Elliptic Pairs II. Euler Class and Relative Index Theorem

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Contents

1 Introduction

In [14], we introduced the notion of an elliptic pair $(\mathcal{M},F)$ on a complex manifold X. Recall that this is the data of a (let us say, right) coherent $\mathcal{D}_X$-module $\mathcal{M}$ and an $\mathbb{R}$-constructible sheaf F (more precisely, objects of derived categories), these data satisfying:

$$\mathrm{char}(\mathcal{M}) \cap SS(F) \subset T^*_X X, \tag{1.1}$$

where $\mathrm{char}(\mathcal{M})$ is the characteristic variety of $\mathcal{M}$, $SS(F)$ is the micro-support of F, (defined in [7]), and $T^*_X X$ is the zero-section of the cotangent bundle to X. More generally, if $f : X \longrightarrow Y$ is a morphism of complex manifolds, we defined the notion of an f-elliptic pair, replacing in (1.1) $\mathrm{char}(\mathcal{M})$ by $\mathrm{char}_f(\mathcal{M})$, the relative characteristic variety.

In [14], we give four basic results on elliptic pairs: we prove a finiteness theorem (coherence of the direct images of $F \otimes \mathcal{M}$, assuming $(\mathcal{M}, F)$ is an f-elliptic pair with proper support), a duality theorem (in the above situation, duality commutes with direct images), a Künneth formula and we prove that microlocalization commutes with direct images.

In this second paper on elliptic pairs, expanding results announced in [12, 13], we will attach a cohomology class to $(\mathcal{M}, F)$ and prove an index formula. More precisely, let $\Lambda_0 = \mathrm{char}(\mathcal{M})$, $\Lambda_1 = SS(F)$, let $d_X = \dim_{\mathbb{C}} X$ and denote by ω_X the dualizing complex on X (hence $\omega_X \simeq \mathbb{C}_X[2d_X]$, since X is oriented). Assuming $(\mathcal{M}, F)$ is elliptic, we construct a cohomology class:

$$\mu\mathrm{eu}(\mathcal{M}, F) \in H^{2d_X}_{\Lambda_0+\Lambda_1}(T^*X; \mathbb{C}_{T^*X}) \qquad (\simeq H^0_{\Lambda_0+\Lambda_1}(T^*X; \pi^{-1}\omega_X))$$

that we call the "microlocal Euler class" of $(\mathcal{M}, F)$. This class is constructed using a diagonal procedure, like in the proof of the Lefschetz formula for constructible sheaves by Kashiwara [6] (see also [7, Chapter IX]), but working here in the framework of $\mathcal{D}$-modules. Set for short:

$$\begin{aligned} \mu\mathrm{eu}(\mathcal{M}) &= \mu\mathrm{eu}(\mathcal{M}, \mathbb{C}_X), \\ \mu\mathrm{eu}(F) &= \mu\mathrm{eu}(\Omega_X, F). \end{aligned}$$

Then the two main results of this paper may be stated as follows.

1) One has the formula:

$$\mu\mathrm{eu}(\mathcal{M}, F) = \mu\mathrm{eu}(\mathcal{M}) *_\mu \mu\mathrm{eu}(F), \tag{1.2}$$

where the operation $*_\mu$:

$$H^0_{\Lambda_0}(T^*X; \pi^{-1}\omega_X) \times H^0_{\Lambda_1}(T^*X; \pi^{-1}\omega_X) \longrightarrow H^0_{\Lambda_0+\Lambda_1}(T^*X; \pi^{-1}\omega_X)$$

is defined by integration along the fibers of the map:

$$s : T^*X \times_X T^*X \longrightarrow T^*X, \quad s(x; \xi_1, \xi_2) = (x; \xi_1 + \xi_2)$$

(this map is proper, thanks to the ellipticity hypothesis).

2) Assume $(\mathcal{M}, F)$ is f-elliptic with proper support. One knows by [14] that $\underline{f}_!(F \otimes \mathcal{M})$ is $\mathcal{D}_Y$-coherent, and we prove the formula:

$$\mu\mathrm{eu}(\underline{f}_!(F \otimes \mathcal{M})) = f_\mu\, \mu\mathrm{eu}(\mathcal{M}, F), \tag{1.3}$$

where f_μ is the morphism:

$$H^0_{\Lambda_0+\Lambda_1}(T^*X; \pi^{-1}\omega_X) \longrightarrow H^0_{f_\pi {}^t f'^{-1}(\Lambda_0+\Lambda_1)}(T^*Y; \pi^{-1}\omega_Y)$$

deduced from the integration morphism $Rf_!\omega_X \longrightarrow \omega_Y$, (see [7, Chapter IX, §3]).

These two theorems will be proved along the same lines as the corresponding results for constructible sheaves (see [7]). We will use various commutative diagrams in derived categories to express the compatibility of the functors involved, and as usual in these matters we do not distinguish between commutative and anti-commutative diagrams. Hence the results should be understood up to sign.

Using these two formulas, we find in particular that if $(\mathcal{M}, F)$ is an elliptic pair with compact support, then:

$$\chi(R\Gamma(X; F \otimes \mathcal{M} \otimes^L_{\mathcal{D}_X} \mathcal{O}_X)) = \int_{T^*X} \mu\mathrm{eu}(\mathcal{M}) \cup \mu\mathrm{eu}(F) \tag{1.4}$$

where $\chi(\cdot)$ denotes the Euler-Poincaré index and $\cup$ the cup product.

If M is a real analytic compact manifold and X is a complexification of M, then $(\mathcal{M}, \mathbb{C}_M)$ is an elliptic pair if and only if $\mathcal{M}$ is elliptic on M in the usual sense. Hence formula (1.4) is similar to the Atiyah-Singer formula [1].

By formula (1.2), we see that to compute $\mu\mathrm{eu}(\mathcal{M}, F)$, it is enough to compute separately $\mu\mathrm{eu}(\mathcal{M})$ and $\mu\mathrm{eu}(F)$. It is easily shown that $\mu\mathrm{eu}(F)$ is nothing but the "characteristic cycle" of F constructed by Kashiwara (loc. cit.). This is a Lagrangian cycle whose calculation is made at generic points and thus offers no difficulties (see [7, Chapter IX, §3]). Hence the remaining problem is to understand $\mu\mathrm{eu}(\mathcal{M})$. At this step our results are essentially conjectural. Assume $\mathcal{M}$ is endowed with a good filtration and denote by $\sigma_\Lambda(\mathcal{M})$ the image of $gr(\mathcal{M})$ in the Grothendieck group of coherent $\mathcal{O}_{T^*X}$-modules supported by Λ, the characteristic variety of $\mathcal{M}$. In the last section we make the two following conjectures (1.5) and (1.6) below:

$$[ch_\Lambda(\sigma_\Lambda(\mathcal{M})) \cup \pi^* td_X(TX)]^j = 0 \quad \text{for} \quad j > 2d_X \tag{1.5}$$

where $ch_\Lambda(\cdot)$ and $td_X(TX)$ denote as usual the local Chern character with support in Λ and the Todd class of X, respectively, and $[\cdot]^j$ is the homogeneous part of degree j in $\oplus_k H^k_\Lambda(T^*X; \mathbb{C}_{T^*X})$,

$$\mu\mathrm{eu}(\mathcal{M}) = [ch_\Lambda(\sigma_\Lambda(\mathcal{M})) \cup \pi^* td_X(TX)]^{2d_X}. \tag{1.6}$$

As an evidence for these conjectures, we prove that both sides of (1.6) are compatible to proper direct images, external products and non-characteristic inverse images, and moreover they coincide in the two extreme cases where $\mathcal{M}$ is holonomic or is induced by a coherent $\mathcal{O}_X$-module.

The Atiyah-Singer theorem, in its K-theoretical version, has recently been generalized to the relative case by Boutet de Monvel and Malgrange [3]. Our results provide a relative index formula in the cohomological setting, and the proof of the above conjectures would give a precise link with the Atiyah-Singer theorem. We hope to come back to these conjectures in a next future.

2 Review on sheaves

In this section, we fix some notations and recall a few results of [7].

Let X be a real analytic manifold. One denotes by $\tau : TX \longrightarrow X$ and $\pi : T^*X \longrightarrow X$ the tangent and cotangent bundles to X, respectively. If Y is a submanifold of X, one denotes by T_YX and T^*_YX the normal and conormal bundles to Y in X, respectively. In particular, T^*_XXdenotes the zero-section of T^*X, that ones identifies to X. If Λ is a subset of T^*X, one denotes by Λ^a its image by the antipodal map.

One denotes by $\delta : X \hookrightarrow X \times X$ the diagonal embedding, and we identify X to its image Δ and T^*X to $T^*_\Delta(X \times X)$ by the first projection defined on $X \times X$ and $T^*(X \times X) \simeq T^*X \times T^*X$, respectively.

If X and Y are two manifolds, one denotes by q_1 and q_2 the first and second projection defined on $X \times Y$.

One denotes by $\mathbf{D}(X)$ the derived category of the category of sheaves of $\mathbb{C}$-vector spaces, and by $\mathbf{D}^{\mathrm{b}}(X)$ the full triangulated subcategory consisting of objects with bounded cohomology. If Z is a subset of X, one denotes by $\mathbb{C}_Z$ the sheaf on X which is constant with stalk $\mathbb{C}$ on Z and zero on $X \setminus Z$.

One denotes by or_X the orientation sheaf on X and by ω_X the dualizing complex on X. Hence:

$$\omega_X \simeq \mathrm{or}_X[\dim X]$$

where $\dim X$ is the real dimension of X. More generally, if f is a morphism from X to Y, one denotes by $\omega_{X/Y}$ the relative dualizing complex. Hence:

$$\omega_{X/Y} \simeq \omega_X \otimes f^{-1}\omega_Y^{\otimes -1}.$$

One denotes by $f^{-1}, Rf_*, Rf_!, f^!, \otimes, R\mathcal{H}om$ the usual classical operations on sheaves and we denote by $\boxtimes$ the external product. We shall use the two duality functors:

$$D'_X F = R\mathcal{H}om(F, \mathbb{C}_X), \tag{2.1}$$
$$D_X F = R\mathcal{H}om(F, \omega_X). \tag{2.2}$$

If there is no risk of confusion, we write D' or D instead of D'_X or D_X.

If F is an object of $\mathbf{D}^{\mathrm{b}}(X)$, one denotes by $SS(F)$ its micro-support, defined in [7], a closed conic involutive subset of T^*X. Moreover, we shall use the functor μ_M of Sato's microlocalization along M. Recall that for F in $\mathbf{D}^{\mathrm{b}}(X)$

$$\mathrm{supp}\,\mu_M(F) \subset T^*_MX \cap SS(F).$$

Now, recall that an object F of $\mathbf{D}^{\mathrm{b}}(X)$ is called weakly $\mathbb{R}$-constructible (w-$\mathbb{R}$-constructible, for short) if there is a subanalytic stratification $X = \bigsqcup_\alpha X_\alpha$ such that for all α, all j, the sheaves $H^j(F)_{|X_\alpha}$ are locally constant. If moreover, for each $x \in X$, each $j \in \mathbb{Z}$, the stalk $H^j(F)_x$ is finite dimensional, one says that F is $\mathbb{R}$-constructible. One denotes by $\mathbf{D}^{\mathrm{b}}_{w-\mathbb{R}-c}(X)$ (resp. $\mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(X)$) the full triangulated subcategory of $\mathbf{D}^{\mathrm{b}}(X)$ consisting of w-$\mathbb{R}$-constructible (resp. $\mathbb{R}$-constructible) objects. It follows from the involutivity of the micro-support that F is w-$\mathbb{R}$-constructible if and only if $SS(F)$ is a closed conic subanalytic Lagrangian subset of T^*X.

Let $f : X \longrightarrow Y$ be a morphism of real analytic manifolds. To f one associates the maps:

$$TX \underset{f'}{\longrightarrow} X \times_Y TY \underset{f_\tau}{\longrightarrow} TY, \tag{2.3}$$
$$T^*X \underset{{}^tf'}{\longleftarrow} X \times_Y T^*Y \underset{f_\pi}{\longrightarrow} T^*Y. \tag{2.4}$$

One says that f is non-characteristic with respect to a closed conic subset Λ of T^*Y if:

$$f_\pi^{-1}(\Lambda) \cap {}^tf'^{-1}(T^*_XX) \subset X \times_Y T^*_YY. \tag{2.5}$$

Let $F \in \mathbf{D}^{\mathrm{b}}(X)$, $G \in \mathbf{D}^{\mathrm{b}}(Y)$. Recall that:

(i) if f is non-characteristic with respect to $SS(G)$, then:

$$SS(f^{-1}G) \subset {}^tf' f_\pi^{-1} SS(G), \tag{2.6}$$

(ii)) if f is proper on $\mathrm{supp}(F)$, then:

$$SS(Rf_*F) \subset f_\pi {}^tf'^{-1}(SS(F)), \tag{2.7}$$

(iii) one has:

$$SS(F \boxtimes G) \subset SS(F) \times SS(G). \tag{2.8}$$

Finally, let us recall some microlocal constructions of [7, Chapter IX] that we shall use.

Let Λ_X and Λ_Y be two closed conic subsets of T^*X and T^*Y, respectively, and consider the diagram:

$$\begin{array}{ccccc} T^*X & \xleftarrow{{}^tf'} & X \times_Y T^*Y & \xrightarrow{f_\pi} & T^*Y \\ \downarrow{\scriptstyle \pi_X} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_Y} \\ X & = & X & \xrightarrow{f} & Y \end{array}$$

Set for short:

$$\begin{aligned} f_\mu(\Lambda_X) &= f_\pi {}^tf'^{-1}(\Lambda_X), & (2.9)\\ f^\mu(\Lambda_Y) &= {}^tf' f_\pi^{-1}(\Lambda_Y). & (2.10) \end{aligned}$$

a) Assume f is proper on $T^*_XX \cap \Lambda_X$, (or equivalently, f_π is proper on Λ_X). Using the morphism:

$$Rf_{\pi !}\pi^{-1}\omega_X \longrightarrow \pi_Y^{-1}Rf_!\omega_X \longrightarrow \pi_Y^{-1}\omega_Y, \tag{2.11}$$

we get the morphisms, for all $j \in \mathbb{Z}$:

$$\begin{aligned} f_\mu : H^j_{\Lambda_X}(T^*X; \pi^{-1}\omega_X) &\longrightarrow H^j_{{}^tf'^{-1}(\Lambda_X)}(X \times_Y T^*Y; \pi^{-1}\omega_X) \\ &\longrightarrow H^j_{f_\mu(\Lambda_X)}(T^*Y; \pi^{-1}\omega_Y). & (2.12) \end{aligned}$$

b) Assume f is non-characteristic for Λ_Y (i.e., ${}^tf'$ is proper on $f_\pi^{-1}(\Lambda_Y)$). Using the natural morphism (see [7]):

$$R^tf'_!\pi^{-1}f^{-1}\omega_Y \longrightarrow \pi_X^{-1}\omega_X, \tag{2.13}$$

we get for all $j\in\mathbb{Z}$, the morphisms:

$$\begin{aligned} f^\mu : H^j_{\Lambda_Y}(T^*Y;\pi^{-1}\omega_Y) &\longrightarrow H^j_{f_\pi^{-1}(\Lambda_Y)}(X\times_Y T^*Y;\pi^{-1}f^{-1}\omega_Y) \\ &\longrightarrow H^j_{f^\mu(\Lambda_Y)}(T^*X;\pi^{-1}\omega_X). \end{aligned} \tag{2.14}$$

Note that the morphism (2.13) may also be obtained as follows. On a manifold Z, there is a natural isomorphism: $\pi_Z^{-1}\omega_Z\simeq\omega_{T^*Z/Z}$. Hence we have the chain of morphisms:

$$\begin{aligned} R^tf'_!\pi^{-1}f^{-1}\omega_Y &\simeq R^tf'_!f_\pi^{-1}\omega_{T^*Y/Y} \\ &\simeq R^tf'_!\omega_{X\times_YT^*Y/X} \\ &\longrightarrow \omega_{T^*X/X} \\ &\simeq \pi_X^{-1}\omega_X. \end{aligned}$$

c) Using the natural isomorphism:

$$\omega_X\boxtimes\omega_Y\simeq\omega_{X\times Y},$$

we get the morphism:

$$\begin{aligned} \boxtimes : H^j_{\Lambda_X}(T^*X;\pi^{-1}\omega_X)\times H^k_{\Lambda_Y}(T^*Y;\pi^{-1}\omega_Y) \\ \longrightarrow H^{j+k}_{\Lambda_X\times\Lambda_Y}(T^*X\times Y;\pi^{-1}\omega_{X\times Y}). \end{aligned} \tag{2.15}$$

d) Let Λ_0 and Λ_1 be two closed conic subsets of T^*X satisfying:

$$\Lambda_0^a\cap\Lambda_1\subset T^*_XX. \tag{2.16}$$

Setting:

$$*_\mu=\delta^\mu\circ\boxtimes$$

we get a morphism:

$$*_\mu : H^j_{\Lambda_0}(T^*X;\pi^{-1}\omega_X)\times H^k_{\Lambda_1}(T^*X;\pi^{-1}\omega_X)\longrightarrow H^{j+k}_{\Lambda_0+\Lambda_1}(T^*X;\pi^{-1}\omega_X). \tag{2.17}$$

Note that the morphism $*_\mu$ (which is not the cup-product) may also be defined as the composite of:

$$\begin{aligned} H^j_{\Lambda_0}(T^*X;\pi^{-1}\omega_X)\times H^k_{\Lambda_1}(T^*X;\pi^{-1}\omega_X) \\ \underset{\delta_\pi^*}{\longrightarrow} H^{j+k}_{\Lambda_0\times_X\Lambda_1}(T^*X\times_XT^*X;\pi^{-1}\omega_X\otimes\omega_X) \\ \underset{{}^t\delta'_*}{\longrightarrow} H^{j+k}_{\Lambda_0+\Lambda_1}(T^*X;\pi^{-1}\omega_X) \end{aligned} \tag{2.18}$$

where δ_π^* is associated to the embedding $T^*X\times_XT^*X\underset{\delta_\pi}{\longrightarrow}T^*X\times T^*X$ and ${}^t\delta'_*$ to the map

$$T^*X\times_XT^*X\underset{{}^t\delta'}{\longrightarrow}T^*X,\quad (x;\xi_1,\xi_2)\mapsto(x;\xi_1+\xi_2).$$

3 Euler class of elliptic pairs

From now on, all manifolds and morphisms of manifolds are complex analytic. If X is a complex manifold, we shall often identify X and $X^{\mathbb{R}}$, the real analytic underlying manifold. We shall also identify $(T^*X)^{\mathbb{R}}$ with $T^*X^{\mathbb{R}}$, as in [7]. We denote by d_X the complex dimension of X. Hence,

$$\dim X^{\mathbb{R}} = 2d_X.$$

Since X is oriented, we identify the orientation sheaf or_X with the constant sheaf $\mathbb{C}_X$, and the dualizing complex ω_X with $\mathbb{C}_X[2d_X]$.

We denote by $\mathcal{O}_X$ the sheaf of holomorphic functions on X, by Ω_X the sheaf of holomorphic d_X-forms and by $\mathcal{D}_X$ the sheaf of rings of (finite order) holomorphic differential operators on X. If Y is another complex manifold and if $\mathcal{F}$ is a sheaf of $\mathcal{O}_{X\times Y}$-modules, one sets:

$$\mathcal{F}^{(0,d_Y)} = \mathcal{F} \otimes_{q_2^{-1}\mathcal{O}_Y} q_2^{-1}\Omega_Y,$$

and one defines similarly $\mathcal{F}^{(d_X,0)}$ or $\mathcal{F}^{(d_X,d_Y)}$.

We shall follow the notations of [7] for $\mathcal{D}$-modules. In particular, $\mathrm{Mod}(\mathcal{D}_X)$ denotes the category of left $\mathcal{D}_X$-modules, $\mathbf{D}(\mathcal{D}_X)$ its derived category, and $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X)$ the full triangulated subcategory of $\mathbf{D}(\mathcal{D}_X)$ consisting of complexes with bounded and coherent cohomology. Replacing $\mathcal{D}_X$ by $\mathcal{D}_X^{op}$, we have similar notations for right $\mathcal{D}_X$-modules. In fact, if there is no risk of confusion, we shall often make no differences between right and left $\mathcal{D}$-modules and write $\mathcal{D}_X$ instead of $\mathcal{D}_X^{op}$.

In the sequel, we will often need to work with bimodule structures. Let k be a field. Recall that if A and B are k-algebras, giving a left (A,B)-bimodule structure on an abelian group M is equivalent to give M a structure of a left $A \otimes_k B$-module. Using this point of view it is easy to extend to bimodules the notions and notations defined usually for modules. For example, we will denote by $\mathrm{Mod}(\mathcal{D}_{X|S} \otimes \mathcal{D}_{X|S})$ the category of left $\mathcal{D}_{X|S}$-bimodules and by $\mathbf{D}(\mathcal{D}_{X|S} \otimes \mathcal{D}_{X|S})$ the corresponding derived category.

The characteristic variety of an object $\mathcal{M}$ of $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X)$ is denoted by $\mathrm{char}(\mathcal{M})$. This is a closed conic involutive analytic subset of T^*X [10], and we have the formula [7, Theorem 11.3.3]:

$$\mathrm{char}(\mathcal{M}) = SS(\mathcal{M} \otimes^L_{\mathcal{D}_X} \mathcal{O}_X). \tag{3.1}$$

As usual, one denotes by $\mathcal{B}_{Z|X}$ the simple holonomic left $\mathcal{D}_X$-module associated to a closed complex submanifold Z of X. We denote by $\underline{f}^{-1}$, $\underline{f}_!$, $\underline{\boxtimes}$ the operations of inverse image, proper direct image, and external product for $\mathcal{D}$-modules, and we denote by $\underline{D}_X$ the dualizing functor. Recall that if $\mathcal{M}$ is a right $\mathcal{D}_X$-module, then

$$\underline{D}_X(\mathcal{M}) = R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{K}_X)$$

where

$$\mathcal{K}_X = \Omega_X[d_X] \otimes_{\mathcal{O}_X} \mathcal{D}_X$$

as a right $\mathcal{D}_X \otimes \mathcal{D}_X$-module. Notice the isomorphism of $\mathcal{D}_X \otimes \mathcal{D}_X^{\mathrm{op}}$-modules:

$$\delta_! \mathcal{D}_X \simeq \mathcal{B}^{(0,d_X)}_{\Delta|X\times X}, \tag{3.2}$$

which induces the isomorphism of $\mathcal{D}_X^{\mathrm{op}} \otimes \mathcal{D}_X^{\mathrm{op}}$-modules:

$$\delta_! \mathcal{K}_X \simeq \mathcal{B}^{(d_X,d_X)}_{\Delta|X\times X}.$$

By this isomorphism, $\mathcal{K}_X$ is naturally endowed with a structure of a right $\delta^{-1}\mathcal{D}_{X\times X}$-module and

$$\delta_! \mathcal{K}_X = \underline{\delta}_! \Omega_X [d_X].$$

Let us recall the notion of an elliptic pair introduced in [14].

Definition 3.1 An elliptic pair $(\mathcal{M}, F)$ on X is the data of $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X^{op})$ and $F \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(X)$ satisfying:

$$\mathrm{char}(\mathcal{M}) \cap SS(F) \subset T^*_X X.$$

The same definition holds for left $\mathcal{D}_X$-modules.

Proposition 3.2 *Let $(\mathcal{M}, F)$ be an elliptic pair on X. Then there are canonical morphisms:*

(i) $\delta_! R\mathcal{H}om_{\mathcal{D}_X}(F \otimes \mathcal{M}, F \otimes \mathcal{M}) \longrightarrow (F \otimes \mathcal{M}) \underline{\boxtimes} (D'F \otimes \underline{D}\mathcal{M}) \otimes^L_{\mathcal{D}_{X\times X}} \mathcal{O}_{X\times X}$,

(ii) $F \otimes \mathcal{M} \underline{\boxtimes} D'F \otimes \underline{D}\mathcal{M} \otimes^L_{\mathcal{D}_{X\times X}} \mathcal{O}_{X\times X} \longrightarrow \delta_! \omega_X$.

Proof: (i) Let $\mathcal{D}_X^\infty$ denote the ring of infinite order holomorphic differential operators. Sato's isomorphism:

$$\mathcal{D}_X^\infty \simeq \delta^! \mathcal{O}^{(0,d_X)}_{X\times X}[d_X]$$

entails the morphism:

$$\delta_! \mathcal{D}_X \longrightarrow \mathcal{O}^{(0,d_X)}_{X\times X}[d_X]. \tag{3.3}$$

Set for short:

$$\mathcal{P} = F \otimes \mathcal{M}.$$

Applying the functor $q_1^{-1}\mathcal{P} \otimes^L_{q_1^{-1}\mathcal{D}_X} \cdot$ to (3.3), then the functor $R\mathcal{H}om_{q_2^{-1}\mathcal{D}_X}(q_2^{-1}\mathcal{P}, \cdot)$, and using the isomorphism:

$$\delta_! R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{P}, \mathcal{P}) \simeq R\mathcal{H}om_{q_2^{-1}\mathcal{D}_X}(q_2^{-1}\mathcal{P}, \delta_! \mathcal{P}),$$

we get the morphism:

$$\delta_! R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{P}, \mathcal{P}) \longrightarrow R\mathcal{H}om_{q_2^{-1}\mathcal{D}_X}(q_2^{-1}\mathcal{P}, q_1^{-1}\mathcal{P} \otimes^L_{q_1^{-1}\mathcal{D}_X} \mathcal{O}^{(0,d_X)}_{X\times X}[d_X]).$$

Then:

$$
\begin{aligned}
& R\mathcal{H}om_{q_2^{-1}\mathcal{D}_X}(q_2^{-1}\mathcal{P}, q_1^{-1}\mathcal{P} \otimes^L_{q_1^{-1}\mathcal{D}_X} \mathcal{O}^{(0,d_X)}_{X\times X}[d_X]) \\
&\quad \simeq \ R\mathcal{H}om(q_2^{-1}F, R\mathcal{H}om_{q_2^{-1}\mathcal{D}_X}(q_2^{-1}\mathcal{M}, q_1^{-1}\mathcal{P} \otimes^L_{q_1^{-1}\mathcal{D}_X} \mathcal{O}^{(0,d_X)}_{X\times X}[d_X])) \\
&\quad \simeq \ R\mathcal{H}om(q_2^{-1}F, \mathcal{P} \underline{\boxtimes}\, \underline{D}_X\mathcal{M} \otimes^L_{\mathcal{D}_X \boxtimes \mathcal{D}_X} \mathcal{O}_{X\times X}) \\
&\quad \simeq \ R\mathcal{H}om(q_2^{-1}F, \mathcal{P} \underline{\boxtimes}\, \underline{D}_X\mathcal{M} \otimes^L_{\mathcal{D}_{X\times X}} \mathcal{O}_{X\times X}).
\end{aligned}
$$

The micro-support of $\mathcal{P} \underline{\boxtimes}\, \underline{D}_X\mathcal{M} \otimes^L_{\mathcal{D}_{X\times X}} \mathcal{O}_{X\times X}$ is contained in $T^*X \times \mathrm{char}(\mathcal{M})$, hence it intersects $SS(q_2^{-1}F)$ inside the zero-section of $T^*(X \times X)$. Using [7, Prop. 5.4.14], we get the isomorphisms:

$$
\begin{aligned}
& R\mathcal{H}om(q_2^{-1}F, \mathcal{P} \underline{\boxtimes}\, \underline{D}_X\mathcal{M} \otimes^L_{\mathcal{D}_{X\times X}} \mathcal{O}_{X\times X}) \\
&\quad \xleftarrow{\sim} \ q_2^{-1}D'F \otimes \left[\mathcal{P} \underline{\boxtimes}\, \underline{D}_X\mathcal{M} \otimes^L_{\mathcal{D}_{X\times X}} \mathcal{O}_{X\times X}\right] \\
&\quad \xleftarrow{\sim} \ (F \otimes \mathcal{M}) \underline{\boxtimes} (D'F \otimes \underline{D}\mathcal{M}) \otimes^L_{\mathcal{D}_{X\times X}} \mathcal{O}_{X\times X}.
\end{aligned}
$$

(ii) Set for short:

$$
L_X = (F \otimes \mathcal{M}) \underline{\boxtimes} (D'F \otimes \underline{D}\mathcal{M}) \otimes^L_{\mathcal{D}_{X\times X}} \mathcal{O}_{X\times X} \tag{3.4}
$$

Using the $\mathcal{D}_{X\times X}$-linear morphism:

$$
F \otimes \mathcal{M} \underline{\boxtimes}\, D'F \otimes \underline{D}\mathcal{M} \longrightarrow \delta_! \mathcal{K}_X,
$$

we get the sequence of morphisms:

$$
\begin{aligned}
L_X \ &\longrightarrow \ \delta_!\mathcal{K}_X \otimes^L_{\mathcal{D}_{X\times X}} \mathcal{O}_{X\times X} \\
&\simeq \ \underline{\delta}_!\Omega_X[d_X] \otimes^L_{\mathcal{D}_{X\times X}} \mathcal{O}_{X\times X} \\
&\simeq \ \delta_!\left[\Omega_X[d_X] \otimes^L_{\mathcal{D}_X} \mathcal{D}_{X\to X\times X} \otimes^L_{\delta^{-1}\mathcal{D}_{X\times X}} \delta^{-1}\mathcal{O}_{X\times X}\right] \\
&\simeq \ \delta_!\Omega_X[d_X] \otimes^L_{\mathcal{D}_X} \mathcal{O}_X \\
&\simeq \ \delta_!\omega_X.
\end{aligned}
$$

□

Using the morphisms defined in the preceding proposition, we can now construct the microlocal Euler class of the elliptic pair $(\mathcal{M}, F)$. Set:

$$
\Lambda = \mathrm{char}(\mathcal{M}) + SS(F)
$$

Then $SS(L_X) \subset \Lambda \times \Lambda^a$ where L_X is defined in (3.4), and

$$
\mathrm{supp}(\mu_\Delta L_X) \subset \Lambda.
$$

By paraphrasing Kashiwara's construction of the characteristic cycle of $\mathbb{R}$-constructible sheaves, [6], we obtain the sequence of morphisms:

$$\begin{array}{rcl} R\mathcal{H}om_{\mathcal{D}_X}(F\otimes\mathcal{M},F\otimes\mathcal{M}) & \longrightarrow & \delta^! L_X \\ & \simeq & R\pi_*\mu_\Delta L_X \\ & \simeq & R\pi_* R\Gamma_\Lambda \mu_\Delta L_X \\ & \longrightarrow & R\pi_* R\Gamma_\Lambda \mu_\Delta \delta_!\omega_X \\ & \simeq & R\pi_* R\Gamma_\Lambda \pi^{-1}\omega_X. \end{array}$$

Applying $H^0R\Gamma(X;\cdot)$, we find the morphism:

$$\mathrm{Hom}_{\mathcal{D}_X}(F\otimes\mathcal{M},F\otimes\mathcal{M}) \longrightarrow H^0_\Lambda(T^*X;\pi^{-1}\omega_X). \tag{3.5}$$

(Recall that

$$H^0(T^*X;\pi^{-1}\omega_X)\simeq H^{2d_X}_\Lambda(T^*X;\mathbb{C}_{T^*X}).)$$

Definition 3.3 Let $(\mathcal{M},F)$ be an elliptic pair. The image of $\mathrm{id}_{F\otimes\mathcal{M}}$ by the morphism (3.5) is the *microlocal Euler class of* $(\mathcal{M},F)$

$$\mu\mathrm{eu}(\mathcal{M},F)\in H^0_{\mathrm{char}(\mathcal{M})+SS(F)}(T^*X;\pi^{-1}\omega_X)$$

Its restriction to the zero-section of T^*X is the *Euler class of* $(\mathcal{M},F)$

$$\mathrm{eu}(\mathcal{M},F)\in H^0_{\mathrm{supp}(\mathcal{M})\cap\mathrm{supp}(F)}(X;\omega_X)$$

If $\mathcal{M}$ is a left $\mathcal{D}_X$-module, we define the microlocal Euler class of $(\mathcal{M},F)$ as being that of $(\Omega_X\otimes_{\mathcal{O}_X}\mathcal{M},F)$. We also introduce the following notations. For $\mathcal{M}\in\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X)$ and $F\in\mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(X)$, we set:

$$\begin{array}{rcl} \mu\mathrm{eu}(\mathcal{M}) & = & \mu\mathrm{eu}(\mathcal{M},\mathbb{C}_X), \\ \mu\mathrm{eu}(F) & = & \mu\mathrm{eu}(\Omega_X,F). \end{array}$$

4 The product formula

Let $(\mathcal{M}$,$F)$ be an elliptic pair on the complex manifold X. Set:

$$\Lambda_0=\mathrm{char}(\mathcal{M}),\quad \Lambda_1=SS(F).$$

Then:

$$\begin{array}{rcl} \mu\mathrm{eu}(\mathcal{M}) & \in & H^0_{\Lambda_0}(T^*X;\pi^{-1}\omega_X), \\ \mu\mathrm{eu}(F) & \in & H^0_{\Lambda_1}(T^*X;\pi^{-1}\omega_X), \\ \mu\mathrm{eu}(\mathcal{M},F) & \in & H^0_{\Lambda_0+\Lambda_1}(T^*X;\pi^{-1}\omega_X). \end{array}$$

The operation $*_\mu$ being that defined in §2, the aim of this section is to prove:

Theorem 4.1 *Let $(\mathcal{M}, F)$ be an elliptic pair. Then:*

$$\mu\mathrm{eu}(\mathcal{M}, F) = \mu\mathrm{eu}(\mathcal{M}) *_\mu \mu\mathrm{eu}(F)$$

The proof decomposes into several steps. In Proposition 4.2 below, and its proof, X will denote a real analytic manifold. In this statement and its proof as well as in the proof of Theorem 5.1, we shall not write the symbol "R" of derived functors, for short; e.g. $\mathcal{H}om(\cdot,\cdot)$ means $R\mathcal{H}om(\cdot,\cdot)$, π_* means $R\pi_*$, etc.

We denote by X_i ($i = 1, 2, 3, 4$) a copy of X and we write

$$(X \times X) \times (X \times X) = X_1 \times X_2 \times X_3 \times X_4.$$

For $J \subset \{1, 2, 3, 4\}$ and any set Z, we introduce the notation

$$\delta_{ij} : \prod_{\ell \in J \setminus \{j\}} X_\ell \times Z \longrightarrow \prod_{\ell \in J} X_\ell \times Z$$

for the diagonal embedding sending $(x_\ell)_{\ell \in J\setminus\{j\}}$ to $(x_\ell)_{\ell \in J}$ with $x_j = x_i$. Similarly, we introduce the notation

$$\delta_{ijk} : \prod_{\ell \in J \setminus \{j,k\}} X_\ell \times Z \longrightarrow \prod_{\ell \in J} X_\ell \times Z$$

for the diagonal embedding sending $(x_\ell)_{\ell \in J\setminus\{j,k\}}$ to $(x_\ell)_{\ell \in J}$ with $x_j = x_k = x_i$. If there is no risk of confusion, we simply write δ for any of these morphisms.

On a product, we denote by q_i the projection to X_i.

We shall make a frequent use of the morphism of functors

$$\delta_{ij}^{-1} \longrightarrow \delta_{ij}^{!} \otimes \omega_X. \tag{4.1}$$

Now, we assume to be given:

$$F \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(X), \quad G \in \mathbf{D}^{\mathrm{b}}(X), \quad H \in \mathbf{D}^{\mathrm{b}}(X \times X).$$

We set:

$$K = G \boxtimes DF$$

$$\tilde{\Lambda}_0 = SS(K), \quad \tilde{\Lambda}_1 = SS(H)$$

$$\Lambda_i = T^*_\Delta(X \times X) \cap \tilde{\Lambda}_i, \quad i = 0, 1$$

We identify $T^*_\Delta(X \times X)$ to T^*X by the first projection. We shall assume:

$$\tilde{\Lambda}_0 \cap \tilde{\Lambda}_1^a \subset T^*_{X\times X}(X \times X). \tag{4.2}$$

Proposition 4.2 *The diagrams below commute.*

$$\begin{tikzcd}
\mathcal{H}om(F,G)\otimes\delta^!H \arrow[rr] \arrow[drr, phantom, "(1)"] & & \mathcal{H}om(F,G\otimes\delta^!H) \arrow[d] \\
\delta^!K\otimes\delta^!H \arrow[u, "\sim"] \arrow[rr] \arrow[ddrr, phantom, "(2)"] & & \mathcal{H}om(F,\delta^!(q_1^{-1}G\otimes H)) \\
 & & \delta^!(K\otimes H\otimes\omega_X^{\otimes -1}) \arrow[u, "\sim"] \\
\pi_*\Gamma_{\Lambda_0}\mu_\Delta K\otimes\pi_*\Gamma_{\Lambda_1}\mu_\Delta H \arrow[uu, "\sim"] \arrow[rr] & & \pi_*\Gamma_{\Lambda_0+\Lambda_1}\mu_\Delta(K\otimes H\otimes\omega_X^{\otimes -1}) \arrow[u, "\sim"]
\end{tikzcd}$$

First, we state three lemmas whose proofs are easy verifications left to the reader.

Lemma 4.3 *The diagram:*

$$\begin{tikzcd}
\mathcal{H}om(F,G)\otimes\delta^!H \arrow[r] \arrow[dr] & \mathcal{H}om(F,G\otimes\delta^!H) \arrow[d] \\
 & \mathcal{H}om(F,\delta^!(q_1^{-1}G\otimes H))
\end{tikzcd}$$

is isomorphic to:

$$\begin{tikzcd}
\delta_{13}^{-1}\delta_{12}^!\delta_{34}^!(K\boxtimes H) \arrow[r] \arrow[dr] & \delta_{12}^!\delta_{13}^{-1}\delta_{34}^!(K\boxtimes H) \arrow[d] \\
 & \delta_{12}^!\delta_{14}^!\delta_{13}^{-1}(K\boxtimes H)
\end{tikzcd}$$

Note that the morphisms

$$\delta_{13}^{-1}\delta_{12}^! \longrightarrow \delta_{12}^!\delta_{13}^{-1}$$

or

$$\delta_{13}^{-1}\delta_{34}^! \longrightarrow \delta_{14}^!\delta_{13}^{-1}$$

are defined as follows. Consider a cartesian square:

$$\begin{tikzcd}
Z_1 \arrow[r, "\lambda_1"'] & Z \\
Z_{12} \arrow[u, "\mu_2"'] \arrow[r, "\mu_1"'] \arrow[ur, phantom, "\Box"] & Z_2 \arrow[u, "\lambda_2"']
\end{tikzcd}$$

Then we have the natural morphism:

$$\mu_2^{-1}\circ\lambda_1^! \longrightarrow \mu_1^!\circ\lambda_2^{-1}$$

defined by:

$$\mu_{1!}\mu_2^{-1}\lambda_1^! \simeq \lambda_2^{-1}\lambda_{1!}\lambda_1^! \longrightarrow \lambda_2^{-1}.$$

Lemma 4.4 *The diagram below commutes:*

$$\begin{tikzcd}
\delta_{12}^{!}\delta_{13}^{-1}\delta_{24}^{-1}(K \boxtimes H) \otimes \omega_X^{\otimes -1} \arrow[r, "\alpha"] \arrow[dr, "\gamma"'] & \delta_{12}^{!}\delta_{14}^{!}\delta_{13}^{-1}(K \boxtimes H) \arrow[d, "\beta"] \\
& \delta_{1234}^{!}(K \boxtimes H) \otimes \omega_X
\end{tikzcd}$$

Moreover, assuming (4.2), α, β and γ are isomorphisms.

Note that α is defined through:

$$\delta_{24}^{-1} \otimes \omega_X^{\otimes -1} \longrightarrow \delta_{24}^{!}$$

and

$$\delta_{12}^{!}\delta_{24}^{!} \simeq \delta_{12}^{!}\delta_{14}^{!},$$

and γ is defined through:

$$\delta_{13}^{-1}\delta_{24}^{-1} \otimes \omega_X^{\otimes -1} \longrightarrow \delta_{13}^{!}\delta_{24}^{!} \otimes \omega_X.$$

Lemma 4.5 *The diagram below commutes:*

$$\begin{tikzcd}
\delta_{13}^{-1}\delta_{12}^{!}\delta_{34}^{!}(K \boxtimes H) \arrow[r] \arrow[dr] & \delta_{12}^{!}\delta_{14}^{!}\delta_{13}^{-1}(K \boxtimes H) \arrow[d] \\
& \delta_{1234}^{!}(K \boxtimes H) \otimes \omega_X
\end{tikzcd}$$

Proof of Proposition 4.2: Diagram (1) obviously commutes. To prove that (2) commutes, we decompose it in the diagram below, after applying Lemma 4.3:

$$\begin{tikzcd}
\delta^{!}K \otimes \delta^{!}H \arrow[rr] \arrow[dr] & & \delta_{12}^{!}\delta_{14}^{!}\delta_{13}^{-1}(K \boxtimes H) \arrow[dl] \\
 & \delta_{1234}^{!}(K \boxtimes H) \otimes \omega_X \arrow[u, phantom, "(6)"] & \delta_{12}^{!}(K \otimes H \otimes \omega_X^{\otimes -1}) \arrow[u, "\sim"'] \\
\pi_*\Gamma_{\Lambda_0}\mu_\Delta K \otimes \pi_*\Gamma_{\Lambda_1}\mu_\Delta H \arrow[uu, "\sim"] \arrow[rr] \arrow[ur] & \arrow[u, phantom, "(4)"] & \pi_*\Gamma_{\Lambda_0+\Lambda_1}\mu_\Delta(K \otimes H \otimes \omega_X^{\otimes -1}) \arrow[u, "\sim"'] \arrow[ul]
\end{tikzcd}$$

(3) (5)

In this diagram, the sub-diagram (6) commutes by Lemma 4.5, the sub-diagram (5) commutes by Lemma 4.4, the sub-diagram (4) commutes by [7, Prop. 4.3.5] and the sub-diagram (3) obviously commutes. Hence the full diagram commutes. □

Proof of Theorem 4.1: We shall apply Proposition 4.2 with $G = F$, $H = \mathcal{M} \boxtimes \underline{D}\mathcal{M} \otimes^L_{\mathcal{D}_{X\times X}} \mathcal{O}_{X\times X}$ (hence $K = F \boxtimes DF$). Note that we have trace morphisms:

$$\begin{aligned} K &\longrightarrow \delta_! \omega_X, \\ H &\longrightarrow \delta_! \omega_X. \end{aligned}$$

Consider the diagrams:

$$\begin{tikzcd}
\mathcal{H}om(F,F) \otimes \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{M}) \arrow[r] \arrow[d] \arrow[dr, phantom, "(7)"] & \mathcal{H}om_{\mathcal{D}_X}(F\otimes\mathcal{M}, F\otimes\mathcal{M}) \arrow[d] \\
\mathcal{H}om(F,F) \otimes \delta^! H \arrow[r] & \mathcal{H}om(F, F\otimes \delta^! H) \\
\delta^! K \otimes \delta^! H \arrow[u, "\sim"] \arrow[dr, phantom, "(8)"] & \delta^!(K \otimes H \otimes \omega_X^{\otimes -1}) \arrow[u, "\sim"'] \\
\pi_*\Gamma_{\Lambda_0}\mu_\Delta K \otimes \pi_*\Gamma_{\Lambda_1}\mu_\Delta H \arrow[r] \arrow[u, "\sim"] \arrow[d] \arrow[dr, phantom, "(9)"] & \pi_*\Gamma_{\Lambda_0+\Lambda_1}\mu_\Delta(K\otimes H\otimes\omega_X^{\otimes -1}) \arrow[u, "\sim"'] \arrow[d] \\
\pi_*\Gamma_{\Lambda_0}\mu_\Delta \delta_!\omega_X \otimes \pi_*\Gamma_{\Lambda_1}\mu_\Delta \delta_!\omega_X \arrow[r] \arrow[d, "\sim"'] \arrow[dr, phantom, "(10)"] & \pi_*\Gamma_{\Lambda_0+\Lambda_1}\mu_\Delta(\delta_!\omega_X \otimes \delta_!\omega_X \otimes \omega_X^{\otimes -1}) \arrow[d, "\sim"'] \\
\pi_*\Gamma_{\Lambda_0}\pi^{-1}\omega_X \otimes \pi_*\Gamma_{\Lambda_1}\pi^{-1}\omega_X \arrow[r] & \pi_*\Gamma_{\Lambda_0+\Lambda_1}\pi^{-1}\omega_X
\end{tikzcd}$$

Diagrams (7) and (10) obviously commute, diagram (8) commutes by Proposition 4.2 and diagram (9) commutes since it is obtained by applying the morphism of functors:

$$\pi_*\Gamma_{\Lambda_0}\mu_\Delta(\cdot) \otimes \pi_*\Gamma_{\Lambda_1}\mu_\Delta(\cdot) \longrightarrow \pi_*\Gamma_{\Lambda_0+\Lambda_1}\mu_\Delta(\cdot \otimes \cdot \otimes \omega_X^{\otimes -1})$$

obtained from [7, Prop. 4.3.5] to $K \longrightarrow \delta_!\omega_X$ and $H \longrightarrow \delta_!\omega_X$. To conclude the proof, it remains to notice that the sequence of morphisms in the second column of the preceding diagrams (7) and (8) is the same as the morphism

$$\delta_!\mathcal{H}om(F\otimes\mathcal{M}, F\otimes\mathcal{M}) \longrightarrow K \otimes H \otimes \omega_X^{\otimes -1} = L_X$$

obtained in 3.2. Then, applying $H^0\mathrm{R}\Gamma(X;\cdot)$ to the preceding diagram, we find the commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}(F,F) \otimes \mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M},\mathcal{M}) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{D}_X}(F\otimes\mathcal{M}, F\otimes\mathcal{M}) \arrow[d] \\
H^0_{\Lambda_0}(T^*X;\pi^{-1}\omega_X) \otimes H^0_{\Lambda_1}(T^*X;\pi^{-1}\omega_X) \arrow[r] & H^0_{\Lambda_0+\Lambda_1}(T^*X;\pi^{-1}\omega_X)
\end{tikzcd}$$

□

5 The direct image formula

Let $f : X \longrightarrow Y$ be a morphism of complex manifolds, and let $(\mathcal{M}, F)$ be an elliptic pair on X. Under suitable conditions that we shall recall now, it is proved in [14] that the direct image $\underline{f}_!(F \otimes \mathcal{M})$ belongs to $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_Y)$. The aim of this section is to prove that in this situation, the microlocal Euler class of this image is the image by the morphism (2.12) of that of $(\mathcal{M}, F)$.

Let us first recall the definition of $\mathrm{char}_f(\mathcal{M})$, the relative characteristic variety of $\mathcal{M}$, (see [11, 14]). If f is smooth, one denotes by $\mathcal{D}_{X|Y}$ the sub-ring of $\mathcal{D}_X$ generated by the vertical vector fields, one locally chooses $\mathcal{M}_0$, a coherent $\mathcal{D}_{X|Y}$-submodule of $\mathcal{M}$ which generates it, and one sets:

$$\mathrm{char}_f(\mathcal{M}) = \mathrm{char}(\mathcal{D}_X \otimes_{\mathcal{D}_{X|Y}} \mathcal{M}_0).$$

One checks easily that this does not depend on the choice of $\mathcal{M}_0$. In the general case (f not necessarily smooth), one decomposes f by its graph as:

$$f : X \underset{i}{\longrightarrow} X \times Y \underset{q}{\longrightarrow} Y$$

and one sets:

$$\mathrm{char}_f(\mathcal{M}) = {}^t i' i_\pi^{-1} \mathrm{char}_q(\underline{i}_! \mathcal{M}).$$

Let $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X)$ and let $F \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(X)$. One says that $(\mathcal{M}, F)$ is f-elliptic if

$$\mathrm{char}_f(\mathcal{M}) \cap SS(F) \subset T^*_X X.$$

Since $\mathrm{char}_f(\mathcal{M})$ contains $\mathrm{char}(\mathcal{M})$, an f-elliptic pair is elliptic. Let $\mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_X)$ denote the full triangulated subcategory of $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X)$ generated by the objects $\mathcal{M}$ such that for all $j \in \mathbb{Z}$ and all compact subset K of X, $H^j(\mathcal{M})$ may be endowed with a good filtration in a neighborhood of K. If $(\mathcal{M}, F)$ is f-elliptic and moreover $\mathcal{M}$ belongs to $\mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_X)$, one says that $(\mathcal{M}, F)$ is a good f-elliptic pair. If moreover f is proper on $\mathrm{supp}\,\mathcal{M} \cap \mathrm{supp}\,F$, one says that $(\mathcal{M}, F)$ has f-proper support. It is proved in [14] that if $(\mathcal{M}, F)$ is a good f-elliptic pair with f-proper support, then $\underline{f}_!(F \otimes \mathcal{M})$ belongs to $\mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_Y)$. Let $\Lambda_0 = \mathrm{char}(\mathcal{M})$, $\Lambda_1 = SS(F)$. We have the canonical morphism:

$$f_\mu : H^0_{\Lambda_0 + \Lambda_1}(T^*X; \pi^{-1}\omega_X) \longrightarrow H^0_{f_\mu(\Lambda_0 + \Lambda_1)}(T^*Y; \pi^{-1}\omega_Y).$$

Theorem 5.1 *Assume $(\mathcal{M}, F)$ is an f-elliptic pair with f-proper support. Then:*

$$\mu\mathrm{eu}(\underline{f}_!(F \otimes \mathcal{M})) = f_\mu\, \mu\mathrm{eu}(\mathcal{M}, F) = f_\mu(\mu\mathrm{eu}(\mathcal{M}) *_\mu \mu\mathrm{eu}(F)).$$

Proof: The proof will decompose into several steps. For short, during this proof, we will not write the symbol "R" or "L" of right or left derived functors. We introduce the notations:

$$\tilde{X} = X \times X, \quad \tilde{f} = f \times f : X \times X \longrightarrow Y \times Y.$$

We denote by δ_X the diagonal embedding $X \hookrightarrow X \times X$, and if there is no risk of confusion, we write δ instead of δ_X. We also set for short:

$$\begin{aligned} L_X &= (F \otimes \mathcal{M}) \boxtimes (D'F \otimes \underline{D}\mathcal{M}) \otimes_{\mathcal{D}_{\tilde{X}}} \mathcal{O}_{\tilde{X}} \\ L_Y &= \underline{f}_!(F \otimes \mathcal{M}) \boxtimes \underline{D}(\underline{f}_!(F \otimes \mathcal{M})) \otimes_{\mathcal{D}_{\tilde{Y}}} \mathcal{O}_{\tilde{Y}} \end{aligned}$$

By the results of [14], we have the isomorphisms:

$$\begin{aligned} \tilde{f}_! L_X &\simeq \tilde{f}_!(F \otimes \mathcal{M} \boxtimes D'F \otimes \underline{D}\mathcal{M} \otimes_{\mathcal{D}_{\tilde{X}}} \mathcal{D}_{\tilde{X} \to \tilde{Y}}) \otimes_{\mathcal{D}_{\tilde{Y}}} \mathcal{O}_{\tilde{Y}} \\ &\xleftarrow{\sim} \underline{f}_!(F \otimes \mathcal{M}) \boxtimes \underline{f}_!(D'F \otimes \underline{D}\mathcal{M}) \otimes_{\mathcal{D}_{\tilde{Y}}} \mathcal{O}_{\tilde{Y}} \\ &\simeq L_Y. \end{aligned}$$

Consider the diagram:

$$\begin{tikzcd}
f_!\mathcal{H}om_{\mathcal{D}_X}(F \otimes \mathcal{M}, F \otimes \mathcal{M}) \arrow[rr] \arrow[d] \arrow[drr, phantom, "(1)"] & & \mathcal{H}om_{\mathcal{D}_Y}(\underline{f}_!(F \otimes \mathcal{M}), \underline{f}_!(F \otimes \mathcal{M})) \arrow[d] \\
f_!\delta^! L_X \arrow[r] & \delta^! \tilde{f}_! L_X \arrow[r] & \delta^! L_Y \\
\pi_* f_{\pi !}{}^t f'^{-1}\Gamma_{\Lambda_X}\mu_{\Delta_X} L_X \arrow[r] \arrow[u, "\sim"] \arrow[d] \arrow[ur, phantom, "(2)"] & \pi_*\Gamma_{\Lambda_Y}\mu_{\Delta_Y}\tilde{f}_! L_X \arrow[r] \arrow[u, "\sim"] \arrow[d] \arrow[ur, phantom, "(3)"] & \pi_*\Gamma_{\Lambda_Y}\mu_{\Delta_Y} L_Y \arrow[u, "\sim"] \arrow[dd] \\
\pi_* f_{\pi !}{}^t f'^{-1}\Gamma_{\Lambda_X}\mu_{\Delta_X}\delta_!\omega_X \arrow[r] \arrow[dd, "\sim"] \arrow[ur, phantom, "(4)"] & \pi_*\Gamma_{\Lambda_Y}\mu_{\Delta_Y}\tilde{f}_!\delta_!\omega_X \arrow[d, "\sim"] \arrow[dr, phantom, "(5)"] & \\
\arrow[r, phantom, "(6)"] & \pi_*\Gamma_{\Lambda_Y}\mu_{\Delta_Y}\delta_! f_!\omega_X \arrow[r] \arrow[d, "\sim"] \arrow[dr, phantom, "(7)"] & \pi_*\Gamma_{\Lambda_Y}\mu_{\Delta_Y}\delta_!\omega_Y \arrow[d, "\sim"] \\
\pi_* f_{\pi !}{}^t f'^{-1}\Gamma_{\Lambda_X}\pi^{-1}\omega_X \arrow[r] & \pi_*\Gamma_{\Lambda_Y}\pi^{-1} f_!\omega_X \arrow[r] & \pi_*\Gamma_{\Lambda_Y}\pi^{-1}\omega_Y
\end{tikzcd}$$

It is enough to prove it is commutative. In fact, applying $H^0\mathrm{R}\Gamma(Y;\cdot)$ to it we get the commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}_X}(F \otimes \mathcal{M}, F \otimes \mathcal{M}) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{D}_Y}(\underline{f}_!(F \otimes \mathcal{M}), \underline{f}_!(F \otimes \mathcal{M})) \arrow[d] \\
H^0_{\Lambda_X}(T^*X; \pi^{-1}\omega_X) \arrow[r] & H^0_{\Lambda_Y}(T^*Y; \pi^{-1}\omega_Y).
\end{tikzcd}$$

Diagram (2) commutes since

$$f_!\delta^! \longrightarrow \delta^! \tilde{f}_!$$

is the restriction to the zero section of:

$$f_{\pi !}{}^t f'^{-1}\mu_{\Delta_X} \longrightarrow \mu_{\Delta_Y}\tilde{f}_!$$

(see [7, Prop. 4.3.4]).

Diagram (3) commutes since it is obtained by applying the natural transformation $\pi_*\Gamma_{\Lambda_Y}\mu_{\Delta_Y} \longrightarrow \delta^!$ to the morphism $\tilde{f}_! L_X \longrightarrow L_Y$.

Diagram (4) commutes. In fact, it is obtained by applying the morphism of functors:

$$\pi_* f_{\pi !}{}^t f'^{-1}\Gamma_{\Lambda_X}\mu_{\Delta_X} \longrightarrow \pi_*\Gamma_{\Lambda_Y}\mu_{\Delta_Y}\tilde{f}_!$$

to $L_X \longrightarrow \delta_!\omega_X$. Diagrams (6) and (7) obviously commute.

Using the base change formula for elliptic pairs of [14] we see that for $\mathcal{P} = \mathcal{Q} = F \otimes \mathcal{M}$ the map

$$\begin{aligned} &\mathcal{H}om_{q_2^{-1}\mathcal{D}_Y}(q_2^{-1}\underline{f}_!\mathcal{P}, q_1^{-1}\underline{f}_!\mathcal{Q} \otimes_{q_1^{-1}\mathcal{D}_Y} \mathcal{O}_{Y\times Y}^{(0,d_Y)}[d_Y]) \\ \longrightarrow\ &\mathcal{H}om_{q_2^{-1}\mathcal{D}_Y}(q_2^{-1}\underline{f}_!\mathcal{P}, f_{2!}(q_1^{-1}\mathcal{Q} \otimes_{q_1^{-1}\mathcal{D}_X} \mathcal{O}_{X\times Y}^{(0,d_Y)}[d_Y])) \end{aligned}$$

appearing in Lemma 5.2 and Lemma 5.3 below is an isomorphism. Moreover, the Künneth formula for elliptic pairs [loc. cit.] shows that the canonical map

$$\underline{f}_!(F\otimes\mathcal{M})\boxtimes\underline{f}_!(D'F\otimes\underline{D}\mathcal{M})\otimes_{\mathcal{D}_{Y\times Y}}\mathcal{O}_{Y\times Y} \longrightarrow \tilde{f}_!(F\otimes\mathcal{M}\boxtimes D'F\otimes\underline{D}\mathcal{M}\otimes_{\mathcal{D}_{X\times X}}\mathcal{O}_{X\times X})$$

is also an isomorphism. Hence, the conjunction of Lemma 5.2 and Lemma 5.3 below, gives the commutativity of diagram (1). In the same way, Lemma 5.4 below shows that diagram (5) commutes. □

Let $f : X \longrightarrow Y$ be a morphism of complex manifolds. We will decompose $\tilde{f} = f\times f$ as

$$\begin{tikzcd} X\times X \arrow[r, "f_1"] & X\times Y \arrow[r, "f_2"] & Y\times Y \\ X \arrow[u, "\delta_X"] \arrow[r, equal] & X \arrow[u, "\delta"] \arrow[r, "f"] & Y \arrow[u, "\delta_Y"] \end{tikzcd}$$

Lemma 5.2 *Let $\mathcal{P}$ and $\mathcal{Q}$ belong to $\mathbf{D}^{\mathrm{b}}(\mathcal{D}_X^{\mathrm{op}})$. Then we have the canonical commutative diagram:*

$$\begin{tikzcd} \tilde{f}_!\delta_{X!}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{P},\mathcal{Q}) \arrow[r] \arrow[dd] & \tilde{f}_!\mathcal{H}om_{q_2^{-1}\mathcal{D}_X}(q_2^{-1}\mathcal{P}, q_1^{-1}\mathcal{Q}\otimes_{q_1^{-1}\mathcal{D}_X}\mathcal{O}_{X\times X}^{(0,d_X)}[d_X]) \arrow[d] \\ & \mathcal{H}om_{q_2^{-1}\mathcal{D}_Y}(q_2^{-1}\underline{f}_!\mathcal{P}, f_{2!}(q_1^{-1}\mathcal{Q}\otimes_{q_1^{-1}\mathcal{D}_X}\mathcal{O}_{X\times Y}^{(0,d_Y)}[d_Y])) \\ \delta_{Y!}\mathcal{H}om_{\mathcal{D}_Y}(\underline{f}_!\mathcal{P},\underline{f}_!\mathcal{Q}) \arrow[r] & \mathcal{H}om_{q_2^{-1}\mathcal{D}_Y}(q_2^{-1}\underline{f}_!\mathcal{P}, q_1^{-1}\underline{f}_!\mathcal{Q}\otimes_{q_1^{-1}\mathcal{D}_Y}\mathcal{O}_{Y\times Y}^{(0,d_Y)}[d_Y]) \arrow[u] \end{tikzcd}$$

Proof: The kernel representation of differential operators induces the morphism of bimodules:

$$\delta_{X!}\mathcal{D}_X \longrightarrow \mathcal{O}_{X\times X}^{(0,d_X)}[d_X].$$

From the relative integration map

$$\underline{f_1}_{|X!}\Omega_{X\times X|X}[d_X] \longrightarrow \Omega_{X\times Y|X}[d_Y],$$

and the Poincaré-Verdier adjunction formula, we deduce the bimodule morphism:

$$\mathcal{O}^{(0,d_X)}_{X\times X}[d_X] \otimes_{q_2^{-1}\mathcal{D}_X} q_2^{-1}\mathcal{D}_{X\to Y} \longrightarrow f_1^!\mathcal{O}^{(0,d_Y)}_{X\times Y}.$$

Hence, we get the chain of bimodule morphisms:

$$\begin{aligned}
\delta_{X!}\mathcal{D}_{X\to Y} &\longrightarrow \delta_{X!}\mathcal{D}_X \otimes_{q_2^{-1}\mathcal{D}_X} q_2^{-1}\mathcal{D}_{X\to Y}\\
&\longrightarrow \mathcal{O}^{(0,d_X)}_{X\times X}[d_X] \otimes_{q_2^{-1}\mathcal{D}_X} q_2^{-1}\mathcal{D}_{X\to Y}\\
&\longrightarrow f_1^!\mathcal{O}^{(0,d_Y)}_{X\times Y}[d_Y].
\end{aligned}$$

This chain of morphisms gives rise to the commutative diagram:

$$\begin{tikzcd}
\delta_{X!}\mathcal{Q} \otimes_{q_2^{-1}\mathcal{D}_X} q_2^{-1}\mathcal{D}_{X\to Y} \arrow[r] \arrow[d] & \delta_{X!}(\mathcal{Q} \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y}) \arrow[d] \\
q_1^{-1}\mathcal{Q} \otimes_{q_1^{-1}\mathcal{D}_X} \mathcal{O}^{(0,d_X)}_{X\times X}[d_X] \otimes_{q_2^{-1}\mathcal{D}_X} q_2^{-1}\mathcal{D}_{X\to Y} \arrow[r] & q_1^{-1}\mathcal{Q} \otimes_{q_1^{-1}\mathcal{D}_X} f_1^!\mathcal{O}^{(0,d_Y)}_{X\times Y}[d_Y].
\end{tikzcd}$$

By adjunction of the tensor product, this gives us the commutative diagram:

$$\begin{tikzcd}
\delta_{X!}\mathcal{Q} \arrow[r] \arrow[d] & \delta_{X!}\mathcal{H}om_{f^{-1}\mathcal{D}_Y}(\mathcal{D}_{X\to Y}, \mathcal{Q} \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y}) \arrow[d] \\
q_1^{-1}\mathcal{Q} \otimes_{q_1^{-1}\mathcal{D}_X} \mathcal{O}^{(0,d_X)}_{X\times X}[d_X] \arrow[r] & \mathcal{H}om_{q_2^{-1}f^{-1}\mathcal{D}_Y}(q_2^{-1}\mathcal{D}_{X\to Y}, q_1^{-1}\mathcal{Q} \otimes_{q_1^{-1}\mathcal{D}_X} f_1^!\mathcal{O}^{(0,d_Y)}_{X\times Y}[d_Y]).
\end{tikzcd}$$

Applying the functor $\mathcal{H}om_{q_2^{-1}\mathcal{D}_X}(q_2^{-1}\mathcal{P}, \cdot)$ to this diagram, we get the commutative diagram:

$$\begin{tikzcd}
\delta_{X!}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{P}, \mathcal{Q}) \arrow[r] \arrow[d] & \delta_{X!}\mathcal{H}om_{f^{-1}\mathcal{D}_Y}(\mathcal{P} \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y}, \mathcal{Q} \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y}) \arrow[d] \\
M_X \arrow[r] & \mathcal{H}om_{q_2^{-1}f^{-1}\mathcal{D}_Y}(q_2^{-1}(\mathcal{P} \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y}), q_1^{-1}\mathcal{Q} \otimes_{q_1^{-1}\mathcal{D}_X} f_1^!\mathcal{O}^{(0,d_Y)}_{X\times Y}[d_Y])
\end{tikzcd}$$

where we have set for short:

$$M_X = \mathcal{H}om_{q_2^{-1}\mathcal{D}_X}(q_2^{-1}\mathcal{P}, q_1^{-1}\mathcal{Q} \otimes_{q_1^{-1}\mathcal{D}_X} \mathcal{O}^{(0,d_X)}_{X\times X}[d_X]).$$

Applying $f_{1!}$ and using the Poincaré-Verdier adjunction formula, we get the commutative diagram:

$$\begin{tikzcd}
f_{1!}\delta_{X!}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{P}, \mathcal{Q}) \arrow[r] \arrow[d] & \delta_!\mathcal{H}om_{f^{-1}\mathcal{D}_Y}(\mathcal{P} \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y}, \mathcal{Q} \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y}) \arrow[d] \\
f_{1!}M_X \arrow[r] & M_{XY}
\end{tikzcd}$$

where we have set for short:

$$M_{XY} = \mathcal{H}om_{q_2^{-1}\mathcal{D}_Y}(q_2^{-1}\underline{f}_!\mathcal{P}, q_1^{-1}\mathcal{Q}\otimes_{q_1^{-1}\mathcal{D}_X}\mathcal{O}^{(0,d_Y)}_{X\times Y}).$$

Finally, apply $f_{2!}$ and note that the diagram below is commutative:

$$\begin{tikzcd}
f_{2!}\delta_!\mathcal{H}om_{f^{-1}\mathcal{D}_Y}(\mathcal{P}\otimes_{\mathcal{D}_X}\mathcal{D}_{X\to Y}, \mathcal{Q}\otimes_{\mathcal{D}_X}\mathcal{D}_{X\to Y}) \arrow[r] \arrow[dd] & \delta_{Y!}\mathcal{H}om_{\mathcal{D}_Y}(\underline{f}_!\mathcal{P}, \underline{f}_!\mathcal{Q}) \arrow[d] \\
 & M_Y \arrow[d] \\
f_{2!}M_{XY} \arrow[r] & \mathcal{H}om_{q_2^{-1}\mathcal{D}_Y}(q_2^{-1}\underline{f}_!\mathcal{P}, f_{2!}(q_1^{-1}\mathcal{Q}\otimes_{q_1^{-1}\mathcal{D}_X}\mathcal{O}^{(0,d_Y)}_{X\times Y}[d_Y])).
\end{tikzcd}$$

Where we have set for short:

$$M_Y = \mathcal{H}om_{q_2^{-1}\mathcal{D}_Y}(q_2^{-1}\underline{f}_!\mathcal{P}, q_1^{-1}\underline{f}_!\mathcal{Q}\otimes_{q_1^{-1}\mathcal{D}_Y}\mathcal{O}^{(0,d_Y)}_{Y\times Y}[d_Y]).$$

□

Lemma 5.3 *Let $\mathcal{P}$ and $\mathcal{Q}$ belong to $\mathbf{D}^{\mathrm{b}}(\mathcal{D}_X^{\mathrm{op}})$. Then, we have the canonical commutative diagram:*

$$\begin{tikzcd}
\tilde{f}_!\mathcal{H}om_{q_2^{-1}\mathcal{D}_X}(q_2^{-1}\mathcal{P}, q_1^{-1}\mathcal{Q}\otimes_{q_1^{-1}\mathcal{D}_X}\mathcal{O}^{(0,d_X)}_{X\times X}[d_X]) \arrow[d] & \tilde{f}_!(\mathcal{Q}\boxtimes\underline{D}\mathcal{P}\otimes_{\mathcal{D}_{X\times X}}\mathcal{O}_{X\times X}) \arrow[l] \\
\mathcal{H}om_{q_2^{-1}\mathcal{D}_Y}(q_2^{-1}\underline{f}_!\mathcal{P}, f_{2!}(q_1^{-1}\mathcal{Q}\otimes_{q_1^{-1}\mathcal{D}_X}\mathcal{O}^{(0,d_Y)}_{X\times Y}[d_Y])) & \underline{f}_!\mathcal{Q}\boxtimes\underline{f}_!\underline{D}\mathcal{P}\otimes_{\mathcal{D}_{Y\times Y}}\mathcal{O}_{Y\times Y} \arrow[u] \arrow[d] \\
\mathcal{H}om_{q_2^{-1}\mathcal{D}_Y}(q_2^{-1}\underline{f}_!\mathcal{P}, q_1^{-1}\underline{f}_!\mathcal{Q}\otimes_{q_1^{-1}\mathcal{D}_Y}\mathcal{O}^{(0,d_Y)}_{Y\times Y}[d_Y]) \arrow[u] & \underline{f}_!\mathcal{Q}\boxtimes\underline{D}\underline{f}_!\mathcal{P}\otimes_{\mathcal{D}_{Y\times Y}}\mathcal{O}_{Y\times Y} \arrow[l]
\end{tikzcd}$$

Proof: Notice that the diagram below is commutative:

$$\begin{tikzcd}
\tilde{f}_![(\mathcal{Q}\boxtimes\underline{D}\mathcal{P})\otimes_{\mathcal{D}_X^{\boxtimes 2}}\mathcal{D}^{\boxtimes 2}_{X\to Y}] \arrow[r] & \tilde{f}_![\mathcal{H}om_{q_2^{-1}\mathcal{D}_X}(q_2^{-1}\mathcal{P}, \mathcal{Q}\boxtimes\mathcal{K}_X)\otimes_{\mathcal{D}_X^{\boxtimes 2}}\mathcal{D}^{\boxtimes 2}_{X\to Y}] \arrow[d] \\
\underline{f}_!\mathcal{Q}\boxtimes\underline{f}_!\underline{D}\mathcal{P} \arrow[u] \arrow[d] & \mathcal{H}om_{q_2^{-1}\mathcal{D}_Y}(q_2^{-1}\underline{f}_!\mathcal{P}, \underline{\tilde{f}}_![(\mathcal{Q}\boxtimes\mathcal{K}_X)\otimes_{\mathcal{D}_X^{\boxtimes 2}}\mathcal{D}^{\boxtimes 2}_{X\to Y}\otimes_{q_2^{-1}\mathcal{D}_X}q_2^{-1}\mathcal{D}_{X\to Y}]) \\
\underline{f}_!\mathcal{Q}\boxtimes\mathcal{H}om_{\mathcal{D}_Y}(\underline{f}_!\mathcal{P}, \underline{\underline{f}}_!\mathcal{K}_X) \arrow[r] \arrow[d] & \mathcal{H}om_{q_2^{-1}\mathcal{D}_Y}(q_2^{-1}\underline{f}_!\mathcal{P}, \underline{f}_!\mathcal{Q}\boxtimes\underline{\underline{f}}_!\mathcal{K}_X) \arrow[u] \arrow[d] \\
\underline{f}_!\mathcal{Q}\boxtimes\underline{D}\underline{f}_!\mathcal{P} \arrow[r] & \mathcal{H}om_{q_2^{-1}\mathcal{D}_Y}(q_2^{-1}\underline{f}_!\mathcal{P}, \underline{f}_!\mathcal{Q}\boxtimes\mathcal{K}_Y)
\end{tikzcd}$$

Thanks to the isomorphism

$$q_2^{-1}\mathcal{K}_X \otimes_{q_2^{-1}\mathcal{D}_X} \mathcal{O}_{X\times X} \longrightarrow \mathcal{O}_{X\times X}^{(0,d_X)}[d_X],$$

and the canonical morphism

$$\mathcal{D}_{X\to Y}^{\boxtimes 2} \otimes_{\mathcal{D}_Y^{\boxtimes 2}} \mathcal{O}_{Y\times Y} \longrightarrow \mathcal{O}_{X\times X},$$

an application of the functor $\cdot \otimes_{\mathcal{D}_Y^{\boxtimes 2}} \mathcal{O}_{Y\times Y}$ to the preceding diagram allows us to conclude. $\square$

Lemma 5.4 *Let $\mathcal{P}$ belong to $\mathbf{D}^{\mathrm{b}}(\mathcal{D}_X^{\mathrm{op}})$. Then, we have the canonical commutative diagram:*

$$\begin{tikzcd}
\tilde{f}_!((\mathcal{P} \underline{\boxtimes} \underline{D}\mathcal{P}) \otimes_{\mathcal{D}_{X\times X}} \mathcal{O}_{X\times X}) \arrow[r] & \tilde{f}_!\delta_{X\,!}\omega_X \arrow[d] \\
(\underline{f}_!\mathcal{P} \underline{\boxtimes} \underline{f}_!\underline{D}\mathcal{P}) \otimes_{\mathcal{D}_{Y\times Y}} \mathcal{O}_{Y\times Y} \arrow[u] \arrow[d] & \delta_{Y\,!}f_!\omega_X \arrow[d] \\
(\underline{f}_!\mathcal{P} \underline{\boxtimes} \underline{D}\underline{f}_!\mathcal{P}) \otimes_{\mathcal{D}_{Y\times Y}} \mathcal{O}_{Y\times Y} \arrow[r] & \delta_{Y\,!}\omega_Y
\end{tikzcd}$$

Proof: Recall that the dualizing complex for $\mathcal{D}$-modules

$$\mathcal{K}_X = \Omega_X[d_X] \otimes_{\mathcal{O}_X} \mathcal{D}_X$$

has a canonical structure of right $\mathcal{D}_X^{\otimes 2}$-module and that

$$\delta_!\mathcal{K}_X \simeq \underline{\delta}_!\Omega[d_X]$$

as $\mathcal{D}_X^{\boxtimes 2}$-modules. Also recall that:

$$\underline{\underline{f}}_!\mathcal{K}_X = f_!(\mathcal{K}_X \otimes_{\mathcal{D}_X^{\otimes 2}} \mathcal{D}_{X\to Y}^{\otimes 2})$$

and that the trace of the duality morphism associated to f is given by the $\mathcal{D}_Y^{\otimes 2}$-linear integration morphism

$$\underline{\underline{f}}_!\mathcal{K}_X \longrightarrow \mathcal{K}_Y.$$

From the construction of this morphism (see [14]) it is clear that we have the canonical commutative $\mathcal{D}_Y^{\boxtimes 2}$-linear diagram:

$$\begin{tikzcd}
\delta_!\underline{\underline{f}}_!\mathcal{K}_X \arrow[r] \arrow[d] & \tilde{\underline{f}}_!\delta_!\mathcal{K}_X \arrow[d] \\
\delta_!\mathcal{K}_Y \arrow[r, equal] & \delta_!\mathcal{K}_Y.
\end{tikzcd}$$

In this diagram, the first horizontal arrow is deduced from the morphism

$$\mathcal{D}^{\boxtimes 2}_{X\to Y} \longrightarrow D_{X^2\to Y^2},$$

the first vertical arrow is deduced from the duality trace map and the second vertical arrow is deduced from the isomorphism

$$\underline{\tilde{f}}_!\underline{\delta}_!\Omega_X[d_X] \xrightarrow{\sim} \underline{\delta}_!\underline{f}_!\Omega_X[d_X]$$

and the integration map

$$\underline{f}_!\Omega_X[d_X] \longrightarrow \Omega_Y[d_Y].$$

Let us consider the commutative diagram:

$$\begin{tikzcd}
\tilde{f}_!(\mathcal{P} \boxtimes \underline{D}\mathcal{P} \otimes_{\mathcal{D}_X^{\boxtimes 2}} \mathcal{D}^{\boxtimes 2}_{X\to Y}) \arrow[r] & \delta_!\underline{\underline{f}}_!\mathcal{K}_X \arrow[r] \arrow[dd, equal] & \underline{\tilde{f}}_!\delta_!\mathcal{K}_X \arrow[ddd] \\
\underline{f}_!\mathcal{P} \boxtimes \underline{f}_!\underline{D}\mathcal{P} \arrow[u] \arrow[d] & & \\
\underline{f}_!\mathcal{P} \boxtimes \mathcal{H}om_{\mathcal{D}_Y}(\underline{f}_!\mathcal{P}, \underline{\underline{f}}_!\mathcal{K}_X) \arrow[r] \arrow[d] & \delta_!\underline{\underline{f}}_!\mathcal{K}_X \arrow[d] & \\
\underline{f}_!\mathcal{P} \boxtimes \mathcal{H}om_{\mathcal{D}_Y}(\underline{f}_!\mathcal{P}, \mathcal{K}_Y) \arrow[r] & \delta_!\mathcal{K}_Y \arrow[r, equal] & \delta_!\mathcal{K}_Y
\end{tikzcd}$$

By scalar extension, it gives rise to the commutative diagram:

$$\begin{tikzcd}
\underline{\tilde{f}}_!(\mathcal{P} \underline{\underline{\boxtimes}} \underline{D}\mathcal{P}) \arrow[r] \arrow[dd] & \underline{\tilde{f}}_!\delta_!\mathcal{K}_X \arrow[dd] \\
\underline{f}_!\mathcal{P} \underline{\underline{\boxtimes}} \underline{f}_!\underline{D}\mathcal{P} \arrow[u, "\alpha"] \arrow[d] & \\
\underline{f}_!\mathcal{P} \underline{\underline{\boxtimes}} \underline{D}\underline{f}_!\mathcal{P} \arrow[r] & \delta_!\mathcal{K}_Y
\end{tikzcd} \tag{5.1}$$

Note that α is an isomorphism by the Künneth formula (see [14]).

Recall that for any holomorphic map $f : X \longrightarrow Y$ and any right $\mathcal{D}_X$-module $\mathcal{M}$, we have

$$\underline{f}_!\mathcal{M} \otimes_{\mathcal{D}_Y} \mathcal{O}_Y \xrightarrow{\sim} f_!(\mathcal{M} \otimes_{\mathcal{D}_X} \mathcal{O}_X).$$

Also recall that the compatibility between the duality morphism for $\mathcal{D}$-modules and the Poincaré-Verdier duality morphism may be expressed by the commutative diagram

$$\begin{tikzcd}
(\underline{f}_!\Omega_X[d_X]) \otimes_{\mathcal{D}_Y} \mathcal{O}_Y \arrow[r] \arrow[d, "\sim"'] & \Omega_Y[d_Y] \otimes_{\mathcal{D}_Y} \mathcal{O}_Y \arrow[dd, "\sim"'] \\
f_!(\Omega_X[d_X] \otimes_{\mathcal{D}_X} \mathcal{O}_X) \arrow[d, "\sim"'] & \\
f_!\omega_X \arrow[r] & \omega_Y .
\end{tikzcd}$$

With these facts in mind, the conclusion follows easily by applying the functor $\cdot \otimes_{\mathcal{D}_{\tilde{Y}}} \mathcal{O}_{\tilde{Y}}$ to the diagram (5.1). $\square$

As a particular case of Theorem 5.1, we get:

Corollary 5.5 *Let $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_X)$ and assume f is proper on $\operatorname{supp} \mathcal{M}$. Then:*

$$\mu\mathrm{eu}(\underline{f}_!\mathcal{M}) = f_\mu(\mu\mathrm{eu}(\mathcal{M})).$$

Now apply these results to the map $f : X \longrightarrow \{\mathrm{pt}\}$. We get:

Corollary 5.6 *Let $(\mathcal{M}, F)$ be a good elliptic pair with compact support, i.e.:*

(i) $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_X)$ *and* $F \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(X)$,

(ii) $\mathrm{char}(\mathcal{M}) \cap SS(F) \subset T^*_X X$,

(iii) $\operatorname{supp} \mathcal{M} \cap \operatorname{supp} F$ *is compact.*

Then the complex $R\Gamma(X; F \otimes \mathcal{M} \otimes^L_{\mathcal{D}_X} \mathcal{O}_X)$ has finite dimensional cohomology and its Euler-Poincaré index is given by the formulas:

$$\begin{aligned} \chi(R\Gamma(X; F \otimes \mathcal{M} \otimes^L_{\mathcal{D}_X} \mathcal{O}_X)) &= \int_X \mathrm{eu}(\mathcal{M}, F) \\ &= \int_X (\mu\mathrm{eu}(\mathcal{M}) *_\mu \mu\mathrm{eu}(F))|_X \\ &= \int_{T^*X} \mu\mathrm{eu}(\mathcal{M}) \cup \mu\mathrm{eu}(F). \end{aligned}$$

Proof: The first formula follows from Theorem 5.1, the second one follows from Theorem 4.1, and the last one from the equality:

$$\int_{T^*X/X} (\lambda_0^a \cup \lambda_1) = (\lambda_0 *_\mu \lambda_1)|_X,$$

which holds for any $\lambda_j \in H^0_{\Lambda_j}(T^*X; \pi^{-1}\omega_X), j = 0, 1$ and whose proof is left to the reader. $\square$

6 Inverse image and external product formulas

Let $f : X \longrightarrow Y$ be a morphism of complex manifolds and let $(\mathcal{N}, G)$ be an elliptic pair on Y. We shall first study its inverse image by f.

Definition 6.1 We shall say that f is non-characteristic for the elliptic pair $(\mathcal{N}, G)$ if f is non characteristic with respect to the set $\mathrm{char}(\mathcal{N}) + SS(G)$ (see (2.5)).

Proposition 6.2 *Assume f is non-characteristic for the elliptic pair $(\mathcal{N}, G)$. Then $(\underline{f}^{-1}\mathcal{N}, f^{-1}G)$ is an elliptic pair in a neighborhood of $f^{-1}(\operatorname{supp} \mathcal{N} \cap \operatorname{supp} G)$.*

Proof: The hypothesis implies that f is non-characteristic with respect to $\mathcal{N}$ and with respect to G in a neighborhood of $\operatorname{supp}\mathcal{N}\cap\operatorname{supp}G$. In particular, $\underline{f}^{-1}\mathcal{N}$ will be $\mathcal{D}_X$-coherent on a neighborhood of $f^{-1}(\operatorname{supp}\mathcal{N}\cap\operatorname{supp}G)$.

Let $(x;\xi)\in\operatorname{char}(\underline{f}^{-1}\mathcal{N})\cap SS(f^{-1}G)$, and let $y=f(x)$. Since f is non-characteristic for $\mathcal{N}$ and for G, one knows [5, 7] that there exist $(y;\eta_0)\in\operatorname{char}(\mathcal{N})$ and $(y;\eta_1)\in SS(G)$ such that ${}^tf'(x).\eta_j=\xi$ for $j=0,1$. Hence ${}^tf'(x).(\eta_0-\eta_1)=0$ which implies by the hypothesis that $\eta_0-\eta_1=0$, hence $\eta_0=\eta_1=0$ since $(\mathcal{N},G)$ is elliptic. □

In view of the above proposition and Theorem 4.1, in order to calculate the microlocal Euler class of $(\underline{f}^{-1}\mathcal{N},f^{-1}G)$, it is enough to calculate separately $\mu\mathrm{eu}(\underline{f}^{-1}\mathcal{N})$ and $\mu\mathrm{eu}(f^{-1}G)$. As we shall see below, the microlocal Euler class of an $\mathbb{R}$-constructible sheaf is nothing but its characteristic cycle, and the functorial properties of this cycle have been studied in [7], where it is proved in particular that it commutes to inverse image (and external product). Hence it is enough to calculate the microlocal Euler class of the inverse image (and external product) of coherent $\mathcal{D}$-modules. Notice that such a situation did not appear when studying direct image, where the result obtained when treating simultaneously both $\mathcal{M}$ and F was much stronger than if we would have assumed f proper on $\operatorname{supp}\mathcal{M}$ and on $\operatorname{supp}F$.

Let $f:X\longrightarrow Y$ be a morphism of complex manifolds. We shall use the notations (2.12), (2.13), (2.15) of §2.

Theorem 6.3 *Let $\mathcal{N}\in\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_Y^{\mathrm{op}})$ and assume f is non-characteristic with respect to $\mathcal{N}$. Then:*

$$\mu\mathrm{eu}(\underline{f}^{-1}\mathcal{N})=f^{\mu}(\mu\mathrm{eu}(\mathcal{N})).$$

Proof: The proof is similar to that of Theorem 5.1, and we shall not give here all details.

Set $\tilde{f}=(f,f):X\times X\longrightarrow Y\times Y$, and decompose $\tilde{f}$ as:

$$\begin{tikzcd}
X\times X \arrow[r, "f_1"] & Y\times X \arrow[r, "f_2"] & Y\times Y \\
\Delta_X \arrow[u, "\delta_X"] \arrow[r, "\sim"] & \Delta \arrow[u, "\delta"] \arrow[r, "f"] & \Delta_Y \arrow[u, "\delta_Y"]
\end{tikzcd}$$

Set:

$$\begin{aligned}
L_Y &= \mathcal{N}\boxtimes\underline{D}\mathcal{N}\otimes^L_{\mathcal{D}_{Y\times Y}}\mathcal{O}_{Y\times Y},\\
L_X &= \underline{f}^{-1}\mathcal{N}\boxtimes\underline{D}\underline{f}^{-1}\mathcal{N}\otimes^L_{\mathcal{D}_{X\times X}}\mathcal{O}_{X\times X},\\
\Lambda_Y &= \operatorname{char}\mathcal{N},\Lambda=f_\pi^{-1}\Lambda_Y,\Lambda_X=f^{\mu}\Lambda_Y.
\end{aligned}$$

Since f is non-characteristic for $\mathcal{N}$, the natural morphism:

$$f_2^{-1}R\mathcal{H}om_{q_2^{-1}\mathcal{D}_Y}(q_2^{-1}\mathcal{N},\mathcal{O}^{(0,d_Y)}_{Y\times Y})\longrightarrow R\mathcal{H}om_{q_2^{-1}\mathcal{D}_X}(q_2^{-1}\underline{f}^{-1}\mathcal{N},\mathcal{O}^{(0,d_X)}_{Y\times X})$$

is an isomorphism.

On the other-hand, the natural $f_1^{-1}\mathcal{D}_Y$-linear morphism:

$$Rf_{1!}(q_1^{-1}\mathcal{D}_{Y\leftarrow X} \otimes^L_{q_1^{-1}\mathcal{D}_X} \mathcal{O}_{X\times X}[d_X]) \longrightarrow \mathcal{O}_{Y\times X}[d_Y]$$

defines the morphism:

$$Rf_{1!}(f_1^{-1}q_1^{-1}\mathcal{N} \otimes^L_{f_1^{-1}q_1^{-1}\mathcal{D}_Y} q_1^{-1}\mathcal{D}_{Y\leftarrow X} \otimes^L_{q_1^{-1}\mathcal{D}_X} \mathcal{O}_{X\times X}[d_X]) \longrightarrow q_1^{-1}\mathcal{N} \otimes^L_{q_1^{-1}\mathcal{D}_Y} \mathcal{O}_{Y\times X}[d_Y],$$

hence the morphism:

$$q_1^{-1}\underline{f}^{-1}\mathcal{N} \otimes^L_{q_1^{-1}\mathcal{D}_X} \mathcal{O}_{X\times X}[d_X] \longrightarrow f_1^!(q_1^{-1}\mathcal{N} \otimes^L_{q_1^{-1}\mathcal{D}_Y} \mathcal{O}_{Y\times X})[d_Y]$$

and this morphism is an isomorphism when f is non-characteristic for $\mathcal{N}$. Combining these two isomorphisms, we get the isomorphism:

$$f_1^! f_2^{-1} L_Y \xrightarrow{\sim} L_X$$

Then, as for Theorem 5.1, the proof is decomposed by proving the commutativity of the diagrams below. Until the end of the proof, we shall not write the symbols "R" or "L" of derived functors, for short.

$$\begin{tikzcd}
f^{-1}\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{N},\mathcal{N}) \arrow[rrr] \arrow[d] & & & \mathcal{H}om_{\mathcal{D}_X}(\underline{f}^{-1}\mathcal{N},\underline{f}^{-1}\mathcal{N}) \arrow[d] \\
f^{-1}\delta_Y^! L_Y \arrow[r] & \delta^! f_2^{-1} L_Y \arrow[r, "\sim"] & \delta_X^! f_1^! f_2^{-1} L_Y \arrow[r] & \delta_X^! L_X \\
\pi_*{}^t f'_! f_\pi^{-1}\Gamma_{\Lambda_Y}\mu_{\Delta_Y} L_Y \arrow[r] \arrow[u, "\sim"] \arrow[d] & \pi_*{}^t f'_! \Gamma_\Lambda \mu_\Delta f_2^{-1} L_Y \arrow[r, "\sim"] \arrow[u, "\sim"] \arrow[d, "\sim"'] & \pi_*\Gamma_{\Lambda_X}\mu_{\Delta_X} f_1^! f_2^{-1} L_Y \arrow[r] \arrow[u, "\sim"] \arrow[d] & \pi_*\Gamma_{\Lambda_X}\mu_{\Delta_X} L_X \arrow[u, "\sim"] \arrow[d] \\
\pi_*{}^t f'_! f_\pi^{-1}\Gamma_{\Lambda_Y}\mu_{\Delta_Y}\delta_{Y!}\omega_Y \arrow[r] & \pi_*\Gamma_\Lambda\mu_\Delta\delta_! f^{-1}\omega_Y \arrow[r] & \pi_*\Gamma_{\Lambda_X}\mu_{\Delta_X}\delta_{X!} f^{-1}\omega_Y \otimes \omega_{X/Y} \arrow[r] & \pi_*\Gamma_{\Lambda_X}\mu_{\Delta_X}\delta_{X!}\omega_X
\end{tikzcd}$$

□

The commutativity of the first diagram will follows from Lemma 6.4 and 6.5 below, and that of the last one from Lemma 6.6 below. Since their proofs follow the same lines as for the direct image, we shall omit them. The other diagrams obviously commute.

Note that in lemmas 6.4, 6.5, 6.6 below, the reversed arrows will become isomorphisms when assuming that $\mathcal{M}$ and $\mathcal{N}$ belong to $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_Y^{op})$ and f is non-characteristic.

Lemma 6.4 *Let $\mathcal{M}$ and $\mathcal{N}$ belong to $\mathbf{D}^{\mathrm{b}}(\mathcal{D}_Y^{op})$. Then the diagram below commutes.*

$$\begin{tikzcd}
f^{-1}\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{N},\mathcal{M}) \arrow[r] \arrow[dddd] & f^{-1}\delta_Y^{!}\mathcal{H}om_{q_2^{-1}\mathcal{D}_Y}(q_2^{-1}\mathcal{N}, q_1^{-1}\mathcal{M}\otimes_{q_1^{-1}\mathcal{D}_Y}\mathcal{O}_{Y\times Y}^{(0,d_Y)}[d_Y]) \arrow[d] \\
& \delta^{!}f_2^{-1}\mathcal{H}om_{q_2^{-1}\mathcal{D}_Y}(q_2^{-1}\mathcal{N}, q_1^{-1}\mathcal{M}\otimes_{q_1^{-1}\mathcal{D}_Y}\mathcal{O}_{Y\times Y}^{(0,d_Y)}[d_Y]) \arrow[d, "\sim"'] \\
& \delta_X^{!}f_1^{!}f_2^{-1}\mathcal{H}om_{q_2^{-1}\mathcal{D}_Y}(q_2^{-1}\mathcal{N}, q_1^{-1}\mathcal{M}\otimes_{q_1^{-1}\mathcal{D}_Y}\mathcal{O}_{Y\times Y}^{(0,d_Y)}[d_Y]) \arrow[d] \\
& \delta_X^{!}f_1^{!}\mathcal{H}om_{q_2^{-1}\mathcal{D}_X}(q_2^{-1}\underline{f}^{-1}\mathcal{N}, q_1^{-1}\mathcal{M}\otimes_{q_1^{-1}\mathcal{D}_Y}\mathcal{O}_{Y\times X}^{(0,d_X)}[d_Y]) \\
\mathcal{H}om_{\mathcal{D}_Y}(\underline{f}^{-1}\mathcal{N},\underline{f}^{-1}\mathcal{M}) \arrow[r] & \delta_X^{!}\mathcal{H}om_{q_2^{-1}\mathcal{D}_X}(q_2^{-1}\underline{f}^{-1}\mathcal{N}, q_1^{-1}\underline{f}^{-1}\mathcal{M}\otimes_{q_1^{-1}\mathcal{D}_X}\mathcal{O}_{X\times X}^{(0,d_X)}[d_X]) \arrow[u]
\end{tikzcd}$$

Lemma 6.5 *Let $\mathcal{M}$ and $\mathcal{N}$ belong to $\mathbf{D}^{\mathrm{b}}(\mathcal{D}_Y^{op})$. Then the diagram below commutes.*

$$\begin{tikzcd}
f^{-1}\delta_Y^{!}\mathcal{H}om_{q_2^{-1}\mathcal{D}_Y}(q_2^{-1}\mathcal{N}, q_1^{-1}\mathcal{M}\otimes_{q_1^{-1}\mathcal{D}_Y}\mathcal{O}_{Y\times Y}^{(0,d_Y)}[d_Y]) \arrow[d] & f^{-1}\delta_Y^{!}\mathcal{M}\,\underline{\boxtimes}\,\underline{D}\mathcal{N}\otimes_{\mathcal{D}_{Y\times Y}}\mathcal{O}_{Y\times Y} \arrow[l] \arrow[d] \\
\delta^{!}f_2^{-1}\mathcal{H}om_{q_2^{-1}\mathcal{D}_Y}(q_2^{-1}\mathcal{N}, q_1^{-1}\mathcal{M}\otimes_{q_1^{-1}\mathcal{D}_Y}\mathcal{O}_{Y\times Y}^{(0,d_Y)}[d_Y]) \arrow[d, "\sim"'] & \delta^{!}f_2^{-1}\mathcal{M}\,\underline{\boxtimes}\,\underline{D}\mathcal{N}\otimes_{\mathcal{D}_{Y\times Y}}\mathcal{O}_{Y\times Y} \arrow[l] \arrow[d, "\sim"'] \\
\delta_X^{!}f_1^{!}f_2^{-1}\mathcal{H}om_{q_2^{-1}\mathcal{D}_Y}(q_2^{-1}\mathcal{N}, q_1^{-1}\mathcal{M}\otimes_{q_1^{-1}\mathcal{D}_Y}\mathcal{O}_{Y\times Y}^{(0,d_Y)}[d_Y]) \arrow[d] & \delta_X^{!}f_1^{!}f_2^{-1}\mathcal{M}\,\underline{\boxtimes}\,\underline{D}\mathcal{N}\otimes_{\mathcal{D}_{Y\times Y}}\mathcal{O}_{Y\times Y} \arrow[l] \arrow[d] \\
\delta_X^{!}f_1^{!}\mathcal{H}om_{q_2^{-1}\mathcal{D}_X}(q_2^{-1}\underline{f}^{-1}\mathcal{N}, q_1^{-1}\mathcal{M}\otimes_{q_1^{-1}\mathcal{D}_Y}\mathcal{O}_{Y\times X}^{(0,d_X)}[d_Y]) & \delta_X^{!}f_1^{!}\mathcal{M}\,\underline{\boxtimes}\,\underline{D}\underline{f}^{-1}\mathcal{N}\otimes_{\mathcal{D}_{Y\times X}}\mathcal{O}_{Y\times X}[d_{Y/X}] \arrow[l] \\
\delta_X^{!}\mathcal{H}om_{q_2^{-1}\mathcal{D}_X}(q_2^{-1}\underline{f}^{-1}\mathcal{N}, q_1^{-1}\underline{f}^{-1}\mathcal{M}\otimes_{q_1^{-1}\mathcal{D}_X}\mathcal{O}_{X\times X}^{(0,d_X)}[d_X]) \arrow[u] & \underline{f}^{-1}\mathcal{M}\,\underline{\boxtimes}\,\underline{D}\underline{f}^{-1}\mathcal{N}\otimes_{\mathcal{D}_{X\times X}}\mathcal{O}_{X\times X} \arrow[l] \arrow[u]
\end{tikzcd}$$

Lemma 6.6 *Let $\mathcal{N}$ belong to $\mathbf{D}^{\mathrm{b}}(\mathcal{D}_Y^{op})$. Then we have the commutative diagram:*

$$\begin{tikzcd}
f_1^{!}f_2^{-1}\mathcal{N}\,\underline{\boxtimes}\,\underline{D}\mathcal{N}\otimes_{\mathcal{D}_{Y\times Y}}\mathcal{O}_{Y\times Y} \arrow[r] & \underline{f}^{-1}\mathcal{N}\,\underline{\boxtimes}\,\underline{D}\underline{f}^{-1}\mathcal{N}\otimes_{\mathcal{D}_{X\times X}}\mathcal{O}_{X\times X} \arrow[dd] \\
f_1^{-1}f_2^{-1}\mathcal{N}\,\underline{\boxtimes}\,\underline{D}\mathcal{N}\otimes_{\mathcal{D}_{Y\times Y}}\mathcal{O}_{Y\times Y}\otimes\omega_{X/Y} \arrow[u] \arrow[d] & \\
\delta_{X!}f^{-1}\omega_Y\otimes\omega_{X/Y} \arrow[r] & \delta_{X!}\omega_X
\end{tikzcd}$$

Now let $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X)$ and $\mathcal{N} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_Y)$.

Theorem 6.7 *One has:*

$$\mu\mathrm{eu}(\mathcal{M} \underline{\boxtimes} \mathcal{N}) = \mu\mathrm{eu}(\mathcal{M}) \boxtimes \mu\mathrm{eu}(\mathcal{N}).$$

Proof: We set:

$$\begin{aligned}
L_X &= \mathcal{M} \underline{\boxtimes}\, \underline{D}\mathcal{M} \otimes^L_{\mathcal{D}_{X\times X}} \mathcal{O}_{X\times X}, \\
L_Y &= \mathcal{N} \underline{\boxtimes}\, \underline{D}\mathcal{N} \otimes^L_{\mathcal{D}_{Y\times Y}} \mathcal{O}_{Y\times Y}, \\
L_{X\times Y} &= (\mathcal{M} \underline{\boxtimes} \mathcal{N}) \underline{\boxtimes} (\underline{D}\mathcal{M} \underline{\boxtimes}\, \underline{D}\mathcal{N}) \otimes^L_{\mathcal{D}_{X\times Y\times X\times Y}} \mathcal{O}_{X\times Y\times X\times Y},
\end{aligned}$$

$$\Lambda_X = \mathrm{char}(\mathcal{M}), \Lambda_Y = \mathrm{char}(\mathcal{N}), \Lambda = \Lambda_X \times \Lambda_Y.$$

Then the diagram below obviously commutes, which completes the proof.

$$\begin{tikzcd}
\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{M}) \boxtimes \mathcal{H}om_{\mathcal{D}_X}(\mathcal{N},\mathcal{N}) \arrow[rr] \arrow[d] & & \mathcal{H}om_{\mathcal{D}_{X\times Y}}(\mathcal{M} \underline{\boxtimes} \mathcal{N}, \mathcal{M} \underline{\boxtimes} \mathcal{N}) \arrow[d] \\
\delta_X^! L_X \boxtimes \delta_Y^! L_Y \arrow[r] & \delta^!(L_X \boxtimes L_Y) \arrow[r] & \delta^! L_{X\times Y} \\
\pi_*\Gamma_{\Lambda_X}\mu_{\Delta_X} L_X \boxtimes \pi_*\Gamma_{\Lambda_Y}\mu_{\Delta_Y} L_Y \arrow[r] \arrow[u, "\sim"] \arrow[d] & \pi_*\Gamma_\Lambda\mu_\Delta(L_X \boxtimes L_Y) \arrow[r] \arrow[u, "\sim"] \arrow[d] & \pi_*\Gamma_\Lambda\mu_\Delta L_{X\times Y} \arrow[u, "\sim"] \arrow[d] \\
\pi_*\Gamma_{\Lambda_X}\mu_{\Delta_X}\delta_!\omega_X \boxtimes \pi_*\Gamma_{\Lambda_Y}\mu_{\Delta_Y}\delta_!\omega_Y \arrow[r] \arrow[d, "\sim"] & \pi_*\Gamma_\Lambda\mu_\Delta\delta_!(\omega_X \boxtimes \omega_Y) \arrow[r] \arrow[d, "\sim"] & \pi_*\Gamma_\Lambda\mu_\Delta\delta_!\omega_{X\times Y} \arrow[d, "\sim"] \\
\pi_*\Gamma_{\Lambda_X}\pi^{-1}\omega_X \boxtimes \pi_*\Gamma_{\Lambda_Y}\pi^{-1}\omega_Y \arrow[r] & \pi_*\Gamma_\Lambda\pi^{-1}(\omega_X \boxtimes \omega_Y) \arrow[r, "\sim"] & \pi_*\Gamma_\Lambda\pi^{-1}\omega_{X\times Y}
\end{tikzcd}$$

$\square$

7 Examples

7.1 Euler class of $\mathbb{R}$-constructible sheaves

Let F be an object of $\mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(X)$, X being still a complex manifold. We shall prove that $\mu\mathrm{eu}(F)$ is nothing but $CC(F)$, the characteristic cycle of F constructed by Kashiwara in [6], (see also [7, Chapter IX]). Recall that $CC(F)$ is obtained as the image of $id_F \in \mathrm{Hom}(F,F)$ in $H^0_\Lambda(T^*X;\pi^{-1}\omega_X)$, (where $\Lambda = SS(F)$), by the sequence of morphisms:

$$\begin{aligned}
R\mathcal{H}om(F,F) &\xleftarrow{\sim} \delta^!(F \boxtimes DF) \\
&\xleftarrow{\sim} R\pi_*R\Gamma_\Lambda\mu_\Delta(F \boxtimes DF) \\
&\longrightarrow R\pi_*R\Gamma_\Lambda\mu_\Delta\delta_!\omega_X \\
&\simeq R\pi_*R\Gamma_\Lambda\pi^{-1}\omega_X.
\end{aligned}$$

Proposition 7.1 *Let $F \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(X)$. Then:*

$$\mu\mathrm{eu}(F) = CC(F)$$

Proof: We start with the commutative diagram:

$$\begin{array}{ccc}
\delta_! \mathbb{C}_X & \xrightarrow{\sim} & \delta_! R\mathcal{H}om_{\mathcal{D}_X}(\Omega_X, \Omega_X) \\
\downarrow & & \downarrow \\
\mathbb{C}_X \boxtimes \omega_X & \xrightarrow{\sim} & \Omega_X \underline{\boxtimes} \underline{D}\Omega_X \otimes^L_{\mathcal{D}_{X\times X}} \mathcal{O}_{X\times X}
\end{array}$$

Tensoring by $q_1^{-1}F$, then applying $R\mathcal{H}om(q_2^{-1}F, \cdot)$, we get the commutative diagram:

$$\begin{array}{ccc}
\delta_! R\mathcal{H}om(F,F) & \xrightarrow{\sim} & \delta_! R\mathcal{H}om_{\mathcal{D}_X}(F\otimes\Omega_X, F\otimes\Omega_X) \\
\downarrow & & \downarrow \\
F \boxtimes DF & \xrightarrow{\sim} & F\otimes\Omega_X \underline{\boxtimes} D'F \otimes \underline{D}\Omega_X \otimes^L_{\mathcal{D}_{X\times X}} \mathcal{O}_{X\times X}
\end{array}$$

Set

$$H = F\otimes\Omega_X \underline{\boxtimes} D'F \otimes \underline{D}\Omega_X \otimes^L_{\mathcal{D}_{X\times X}} \mathcal{O}_{X\times X}$$

We have a commutative diagram:

$$\begin{array}{ccc}
F \boxtimes DF & \longrightarrow & H \\
\downarrow & & \downarrow \\
\delta_!\omega_X & = & \delta_!\omega_X .
\end{array}$$

Hence we have a commutative diagram, in which $\Lambda = SS(F)$:

$$\begin{array}{ccc}
R\mathcal{H}om(F,F) & \xrightarrow{\sim} & R\mathcal{H}om_{\mathcal{D}_X}(F\otimes\Omega_X, F\otimes\Omega_X) \\
\downarrow & & \downarrow \\
\delta^! F \boxtimes DF & \xrightarrow{\sim} & \delta^! H \\
\uparrow\wr & & \uparrow\wr \\
R\pi_* R\Gamma_\Lambda \mu_\Delta (F\boxtimes DF) & \xrightarrow{\sim} & R\pi_* R\Gamma_\Lambda \mu_\Delta H \\
\downarrow & & \downarrow \\
R\pi_* R\Gamma_\Lambda \mu_\Delta \delta_!\omega_X & = & R\pi_* R\Gamma_\Lambda \delta_!\omega_X \\
\downarrow & & \downarrow \\
R\pi_* R\Gamma_\Lambda \pi^{-1}\omega_X & = & R\pi_* R\Gamma_\Lambda \pi^{-1}\omega_X .
\end{array}$$

The result follows by applying the functor $H^0 R\Gamma(X;\cdot)$. □

7.2 Euler class of $\mathcal{D}$-modules and $\mathcal{E}$-modules

Let us first recall the construction of the microlocal Euler class of a coherent $\mathcal{D}_X$-module, which of course, is a little easier than that of an elliptic pair.

Let $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{coh}(\mathcal{D}_X)$, and let $\Lambda = \mathrm{char}(\mathcal{M})$. The isomorphism of $(\mathcal{D}_X, \mathcal{D}_X)$-bimodules

$$\mathcal{D}_X \simeq \mathcal{B}^{(0,d_X)}_{\Delta|X\times X}$$

gives rise to the chain of morphisms:

$$\begin{aligned}
R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{M}) &\simeq R\mathcal{H}om_{q_2^{-1}\mathcal{D}_X}(q_2^{-1}\mathcal{M}, q_1^{-1}\mathcal{M}\otimes_{q_1^{-1}\mathcal{D}_X}\mathcal{B}^{(0,d_X)}_{\Delta|X\times X})\\
&\simeq (\mathcal{M}\boxtimes \underline{D}\mathcal{M})\otimes^L_{\mathcal{D}_{X\times X}}\mathcal{B}_{\Delta|X\times X}[-d_X]\\
&\longrightarrow R\pi_*R\Gamma_\Lambda\mu_\Delta(\mathcal{M}\boxtimes\underline{D}\mathcal{M}\otimes^L_{\mathcal{D}_{X\times X}}\mathcal{O}_{X\times X})\\
&\longrightarrow R\pi_*R\Gamma_\Lambda\mu_\Delta\delta_!\omega_X\\
&\simeq R\pi_*R\Gamma_\Lambda\pi^{-1}\omega_X.
\end{aligned}$$

This defines the morphism:

$$\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M},\mathcal{M})\longrightarrow H^0_\Lambda(T^*X;\pi^{-1}\omega_X)$$

and since this morphism is obviously the same as that constructed for elliptic pairs in §3, $\mu\mathrm{eu}(\mathcal{M})$ is the image of $\mathrm{id}_\mathcal{M}$.

Let $\mathcal{E}_X$ denote the sheaf on T^*X of finite order microdifferential operators of [10] (see also [11] for a detailed exposition). We shall adapt our construction of the microlocal Euler class to the case of coherent $\mathcal{E}_X$-modules.

One denotes by $\mathcal{C}_{\Delta|X\times X}$ the simple holonomic $\mathcal{E}_{X\times X}$-module associated to the diagonal embedding $\Delta\hookrightarrow X\times X$, the "microlocalization" of the $\mathcal{D}_X$-module $\mathcal{B}_{\Delta|X\times X}$ encountered above. Isomorphism (3.2) entails the isomorphism of $(\mathcal{E}_X,\mathcal{E}_X)$-bimodules:

$$\mathcal{E}_X\simeq\mathcal{C}^{(0,d_X)}_{\Delta|X\times X}. \tag{7.1}$$

Consider a coherent right $\mathcal{E}_X$-module $\mathcal{N}$ defined on an open subset U of T^*X (or more generally an object of the derived category $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{E}_X^{\mathrm{op}}|_U)$). One can adapt to this situation the construction of the microlocal Euler class of elliptic pairs. Set

$$\underline{D}\mathcal{N}=R\mathcal{H}om_{\mathcal{E}_X}(\mathcal{N},\mathcal{E}_X\otimes_{\pi^{-1}\mathcal{O}_X}\pi^{-1}\Omega_X[d_X])$$

and let $\Lambda=\mathrm{supp}\,\mathcal{N}$. Morphism (7.1) gives rise to the chain of morphisms:

$$\begin{aligned}
R\mathcal{H}om_{\mathcal{E}_X}(\mathcal{N},\mathcal{N}) &\simeq R\mathcal{H}om_{q_2^{-1}\mathcal{E}_X}(q_2^{-1}\mathcal{N}, q_1^{-1}\mathcal{N}\otimes_{q_1^{-1}\mathcal{E}_X}\mathcal{C}^{(0,d_X)}_{\Delta|X\times X})\\
&\simeq \mathcal{N}\boxtimes\underline{D}\mathcal{N}\otimes^L_{\mathcal{E}_{X\times X}}\mathcal{C}_{\Delta|X\times X}[-d_X]\\
&\xleftarrow{\sim} \mathrm{R}\Gamma_\Lambda(\mathcal{N}\boxtimes\underline{D}\mathcal{N}\otimes^L_{\mathcal{E}_{X\times X}}\mathcal{C}_{\Delta|X\times X}[-d_X])\\
&\longrightarrow \mathrm{R}\Gamma_\Lambda(\mathcal{C}^{(2d_X)}_{\Delta|X\times X}\otimes^L_{\mathcal{E}_{X\times X}}\mathcal{C}_{\Delta|X\times X})\\
&\simeq \mathrm{R}\Gamma_\Lambda\pi^{-1}\omega_X.
\end{aligned}$$

Applying the functor $H^0R\Gamma(U;\cdot)$, we obtain the morphism:

$$\mathrm{Hom}_{\mathcal{E}_X}(\mathcal{N},\mathcal{N})\longrightarrow H^0_\Lambda(U;\pi^{-1}\omega_X). \tag{7.2}$$

Definition 7.2 Let $\mathcal{N}\in\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{E}_X^{\mathrm{op}}|_U)$. The image of $\mathrm{id}_\mathcal{N}$ by the morphism (7.2) is called the microlocal Euler class of $\mathcal{N}$ and is denoted $\mu\mathrm{eu}(\mathcal{N})$.

This definition is clearly compatible to that we have made for $\mathcal{D}$-modules, which implies that if U is open in T^*X and if $\mathcal{M}$ belongs to $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X)$, then:

$$\mu\mathrm{eu}(\mathcal{E}_X \otimes_{\pi^{-1}\mathcal{D}_X} \pi^{-1}\mathcal{M}|_U) = \mu\mathrm{eu}(\mathcal{M})|_U. \tag{7.3}$$

Remark 7.3 Let $\mathcal{E}_X^{\mathbb{R}}$ denote the sheaf of microlocal operators constructed in [10]. Recall that it is defined as:

$$\mathcal{E}_X^{\mathbb{R}} = \mu_\Delta \mathcal{O}_{X\times X}^{(0,d_X)}[d_X]. \tag{7.4}$$

If $\mathcal{N} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{E}_X^{\mathrm{op}}|_U)$, we set

$$\mathcal{N}^{\mathbb{R}} = \mathcal{N} \otimes_{\mathcal{E}_X} \mathcal{E}_X^{\mathbb{R}}.$$

Then replacing $\mathcal{C}_{\Delta|X\times X}$ by $\mathcal{C}^{\mathbb{R}}_{\Delta|X\times X}$ in the above construction, one sees it would be possible to define directly the microlocal Euler class of $\mathcal{N}^{\mathbb{R}}$ for $\mathcal{N}^{\mathbb{R}}$ perfect. Using the isomorphism

$$\mathcal{C}^{(2d_X)}_{\Delta|X\times X} \otimes^L_{\mathcal{E}_{X\times X}} \mathcal{C}_{\Delta|X\times X} \simeq \mathcal{C}^{(2d_X)}_{\Delta|X\times X} \otimes^L_{\mathcal{E}_{X\times X}} \mathcal{C}^{\mathbb{R}}_{\Delta|X\times X},$$

one gets that $\mu\mathrm{eu}(\mathcal{N}) = \mu\mathrm{eu}(\mathcal{N}^{\mathbb{R}})$. In particular, $\mu\mathrm{eu}(\mathcal{N})$ depends only on $\mathcal{N}^{\mathbb{R}}$.

7.3 Euler class of holonomic modules

Let $\mathcal{N}$ be a holonomic $\mathcal{E}_X$-module defined on an open subset U of T^*X, and let Λ denotes its support (i.e., its characteristic variety). Then Λ is a closed complex analytic Lagrangian subset of U, conic for the action of $\mathbb{C}^\times$ on T^*X and there is a complex conic smooth submanifold $\Lambda_0 \subset \Lambda$ which is open and dense in Λ. Let $\Lambda_0 = \bigsqcup_\alpha \Lambda_\alpha$, the Λ_α's being locally closed smooth and connected.

On each Λ_α, the $\mathcal{E}_X$-module $\mathcal{N}$ has a well-defined multiplicity m_α, defined by Kashiwara in [5]. Moreover, each Λ_α is closed in

$$U' := U \setminus (\Lambda \setminus \bigsqcup_\alpha \Lambda_\alpha)$$

and defines a Lagrangian cycle $[\Lambda_\alpha]$ in U'. Since $U \setminus U'$ has real codimension at least two in U, the sum $\sum_\alpha m_\alpha[\Lambda_\alpha]$ defines a Lagrangian cycle on U supported by Λ. Let us denote it by $CC(\mathcal{N})$. Then:

$$CC(\mathcal{N}) \in H^0_\Lambda(U; \pi^{-1}\omega_X).$$

Proposition 7.4 *Let $\mathcal{N}$ be a holonomic $\mathcal{E}_X$-module. Then:*

$$\mu\mathrm{eu}(\mathcal{N}) = CC(\mathcal{N}).$$

Proof: Since both terms of the formula are Lagrangian cycles, it is enough to prove the result at generic points of Λ. Hence we may assume $\Lambda = T^*_Z X \cap U$, where Z is a closed complex submanifold of X. Since $\mu\mathrm{eu}(\mathcal{N})$ depends only on $\mathcal{N}^{\mathbb{R}}$ (see Remark 7.3), we may assume that $\mathcal{N}$ is a finite direct sum of sheaves $\mathcal{C}_{Z|X} \otimes_{\mathcal{O}_X} \Omega_X$ on U. Hence, it remains to prove the formula:

$$\mu\mathrm{eu}(\mathcal{B}_{Z|X}) = [T^*_Z X] \tag{7.5}$$

This equality is a corollary of our preceding results. In fact, consider the embedding $i : Z \hookrightarrow X$ and the projection $a : Z \longrightarrow \{\mathrm{pt}\}$. Then $\mathcal{B}_{Z|X} = \underline{i}_*(\mathcal{O}_Z)$, and $\mathcal{O}_Z = \underline{a}^{-1}(\mathbb{C}_{\{\mathrm{pt}\}})$. Since the Lagrangian cycle $[T^*_Z X]$ is the direct image by the map i of the inverse image by a of the Lagrangian cycle $\mathbb{C}$ on the manifold $\{\mathrm{pt}\}$, the result follows from Theorems 5.1 and 6.3. □

Corollary 7.5 *Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module. Then*

$$\mu\mathrm{eu}(\mathcal{M}) = \mu\mathrm{eu}(\mathcal{M} \otimes^L_{\mathcal{D}_X} \mathcal{O}_X).$$

In other words, the microlocal Euler class of a holonomic $\mathcal{D}_X$-module is the same as that of the complex of its holomorphic solutions. (Recall that this last complex is constructible by [4].)

Proof: The result follows from Proposition 3.2 and the equality

$$CC(\mathcal{M}) = CC(\mathcal{M} \otimes^L_{\mathcal{D}_X} \mathcal{O}_X)$$

proved in [5], but it can also be obtained directly, by considering the commutative diagram below.

$$\begin{tikzcd}
R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{M}) \arrow[r] \arrow[d] & R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M} \otimes^L_{\mathcal{D}_X} \mathcal{O}_X, \mathcal{M} \otimes^L_{\mathcal{D}_X} \mathcal{O}_X) \arrow[d] \\
\delta^!(\mathcal{M} \underline{\boxtimes}\, \underline{D}\mathcal{M} \otimes^L_{\mathcal{D}_{X\times X}} \mathcal{O}_{X\times X}) \arrow[d] & \delta^!((\mathcal{M} \otimes^L_{\mathcal{D}_X} \mathcal{O}_X) \boxtimes (\underline{D}\mathcal{M} \otimes^L_{\mathcal{D}_X} \mathcal{O}_X)) \arrow[l, "\sim"'] \arrow[d] \\
\Omega_X[d_X] \otimes^L_{\mathcal{D}_X} \mathcal{O}_X \arrow[r, "\sim"] & \omega_X
\end{tikzcd}$$

□

7.4 Euler class of $\mathcal{O}$-modules

Consider a coherent $\mathcal{O}_X$-module $\mathcal{F}$. To it, one can associate the right coherent $\mathcal{D}_X$-module $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{D}_X$. We shall show that the Euler class of this $\mathcal{D}_X$-module is the natural image of a cohomology class which belongs to $H^{d_X}_{\mathrm{supp}\,\mathcal{F}}(X;\Omega_X)$. For that purpose, let us introduce the following notations.

Let $\mathcal{F}$ and $\mathcal{G}$ be two $\mathcal{O}_X$-modules. We set:

$$\begin{aligned}
D_{\mathcal{O}}\mathcal{F} &= R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \Omega_X[d_X]), \\
\mathcal{F} \boxtimes_{\mathcal{O}} \mathcal{G} &= \mathcal{O}_{X\times X} \otimes_{\mathcal{O}_X \boxtimes \mathcal{O}_X} (\mathcal{F} \boxtimes \mathcal{G}).
\end{aligned}$$

Now assume $\mathcal{F}$ is $\mathcal{O}_X$-coherent and consider the chain of morphisms:

$$R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{F}) \longrightarrow \delta^!(\mathcal{F} \boxtimes_{\mathcal{O}} D_{\mathcal{O}}\mathcal{F}) \longrightarrow \mathcal{F} \otimes^L_{\mathcal{O}} D_{\mathcal{O}}\mathcal{F} \longrightarrow \Omega_X[d_X].$$

This defines:

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{F}) \longrightarrow H^{d_X}_{\operatorname{supp}\mathcal{F}}(X;\Omega_X). \tag{7.6}$$

Definition 7.6 Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module and let S denote its support. The image of $id_{\mathcal{F}}$ in $H^{d_X}_S(X;\Omega_X)$ by the morphism (7.6) is called the *holomorphic Euler class* of $\mathcal{F}$ and denoted by $\mathrm{eu}_{\mathcal{O}}(\mathcal{F})$.

The natural morphism $\Omega_X[d_X] \longrightarrow \omega_X$ defines the morphism:

$$\alpha : H^{d_X}_S(X;\Omega_X) \longrightarrow H^0_S(X;\omega_X). \tag{7.7}$$

Proposition 7.7 *Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then $\mathrm{eu}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{D}_X)$ is the image of $\mathrm{eu}_{\mathcal{O}}(\mathcal{F})$ by the morphism (7.7).*

Proof: We start with the commutative diagram:

$$\begin{tikzcd}
\delta_!\mathcal{O}_X \arrow[r] \arrow[d] & \delta_!\mathcal{D}_X \arrow[d] \\
\mathcal{O}_X \boxtimes_{\mathcal{O}} D_{\mathcal{O}}\mathcal{O}_X \arrow[r] & (\mathcal{D}_X \boxtimes \underline{D}\mathcal{D}_X) \otimes^L_{\mathcal{D}_X \boxtimes \mathcal{D}_X} \mathcal{O}_{X\times X}.
\end{tikzcd}$$

Applying the functor $q_1^{-1}\mathcal{F} \otimes_{q_1^{-1}\mathcal{O}_X} \cdot$, then the functor $R\mathcal{H}om_{q_2^{-1}\mathcal{O}_X}(q_2^{-1}\mathcal{F}, \cdot)$, we get the commutative diagram:

$$\begin{tikzcd}
\delta_! R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{F}) \arrow[r] \arrow[d] & \delta_! R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{D}_X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{D}_X) \arrow[d] \\
\mathcal{F} \boxtimes_{\mathcal{O}} D_{\mathcal{O}}\mathcal{F} \arrow[r] & (\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{D}_X) \boxtimes \underline{D}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{D}_X) \otimes^L_{\mathcal{D}_X \boxtimes \mathcal{D}_X} \mathcal{O}_{X\times X}.
\end{tikzcd}$$

Set

$$H = (\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{D}_X) \underline{\boxtimes}\, \underline{D}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{D}_X) \otimes^L_{\mathcal{D}_{X\times X}} \mathcal{O}_{X\times X}.$$

Then

$$H \simeq (\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{D}_X) \boxtimes \underline{D}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{D}_X) \otimes^L_{\mathcal{D}_X \boxtimes \mathcal{D}_X} \mathcal{O}_{X\times X}.$$

On the other hand, we have the commutative diagram:

$$\begin{tikzcd}
\delta^{-1}\mathcal{F} \boxtimes_{\mathcal{O}} D_{\mathcal{O}}\mathcal{F} \arrow[r] \arrow[d] & \delta^{-1}H \arrow[d] \\
\Omega_X[d_X] \arrow[r] & \omega_X.
\end{tikzcd}$$

Hence, we get the commutative diagram:

$$\begin{tikzcd}
R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{F}) \arrow[r] \arrow[d] & R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{D}_X, \mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{D}_X) \arrow[d] \\
\delta^!(\mathcal{F}\boxtimes_{\mathcal{O}} D_{\mathcal{O}_X}\mathcal{F}) \arrow[r] \arrow[d] & \delta^! H \arrow[d] \\
\Omega_X[d_X] \arrow[r] & \omega_X
\end{tikzcd}$$

which completes the proof. □

Remark 7.8 The holomorphic Euler class of a coherent $\mathcal{O}_X$-module is known for long, and O'Brian, Toledo and Tong [9] have proved that this class can be obtained as the term of degree d_X of the product of the Chern character of $\mathcal{F}$ by the Todd class of X. See §7 below for further comments on this point.

Remark 7.9 One should not confuse the holomorphic Euler class $\mathrm{eu}_{\mathcal{O}_X}(\cdot)$ and the Euler class $\mathrm{eu}(\cdot)$. For example, $\alpha(\mathrm{eu}_{\mathcal{O}_X}(\mathcal{O}_X)) = \mathrm{eu}(\mathcal{D}_X)$ and $\mathrm{eu}(\mathcal{O}_X) = \mathrm{eu}(\mathbb{C}_X)$. If one chooses $X = P^1(\mathbb{C})$, it follows from Theorem 5.1 that:

$$\int_X \mathrm{eu}(\mathcal{O}_X) = 2, \tag{7.8}$$

$$\int_X \mathrm{eu}_{\mathcal{O}_X}(\mathcal{O}_X) = 1. \tag{7.9}$$

This example also shows that the diagram below is not commutative.

$$\begin{tikzcd}
\mathbb{C}_X \arrow[r] \arrow[d] & \mathcal{O}_X \arrow[d] \\
\omega_X & \Omega_X[d_X] \arrow[l]
\end{tikzcd}$$

Here, the first and second vertical arrows are defined by

$$\delta_! \mathbb{C}_X \longrightarrow \mathbb{C}_X \boxtimes \omega_X,$$

and

$$\delta_! \mathcal{O}_X \longrightarrow \mathcal{O}_X \boxtimes_{\mathcal{O}} D_{\mathcal{O}}\mathcal{O}_X,$$

respectively, as in the proofs of Propositions 3.2 and 3.8.

Remark 7.10 Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module and denote by S its support. Then $\mathrm{char}(\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{D}_X) = \pi^{-1}S$, hence:

$$\mu\mathrm{eu}(\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{D}_X) = \pi^*\,\mathrm{eu}(\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{D}_X),$$

where π^* is the isomorphism:

$$H^0_S(X;\omega_X) \xrightarrow{\sim} H^0_{\pi^{-1}S}(T^*X;\pi^{-1}\omega_X).$$

8 A conjectural link with Chern classes

Let X be a complex manifold, Z a complex analytic subset and denote by $K_Z^{an}(X)$ the Grothendieck group of the full subcategory of $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{O}_X)$ consisting of objects supported by Z. In this section we shall assume to be constructed a local Chern character:

$$ch_Z : K_Z^{an}(X) \longrightarrow \bigoplus_j H_Z^{2j}(X; \mathbb{C}_X)$$

such that if we define the local Euler character by the formula:

$$\mathrm{eu}_Z(\mathcal{F}) = ch_Z(\mathcal{F}) \cup td_X(TX)$$

(where $\cup$ is the cup product and $td_X(\cdot)$ is the Todd class), then the local Chern character is compatible to external product and inverse image and the local Euler character is compatible to external product and proper direct image, this last point being what we shall refer as the Grothendieck-Riemann-Roch theorem. Such a construction does exists in the algebraic case (see [2]). In the analytic case, one can construct $ch_Z(\cdot)$ after shrinking X (see [3]). More precisely, let X' be an open relatively compact subset of X. Then one defines the natural morphism:

$$\rho : K_Z^{an}(X) \longrightarrow K_Z^{top}(X')$$

by realification. If $\mathcal{F}$ is a bounded complex of coherent $\mathcal{O}_X$-modules, we associate to it the complex $\mathcal{F}^{\mathbb{R}} := \mathcal{A}_{X^{\mathbb{R}}} \otimes_{\mathcal{O}_X} \mathcal{F}$, where $\mathcal{A}_{X^{\mathbb{R}}}$ denotes the sheaf of real analytic functions on the real analytic manifold $X^{\mathbb{R}}$ underlying X. Applying Cartan's theorem "A" (on the closure of X') we see that $\mathcal{F}^{\mathbb{R}}$ defines an element of $K_Z^{top}(X')$. Unfortunately, the Grothendieck-Riemann-Roch theorem (with supports) has, to our knowledge, never been written in this case. Hence the results of this section should be considered as conjectural, or should be stated with suitable modifications (e.g. assuming we work in the algebraic category).

Now consider a left coherent $\mathcal{D}_X$-module $\mathcal{M}$ endowed with a good filtration and whose characteristic variety is contained in a closed conic analytic subset Λ of T^*X. Let $gr(\mathcal{M})$ denote the associated graded module and set:

$$\tilde{gr}(\mathcal{M}) = \mathcal{O}_{T^*X} \otimes_{\pi^{-1}gr(\mathcal{D}_X)} \pi^{-1}gr(\mathcal{M}).$$

Note that the element $\sigma_\Lambda(\mathcal{M})$ of $K_\Lambda^{an}(T^*X)$ defined by $\tilde{gr}(\mathcal{M})$ locally depends only on $\mathcal{M}$, not on the choice of the good filtration [5].

Let $f : X \longrightarrow Y$ be a morphism of complex manifolds. We shall use the notations introduced in §2, in particular in (2.9), (2.12), (2.13) and (2.15).

First consider a closed conic subset Λ_Y of T^*Y, and assume f is non-characteristic with respect to Λ_Y (i.e. ${}^tf'$ is proper on $f_\pi^{-1}(\Lambda_Y)$). Then the morphisms:

$$f_\pi^* : K_{\Lambda_Y}^{an}(T^*Y) \longrightarrow K_{f_\pi^{-1}\Lambda_X}^{an}(X \times_Y T^*Y),$$

and
$$ {}^t f'_* : K^{an}_{f_\pi^{-1}\Lambda_Y}(X \times_Y T^*Y) \longrightarrow K^{an}_{f^\mu(\Lambda_Y)}(T^*X) $$
are well-defined and if $\mathcal{N}$ is a left coherent $\mathcal{D}_Y$-module whose characteristic variety is contained in Λ_Y, it follows from Kashiwara [5] that:
$$ {}^t f'_* f_\pi^* \sigma_{\Lambda_Y}(\mathcal{N}) = \sigma_{f^\mu(\Lambda_Y)}(\underline{f}^{-1}\mathcal{N}). \tag{8.1} $$
Similarly if f is proper on $\Lambda_X \cap T^*_X X$ (i.e. f_π is proper on ${}^t f'^{-1}(\Lambda_Y)$), then:
$$ {}^t f'^* : K^{an}_{\Lambda_X}(T^*X) \longrightarrow K^{an}_{{}^t f'^{-1}(\Lambda_X)}(X \times_Y T^*Y) $$
and
$$ f_{\pi *} : K^{an}_{{}^t f'^{-1}(\Lambda_Y)}(X \times_Y T^*X) \longrightarrow K^{an}_{f_\mu(\Lambda_X)}(T^*Y) $$
are well-defined, and it is shown in Laumon [8] that if $\mathcal{M}$ is a right good $\mathcal{D}_X$-module whose characteristic variety is contained in Λ_X, then:
$$ f_{\pi *} {}^t f'^* \sigma_{\Lambda_X}(\mathcal{M}) = \sigma_{f_\mu(\Lambda_X)}(\underline{f}_!\mathcal{M}). \tag{8.2} $$
Finally one shows easily that:
$$ \sigma_{\Lambda_X}(\mathcal{M}) \boxtimes \sigma_{\Lambda_Y}(\mathcal{N}) = \sigma_{\Lambda_X \times \Lambda_Y}(\mathcal{M} \underline{\boxtimes} \mathcal{N}). \tag{8.3} $$

Using the Riemann-Roch-Grothendieck Theorem at the level of cotangent bundles, Laumon (loc.cit.) has deduced from (8.2) a formula which computes the Chern characters of $\sigma_{f_\mu(\Lambda_X)}(\underline{f}_!\mathcal{M})$ from that of $\sigma_{\Lambda_X}(\mathcal{M})$. In order to get a class which behaves well both under direct and inverse images, we introduce the following:

Definition 8.1 Let $\mathcal{M}$ (resp. $\mathcal{N}$) be a right (resp. left) coherent $\mathcal{D}_X$-module endowed with a good filtration and whose characteristic variety is contained in a closed conic analytic subset Λ of T^*X. We define the microlocal Chern character of $\mathcal{M}$ and $\mathcal{N}$ along Λ as:
$$ \begin{aligned} \mu ch_\Lambda(\mathcal{M}) &= ch_\Lambda(\sigma_\Lambda(\mathcal{M})) \cup \pi^* td_X(TX), \\ \mu ch_\Lambda(\mathcal{N}) &= ch_\Lambda(\sigma_\Lambda(\mathcal{N})) \cup \pi^* td_X(T^*X). \end{aligned} $$
We denote by $\mu ch^j_\Lambda(\mathcal{M})$ the component of $\mu ch_\Lambda(\mathcal{M})$ in $H^j_\Lambda(T^*X; \mathbb{C}_{T^*X})$, and similarly for $\mathcal{N}$.

This definition is motivated by the two following statements.

Proposition 8.2 *The microlocal Chern character of the right $\mathcal{D}_X$-module $\mathcal{M} \otimes_{\mathcal{O}_X} \Omega_X$ is the microlocal Chern character of the left $\mathcal{D}_X$-module $\mathcal{M}$. In other words, if $\mathcal{M}$ is a left $\mathcal{D}_X$-module:*
$$ \mu ch_\Lambda(\mathcal{M} \otimes_{\mathcal{O}_X} \Omega_X) = \mu ch_\Lambda(\mathcal{M}). $$

Proof: Recall that if E is a complex vector bundle of rank d, then denoting by $c_1(\cdot)$ the first Chern class:

$$\begin{aligned} td_X(E^*) &= e^{c_1(E^*)} \cup td_X(E), \\ ch(\Lambda^d E) &= e^{c_1(E)}. \end{aligned}$$

Choosing $E = TX$, we get:

$$td_X(T^*X) = ch(\Omega_X) \cup td_X(TX),$$

hence:

$$\begin{aligned} ch_\Lambda(\sigma_\Lambda(\mathcal{M} \otimes_{\mathcal{O}_X} \Omega_X)) \cup \pi^* td_X(TX) &= ch_\Lambda(\sigma_\Lambda(\mathcal{M})) \cup \pi^* ch(\Omega_X) \cup \pi^* td_X(TX) \\ &= ch_\Lambda(\sigma_\Lambda(\mathcal{M})) \cup \pi^* td_X(T^*X). \end{aligned}$$

□

Theorem 8.3 *Let $\mathcal{M}$ (resp. $\mathcal{N}$) be a coherent $\mathcal{D}_X$-module (resp. $\mathcal{D}_Y$-module) endowed with a good filtration, and let Λ_X (resp. Λ_Y) denote its characteristic variety.*

(i) Assume f in non-characteristic for $\mathcal{N}$. Then:

$$f^\mu(\mu ch_{\Lambda_Y}(\mathcal{N})) = \mu ch_{f^\mu(\Lambda_Y)}(\underline{f}^{-1}\mathcal{N}).$$

(ii) Assume f is proper on $\operatorname{supp}\mathcal{M}$. *Then:*

$$f_\mu(\mu ch_{\Lambda_X}(\mathcal{M})) = \mu ch_{f_\mu(\Lambda_X)}(\underline{f}_*\mathcal{M}).$$

(iii) One has:

$$\mu ch_{\Lambda_X}(\mathcal{M}) \boxtimes \mu ch_{\Lambda_Y}(\mathcal{N}) = \mu ch_{\Lambda_X \times \Lambda_Y}(\mathcal{M} \boxtimes \mathcal{N}).$$

Notice that in the above statements (i) and (ii), $\mathcal{M}$ or $\mathcal{N}$ can either be a right or a left $\mathcal{D}$-module (of course, in (iii) they need to be of the same type). This follows from Proposition 8.2 since

$$\underline{f}_!(\mathcal{M} \otimes_{\mathcal{O}_X} \Omega_X) = (\underline{f}_!\mathcal{M}) \otimes_{\mathcal{O}_Y} \Omega_Y,$$

and similarly for inverse images.

Proof: In the course of the proof we shall sometimes use the following notations: if W is a manifold, we set for short

$$td(W) = td_W(TW).$$

Then recall that if $p : E \longrightarrow W$ is a complex vector bundle on W, one has:

$$td(E) = td_E(TE) = p^* td_W(E) \cup p^* td(W),$$

which follows from the exact sequence of vector bundles on W:

$$0 \longrightarrow p^{-1}E \longrightarrow TE \longrightarrow p^{-1}TW \longrightarrow 0.$$

Also recall the diagram associated to f:

$$\begin{tikzcd} T^*X \arrow[d, "\pi_X"'] & X \times_Y T^*Y \arrow[l, "{}^t f'"'] \arrow[r, "f_\pi"] \arrow[d, "\pi"] & T^*Y \arrow[d, "\pi_Y"] \\ X \arrow[r, equal] & X \arrow[r, "f"] & Y \end{tikzcd}$$

(i) We may assume $\mathcal{N}$ is a left $\mathcal{D}_Y$-module. Using (8.1) and the Riemann-Roch-Grothendieck theorem applied to the map ${}^t f'$, we obtain:

$$\begin{aligned} ch_{f^\mu(\Lambda_Y)}\sigma_{f^\mu(\Lambda_Y)}(\underline{f}^{-1}\mathcal{N}) \cup td(T^*X) &= ch_{f^\mu(\Lambda_Y)}[{}^t f'_! f_\pi^* \sigma_{\Lambda_Y}(\mathcal{N})] \cup td(T^*X) \\ &= {}^t f'_! [f_\pi^* ch_{\Lambda_Y}\sigma_{\Lambda_Y}(\mathcal{N}) \cup td(X \times_Y T^*Y)]. \end{aligned}$$

Hence:

$$\begin{aligned} & ch_{f^\mu(\Lambda_Y)}\sigma_{f^\mu(\Lambda_Y)}(\underline{f}^{-1}\mathcal{N}) \cup \pi_X^* td_X(T^*X) \cup \pi_X^* td_X(TX) \\ &\quad = {}^t f'_! [f_\pi^* ch_{\Lambda_Y}\sigma_{\Lambda_Y}(\mathcal{N}) \cup \pi^* td_X(TX) \cup \pi^* f^* td_Y(T^*Y)] \\ &\quad = {}^t f'_! f_\pi^* [ch_{\Lambda_Y}\sigma_{\Lambda_Y}(\mathcal{N}) \cup \pi_Y^* td_Y(T^*Y)] \cup \pi_X^* td_X(TX) \end{aligned}$$

and the result follows since $td_X(TX)$ has an inverse.
(ii) We may assume $\mathcal{M}$ is a right $\mathcal{D}_X$-module. Using (8.2) and the Riemann-Roch-Grothendieck theorem, we get:

$$\begin{aligned} ch_{f_\mu(\Lambda_X)}\sigma_{f_\mu}(\underline{f}_!(\mathcal{M}) \cup td(T^*Y)) &= ch_{f_\mu(\Lambda_X)}[f_{\pi !}{}^t f'^* \sigma_{\Lambda_X}(\mathcal{M})] \cup td(T^*Y) \\ &= f_{\pi !}[{}^t f'^* ch_{\Lambda_X}\sigma_{\Lambda_X}(\mathcal{M}) \cup td(X \times_Y T^*Y)]. \end{aligned}$$

Hence:

$$\begin{aligned} & ch_{f_\mu(\Lambda_X)}\sigma_{f_\mu(\Lambda_X)}(\underline{f}_!\mathcal{M}) \cup \pi_Y^* td_Y(T^*Y) \cup \pi_Y^* td_Y(TY) \\ &\quad = f_{\pi !}[{}^t f'^* ch_{\Lambda_X}\sigma_{\Lambda_X}(\mathcal{M}) \cup \pi^* td_X(TX) \cup \pi^* f^* td_Y(T^*Y)] \\ &\quad = f_{\pi !}{}^t f'^* [ch_{\Lambda_X}\sigma_{\Lambda_X}(\mathcal{M}) \cup \pi_X^* td_X(TX)] \cup \pi_Y^* td_Y(T^*Y) \end{aligned}$$

and the result follows since $td_Y(T^*Y)$ is invertible.
(iii) follows from (8.3) and the fact that $ch(\cdot)$ commutes to external product. □

As a corollary we get that if $\mathcal{M}$ and $\mathcal{N}$ are two $\mathcal{D}_X$-modules with characteristic variety contained in Λ_0 and Λ_1 respectively, and if $\Lambda_0 \cap \Lambda_1 \subset T_X^*X$, then:

$$\mu ch_{(\Lambda_0+\Lambda_1)}(\mathcal{M} \otimes_{O_X}^L \mathcal{N}) = \mu ch_{\Lambda_0}(\mathcal{M}) *_\mu \mu ch_{\Lambda_1}(\mathcal{N}).$$

In view of Theorem 8.3, the microlocal Chern character has the same functorial properties as the microlocal Euler class.

Let us come back to the situation of Definition 8.1 and set:

$$c_\Lambda = \operatorname{codim}_{\mathbb{C}} \Lambda.$$

It is clear that:

$$\mu ch^j_\Lambda(\mathcal{M}) = 0 \quad \text{for} \quad j < 2c_\Lambda.$$

Proposition 8.4 *The component of degree $2c_\Lambda$ of the microlocal Chern character of $\mathcal{M}$, that is, $\mu ch^{2c_\Lambda}_\Lambda(\mathcal{M})$, is the characteristic cycle of $\mathcal{M}$ along Λ. In particular if $\mathcal{M}$ is holonomic:*

$$\mu ch^{2d_X}_\Lambda(\mathcal{M}) = \mu\mathrm{eu}(\mathcal{M}).$$

Proof: Since $\mu ch^j_\Lambda(\mathcal{M})$ is zero for $j < 2c_\Lambda$,we have

$$\mu ch^{2c_\Lambda}_\Lambda(\mathcal{M}) = [ch_\Lambda(\sigma_\Lambda(\tilde{gr}(\mathcal{M})))]^{2c_\Lambda},$$

and it is well-known that the term on the right-hand side is the analytic cycle on Λ of the $\mathcal{O}_{T^*X}$-coherent module $\tilde{gr}(\mathcal{M})$, that is, the characteristic cycle of $\mathcal{M}$. □

Now we make the following conjectures:

Conjecture 8.5 *(i) $\mu ch^j_\Lambda(\mathcal{M}) = 0$ for $j \notin [2c_\Lambda, 2d_X]$,*

(ii) $\mu ch^{2d_X}_\Lambda(\mathcal{M}) = \mu\mathrm{eu}(\mathcal{M})$.

By Proposition 8.4, Conjecture 8.5 (ii) is true for holonomic $\mathcal{D}_X$-modules. Moreover it follows from Remark 7.10 and the work of O'Brian-Toledo-Tong [9] that Conjecture 8.5 is true for induced $\mathcal{D}_X$-modules, i.e., for modules of the type $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{D}_X$, $\mathcal{F}$ being $\mathcal{O}_X$-coherent.

Example 8.6 Let M be a compact n-dimensional real analytic manifold, X a complexification of M, $\mathcal{M}$ a right coherent $\mathcal{D}_X$-module, elliptic on M. By Corollary 5.6, we have:

$$\chi(R\Gamma(M; \mathcal{M} \otimes^L_{\mathcal{D}_X} \mathcal{O}_X)) = \int_{T^*X} \mu\mathrm{eu}(\mathcal{M}) \cup \mu\mathrm{eu}(\mathbb{C}_M).$$

Denote by σ_M the zero-section embedding $M \hookrightarrow T^*_M X$ and by j the embedding $T^*_M X \hookrightarrow T^*X$. Since $T^*_M X \cap \operatorname{char}(\mathcal{M})$ is contained in M, we get:

$$\int_{T^*X} \mu\mathrm{eu}(\mathcal{M}) \cup \mu\mathrm{eu}(\mathbb{C}_M) = \int_{T^*_M X} j^* \mu\mathrm{eu}(\mathcal{M}) = \int_M \sigma^*_M j^* \mu\mathrm{eu}(\mathcal{M}).$$

Now assume Conjecture 8.5 (ii) is true. We get, with $\Lambda = \operatorname{char}(\mathcal{M})$:

$$\begin{aligned} \chi(R\Gamma(M; \mathcal{M} \otimes^L_{\mathcal{D}_X} \mathcal{O}_X)) &= \int_M \sigma^*_M j^*[ch(\sigma_\Lambda(\mathcal{M})) \cup \pi^* td_X(TX)] \\ &= \int_M \sigma^*_M [j^* ch(\sigma_\Lambda(\mathcal{M}))] \cup td_M(TM^{\mathbb{C}}). \end{aligned}$$

This is the classical Atiyah-Singer index formula.

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