,(K, \alpha_X F)$,
(ii) $R\mathcal{H}om\,(\beta_X K \otimes G, F) \simeq R\mathcal{H}om\,(K, R\mathcal{H}om\,(G, F))$,
(iii) $\mathrm{RHom}_{D^+(k_X)}(K, R\mathcal{H}om_{\mathrm{I}(k_X)}(G, F)) \simeq \mathrm{RHom}_{D^+(\mathrm{I}(k_X))}(\beta_X K \otimes G, F)$,
(iv) $R\mathcal{I}hom\,(K, F \otimes \beta_X K') \simeq R\mathcal{I}hom\,(K, F) \otimes \beta_X K'$,
(v) $R\mathcal{H}om\,(K, F \otimes \beta_X K') \simeq R\mathcal{H}om\,(K, F) \otimes K'$.

Proof. — (i) By Corollary 5.1.8, we can find a quasi-injective $F' \in K^+(\mathrm{I}(k_X))$ and a quasi-isomorphism $F \to F'$ such that $R\mathcal{H}om\,(\beta_X K, F)$ is represented by $\mathcal{H}om\,(\beta_X K, F') \simeq \mathcal{H}om\,(K, \alpha_X F')$. Moreover since $\alpha_X(F') \simeq \mathcal{H}om\,(k_X, F')$, we may assume that $\alpha_X(F')$ is injective. Hence $\mathcal{H}om\,(K, \alpha_X F')$ represents $R\mathcal{H}om\,(K, \alpha_X F)$.

(ii) By (i) we have

$$\begin{aligned} R\mathcal{H}om\,(\beta_X K \otimes G, F) &\simeq R\mathcal{H}om\,(\beta_X K, R\mathcal{I}hom\,(G, F)) \\ &\simeq R\mathcal{H}om\,(K, \alpha_X R\mathcal{I}hom\,(G, F)) \\ &\simeq R\mathcal{H}om\,(K, R\mathcal{H}om\,(G, F)). \end{aligned}$$

(iii) follows from (ii) by applying $\mathrm{R}\Gamma(X; \cdot\,)$.

(iv) We may assume that $K \in \mathcal{P}$ and $F \in \mathcal{I}_q$. Then the result follows from Proposition 4.2.22 since $F \otimes \beta_X K \in \mathcal{I}_q$ by Proposition 4.2.21.

(v) follows from (iv) by applying the functor α_X. q.e.d.

Proposition 5.1.11. — *Let $G \in D^-(k_X)$. Let $\{F_i\}_i$ be a small filtrant inductive system in $\mathrm{I}(k_X)$. Then we have*

$$\begin{aligned} \text{“}\varinjlim_i\text{”}\, H^k R\mathcal{I}hom\,(G, F_i) &\xrightarrow{\sim} H^k R\mathcal{I}hom\,(G, \text{“}\varinjlim_i\text{”}\, F_i)\,, \\ \varinjlim_i H^k R\mathcal{H}om\,(G, F_i) &\xrightarrow{\sim} H^k R\mathcal{H}om\,(G, \text{“}\varinjlim_i\text{”}\, F_i)\,. \end{aligned}$$

Moreover, if G has compact support, then

$$\varinjlim_i \mathrm{Hom}_{D(k_X)}(G, F_i) \xrightarrow{\sim} \mathrm{Hom}_{D(\mathrm{I}(k_X))}(G, \text{“}\varinjlim_i\text{”}\, F_i).$$

Proof. — We may assume that $G \in \mathrm{Mod}(k_X)$. Then the assertion follows from Theorem 1.5.6 and Corollary 5.1.9. q.e.d.

5.2. External operations

Let $f\colon X \to Y$ be a continuous map. Recall that X and Y are assumed to be Hausdorff, locally compact and with a countable base of open subsets.

Direct images. —

Proposition 5.2.1. — (i) *The category $\mathcal{I}_q(k_X)$ is f_*-injective.*
(ii) *If $F \in \mathcal{I}_q(k_X)$, then $f_*F \in \mathcal{I}_q(k_Y)$.*

Proof. — (i) Consider an exact sequence $0 \to F' \to F \to F'' \to 0$ in $\mathrm{I}(k_X)$, with F' quasi-injective. In order to prove that the sequence obtained by applying the functor f_* remains exact, it is enough to prove that for $H \in \mathrm{Mod}^c(k_Y)$, the sequence

$$0 \to \mathrm{Hom}_{\mathrm{I}(k_Y)}(H, f_*F') \to \mathrm{Hom}_{\mathrm{I}(k_Y)}(H, f_*F) \to \mathrm{Hom}_{\mathrm{I}(k_Y)}(H, f_*F'') \to 0$$

is exact. This follows immediately from the adjunction formula of Proposition 4.3.4 and Corollary 4.2.20.
(ii) A similar argument proves the exactness of the functor $\mathrm{Hom}_{\mathrm{I}(k_Y)}(\,\cdot\,, f_*F)$ on the category $\mathrm{Mod}^c(k_Y)$ by Proposition 4.2.19. q.e.d.

Recall that if $\mathcal{C}$ is an abelian category with enough injectives, its cohomological dimension is the smallest $n \in \mathbb{N} \sqcup \infty$ such that any object $F \in \mathcal{C}$ has an injective resolution of length $\leq n$.

Corollary 5.2.2. — (i) *The derived functor $Rf_*\colon D^+(\mathrm{I}(k_X)) \to D^+(\mathrm{I}(k_Y))$ is well-defined.*
(ii) *If $\mathrm{Mod}(k_X)$ has finite cohomological dimension, then Rf_* induced a functor $Rf_*\colon D^b(\mathrm{I}(k_X)) \to D^b(\mathrm{I}(k_Y))$.*
(iii) *If $g\colon Y \to Z$ is another continuous map, then $R(g\circ f)_* \simeq Rg_* \circ Rf_*$.*

Proposition 5.2.3. — *For $G \in D^-(\mathrm{I}(k_Y))$ and $F \in D^+(\mathrm{I}(k_X))$, one has the isomorphisms*

$$\begin{aligned} \mathrm{RHom}_{\mathrm{I}(k_Y)}(G, Rf_*F) &\simeq \mathrm{RHom}_{\mathrm{I}(k_X)}(f^{-1}G, F), \\ R\mathcal{H}om\,(G, Rf_*F) &\simeq Rf_*R\mathcal{H}om\,(f^{-1}G, F), \\ R\mathcal{I}hom\,(G, Rf_*F) &\simeq Rf_*R\mathcal{I}hom\,(f^{-1}G, F). \end{aligned}$$

Proof. — We may assume that $G \in K^-(\mathcal{P}(k_Y))$. We can take a quasi-isomorphism $F \to F'$ with a quasi-injective F'. Then applying Theorem 5.1.3, the result follows from the non derived case. q.e.d.

Theorem 5.2.4. — *The functors Rf_* and f^{-1} are adjoint. More precisely, for $F \in D^+(\mathrm{I}(k_X))$ and $G \in D^+(\mathrm{I}(k_Y))$ one has the isomorphism*

$$\mathrm{Hom}_{D^+(\mathrm{I}(k_Y))}(G, Rf_*F) \simeq \mathrm{Hom}_{D^+(\mathrm{I}(k_X))}(f^{-1}G, F).$$

Proof. — We have the chain of isomorphisms

$$\begin{aligned}\mathrm{Hom}_{D^+(\mathrm{I}(k_Y))}(G, Rf_*F) &\simeq \varinjlim_{F \xrightarrow[qis]{} F',\, G' \xrightarrow[qis]{} G} \mathrm{Hom}_{K^+(\mathrm{I}(k_Y))}(G', f_*F') \\ &\simeq \varinjlim_{F \xrightarrow[qis]{} F',\, G' \xrightarrow[qis]{} G} \mathrm{Hom}_{K^+(\mathrm{I}(k_X))}(f^{-1}G', F') \\ &\simeq \mathrm{Hom}_{D^+(\mathrm{I}(k_X))}(f^{-1}G, F).\end{aligned}$$

q.e.d.

Proposition 5.2.5. — *There are natural isomorphisms and morphisms*

(i) $\iota_Y Rf_* \xrightarrow{\sim} Rf_*\iota_X$,
(ii) $\alpha_Y Rf_* \xrightarrow{\sim} Rf_*\alpha_X$,
(iii) $\beta_Y Rf_* \to Rf_*\beta_X$.

Proof. — (i) To prove that for $F \in D^+(k_X)$, $\iota_Y Rf_*F \simeq Rf_*\iota_X F$, we may assume that $F \in K^+(\mathrm{Mod}(k_X))$ and that F is injective. Then $\iota_X F \in K^+(\mathcal{I}_q(k_X))$, and the result follows.

(ii) To prove that for $F \in D^+(\mathrm{I}(k_X))$, $\alpha_Y Rf_*F \simeq Rf_*\alpha_X F$, we may assume that $F \in K^+(\mathcal{I}_q(k_X))$. Then $\alpha_X F$ is soft, hence f_*-acyclic and the result follows.

(iii) We have $f^{-1}\beta_Y Rf_* \simeq \beta_X f^{-1}Rf_* \to \beta_X$, and the result follows by adjunction (i.e. by Theorem 5.2.4). q.e.d.

Proper direct images. —

Proposition 5.2.6. — (i) *The category $\mathcal{I}_q(k_X)$ is injective with respect to $f_{!!}$, and the functor $f_{!!}$ admits a right derived functor $Rf_{!!} : D^+(\mathrm{I}(k_X)) \to D^+(\mathrm{I}(k_Y))$. Moreover, for each k, the functor $R^kf_{!!} \colon \mathrm{I}(k_X) \to \mathrm{I}(k_Y)$ commutes with "$\varinjlim$".*

(ii) *There is a natural morphism of functors $Rf_{!!} \to Rf_*$. If $F \in D^+(\mathrm{I}(k_X))$ has proper support over Y, then this morphism induces an isomorphism $Rf_{!!}F \xrightarrow{\sim} Rf_*F$.*
(iii) *If $K \in \mathrm{Mod}(k_X)$ is soft, then $R^kf_{!!}(F \otimes K) = 0$ for $k \neq 0$ and $F \in \mathrm{I}(k_X)$.*
(iv) *If $F \in D^+(k_X)$ and f is proper on* $\mathrm{supp}\,F$, *then $Rf_{!!}F \simeq Rf_!F$.*
(v) *The functor $f_{!!}$ sends $\mathcal{I}_q(k_X)$ to $\mathcal{I}_q(k_Y)$. Moreover, if $g\colon Y \to Z$ is another continuous map, then $R(g \circ f)_{!!} \simeq Rg_{!!} \circ Rf_{!!}$.*
(vi) *One has the isomorphism $\alpha_Y \circ Rf_{!!} \xrightarrow{\sim} Rf_! \circ \alpha_X$.*
(vii) *One has a natural morphism $\beta_Y Rf_! \to Rf_{!!}\beta_X$, and this morphism is an isomorphism when f is an open embedding.*

Proof. — (i) The functor $f_{!!}\colon \mathrm{I}(k_X) \to \mathrm{I}(k_Y)$ is of the type $If_!$, where one denotes again by $f_!\colon \mathrm{Mod}^c(k_X) \to \mathrm{Mod}^c(k_Y)$ the restriction of the usual functor $f_!$. Hence we can apply Theorem 1.5.6.

(ii) Let $F \in K^+(\mathrm{I}(k_X))$. Proposition 4.3.16 implies the existence of a morphism $f_{!!}F \to f_*F$ functorial with respect to F. Therefore, we obtain the morphism $Rf_{!!} \to Rf_*$. If the support of $F \in D^+(\mathrm{I}(k_X))$ is proper over Y, then we can choose its representative in $K^+(\mathcal{I}_q(k_X))$ whose support is proper over Y. Then Proposition 4.3.16 implies that $Rf_{!!}F \to Rf_*F$ is an isomorphism.

(iii) Let $F = \text{“}\varinjlim_i\text{”}\, F_i \in \mathrm{I}(k_X)$ with $F_i \in \mathrm{Mod}^c(k_X)$. One has

$$R^k f_{!!}(F \otimes K) \simeq \text{“}\varinjlim_i\text{”}\, R^k f_!(F_i \otimes K) = 0 \quad \text{for } k \neq 0.$$

Here, the first isomorphism follows from (i) and (ii). The vanishing result follows from the fact that $F_i \otimes K$ is soft by Exercise 4.7.

(iv) By (ii) and the corresponding result in sheaf theory, one has $Rf_{!!}F \xrightarrow{\sim} Rf_*F \xleftarrow{\sim} Rf_!F$.

(v) Let $F = \text{“}\varinjlim_i\text{”}\, F_i$ with the F_i's injective with compact support. Then $f_{!!}F \simeq \text{“}\varinjlim_i\text{”}\, f_*F_i$, and the sheaves f_*F_i are injective.

(vi) If $F \in \mathcal{I}_q(k_X)$, then $\alpha_X F$ is soft, hence $f_!$-acyclic.

(vii) The isomorphism $\mathrm{id} \xrightarrow{\sim} \alpha_X\beta_X$ defines $Rf_! \to Rf_!\alpha_X\beta_X \simeq \alpha_Y Rf_{!!}\beta_X$, and the result follows by adjunction (i.e. by Theorem 3.3.26). When f is an open embedding, $f_!$ and $f_{!!}$ are exact, and the isomorphism follows from Proposition 4.3.17 (ii). q.e.d.

Theorem 5.2.7. — *Let $F \in D^+(\mathrm{I}(k_X))$ and $G \in D^+(\mathrm{I}(k_Y))$. Then*

$$G \otimes Rf_{!!}F \simeq Rf_{!!}(f^{-1}G \otimes F).$$

Proof. — We may assume that F is quasi-injective. In this case, the left hand side is isomorphic to $G \otimes f_{!!}F$. By Proposition 4.3.14, $G \otimes f_{!!}F \simeq f_{!!}(f^{-1}G \otimes F)$. On the other hand, $f_{!!}(f^{-1}G \otimes F)$ represents $Rf_{!!}(f^{-1}G \otimes F)$ by Proposition 5.2.6 (iii). q.e.d.

The next result has no counterpart in classical sheaf theory.

Lemma 5.2.8. — (i) *For $G \in D^-(\mathrm{I}(k_Y))$ and $F \in D^+(\mathrm{I}(k_X))$, there are natural morphisms*

$$\begin{aligned} Rf_{!!}R\mathcal{I}hom\,(f^{-1}G, F) &\to R\mathcal{I}hom\,(G, Rf_{!!}F)\,, \\ Rf_!R\mathcal{H}om\,(f^{-1}G, F) &\to R\mathcal{H}om\,(G, Rf_{!!}F)\,. \end{aligned}$$

(ii) *If $G \in D^-(k_Y)$, these morphisms are isomorphisms.*

Proof. — The second morphism being obtained from the first one by applying α, we shall prove the assertions for the first morphism. We have a chain of morphisms:

$$G \otimes Rf_{!!}R\mathcal{I}hom\,(f^{-1}G, F) \simeq Rf_{!!}(f^{-1}G \otimes R\mathcal{I}hom\,(f^{-1}G, F)) \to Rf_{!!}F.$$

Then by the adjunction, we obtain the first morphism.

In order to see (ii), we may assume $G \in \mathrm{Mod}(k_Y)$ and $F \in \mathcal{I}_q(k_X)$. In this case, the morphisms above reduce to $f_{!!}\mathcal{I}hom\,(f^{-1}G, F) \to \mathcal{I}hom\,(G, f_{!!}F)$. This morphism is an isomorphism by Corollary 4.3.15. q.e.d.

Consider now a Cartesian square of topological spaces

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\Box"] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y. \end{tikzcd}$$

Theorem 5.2.9. — *There is a natural isomorphism of functors from $D^+(\mathrm{I}(k_X))$ to $D^+(\mathrm{I}(k_{Y'}))$:*

$$Rf'_{!!}{g'}^{-1} \simeq g^{-1}Rf_{!!}\,.$$

Proof. — By Proposition 4.3.18, there is an isomorphism $f'_{!!}{g'}^{-1} \simeq g^{-1}f_{!!}$. The right hand side admits a right derived functor, and $R(g^{-1}f_{!!}) \simeq g^{-1}Rf_{!!}$. Hence we get the morphism of functors $g^{-1}Rf_{!!} \to (Rf'_{!!}){g'}^{-1}$. Then it is enough to prove that for each k and each $F \in \mathrm{I}(k_X)$, it induces an isomorphism $g^{-1}R^kf_{!!}F \to R^kf'_{!!}{g'}^{-1}F$. Since both sides commute with "$\varinjlim$" in view of Proposition 5.2.6 (i), we may assume that $F \in \mathrm{Mod}^c(k_X)$. In this case, the result follows from the corresponding one for sheaves.

q.e.d.

5.3. Duality

As in the preceding section, consider a continuous map $f\colon X \to Y$. We shall extend the classical Poincaré-Verdier duality to ind-sheaves and construct a right adjoint functor to $f_{!!}$. We refer to [**10**] Chapter III for an exposition of the classical case.

Recall that the cohomological dimension of the functor $f_!$ is the smallest $d \in \mathbb{N} \sqcup \infty$ such that $R^jf_!F = 0$ if $j > d$ for all $F \in \mathrm{Mod}(k_X)$. A sheaf F on X is called f-soft if for any $x \in X$, the sheaf $F|_{f^{-1}(x)}$ is c-soft. The functor $f_!$ has cohomological dimension $\leq d$ if and only if any sheaf on X has an f-soft resolution of length $\leq d$.

From now on, we shall make the following hypothesis:

(5.3.1) the functor $f_!$ has finite cohomological dimension.

Lemma 5.3.1. — *The two functors $f_{!!}$ and $f_!$ have the same cohomological dimension.*

Proof. — Denote by d the cohomological dimension of $f_!$. Let $F \simeq$ "$\varinjlim\limits_i$" F_i, with $F_i \in \mathrm{Mod}^c(k_X)$. Then $R^k f_{!!} F \simeq$ "$\varinjlim\limits_i$" $R^k f_! F_i = 0$ for $j > d$. Hence, the cohomological dimension of $f_{!!}$ is less than or equal to that of $f_!$.

The other estimate follows from $R^k f_! F \simeq \alpha_Y R^k f_{!!} F$ for $F \in \mathrm{Mod}(k_X)$. q.e.d.

Let K be an f-soft sheaf on X and let $G \in \mathrm{Mod}(k_Y)$. One introduces the presheaf on X:

$$f_K^! G : U \mapsto \mathrm{Hom}_{k_Y}(f_! K_U, G).$$

Proposition 5.3.2. — (i) *For $F \in \mathrm{Mod}(k_X)$, $F \otimes K$ is f-soft. In particular, the functor $F \mapsto f_!(F \otimes K_U)$ is exact,*

(ii) *if G is injective, the presheaf $f_K^! G$ is an injective sheaf,*

(iii) *if G is injective and $F \in \mathrm{Mod}^c(k_X)$, then*

$$\mathrm{Hom}_{k_X}(F, f_K^! G) \simeq \mathrm{Hom}_{k_Y}(f_!(K \otimes F), G).$$

We refer to Exercise 4.7 and to loc. cit. for a proof.

We shall extend the functor $f_K^!$ to ind-sheaves. Let $G \in \mathrm{I}(k_Y)$. We define $f_K^! G \in \mathrm{Mod}^c(k_X)^{\wedge, add}$ by the formula:

$$f_K^! G(F) = \mathrm{Hom}_{\mathrm{I}(k_Y)}(f_!(K \otimes F), G) \quad \text{for } F \in \mathrm{Mod}^c(k_X).$$

Lemma 5.3.3. — (i) *$f_K^! G$ belongs to $\mathrm{I}(k_X)$.*

(ii) *The functor $f_K^! \colon \mathrm{I}(k_Y) \to \mathrm{I}(k_X)$ commutes with filtrant inductive limits and with projective limits (in particular, it is left exact).*

(iii) *If G is quasi-injective, then $f_K^! G$ is quasi-injective.*

(iv) *The category $\mathcal{I}_q(k_Y)$ is $f_K^!$-injective.*

(v) *For $F \in \mathrm{I}(k_X)$ and $G \in \mathrm{I}(k_Y)$, one has*

$$\mathrm{Hom}_{\mathrm{I}(k_X)}(F, f_K^! G) \simeq \mathrm{Hom}_{\mathrm{I}(k_Y)}(f_{!!}(K \otimes F), G).$$

Proof. — (i) Clearly, the functor $f_K^! \colon \mathrm{I}(k_Y) \to \mathrm{Mod}^c(k_X)^{\wedge, add}$ commutes with filtrant inductive limits and with projective limits. Hence, to prove (i), we may assume that G is an injective sheaf. Then the result follows from Proposition 5.3.2.

(ii) is obvious.

(iii) follows from the fact that the functor $F \mapsto f_!(K \otimes F)$ is exact.

(iv) Consider an exact sequence $0 \to G' \to G \to G'' \to 0$ of quasi-injective sheaves, and let us apply the functor $f_K^!$ to this sequence. To check that the sequence we have obtained is exact, it is enough to prove that it is exact after applying the functor $\mathrm{Hom}_{\mathrm{I}(k_X)}(S, \cdot\,)$, with $S \in \mathrm{Mod}^c(k_X)$. This is clear by the definition of $f_K^!$.

(v) Let $F \simeq$ "$\varinjlim_i$" F_i with $F_i \in \mathrm{Mod}^c(k_X)$. One has the chain of isomorphisms

$$\begin{aligned}\mathrm{Hom}_{\mathrm{I}(k_X)}(F, f^!_K G) &\simeq \varprojlim_i \mathrm{Hom}_{\mathrm{I}(k_X)}(F_i, f^!_K G) \simeq \varprojlim_i \mathrm{Hom}_{\mathrm{I}(k_Y)}(f_!(K\otimes F_i), G)\\ &\simeq \mathrm{Hom}_{\mathrm{I}(k_Y)}(\text{“}\varinjlim_i\text{”}\, f_!(K\otimes F_i), G) \simeq \mathrm{Hom}_{\mathrm{I}(k_Y)}(f_{!!}(K\otimes F), G).\end{aligned}$$

q.e.d.

Theorem 5.3.4. — *The functor $Rf_{!!}\colon D^+(\mathrm{I}(k_X)) \to D^+(\mathrm{I}(k_Y))$ admits a right adjoint, denoted by $f^!$, that is:*

$$\mathrm{Hom}_{D^+(\mathrm{I}(k_Y))}(Rf_{!!}F, G) \simeq \mathrm{Hom}_{D^+(\mathrm{I}(k_X))}(F, f^!G).$$

Moreover, $\iota_X f^! \simeq f^! \iota_Y$. (In other words, the restriction of this new functor to the derived category of sheaves coincides with the classical functor $f^!$.)

Proof. — There exists a complex of f-soft sheaves $K \in K^b(\mathrm{Mod}(k_X))$ qis to the sheaf k_X. By Lemma 5.3.3, for $F \in K^+(\mathrm{I}(k_X))$ and $G \in K^+(\mathrm{I}(k_Y))$ one has the isomorphism

$$\mathrm{Hom}_{K^+(\mathrm{I}(k_X))}(F, f^!_K G) \simeq \mathrm{Hom}_{K^+(\mathrm{I}(k_Y))}(f_{!!}(F\otimes K), G).$$

Let $G \in K^+(\mathrm{I}(k_Y))$ and assume that all components of the complex G are quasi-injective. Let us prove that $f^!_K G$ represents $f^!G$. We have the chain of isomorphisms

$$\begin{aligned}\mathrm{Hom}_{D^+(\mathrm{I}(k_Y))}(Rf_{!!}F, G) &\simeq \varinjlim_{F' \xrightarrow[qis]{} F,\, G \xrightarrow[qis]{} G'} \mathrm{Hom}_{K^+(\mathrm{I}(k_Y))}(f_{!!}(F'\otimes K), G')\\ &\simeq \varinjlim_{F' \xrightarrow[qis]{} F,\, G \xrightarrow[qis]{} G'} \mathrm{Hom}_{K^+(\mathrm{I}(k_X))}(F', f^!_K G')\\ &\simeq \mathrm{Hom}_{D^+(\mathrm{I}(k_X))}(F, f^!_K G).\end{aligned}$$

Here, we have used the fact that $f_{!!}(F'\otimes K)$ represents $Rf_{!!}(F\otimes K)$ and the fact that $f^!_K G'$ is qis to $f^!_K G$ if G' is quasi-injective.

By its construction the new functor $f^!$ coincides with the classical one when restricted to the derived category of sheaves. q.e.d.

Corollary 5.3.5. — *Let $F \in D^b(\mathrm{I}(k_X))$ and $G \in D^+(\mathrm{I}(k_Y))$. Then*

(i) $R\mathcal{I}hom\,(Rf_{!!}F, G) \simeq Rf_* R\mathcal{I}hom\,(F, f^!G)$,
(ii) $R\mathcal{H}om\,(Rf_{!!}F, G) \simeq Rf_* R\mathcal{H}om\,(F, f^!G)$.

Proof. — Let $S \in D^+(\mathrm{I}(k_Y))$. One has the chain of isomorphisms

$$\begin{aligned}
\mathrm{Hom}_{D^+(\mathrm{I}(k_Y))}(S, R\mathcal{I}hom\,(Rf_{!!}F, G)) &\simeq \mathrm{Hom}_{D^+(\mathrm{I}(k_Y))}(S \otimes Rf_{!!}F, G) \\
&\simeq \mathrm{Hom}_{D^+(\mathrm{I}(k_Y))}(Rf_{!!}(f^{-1}S \otimes F), G) \\
&\simeq \mathrm{Hom}_{D^+(\mathrm{I}(k_X))}(f^{-1}S \otimes F, f^!G) \\
&\simeq \mathrm{Hom}_{D^+(\mathrm{I}(k_X))}(f^{-1}S, R\mathcal{I}hom\,(F, f^!G)) \\
&\simeq \mathrm{Hom}_{D^+(\mathrm{I}(k_Y))}(S, Rf_*R\mathcal{I}hom\,(F, f^!G)).
\end{aligned}$$

The second formula follows by applying α_X. q.e.d.

Proposition 5.3.6. — *Let $\{G_i\}_i$ be a small filtrant inductive system in $\mathrm{I}(k_Y)$. Then for $k \in \mathbb{Z}$ one has*

$$H^k(f^!(\text{“}\varinjlim_i\text{”}\, G_i)) \simeq \text{“}\varinjlim_i\text{”}\, H^k(f^!(G_i)).$$

Proof. — Since each G_i is a small inductive limit of sheaves, we may assume from the beginning that all G_i's are sheaves. Denote by $G_i^\bullet$ the canonical injective resolution of G_i. Then $\{G_i^\bullet\}_i$ is an inductive system of complexes of injective sheaves. Denote by K a bounded f-soft resolution of the constant sheaf k_X, as in the proof of Theorem 5.3.4. Then

$$\begin{aligned}
H^k(f^!(\text{“}\varinjlim_i\text{”}\, G_i)) &\simeq H^k(f^!(\text{“}\varinjlim_i\text{”}\, G_i^\bullet)) \simeq H^k(f^!_K(\text{“}\varinjlim_i\text{”}\, G_i^\bullet)) \\
&\simeq \text{“}\varinjlim_i\text{”}\, H^k(f^!_K(G_i^\bullet)) \simeq \text{“}\varinjlim_i\text{”}\, H^k(f^!(G_i)).
\end{aligned}$$

Here we have used the fact that if G is quasi-injective then $f^!_K(G)$ represents $f^!(G)$, and $f^!_K$ commutes with “$\varinjlim$”. q.e.d.

Proposition 5.3.7. — (i) *Let $G_1, G_2 \in D^+(\mathrm{I}(k_Y))$. There is a natural morphism*

$$f^!G_1 \otimes f^{-1}G_2 \to f^!(G_1 \otimes G_2).$$

(ii) *Assume that, locally on X, f is isomorphic to the projection $Y \times \mathbb{R}^n \to Y$. Then for $G \in D^+(\mathrm{I}(k_Y))$, we have the isomorphism*

$$f^{-1}G \otimes f^!k_Y \xrightarrow{\sim} f^!G.$$

In particular, if $f \colon X \to Y$ is an open embedding, then $f^! \simeq f^{-1}$.

(iii) *Assume that $f \colon X \to Y$ is a closed embedding. Then*

$$f^! \simeq f^{-1}R\mathcal{I}hom\,((k_Y)_X,\ \cdot\ ).$$

Moreover, $\mathrm{id} \xrightarrow{\sim} f^!Rf_{!!}$.

Proof. — (i) Consider the morphisms

$$\begin{aligned}
Rf_{!!}(f^!G_1 \otimes f^{-1}G_2) &\simeq Rf_{!!}f^!G_1 \otimes G_2 \\
&\to G_1 \otimes G_2.
\end{aligned}$$

By adjunction, we get the desired morphism.

(ii) The morphism is constructed in (i) and it is known that it is an isomorphism when $G \in D^b(k_Y)$. To check that it is an isomorphism for a general G, by "dévissage", we may assume that $G \in \mathrm{I}(k_Y)$. Let $G \simeq \text{“}\varinjlim_i\text{”}\, G_i$, with $G_i \in \mathrm{Mod}(k_Y)$. We have

$$\begin{aligned} H^k(f^{-1}G \otimes f^! k_Y) &\simeq H^k((f^{-1}\text{“}\varinjlim_i\text{”}\, G_i) \otimes f^! k_Y) \simeq \text{“}\varinjlim_i\text{”}\, H^k(f^{-1}G_i \otimes f^! k_Y) \\ &\simeq \text{“}\varinjlim_i\text{”}\, H^k(f^! G_i) \simeq H^k(f^!\,\text{“}\varinjlim_i\text{”}\, G_i). \end{aligned}$$

(iii) By Corollary 5.3.5, we have for $G \in D^b(\mathrm{I}(k_Y))$

$$Rf_* f^! G \simeq Rf_* R\mathcal{I}hom\,(k_X, f^! G) \simeq R\mathcal{I}hom\,(f_* k_X, G).$$

Hence, $f^! G \simeq f^{-1}Rf_* f^! G \simeq f^{-1}R\mathcal{I}hom\,(f_* k_X, G)$.

Now, let $F \in D^b(\mathrm{I}(k_X))$. We get

$$\begin{aligned} f^! Rf_* F &\simeq f^{-1}R\mathcal{I}hom\,(f_* k_X, Rf_* F) \simeq f^{-1}Rf_* R\mathcal{I}hom\,(f^{-1}f_* k_X, F) \\ &\simeq f^{-1}Rf_* F \simeq F. \end{aligned}$$

q.e.d.

Proposition 5.3.8. — *Let $K \in D^-(\mathrm{I}(k_Y))$ and $G \in D^+(\mathrm{I}(k_Y))$. Then*

$$R\mathcal{I}hom\,(f^{-1}K, f^! G) \xrightarrow{\sim} f^! R\mathcal{I}hom\,(K, G).$$

Proof. — For any $F \in D^+(\mathrm{I}(k_X))$, one has the chain of isomorphisms

$$\begin{aligned} &\mathrm{Hom}_{D^+(\mathrm{I}(k_X))}(F, f^! R\mathcal{I}hom\,(K,G)) \simeq \mathrm{Hom}_{D^+(\mathrm{I}(k_Y))}(Rf_{!!}F, R\mathcal{I}hom\,(K,G)) \\ &\simeq \mathrm{Hom}_{D^+(\mathrm{I}(k_Y))}(K \otimes Rf_{!!}F, G) \simeq \mathrm{Hom}_{D^+(\mathrm{I}(k_Y))}(Rf_{!!}(f^{-1}K \otimes F), G) \\ &\simeq \mathrm{Hom}_{D^+(\mathrm{I}(k_X))}(f^{-1}K \otimes F, f^! G) \simeq \mathrm{Hom}_{D^+(\mathrm{I}(k_X))}(F, R\mathcal{I}hom\,(f^{-1}K, f^! G)). \end{aligned}$$

q.e.d.

Proposition 5.3.9. — *There are a morphism and an isomorphism of functors*

(i) $\alpha_X f^! \to f^! \alpha_Y$,

(ii) $f^{-1}\beta_Y(\,\cdot\,) \otimes f^! k_Y \xrightarrow{\sim} f^! \beta_Y(\,\cdot\,)$.

Proof. — (i) By Theorem 5.3.4, there is a natural morphism $Rf_{!!}f^! \to \mathrm{id}$. This defines $Rf_! \alpha_X f^! \simeq \alpha_Y Rf_{!!} f^! \to \alpha_Y$, and the result follows by adjunction.

(ii) Let $G \in D^+(k_Y)$. Using the morphism $Rf_{!!}f^! k_Y \to k_Y$ we have the chain of morphisms

$$\begin{aligned} Rf_{!!}(f^{-1}\beta_Y G \otimes f^! k_Y) &\simeq \beta_Y G \otimes Rf_{!!} f^! k_Y \\ &\to \beta_Y G \otimes k_Y. \end{aligned}$$

We get the morphism $f^{-1}\beta_Y G \otimes f^! k_Y \to f^! \beta_Y G$ by adjunction. To prove it is an isomorphism, we take $S \in D^b(\mathrm{Mod}^c(k_X))$ and check the chain of isomorphisms

$$\begin{aligned}
\mathrm{Hom}_{D^+(\mathrm{I}(k_X))}(S, f^{-1}\beta_Y G \otimes f^! k_Y) &\simeq H^0\mathrm{R}\Gamma\Big(X; R\mathcal{H}om\,(S, \beta_X f^{-1}G \otimes f^! k_Y)\Big) \\
&\simeq H^0\mathrm{R}\Gamma\Big(X; R\mathcal{H}om\,(S, f^! k_Y) \otimes f^{-1}G\Big) \\
&\simeq H^0\mathrm{R}\Gamma\Big(Y; Rf_!\big(R\mathcal{H}om\,(S, f^! k_Y) \otimes f^{-1}G\big)\Big) \\
&\simeq H^0\mathrm{R}\Gamma\Big(Y; Rf_! R\mathcal{H}om\,(S, f^! k_Y) \otimes G\Big) \\
&\simeq H^0\mathrm{R}\Gamma\Big(Y; Rf_* R\mathcal{H}om\,(S, f^! k_Y) \otimes G\Big) \\
&\simeq H^0\mathrm{R}\Gamma\Big(Y; R\mathcal{H}om\,(Rf_{!!}S, k_Y) \otimes G\Big) \\
&\simeq H^0\mathrm{R}\Gamma\Big(Y; R\mathcal{H}om\,(Rf_{!!}S, \beta_Y G)\Big) \\
&\simeq \mathrm{Hom}_{D^+(\mathrm{I}(k_Y))}(Rf_{!!}S, \beta_Y G) \\
&\simeq \mathrm{Hom}_{D^+(\mathrm{I}(k_X))}(S, f^! \beta_Y G).
\end{aligned}$$

q.e.d.

Consider a Cartesian square of topological spaces

(5.3.2)
$$\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\Box"] & Y' \arrow[d, "g"] \\
X \arrow[r, "f"] & Y.
\end{tikzcd}$$

Theorem 5.3.10. — *There is a natural isomorphism of functors*

$$Rf'_* g'^! \xrightarrow{\sim} g^! Rf_*.$$

Proof. — It is enough to prove the isomorphism

$$\mathrm{RHom}_{\mathrm{I}(k_{Y'})}(K, Rf'_* g'^! F) \simeq \mathrm{RHom}_{\mathrm{I}(k_{Y'})}(K, g^! Rf_* F)$$

for $K \in D^+(\mathrm{I}(k_{Y'}))$ and $F \in D^+(\mathrm{I}(k_X))$. By adjunction, this follows from the isomorphism $Rg'_{!!} f'^{-1} K \simeq f^{-1} Rg_{!!} K$ given by Theorem 5.2.9. q.e.d.

Theorem 5.3.11. — *There is a natural isomorphism of functors*

$$Rf'_{!!} g'^! \xrightarrow{\sim} g^! Rf_{!!}.$$

Note that this isomorphism has no counterpart in sheaf theory.

Proof. — The morphism is constructed by the chain of morphisms $Rg_{!!} Rf'_{!!} g'^! \simeq Rf_{!!} Rg'_{!!} g'^! \to Rf_{!!}$ and by adjunction. Hence, it is enough to prove the isomorphism

$$\mathrm{RHom}_{\mathrm{I}(k_{Y'})}(K, Rf'_{!!} g'^! F) \xrightarrow{\sim} \mathrm{RHom}_{\mathrm{I}(k_{Y'})}(K, g^! Rf_{!!} F)$$

for $K \in \mathrm{Mod}^c(k_{Y'})$ and $F \in D^+(\mathrm{I}(k_X))$. Since K has compact support, one has the isomorphisms

$$\begin{aligned}\mathrm{RHom}_{\mathrm{I}(k_{Y'})}(K, Rf'_{!!}g'^{!}F) &\simeq \mathrm{R}\Gamma(Y'; R\mathcal{H}om\,(K, Rf'_{!!}g'^{!}F)) \\ &\simeq \mathrm{R}\Gamma(Y; Rg_! R\mathcal{H}om\,(K, Rf'_{!!}g'^{!}F)).\end{aligned}$$

Now, using Lemma 5.2.8 and the fact that g' is proper on $\mathrm{supp}(f'^{-1}K)$, we find the chain of isomorphisms

$$\begin{aligned}Rg_! R\mathcal{H}om\,(K, Rf'_{!!}g'^{!}F) &\simeq Rg_! Rf'_! R\mathcal{H}om\,(f'^{-1}K, g'^{!}F) \\ &\simeq Rf_! Rg'_! R\mathcal{H}om\,(f'^{-1}K, g'^{!}F) \\ &\simeq Rf_! Rg'_* R\mathcal{H}om\,(f'^{-1}K, g'^{!}F) \\ &\simeq Rf_! R\mathcal{H}om\,(Rg'_{!!}f'^{-1}K, F) \\ &\simeq Rf_! R\mathcal{H}om\,(f^{-1}Rg_{!!}K, F) \\ &\simeq R\mathcal{H}om\,(Rg_{!!}K, Rf_{!!}F) \\ &\simeq Rg_* R\mathcal{H}om\,(K, g^{!}Rf_{!!}F).\end{aligned}$$

Therefore we get

$$\begin{aligned}\mathrm{RHom}_{\mathrm{I}(k_{Y'})}(K, Rf'_{!!}g'^{!}F) &\simeq \mathrm{R}\Gamma(Y; Rg_! R\mathcal{H}om\,(K, Rf'_{!!}g'^{!}F)) \\ &\simeq \mathrm{R}\Gamma(Y; Rg_* R\mathcal{H}om\,(K, g^{!}Rf_{!!}F)) \\ &\simeq \mathrm{R}\Gamma(Y'; R\mathcal{H}om\,(K, g^{!}Rf_{!!}F)) \\ &\simeq \mathrm{RHom}_{\mathrm{I}(k_{Y'})}(K, g^{!}Rf_{!!}F).\end{aligned}$$

q.e.d.

We may summarize the commutativity of the various functors we have introduced in the table below. Here, "$\circ$" means that the functors commute, and "$\times$" they do not. Examples showing that the functors do not commute are given in Exercises 4.2, 4.3, 4.4, 4.5, 4.6, 5.1, 5.2.

	ι	α	β	$\varinjlim$	$\varprojlim$
$\otimes$	$\circ$	$\circ$	$\circ$	$\circ$	$\times$
f^{-1}	$\circ$	$\circ$	$\circ$	$\circ$	$\times$
f_*	$\circ$	$\circ$	$\times$	$\times$	$\circ$
$f_{!!}$	$\times$	$\circ$	$\times$	$\circ$	$\times$
$f^!$	$\circ$	$\times$	$\times$	$\circ$	
$\varinjlim$	$\times$	$\circ$	$\circ$		
$\varprojlim$	$\circ$	$\circ$	$\times$		

5.4. Ring action I

For the reader's convenience, we define the standard notions of a ring or a module in the case of the category $\mathrm{I}(k_X)$, although such constructions make sense and are classical in the more general framework of tensor categories. Recall that k is assumed to be a field.

Definition 5.4.1. — (i) A ring in $\mathrm{I}(k_X)$ (or "an ind-k_X-algebra", or simply "an ind-ring") is the data of an object $\mathcal{A} \in \mathrm{I}(k_X)$ and morphisms $\mu_{\mathcal{A}} \colon \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ and $\varepsilon_{\mathcal{A}} \colon k_X \to \mathcal{A}$, such that the diagrams below are commutative.

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\sim"] \arrow[d, "\mathrm{id}_{\mathcal{A}}"'] & k_X \otimes \mathcal{A} \arrow[d, "\varepsilon_{\mathcal{A}} \otimes \mathcal{A}"] \\ \mathcal{A} & \mathcal{A} \otimes \mathcal{A} \arrow[l, "\mu_{\mathcal{A}}"'] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{A} \arrow[r, "\sim"] \arrow[d, "\mathrm{id}_{\mathcal{A}}"'] & \mathcal{A} \otimes k_X \arrow[d, "\mathcal{A} \otimes \varepsilon_{\mathcal{A}}"] \\ \mathcal{A} & \mathcal{A} \otimes \mathcal{A} \arrow[l, "\mu_{\mathcal{A}}"'] \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \arrow[r, "\mu_{\mathcal{A}} \otimes \mathcal{A}"] \arrow[d, "\mathcal{A} \otimes \mu_{\mathcal{A}}"'] & \mathcal{A} \otimes \mathcal{A} \arrow[d, "\mu_{\mathcal{A}}"] \\ \mathcal{A} \otimes \mathcal{A} \arrow[r, "\mu_{\mathcal{A}}"] & \mathcal{A}. \end{tikzcd}$$

(ii) A left $\mathcal{A}$-module (or simply, an $\mathcal{A}$-module or "an ind-module") is the data of an object $M \in \mathrm{I}(k_X)$ and a morphism $\mu_M \colon A \otimes M \to M$ such that the diagrams below are commutative.

$$\begin{tikzcd} \mathcal{A} \otimes \mathcal{A} \otimes M \arrow[r, "\mu_{\mathcal{A}} \otimes M"] \arrow[d, "\mathcal{A} \otimes \mu_M"'] & \mathcal{A} \otimes M \arrow[d, "\mu_M"] \\ \mathcal{A} \otimes M \arrow[r, "\mu_M"] & M \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "\sim"] \arrow[d, "\mathrm{id}_{\mathcal{A}}"'] & k_X \otimes M \arrow[d, "\varepsilon_{\mathcal{A}} \otimes M"] \\ M & \mathcal{A} \otimes M \arrow[l, "\mu_M"'] \end{tikzcd}$$

Notation 5.4.2. — (i) One denotes by $\nu_M \colon M \to \mathcal{I}hom\,(\mathcal{A}, M)$ the morphism deduced from μ_M by the isomorphism

(5.4.1) $$\mathrm{Hom}_{\mathrm{I}(k_X)}(\mathcal{A} \otimes M, M) \simeq \mathrm{Hom}_{\mathrm{I}(k_X)}(M, \mathcal{I}hom\,(\mathcal{A}, M)).$$

(ii) One denotes by $e_M \colon M \to \mathcal{A} \otimes M$ the composition $M \xrightarrow{\sim} k_X \otimes M \xrightarrow[\varepsilon_{\mathcal{A}} \otimes M]{} \mathcal{A} \otimes M$.

(iii) One denotes by $e_M^* \colon \mathcal{I}hom\,(\mathcal{A}, M) \to M$ the composition

$$\mathcal{I}hom\,(\mathcal{A}, M) \xrightarrow[\mathcal{I}hom\,(\varepsilon_{\mathcal{A}}, M)]{} \mathcal{I}hom\,(k_X, M) \simeq M\,.$$

Remark 5.4.3. — In the classical case of a module M over a ring A, the analogous morphisms of μ_M, ν_M, e_M and e_M^* are the morphisms $a \otimes m \mapsto am$, $m \mapsto (a \mapsto am)$, $m \mapsto 1 \otimes m$ and $\varphi \mapsto \varphi(1)$, respectively.

Definition 5.4.4. — A morphism of $\mathcal{A}$-modules from M to N is a morphism $u \in \mathrm{Hom}_{\mathrm{I}(k_X)}(M, N)$ such that the diagram below commutes

$$\begin{tikzcd} \mathcal{A} \otimes M \arrow[r, "\mathcal{A}\otimes u"] \arrow[d, "\mu_M"] & \mathcal{A} \otimes N \arrow[d, "\mu_N"] \\ M \arrow[r, "u"] & N. \end{tikzcd}$$

If u is such a morphism, we shall say that "u is $\mathcal{A}$-linear".

By considering the $\mathcal{A}$-modules and the morphisms of $\mathcal{A}$-modules, one gets a category, which we denote by $\mathrm{I}(\mathcal{A})$.

Lemma 5.4.5. — (i) *The correspondence $U \mapsto \mathrm{I}(\mathcal{A}|_U)$ is a proper stack of k-abelian categories.*

(ii) *The natural functor $\mathrm{I}(\mathcal{A}) \to \mathrm{I}(k_X)$ is exact and faithful.*

The proof is left as an exercise.

As usual, one denotes by

$$\mathcal{H}om_{\mathrm{I}(\mathcal{A})} : \mathrm{I}(\mathcal{A})^{\mathrm{op}} \times \mathrm{I}(\mathcal{A}) \to \mathrm{Mod}(k_X)$$
$$\mathrm{Hom}_{\mathrm{I}(\mathcal{A})} : \mathrm{I}(\mathcal{A})^{\mathrm{op}} \times \mathrm{I}(\mathcal{A}) \to \mathrm{Mod}(k)$$

the natural functors.

Definition 5.4.6. — (a) One denotes by $\mathcal{A}^{\mathrm{op}}$ the object $\mathcal{A}$ endowed with the morphisms $\varepsilon_{\mathcal{A}^{\mathrm{op}}} := \varepsilon_{\mathcal{A}}$ and $\mu_{\mathcal{A}^{\mathrm{op}}} := \mu_{\mathcal{A}} \circ v$, where $v \colon \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ is the morphism corresponding to $a \otimes b \mapsto b \otimes a$.

(b) An $\mathcal{A}^{\mathrm{op}}$-module is called a right $\mathcal{A}$-module.

Note that $\mathcal{A}$ is both a left and right $\mathcal{A}$-module.

Example 5.4.7. — (i) If $\mathcal{A}$ is a sheaf of k_X-algebras, then $\mathrm{I}(\mathcal{A}) \simeq \mathrm{Ind}(\mathrm{Mod}^{\mathrm{c}}(\mathcal{A}))$ (see Exercise 5.3).

(ii) If $\mathcal{A}$ is a sheaf of k_X-algebras, then $\beta_X \mathcal{A}$ is a ring in $\mathrm{I}(k_X)$, and if M is a sheaf of $\mathcal{A}$-modules, then $\beta_X M$ is a $\beta_X \mathcal{A}$-module. This follows immediately from the fact that β_X commutes with $\otimes$. Note that, with the notations of Exercise 3.4, one has the equivalence $\mathrm{Mod}(\mathcal{A}, \mathrm{I}(k_X)) \simeq \mathrm{I}(\beta_X \mathcal{A})$, because $\mathrm{Hom}_{k_X}(\mathcal{A}, \mathcal{H}om\,(M, M)) = \mathrm{Hom}_{\mathrm{I}(k_X)}(\beta \mathcal{A} \otimes M, M)$.

Consider the two sequences of morphisms

(5.4.2) $\mathcal{A} \otimes \mathcal{A} \otimes M \underset{d}{\to} \mathcal{A} \otimes M \underset{\mu_M}{\longrightarrow} M \to 0 \qquad$ where $d = \mu_{\mathcal{A}} \otimes M - \mathcal{A} \otimes \mu_M$,

(5.4.3) $0 \to M \underset{\nu_M}{\longrightarrow} \mathcal{I}hom\,(\mathcal{A}, M) \underset{d}{\to} \mathcal{I}hom\,(\mathcal{A} \otimes \mathcal{A}, M) \simeq \mathcal{I}hom\,(\mathcal{A}, \mathcal{I}hom\,(\mathcal{A}, M))$,

where $d = \mathcal{I}hom\,(\mu_{\mathcal{A}}, M) - \mathcal{I}hom\,(\mathcal{A}, \nu_M)$.

Clearly, $\mu_M \circ d = 0$ in (5.4.2), and $d \circ \nu_M = 0$ in (5.4.3).

Lemma 5.4.8. — *The two complexes* (5.4.2) *and* (5.4.3) *are exact.*

Proof. — (i) The exactness of the complex (5.4.2) follows from the existence of e_M. In fact,

$$\mu_M \circ e_M = \mathrm{id}_M,\ d \circ e_{\mathcal{A}\otimes M} + e_M \circ \mu_M = \mathrm{id}_{\mathcal{A}\otimes M}\,.$$

(Here, we regard $\mathcal{A} \otimes M$ as an $\mathcal{A}$-module via the $\mathcal{A}$-module structure of $\mathcal{A}$.)

(ii) Similarly,

$$e^*_M \circ \nu_M = \mathrm{id}_M,\ e^*_{\mathcal{A}\otimes M} \circ d + \nu_M \circ e^*_M = \mathrm{id}_{\mathcal{I}hom\,(\mathcal{A},M)}\,.$$

q.e.d.

Definition 5.4.9. — (i) One defines the bifunctor

$$\begin{aligned} \cdot \otimes_{\mathcal{A}} \cdot \quad &: \quad \mathrm{I}(\mathcal{A}^{\mathrm{op}}) \times \mathrm{I}(\mathcal{A}) \to \mathrm{I}(k_X) \\ M \otimes_{\mathcal{A}} N \quad &:= \quad \mathrm{coker}(M \otimes \mathcal{A} \otimes N \underset{d}{\to} M \otimes N) \\ &\qquad\qquad \text{where } d = \mu_M \otimes N - M \otimes \mu_N. \end{aligned}$$

(ii) One defines the bifunctor

$$\begin{aligned} \mathcal{I}hom_{\mathcal{A}}(\,\cdot\,,\,\cdot\,) \quad &: \quad \mathrm{I}(\mathcal{A})^{\mathrm{op}} \times \mathrm{I}(\mathcal{A}) \to \mathrm{I}(k_X) \\ \mathcal{I}hom_{\mathcal{A}}(M,N) \quad &:= \quad \ker(\mathcal{I}hom\,(M,N) \underset{d}{\to} \mathcal{I}hom\,(\mathcal{A}\otimes M, N)) \\ &\qquad\qquad \text{where } d = \mathcal{I}hom\,(\mu_M, N) - \mathcal{I}hom\,(M, \nu_N). \end{aligned}$$

Remark 5.4.10. — One shall not confuse the functor $\mathcal{I}hom_{\mathcal{A}}\colon \mathrm{I}(\mathcal{A})^{\mathrm{op}} \times \mathrm{I}(\mathcal{A}) \to \mathrm{I}(k_X)$ and the functor $\mathcal{H}om_{\mathrm{I}(\mathcal{A})}\colon \mathrm{I}(\mathcal{A})^{\mathrm{op}} \times \mathrm{I}(\mathcal{A}) \to \mathrm{Mod}(k_X)$. Recall that when $\mathcal{A} = k_X$ we simply write $\mathcal{I}hom$ and $\mathcal{H}om$ instead of $\mathcal{I}hom_{k_X}$ and $\mathcal{H}om_{\mathrm{I}(k_X)}$.

Note that the functor $\otimes_{\mathcal{A}}$ is right exact, the functor $\mathcal{I}hom_{\mathcal{A}}$ is left exact, and by Lemma 5.4.8, one has the isomorphisms

$$\begin{aligned} \mathcal{A} \otimes_{\mathcal{A}} M \quad &\simeq \quad M, \\ \mathcal{I}hom_{\mathcal{A}}(\mathcal{A}, M) \quad &\simeq \quad M. \end{aligned}$$

Proposition 5.4.11. — *One has the isomorphism*

$$\alpha_X \mathcal{I}hom_{\mathcal{A}}(M,N) \simeq \mathcal{H}om_{\mathrm{I}(\mathcal{A})}(M,N).$$

Proof. — The left hand side is isomorphic to $\ker(\mathcal{H}om\,(M,N) \underset{d}{\to} \mathcal{H}om\,(\mathcal{A}\otimes M, N))$ where $d = \mathcal{H}om\,(\mu_M, N) - \mathcal{H}om\,(M, \nu_N)$. Let $\lambda\colon \mathcal{H}om_{\mathcal{A}}(M,N) \to \mathcal{H}om\,(M,N)$. It is enough to check that λ induces an isomorphism $\Gamma(U; \mathcal{H}om_{\mathrm{I}(\mathcal{A})}(M,N)) \to \ker(d|_U)$ on each open $U \subset X$. We may assume $U = X$. In this case, one checks the isomorphism

$$\mathrm{Hom}_{\mathrm{I}(\mathcal{A})}(M,N) \simeq \ker(\mathrm{Hom}_{\mathrm{I}(k_X)}(M,N) \underset{d}{\to} \mathrm{Hom}_{\mathrm{I}(k_X)}(\mathcal{A}\otimes M, N)).$$

Indeed, d is the morphism $u \mapsto (\mu_M \circ u - u \circ \nu_N)$, and u is $\mathcal{A}$-linear if and only if $u \circ \nu_N = \mu_N \circ \mathrm{id}_{\mathcal{A}} \otimes u$. This is better visualized by the diagram

$$\begin{tikzcd}
 & & \mathrm{Hom}_{\mathrm{I}(k_X)}(\mathcal{A} \otimes M, N) \arrow[dd, dash, "\sim"] \\
\mathrm{Hom}_{\mathrm{I}(\mathcal{A})}(M, N) \arrow[r] & \mathrm{Hom}_{\mathrm{I}(k_X)}(M, N) \arrow[ur] \arrow[dr] & \\
 & & \mathrm{Hom}_{\mathrm{I}(k_X)}(M, \mathcal{I}hom\,(\mathcal{A}, N)).
\end{tikzcd}$$

q.e.d.

Sometimes, one has to consider various rings in $\mathrm{I}(k_X)$.

Proposition 5.4.12. — *Let $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ be three rings in $\mathrm{I}(k_X)$.*

(i) *The functor $\otimes_{\mathcal{A}_2}$ induces a functor*

$$\mathrm{I}(\mathcal{A}_1 \otimes \mathcal{A}_2^{\mathrm{op}}) \times \mathrm{I}(\mathcal{A}_2 \otimes \mathcal{A}_3^{\mathrm{op}}) \to \mathrm{I}(\mathcal{A}_1 \otimes \mathcal{A}_3^{\mathrm{op}}).$$

(ii) *The functor $\mathcal{I}hom_{\mathcal{A}_1}$ induces a functor*

$$\mathrm{I}(\mathcal{A}_1 \otimes \mathcal{A}_2)^{\mathrm{op}} \times \mathrm{I}(\mathcal{A}_1 \otimes \mathcal{A}_3) \to \mathrm{I}(\mathcal{A}_2^{\mathrm{op}} \otimes \mathcal{A}_3).$$

The proof is straightforward.

Proposition 5.4.13. — *Let $\mathcal{A}$ be a ring in $\mathrm{I}(k_X)$, let $M \in \mathrm{I}(\mathcal{A})$ and let $N \in \mathrm{Mod}(\alpha\mathcal{A})$. There are natural isomorphisms in $\mathrm{I}(k_X)$:*

$$\mathcal{H}om_{\alpha\mathcal{A}}(\alpha M, N) \simeq \mathcal{H}om_{\mathrm{I}(\mathcal{A})}(M, \iota N) \simeq \mathcal{I}hom_{\mathcal{A}}(M, \iota N).$$

The proof is straightforward.

Proposition 5.4.14. — *Let $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$, $\mathcal{A}_4$ be four rings in $\mathrm{I}(k_X)$. There is a natural isomorphism in $\mathrm{I}(\mathcal{A}_1 \otimes A_4^{\mathrm{op}})$:*

$$({}_1M_2 \otimes_{\mathcal{A}_2} {}_2M_3) \otimes_{\mathcal{A}_3} {}_3M_4 \simeq {}_1M_2 \otimes_{\mathcal{A}_2} ({}_2M_3 \otimes_{\mathcal{A}_3} {}_3M_4).$$

where ${}_iM_j$ means that M is an $\mathcal{A}_i \otimes A_j^{\mathrm{op}}$-module.

The proof is straightforward.

Proposition 5.4.15. — *Let $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4$ be four rings in $\mathrm{I}(k_X)$. There is a natural isomorphism in $\mathrm{I}(\mathcal{A}_3 \otimes A_4^{\mathrm{op}})$:*

$$\mathcal{I}hom_{\mathcal{A}_2}({}_2M_3, \mathcal{I}hom_{\mathcal{A}_1}({}_1M_2, {}_1M_4)) \simeq \mathcal{I}hom_{\mathcal{A}_1}({}_1M_2 \otimes_{\mathcal{A}_2} {}_2M_3, {}_1M_4).$$

Here, ${}_iM_j \in \mathrm{I}(\mathcal{A}_i \otimes \mathcal{A}_j^{\mathrm{op}})$.

The proof is straightforward.

We shall now construct the derived functors of $\otimes_{\mathcal{A}}$ and $\mathcal{I}hom_{\mathcal{A}}$.

Definition 5.4.16. — (i) $\mathcal{F}r(\mathcal{A})$ is the full subcategory of $\mathrm{I}(\mathcal{A})$ consisting of the objects isomorphic to $\mathcal{A} \otimes F$ for some $F \in \mathrm{I}(k_X)$.

(ii) For a small subset S of $\mathrm{I}(k_X)$, let $\mathcal{I}_q(\mathcal{A}, S)$ be the full subcategory of $\mathrm{I}(\mathcal{A})$ consisting of the objects isomorphic to $\mathcal{I}hom(\mathcal{A}, I)$ for some $I \in \mathcal{J}(S)$.

Recall that $\mathcal{J}(S)$ is given in Definition 5.1.7. We saw in Corollary 5.1.8 that $\mathcal{J}(S)$ is cogenerating in $\mathrm{I}(k_X)$.

Lemma 5.4.17. — (i) *The category $\mathcal{F}r(\mathcal{A})$ is generating in $\mathrm{I}(\mathcal{A})$.*

(ii) *For a small subset S of $\mathrm{I}(k_X)$, the category $\mathcal{I}_q(\mathcal{A}, S)$ is cogenerating in $\mathrm{I}(\mathcal{A})$.*

Proof. — (i) Let $M \in \mathrm{I}(\mathcal{A})$. Then $\mathcal{A} \otimes M \xrightarrow[\mu_M]{} M$ is an $\mathcal{A}$-linear epimorphism.

(ii) Let $M \in \mathrm{I}(\mathcal{A})$. Since $\mathcal{J}(S)$ is cogenerating, there exists a monomorphism $M \rightarrowtail I$ in $\mathrm{I}(k_X)$ with $I \in \mathcal{J}(S)$. Then the composition $M \to \mathcal{I}hom(\mathcal{A}, M) \to \mathcal{I}hom(\mathcal{A}, I)$ is an $\mathcal{A}$-linear monomorphism. q.e.d.

Lemma 5.4.18. — *For an $\mathcal{A}$-linear epimorphism $M \twoheadrightarrow \mathcal{A} \otimes F''$ with $M \in \mathrm{I}(\mathcal{A})$ and $F'' \in \mathrm{I}(k_X)$, there exists an exact sequence $0 \to F' \to F \to F'' \to 0$ in $\mathrm{I}(k_X)$ such that the morphism $\mathcal{A} \otimes F \to \mathcal{A} \otimes F''$ factors through M in $\mathrm{I}(\mathcal{A})$.*

Proof. — The morphism $\varepsilon_{\mathcal{A}}$ defines the morphism $F'' \to \mathcal{A} \otimes F''$. Let F be the fiber product of M and F'' over $\mathcal{A} \otimes F''$. Then $F \twoheadrightarrow F''$ is an epimorphism in $\mathrm{I}(k_X)$ and the composition $F \to F'' \to \mathcal{A} \otimes F''$ factors through M in $\mathrm{I}(k_X)$. Setting $F' = \ker(F \to F'')$, we get the result. q.e.d.

Theorem 5.4.19. — (a) *For any $M \in \mathrm{I}(\mathcal{A}_1 \otimes \mathcal{A}_2^{\mathrm{op}})$, the family $\mathcal{F}r(\mathcal{A}_2 \otimes \mathcal{A}_3^{\mathrm{op}})$ is projective with respect to the functor $M \otimes_{\mathcal{A}_2} \cdot : \mathrm{I}(\mathcal{A}_2 \otimes \mathcal{A}_3^{\mathrm{op}}) \to \mathrm{I}(\mathcal{A}_1 \otimes \mathcal{A}_3^{\mathrm{op}})$.*

(b) *The functor below is well-defined:*

$$\overset{L}{\underset{\mathcal{A}_2}{\otimes}} : D^-(\mathrm{I}(\mathcal{A}_1 \otimes \mathcal{A}_2^{\mathrm{op}})) \times D^-(\mathrm{I}(\mathcal{A}_2 \otimes \mathcal{A}_3^{\mathrm{op}})) \to D^-(\mathrm{I}(\mathcal{A}_1 \otimes \mathcal{A}_3^{\mathrm{op}})).$$

(c) *For four ind-rings $\mathcal{A}_\nu$ $(\nu = 1, \dots, 4)$, we have a canonical isomorphism in $D^-(\mathrm{I}(\mathcal{A}_1 \otimes \mathcal{A}_4^{\mathrm{op}}))$:*

$$({}_1M_2 \overset{L}{\underset{\mathcal{A}_2}{\otimes}} {}_2M_3) \overset{L}{\underset{\mathcal{A}_3}{\otimes}} {}_3M_4 \simeq {}_1M_2 \overset{L}{\underset{\mathcal{A}_2}{\otimes}} ({}_2M_3 \overset{L}{\underset{\mathcal{A}_3}{\otimes}} {}_3M_4).$$

Here ${}_iM_j \in D^-(\mathrm{I}(\mathcal{A}_i \otimes \mathcal{A}_j^{\mathrm{op}}))$.

Proof. — (a) Let $N = \mathcal{A}_2 \otimes \mathcal{A}_3^{\mathrm{op}} \otimes F$ with $F \in \mathrm{I}(k_X)$. Then $M \otimes_{\mathcal{A}_2} N \simeq \mathcal{A}_1 \otimes \mathcal{A}_3^{\mathrm{op}} \otimes F$ is exact in $F \in \mathrm{I}(k_X)$. Hence the assertion follows from the preceding lemma and Theorem 1.4.3.

(b) follows from (a).

To prove (c) we may assume ${}_2M_3 \in \mathcal{F}r(\mathcal{A}_2 \otimes \mathcal{A}_3^{\mathrm{op}})$. Then the assertion follows from Proposition 5.4.14. q.e.d.

Lemma 5.4.20. — *Let S be a small subset in $\mathrm{I}(k_X)$. For any $\mathcal{A}$-linear monomorphism $\mathcal{I}hom(\mathcal{A}, I') \rightarrowtail M$ with $M \in \mathrm{I}(\mathcal{A})$ and $I' \in \mathcal{J}(S)$, there exists an exact sequence $0 \to I' \to I \to I'' \to 0$ in $\mathrm{I}(k_X)$ with $I,\ I'' \in \mathcal{J}(S)$ such that $\mathcal{I}hom(\mathcal{A}, I') \to \mathcal{I}hom(\mathcal{A}, I)$ factors through M in $\mathrm{I}(\mathcal{A})$.*

Proof. — The proof is dual to that of Lemma 5.4.18. The morphism $\varepsilon_{\mathcal{A}}$ defines the morphism $\mathcal{I}hom(\mathcal{A}, I') \to I'$. Let N be the fiber coproduct of I' and M over $\mathcal{I}hom(\mathcal{A}, I')$. Then $I' \to N$ is a monomorphism in $\mathrm{I}(k_X)$. Since $\mathcal{J}(S)$ is cogenerating, there is a monomorphism $N \to I$ with $I \in \mathcal{J}(S)$. The cokernel I'' of the monomorphism $I' \to I$ belongs to $\mathcal{J}(S)$. On the other hand, the composition $\mathcal{A} \otimes M \xrightarrow[\mu_M]{} M \to N \to I$ gives a morphism $M \to \mathcal{I}hom(\mathcal{A}, I)$ in $\mathrm{I}(\mathcal{A})$. We can easily see that the morphism $\mathcal{I}hom(\mathcal{A}, I') \to \mathcal{I}hom(\mathcal{A}, I)$ coincides with the composition of $\mathcal{I}hom(\mathcal{A}, I') \to M$ and $M \to \mathcal{I}hom(\mathcal{A}, I)$ in $\mathrm{I}(\mathcal{A})$. q.e.d.

Theorem 5.4.21. — *Let $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$ be three rings in $\mathrm{I}(k_X)$.*

(a) *For any small subset T of $K^-(\mathcal{A}_1 \otimes \mathcal{A}_2^{\mathrm{op}})$, there exists a small subset S of $\mathrm{I}(k_X)$ such that, for any ${}_1M_2 \in T$, the category $\mathcal{I}_q(\mathcal{A}_1 \otimes \mathcal{A}_3^{\mathrm{op}}, S)$ is injective with respect to the functors:*

$$\begin{aligned}
\mathcal{I}hom_{\mathcal{A}_1}({}_1M_2, \cdot\,) &\ :\ \mathrm{I}(\mathcal{A}_1 \otimes \mathcal{A}_3^{\mathrm{op}}) \to \mathrm{I}(\mathcal{A}_2 \otimes \mathcal{A}_3^{\mathrm{op}}),\\
\mathcal{H}om_{\mathrm{I}(\mathcal{A}_1)}({}_1M_2, \cdot\,) &\ :\ \mathrm{I}(\mathcal{A}_1 \otimes \mathcal{A}_3^{\mathrm{op}}) \to \mathrm{Mod}(\alpha_X(\mathcal{A}_2 \otimes \mathcal{A}_3^{\mathrm{op}})),\\
\mathrm{Hom}_{\mathrm{I}(\mathcal{A}_1)}({}_1M_2, \cdot\,) &\ :\ \mathrm{I}(\mathcal{A}_1 \otimes \mathcal{A}_3^{\mathrm{op}}) \to \mathrm{Mod}(k).
\end{aligned}$$

(b) *The functors below are well-defined:*

$$\begin{aligned}
R\mathcal{I}hom_{\mathcal{A}_1} &: D^-(\mathrm{I}(\mathcal{A}_1 \otimes \mathcal{A}_2^{\mathrm{op}}))^{\mathrm{op}} \times D^+(\mathrm{I}(\mathcal{A}_1 \otimes \mathcal{A}_3^{\mathrm{op}})) \to D^+(\mathrm{I}(\mathcal{A}_2 \otimes \mathcal{A}_3^{\mathrm{op}}))\,,\\
R\mathcal{H}om_{\mathrm{I}(\mathcal{A}_1)} &: D^-(\mathrm{I}(\mathcal{A}_1 \otimes \mathcal{A}_2^{\mathrm{op}}))^{\mathrm{op}} \times D^+(\mathrm{I}(\mathcal{A}_1 \otimes \mathcal{A}_3^{\mathrm{op}}))\\
&\qquad\qquad \to D^+(\mathrm{Mod}(\alpha_X(\mathcal{A}_2 \otimes \mathcal{A}_3^{\mathrm{op}})))\\
\mathrm{RHom}_{\mathrm{I}(\mathcal{A}_1)} &: D^-(\mathrm{I}(\mathcal{A}_1 \otimes \mathcal{A}_2^{\mathrm{op}}))^{\mathrm{op}} \times D^+(\mathrm{I}(\mathcal{A}_1 \otimes \mathcal{A}_3^{\mathrm{op}})) \to D^+(\mathrm{Mod}(k)).
\end{aligned}$$

(c) *For $M \in K^-(\mathrm{I}(\mathcal{A}_1 \otimes \mathcal{A}_2^{\mathrm{op}}))$, we have*

$$\begin{aligned}
R\mathcal{I}hom_{\mathcal{A}_1}(M, N) &\simeq \text{“}\varinjlim\text{”}\, \mathcal{I}hom_{\mathcal{A}_1}(M, N'),\\
R\mathcal{H}om_{\mathrm{I}(\mathcal{A}_1)}(M, N) &\simeq \text{“}\varinjlim\text{”}\, \mathcal{H}om_{\mathrm{I}(\mathcal{A}_1)}(M, N'),\\
R\mathrm{Hom}_{\mathrm{I}(\mathcal{A}_1)}(M, N) &\simeq \text{“}\varinjlim\text{”}\, \mathrm{Hom}_{\mathrm{I}(\mathcal{A}_1)}(M, N').
\end{aligned}$$

Here the inductive limits are taken in the category of ind-objects in the derived categories, and $N \to N'$ ranges over the family of qis in $K^+(\mathrm{I}(\mathcal{A}_1 \otimes \mathcal{A}_3^{\mathrm{op}}))$.

Proof. — We shall only treat the functor $\mathcal{I}hom_{\mathcal{A}_1}$, the other cases being similar.
(a) We shall take S such that

– $\mathcal{A}_1 \otimes \mathcal{A}_3^{\mathrm{op}} \in S$,
– $\mathcal{A}_3^{\mathrm{op}} \otimes {}_1M_2 \in S$ for any ${}_1M_2 \in T$.

By Theorem 1.4.3 and the preceding lemma, it is enough to show that for an exact sequence $0 \to I' \to I \to I'' \to 0$ in $\mathcal{J}(S)$,

$$0 \to \mathcal{I}hom_{\mathcal{A}_1}({}_1M_2, \mathcal{I}hom(\mathcal{A}_1 \otimes \mathcal{A}_3^{\mathrm{op}}, I')) \to \mathcal{I}hom_{\mathcal{A}_1}({}_1M_2, \mathcal{I}hom(\mathcal{A}_1 \otimes \mathcal{A}_3^{\mathrm{op}}, I)) \\ \to \mathcal{I}hom_{\mathcal{A}_1}({}_1M_2, \mathcal{I}hom(\mathcal{A}_1 \otimes \mathcal{A}_3^{\mathrm{op}}, I'')) \to 0$$

is exact for any ${}_1M_2 \in T$. This follows from $\mathcal{I}hom_{\mathcal{A}_1}({}_1M_2, \mathcal{I}hom(\mathcal{A}_1 \otimes \mathcal{A}_3^{\mathrm{op}}, I)) \simeq \mathcal{I}hom(\mathcal{A}_3^{\mathrm{op}} \otimes {}_1M_2, I)$ and $H^1(R\mathcal{I}hom(\mathcal{A}_3^{\mathrm{op}} \otimes {}_1M_2, I')) = 0$.
(b) It remains to show that for any ${}_1M_2 \in K^-(\mathcal{A}_1 \otimes \mathcal{A}_2^{\mathrm{op}})$ quasi-isomorphic to 0, $\mathcal{I}hom_{\mathcal{A}_1}({}_1M_2, N)$ is also quasi-isomorphic to 0 for any $N \in K^+(\mathcal{I}_q(\mathcal{A}_1 \otimes \mathcal{A}_3^{\mathrm{op}}, S))$ if we take S big enough. By standard arguments, it is enough to show that for any exact sequence $0 \to {}_1M_2' \to {}_1M_2 \to {}_1M_2'' \to 0$, the sequence $0 \to \mathcal{I}hom_{\mathcal{A}_1}({}_1M_2'', N) \to \mathcal{I}hom_{\mathcal{A}_1}({}_1M_2, N) \to \mathcal{I}hom_{\mathcal{A}_1}({}_1M_2', N) \to 0$ is exact for any $N \in \mathcal{I}_q(\mathcal{A}_1 \otimes \mathcal{A}_3^{\mathrm{op}}, S)$. Writing $N \simeq \mathcal{I}hom(\mathcal{A}_1 \otimes \mathcal{A}_3^{\mathrm{op}}, I)$ with $I \in \mathcal{J}(S)$, this follows from $\mathcal{I}hom_{\mathcal{A}_1}({}_1M_2, N) \simeq \mathcal{I}hom(\mathcal{A}_3^{\mathrm{op}} \otimes {}_1M_2, I)$ and $H^1(R\mathcal{I}hom(\mathcal{A}_3^{\mathrm{op}} \otimes {}_1M_2'', I)) = 0$. q.e.d.

In order to show the relations between $\otimes$ and $\mathcal{I}hom$, let us prove the following lemma.

Lemma 5.4.22. — *Let $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$ be three rings in $\mathrm{I}(k_X)$, and let S_{12} and S_{23} be small subsets of $\mathrm{I}(k_X)$. Then there exists a small subset S_{13} of $\mathrm{I}(k_X)$ such that $\mathcal{I}hom_{\mathcal{A}_1}(M, N)$ belongs to $\mathcal{I}_q(\mathcal{A}_2 \otimes \mathcal{A}_3^{\mathrm{op}}, S_{23})$ for any $M \in \mathrm{I}(\mathcal{A}_1 \otimes \mathcal{A}_2^{\mathrm{op}})$ of the form $\mathcal{A}_1 \otimes \mathcal{A}_2^{\mathrm{op}} \otimes K$ with $K \in S_{12}$ and any $N \in \mathcal{I}_q(\mathcal{A}_1 \otimes \mathcal{A}_3^{\mathrm{op}}, S_{13})$.*

Proof. — Take S_{13} such that $\mathcal{A}_1 \otimes \mathcal{A}_3 \in S_{13}$, $S_{12} \subset S_{13}$, and $F \otimes K \in S_{13}$ for any $F \in S_{23}$ and $K \in S_{12}$. Write $N = \mathcal{I}hom(\mathcal{A}_1 \otimes \mathcal{A}_3^{\mathrm{op}}, I)$ with $I \in \mathcal{J}(S_{13})$. Then we have

$$\mathcal{I}hom_{\mathcal{A}_1}(M, N) \simeq \mathcal{I}hom(\mathcal{A}_2 \otimes \mathcal{A}_3^{\mathrm{op}} \otimes K, I) \simeq \mathcal{I}hom(\mathcal{A}_2 \otimes \mathcal{A}_3^{\mathrm{op}}, \mathcal{I}hom(K, I)).$$

Hence it is enough to show that $\mathcal{I}hom(K, I)$ belongs to $\mathcal{J}(S_{23})$. Since $K \in S_{12} \subset S_{13}$, we have $R\mathcal{I}hom(K, I) \simeq \mathcal{I}hom(K, I)$, and we conclude, for any $F \in S_{23}$

$$R\mathcal{I}hom(F, \mathcal{I}hom(K, I)) \simeq R\mathcal{I}hom(F \otimes K, I) \simeq \mathcal{I}hom(F \otimes K, I).$$

q.e.d.

Theorem 5.4.23. — *Let $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$, $\mathcal{A}_4$ be four rings in $\mathrm{I}(k_X)$. Then there is a natural isomorphism in $D^+(\mathrm{I}(\mathcal{A}_3 \otimes \mathcal{A}_4^{\mathrm{op}}))$*

$$R\mathcal{I}hom_{\mathcal{A}_2}({}_2M_3, R\mathcal{I}hom_{\mathcal{A}_1}({}_1M_2, {}_1N_4)) \simeq R\mathcal{I}hom_{\mathcal{A}_1}({}_1M_2 \overset{L}{\underset{\mathcal{A}_2}{\otimes}} {}_2M_3, {}_1N_4).$$

Here, ${}_iM_j \in D^-(\mathrm{I}(\mathcal{A}_i \otimes \mathcal{A}_j^{\mathrm{op}}))$ and ${}_1N_4 \in D^+(\mathrm{I}(\mathcal{A}_1 \otimes \mathcal{A}_4^{\mathrm{op}}))$.

Proof. — We may assume that ${}_1M_2$ is a complex in $\mathcal{F}r(\mathcal{A}_1 \otimes \mathcal{A}_2^{\mathrm{op}})$. Then, taking S_{24} big enough, we have $R\mathcal{I}hom_{\mathcal{A}_2}({}_2M_3, I) \simeq \mathcal{I}hom_{\mathcal{A}_2}({}_2M_3, I)$ for any $I \in \mathcal{I}_q(\mathcal{A}_2 \otimes \mathcal{A}_4^{\mathrm{op}}, S_{24})$. Taking S_{14} big enough and assuming ${}_1N_4 \in \mathcal{I}_q(\mathcal{A}_1 \otimes \mathcal{A}_4^{\mathrm{op}}, S_{14})$, $R\mathcal{I}hom_{\mathcal{A}_1}({}_1M_2, {}_1N_4)$ is represented by $\mathcal{I}hom_{\mathcal{A}_1}({}_1M_2, {}_1N_4)$. Furthermore, by the last lemma, we may assume that the last term belongs to $\mathcal{I}_q(\mathcal{A}_2 \otimes \mathcal{A}_4^{\mathrm{op}}, S_{24})$. Hence we have

$$R\mathcal{I}hom_{\mathcal{A}_2}({}_2M_3, R\mathcal{I}hom_{\mathcal{A}_1}({}_1M_2, {}_1N_4)) \simeq \mathcal{I}hom_{\mathcal{A}_2}({}_2M_3, \mathcal{I}hom_{\mathcal{A}_1}({}_1M_2, {}_1N_4)).$$

On the other hand, $R\mathcal{I}hom_{\mathcal{A}_1}({}_1M_2 \otimes_{\mathcal{A}_2} {}_2M_3, {}_1N_4)$ is represented by $\mathcal{I}hom_{\mathcal{A}_1}({}_1M_2 \otimes_{\mathcal{A}_2} {}_2M_3, {}_1N_4)$ and ${}_1M_2 \overset{L}{\underset{\mathcal{A}_2}{\otimes}} {}_2M_3$ is represented by ${}_1M_2 \otimes_{\mathcal{A}_2} {}_2M_3$. Then it is enough to apply Proposition 5.4.15. q.e.d.

5.5. Ring action II

In this section we shall extend some results of section 5.2, replacing the base ring k by ind-rings.

Since the formalism is similar to that developed previously, we shall not give any proof.

We consider the following situation: $f : X \to Y$ is a continuous map and $\mathcal{B}$ is an ind-ring on Y. We shall assume for simplicity:

(5.5.1) the cohomological dimension of $\mathrm{Mod}(k_X)$ is finite.

Theorem 5.5.1. — *In* (i)–(iii) *below,* $D^\dagger$ *is* D, D^b, D^+ *or* D^-.
(i) *The functor* $f^{-1} \colon \mathrm{I}(k_Y) \to \mathrm{I}(k_X)$ *induces a functor* $f^{-1} : D^\dagger(\mathrm{I}(\mathcal{B})) \to D^\dagger(\mathrm{I}(f^{-1}\mathcal{B}))$,
(ii) *The functor* $f_* \colon \mathrm{I}(k_X) \to \mathrm{I}(k_Y)$ *induces a functor* $Rf_* \colon D^\dagger(\mathrm{I}(f^{-1}\mathcal{B})) \to D^\dagger(\mathrm{I}(\mathcal{B}))$,
(iii) *The functor* $f_{!!} \colon \mathrm{I}(k_X) \to \mathrm{I}(k_Y)$ *induces a functor* $Rf_{!!} \colon D^\dagger(\mathrm{I}(f^{-1}\mathcal{B})) \to D^\dagger(\mathrm{I}(\mathcal{B}))$.

Theorem 5.5.2. — *For* $G \in D^-(\mathrm{I}(\mathcal{B}))$ *and* $F \in D^+(\mathrm{I}(f^{-1}\mathcal{B}))$, *one has the isomorphisms*

$$\begin{aligned}
\mathrm{RHom}_{\mathrm{I}(\mathcal{B})}(G, Rf_*F) &\simeq \mathrm{RHom}_{\mathrm{I}(f^{-1}\mathcal{B})}(f^{-1}G, F),\\
R\mathcal{H}om_{\mathrm{I}(\mathcal{B})}(G, Rf_*F) &\simeq Rf_*R\mathcal{H}om_{\mathrm{I}(f^{-1}\mathcal{B})}(f^{-1}G, F),\\
R\mathcal{I}hom_{\mathcal{B}}(G, Rf_*F) &\simeq Rf_*R\mathcal{I}hom_{f^{-1}\mathcal{B}}(f^{-1}G, F).
\end{aligned}$$

Theorem 5.5.3. — *The functors* f^{-1} *and* Rf_* *are adjoint. More precisely, for* $F \in D^+(\mathrm{I}(f^{-1}\mathcal{B}))$ *and* $G \in D^+(\mathrm{I}(\mathcal{B}))$, *one has the isomorphism*

$$\mathrm{Hom}_{D^+(\mathrm{I}(f^{-1}\mathcal{B}))}(f^{-1}G, F) \simeq \mathrm{Hom}_{D^+(\mathrm{I}(\mathcal{B}))}(G, Rf_*F).$$

Theorem 5.5.4. — *For* $F \in D^-(\mathrm{I}(f^{-1}\mathcal{B}))$ *and* $G \in D^-(\mathrm{I}(\mathcal{B}^{\mathrm{op}}))$, *one has the isomorphism*

$$G \overset{L}{\underset{\mathcal{B}}{\otimes}} Rf_{!!}F \simeq Rf_{!!}(f^{-1}G \overset{L}{\underset{f^{-1}\mathcal{B}}{\otimes}} F).$$

Theorem 5.5.5. — *Consider the Cartesian square (5.3.2). There is a natural isomorphism of functors from $D^+(\mathrm{I}(f^{-1}\mathcal{B}))$ to $D^+(\mathrm{I}(g^{-1}\mathcal{B}))$:*

$$Rf'_{!!}g'^{-1} \simeq g^{-1}Rf_{!!}\,.$$

Theorem 5.5.6. — *The functor $Rf_{!!}\colon D^+(\mathrm{I}(f^{-1}\mathcal{B})) \to D^+(\mathrm{I}(\mathcal{B}))$ admits a right adjoint, denoted by $f^!$. More precisely, for $F \in D^+(\mathrm{I}(f^{-1}\mathcal{B}))$ and $G \in D^+(\mathrm{I}(\mathcal{B}))$ one has:*

$$\mathrm{Hom}_{D^+(\mathrm{I}(\mathcal{B}))}(Rf_{!!}F, G) \simeq \mathrm{Hom}_{D^+(\mathrm{I}(f^{-1}\mathcal{B}))}(F, f^!G).$$

5.6. Action of $\beta\mathcal{A}$

In this section we give some formulas in the particular case where the ind-ring is of the type $\beta\mathcal{A}$ with $\mathcal{A}$ a sheaf of k-algebras as in Example 5.4.7 (ii). We shall assume

$$\begin{aligned}&\text{the cohomological dimension of } \mathrm{Mod}(k_X) \text{ is finite,}\\ &\text{the flat dimension of } \mathcal{A} \text{ is finite.}\end{aligned}$$

In the sequel we shall write β instead of $\beta_X\colon \mathrm{Mod}(k_X) \to \mathrm{I}(k_X)$, for short. Note that $\mathrm{I}(\beta\mathcal{A}) \simeq \mathrm{Mod}(\mathcal{A}, \mathrm{I}(k_X))$ and β induces an exact functor (still denoted by β)

$$\beta\colon \mathrm{Mod}(\mathcal{A}) \to \mathrm{I}(\beta\mathcal{A}).$$

Theorem 5.6.1. — *Let $\mathcal{A}$ be a sheaf of k_X-algebras, let $F \in D^b(k_X)$, let $K \in D^b(\mathcal{A})$, let $M \in D^b(\mathrm{I}(\beta\mathcal{A}^{\mathrm{op}}))$ and let $N \in D^b(\mathrm{I}(\beta\mathcal{A}))$. Then one has the isomorphisms:*

(i) $\alpha(M \overset{L}{\underset{\beta\mathcal{A}}{\otimes}} N) \simeq \alpha(M) \overset{L}{\underset{\mathcal{A}}{\otimes}} \alpha(N)$,

(ii) $R\mathcal{I}hom\,(F, M) \overset{L}{\underset{\beta\mathcal{A}}{\otimes}} \beta K \xrightarrow{\sim} R\mathcal{I}hom\,(F, M \overset{L}{\underset{\beta\mathcal{A}}{\otimes}} \beta K)$,

(iii) $R\mathcal{H}om\,(F, M) \overset{L}{\underset{\mathcal{A}}{\otimes}} K \xrightarrow{\sim} R\mathcal{H}om\,(F, M \overset{L}{\underset{\beta\mathcal{A}}{\otimes}} \beta K)$.

Proof. — (i) For $M \in \mathrm{I}(\beta\mathcal{A}^{\mathrm{op}})$ and $N \in \mathrm{I}(\beta\mathcal{A})$, one checks easily the formula $\alpha(M \otimes_{\beta\mathcal{A}} N) \simeq \alpha M \otimes_{\mathcal{A}} \alpha N$. Hence

$$\begin{aligned}\alpha(M \overset{L}{\underset{\beta\mathcal{A}}{\otimes}} N) &\simeq \underset{M'\to, M, N'\to N}{\text{“}\varprojlim\text{”}} \alpha(M' \otimes_{\beta\mathcal{A}} N')\\ &\simeq \underset{M'\to M,\, N'\to N}{\text{“}\varprojlim\text{”}} \big(\alpha M' \otimes_{\mathcal{A}} \alpha N'\big) \simeq \alpha M \overset{L}{\underset{\mathcal{A}}{\otimes}} \alpha N.\end{aligned}$$

Here the projective limits range over the family of quasi-isomorphisms $M' \to M$ and $N' \to N$.

(ii) There is a canonical morphism $R\mathcal{I}hom\,(F, M) \overset{L}{\underset{\beta\mathcal{A}}{\otimes}} \beta K \to R\mathcal{I}hom\,(F, M \overset{L}{\underset{\beta\mathcal{A}}{\otimes}} \beta K)$. In order to prove that it is an isomorphism, it is enough to check that it induces an isomorphism on the cohomology objects. Since cohomology commutes with inductive limits, we may reduce to the case where $K = \mathcal{A}_U$ for an open subset U of X. We then have the chain of isomorphisms

$$\begin{aligned} R\mathcal{I}hom\,(F,M) \overset{L}{\underset{\beta\mathcal{A}}{\otimes}} \beta\mathcal{A}_U &\simeq R\mathcal{I}hom\,(F,M) \otimes_{k_X} {}_U k \\ &\simeq R\mathcal{I}hom\,(F, M \otimes_{k_X} {}_U k) \\ &\simeq R\mathcal{I}hom\,(F, M \overset{L}{\underset{\beta\mathcal{A}}{\otimes}} \beta\mathcal{A}_U). \end{aligned}$$

(iii) The third isomorphism follows by applying α_X. q.e.d.

Theorem 5.6.2. — *Let $\mathcal{A}$ be a sheaf of k_X-algebras, let $N \in D^b(\mathcal{A}^{\mathrm{op}})$, let $M \in D^b(\mathcal{A})$ and let $K \in D^b(\mathrm{I}(\beta\mathcal{A}))$. There are natural isomorphisms*

(i) $R\mathcal{H}om_{\mathcal{A}}(\alpha K, M) \simeq R\mathcal{I}hom_{\beta\mathcal{A}}(K,M) \simeq R\mathcal{H}om_{\mathrm{I}(\beta\mathcal{A})}(K,M)$,

(ii) $R\mathcal{H}om_{\mathrm{I}(\beta\mathcal{A})}(\beta M, K) \simeq R\mathcal{H}om_{\mathcal{A}}(M, \alpha K)$,

(iii) $\beta(N \overset{L}{\underset{\mathcal{A}}{\otimes}} M) \simeq \beta N \overset{L}{\underset{\beta\mathcal{A}}{\otimes}} \beta M$.

Proof. — (i) Let us denote by $\mathcal{I}nj(\mathcal{A})$ the category of injective $\mathcal{A}$-modules. Then ι_X sends $\mathcal{I}nj(\mathcal{A})$ to the set of injective objects in $\mathrm{I}(\beta\mathcal{A})$, and the result follows from Proposition 5.4.13.
(ii) The isomorphism

$$\mathcal{H}om_{\mathrm{I}(\beta\mathcal{A})}(\beta M, K) \simeq \mathcal{H}om_{\mathcal{A}}(M, \alpha K) \tag{5.6.1}$$

follows immediately from $\mathcal{H}om\,(\beta M, K) \simeq \mathcal{H}om\,(M, \alpha K)$. Taking $S \subset \mathrm{I}(k_X)$ big enough and assuming that the components of K belong to $\mathcal{I}_q(\beta\mathcal{A}, S)$, we have

$$\begin{aligned} R\mathcal{H}om_{\mathrm{I}(\beta\mathcal{A})}(\beta M, K) &\simeq \mathcal{H}om_{\mathrm{I}(\beta\mathcal{A})}(\beta M, K) \\ &\simeq \mathcal{H}om_{\mathcal{A}}(M, \alpha K). \end{aligned}$$

Then the results follows since αK is a complex of injective $\mathcal{A}$-modules.
(iii) Let $K \in D^b(\mathrm{I}(k_X))$. Using the result of (i), we get the chain of isomorphisms

$$\begin{aligned} &\mathrm{Hom}_{D^b(\mathrm{I}(k_X))}(\beta(N \overset{L}{\underset{\mathcal{A}}{\otimes}} M), K) \simeq \mathrm{Hom}_{D^b(k_X)}(N \overset{L}{\underset{\mathcal{A}}{\otimes}} M, \alpha K) \\ &\quad\simeq \mathrm{Hom}_{D^b(\mathcal{A})}(M, R\mathcal{H}om\,(N, \alpha K)) \simeq \mathrm{Hom}_{D^b(\mathcal{A})}(M, \alpha(R\mathcal{I}hom\,(\beta N, K))) \\ &\quad\simeq \mathrm{Hom}_{D^b(\mathrm{I}(\beta\mathcal{A}))}(\beta M, R\mathcal{I}hom\,(\beta N, K)) \simeq \mathrm{Hom}_{D^b(\mathrm{I}(k_X))}(\beta M \overset{L}{\underset{\beta\mathcal{A}}{\otimes}} \beta N, K). \end{aligned}$$

q.e.d.

Theorem 5.6.3. — *Let $\mathcal{B}$ be a sheaf of k_Y-algebras, let $L \in D^b(\mathrm{I}(\beta\mathcal{B}^{\mathrm{op}}))$ and $N \in D^b(\mathcal{B})$. Then we have the isomorphism*

$$f^!(L \overset{L}{\underset{\beta\mathcal{B}}{\otimes}} \beta N) \simeq f^! L \overset{L}{\underset{\beta f^{-1}\mathcal{B}}{\otimes}} \beta f^{-1} N.$$

Proof. — We have the morphisms

$$Rf_{!!}(f^! L \overset{L}{\underset{\beta f^{-1}\mathcal{B}}{\otimes}} \beta f^{-1} N) \simeq Rf_{!!} f^! L \overset{L}{\underset{\beta\mathcal{B}}{\otimes}} \beta N \to L \overset{L}{\underset{\beta\mathcal{B}}{\otimes}} \beta N.$$

By adjunction, we get the morphism $f^! L \overset{L}{\underset{\beta f^{-1}\mathcal{B}}{\otimes}} \beta f^{-1} N \to f^!(L \overset{L}{\underset{\beta\mathcal{B}}{\otimes}} \beta N)$. Let $K \in \mathrm{Mod}^c(k_X)$. We have the chain of isomorphisms

$$\begin{aligned} Rf_* R\mathcal{H}om\,(K, f^! L \overset{L}{\underset{\beta f^{-1}\mathcal{B}}{\otimes}} \beta f^{-1} N) &\simeq Rf_!(R\mathcal{H}om\,(K, f^! L) \overset{L}{\underset{f^{-1}\mathcal{B}}{\otimes}} f^{-1} N) \\ &\simeq Rf_! R\mathcal{H}om\,(K, f^! L) \overset{L}{\underset{\mathcal{B}}{\otimes}} N \simeq R\mathcal{H}om\,(Rf_{!!} K, L) \overset{L}{\underset{\mathcal{B}}{\otimes}} N \\ &\simeq R\mathcal{H}om\,(Rf_{!!} K, L \overset{L}{\underset{\beta\mathcal{B}}{\otimes}} \beta N) \simeq Rf_* R\mathcal{H}om\,(K, f^!(L \overset{L}{\underset{\beta\mathcal{B}}{\otimes}} \beta N)). \end{aligned}$$

This implies the isomorphism

$$\mathrm{Hom}_{D^b(\mathrm{I}(k_X))}(K, f^!(L \overset{L}{\underset{\beta\mathcal{B}}{\otimes}} \beta N)) \simeq \mathrm{Hom}_{D^b(\mathrm{I}(k_X))}(K, f^! L \overset{L}{\underset{\beta f^{-1}\mathcal{B}}{\otimes}} \beta f^{-1} N),$$

whence the result. q.e.d.

Exercises to Chapter 5

Exercise 5.1. — Let $Y = \mathbb{R}$, $X = \{0\}$ and denote by $f\colon X \to Y$ the embedding. Let $G = {}_{\{0\}}(k_Y) \in \mathrm{I}(k_Y)$. Prove that $f^! G \simeq k_X[-1]$ and $f^! \alpha_Y G \simeq k_X$. Hence, $f^!$ does not commute with α.

Exercise 5.2. — Let $f\colon X \to Y$ be as in Exercise 5.1 and let $G = k_{\{0\}} \in \mathrm{Mod}(k_Y)$. Prove that $f^! \beta_Y G \simeq k_X[-1]$ and $\beta_X f^! G \simeq k_X$. Hence, $f^!$ does not commute with β.

Exercise 5.3. — Let $\mathcal{A}$ denote a sheaf of k_X-algebras. We also regard $\mathcal{A}$ as a ring in $\mathrm{I}(k_X)$, and we denote by $\varphi\colon \mathrm{Ind}(\mathrm{Mod}^c(\mathcal{A})) \to \mathrm{I}(\mathcal{A})$ the canonical functor. Prove that φ is an equivalence of categories as follows.
(i) Construct the functor $\tau\colon \mathrm{I}(k_X) \to \mathrm{Ind}(\mathrm{Mod}^c(\mathcal{A}))$ such that $\varphi \circ \tau(F) \simeq \mathcal{A} \otimes F$ for $F \in \mathrm{I}(k_X)$.
(ii) For $M \in \mathrm{I}(\mathcal{A})$, construct a morphism $\tilde{d}(M)\colon \tau(\mathcal{A} \otimes M) \to \tau(M)$ in $\mathrm{Ind}(\mathrm{Mod}^c(\mathcal{A}))$ such that $\varphi(\tilde{d}(M))$ coincides with d in (5.4.2).
(iii) Define $\psi\colon \mathrm{I}(\mathcal{A}) \to \mathrm{Ind}(\mathrm{Mod}^c(\mathcal{A}))$ as $\psi = \operatorname{coker} \tilde{d}$. Prove that the two functors φ and ψ are a quasi-inverse to each other.

CHAPTER 6

CONSTRUCTION OF IND-SHEAVES

In this chapter we use Grothendieck topologies in order to construct ind-sheaves. We shall assume that k is a field and that X is a Hausdorff locally compact space with a countable base of open subsets.

6.1. $\mathcal{T}$-topology

We consider a family $\mathcal{T}$ of open subsets of X. We set

$$\begin{aligned} \mathcal{T}_c &= \{U \in \mathcal{T};\ U \text{ is relatively compact}\}, \\ \mathcal{T}_0 &= \{U \in \mathcal{T}_c;\ U \text{ is connected}\}. \end{aligned}$$

We shall consider some of the hypotheses (6.1.1) and (6.1.2) below.

$$(6.1.1)\quad \begin{cases} \text{(i)} & U, V \in \mathcal{T} \text{ implies } U \cap V \in \mathcal{T}, \\ \text{(ii)} & U \text{ and } V \text{ belong to } \mathcal{T} \text{ if and only if } U \cap V \text{ and } U \cup V \text{ belong to } \mathcal{T}, \\ \text{(iii)} & U \setminus V \text{ has finitely many connected components for every } U, V \in \mathcal{T}_c, \\ \text{(iv)} & \mathcal{T}_c \text{ is a covering of } X, \text{ and } \varnothing,\ X \in \mathcal{T}. \end{cases}$$

$$(6.1.2)\quad \begin{array}{l} \text{for any } x \in X,\ \{U \in \mathcal{T}; x \in U\} \text{ is a neighborhood system of } x, \\ \text{(i.e. } \mathcal{T} \text{ is a basis of the topology).} \end{array}$$

Note that assuming (6.1.1), every $U \in \mathcal{T}_c$ has finitely many connected components, each of which belongs to $\mathcal{T}_c$.

We regard $\mathcal{T}$ as a subcategory of $\mathrm{Op}(X)$. Assuming (6.1.1) (i), $\mathcal{T}$ admits products and fiber products. Moreover hypothesis (2.3.2) of Chapter 2 is clearly satisfied.

Definition 6.1.1. — Assume (6.1.1) (i).

(i) We denote by $X_{\mathcal{T}}$ the category $\mathcal{T}$ endowed with the topology induced by X_{lf} (see Example 2.1.4). Hence $S \subset \mathcal{T} \cap \mathrm{Op}(U)$ is a covering of $U \in \mathcal{T}$ if for any compact K of X, there exists a finite subset $S_0 \subset S$ such that $K \cap (\cup_{V \in S_0} V) = K \cap U$.

(ii) For $U \in \mathrm{Op}(X)$, we set $\mathcal{T} \cap U := \{V \cap U; V \in \mathcal{T}\}$.

(iii) For $U \in \mathrm{Op}(X)$, we denote by $U_{\mathcal{T}}$ the category $\mathcal{T} \cap U$ endowed with the topology induced by U_{lf}. We denote by $i_{U_{\mathcal{T}}} \colon U_{\mathcal{T}} \to X_{\mathcal{T}}$ the natural functor of sites associated with $\mathcal{T} \ni V \mapsto V \cap U \in \mathcal{T} \cap U$.

(iv) For $U \in \mathcal{T}$, we denote by $U_{X\mathcal{T}}$ the category $\mathcal{T} \cap U$ endowed with the topology induced by $X_{\mathcal{T}}$. A covering of V in $U_{X\mathcal{T}}$ is a covering in $X_{\mathcal{T}}$. We denote by $i_{U_{X\mathcal{T}}} \colon U_{X\mathcal{T}} \to X_{\mathcal{T}}$ the natural morphism of sites $\mathcal{T} \ni V \mapsto V \cap U$ and by $j_{U_{X\mathcal{T}}} \colon X_{\mathcal{T}} \to U_{X\mathcal{T}}$ the morphism of sites $\mathcal{T} \cap U \ni V \mapsto V \in \mathcal{T}$.

Notation 6.1.2. — (i) We shall often write $k_{\mathcal{T}}$ instead of $k_{X_{\mathcal{T}}}$ and hence $\mathrm{Mod}(k_{\mathcal{T}})$ instead of $\mathrm{Mod}(k_{X_{\mathcal{T}}})$.
(ii) We denote by $\rho \colon X \to X_{\mathcal{T}}$ the natural morphism of sites.

As already mentioned in Chapter 2, if $U \in \mathcal{T}$, then $j_{U_{X\mathcal{T}}*} \simeq i^{-1}_{U_{X\mathcal{T}}}$. Hence, we set for $U \in \mathcal{T}$:

$$i_{U_{X\mathcal{T}}!} = j_{U_{X\mathcal{T}}}{}^{-1} : \mathrm{Mod}(k_{U_{X\mathcal{T}}}) \to \mathrm{Mod}(k_{X_{\mathcal{T}}}).$$

If $F \in \mathrm{Mod}(k_{X_{\mathcal{T}}})$, we also write for short

$$\begin{aligned} F|_{U_{X\mathcal{T}}} &= i_{U_{X\mathcal{T}}}{}^{-1}F \in \mathrm{Mod}(k_{U_{X\mathcal{T}}}), \\ F_{U_{X\mathcal{T}}} &= i_{U_{X\mathcal{T}}!}i_{U_{X\mathcal{T}}}{}^{-1}F, \\ F|_{U_{\mathcal{T}}} &= i_{U_{\mathcal{T}}}{}^{-1}F \in \mathrm{Mod}(k_{U_{\mathcal{T}}}), \end{aligned}$$

and we keep the same notation $F|_{U_{X\mathcal{T}}}$ or $F|_{U_{\mathcal{T}}}$ if $F \in \mathrm{Mod}(k_{V_{X\mathcal{T}}})$ with $U \subset V$.

If U, $X \in \mathcal{T}$, we have

$$\begin{aligned} k_{\mathcal{T}} &:= k_{X_{\mathcal{T}}} = a_{\mathcal{T}}^{-1}k_{\mathrm{pt}} \text{ where } a_{\mathcal{T}} \colon X_{\mathcal{T}} \to \{\mathrm{pt}\}, \\ k_{\mathcal{T}U} &:= i_{U_{X\mathcal{T}}!}k_{U_{X\mathcal{T}}} \simeq (k_{\mathcal{T}})_{U_{X\mathcal{T}}} \in \mathrm{Mod}(k_{\mathcal{T}}). \end{aligned}$$

Note that the natural morphism $k_{\mathcal{T}U} \to k_{\mathcal{T}}$ is a monomorphism.

6.2. $\mathcal{T}$-coherent sheaves

In this section, we assume that $\mathcal{T}$ satisfies (6.1.1).

Notation 6.2.1. — If Z is a locally closed subset of X, we shall often write k_Z instead of k_{XZ}.

We introduce the category $\mathcal{K}(\mathcal{T})$ by setting

$$\begin{aligned} \mathrm{Ob}(\mathcal{K}(\mathcal{T})) &= \{(I, \{U_i\}_{i \in I}); I \text{ finite }, U_i \in \mathcal{T} \setminus \{\varnothing\}\}. \\ \mathrm{Hom}_{\mathcal{K}(\mathcal{T})}((I, \{U_i\}_i), (J, \{V_j\}_j)) &= \{(a_{ji})_{j \in J, i \in I}; a_{ji} \in k, a_{ji} \neq 0 \Rightarrow U_i \subset V_j\}. \end{aligned}$$

The composition is defined as follows. Let $\varphi = (a_{ji})_{j\in J, i\in I}\colon (I, \{U_i\}_i) \to (J, \{V_j\}_j)$ and $\psi = (b_{kj})_{k\in K, j\in J} : (J, \{V_j\}_j) \to (K, \{W_k\}_k)$. Then $\psi \circ \varphi = (c_{ki})_{k\in K, i\in I}$, with

$$c_{ki} = \sum_j b_{kj} a_{ji}.$$

We define naturally the full subcategories $\mathcal{K}(\mathcal{T}_c)$ and $\mathcal{K}(\mathcal{T}_0)$ of $\mathcal{K}(\mathcal{T})$.

The functor $\mathcal{K}(\mathcal{T}) \to \mathrm{Mod}(k_X)$ which associates the sheaf $\bigoplus_i k_{U_i}$ to $(I, \{U_i\}_i)$ is faithful and we shall identify $\mathcal{K}(\mathcal{T})$ with a subcategory of $\mathrm{Mod}(k_X)$. Note that this functor is not fully faithful in general since the open sets which belong to $\mathcal{T}$ are not necessarily connected. This functor is fully faithful when restricted to $\mathcal{K}(\mathcal{T}_0)$.

Now recall that if $\mathcal{C}$ is an abelian category, $\mathcal{J}$ an additive subcategory, and $F \in \mathcal{C}$, one says that:

(i) F is $\mathcal{J}$-finite if there exists an epimorphism $G \twoheadrightarrow F$ with $G \in \mathcal{J}$,

(ii) F is $\mathcal{J}$-pseudo-coherent if if for any morphism $G \underset{\varphi}{\to} F$ with $G \in \mathcal{J}$, $\ker\varphi$ is $\mathcal{J}$-finite,

(iii) F is $\mathcal{J}$-coherent if F is both $\mathcal{J}$-finite and $\mathcal{J}$-pseudo-coherent.

Note that (ii) is equivalent to the same condition with "$G \in \mathcal{J}$" replaced by "G is $\mathcal{J}$-finite".

One denotes by $\mathrm{Coh}(\mathcal{J})$ the full subcategory of $\mathcal{C}$ consisting of $\mathcal{J}$-coherent objects.

Then one easily proves that the category $\mathrm{Coh}(\mathcal{J})$ is additive and stable by kernels (see [**13**]).

We apply these constructions with $\mathcal{C} = \mathrm{Mod}(k_X)$ and $\mathcal{J} = \mathcal{K}(\mathcal{T}_c)$. We shall say that a sheaf F is $\mathcal{T}_c$-finite (resp. pseudo-coherent, resp. coherent) instead of $\mathcal{K}(\mathcal{T}_c)$-finite (resp. pseudo-coherent, resp. coherent).

One denotes by $\mathrm{Coh}(\mathcal{T}_c)$ the full subcategory of $\mathrm{Mod}(k_X)$ consisting of $\mathcal{T}_c$-coherent objects. Note that $\mathrm{Coh}(\mathcal{T}_c) = \mathrm{Coh}(\mathcal{T}_0)$.

Theorem 6.2.2. — *The subcategory* $\mathrm{Coh}(\mathcal{T}_c)$ *is stable by kernels, cokernels and finite direct sums. In other words, it is abelian and the natural functor* $\mathrm{Coh}(\mathcal{T}_c) \to \mathrm{Mod}(k_X)$ *is exact. Moreover* $\mathrm{Coh}(\mathcal{T}_c)$ *contains* $\mathcal{K}(\mathcal{T}_c)$.

We know that $\mathrm{Coh}(\mathcal{T}_c)$ is stable by kernels and finite direct sums. The proof that it is stable by cokernels is given in Lemmas 6.2.3–6.2.8 below.

Lemma 6.2.3. — *Let* $0 \to F' \to F \to F''$ *be an exact sequence and assume that* F' *and* F'' *are* $\mathcal{T}_c$*-pseudo-coherent. Then* F *is* $\mathcal{T}_c$*-pseudo-coherent.*

Proof. — We shall show that for any morphism $\varphi\colon G \to F$ with a $\mathcal{T}_c$-finite G, $\ker(\varphi)$ is $\mathcal{T}_c$-finite. It follows easily from $\ker(\varphi) = \ker(\ker(G \to F'') \to F')$. q.e.d.

Notation 6.2.4. — Let Z and Z' be a pair of locally closed subsets such that $Z \cap Z'$ is closed in Z and open in Z'. The morphism $k_Z \to k_{Z'}$ defined as the composition

$k_Z \to k_{Z\cap Z'} \to k_{Z'}$ will be denoted by $1_{Z\to Z'}$. Note that $1_{Z'\to Z''} \circ 1_{Z\to Z'} = 1_{Z\to Z''}$ does not hold in general.

Lemma 6.2.5. — *Consider a morphism $\varphi\colon k_U \to k_{X\setminus V}$ with $U, V \in \mathcal{T}_c$. Then there exists a finite open covering $U = \bigcup_i U_i$ with $U_i \in \mathcal{T}_c$ and a commutative diagram*

$$\begin{tikzcd} \bigoplus_i k_{U_i} \arrow[d] \arrow[dr, "\psi"] & \\ k_U \arrow[r, "\varphi"] & k_{X\setminus V} \end{tikzcd}$$

where the vertical arrow is given by $1_{U_i\to U}$ and ψ is given by $a_i 1_{U_i\to X\setminus V}$ for some $a_i \in k$.

Proof. — Let $U \setminus V = \bigsqcup_{i\in I} Z_i$ be the decomposition into connected components. Then I is finite by (6.1.1) (iii), and the Z_i's are connected and closed in U. Set $U_i = (U \cap V) \cup Z_i = U \setminus (\cap_{j\neq i} Z_j)$. Hence U_i is open, $U_i \cup \bigcup_{j\neq i} U_j = U$ and $U_i \cap (\bigcup_{j\neq i} U_j) = U \cap V$. Therefore $U_i \in \mathcal{T}_c$ for all i by (6.1.1) (ii). Consider the composition $k_{U_i} \to k_U \to k_{X\setminus V}$. Since $U_i \setminus V = Z_i$, it factors through k_{Z_i}. Since Z_i is connected, it is given by $a_i 1_{U_i\to X\setminus V}$ for some $a_i \in k$. q.e.d.

Lemma 6.2.6. — *Let F be a subsheaf of k_U with $U \in \mathcal{T}_c$, and assume that F is $\mathcal{T}_c$-finite. Then there exists $V \in \mathcal{T}_c$, $V \subset U$ such that $F \simeq k_V$.*

Proof. — By the hypothesis, there exists an epimorphism $\bigoplus_i k_{U_i} \twoheadrightarrow F$ with $U_i \in \mathcal{T}_c$ and we may assume each U_i connected. The composition $k_{U_i} \to F \to k_U$ is given by $a_i 1_{U_i\to U}$ with $a_i \in k$. Let $V = \bigcup_{i, a_i\neq 0} U_i$. Then V belongs to $\mathcal{T}_c$ and $F \simeq k_V$. q.e.d.

Lemma 6.2.7. — *Let V and W belong to $\mathcal{T}_c$. Then $k_{V\setminus W}$ is $\mathcal{T}_c$-coherent.*

Proof. — Clearly $k_{V\setminus W}$ is $\mathcal{T}_c$-finite. Let $S \in \mathcal{K}(\mathcal{T}_c)$ and $\varphi\colon S \to k_{V\setminus W}$. Let us show that $\ker\varphi$ is $\mathcal{T}_c$-finite. By Lemma 6.2.5, there exist $S' = \bigoplus_{i\in I} k_{U_i}$ with a finite index set I and $U_i \in \mathcal{T}_c$, and an epimorphism $f\colon S' \twoheadrightarrow S$ such that $\psi = \varphi \circ f\colon S' \to k_{V\setminus W}$ is given by $a_i 1_{U_i\to V\setminus W}$ for $a_i \in k$. As $\ker\psi \to \ker\varphi$ is an epimorphism, in order to see that $\ker\varphi$ is $\mathcal{T}_c$-finite, it is enough to show that $\ker\psi$ is $\mathcal{T}_c$-finite. Set $J_0 = \{i \in I\,;\, a_i = 0\}$. We define the morphism

$$g : \bigoplus_{i\in J_0} k_{U_i} \oplus \bigoplus_{i\in I} k_{U_i\cap W} \oplus \bigoplus_{i\neq j\in I} k_{U_i\cap U_j} \to S'$$

as follows: $g|_{k_{U_i}}$ is the natural morphism $k_{U_i} \to S'$ for $i \in J_0$, $g|_{k_{U_i\cap W}}$ is given by $1_{U_i\cap W\to U_i}$ for $i \in I$, and $g|_{k_{U_i\cap U_j}}$ is the composition of $k_{U_i\cap U_j} \to k_{U_i} \oplus k_{U_j}$ given by $(a_j 1_{U_i\cap U_j\to U_i}, -a_i 1_{U_i\cap U_j\to U_j})$ and $k_{U_i} \oplus k_{U_j} \to S'$.

Clearly the image of g coincides with $\ker\psi$. Therefore, $\ker\psi$ is $\mathcal{T}_c$-finite. q.e.d.

Lemma 6.2.8. — *Let F be the quotient of a $\mathcal{T}_c$-coherent sheaf by a $\mathcal{T}_c$-coherent subsheaf. Then F is $\mathcal{T}_c$-coherent.*

Proof. — It is enough to show that F is $\mathcal{T}_c$-pseudo-coherent. Let us consider an exact sequence $0 \to N \to L \to F \to 0$ with $\mathcal{T}_c$-coherent N and L. There exists an epimorphism $L_0 = \oplus_{i=1}^n k_{U_i} \to L$ with $U_i \in \mathcal{T}_c$. Since the kernel N_0 of $L_0 \to F$ is the image of $\ker(N \oplus L_0 \to L)$, N_0 is $\mathcal{T}_c$-finite and hence $\mathcal{T}_c$-coherent. Hence we may assume $L = \oplus_{i=1}^n k_{U_i}$ with $U_i \in \mathcal{T}_c$ from the beginning. We shall argue by induction on n.

Case $n = 1$. N is of the form k_V for some $V \in \mathcal{T}_c$ by Lemma 6.2.6 and hence $F \simeq k_{U_1 \setminus V}$ is $\mathcal{T}_c$-coherent by Lemma 6.2.7.

Case $n > 1$. Set $L_1 = k_{U_1} \subset L$ and $L_2 = L/L_1 \simeq \oplus_{i=2}^n k_{U_i}$. Let F_1 be the image of $L_1 \to F$. Then we have a commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & N_1 \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & N_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L_1 \arrow[r] \arrow[d] & L \arrow[r] \arrow[d] & L_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F_1 \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & F_2 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Since N_1 is the kernel of $L_1 \oplus N \to L$, N_1 is $\mathcal{T}_c$-coherent. The sheaf N_2 is $\mathcal{T}_c$-finite because it is a quotient of N, and it is $\mathcal{T}_c$-pseudo-coherent because it is a submodule of L_2. Hence N_2 is also $\mathcal{T}_c$-coherent. Therefore F_1 and F_2 are $\mathcal{T}_c$-pseudo-coherent by the induction hypothesis, and Lemma 6.2.3 implies that F is $\mathcal{T}_c$-pseudo-coherent. q.e.d.

This completes the proof of Theorem 6.2.2.

6.3. $\mathcal{T}$-sheaves I

In this section, we assume that $\mathcal{T}$ satisfies (6.1.1).

Proposition 6.3.1. — *Let $U \in \mathcal{T}$. Then $k_{\mathcal{T}U} \xrightarrow{\sim} \rho_* k_{XU}$ and $\rho^{-1} k_{\mathcal{T}U} \xrightarrow{\sim} k_{XU}$.*

Proof. — (i) The isomorphism $\rho^{-1} k_{\mathcal{T}U} \xrightarrow{\sim} k_{XU}$ follows from (2.3.5).
(ii) Let us first prove the isomorphism $k_{\mathcal{T}} \simeq \rho_* k_X$. Denote by F the presheaf on $\mathcal{T}$ defined by $F(\varnothing) = 0$ and $F(U) = k$ for $U \neq \varnothing$, $U \in \mathcal{T}$. One has a monomorphism

of presheaves $F \to \rho_* k_X$. Hence F is separated, $k_{\mathcal{T}} \simeq F^+$ and $F^+ \hookrightarrow \rho_* k_X$. Let $U \neq \varnothing, U \in \mathcal{T}$ with U connected. We have the sequence of arrows

$$k = F(U) \hookrightarrow F^+(U) \hookrightarrow \rho_* k_X(U) \simeq k_X(U) = k.$$

Therefore $k_{\mathcal{T}}(U) \simeq \rho_* k_X(U)$ for any connected U, and the result follows since $\mathcal{T}_0$ is a covering of X.

(iii) We can now prove that the natural morphism $k_{\mathcal{T}U} \to \rho_* k_{XU}$ is an isomorphism. Consider the diagram

$$\begin{tikzcd} k_{\mathcal{T}U} \arrow[r] \arrow[d] & \rho_* k_{XU} \arrow[d] \\ k_{\mathcal{T}} \arrow[r, "\sim"] & k_X. \end{tikzcd}$$

Since both vertical arrows are monomorphisms, $k_{\mathcal{T}U}$ is a subsheaf of $\rho_* k_{XU}$. Let $V \in \mathcal{T}_0$. If $V \subset U$, then $k_{\mathcal{T}U}(V) \simeq k_{\mathcal{T}}(V) \simeq \rho_* k_X(V) \simeq \rho_* k_{XU}(V)$. If V is not contained in U, $\rho_* k_{XU}(V) = 0$, which implies $k_{\mathcal{T}U}(V) = 0$. q.e.d.

Proposition 6.3.2. — *Denote by* $\rho_{c*} : \mathrm{Coh}(\mathcal{T}_c) \to \mathrm{Mod}(k_{\mathcal{T}})$ *the restriction of the functor* ρ_* *to* $\mathrm{Coh}(\mathcal{T}_c)$*. Then* ρ_{c*} *is exact and fully faithful, and* $\rho^{-1}\rho_{c*}$ *is isomorphic to the canonical functor* $\mathrm{Coh}(\mathcal{T}_c) \to \mathrm{Mod}(k_X)$*.*

Proof. — (i) Let us prove that ρ_{c*} is exact. Consider an exact sequence $0 \to G \to S \xrightarrow[\psi]{} F \to 0$ in $\mathrm{Coh}(\mathcal{T}_c)$ and let us apply the functor ρ_*. We already know that this functor is left exact. Hence it remains to show that $\rho_*(S) \to \rho_*(F)$ is an epimorphism.

Let $U \in \mathcal{T}_c$ and let $s \in \Gamma(U; \rho_* F) \simeq \mathrm{Hom}_{k_X}(k_{XU}, F)$. Set $S' = S \times_F k_{XU}$. Then $S' \in \mathrm{Coh}(\mathcal{T}_c)$ and moreover, $S' \to k_{XU}$ is an epimorphism. Since S' is $\mathcal{K}(\mathcal{T}_c)$-finite, there exists an epimorphism $\varphi : \oplus_{i \in I} k_{XU_i} \twoheadrightarrow S'$ with I finite. We may assume further that $U_i \in \mathcal{T}_0$. The composition $k_{XU_i} \to S' \to k_{XU}$ is given by $a_i 1_{U_i \to U}$, with $a_i \in k$. Let $I_0 = \{i \in I; a_i \neq 0\}$. Then we may assume that $a_i = 1$ for $i \in I_0$, and $U = \cup_{i \in I_0} U_i$. We get the commutative diagram

$$\begin{tikzcd} \oplus_{i \in I_0} k_{XU_i} \arrow[d] \arrow[dr, two heads] & \\ S' = S \times_F k_{XU} \arrow[r, two heads] \arrow[d] & k_{XU} \arrow[d, "s"] \\ S \arrow[r, two heads, "\psi"'] & F \end{tikzcd}$$

The composition $k_{XU_i} \to S' \to S$ defines $t_i \in \mathrm{Hom}_{k_X}(k_{XU_i}, S) = \Gamma(U_i; S)$. Since the diagram above commutes, we have $\psi(t_i) = s|_{U_i}$. It remains to apply Proposition 2.1.12.

(ii) it is enough to prove the isomorphism $\rho^{-1}\rho_* F \xrightarrow{\sim} F$ for any $F \in \mathrm{Coh}(\mathcal{T}_c)$. Since $\rho^{-1}\rho_{c*}$ is exact, we may reduce to the case where $F = k_{XU}$. Then apply Proposition 6.3.1. q.e.d.

Proposition 6.3.3. — *Let $G \in \mathrm{Coh}(\mathcal{T}_c)$ and let $\{F_i\}$ be an inductive system in $\mathrm{Mod}(k_{\mathcal{T}})$ indexed by a small filtrant category I. Then the natural morphism*

$$\varinjlim_i \mathrm{Hom}_{k_{\mathcal{T}}}(\rho_* G, F_i) \to \mathrm{Hom}_{k_{\mathcal{T}}}(\rho_* G, \varinjlim_i F_i)$$

is an isomorphism.

Proof. — (i) First we assume that $G = k_{XU}$ with $U \in \mathcal{T}_c$. By Proposition 6.3.1, we are reduced to prove the isomorphism $\varinjlim_i (F_i(U)) \xrightarrow{\sim} (\varinjlim_i F_i)(U)$.

Denote by F the inductive limit in $\mathrm{Psh}(k_{\mathcal{T}})$. Then F is a separated presheaf on $X_{\mathcal{T}}$. Hence it is enough to prove that $F(U) \to F^+(U)$ is an isomorphism. Let $S \in \mathrm{Cov}(U)$. For each $i \in I$ we have isomorphisms $F_i(U) \xrightarrow{\sim} F_i(S)$. If S is finite, we deduce the isomorphism $F(U) \xrightarrow{\sim} F(S)$. Since the family of finite covering is cofinal in $\mathrm{Cov}(U)^{\mathrm{op}}$, we obtain $F(U) \xrightarrow{\sim} F^+(U)$.
(ii) There exists an exact sequence $G_1 \to G_0 \to G \to 0$, with each G_i $(i = 0, 1)$ a finite direct sum of sheaves of the type k_{XU}. Since ρ_* is exact on $\mathrm{Coh}(\mathcal{T}_c)$ and Hom is left exact, the result follows. q.e.d.

Proposition 6.3.4. — *Let $F \in \mathrm{Mod}(k_{\mathcal{T}})$. There exist a small and filtrant category I and a functor $I \to \mathrm{Coh}(\mathcal{T}_c), i \mapsto F_i$ such that $F \simeq \varinjlim_i \rho_* F_i$.*

Proof. — Let $F \in \mathrm{Mod}(k_{\mathcal{T}})$. Define

$$I_0 = \{(U, s); U \in \mathcal{T}_c, s \in F(U)\}$$
$$G_0 = \oplus_{(U,s) \in I_0} k_{\mathcal{T}U}$$

Since $F(U) \simeq \mathrm{Hom}_{k_{\mathcal{T}}}(k_{\mathcal{T}U}, F)$, $s \in F(U)$ defines a morphism $\varphi_{U,s} : k_{\mathcal{T}U} \to F$. Let $\varphi = \oplus_{(U,s) \in I_0} \varphi_{U,s}$. Since $\mathcal{T}_c$ is a covering of X, we find that $\varphi : G_0 \to F$ is an epimorphism. Replacing F by $\ker \varphi$, we find an object $G_1 = \oplus_{(V,t) \in I_1} k_{\mathcal{T}V}$ and an exact sequence $G_1 \to G_0 \to F \to 0$. For $J_0 \subset I_0$, set for short $G_{J_0} = \oplus_{(U,s) \in J_0} k_{\mathcal{T}U}$ and define similarly G_{J_1}. Define

$$J = \{(J_1, J_0); J_k \subset I_k, J_k \text{ is finite, } \mathrm{im}\, \varphi|_{G_{J_1}} \subset G_{J_0}\}.$$

Then J is filtrant and

$$F \simeq \varinjlim_{(J_1, J_0) \in J} \mathrm{coker}(G_{J_1} \to G_{J_0}).$$

q.e.d.

Let us write $\mathrm{I}(\mathrm{Coh}(\mathcal{T}_c))$ instead of $\mathrm{Ind}(\mathrm{Coh}(\mathcal{T}_c))$ for short.

We shall extend the functor $\rho_{c*}\colon \mathrm{Coh}(\mathcal{T}_c) \to \mathrm{Mod}(k_{\mathcal{T}})$ by setting:

$$\begin{aligned} \lambda\colon \mathrm{I}(\mathrm{Coh}(\mathcal{T}_c)) &\to \mathrm{Mod}(k_{\mathcal{T}}) \\ \lambda(\text{“}\varinjlim_i\text{”}\, F_i) &= \varinjlim_i \rho_{c*}(F_i). \end{aligned} \tag{6.3.1}$$

By Proposition 6.3.3 it is equivalent to define $\lambda(F)$ (with $F \in \mathrm{I}(\mathrm{Coh}(\mathcal{T}_c))$) by the formula

$$\Gamma(U;\lambda(F)) = \mathrm{Hom}_{\mathrm{I}(\mathrm{Coh}(\mathcal{T}_c))}(k_{XU}, F). \tag{6.3.2}$$

Theorem 6.3.5. — *The functor λ in* (6.3.1) *is an equivalence of abelian categories.*

Proof. — The functor λ is essentially surjective by Proposition 6.3.4. Let us prove that it is fully faithful. Let $F, G \in \mathrm{I}(\mathrm{Coh}(\mathcal{T}_c))$. We may assume $F = \text{“}\varinjlim_i\text{”}\, F_i$, $G = \text{“}\varinjlim_j\text{”}\, G_j$, with $F_i, G_j \in \mathrm{Coh}(\mathcal{T}_c)$. Applying Proposition 6.3.3 and the fact that ρ_{c*} is fully faithful, we get the chain of isomorphisms

$$\begin{aligned} \mathrm{Hom}_{\mathrm{I}(\mathrm{Coh}(\mathcal{T}_c))}(F,G) &\simeq \varprojlim_j \varinjlim_i \mathrm{Hom}_{\mathrm{Coh}(\mathcal{T}_c)}(G_j, F_i) \\ &\simeq \varprojlim_j \varinjlim_i \mathrm{Hom}_{k_{\mathcal{T}}}(\rho_{c*}G_j, \rho_{c*}F_i) \\ &\simeq \varprojlim_j \mathrm{Hom}_{k_{\mathcal{T}}}(\rho_{c*}G_j, \varinjlim_i \rho_{c*}F_i) \\ &\simeq \mathrm{Hom}_{k_{\mathcal{T}}}(\varinjlim_j \rho_{c*}G_j, \varinjlim_i \rho_{c*}F_i). \end{aligned}$$

q.e.d.

Remark 6.3.6. — Note that the natural functor *for* : $\mathrm{Mod}(k_{\mathcal{T}}) \to \mathrm{Mod}(k_{\mathcal{T}_c})$ is an equivalence of categories. Indeed, for $F \in \mathrm{Mod}(k_{\mathcal{T}_c})$ and $V \in \mathcal{T}$ set

$$\widetilde{F}(V) = \varprojlim_{U \in \mathcal{T}_c} F(U \cap V).$$

Then $\widetilde{F} \in \mathrm{Mod}(k_{\mathcal{T}})$ and the functor $F \mapsto \widetilde{F}$ is quasi-inverse to the functor *for*.
Moreover, since $\mathrm{Coh}(\mathcal{T}_c)$ is small, there is an equivalence of categories $\mathrm{I}(\mathrm{Coh}(\mathcal{T}_c)) \simeq (\mathrm{Coh}(\mathcal{T}_c))^{\wedge,k-add,l}$, where the term on the right hand side stands for the category of k-additive left exact contravariant functors on $\mathrm{Coh}(\mathcal{T}_c)$ with values in $\mathrm{Mod}(k)$. Therefore we get the equivalences

$$\mathrm{Mod}(k_{\mathcal{T}}) \simeq \mathrm{Mod}(k_{\mathcal{T}_c}) \simeq \mathrm{I}(\mathrm{Coh}(\mathcal{T}_c)) \simeq (\mathrm{Coh}(\mathcal{T}_c))^{\wedge,k-add,l}. \tag{6.3.3}$$

6.4. Construction of ind-sheaves

In this section, we assume that $\mathcal{T}$ satisfies (6.1.1).

Proposition 6.4.1. — *Let $F \in \mathrm{Psh}(k_{\mathcal{T}_c})$. Assume:*

(i) $F(\varnothing) = 0$,

(ii) *for any U and V in $\mathcal{T}_c$, the sequence $0 \to F(U \cup V) \to F(U) \oplus F(V) \to F(U \cap V)$ is exact.*

Then $F \in \mathrm{Mod}(k_{\mathcal{T}_c})$ and there is a unique $\widetilde{F} \in \mathrm{Mod}(k_{\mathcal{T}})$ which satisfies $\widetilde{F}(U) \simeq F(U)$ for all $U \in \mathcal{T}_c$.

Proof. — By Remark 6.3.6, it is enough to prove the first assertion. Let $\{U_j; 1 \le j \le n\}$ be a finite family in $\mathcal{T}_c$. We shall show that the sequence below is exact:

$$0 \to F(\bigcup_{1 \le k \le n} U_k) \to \bigoplus_{1 \le k \le n} F(U_k) \underset{\varphi}{\to} \bigoplus_{1 \le i < j \le n} F(U_{ij}).$$

Here, the morphism φ sends $(s_k)_{1 \le k \le n}$ to $(t_{ij})_{1 \le i < j \le n}$ by $t_{ij} = s_i|_{U_{ij}} - s_j|_{U_{ij}}$.

For $n \le 1$ this is trivial, and for $n = 2$ this is the hypothesis. For $n > 2$ assume the result is proved for $j \le n-1$. Setting $U' = \cup_{1 \le k < n} U_k$, the commutative diagram below is exact by the induction hypothesis.

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & F(U' \cup U_n) \arrow[r] & F(U') \oplus F(U_n) \arrow[r] \arrow[d] & F(U' \cap U_n) \arrow[d] \\
 & & \big(\oplus_{i<n}(F(U_i))\big) \oplus F(U_n) \arrow[r] \arrow[d] & \oplus_{i<n} F(U_{in}) \\
 & & \oplus_{i<j<n} F(U_{ij}) &
\end{tikzcd}$$

The assertion follows. q.e.d.

Proposition 6.4.2. — *Let $F \in \mathrm{Mod}(k_{\mathcal{T}_c})$ and assume that for any U, V in $\mathcal{T}_c$ with $U \subset V$, the sequence $F(V) \to F(U) \to 0$ is exact. Then F is quasi-injective in $\mathrm{I}(\mathrm{Coh}(\mathcal{T}_c))$, i.e., the functor $\mathrm{Hom}_{k_{\mathcal{T}_c}}(\cdot, F)$ is exact on $\mathrm{Coh}(\mathcal{T}_c)$.*

Proof. — If $G \in \mathrm{Coh}(\mathcal{T}_c)$, we shall write $F(G)$ instead of $\mathrm{Hom}_{k_{\mathcal{T}}}(G, F)$ for short.

Let $G' \rightarrowtail G$ be a monomorphism in $\mathrm{Coh}(\mathcal{T}_c)$ and let us prove that $F(G) \to F(G')$ is an epimorphism. There exists an epimorphism $\bigoplus_{i=1}^n k_{U_i} \twoheadrightarrow G$, with $U_i \in \mathcal{T}_c$. Define $G'_j := \mathrm{im}\Big(G' \oplus (\bigoplus_{i=1}^j k_{U_i}) \to G\Big)$. This is an increasing sequence of subobjects of G. It is enough to prove that the monomorphisms $G'_j \rightarrowtail G'_{j+1}$ give rise to the epimorphisms $F(G'_{j+1}) \to F(G'_j)$. Hence, we may assume from the beginning that there exists

$U \in \mathcal{T}_c$ and a morphism $\varphi\colon k_U \to G$ such that $G' \oplus k_U \to G$ is an epimorphism. Consider the commutative exact diagram

$$\begin{tikzcd} & & 0 \arrow[d] & 0 \arrow[d] & \\ & & G' \arrow[r, equal] \arrow[d] & G' \arrow[d] & \\ & K \arrow[r] \arrow[d, equal] & G' \oplus k_U \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & K \arrow[r] & k_U \arrow[r] \arrow[d] & G'' \arrow[r] \arrow[d] & 0 \\ & & 0 & 0 & \end{tikzcd}$$

Then G'' is $\mathcal{T}_c$-coherent, as well as K. Since K is a sub-object of k_U, it is isomorphic to k_V for some $V \in \mathcal{T}_c$. Applying the left exact functor $\mathrm{Hom}_{k_{\mathcal{T}}}(\,\cdot\,, F)$ to this diagram, the middle column will remain split exact, and the rows will remain exact in view of the hypothesis. Hence, the whole diagram will remain exact. q.e.d.

Let us denote by $\iota_{\mathcal{T}}\colon \mathrm{Coh}(\mathcal{T}_c) \to \mathrm{Mod}(k_X)$ the natural functor. It gives rise to a functor $\mathrm{I}(\mathrm{Coh}(\mathcal{T}_c)) \to \mathrm{I}(k_X)$ and hence a functor $I_{\mathcal{T}}\colon \mathrm{Mod}(k_{\mathcal{T}}) \to \mathrm{I}(k_X)$. This functor is exact and commutes with inductive limits.

Since $\mathrm{I}(\mathrm{Coh}(\mathcal{T}_c))$ is equivalent to the category of left exact functors from $\mathrm{Coh}(\mathcal{T}_c)^{\mathrm{op}}$ to $\mathrm{Mod}(k)$, we may also define a functor $J_{\mathcal{T}}\colon \mathrm{I}(k_X) \to \mathrm{I}(\mathrm{Coh}(\mathcal{T}_c)) \xrightarrow{\sim} \mathrm{Mod}(k_{\mathcal{T}})$ by setting for $F \in \mathrm{I}(k_X)$ and $K \in \mathrm{Coh}(\mathcal{T}_c)$, $\mathrm{Hom}_{\mathrm{I}(\mathrm{Coh}(\mathcal{T}_c))}(K, J_{\mathcal{T}}F) = \mathrm{Hom}_{\mathrm{I}(k_X)}(\iota_X \iota_{\mathcal{T}} K, F)$, or equivalently, $\Gamma(U; J_{\mathcal{T}}(F)) = \mathrm{Hom}_{\mathrm{I}(k_X)}(k_{XU}, F)$ for $U \in \mathcal{T}$. The functor $J_{\mathcal{T}}$ commutes with filtrant inductive limits.

Proposition 6.4.3. — *The functor $J_{\mathcal{T}}$ is right adjoint to $I_{\mathcal{T}}$. In other words, for $F \in \mathrm{I}(k_X)$ and $G \in \mathrm{I}(\mathrm{Coh}(\mathcal{T}_c)) \simeq \mathrm{Mod}(k_{\mathcal{T}})$ one has*

$$\mathrm{Hom}_{k_{\mathcal{T}}}(G, J_{\mathcal{T}}F) \simeq \mathrm{Hom}_{\mathrm{I}(k_X)}(I_{\mathcal{T}}G, F).$$

Proof. — Let $G \simeq \varinjlim_i \rho_{c*} K_i$ with $K_i \in \mathrm{Coh}(\mathcal{T}_c)$. Using the isomorphism $I_{\mathcal{T}} \circ \rho_{c*} \simeq \iota_X \circ \iota_{\mathcal{T}}$ and the fact that $I_{\mathcal{T}}$ commutes with inductive limits, we get the chain of isomorphisms

$$\begin{aligned}
\operatorname{Hom}_{k_{\mathcal{T}}}(G, J_{\mathcal{T}}F) &\simeq \varprojlim_i \operatorname{Hom}_{k_{\mathcal{T}}}(\rho_{c*}K_i, J_{\mathcal{T}}F)\\
&\simeq \varprojlim_i \operatorname{Hom}_{\mathrm{I}(k_X)}(\iota_X\iota_{\mathcal{T}}K_i, F)\\
&\simeq \operatorname{Hom}_{\mathrm{I}(k_X)}(\text{“}\varinjlim_i\text{''}\, I_{\mathcal{T}}\rho_{c*}K_i, F)\\
&\simeq \operatorname{Hom}_{\mathrm{I}(k_X)}(I_{\mathcal{T}}G, F).
\end{aligned}$$

q.e.d.

We shall now compare those functors.

Note that the functors $\iota_X, \rho_{c*}, \iota_{\mathcal{T}}, I_{\mathcal{T}}$ are fully faithful and exact, the functors ρ^{-1}, α_X are exact and commute with inductive limits, and ρ_* is left exact. Moreover

$$\begin{aligned}
\iota_{\mathcal{T}} &\simeq \rho^{-1}\circ\rho_{c*},\\
\rho_{c*} &= \rho_*\circ\iota_{\mathcal{T}},\\
\iota_X\circ\iota_{\mathcal{T}} &\simeq I_{\mathcal{T}}\circ\rho_{c*},\\
\rho^{-1} &\simeq \alpha_X\circ I_{\mathcal{T}},\\
J_{\mathcal{T}}\circ\iota_X &\simeq \rho_*,\\
J_{\mathcal{T}}\circ I_{\mathcal{T}} &\simeq \mathrm{id}_{\mathrm{Mod}(k_{\mathcal{T}})}.
\end{aligned}$$

One shall be aware that the formula $\iota_X \simeq I_{\mathcal{T}}\circ\rho_*$ is false in general.

We may represent these isomorphisms by the commutative diagrams below.

$$\begin{tikzcd}
\mathrm{Coh}(\mathcal{T}_c) \arrow[r, "\iota_{\mathcal{T}}"] \arrow[d, "\rho_{c*}"'] & \mathrm{Mod}(k_X) \arrow[d, "\iota_X"] \\
\mathrm{Mod}(k_{\mathcal{T}}) \arrow[r, "I_{\mathcal{T}}"] & \mathrm{I}(k_X)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Coh}(\mathcal{T}_c) \arrow[r, "\iota_{\mathcal{T}}"] \arrow[d, "\rho_{c*}"'] & \mathrm{Mod}(k_X) \arrow[dl, shift left, "\rho_*"] \\
\mathrm{Mod}(k_{\mathcal{T}}) \arrow[ur, shift left, "\rho^{-1}"] &
\end{tikzcd}$$

$$\begin{tikzcd}
& \mathrm{Mod}(k_X) \arrow[dl, "\rho_*"'] \arrow[d, "\iota_X"] \\
\mathrm{Mod}(k_{\mathcal{T}}) & \mathrm{I}(k_X) \arrow[l, "J_{\mathcal{T}}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
& \mathrm{Mod}(k_X) \\
\mathrm{Mod}(k_{\mathcal{T}}) \arrow[ur, "\rho^{-1}"] \arrow[r, "I_{\mathcal{T}}"] & \mathrm{I}(k_X) \arrow[u, "\alpha_X"']
\end{tikzcd}$$

6.5. Construction II

In this section, we assume that $\mathcal{T}$ satisfies (6.1.1).

We shall need the following generalization of some of the preceding results.

Proposition 6.5.1. — *Let $\mathcal{C}$ be a k-abelian category and let $F : \mathcal{T}_c^{\mathrm{op}} \to \mathcal{C}$ be a functor.*

(a) *Assume*

(i) $F(\varnothing) = 0$,

(ii) *for any U and V in $\mathcal{T}_c$, the sequence* $0 \to F(U \cup V) \to F(U) \oplus F(V) \to F(U \cap V)$ *is exact in* $\mathcal{C}$.

Then there is a unique (up to isomorphism) k-additive left exact functor $\widetilde{F} : \mathrm{Coh}(\mathcal{T}_c)^{\mathrm{op}} \to \mathcal{C}$ *which satisfies* $\widetilde{F}(k_{XU}) \simeq F(U)$ *for all* $U \in \mathcal{T}_c$.

(b) *Assume moreover that*

$$\text{for any } U,\ V \in \mathcal{T}_c \text{ with } U \subset V,\ F(V) \to F(U) \text{ is an epimorphism.}$$

Then $\widetilde{F}$ is exact.

Proof. — (a) For $S \in \mathcal{C}$, define $F_S \in \mathrm{Psh}(k_{\mathcal{T}})$ by

$$F_S(U) = \mathrm{Hom}_{\mathcal{C}}(S, F(U)).$$

Applying Proposition 6.4.1 and Theorem 6.3.5, we get a unique $\widetilde{F_S} \in \mathrm{Coh}(\mathcal{T}_c)^{\wedge, k-add, l}$ which extends F_S. Clearly, the correspondence $S \mapsto \widetilde{F_S}$ is functorial, and thus defines a left exact functor $\widetilde{F} : \mathrm{Coh}(\mathcal{T}_c)^{\mathrm{op}} \to \mathcal{C}^{\wedge}$.

It remains to show that $\widetilde{F}$ takes its values in $\mathcal{C}$. Let $G \in \mathrm{Coh}(\mathcal{T}_c)$. There exists an exact sequence $G_1 \to G_0 \to G \to 0$ with G_1 and G_0 finite direct sums of sheaves of the type k_{XU}. Since the sequence $0 \to \widetilde{F}(G) \to \widetilde{F}(G_0) \to \widetilde{F}(G_1)$ is exact and $\widetilde{F}(G_i) \in \mathcal{C}$ for $i = 0, 1$, the result follows.

(b) The proof is similar to that of Proposition 6.4.2. q.e.d.

Corollary 6.5.2. — *Let $G : \mathcal{T}_c \to \mathrm{Mod}(k)$ be a functor.*

(a) *Assume*

(i) $G(\varnothing) = 0$,

(ii) *for any U and V in $\mathcal{T}_c$, the sequence* $G(U \cap V) \to G(U) \oplus G(V) \to G(U \cup V) \to 0$ *is exact.*

Then there is a unique (up to isomorphism) k-additive right exact functor $\widetilde{G} : \mathrm{Coh}(\mathcal{T}_c) \to \mathrm{Mod}(k)$ *which satisfies:* $\widetilde{G}(k_{XU}) \simeq G(U)$ *for all* $U \in \mathcal{T}_c$.

(b) *Assume moreover*

$$\text{for any } U,\ V \in \mathcal{T}_c \text{ with } U \subset V,\ G(U) \to G(V) \text{ is a monomorphism.}$$

Then $\widetilde{G}$ is exact.

Proof. — Apply Proposition 6.5.1 with $F = G^{\mathrm{op}} : \mathrm{Coh}(\mathcal{T}_c)^{\mathrm{op}} \to \mathrm{Mod}(k)^{\mathrm{op}}$. q.e.d.

6.6. $\mathcal{T}$-sheaves II

In this section, we assume (6.1.1) and (6.1.2).

Proposition 6.6.1. — *One has* $\rho^{-1} \circ \rho_* \xrightarrow{\sim} \mathrm{id}$ *and the functor ρ_* is fully faithful.*

Proof. — Let $x \in X$. We have the chain of isomorphisms

$$(\rho^{-1}\rho_* F)_x \simeq \varinjlim_{x \in U} \rho^{-1}\rho_* F(U) \simeq \varinjlim_{x \in U \in \mathcal{T}} \rho_* F(U) \simeq \varinjlim_{x \in U \in \mathcal{T}} F(U) \simeq F_x.$$

The second assertion follows. q.e.d.

Proposition 6.6.2. — *Let $F \in \mathrm{Mod}(k_{\mathcal{T}})$ and let U be an open subset of X. Then*

$$\Gamma(U;\rho^{-1}F) \simeq \varprojlim_{V \subset\subset U, V\in\mathcal{T}} \Gamma(V;F).$$

Proof. — We may assume $F = \varinjlim_i \rho_{c*}F_i$ with $F_i \in \mathrm{Coh}(\mathcal{T}_c)$. Then one has

$$\rho^{-1}F \simeq \varinjlim_i \rho^{-1}\rho_{c*}F_i \simeq \varinjlim_i F_i.$$

Since $\overline{V}$ is compact, one has

$$\Gamma(\overline{V};\rho^{-1}F) \simeq \varinjlim_i \Gamma(\overline{V};\rho^{-1}F_i).$$

Therefore,

$$\begin{aligned}
\Gamma(U;\rho^{-1}F) &\simeq \varprojlim_{V\subset\subset U, V\in\mathcal{T}_c} \Gamma(\overline{V};\rho^{-1}F) \simeq \varprojlim_{V\subset\subset U, V\in\mathcal{T}_c} \varinjlim_i \Gamma(\overline{V};\rho^{-1}F_i)\\
&\simeq \varprojlim_{V\subset\subset U, V\in\mathcal{T}_c} \varinjlim_i \Gamma(V;\rho^{-1}F_i) \simeq \varprojlim_{V\subset\subset U, V\in\mathcal{T}_c} \varinjlim_i \Gamma(V;F_i)\\
&\simeq \varprojlim_{V\subset\subset U, V\in\mathcal{T}_c} \Gamma(V;F),
\end{aligned}$$

where the last isomorphism follows from Proposition 6.3.3. q.e.d.

Proposition 6.6.3. — (i) *The functor ρ^{-1} admits a left adjoint, which we denote by $\rho_!$:*

$$\mathrm{Hom}_{k_{\mathcal{T}}}(\rho_! F, G) \simeq \mathrm{Hom}_{k_X}(F, \rho^{-1}G)$$

for $F \in \mathrm{Mod}(k_X)$ and $G \in \mathrm{Mod}(k_{\mathcal{T}})$.

(ii) *For $F \in \mathrm{Mod}(k_X)$, $\rho_!(F)$ is the sheaf associated with the presheaf $\mathcal{T} \ni U \mapsto F(\overline{U})$.*

(iii) *For $U \in \mathrm{Op}(X)$ one has*

$$\rho_!(k_{XU}) \simeq \varinjlim_{V\subset\subset U, V\in\mathcal{T}} k_{\mathcal{T}V}.$$

Proof. — (i)–(ii) Denote by $\widetilde{F} \in \mathrm{Psh}(k_{\mathcal{T}_c})$ the presheaf $U \mapsto F(\overline{U})$. First, we construct morphisms functorial with respect to $G \in \mathrm{Mod}(k_{\mathcal{T}})$:

$$\mathrm{Hom}_{\mathrm{Psh}(k_{\mathcal{T}_c})}(\widetilde{F}, G) \underset{\eta}{\overset{\xi}{\rightleftarrows}} \mathrm{Hom}_{k_X}(F, \rho^{-1}G).$$

Define ξ as follows. Let $\varphi : \widetilde{F} \to G$ and $U \in \mathrm{Op}(X)$. Then $\xi(\varphi)(U) : F(U) \to (\rho^{-1}G)(U)$ is given by the sequence of morphisms

$$F(U) \simeq \varprojlim_{V\subset\subset U, V\in\mathcal{T}} F(\overline{V}) = \varprojlim_{V\subset\subset U, V\in\mathcal{T}} \widetilde{F}(V) \xrightarrow{\varphi} \varprojlim_{V\subset\subset U, V\in\mathcal{T}} G(V) \simeq (\rho^{-1}G)(U).$$

(Here we have used Proposition 6.6.2.) Define η as follows. Let $\psi : F \to \rho^{-1}G$ and $U \in \mathcal{T}_c$. Then $\eta(\psi)(U) : \widetilde{F}(U) \to G(U)$ is given by the chain of morphisms

$$\widetilde{F}(U) = F(\overline{U}) \simeq \varinjlim_{\overline{U} \subset V \in \mathcal{T}_c} F(V) \xrightarrow{\psi} \varinjlim_{\overline{U} \subset V \in \mathcal{T}_c} \rho^{-1}G(V) \to G(U).$$

One checks easily that ξ and η are an inverse to each other. Since

$$\operatorname{Hom}_{\operatorname{Psh}(k_{\mathcal{T}_c})}(\widetilde{F}, G) \simeq \operatorname{Hom}_{k_{\mathcal{T}_c}}(\widetilde{F}^{++}, G) \simeq \operatorname{Hom}_{k_{\mathcal{T}}}(\widetilde{F}^{++}, G),$$

we get (i) and (ii).
(iii) By (i) and Proposition 6.6.2, one has for $F \in \operatorname{Mod}(k_{\mathcal{T}})$:

$$\begin{aligned}
\operatorname{Hom}_{k_{\mathcal{T}}}(\rho_!(k_{XU}), F) &\simeq \operatorname{Hom}_{k_X}(k_{XU}, \rho^{-1}F) \\
&\simeq \varprojlim_{V \subset\subset U, V \in \mathcal{T}} F(V) \\
&\simeq \varprojlim_{V \subset\subset U, V \in \mathcal{T}} \operatorname{Hom}_{k_{\mathcal{T}}}(k_{\mathcal{T}V}, F) \\
&\simeq \operatorname{Hom}_{k_{\mathcal{T}}}(\varinjlim_{V \subset\subset U, V \in \mathcal{T}} k_{\mathcal{T}V}, F).
\end{aligned}$$

q.e.d.

Proposition 6.6.4. — (i) *The functor $\rho_!$ is exact and commutes with inductive limits.*
(ii) *The functor $\rho_!$ commutes with tensor products.*

Proof. — (i) Let us prove that $\rho_!$ is left exact, the other assertions being obvious by adjunction. For $F \in \operatorname{Mod}(k_X)$, denote by $\widetilde{F}$ the presheaf on $\mathcal{T}_c$ given by $\widetilde{F}(U) = F(\overline{U})$. Then $\rho_! F \simeq (\widetilde{F})^{++}$ by Proposition 6.6.3 (ii). Since the functors $F \mapsto \widetilde{F}$ and $G \mapsto G^{++}$ are left exact, the result follows.
(ii) Let $F, G \in \operatorname{Mod}(k_X)$. The morphism $F(\overline{U}) \otimes G(\overline{U}) \to (F \otimes G)(\overline{U})$ gives a morphism in $\operatorname{Mod}(k_{\mathcal{T}})$

$$\rho_!(F) \otimes \rho_!(G) \to \rho_!(F \otimes G)$$

by Proposition 6.6.3 (ii). Let us show that it is an isomorphism. Since $\rho_!$ commutes with inductive limits, we may reduce the proof to the case when $F = k_{XU}$ and $G = k_{XV}$. Then the result follows from Proposition 6.6.3 (iii). q.e.d.

Proposition 6.6.5. — *One has the isomorphisms of functors*

(i) $\alpha_X \simeq \rho^{-1} \circ J_{\mathcal{T}}$,
(ii) $\beta_X \simeq I_{\mathcal{T}} \circ \rho_!$.

Proof. — (i) For $U \in \mathrm{Op}(X)$, and $F \in \mathrm{I}(k_X)$, one has the chain of isomorphisms

$$\begin{aligned}\Gamma(U;\rho^{-1}J_{\mathcal{T}}F) &\simeq \varinjlim_{V\subset\subset U, V\in\mathcal{T}_c} (J_{\mathcal{T}}F)(V)\\ &\simeq \varinjlim_{V\subset\subset U, V\in\mathcal{T}_c} \mathrm{Hom}_{\mathrm{I}(k_X)}(k_{XV},F)\\ &\simeq \mathrm{Hom}_{\mathrm{I}(k_X)}(\beta_X k_{XU},F)\\ &\simeq (\alpha_X F)(U).\end{aligned}$$

(ii) follows from (i) by taking the left adjoint functors. q.e.d.

This is visualized by the diagrams

$$\begin{tikzcd} & \mathrm{Mod}(k_X) \arrow[dl, "\rho_!"'] \arrow[d, "\beta_X"] \\ \mathrm{Mod}(k_{\mathcal{T}}) \arrow[r, "I_{\mathcal{T}}"'] & \mathrm{I}(k_X)\end{tikzcd} \qquad \begin{tikzcd} & \mathrm{Mod}(k_X) \\ \mathrm{Mod}(k_{\mathcal{T}}) \arrow[ur, "\rho^{-1}"] & \mathrm{I}(k_X) \arrow[l, "J_{\mathcal{T}}"] \arrow[u, "\alpha_X"']\end{tikzcd}$$

Definition 6.6.6. — For $F, G \in \mathrm{Mod}(k_{\mathcal{T}})$, we denote by $\mathcal{H}om_{k_X}(F,G)$ the presheaf on X, $\mathrm{Op}(X) \ni U \mapsto \mathrm{Hom}_{k_{U_{\mathcal{T}}}}(F|_{U_{\mathcal{T}}}, G|_{U_{\mathcal{T}}})$. (Recall that the site $U_{\mathcal{T}}$ is introduced in Definition 6.1.1 (iii).)

Proposition 6.6.7. — *The presheaf $\mathcal{H}om_{k_X}(F,G)$ is a sheaf on X. Moreover*

$$\mathcal{H}om_{k_X}(F,G) \simeq \rho^{-1}\mathcal{H}om_{k_{\mathcal{T}}}(F,G).$$

Proof. — Let U be an open subset of X. Using Proposition 6.6.2 it is enough to prove that

$$\mathrm{Hom}_{k_{U_{\mathcal{T}}}}(F|_{U_{\mathcal{T}}}, G|_{U_{\mathcal{T}}}) \simeq \varprojlim_{V\subset\subset U, V\in\mathcal{T}} \mathrm{Hom}_{k_{V_{X\mathcal{T}}}}(F|_{V_{X\mathcal{T}}}, G|_{V_{X\mathcal{T}}}).$$

This follows from the fact that the topologies induced on V by $X_{\mathcal{T}}$ and by $U_{\mathcal{T}}$ are the same. q.e.d.

6.7. Ring action

In this section, we make hypotheses (6.1.1) and (6.1.2).

One defines naturally the notion of a sheaf of rings in $\mathrm{Mod}(k_{\mathcal{T}})$, as well as the notion of modules over such a sheaf of rings. Note that if $\mathcal{A}$ is a ring in $\mathrm{Mod}(k_X)$, then $\rho_*\mathcal{A}$ and $\rho_!\mathcal{A}$ are rings in $\mathrm{Mod}(k_{\mathcal{T}})$.

Consider a sheaf $\mathcal{A}$ of unitary k-algebras on X. Let $\widetilde{\mathcal{A}}$ be the presheaf on $\mathcal{T}$ that associates $\Gamma(\overline{U};\mathcal{A})$ to $U \in \mathcal{T}$. Let F be a presheaf on $\mathcal{T}$ and assume that for each

$U \in \mathcal{T}$, $F(U)$ is an $\widetilde{\mathcal{A}}(U)$-module and the restriction morphisms are $\widetilde{\mathcal{A}}$-linear, that is, for $V \subset U$, $U, V \in \mathcal{T}$, the diagram below is commutative:

$$\begin{tikzcd}
\Gamma(\overline{U};\mathcal{A}) \otimes F(U) \arrow[r] \arrow[d] & F(U) \arrow[d] \\
\Gamma(\overline{V};\mathcal{A}) \otimes F(V) \arrow[r] & F(V).
\end{tikzcd}$$

In such a situation, we shall say that F is a presheaf of $\widetilde{\mathcal{A}}$-modules on $\mathcal{T}$.

Proposition 6.7.1. — *Let $\mathcal{A}$ be a sheaf of k-algebras on X and let F be a presheaf of $\widetilde{\mathcal{A}}$-modules on $\mathcal{T}$. Then $F^{++} \in \mathrm{Mod}(\rho_! \mathcal{A})$.*

Proof. — Let $U \in \mathcal{T}$ and let $a \in \Gamma(\overline{U};\mathcal{A})$. Then a defines an endomorphism of $F|_{U_{X\mathcal{T}}}$, hence an endomorphism of $(F^{++})|_{U_{X\mathcal{T}}} \simeq (F|_{U_{X\mathcal{T}}})^{++}$. Therefore, we have a morphism of presheaves of algebras on $\mathcal{T}$, $\widetilde{\mathcal{A}} \to \mathcal{E}nd_{k_{\mathcal{T}}}(F^{++})$. This morphism defines a morphism of sheaves $\widetilde{\mathcal{A}}^{++} \to \mathcal{E}nd_{k_{\mathcal{T}}}(F^{++})$, and the result follows since $\widetilde{\mathcal{A}}^{++} \simeq \rho_! \mathcal{A}$ by Proposition 6.6.3 (ii). q.e.d.

CHAPTER 7

IND-SHEAVES ON ANALYTIC MANIFOLDS

Applying the preceding constructions, we shall define various ind-sheaves associated with spaces of holomorphic functions. This is a reformulation in the language of ind-sheaves of previous results in [**8**] and [**11**].

7.1. Subanalytic sites

In this chapter, X will be a real analytic manifold and k is a field. We refer to [**10**] for an exposition on the notions of subanalytic subsets and $\mathbb{R}$-constructible sheaves.

Let $\mathcal{T}$ denote the family of open subanalytic subsets of X. Then hypotheses (6.1.1) and (6.1.2) are satisfied.

Definition 7.1.1. — We call the site $X_{\mathcal{T}}$ the subanalytic site on X and denote it by X_{sa}.

We denote by $\mathbb{R}\text{-}\mathrm{C}(k_X)$ the abelian category of $\mathbb{R}$-constructible sheaves of k-vector spaces on X, and by $\mathbb{R}\text{-}\mathrm{C}^c(k_X)$ the full abelian subcategory of sheaves with compact support. Hence, the category $\mathrm{Coh}(\mathcal{T}_c)$ coincides with $\mathbb{R}\text{-}\mathrm{C}^c(k_X)$. Set

$$\mathrm{I}_{\mathbb{R}-\mathrm{c}}(k_X) = \mathrm{Ind}(\mathbb{R}\text{-}\mathrm{C}^c(k_X)).$$

Applying Theorem 6.3.5, we obtain the equivalence

$$\mathrm{I}_{\mathbb{R}-\mathrm{c}}(k_X) \simeq \mathrm{Mod}(k_{X_{sa}}).$$

In other words, ind-$\mathbb{R}$-constructible sheaves are "usual sheaves" on the subanalytic site.

Denote by $D^b_{\mathbb{R}-c}(k_X)$ the full triangulated category of $D^b(k_X)$ consisting of objects with $\mathbb{R}$-constructible cohomology (i.e., cohomology in $\mathbb{R}\text{-}\mathrm{C}(k_X)$). A theorem of [**8**] asserts that the natural functor $D^b(\mathbb{R}\text{-}\mathrm{C}(k_X)) \to D^b_{\mathbb{R}-c}(k_X)$ is an equivalence (see also [**10**] Theorem 8.1.11).

We denote by $D^b_{\mathrm{I}\,\mathbb{R}-\mathrm{c}}(\mathrm{I}(k_X))$ the full subcategory of $D^b(\mathrm{I}(k_X))$ consisting of objects with cohomology in $\mathrm{I}_{\mathbb{R}-\mathrm{c}}(k_X)$. Since $\mathbb{R}\text{-}\mathrm{C}^c(k_X)$ is a full subcategory of $\mathrm{Mod}(k_X)$ stable by extension, $\mathrm{I}_{\mathbb{R}-\mathrm{c}}(k_X)$ is a full subcategory of $\mathrm{I}(k_X)$ stable by extension, and $D^b_{\mathrm{I}\,\mathbb{R}-\mathrm{c}}(\mathrm{I}(k_X))$ is triangulated. The exact functor $I_{\mathcal{T}} : \mathrm{I}_{\mathbb{R}-\mathrm{c}}(k_X) \to \mathrm{I}(k_X)$ induces a triangulated functor

$$I_{\mathcal{T}} : D^b(\mathrm{I}_{\mathbb{R}-\mathrm{c}}(k_X)) \to D^b_{\mathrm{I}\,\mathbb{R}-\mathrm{c}}(\mathrm{I}(k_X)). \tag{7.1.1}$$

Theorem 7.1.2. — *The functor $I_{\mathcal{T}}$ in* (7.1.1) *is an equivalence of triangulated categories.*

Proof. — By dévissage, it is enough to prove that for $F, G \in \mathrm{I}_{\mathbb{R}-\mathrm{c}}(k_X)$, the natural morphism (7.1.2) below is an isomorphism

$$\mathrm{Hom}_{D^b(\mathrm{I}_{\mathbb{R}-\mathrm{c}}(k_X))}(F, G[n]) \to \mathrm{Hom}_{D^b(\mathrm{I}(k_X))}(I_{\mathcal{T}}F, I_{\mathcal{T}}G[n]). \tag{7.1.2}$$

We may reduce to the case when $F = \text{“}\bigoplus_{i\in I}\text{”}\, k_{XU_i}$ with a small set I and $U_i \in \mathcal{T}_c$. Since

$$\mathrm{Hom}_{D^b(\mathrm{I}(k_X))}(\text{“}\bigoplus_i\text{”}\, F_i, G[n]) \simeq \prod_i \mathrm{Hom}_{D^b(\mathrm{I}(k_X))}(F_i, G[n])$$

and there is a similar formula with $\mathrm{Hom}_{D^b(\mathrm{I}(k_X))}$ replaced by $\mathrm{Hom}_{D^b(\mathrm{I}_{\mathbb{R}-\mathrm{c}}(k_X))}$, we are reduced to prove the isomorphism (7.1.2) when $F = k_{XU}$, with $U \in \mathcal{T}_c$. Let $G = \text{“}\varinjlim_j\text{”}\, G_j$, with $G_j \in \mathbb{R}\text{-}\mathrm{C}^c(k_X)$. By Corollary 1.5.7 and Proposition 5.1.11, we may reduce to the case where $G \in \mathbb{R}\text{-}\mathrm{C}^c(k_X)$. In this case we have:

$$\begin{aligned}\mathrm{Hom}_{D^b(\mathrm{I}_{\mathbb{R}-\mathrm{c}}(k_X))}(k_{\mathcal{T}U}, G[n]) &\simeq \mathrm{Hom}_{D^b(\mathbb{R}\text{-}\mathrm{C}(k_X))}(k_{XU}, G[n])\\ \mathrm{Hom}_{D^b(\mathrm{I}(k_X))}(k_{XU}, G[n]) &\simeq \mathrm{Hom}_{D^b(k_X)}(k_{XU}, G[n])\end{aligned}$$

and the result follows since the functor $D^b(\mathbb{R}\text{-}\mathrm{C}(k_X)) \to D^b(k_X)$ is fully faithful.
q.e.d.

Lemma 7.1.3. — *Let $f\colon X \to Y$ be a real analytic map and let $F \in D^b_{\mathbb{R}-c}(k_X)$. The functors below are well defined:*

(i) $Rf_{!!}\colon D^b_{\mathrm{I}\,\mathbb{R}-\mathrm{c}}(\mathrm{I}(k_X)) \to D^b_{\mathrm{I}\,\mathbb{R}-\mathrm{c}}(\mathrm{I}(k_Y))$,
(ii) $f^!\colon D^b_{\mathrm{I}\,\mathbb{R}-\mathrm{c}}(\mathrm{I}(k_Y)) \to D^b_{\mathrm{I}\,\mathbb{R}-\mathrm{c}}(\mathrm{I}(k_X))$,
(iii) $\otimes\colon D^b_{\mathrm{I}\,\mathbb{R}-\mathrm{c}}(\mathrm{I}(k_X)) \times D^b_{\mathrm{I}\,\mathbb{R}-\mathrm{c}}(\mathrm{I}(k_X)) \to D^b_{\mathrm{I}\,\mathbb{R}-\mathrm{c}}(\mathrm{I}(k_X))$,
(iv) $R\mathcal{I}hom\,(F, \cdot\,)\colon D^b_{\mathrm{I}\,\mathbb{R}-\mathrm{c}}(\mathrm{I}(k_X)) \to D^b_{\mathrm{I}\,\mathbb{R}-\mathrm{c}}(\mathrm{I}(k_X))$,
(v) $\beta\colon D^b(k_X) \to D^b(\mathrm{I}_{\mathbb{R}-\mathrm{c}}(k_X))$.

Proof. — (i) Let $F \in D^b_{\mathrm{I}\,\mathbb{R}-\mathrm{c}}(\mathrm{I}(k_X))$. By “dévissage”, we may assume F in degree 0. Let $F \simeq \text{“}\varinjlim_i\text{”}\, F_i$, with $F_i \in \mathbb{R}\text{-}\mathrm{C}^c(k_X)$. Since $H^jRf_{!!}F_i \in \mathbb{R}\text{-}\mathrm{C}^c(k_Y)$, it remains to notice that the functor $H^jRf_{!!}$ commutes with “$\varinjlim$”.

(ii)–(iii) The proof is similar.

(iv) Let $G \in D^b_{\mathrm{I}\,\mathbb{R}-\mathrm{c}}(\mathrm{I}(k_X))$. By dévissage, we reduce to the case where $G \in \mathrm{I}_{\mathbb{R}-\mathrm{c}}(k_X)$ and $F \in \mathbb{R}\text{-}\mathrm{C}(k_X)$. Let $G \simeq \text{“}\varinjlim_i\text{”}\, G_i$ with $G_i \in \mathbb{R}\text{-}\mathrm{C}^c(k_X)$. Then we have

$$H^j R\mathcal{I}hom\,(F,G) \simeq \text{“}\varinjlim_i\text{”}\, H^j R\mathcal{H}om\,(F,G_i)$$

and $H^j R\mathcal{H}om\,(F,G_i) \in \mathbb{R}\text{-}\mathrm{C}^c(k_X)$.
(v) The functor $\beta\colon \mathrm{Mod}(k_X) \to \mathrm{I}(k_X)$ is the composition $I_{\mathcal{T}} \circ \rho_!$. q.e.d.

Proposition 7.1.4. — *Let $u\colon F \to G$ be a morphism in $D^b_{\mathrm{I}\,\mathbb{R}-\mathrm{c}}(\mathrm{I}(k_X))$. Then u is an isomorphism if and only if for any $K \in \mathbb{R}\text{-}\mathrm{C}^c(k_X)$, u induces an isomorphism $R\mathcal{H}om\,(K,F) \xrightarrow{\sim} R\mathcal{H}om\,(K,G)$.*

Proof. — Consider a distinguished triangle $F \to G \to L \xrightarrow[+1]{}$ and assume that for any $K \in \mathbb{R}\text{-}\mathrm{C}^c(k_X)$, $R\mathcal{H}om\,(K,L) = 0$. Let $k \in \mathbb{Z}$ such that $H^j(L) = 0$ for $j < k$. Then $\mathcal{H}om\,(K, H^k(L)) \simeq H^k R\mathcal{H}om\,(K,L) = 0$. Hence, $\mathrm{Hom}\,(K, H^k(L)) = 0$, and this implies $H^k(L) = 0$ since $H^k(L) \in \mathrm{I}_{\mathbb{R}-\mathrm{c}}(k_X)$. q.e.d.

Proposition 7.1.5. — *Let $K \in K^b(\mathbb{R}\text{-}\mathrm{C}(k_X))$ and let $F \in K^b(\mathrm{I}_{\mathbb{R}-\mathrm{c}}(k_X))$. Then*

$$H^k R\mathcal{H}om\,(K,F) \simeq \varinjlim_{K' \to K} H^k \mathcal{H}om\,(K',F)$$

where $K' \in K^b(\mathbb{R}\text{-}\mathrm{C}(k_X))$ ranges through the family of complexes qis to K.

Proof. — We may reduce to the case where $F \in \mathrm{I}_{\mathbb{R}-\mathrm{c}}(k_X)$, then to the case where $F \in \mathbb{R}\text{-}\mathrm{C}(k_X)$. Then the result follows from the equivalence $D^b(\mathbb{R}\text{-}\mathrm{C}(k_X)) \simeq D^b_{\mathbb{R}-c}(k_X)$. q.e.d.

Corollary 7.1.6. — *Let $F \in \mathrm{I}_{\mathbb{R}-\mathrm{c}}(k_X)$ and assume that the functor $\mathcal{H}om\,(\cdot,F)$ is exact on the category $\mathbb{R}\text{-}\mathrm{C}^c(k_X)$. Then one has $H^j R\mathcal{H}om\,(K,F) = 0$ for $j \neq 0$ and any $K \in \mathbb{R}\text{-}\mathrm{C}(k_X)$.*

7.2. Some classical ind-sheaves

From now on, the base field k is $\mathbb{C}$. Denote by X a real analytic manifold.

Notation 7.2.1. — (i) We denote by $d_X^{\mathbb{R}}$ the dimension of X.
(ii) As usual, we denote by $\mathcal{C}_X^\infty$ (resp. $\mathcal{C}_X^\omega$) the sheaf of complex functions of class $\mathcal{C}^\infty$ (resp. real analytic), by $\mathcal{D}b_X$ (resp. $\mathcal{B}_X$) the sheaf of Schwartz's distributions (resp. Sato's hyperfunctions), and by $\mathcal{D}_X$ the sheaf of analytic finite-order differential operators. We also use the notation $\mathcal{A}_X = \mathcal{C}_X^\omega$.
(iii) We denote by Ω_X^p the sheaf of p-differential forms with coefficients in $\mathcal{A}_X$ (hence, $\Omega_X^0 = \mathcal{A}_X$) and by $\Omega_X^\bullet$ the De Rham complex with coefficients in $\mathcal{A}_X$, that is, the complex

$$0 \to \Omega_X^0 \to \cdots \to \Omega_X^{d_X^{\mathbb{R}}} \to 0$$

We also set $\Omega_X := \Omega_X^{d_X^{\mathbb{R}}}$.

An important property of subanalytic subsets is given by the lemma below.

Lemma 7.2.2. — *Let U and V be two open subanalytic subsets of $\mathbb{R}^n$, and K a compact subset of $\mathbb{R}^n$. Denote by $\operatorname{dist}(x, K \setminus U)$ the distance from $x \in \mathbb{R}^n$ to $K \setminus U$. Then there exist a positive integer N and $C > 0$ such that*

$$\operatorname{dist}\big(x, K \setminus (U \cup V)\big)^N \leq C\big(\operatorname{dist}(x, K \setminus U) + \operatorname{dist}(x, K \setminus V)\big) \quad \text{for any } x \in K.$$

Let U be an open subset of X. One sets $\mathcal{C}_X^\infty(U) = \Gamma(U; \mathcal{C}_X^\infty)$.

Definition 7.2.3. — Let $f \in \mathcal{C}_X^\infty(U)$. One says that f has *polynomial growth* at $p \in X$ if it satisfies the following condition. For a local coordinate system $(x_1, \dots, x_n)$ around p, there exist a sufficiently small compact neighborhood K of p and a positive integer N such that

$$\sup_{x \in K \cap U} \big(\operatorname{dist}(x, K \setminus U)\big)^N |f(x)| < \infty\,. \tag{7.2.1}$$

It is obvious that f has polynomial growth at any point of U. We say that f is *tempered* at p if all its derivatives have polynomial growth at p. We say that f is tempered if it is tempered at any point.

For an open subanalytic set U in X, denote by $\mathcal{C}_X^{\infty,t}(U)$ the subspace of $\mathcal{C}_X^\infty(U)$ consisting of tempered functions. Denote by $\mathcal{D}b_X^t(U)$ the space of tempered distributions on U, defined by the exact sequence

$$0 \to \Gamma_{X \setminus U}(X; \mathcal{D}b_X) \to \Gamma(X; \mathcal{D}b_X) \to \mathcal{D}b_X^t(U) \to 0.$$

For a closed subanalytic subset S in X, denote by $\mathcal{I}_{X,S}^\infty$ the subsheaf of $\mathcal{C}_X^\infty$ consisting of functions which vanish up to infinite order on S.

In [**8**], [**11**], one introduces the sheaves:

$$\begin{aligned} T\mathcal{H}om\,(\mathbb{C}_U, \mathcal{C}_X^\infty) &:= V \mapsto \mathcal{C}_V^{\infty,t}(U \cap V), \\ T\mathcal{H}om\,(\mathbb{C}_U, \mathcal{D}b_X) &:= V \mapsto \mathcal{D}b_V^t(U \cap V), \\ \mathbb{C}_U \overset{\mathrm{w}}{\otimes} \mathcal{C}_X^\infty &:= V \mapsto \Gamma(V; \mathcal{I}_{V, V \setminus U}^\infty). \end{aligned}$$

As a consequence of the theorems of Lojasiewicz [**14**] (see also Malgrange [**15**]) one gets the following

Lemma 7.2.4. — *Let U and V be two open subanalytic subsets of X. The sequences below are exact:*

$$\begin{aligned} &0 \to \mathcal{C}_X^{\infty,t}(U \cup V) \to \mathcal{C}_X^{\infty,t}(U) \oplus \mathcal{C}_X^{\infty,t}(V) \to \mathcal{C}_X^{\infty,t}(U \cap V) \to 0, \\ &0 \to \mathcal{D}b_X^t(U \cup V) \to \mathcal{D}b_X^t(U) \oplus \mathcal{D}b_X^t(V) \to \mathcal{D}b_X^t(U \cap V) \to 0, \\ &0 \to \Gamma(X; \mathbb{C}_{U \cap V} \overset{\mathrm{w}}{\otimes} \mathcal{C}_X^\infty) \to \Gamma(X; \mathbb{C}_U \overset{\mathrm{w}}{\otimes} \mathcal{C}_X^\infty) \oplus \Gamma(X; \mathbb{C}_V \overset{\mathrm{w}}{\otimes} \mathcal{C}_X^\infty) \\ &\qquad\qquad \to \Gamma(X; \mathbb{C}_{U \cup V} \overset{\mathrm{w}}{\otimes} \mathcal{C}_X^\infty) \to 0. \end{aligned}$$

Applying Proposition 6.4.1, we find that $\mathcal{C}_X^{\infty,t}$ and $\mathcal{D}b_X^t$ are sheaves on X_{sa}. Moreover the functor $\mathcal{D}b_X^t(\,\cdot\,)$ is exact on the category $\mathbb{R}\text{-}\mathrm{C}^c(\mathbb{C}_X)^{\mathrm{op}}$ by Proposition 6.4.2.

Applying Corollary 6.5.2, the functor

$$\begin{aligned} \cdot \overset{\mathrm{w}}{\otimes} \mathcal{C}_X^\infty : \mathcal{T} &\to \mathrm{Mod}(\mathcal{D}_X) \\ U &\mapsto \mathbb{C}_U \overset{\mathrm{w}}{\otimes} \mathcal{C}_X^\infty \end{aligned}$$

extends to the category $\mathbb{R}\text{-}\mathrm{C}^c(\mathbb{C}_X)$ as an exact functor. We may thus define the left exact functor:

$$\begin{aligned} \mathcal{C}_X^{\infty,\mathrm{w}}(\,\cdot\,) : \mathbb{R}\text{-}\mathrm{C}^c(\mathbb{C}_X)^{\mathrm{op}} &\to \mathrm{Mod}(\mathbb{C}) \\ \mathcal{C}_X^{\infty,\mathrm{w}}(F) &= \Gamma(X; H^0(D'F) \overset{\mathrm{w}}{\otimes} \mathcal{C}_X^\infty), \end{aligned}$$

where we set

$$D'F = R\mathcal{H}om(F, \mathbb{C}_X),$$

and hence $H^0(D'F) = \mathcal{H}om(F, \mathbb{C}_X)$. Therefore, $\mathcal{C}_X^{\infty,\mathrm{w}}$ is a sheaf on X_{sa}.

Applying Proposition 6.7.1, the above sheaves on X_{sa} are $\rho_!\mathcal{D}_X$-modules, hence their images by $I_{\mathcal{T}}$ belong to $\mathrm{I}(\beta\mathcal{D}_X)$:

$$\mathcal{C}_X^{\infty,t}, \mathcal{D}b_X^t, \mathcal{C}_X^{\infty,\mathrm{w}} \in \mathrm{I}(\beta\mathcal{D}_X).$$

Definition 7.2.5. — We call $\mathcal{C}_X^{\infty,t}$ (resp. $\mathcal{D}b_X^t$, $\mathcal{C}_X^{\infty,\mathrm{w}}$) the ind-sheaf of tempered $\mathcal{C}^\infty$-functions (resp. tempered distributions, Whitney $\mathcal{C}^\infty$-functions).

These ind-sheaves satisfy for $F \in \mathbb{R}\text{-}\mathrm{C}^c(\mathbb{C}_X)$

$$\mathrm{Hom}_{\mathrm{I}(k_X)}(F, \mathcal{C}_X^{\infty,t}) \simeq \Gamma(X; T\mathcal{H}om(F, \mathcal{C}_X^\infty)), \tag{7.2.2}$$

$$\mathrm{Hom}_{\mathrm{I}(k_X)}(F, \mathcal{D}b_X^t) \simeq \Gamma(X; T\mathcal{H}om(F, \mathcal{D}b_X)), \tag{7.2.3}$$

$$\mathrm{Hom}_{\mathrm{I}(k_X)}(F, \mathcal{C}_X^{\infty,\mathrm{w}}) \simeq \Gamma(X; H^0(D'F) \overset{\mathrm{w}}{\otimes} \mathcal{C}_X^\infty). \tag{7.2.4}$$

Replacing the "Whitney tensor product" $\cdot \overset{\mathrm{w}}{\otimes} \mathcal{C}_X^\infty$ with the usual tensor product, we get the left exact functor (defined on the whole category $\mathrm{Mod}^c(\mathbb{C}_X)$):

$$\mathcal{C}_X^{\infty,\omega}(F) = \Gamma(X; H^0(D'F) \otimes \mathcal{C}_X^\infty).$$

Hence we get the ind-sheaf, $\mathcal{C}_X^{\infty,\omega}$ which is nothing but the ind-sheaf $\beta_X(\mathcal{C}_X^\infty)$ and

$$\mathcal{C}_X^{\infty,\omega} \in \mathrm{I}(\beta\mathcal{D}_X).$$

Proposition 7.2.6. — *Assume* $F \in D^b_{\mathbb{R}-c}(\mathbb{C}_X)$. *(In* (iv) *one may assume* $F \in D^b(\mathbb{C}_X)$.*) Then*

(i) $R\mathcal{H}om(F, \mathcal{D}b_X^t) \simeq T\mathcal{H}om(F, \mathcal{D}b_X)$,
(ii) $R\mathcal{H}om(F, \mathcal{C}_X^{\infty,t}) \simeq RT\mathcal{H}om(F, \mathcal{C}_X^\infty)$,
(iii) $R\mathcal{H}om(F, \mathcal{C}_X^{\infty,\mathrm{w}}) \simeq (D'F) \overset{\mathrm{w}}{\otimes} \mathcal{C}_X^\infty$,
(iv) $R\mathcal{H}om(F, \mathcal{C}_X^{\infty,\omega}) \simeq (D'F) \otimes \mathcal{C}_X^\infty$.

Proof. — (i) For $F \in \mathbb{R}\text{-C}(\mathbb{C}_X)$, the isomorphism (7.2.3) implies $\mathcal{H}om\,(F, \mathcal{D}b^t_X) \simeq T\mathcal{H}om\,(F, \mathcal{D}b_X)$. Then the result follows from Corollary 7.1.6, since $T\mathcal{H}om\,(\cdot, \mathcal{D}b_X)$ is exact on the category $\mathbb{R}\text{-C}^c(\mathbb{C}_X)$.
(ii) follows from (7.2.2) and Proposition 7.1.5. Indeed, $RT\mathcal{H}om\,(F, \mathcal{C}^\infty_X)$ is by definition $\varinjlim_{F' \to F} T\mathcal{H}om\,(F', \mathcal{C}^\infty_X)$. Here $F' \to F$ ranges over the family of qis with $F' \in K^b(\mathbb{R}\text{-C}(\mathbb{C}_X))$, and $\varinjlim$ is taken in $\mathrm{Ind}(D^b(\mathbb{C}_X))$.

(iii) For $F \in \mathbb{R}\text{-C}(C_X)$, the isomorphism (7.2.4) implies $\mathcal{H}om\,(F, \mathcal{C}^{\infty,\mathrm{w}}_X) \simeq H^0(D'F) \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_X$. Now let $F \in D^b_{\mathbb{R}-c}(\mathbb{C}_X)$. We represent it by an object $F \in K^b(\mathbb{R}\text{-C}(\mathbb{C}_X))$. We have the chain of morphisms

$$\begin{aligned}
(D'F) \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_X &\simeq \text{“}\varinjlim_{F' \to F}\text{”}\, \mathcal{H}om\,(F', \mathbb{C}_X) \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_X \\
&\simeq \text{“}\varinjlim_{F' \to F}\text{”}\, \mathcal{H}om\,(F', \mathcal{C}^{\infty,\mathrm{w}}_X) \\
&\to \text{“}\varinjlim_{F' \to F}\text{”}\, R\mathcal{H}om\,(F', \mathcal{C}^{\infty,\mathrm{w}}_X) \\
&\simeq R\mathcal{H}om\,(F, \mathcal{C}^{\infty,\mathrm{w}}_X)
\end{aligned}$$

Here $F' \to F$ ranges over the family of qis, with $F' \in K^b(\mathbb{R}\text{-C}(\mathbb{C}_X))$. We have

$$H^j(\text{“}\varinjlim_{F' \to F}\text{”}\, \mathcal{H}om\,(F', \mathcal{C}^{\infty,\mathrm{w}}_X)) \simeq \text{“}\varinjlim_{F' \to F}\text{”}\, H^j(\mathcal{H}om\,(F', \mathcal{C}^{\infty,\mathrm{w}}_X)).$$

Applying Proposition 7.1.5, we get the isomorphism

$$H^j((D'F) \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_X) \xrightarrow{\sim} H^j(R\mathcal{H}om\,(F, \mathcal{C}^{\infty,\mathrm{w}}_X))$$

and the result follows.
(iv) follows from Proposition 5.1.10 (v). q.e.d.

There is a chain of morphisms

$$\mathcal{C}^{\infty,\omega}_X \to \mathcal{C}^{\infty,\mathrm{w}}_X \to \mathcal{C}^{\infty,t}_X \to \mathcal{D}b^t_X \to \mathcal{D}b_X \to \mathcal{B}_X.$$

7.3. Ind-sheaves associated with holomorphic functions

Let X be a complex manifold with structure sheaf $\mathcal{O}_X$.

Notation 7.3.1. — We shall mainly follow the notations of [**10**].
(i) We denote by $\overline{X}$ the complex conjugate manifold and by $X^{\mathbb{R}}$ the underlying real analytic manifold, identified with the diagonal of $X \times \overline{X}$.
(ii) We denote by d_X the complex dimension of X, by Ω^p_X the sheaf of p-differential forms with coefficients in $\mathcal{O}_X$ (hence, $\Omega^0_X = \mathcal{O}_X$) and by $\Omega^\bullet_X$ the De Rham complex with coefficients in $\mathcal{O}_X$. We also set $\Omega_X = \Omega^{d_X}_X$. One should not confuse Ω^p_X and $\Omega^p_{X^{\mathbb{R}}}$.

(iii) We denote by $\mathcal{D}_X$ the sheaf of rings of finite-order holomorphic differential operators, not to be confused with $\mathcal{D}_{X^{\mathbb{R}}}$.
(iv) As usual, $D^b_{\mathbb{C}-c}(\mathbb{C}_X)$ denotes the full triangulated subcategory of $D^b(\mathbb{C}_X)$ consisting of complexes with $\mathbb{C}$-constructible cohomology.

Let $F \in D^b_{\mathbb{R}-c}(\mathbb{C}_X)$. Recall that in [**8**] and [**11**], one introduces the objects:

$$\begin{aligned} T\mathcal{H}om\,(F, \mathcal{O}_X) &= R\mathcal{H}om_{\mathcal{D}_{\overline{X}}}(\mathcal{O}_{\overline{X}}, T\mathcal{H}om\,(F, \mathcal{D}b_X)), \\ F \overset{\mathrm{w}}{\otimes} \mathcal{O}_X &= R\mathcal{H}om_{\mathcal{D}_{\overline{X}}}(\mathcal{O}_{\overline{X}}, F \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_X). \end{aligned}$$

Moreover, one has the canonical isomorphism

$$R\mathcal{H}om_{\mathcal{D}_{\overline{X}}}(\mathcal{O}_{\overline{X}}, RT\mathcal{H}om\,(F, \mathcal{C}^\infty_X)) \xrightarrow{\sim} T\mathcal{H}om\,(F, \mathcal{O}_X). \tag{7.3.1}$$

For $\lambda = t, \mathrm{w}, \omega$, one defines the objects $\mathcal{O}^\lambda_X \in D^b(\mathrm{I}(\beta\mathcal{D}_X))$ by the formulas:

$$\begin{aligned} \mathcal{O}^t_X &:= R\mathcal{I}hom_{\beta\mathcal{D}_{\overline{X}}}(\beta\mathcal{O}_{\overline{X}}, \mathcal{D}b^t_{X^{\mathbb{R}}}) \xleftarrow{\sim} R\mathcal{I}hom_{\beta\mathcal{D}_{\overline{X}}}(\beta\mathcal{O}_{\overline{X}}, \mathcal{C}^{\infty,t}_X), \\ \mathcal{O}^{\mathrm{w}}_X &:= R\mathcal{I}hom_{\beta\mathcal{D}_{\overline{X}}}(\beta\mathcal{O}_{\overline{X}}, \mathcal{C}^{\infty,\mathrm{w}}_X), \\ \mathcal{O}^\omega_X &:= R\mathcal{I}hom_{\beta\mathcal{D}_{\overline{X}}}(\beta\mathcal{O}_{\overline{X}}, \mathcal{C}^{\infty,\omega}_X). \end{aligned}$$

Of course, $\mathcal{O}^\omega_X \simeq \beta_X\mathcal{O}_X$. Moreover the first isomorphism follows from (7.3.1).

Proposition 7.3.2. — *Let $F \in D^b_{\mathbb{R}-c}(\mathbb{C}_X)$. Then*

$$\begin{aligned} R\mathcal{H}om\,(F, \mathcal{O}^t_X) &\simeq T\mathcal{H}om\,(F, \mathcal{O}_X), \\ R\mathcal{H}om\,(F, \mathcal{O}^{\mathrm{w}}_X) &\simeq (D'F) \overset{\mathrm{w}}{\otimes} \mathcal{O}_X, \\ R\mathcal{H}om\,(F, \mathcal{O}^\omega_X) &\simeq (D'F) \otimes \mathcal{O}_X. \end{aligned}$$

Proof. — This follows immediately from Proposition 7.2.6. q.e.d.

Note that we have a chain of morphisms

$$\mathcal{O}^\omega_X \to \mathcal{O}^{\mathrm{w}}_X \to \mathcal{O}^t_X \to \mathcal{O}_X.$$

Notation 7.3.3. — Let $\mathcal{L}$ be a locally free $\mathcal{O}_X$-module of finite rank, and let $\lambda = t, \mathrm{w}, \omega$. We set

$$\mathcal{L}^\lambda = (\beta\mathcal{L}) \underset{\beta\mathcal{O}_X}{\otimes} \mathcal{O}^\lambda_X.$$

Remark 7.3.4. — (i) One shall be aware that on a complex manifold X of dimension $n > 1$, the object $\mathcal{O}^t_X \in D^b(\mathrm{I}(\mathbb{C}_X))$ is not concentrated in degree 0. Indeed consider the Dolbeault complex

$$0 \to \mathcal{D}b^{t,(0,0)}_X \xrightarrow[\overline{\partial}]{} \cdots \xrightarrow[\overline{\partial}]{} \mathcal{D}b^{t,(0,n)}_X \to 0 \tag{7.3.2}$$

and suppose that this complex is exact at degree p. Let U be an open subanalytic subset of X and consider the diagram

$$\begin{tikzcd} F \arrow[r, dotted, "h"] \arrow[d, dotted] & \mathbb{C}_U \arrow[d] \arrow[dr, "0"] & \\ \mathcal{D}b_X^{t,(0,p-1)} \arrow[r, "\overline{\partial}"'] & \mathcal{D}b_X^{t,(0,p)} \arrow[r, "\overline{\partial}"'] & \mathcal{D}b_X^{t,(0,p+1)} \end{tikzcd}$$

Applying Proposition 1.3.2, the dotted arrows can be completed to a commutative diagram with $F \in \mathbb{R}\text{-}\mathrm{C}^c(\mathbb{C}_X)$ and an epimorphism h. It means that, for any $s \in \mathrm{Hom}_{\mathrm{I}(\mathbb{C}_X)}(\mathbb{C}_U, \mathcal{D}b_X^{t,(0,p)}) = \mathcal{D}b_X^{t,(0,p)}(U)$ satisfying the equation $\overline{\partial}s = 0$, there exists $t \in \mathrm{Hom}_{\mathrm{I}(\mathbb{C}_X)}(F, \mathcal{D}b_X^{t,(0,p-1)})$ such that $\overline{\partial}t = s$. We may assume that F is a finite direct sum of sheaves $\mathbb{C}_{U_j}, j \in J$, with U_j open subanalytic and $U = \cup_j U_j$.

Therefore, there exist $t_j \in \mathcal{D}b_X^{t,(0,p-1)}(U_j), j \in J$ solution of $\overline{\partial}t_j = s$ on U_j. If $n > 1$, this is not possible for a suitable choice of U.
(ii) The same argument shows that $R\rho_*\mathcal{O}_X$ is not concentrated in degree 0 for $n > 1$. (Recall that ρ is the natural morphism of sites $X \to X_{sa}$.)

7.4. Operations on $\mathcal{O}^t$

As an application, let us prove the adjunction formula for integral transforms of [**11**] in the framework of ind-sheaves. (For the case of sheaves and $\mathcal{D}$-modules without growth conditions, refer to [**4**].)

We shall follow the notations of [**11**] with an exception: if $f\colon X \to Y$ is a morphism of complex manifolds, we denote by $\mathrm{D}f^{-1}$ and $\mathrm{D}f_!$ the inverse and proper direct images in the derived categories of $\mathcal{D}$-modules.

Following [**12**], we say that an $\mathcal{O}_X$-module $\mathcal{F}$ is quasi-good if for any relatively compact open subset U, $\mathcal{F}|_U$ is a union of an increasing sequence of coherent $\mathcal{O}_X|_U$-submodules. A $\mathcal{D}_X$-module is called quasi-good if it is quasi-good as an $\mathcal{O}_X$-module. Recall (loc. cit.) that the full subcategory of $\mathrm{Mod}(\mathcal{O}_X)$ consisting of quasi-good modules is stable by kernels, cokernels and extension.

We denote by $D^b_{q-good}(\mathcal{D}_X)$ the full triangulated subcategory of $D^b(\mathcal{D}_X)$ consisting of objects with quasi-good cohomology and by $D^b_{r-h}(\mathcal{D}_X)$ the full triangulated subcategory of $D^b(\mathcal{D}_X)$ consisting of objects with regular holonomic cohomology.

Theorem 7.4.1. — *Let $f\colon X \to Y$ be a holomorphic map and let $\mathcal{N} \in D^b(\mathcal{D}_Y)$. Then there exists a natural isomorphism in $D^b(\mathrm{I}(\mathbb{C}_X))$:*

$$\Omega^t_X \overset{L}{\underset{\beta\mathcal{D}_X}{\otimes}} \beta(\mathrm{D}f^{-1}\mathcal{N})[d_X] \xrightarrow{\sim} f^!(\Omega^t_Y \overset{L}{\underset{\beta\mathcal{D}_Y}{\otimes}} \beta\mathcal{N})[d_Y]. \tag{7.4.1}$$

We need some preliminary results.

Lemma 7.4.2. — *Let $f\colon X \to Y$ be a morphism of real analytic manifolds and let $\mathcal{F}$ be a locally free $\mathcal{A}_X$-module of finite rank. Then $R^k f_{!!}(\mathcal{D}b^t_X \otimes_{\beta\mathcal{A}_X} \beta\mathcal{F}) = 0$ for $k \neq 0$.*

Proof. — Since $Rf_{!!}(\mathcal{D}b^t_X \otimes_{\beta\mathcal{A}_X} \beta\mathcal{F})$ belongs to $D^b_{\mathrm{I}\mathbb{R}-\mathrm{c}}(\mathbb{C}_Y)$, it is enough to check that for any $G \in \mathbb{R}\text{-}\mathrm{C}^c(\mathbb{C}_Y)$, the complex $\mathrm{RHom}_{\mathrm{I}(\mathbb{C}_Y)}(G, Rf_{!!}(\mathcal{D}b^t_X \otimes_{\beta\mathcal{A}_X} \beta\mathcal{F}))$ is concentrated in degree 0. Consider the chain of isomorphisms

$$\begin{aligned}
\mathrm{RHom}_{\mathrm{I}(\mathbb{C}_Y)}(G, Rf_{!!}(\mathcal{D}b^t_X \otimes_{\beta\mathcal{A}_X} \beta\mathcal{F})) &\simeq \mathrm{R}\Gamma_c(Y; R\mathcal{H}om\,(G, Rf_{!!}(\mathcal{D}b^t_X \otimes_{\beta\mathcal{A}_X} \beta\mathcal{F})))\\
&\simeq \mathrm{R}\Gamma_c(Y; Rf_! R\mathcal{H}om\,(f^{-1}G, \mathcal{D}b^t_X \otimes_{\beta\mathcal{A}_X} \beta\mathcal{F}))\\
&\simeq \mathrm{R}\Gamma_c(X; R\mathcal{H}om\,(f^{-1}G, \mathcal{D}b^t_X) \otimes_{\mathcal{A}_X} \mathcal{F})\\
&\simeq \mathrm{R}\Gamma_c(X; T\mathcal{H}om\,(f^{-1}G, \mathcal{D}b_X) \otimes_{\mathcal{A}_X} \mathcal{F}).
\end{aligned}$$

Since $T\mathcal{H}om\,(f^{-1}G, \mathcal{D}b_X)$ is soft, these complexes are concentrated in degree 0. q.e.d.

Recall that $\Omega^\bullet_X$ denotes the De Rham complex on X.

Lemma 7.4.3. — *Let $f\colon X \to Y$ be a morphism of oriented real analytic manifolds. There is a natural morphism in $C^b(\mathrm{I}(\mathbb{C}_Y))$*

$$f_{!!}(\mathcal{D}b^t_X \otimes_{\beta\mathcal{A}_X} \beta\Omega^\bullet_X)\,[d^{\mathbb{R}}_X] \to \mathcal{D}b^t_Y \otimes_{\beta\mathcal{A}_Y} \beta\Omega^\bullet_Y\,[d^{\mathbb{R}}_Y].$$

Proof. — Let $G \in \mathbb{R}\text{-}\mathrm{C}^c(\mathbb{C}_Y)$. Using Proposition 4.3 of [**11**], one gets the morphisms

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{I}(\mathbb{C}_Y)}(G, f_{!!}(\mathcal{D}b^t_X \otimes_{\beta\mathcal{A}_X} \beta\Omega_X^{d^{\mathbb{R}}_X - i})) &\simeq \Gamma_c(X; T\mathcal{H}om\,(f^{-1}G, \mathcal{D}b_X \otimes_{\mathcal{A}_X} \Omega_X^{d^{\mathbb{R}}_X - i}))\\
&\to \Gamma_c(Y; T\mathcal{H}om\,(G, \mathcal{D}b_Y \otimes_{\mathcal{A}_Y} \Omega_Y^{d^{\mathbb{R}}_Y - i})).
\end{aligned}$$

q.e.d.

Lemma 7.4.4. — *Let $f\colon X \to Y$ be a morphism of complex manifolds. There is a natural morphism in $D^b(\mathrm{I}(\beta\mathcal{D}^{\mathrm{op}}_Y))$:*

$$Rf_{!!}(\Omega^t_X \overset{L}{\underset{\beta\mathcal{D}_X}{\otimes}} \beta\mathcal{D}_{X\to Y})[d_X] \to \Omega^t_Y[d_Y]. \tag{7.4.2}$$

Proof. — First, let us recall how to construct a $\mathcal{D}_X$-free resolution of the $\mathcal{D}_X \otimes f^{-1}\mathcal{D}^{\mathrm{op}}_Y$-module $\mathcal{D}_{X\to Y}$. Denote by Θ_X the sheaf of holomorphic vector fields on X and by $\bigwedge^\bullet \Theta_X$ its exterior algebra. For a left $\mathcal{D}_X$-module $\mathcal{M}$, denote by $Sp^\bullet(\mathcal{M})$ the Spencer resolution of $\mathcal{M}$

$$Sp^\bullet(\mathcal{M}) := \mathcal{D}_X \otimes_{\mathcal{O}_X} \bigwedge^{\bullet} \Theta_X \otimes_{\mathcal{O}_X} \mathcal{M}$$

where the differential is given by

$$
\begin{aligned}
& d(P\otimes(v_1\wedge\cdots\wedge v_p)\otimes u)= \\
& \qquad \sum_{i=1}^{p}(-1)^{i-1}Pv_i\otimes(v_1\wedge\cdots\wedge\widehat{v_i}\wedge\cdots\wedge v_p)\otimes u \\
& \qquad -\sum_{i=1}^{p}(-1)(-1)^{i-1}P\otimes(v_1\wedge\cdots\wedge\widehat{v_i}\wedge\cdots\wedge v_p)\otimes v_iu \\
& \qquad +\sum_{1\le i<j\le p}(-1)^{i+j}P\otimes([v_i,v_j]\wedge v_1\wedge\cdots\wedge\widehat{v_i}\wedge\cdots\wedge\widehat{v_j}\wedge\cdots\wedge v_p)\otimes u.
\end{aligned}
$$

It is well-known that the natural morphism $Sp^\bullet(\mathcal{M})\to\mathcal{M}$ is a qis of $\mathcal{D}_X$-modules. Applying this to $\mathcal{M}=\mathcal{D}_{X\to Y}$, we get the qis

$$
\begin{aligned}
\mathcal{D}_{X\to Y} &\leftarrow \mathcal{D}_X\otimes_{\mathcal{O}_X}\bigwedge^{\bullet}\Theta_X\otimes_{\mathcal{O}_X}\mathcal{D}_{X\to Y} \\
&\simeq \mathcal{D}_X\otimes_{\mathcal{O}_X}\bigwedge^{\bullet}\Theta_X\otimes_{f^{-1}\mathcal{O}_Y}f^{-1}\mathcal{D}_Y.
\end{aligned}
$$

Hence we get the isomorphisms in $D^b(\beta f^{-1}\mathcal{D}_Y^{\mathrm{op}})$:

$$
\begin{aligned}
\Omega_X^t\overset{L}{\underset{\beta\mathcal{D}_X}{\otimes}}\beta\mathcal{D}_{X\to Y} &\simeq \Omega_X^t\underset{\beta\mathcal{D}_X}{\otimes}\beta(\mathcal{D}_X\otimes_{\mathcal{O}_X}\bigwedge^{\bullet}\Theta_X\otimes_{f^{-1}\mathcal{O}_Y}f^{-1}\mathcal{D}_Y) \\
&\simeq \Omega_X^t\underset{\beta\mathcal{O}_X}{\otimes}\beta(\bigwedge^{\bullet}\Theta_X\otimes_{f^{-1}\mathcal{O}_Y}f^{-1}\mathcal{D}_Y) \\
&\simeq \mathcal{O}_X^t\underset{\beta\mathcal{O}_X}{\otimes}\beta(\Omega_X^{\bullet}\otimes_{f^{-1}\mathcal{O}_Y}f^{-1}\mathcal{D}_Y)[d_X] \\
&\simeq \mathcal{D}b_X^{t(0,\bullet)}\underset{\beta\mathcal{O}_X}{\otimes}\beta(\Omega_X^{\bullet}\otimes_{f^{-1}\mathcal{O}_Y}f^{-1}\mathcal{D}_Y)[d_X] \\
&\simeq \mathcal{D}b_X^t\underset{\beta\mathcal{A}_{X^{\mathbb{R}}}}{\otimes}\beta(\Omega_{X^{\mathbb{R}}}^{\bullet;\bullet}\otimes_{f^{-1}\mathcal{O}_Y}f^{-1}\mathcal{D}_Y)[d_X].
\end{aligned}
$$

Applying $Rf_{!!}$ we get the desired morphism in $D^b(\mathrm{I}(\beta\mathcal{D}_Y))$:

$$
\begin{aligned}
Rf_{!!}(\Omega_X^t\overset{L}{\underset{\beta\mathcal{D}_X}{\otimes}}\beta\mathcal{D}_{X\to Y})\,[d_X] &\simeq f_{!!}(\mathcal{D}b_X^t\otimes_{\mathcal{A}_{X^{\mathbb{R}}}}\beta(\Omega_{X^{\mathbb{R}}}^{\bullet;\bullet}\otimes_{f^{-1}\mathcal{O}_Y}f^{-1}\mathcal{D}_Y))\,[2d_X] \\
&\to \mathcal{D}b_Y^t\otimes_{\mathcal{A}_{Y^{\mathbb{R}}}}\beta(\Omega_{Y^{\mathbb{R}}}^{\bullet;\bullet}\otimes_{\mathcal{O}_Y}\mathcal{D}_Y)[2d_Y] \\
&\simeq \mathcal{O}_Y^t\otimes_{\beta\mathcal{O}_Y}\beta(\Omega_Y^{\bullet}\otimes_{\mathcal{O}_Y}\mathcal{D}_Y)\,[2d_Y] \\
&\simeq \mathcal{O}_Y^t\overset{L}{\underset{\beta\mathcal{O}_Y}{\otimes}}\beta\Omega_Y\,[d_Y].
\end{aligned}
$$

q.e.d.

By adjunction, (7.4.2) gives the morphism in $D^b(\mathrm{I}(\beta f^{-1}\mathcal{D}_Y^{\mathrm{op}}))$:

$$\Omega_X^t \overset{L}{\underset{\beta\mathcal{D}_X}{\otimes}} \beta\mathcal{D}_{X\to Y}[d_X] \to f^!\Omega_Y^t[d_Y]. \tag{7.4.3}$$

Lemma 7.4.5. — *The morphism* (7.4.3) *is an isomorphism.*

Proof. — For any $F \in \mathbb{R}\text{-}\mathrm{C}^c(\mathbb{C}_X)$, we have the isomorphisms

$$\begin{aligned}
&\mathrm{RHom}_{\mathrm{I}(\mathbb{C}_X)}(F, \Omega_X^t \overset{L}{\underset{\beta\mathcal{D}_X}{\otimes}} \beta\mathcal{D}_{X\to Y})[d_X] \\
&\qquad \simeq \mathrm{R}\Gamma_c(X; T\mathcal{H}om\,(F, \Omega_X) \overset{L}{\underset{\mathcal{D}_X}{\otimes}} \mathcal{D}_{X\to Y})[d_X] \\
&\qquad \simeq \mathrm{R}\Gamma_c(Y; Rf_!(T\mathcal{H}om\,(F, \Omega_X) \overset{L}{\underset{\mathcal{D}_X}{\otimes}} \mathcal{D}_{X\to Y}))[d_X] \\
&\qquad \simeq \mathrm{R}\Gamma(Y; T\mathcal{H}om\,(Rf_!F, \Omega_Y))[d_Y] \\
&\qquad \simeq \mathrm{RHom}_{\mathrm{I}(\mathbb{C}_Y)}(Rf_{!!}F, \Omega_Y^t)[d_Y] \\
&\qquad \simeq \mathrm{RHom}_{\mathrm{I}(\mathbb{C}_X)}(F, f^!\Omega_Y^t)[d_Y].
\end{aligned}$$

Here, the third isomorphism follows from Theorem 5.7 in [**11**]. q.e.d.

Proof of Theorem 7.4.1.

Consider the chain of isomorphisms

$$\begin{aligned}
f^!(\Omega_Y^t \overset{L}{\underset{\beta\mathcal{D}_Y}{\otimes}} \beta\mathcal{N})[d_Y] &\simeq f^!\Omega_Y^t \overset{L}{\underset{\beta f^{-1}\mathcal{D}_Y}{\otimes}} \beta f^{-1}\mathcal{N}[d_Y] \\
&\simeq \Omega_X^t \overset{L}{\underset{\beta\mathcal{D}_X}{\otimes}} \beta\mathcal{D}_{X\to Y} \overset{L}{\underset{\beta f^{-1}\mathcal{D}_Y}{\otimes}} \beta f^{-1}\mathcal{N}[d_X] \\
&\simeq \Omega_X^t \overset{L}{\underset{\beta\mathcal{D}_X}{\otimes}} \beta(\mathrm{D}f^{-1}\mathcal{N})[d_X].
\end{aligned}$$

In the first isomorphism, we have used Theorem 5.6.3.

By the equivalence of left $\mathcal{D}$-modules and right $\mathcal{D}$-modules, (7.4.3) gives the morphism in $D^b(\mathrm{I}(\beta f^{-1}\mathcal{D}_Y))$:

$$\beta\mathcal{D}_{Y\leftarrow X} \overset{L}{\underset{\beta\mathcal{D}_X}{\otimes}} \mathcal{O}_X^t[d_X] \xrightarrow{\sim} f^!\mathcal{O}_Y^t[d_Y]. \tag{7.4.4}$$

Theorem 7.4.6. — *Let $f\colon X \to Y$ be a holomorphic map and let $\mathcal{M}$ be an object of $D^b_{q-good}(\mathcal{D}_X)$ such that* $\operatorname{supp}\mathcal{M}$ *is proper over Y. Then there exists a natural isomorphism in $D^b(\mathrm{I}(\mathbb{C}_Y))$:*

$$Rf_{!!}(\Omega_X^t \overset{L}{\underset{\beta\mathcal{D}_X}{\otimes}} \beta\mathcal{M}) \simeq \Omega_Y^t \overset{L}{\underset{\beta\mathcal{D}_Y}{\otimes}} \beta\mathrm{D}f_!\mathcal{M}. \tag{7.4.5}$$

We need some lemmas.

Lemma 7.4.7. — *Let $f\colon X \to Y$ be a closed embedding of complex manifolds. There is a natural isomorphism in $D^b(\mathrm{I}(\beta\mathcal{D}_X))$*

$$\beta\mathcal{D}_{X\to Y} \overset{L}{\underset{\beta f^{-1}\mathcal{D}_Y}{\otimes}} f^{-1}\mathcal{O}^t_Y \simeq \mathcal{O}^t_X.$$

Proof. — We have

$$\beta\mathcal{D}_{X\to Y} \overset{L}{\underset{\beta f^{-1}\mathcal{D}_Y}{\otimes}} f^{-1}\mathcal{O}^t_Y \simeq f^{-1}Rf_*(\beta\mathcal{D}_{X\to Y} \overset{L}{\underset{\beta\mathcal{D}_Y}{\otimes}} f^{-1}\mathcal{O}^t_X).$$

Since f is a closed embedding, we have

$$\begin{aligned}
\beta\mathcal{D}_{X\to Y} \overset{L}{\underset{\beta f^{-1}\mathcal{D}_Y}{\otimes}} f^{-1}\mathcal{O}^t_Y &\simeq f^!Rf_*(\beta\mathcal{D}_{X\to Y} \overset{L}{\underset{\beta f^{-1}\mathcal{D}_Y}{\otimes}} f^{-1}\mathcal{O}^t_Y) \\
&\simeq f^!(\beta Rf_*\mathcal{D}_{X\to Y} \overset{L}{\underset{\beta\mathcal{D}_Y}{\otimes}} \mathcal{O}^t_Y) \\
&\simeq \beta\mathcal{D}_{X\to Y} \overset{L}{\underset{\beta f^{-1}\mathcal{D}_Y}{\otimes}} f^!\mathcal{O}^t_Y \\
&\simeq \beta\mathcal{D}_{X\to Y} \overset{L}{\underset{\beta f^{-1}\mathcal{D}_Y}{\otimes}} \beta\mathcal{D}_{Y\leftarrow X} \overset{L}{\underset{\beta\mathcal{D}_X}{\otimes}} \mathcal{O}^t_X[d_X - d_Y] \\
&\simeq \mathcal{O}^t_X.
\end{aligned}$$

Here the fourth isomorphism follows from (7.4.4), and the last isomorphism follows from $\mathcal{D}_{X\to Y} \overset{L}{\underset{\mathcal{D}_Y}{\otimes}} \mathcal{D}_{Y\leftarrow X}[d_X - d_Y] \simeq \mathcal{D}_X$. q.e.d.

Lemma 7.4.8. — *Let $f\colon X \to Y$ be a smooth morphism of complex manifolds. There is a natural isomorphism in $D^b(\mathrm{I}(\beta f^{-1}\mathcal{D}_Y))$*

$$R\mathcal{I}hom_{\beta\mathcal{D}_X}(\beta\mathcal{D}_{X\to Y}, \mathcal{O}^t_X) \simeq f^{-1}\mathcal{O}^t_Y.$$

Proof. — Consider the chain of isomorphisms

$$\begin{aligned}
R\mathcal{I}hom_{\beta\mathcal{D}_X}(\beta\mathcal{D}_{X\to Y}, \mathcal{O}^t_X) &\simeq \beta R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_{X\to Y}, \mathcal{D}_X) \overset{L}{\underset{\beta\mathcal{D}_X}{\otimes}} \mathcal{O}^t_X \\
&\simeq \beta\mathcal{D}_{Y\leftarrow X} \overset{L}{\underset{\beta\mathcal{D}_X}{\otimes}} \mathcal{O}^t_X[d_Y - d_X] \\
&\simeq f^!\mathcal{O}^t_Y[2d_Y - 2d_X] \\
&\simeq f^{-1}\mathcal{O}^t_Y.
\end{aligned}$$

Here the third isomorphism is given by (7.4.4), and we have used the hypothesis that f is smooth and Proposition 5.3.7 to prove the last isomorphism. q.e.d.

Lemma 7.4.9. — *Let $f\colon X \to Y$ be a morphism of complex manifolds. There is a natural morphism in $D^b(\mathrm{I}(\beta\mathcal{D}_X))$:*

$$\beta\mathcal{D}_{X\to Y} \overset{L}{\underset{\beta f^{-1}\mathcal{D}_Y}{\otimes}} f^{-1}\mathcal{O}^t_Y \to \mathcal{O}^t_X,$$

or equivalently there is a morphism in $D^b(\mathrm{I}(\beta\mathcal{D}_X^{\mathrm{op}}))$*:*

$$f^{-1}\Omega_Y^t \overset{L}{\underset{\beta f^{-1}\mathcal{D}_Y}{\otimes}} \beta\mathcal{D}_{Y\leftarrow X} \to \Omega_X^t.$$

Proof. — When f is a closed embedding, we have already constructed this morphism and shown it is an isomorphism. Hence, we may assume that f is smooth. Then we have the morphisms

$$\begin{aligned} \beta\mathcal{D}_{X\to Y} \overset{L}{\underset{\beta f^{-1}\mathcal{D}_Y}{\otimes}} f^{-1}\mathcal{O}_Y^t &\simeq \beta\mathcal{D}_{X\to Y} \overset{L}{\underset{\beta f^{-1}\mathcal{D}_Y}{\otimes}} R\mathcal{I}hom_{\beta\mathcal{D}_X}(\beta\mathcal{D}_{X\to Y}, \mathcal{O}_X^t) \\ &\to \mathcal{O}_X^t. \end{aligned}$$

q.e.d.

Lemma 7.4.10. — *Let* $f\colon X \to Y$ *be a morphism of complex manifolds, and let* $\mathcal{M} \in D^b(\mathcal{D}_X)$*. Then there is a natural morphism in* $D^b(\mathrm{I}(\mathbb{C}_Y))$

$$\Omega_Y^t \overset{L}{\underset{\beta\mathcal{D}_Y}{\otimes}} \beta(\mathrm{D}f_!\mathcal{M}) \to Rf_{!!}(\Omega_X^t \overset{L}{\underset{\beta\mathcal{D}_X}{\otimes}} \beta\mathcal{M}). \tag{7.4.6}$$

Proof. — Consider the chain of morphisms

$$\begin{aligned} \Omega_Y^t \overset{L}{\underset{\beta\mathcal{D}_Y}{\otimes}} \beta(\mathrm{D}f_!\mathcal{M}) &\simeq \Omega_Y^t \overset{L}{\underset{\beta\mathcal{D}_Y}{\otimes}} \beta(Rf_!(\mathcal{D}_{Y\leftarrow X} \overset{L}{\underset{\mathcal{D}_X}{\otimes}} \mathcal{M})) \\ &\to \Omega_Y^t \overset{L}{\underset{\beta\mathcal{D}_Y}{\otimes}} Rf_{!!}(\beta(\mathcal{D}_{Y\leftarrow X} \overset{L}{\underset{\mathcal{D}_X}{\otimes}} \mathcal{M})) \\ &\simeq Rf_{!!}(f^{-1}\Omega_Y^t \overset{L}{\underset{\beta f^{-1}\mathcal{D}_Y}{\otimes}} \beta(\mathcal{D}_{Y\leftarrow X} \overset{L}{\underset{\mathcal{D}_X}{\otimes}} \mathcal{M})) \\ &\simeq Rf_{!!}((f^{-1}\Omega_Y^t \overset{L}{\underset{\beta f^{-1}\mathcal{D}_Y}{\otimes}} \beta\mathcal{D}_{Y\leftarrow X}) \overset{L}{\underset{\beta\mathcal{D}_X}{\otimes}} \beta\mathcal{M}) \\ &\to Rf_{!!}(\Omega_X^t \overset{L}{\underset{\beta\mathcal{D}_X}{\otimes}} \beta\mathcal{M}). \end{aligned}$$

q.e.d.

Proof of Theorem 7.4.6. The morphism is constructed in Lemma 7.4.10. To check that it is an isomorphism, take $G \in \mathbb{R}\text{-}\mathrm{C}^c(\mathbb{C}_Y)$. We have the chain of isomorphisms

$$\begin{aligned}
\mathrm{RHom}_{\mathrm{I}(\mathbb{C}_Y)}(G, \Omega_Y^t \overset{L}{\underset{\beta\mathcal{D}_Y}{\otimes}} \beta(\mathrm{D}f_!\mathcal{M})) &\simeq \mathrm{R}\Gamma_c(Y; T\mathcal{H}om\,(G, \Omega_Y) \overset{L}{\underset{\mathcal{D}_Y}{\otimes}} \mathrm{D}f_!\mathcal{M}) \\
&\simeq \mathrm{R}\Gamma_c(Y; Rf_!(T\mathcal{H}om\,(f^{-1}G, \Omega_X) \overset{L}{\underset{\mathcal{D}_X}{\otimes}} \mathcal{M})) \\
&\simeq \mathrm{R}\Gamma_c(X; T\mathcal{H}om\,(f^{-1}G, \Omega_X) \overset{L}{\underset{\mathcal{D}_X}{\otimes}} \mathcal{M}) \\
&\simeq \mathrm{RHom}_{\mathrm{I}(\mathbb{C}_X)}(f^{-1}G, \Omega_X^t \overset{L}{\underset{\beta\mathcal{D}_X}{\otimes}} \beta\mathcal{M}) \\
&\simeq \mathrm{RHom}_{\mathrm{I}(\mathbb{C}_Y)}(G, Rf_{!!}(\Omega_X^t \overset{L}{\underset{\beta\mathcal{D}_X}{\otimes}} \beta\mathcal{M})).
\end{aligned}$$

Here the second isomorphism is given by Theorem 7.3 in [**11**]. This completes the proof of Theorem 7.4.6.

Recall that if $F \in \mathrm{I}(k_X)$, there is a natural morphism $F \to \iota\alpha F$. Also recall that we do not write ι, for short.

Lemma 7.4.11. — *For $\mathcal{L} \in D^b_{r-h}(\mathcal{D}_X)$, the morphisms $R\mathcal{I}hom_{\beta\mathcal{D}_X}(\beta\mathcal{L}, \mathcal{O}_X^t) \to \alpha R\mathcal{I}hom_{\beta\mathcal{D}_X}(\beta\mathcal{L}, \mathcal{O}_X^t) \to R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{L}, \mathcal{O}_X)$ are isomorphisms.*

Proof. — Since these objects belong to $D^b_{\mathrm{I}\mathbb{R}-c}(\mathrm{I}(\mathbb{C}_X))$, the result follows from the chain of isomorphisms below, where $F \in \mathbb{R}\text{-}\mathrm{C}^c(\mathbb{C}_X)$.

$$\begin{aligned}
R\mathcal{H}om\,(F, R\mathcal{I}hom_{\beta\mathcal{D}_X}(\beta\mathcal{L}, \mathcal{O}_X^t)) &\simeq R\mathcal{H}om_{\mathrm{I}(\beta\mathcal{D}_X)}(\beta\mathcal{L}, R\mathcal{I}hom\,(F, \mathcal{O}_X^t)) \\
&\simeq R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{L}, \alpha R\mathcal{I}hom\,(F, \mathcal{O}_X^t)) \\
&\simeq R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{L}, R\mathcal{H}om\,(F, \mathcal{O}_X^t)) \\
&\simeq R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{L}, T\mathcal{H}om\,(F, \mathcal{O}_X)) \\
&\simeq R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{L}, R\mathcal{H}om\,(F, \mathcal{O}_X)) \\
&\simeq R\mathcal{H}om\,(F, R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{L}, \mathcal{O}_X)).
\end{aligned}$$

Here the fifth isomorphism follows from [**8**]. q.e.d.

Theorem 7.4.12. — *Let $\mathcal{L} \in D^b_{r-h}(\mathcal{D}_X)$, and set $L = R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{L}, \mathcal{O}_X)$. Then there exists a natural isomorphism in $D^b(\mathrm{I}(\mathbb{C}_X))$:*

$$\Omega_X^t \overset{L}{\underset{\beta\mathcal{O}_X}{\otimes}} \beta\mathcal{L} \xrightarrow{\sim} R\mathcal{I}hom\,(L, \Omega_X^t). \tag{7.4.7}$$

Proof. — First, let us construct the morphism. Since $L \simeq R\mathcal{I}hom_{\beta\mathcal{D}_X}(\beta\mathcal{L}, \beta\mathcal{O}_X)$ by the preceding lemma, we have the morphism $L \otimes \beta\mathcal{L} \to \beta\mathcal{O}_X$ from which we deduce the morphism:

$$\Omega^t_X \underset{\beta\mathcal{O}_X}{\overset{L}{\otimes}} \beta\mathcal{L} \otimes L \quad\to\quad \Omega^t_X \underset{\beta\mathcal{O}_X}{\overset{L}{\otimes}} \beta\mathcal{O}_X \simeq \Omega^t_X.$$

The morphism (7.4.7) is obtained by adjunction. To prove that it is an isomorphism, let us take $F \in \mathbb{R}\text{-}\mathrm{C}^c(\mathbb{C}_X)$. We have the chain of isomorphisms

$$\begin{aligned}
\mathrm{RHom}_{\mathrm{I}(\mathbb{C}_X)}(F, \Omega^t_X \underset{\beta\mathcal{O}_X}{\overset{L}{\otimes}} \beta\mathcal{L}) &\simeq \mathrm{R}\Gamma(X; T\mathcal{H}om\,(F, \Omega_X) \underset{\mathcal{O}_X}{\overset{L}{\otimes}} \mathcal{L})\\
&\simeq \mathrm{R}\Gamma(X; T\mathcal{H}om\,(F \otimes L, \Omega_X))\\
&\simeq \mathrm{RHom}_{\mathrm{I}(\mathbb{C}_X)}(F \otimes L, \Omega^t_X)\\
&\simeq \mathrm{RHom}_{\mathrm{I}(\mathbb{C}_X)}(F, R\mathcal{I}hom\,(L, \Omega^t_X)).
\end{aligned}$$

Here, the second isomorphism follows from a theorem of Björk [**2**] (see also Theorem 10.7 in [**11**]). q.e.d.

Consider three complex manifolds X, S, Y and a correspondence:

$$\begin{tikzcd} & S \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd}$$

Let $\mathcal{M} \in D^b_{q-good}(\mathcal{D}_X)$ and $\mathcal{L} \in D^b_{r-h}(\mathcal{D}_S)$. We set $L = R\mathcal{H}om_{\mathcal{D}_S}(\mathcal{L}, \mathcal{O}_S)$.
We make the hypothesis:

(7.4.8) $\qquad f^{-1}\operatorname{supp}(\mathcal{M}) \cap \operatorname{supp}(\mathcal{L})$ is proper over Y.

Let $G \in D^b(\mathrm{I}(\mathbb{C}_Y))$. We define

$$\begin{aligned}
\mathcal{M} \circ \mathcal{L} &= \mathrm{D}g_!(\mathrm{D}f^{-1}\mathcal{M} \underset{\mathcal{O}_S}{\overset{L}{\otimes}} \mathcal{L}),\\
L \circ G &= Rf_{!!}(L \otimes g^{-1}G).
\end{aligned}$$

Theorem 7.4.13. — *One has the isomorphism*

$$\mathrm{RHom}_{\mathrm{I}(\mathbb{C}_X)}(L \circ G, \Omega^t_X[d_X] \underset{\beta\mathcal{D}_X}{\overset{L}{\otimes}} \beta\mathcal{M})) \simeq \mathrm{RHom}_{\mathrm{I}(\mathbb{C}_Y)}(G, \Omega^t_Y[d_S] \underset{\beta\mathcal{D}_Y}{\overset{L}{\otimes}} \beta(\mathcal{M} \circ \mathcal{L})).$$

Proof. — We have the chain of isomorphisms

$$\begin{aligned}
&\mathrm{RHom}_{\mathrm{I}(\mathbb{C}_X)}\Big(Rf_{!!}(L\otimes g^{-1}G), \Omega^t_X[d_X]\overset{L}{\underset{\beta\mathcal{D}_X}{\otimes}}\beta\mathcal{M}\Big)\\
&\simeq \mathrm{RHom}_{\mathrm{I}(\mathbb{C}_S)}\Big(L\otimes g^{-1}G, f^!(\Omega^t_X[d_X]\overset{L}{\underset{\beta\mathcal{D}_X}{\otimes}}\beta\mathcal{M})\Big)\\
&\simeq \mathrm{RHom}_{\mathrm{I}(\mathbb{C}_S)}\Big(L\otimes g^{-1}G, \Omega^t_S[d_S]\overset{L}{\underset{\beta\mathcal{D}_S}{\otimes}}\beta(\mathrm{D}f^{-1}\mathcal{M})\Big)\\
&\simeq \mathrm{RHom}_{\mathrm{I}(\mathbb{C}_S)}\Big(g^{-1}G, R\mathcal{I}hom\,(L, \Omega^t_S[d_S]\overset{L}{\underset{\beta\mathcal{D}_S}{\otimes}}\beta(\mathrm{D}f^{-1}\mathcal{M}))\Big)\\
&\simeq \mathrm{RHom}_{\mathrm{I}(\mathbb{C}_S)}\Big(g^{-1}G, R\mathcal{I}hom\,(L, \Omega^t_S[d_S])\overset{L}{\underset{\beta\mathcal{D}_S}{\otimes}}\beta(\mathrm{D}f^{-1}\mathcal{M})\Big)\\
&\simeq \mathrm{RHom}_{\mathrm{I}(\mathbb{C}_S)}\Big(g^{-1}G, (\Omega^t_S[d_S]\overset{L}{\underset{\beta\mathcal{O}_S}{\otimes}}\beta\mathcal{L})\overset{L}{\underset{\beta\mathcal{D}_S}{\otimes}}\beta(\mathrm{D}f^{-1}\mathcal{M})\Big)\\
&\simeq \mathrm{RHom}_{\mathrm{I}(\mathbb{C}_Y)}\Big(G, Rg_*(\Omega^t_S[d_S]\overset{L}{\underset{\beta\mathcal{D}_S}{\otimes}}(\beta\mathcal{L}\overset{L}{\underset{\beta\mathcal{O}_S}{\otimes}}\beta\mathrm{D}f^{-1}\mathcal{M}))\Big)\\
&\simeq \mathrm{RHom}_{\mathrm{I}(\mathbb{C}_Y)}\Big(G, Rg_{!!}(\Omega^t_S[d_S]\overset{L}{\underset{\beta\mathcal{D}_S}{\otimes}}\beta(\mathcal{L}\overset{L}{\underset{\mathcal{O}_S}{\otimes}}\mathrm{D}f^{-1}\mathcal{M}))\Big)\\
&\simeq \mathrm{RHom}_{\mathrm{I}(\mathbb{C}_Y)}\Big(G, \Omega^t_Y\overset{L}{\underset{\beta\mathcal{D}_Y}{\otimes}}\beta(\mathrm{D}g_!(\mathcal{L}\overset{L}{\underset{\mathcal{O}_S}{\otimes}}\mathrm{D}f^{-1}\mathcal{M}))\Big)[d_S]\\
&\simeq \mathrm{RHom}_{\mathrm{I}(\mathbb{C}_Y)}\Big(G, \Omega^t_Y[d_S]\overset{L}{\underset{\beta\mathcal{D}_Y}{\otimes}}\beta(\mathcal{M}\circ\mathcal{L})\Big).
\end{aligned}$$

q.e.d.

Remark 7.4.14. — It could be interesting to study the sheaves on X_{sa} associated to the Sobolev presheaves on a real analytic manifold X, and to endow $\mathcal{O}^t_X$ with a Sobolev filtration on a complex manifold X.

Remark 7.4.15. — In this paper we have not considered the "microlocal" point of view, in the line of [**17**]. In a forthcoming paper, we shall apply the theory of ind-sheaves to Sato's microlocalization and construct a functor μ_X from ind-sheaves on X to ind-sheaves on T^*X. When applied to the ind-sheaves $\mathcal{O}^t_X$ or $\mathcal{O}^{\mathrm{w}}_X$, this will provide an alternative approach to the constructions of [**1**], [**3**]. We shall also study the micro-support of ind-sheaves, in the line of [**10**].

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