Quantization of real conic Lagrangian manifolds: a survey

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Abstract

We will recall classical results on quantization of conic Lagrangian submanifolds of cotangent bundles in the real case. These Notes contain nothing really new or original although some results of § 4 do not appear in the literature.

Contents

1 Sheaves

In all these Notes, references are made to [KS90].

Main notations

We choose a field $\mathbf{k}$ although many results extend when replacing $\mathbf{k}$ with a unital commutative ring with finite global dimension ($\mathbf{k}$ being Noetherian when considering constructible sheaves).

In these Notes, M will denote a real manifold of class C^∞. We denote by $\mathrm{D}(\mathbf{k}_M)$ the derived category of sheaves of $\mathbf{k}$-modules on M and by $\mathrm{D^b}(\mathbf{k}_M)$ the bounded derived category. We shall also consider the subcategory $\mathrm{D^{lb}}(\mathbf{k}_M)$ of $\mathrm{D}(\mathbf{k}_M)$ consisting of objects "locally bounded".

When M is real analytic, we denote by $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_M)$ the full triangulated subcategory of $\mathrm{D^b}(\mathbf{k}_M)$ consisting of $\mathbb{R}$-constructible sheaves. If $M = \mathrm{pt}$, one has $\mathrm{D^b}(\mathbf{k}_{\mathrm{pt}}) \simeq \mathrm{D^b}(\mathbf{k})$, the bounded derived category of $\mathbf{k}$-modules, and $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{\mathrm{pt}}) \simeq \mathrm{D}^{\mathrm{b}}_f(\mathbf{k})$, the subcategory of $\mathrm{D^b}(\mathbf{k})$ consisting of complexes with finite dimensional cohomology.

For a locally closed subset $A \subset M$, we denote by $\mathbf{k}_A$ the constant sheaf on A extended by 0 on $M \setminus A$. We define more generally the sheaf V_A for $V \in \mathrm{D^b}(\mathbf{k})$.

We denote by ω_M the dualizing complex on M. Recall that $\omega_M \simeq \mathrm{or}_M\,[\dim M]$, where or_M is the orientation sheaf on M and $\dim M$ the dimension of M. For a sheaf F (*i.e.,* an object of $\mathrm{D^b}(\mathbf{k}_M)$) one sets

$$\mathrm{D}'_M F = \mathrm{R}\mathscr{H}om\,(F, \mathbf{k}_M), \quad \mathrm{D}_M F = \mathrm{R}\mathscr{H}om\,(F, \omega_M).$$

We denote as usual by $\pi \colon T^*M \to M$ the cotangent bundle to M and for a smooth submanifold $S \subset M$, we denote by $T^*_S M$ the conormal bundle to S in M. We identify $T^*_M M$ with the zero-section of T^*M and we set $\dot{T}^*M := T^*M \setminus T^*_M M$.

The sphere cotangent bundle S^*M is the quotient of $\dot{T}^*M$ by the $\mathbb{R}^+$-action. It is a contact manifold and one denotes by ρ the map

$$\rho \colon \dot{T}^*M \to S^*M := \dot{T}^*M/\mathbb{R}^+.$$

When $M = N \times \mathbb{R}$, the manifold $(T^*N) \times \mathbb{R}$ is a contact manifold, for the contact form $dt + \xi/\tau dx$ and one considers the map

$$\rho \colon T^*_{\tau \neq 0}(N \times \mathbb{R}) \to (T^*N) \times \mathbb{R}, \quad (x, t; \xi, \tau) \mapsto ((x; \xi/\tau), t).$$

Microsupport

References for this subsection are made to [KS90, § 5.1, § 6.1, § 6.5].

We shall recall the definition of the microsupport (or singular support) $\mathrm{SS}(F)$ of a sheaf F

Definition 1.1 (See [KS90, Def. 5.1.2])**.** Let $F \in \mathrm{D^b}(\mathbf{k}_M)$ and let $p \in T^*M$. One says that $p \notin \mathrm{SS}(F)$ if there exists an open neighborhood U of p such that for any $x_0 \in M$ and any real C^1-function φ on M defined in a neighborhood of x_0 with $(x_0; d\varphi(x_0)) \in U$, one has $\mathrm{R}\Gamma_{\{x;\varphi(x) \geq \varphi(x_0)\}}(F)_{x_0} \simeq 0$.

In other words, $p \notin \mathrm{SS}(F)$ if the sheaf F has no cohomology supported by "half-spaces" whose conormals are contained in a neighborhood of p.

- By its construction, the microsupport is $\mathbb{R}^+$-conic, that is, invariant by the action of $\mathbb{R}^+$ on T^*M.
- $\mathrm{SS}(F) \cap T^*_M M = \pi_M(\mathrm{SS}(F)) = \mathrm{supp}(F)$, where $\mathrm{supp}(\cdot)$ denotes the support.
- The microsupport satisfies the triangular inequality: if $F_1 \to F_2 \to F_3 \xrightarrow{+1}$ is a distinguished triangle in $\mathrm{D^b}(\mathbf{k}_M)$, then $\mathrm{SS}(F_i) \subset \mathrm{SS}(F_j) \cup \mathrm{SS}(F_k)$ for all $i, j, k \in \{1, 2, 3\}$ with $j \neq k$.

Example 1.2. (i) If M is connected, then $\mathrm{SS}(F) = T^*_M M$ if and only if $H^j(F)$ is a locally constant sheaf on M for all $j \in \mathbb{Z}$ and $F \neq 0$.
(ii) If N is a closed submanifold of M and $F = \mathbf{k}_N$, then $\mathrm{SS}(F) = T^*_N M$, the conormal bundle to N in M.

(iii) Let φ be a C^1-function such that $d\varphi(x) \neq 0$ whenever $\varphi(x) = 0$. Let $U = \{x \in M; \varphi(x) > 0\}$ and let $Z = \{x \in M; \varphi(x) \geq 0\}$. Then

$$\begin{aligned}
\mathrm{SS}(\mathbf{k}_U) &= U \times_M T^*_M M \cup \{(x; \lambda d\varphi(x)); \varphi(x) = 0, \lambda \leq 0\},\\
\mathrm{SS}(\mathbf{k}_Z) &= Z \times_M T^*_M M \cup \{(x; \lambda d\varphi(x)); \varphi(x) = 0, \lambda \geq 0\}.
\end{aligned}$$

For a precise definition of being involutive (or co-isotropic), we refer to [KS90, Def. 6.5.1]

Theorem 1.3. *Let $F \in \mathrm{D^b}(\mathbf{k}_M)$. Then its microsupport $\mathrm{SS}(F)$ is involutive.*

Notation 1.4. One denotes by $\mathrm{DLoc}(\mathbf{k}_M)$ the full triangulated subcategory of $\mathrm{D^b}(\mathbf{k}_M)$ consisting of sheaves with microsupport contained in the zero-section $T^*_M M$.

Kernels

References for this subsection are made to [KS90, §3.6].

Let M_i $(i = 1, 2, 3)$ be manifolds. For short, we write $M_{ij} := M_i \times M_j$ $(1 \leq i, j \leq 3)$ and $M_{123} = M_1 \times M_2 \times M_3$. We denote by q_i the projection $M_{ij} \to M_i$ or the projection $M_{123} \to M_i$ and by q_{ij} the projection $M_{123} \to M_{ij}$. Similarly, we denote by p_i the projection $T^*M_{ij} \to T^*M_i$ or the projection $T^*M_{123} \to T^*M_i$ and by p_{ij} the projection $T^*M_{123} \to T^*M_{ij}$. We also need to introduce the map p_{12^a}, the composition of p_{12} and the antipodal map on T^*M_2.

Let $\Lambda_1 \subset T^*M_{12}$ and $\Lambda_2 \subset T^*M_{23}$. We set

$$\Lambda_1 \overset{a}{\circ} \Lambda_2 := p_{13}({p_{12^a}}^{-1}\Lambda_1 \cap p_{23}^{-1}\Lambda_2). \tag{1.1}$$

We consider the operation of convolution of kernels:

$$\begin{aligned}
\underset{2}{\circ}\colon \mathrm{D^b}(\mathbf{k}_{M_{12}}) \times \mathrm{D^b}(\mathbf{k}_{M_{23}}) &\to \mathrm{D^b}(\mathbf{k}_{M_{13}})\\
(K_1, K_2) &\mapsto K_1 \underset{2}{\circ} K_2 := \mathrm{R}q_{13!}(q_{12}^{-1}K_1 \overset{\mathrm{L}}{\otimes} q_{23}^{-1}K_2).
\end{aligned}$$

Let $\Lambda_i = \mathrm{SS}(K_i) \subset T^*M_{i,i+1}$ and assume that

$$\begin{cases}
\text{(i) } q_{13} \text{ is proper on } q_{12}^{-1}\operatorname{supp}(K_1) \cap q_{23}^{-1}\operatorname{supp}(K_2),\\
\text{(ii) } p_{12^a}^{-1}\Lambda_1 \cap p_{23}^{-1}\Lambda_2 \cap (T^*_{M_1}M_1 \times T^*M_2 \times T^*_{M_3}M_3)\\
\qquad\qquad \subset T^*_{M_1 \times M_2 \times M_3}(M_1 \times M_2 \times M_3).
\end{cases} \tag{1.2}$$

It follows from the functorial properties of the microsupport (that we have not recalled here) that under the assumption (1.2) we have:

$$\mathrm{SS}(K_1 \underset{2}{\circ} K_2) \subset \Lambda_1 \overset{a}{\circ} \Lambda_2. \tag{1.3}$$

If there is no risk of confusion, we write $\circ$ instead of $\underset{2}{\circ}$.

2 Simple sheaves

Localization

Let A be a subset of T^*M and let $Z = T^*M \setminus A$. The full subcategory $\mathrm{D}^{\mathrm{b}}_Z(\mathbf{k}_M)$ of $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_M)$ consisting of sheaves F such that $\mathrm{SS}(F) \subset Z$ is triangulated. One sets

$$\mathrm{D}^{\mathrm{b}}(\mathbf{k}_M; A) := \mathrm{D}^{\mathrm{b}}(\mathbf{k}_M)/\mathrm{D}^{\mathrm{b}}_Z(\mathbf{k}_M), \tag{2.1}$$

the localization of $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_M)$ by $\mathrm{D}^{\mathrm{b}}_Z(\mathbf{k}_M)$. Hence, the objects of $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_M; A)$ are those of $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_M)$ but a morphism $u\colon F_1 \to F_2$ in $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_M)$ becomes an isomorphism in $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_M; A)$ if, after embedding this morphism in a distinguished triangle $F_1 \to F_2 \to F_3 \xrightarrow{+1}$, one has $\mathrm{SS}(F_3) \cap A = \varnothing$.

When $A = \{p\}$ for some $p \in T^*M$, one writes $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_M; p)$ instead of $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_M; \{p\})$.

Assume that A is locally closed. We say that two subsets S_1, S_2 of T^*M define the same germ along A if $S_1 \cap U = S_2 \cap U$ for some neighborhood U of A. Let $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_M; A)$. Representing F with an object $\widetilde{F} \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_M)$, the germ of $\mathrm{SS}(\widetilde{F})$ along A depends only on F and is again called the micro-support of F. If S is a germ of subset along A, one denotes by

$$\mathrm{D}^{\mathrm{b}}_S(\mathbf{k}_M; A) \tag{2.2}$$

the full triangulated category of $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_M; A)$ consisting of objects micro-supported by S.

The functor μhom

References for this subsection are made to [KS90, §4.4, §6.2 §7.2].

The functor of microlocalization along a submanifold has been introduced by Mikio Sato in the 70's (see [SKK73]) and has been at the origin of what is now called "microlocal analysis". A variant of this functor, the bifunctor

$$\mu hom\colon \mathrm{D}^{\mathrm{b}}(\mathbf{k}_M)^{\mathrm{op}} \times \mathrm{D}^{\mathrm{b}}(\mathbf{k}_M) \to \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{T^*M}) \tag{2.3}$$

has been constructed in [KS90]. It satisfies

$$\mathrm{R}\mathscr{H}om\,(G, F) \simeq \mathrm{R}\pi_* \mu hom(G, F). \tag{2.4}$$

Moreover, similarly as for $\mathrm{R}\mathscr{H}om$, there is a composition morphism

$$\mu hom(H, G) \otimes \mu hom(G, F) \to \mu hom(H, F).$$

In order to describe the microsupport of $\mu hom(G, F)$, recall that for a manifold X and two subsets $A, B \subset X$, $\mathrm{C(A, B)}$ denotes the *Whithney normal cone*, a closed subset of TX. When $X = T^*M$, one identifies TX and T^*X via the Hamiltonian isomorphism.

Theorem 2.1 (See [KS90, Cor. 6.4.3])**.** *Let $F, G \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_M)$. Then*

$$\mathrm{SS}(\mu hom(G, F)) \subset \mathrm{C(SS(F), SS(G))}. \tag{2.5}$$

In particular, $\operatorname{supp}(\mu hom(F,G)) \subset \mathrm{SS}(F) \cap \mathrm{SS}(G)$ and the functor μhom describes in some sense the microlocal morphisms. More precisely, for U open in T^*M, the sequence of morphisms

$$\begin{aligned}\operatorname{Hom}(G,F) &\simeq H^0\mathrm{R}\Gamma(M;\mathrm{R}\mathscr{H}om(G,F)\\ &\simeq H^0\mathrm{R}\Gamma(T^*M;\mu hom(G,F))\\ &\to H^0\mathrm{R}\Gamma(U;\mu hom(G,F))\end{aligned}$$

define the morphism

$$\operatorname{Hom}_{\mathrm{D^b}(\mathbf{k}_M;U)}(G,F) \to \mathrm{R}\Gamma(U;\mu hom(G,F)). \tag{2.6}$$

In particular, μhom induces a bifunctor:

$$\mu hom\colon \mathrm{D^b}(\mathbf{k}_M;U)^{\mathrm{op}} \times \mathrm{D^b}(\mathbf{k}_M;U) \to \mathrm{D^b}(\mathbf{k}_U).$$

The morphism (2.6) is not an isomorphism, but it induces an isomorphism at each $p \in T^*M$:

Theorem 2.2 (See [KS90, Th. 6.1.2])**.** *Let $p \in T^*M$. Then*

$$\operatorname{Hom}_{\mathrm{D^b}(\mathbf{k}_M;p)}(G,F) \simeq H^0(\mu hom(G,F)_p).$$

Pure and simple sheaves

Let S be a smooth submanifold of M and let $\Lambda = T^*_SM$. Let $p \in \Lambda, p \notin T^*_MM$ and let $F \in \mathrm{D^b}(\mathbf{k}_M;p)$. Let us say that F is pure at p if $F \simeq V_S[d]$ for some $\mathbf{k}$-module V and some shift d and let us say that F is simple if moreover V has dimension one. A natural question is to generalize this definition to the case where Λ is a smooth Lagrangian submanifold of $\dot{T}^*M$ but is no more necessarily a conormal bundle. Another natural question is to calculate the shift d, which will be discussed in § 3.

Notation 2.3. Let $\Lambda \subset \dot{T}^*M$ be a smooth $\mathbb{R}^+$-conic Lagrangian *locally closed* submanifold. According to (2.2), $\mathrm{D}^{\mathrm{b}}_{\Lambda}(\mathbf{k}_M;\Lambda)$ denote the full triangulated subcategory of $\mathrm{D^b}(\mathbf{k}_M;\Lambda)$ consisting of objects micro-supported by Λ. Such an object may be represented by an object of $\mathrm{D}^{\mathrm{b}}_{\Lambda}(\mathbf{k}_M;U)$ where U is open in T^*M and Λ is closed in U.

Applying Theorem 2.1, we get

Corollary 2.4. *The functor μhom induces a functor*

$$\mu hom\colon \mathrm{D}^{\mathrm{b}}_{\Lambda}(\mathbf{k}_M;\Lambda)^{\mathrm{op}} \times \mathrm{D}^{\mathrm{b}}_{\Lambda}(\mathbf{k}_M;\Lambda) \to \mathrm{DLoc}(\mathbf{k}_\Lambda).$$

Lemma 2.5. *Let $L \in \mathrm{D}^{\mathrm{b}}_{\Lambda}(\mathbf{k}_M;\Lambda)$. There is a natural morphism $\mathbf{k}_\Lambda \to \mu hom(L,L)|_\Lambda$.*

Proof. Let U be an open neighborhood of Λ as in Notation 2.3. The morphism $\mathbf{k}_M \to \mathrm{R}\mathscr{H}om(L,L) \simeq \mathrm{R}\pi_*\mu hom(L,L)$ defines the morphism $\mathbf{k}_{T^*M} \to \mu hom(L,L)$, hence the morphism $\mathbf{k}_U \to \mu hom(L,L)|_U$. Since $\mu hom(L,L)|_U$ is supported by Λ, this last morphism factorizes through $\mathbf{k}_\Lambda$. □

Definition 2.6. Let $L \in \mathrm{D}^{\mathrm{b}}_{\Lambda}(\mathbf{k}_M;\Lambda)$.

(a) One says that L is pure on Λ if $\mu hom(L,L)|_{\Lambda}$ is concentrated in degree 0. One denotes by $\mathrm{Pure}(\Lambda,\mathbf{k})$ the subcategory of $\mathrm{D}^{\mathrm{b}}_{\Lambda}(\mathbf{k}_M;\Lambda)$ consisting of pure sheaves.

(b) One says that L simple on Λ if $\mathbf{k}_{\Lambda} \xrightarrow{\sim} \mu hom(L,L)|_{\Lambda}$. One denotes by $\mathrm{Simple}(\Lambda,\mathbf{k})$ the subcategory of $\mathrm{D}^{\mathrm{b}}_{\Lambda}(\mathbf{k}_M;\Lambda)$ consisting of simple sheaves.

Of course, the categories $\mathrm{Pure}(\Lambda,\mathbf{k})$ and $\mathrm{Simple}(\Lambda,\mathbf{k})$ are not additive in general.

Remark 2.7. Let $L \in \mathrm{Simple}(\Lambda,\mathbf{k})$. Then the functor

$$\mu hom(L,\bullet)\colon \mathrm{Pure}(\Lambda,\mathbf{k}) \to \mathrm{DLoc}(\mathbf{k}_{\Lambda}) \tag{2.7}$$

is well-defined. One shall be aware of the following facts.
(i) The category $\mathrm{Simple}(\Lambda,\mathbf{k})$ may be empty (see Prop. 4.9).
(ii) The functor in (2.7) is not fully faithful in general (see § 4 or [KS90, Ex. VI.6]).

Proposition 2.8 (see [KS90, Cor.7.5.4])**.** *Assume that Λ is connected. If $L \in \mathrm{D}^{\mathrm{b}}_{\Lambda}(\mathbf{k}_M;\Lambda)$ is pure (resp. simple) in a neighborhood of $p \in \Lambda$, then L is pure (resp. simple) on Λ.*

Proof. One know by Corollary 2.4 that $\mathscr{L} := \mu hom(L,L)$ is a local system on Λ. If $\mathscr{L}$ is concentrated in degree 0 at some point (resp. is of rank one) then the same property will hold at any point of Λ. □

Lagrangian/Legendrian

Denote by t a coordinate on $\mathbb{R}$ and by $(t;\tau)$ the associated coordinates on $T^*\mathbb{R}$. We denote by $T^*_{\tau>0}\mathbb{R}$ the set $\{(t;\tau) \in T^*\mathbb{R}; \tau > 0\}$.

Consider the case where $M = N \times \mathbb{R}$ and Λ is a closed conic Lagrangian submanifold of $T^*N \times T^*_{\tau>0}\mathbb{R}$. In this case we consider the map

$$T^*N \times T^*_{\tau>0}\mathbb{R} \to (T^*N) \times \mathbb{R}, \quad (x,t;\xi,\tau) \mapsto (x,t;\xi/\tau). \tag{2.8}$$

Let Λ_0 denote the image of Λ. This is a Legendrian closed submanifold of the contact manifold $(T^*N) \times \mathbb{R}$.

We get the the notions of pure and simple sheaves along the Legendrian submanifold Λ_0 of $(T^*N) \times \mathbb{R}$.

Quantized contact transformations

References for this subsection are made to [KS90, §7.2].

Consider two manifolds M_1 and M_2, two conic open subsets $U_1 \subset T^*M_1$ and $U_2 \subset T^*M_2$ and a homogeneous contact transformation χ:

$$\dot{T}^*M_2 \supset U_2 \xrightarrow[\chi]{\sim} U_1 \subset \dot{T}^*M_1. \tag{2.9}$$

Denote by U_2^a the image of V by the antipodal map a_{M_2} on T^*M_2 and by Λ the image of the graph of χ by $\mathrm{id}_{U_1} \times a_{M_2}$. Hence Λ is a conic Lagrangian submanifold of $U_1 \times U_2^a$. Consider $K \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{M_1\times M_2})$ and the hypotheses

$$(2.10)\qquad \begin{cases} K \text{ is cohomologically constructible (see [KS90, Def. 3.4.1])},\\ \mathrm{SS}(K)\cap(U_1\times U_2^a)=\Lambda,\\ K \text{ is simple along } \Lambda,\\ (p_1^{-1}U_1\cup {p_2}^{-1}U_2^a)\cap \mathrm{SS}(K)\subset\Lambda. \end{cases}$$

Theorem 2.9 (See [KS90, Th. 7.2.1])**.** *If K satisfies the hypotheses (2.10), then the functor $K\circ$ induces an equivalence*

$$(2.11)\qquad K\circ\colon \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{M_2};U_2)\xrightarrow{\sim}\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{M_1};U_1).$$

Moreover, for $G_1, G_2 \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{M_2};U_2)$

$$(2.12)\qquad \chi_*(\mu hom(G_1,G_2)|_V)\xrightarrow{\sim}\mu hom(K\circ G_1,K\circ G_2)|_{U_1}.$$

One calls (χ, K) a quantized contact transformation (a QCT, for short).

Given χ and $q\in U_2$, $p=\chi(q)\in U_1$, there exists such a QCT after replacing U_1 and U_2 by sufficiently small neighborhoods of p and q.

Corollary 2.10. *Consider a homogeneous contact transformation $\chi\colon T^*M_1\supset U_1\xrightarrow{\sim} U_2\subset T^*M_2$. Then for any $p\in U_1$, there exists a conic open neighborhood W of p in U_1 and a quantized contact transform $(\chi|_W, K)$ where $\chi|_W\colon W\xrightarrow{\sim}\chi(W)$ is the restriction of χ.*

Proof. Locally any contact transform χ is the composition $\chi_1\circ\chi_2$ where the graph of each χ_i ($i=1,2$) is the Lagrangian manifold associated with the conormal to a hypersurface S. In this case, one can choose $K=\mathbf{k}_S$. □

Corollary 2.11. *Let Λ_i be a conic smooth Lagrangian submanifold of $\dot{T}^*M_i$ closed in U_i ($i=1,2$) with $\chi(\Lambda_2)=\Lambda_1$. Then $K\circ$ induces an equivalence* $\mathrm{Pure}(\Lambda_2,\mathbf{k})\xrightarrow{\sim}\mathrm{Pure}(\Lambda_1,\mathbf{k})$ *and similarly when* Pure *is replaced with* Simple*.*

Invariance by Hamiltonian isotopies

References for this subsection are made to [GKS12].

Let M be a real manifold of class C^∞ and I an open interval of $\mathbb{R}$ containing the origin. We consider a C^∞-map $\Phi\colon \dot{T}^*M\times I\to\dot{T}^*M$. Setting $\varphi_t=\Phi(\,\bullet\,,t)$ ($t\in I$), we shall always assume

$$(2.13)\qquad \begin{cases}\varphi_t \text{ is a homogeneous symplectic isomorphism for each } t\in I,\\ \varphi_0=\mathrm{id}_{\dot{T}^*M}.\end{cases}$$

Let us recall here some classical facts. Set

$$\begin{aligned} v_\Phi &:= \frac{\partial \Phi}{\partial t} \colon \dot{T}^*M \times I \to T\dot{T}^*M, \\ f &= \langle \alpha_M, v_\Phi \rangle \colon \dot{T}^*M \times I \to \mathbb{R},\ f_t = f(\cdot, t). \end{aligned}$$

Denote by H_g the Hamiltonian flow of a function $g\colon \dot{T}^*M \to \mathbb{R}$. Then

$$\frac{\partial \Phi}{\partial t} = H_{f_t}.$$

In other words, the homogeneous isotopy Φ is Hamiltonian.

In this situation, there exists a unique conic Lagrangian submanifold $\widetilde{\Lambda}$ of $\dot{T}^*M \times \dot{T}^*M \times T^*I$ closed in $\dot{T}^*(M \times M \times I)$ such that $\widetilde{\Lambda} \circ T_t^*I$ is the graph of φ_t.

Theorem 2.12 (See [GKS12, Th. 3.7])**.** *We consider $\Phi\colon \dot{T}^*M \times I \to \dot{T}^*M$ and we assume that it satisfies hypothesis (2.13). Then there exists $K \in \mathrm{D}^{\mathrm{lb}}(\mathbf{k}_{M\times M\times I})$ satisfying the following conditions.*

(a) $\mathrm{SS}(K) \subset \Lambda \cup T^*_{M\times M\times I}(M \times M \times I)$,

(b) $K_0 \simeq \mathbf{k}_\Delta$,

(c) *both projections* $\mathrm{supp}(K) \rightrightarrows M \times I$ *are proper,*

(d) $K_t \circ K_t^{-1} \simeq K_t^{-1} \circ K_t \simeq \mathbf{k}_\Delta$ *for all* $t \in I$.

Moreover,

(i) *such a K satisfying the conditions (a)–(b) is unique up to a unique isomorphism,*

(ii) *K is simple along Λ and K_t is simple along Λ_t for $t \in I$.*

Note that the conclusion (ii) was not explicitly stated in loc. cit. but follows immediately from the proof.

Applying Theorems 2.12 and 2.9 we get:

Corollary 2.13. *Let Φ be an homogeneous Hamiltonian isotopy and let K be as in Theorem 2.12. Let $\Lambda_0 \subset \dot{T}^*M$ be a closed smooth Lagrangian submanifold and set $\Lambda_t = \varphi_t(\Lambda_0)$. Then the functor $K_t \circ (\bullet)$ induces an equivalence* $\mathrm{Pure}(\Lambda_t, \mathbf{k}) \simeq \mathrm{Pure}(\Lambda_0, \mathbf{k})$ *and similarly when* Pure *is replaced with* Simple.

Roughly speaking, Corollary 2.13 asserts that the categories $\mathrm{Pure}(\Lambda, \mathbf{k})$ and $\mathrm{Simple}(\Lambda, \mathbf{k})$ are invariant by homogeneous Hamiltonian isotopies.

3 Maslov index and shift

References for this subsection are made to [KS90, §7.5].

The constructions which appear in this section make an intensive use of the Maslov index $\tau(\lambda_1, \lambda_2, \lambda_3)$ of three Lagrangian planes λ_i ($i = 1, 2, 3$) in a real symplectic vector space (E, σ). We refer to [KS90, Appendix] for a detailed exposition, simply recalling its definition, namely

$$(3.1) \qquad \begin{cases} \tau_E(\lambda_1, \lambda_2, \lambda_3) \text{ is the signature of the quadratic form on } \lambda_1 \oplus \lambda_2 \oplus \lambda_3 \\ \text{given by } q(x_1, x_2, x_3) = \sigma(x_1, x_2) + \sigma(x_2, x_3) + \sigma(x_3, x_1). \end{cases}$$

If a quadratic form has n^+ positive and n^- negative eigenvalues, then its signature is $n^+ - n^-$.

For a function φ on M we denote by Λ_φ the (non conic) Lagrangian submanifold of T^*M given by

$$\Lambda_\varphi := \{(x; d\varphi(x)); x \in M\}.$$

Let Λ be a conic smooth Lagrangian locally closed submanifold of T^*M and let $p \in \Lambda$. One says that φ is transversal to Λ at p if $\varphi(\pi_M(p)) = 0$ and the manifolds Λ and Λ_φ intersect transversally at p. We define the Lagrangian planes in T_pT^*M:

$$\lambda_0(p) = T_p(\pi_M^{-1}\pi_M(p)), \quad \lambda_\Lambda(p) = T_p\Lambda, \quad \lambda_\varphi(p) = T_p\Lambda_\varphi.$$

We set

$$(3.2) \qquad \tau_\varphi(p) = \tau_E(\lambda_0(p), \lambda_\Lambda(p), \lambda_\varphi(p)) \text{ where } E = T_pT^*M.$$

Since $p = (x; \xi) \in \Lambda_\varphi$, we have $d\varphi(x) = \xi \neq 0$ ($p \in \Lambda \subset \dot{T}^*M$). If φ is transversal to Λ at $p \notin T^*_MM$ then $d\varphi(\pi_M(p)) \neq 0$. Consider the smooth hypersurface $S = \{x \in M; \varphi(x) = 0\}$ and denote by $\lambda_{\varphi=0}(p)$ the tangent plane to T^*_SM at p. Then we also have (see [KS90, eq. (7.5.5)]):

$$(3.3) \qquad \tau_\varphi(p) = \tau(\lambda_0(p), \lambda_\Lambda(p), \lambda_{\varphi=0}(p)).$$

Example 3.1. Consider a local coordinate system $x = (x', x'')$ on M where $x = (x_1, \dots, x_n)$, $x' = (x_1, \dots, x_l)$ and denote by $(x; \xi) = (x', x''; \xi', \xi'')$ the associated coordinates on T^*M. For $p \in T^*M$, we still denote by $(x; \xi) = (x', x''; \xi', \xi'')$ the coordinates on T_pT^*M. Let $S = \{x'' = 0\}$, $p = (0; dx_n)$ and let $\varphi \colon M \to \mathbb{R}$ be a C^2-functions with $d\varphi(0) \neq 0$. We have

$$\begin{aligned} \Lambda_\varphi &= \{(x; \xi); \xi_j = \partial\varphi/\partial x_j\}, \\ T_p\Lambda_\varphi &= \{(x; \xi); \xi_j = \sum_{k=1}^n \partial^2_{x_jx_k}\varphi(0) \cdot x_k\}, \\ T_pT^*_SM &= \{(x; \xi); x'' = \xi' = 0\}. \end{aligned}$$

Then φ is transversal to S at p if and only if the intersection $T_p\Lambda_\varphi \cap T_pT^*_SM$ is $\{0\}$ and thus, if and only if the matrix $\partial^2_{x'x'}\varphi(0)$ is non degenerate. By the Morse Lemma, we may assume after a change of coordinates that $\varphi|_S = \sum_{j=1}^{l} a_j x_j^2$, $a_j \in \mathbb{R}$, $a_j \neq 0$. Then, setting $E = T_pT^*M$, $E = E' \oplus E''$ with $(x';\xi') \in E'$,

$$\begin{aligned}\tau_\varphi(p) &= \tau_E(\{x=0\},\{x''=\xi'=0\},\{\xi = \partial^2_{x',x'}\varphi(0)\cdot x')\} \\ &= \tau_{E'}(\{x'=0\},\{\xi'=0\},\{\xi' = \partial^2_{x',x'}\varphi(0)\cdot x'\}) \\ &= -sign(\partial^2_{x',x'}\varphi(0)) = \#\{j; a_j<0\} - \#\{j; a_j>0\}.\end{aligned}$$

(See [KS90, Prop. A.3.6] for details.)

Lemma 3.2. *Let Λ be a conic smooth Lagrangian locally closed submanifold of $\dot{T}^*M$, let $p \in \Lambda$ and let φ be transversal to Λ at p. The property that $\mathrm{R}\Gamma_{\varphi\geq 0}(F)_{\pi_M(p)}$ is concentrated in a single degree does not depend on the choice of φ and is invariant by QCT. More precisely, for another φ' transversal to Λ, we have $\mathrm{R}\Gamma_{\varphi'\geq 0}(F)_{\pi_M(p)} \simeq \mathrm{R}\Gamma_{\varphi\geq 0}(F)_{\pi_M(p)}[\frac{1}{2}(\tau_\varphi(p) - \tau_{\varphi'}(p))]$.*

Proof. See [KS90, Prop. 7.5.3, 7.5.6]. □

Lemma 3.3. *Let $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_M;p)$ supported by Λ, let $\varphi\colon M \to \mathbb{R}$ be transversal to Λ at $p \notin T^*_MM$ and set $V = \mathrm{R}\Gamma_{\varphi\geq 0}(F)_{\pi_M(p)}$. Then F is pure at p if and only if V is concentrated in a single degree, say $-j$, and F is simple if moreover $H^j(V)$ has rank one.*

Proof. (i) By Corollary 2.11, the property of being pure or simple is invariant by QCT. Hence, applying Lemma 3.2 we may assume from the beginning that $\Lambda = T^*_SM$ is the conormal bundle to a hypersurface S.
(ii) By [KS90, Prop. 6.6.1], there exists $W \in \mathrm{D}^{\mathrm{b}}(\mathbf{k})$ such that $F \simeq W_S$ in $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_M;p)$. Then $\mu hom(F,F)_p \simeq \mathrm{RHom}\,(W,W)$ and this complex is concentrated in degree 0 if and only if W is concentrated in a single degree. Moreover, $\mathrm{Hom}\,(W,W)$ is of rank one if and only if so is W. To conclude, it remains to calculate $V = \mathrm{R}\Gamma_{\varphi\geq 0}(W_M)_{\pi_M(p)}$. This is left to the reader, using Example 3.1. □

Definition 3.4. (See [KS90, Def. 7.5.4]) Let $d \in \frac{1}{2}\dim(\lambda_0(p)\cap\lambda_\Lambda(p)) + \mathbb{Z}$. Let $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_M;p)$ supported by Λ and let $V \in \mathrm{D}^{\mathrm{b}}_f(\mathbf{k})$.

(a) One says that F is of type V with shift d at p if

$$\mathrm{R}\Gamma_{\varphi\geq 0}(F)_{\pi_M(p)} \simeq V\,[d - \frac{n}{2} - \frac{1}{2}\tau_\varphi(p)]. \tag{3.4}$$

(b) If $H^j(V) = 0$ for $j \neq 0$ one says that F is pure with shift d. If moreover V is a free $\mathbf{k}$-module of rank one, one says that F is simple with shift d.

It follows from Lemma 3.3 that the notions of being pure or simple sheaves introduced in Definition 3.4 coincide with those of Definition 2.6. It is proved in loc. cit. that the definition of the shift does not depend on the choice of φ.

- If F is pure with of type V with shift d along Λ and $\Lambda = T^*_SM$ for some hypersurface S, then there exists a sheaf G simple of shift d along Λ and an isomorphism $F \simeq V_S \otimes G$ in $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_M;p)$.

- If F is of type V with shift d at p, then F is of type $V[-j]$ with shift $d+j$ and $F[j]$ is of type V with shift $d+j$ at p.

Notation 3.5. Fix some point $p \in \Lambda$ and let $d \in \frac{1}{2}\dim(\lambda_0(p) \cap \lambda_\Lambda(p)) + \mathbb{Z}$. One denotes by $\mathrm{Pure}_{p,d}(\Lambda, \mathbf{k})$ the additive full subcategory of $\mathrm{Pure}(\Lambda, \mathbf{k})$ consisting of sheaves which are pure of shift d at p. We define similarly $\mathrm{Simple}_{p,d}(\Lambda, \mathbf{k})$.

Example 3.6. (See [KS90, Exa. 7.5.5].) If S is a closed submanifold of M, the sheaf $\mathbf{k}_S$ on M is simple with shift $\frac{1}{2}\operatorname{codim}_M S$ at each $p \in T^*_S M$.

To check this point, we choose a local coordinate system $x = (x', x'')$ as in Example 3.1. Hence, $S = \{x'' = 0\}$, $p = (0; dx_n)$. Choose $\varphi(x) = x_n + \sum_{j=1}^{l} x_j^2$. Then φ is transversal to $T^*_S M$ at p. We have

$$(\mathrm{R}\Gamma_{\varphi \geq 0}(\mathbf{k}_S))_0 \simeq \mathbf{k}.$$

Since $\tau_\varphi(p) = -l$, we find by (3.4) that $d = \frac{1}{2}(n-l) = \frac{1}{2}\operatorname{codim}_M S$.
Let $\psi\colon M \to \mathbb{R}$ be a C^2 function with $d\psi(x) \neq 0$ on the hypersurface $S = \{x \in M; \psi(x) = 0\}$. Set $Z = \{x \in M; \psi(x) \geq 0\}$, $U = \{x \in M; \psi(x) < 0\}$, $p = (x_0; d\psi(x_0))$ for some x_0 with $\psi(x_0) = 0$. Then at p (that is, in the category $\mathrm{D^b}(\mathbf{k}_M; p)$) one has $\mathbf{k}_Z \simeq \mathbf{k}_S \simeq \mathbf{k}_U[1]$. Applying the above result, we get that $\mathbf{k}_Z$ is simple with shift $\frac{1}{2}$ at p and $\mathbf{k}_U$ is simple with shift $-\frac{1}{2}$ at p.

Example 3.7. Assume X is a complex manifold and $F \in \mathrm{D^b_{\mathbb{C}c}}(\mathbf{k}_X)$, that is, F is $\mathbb{C}$-constructible. Then F is perverse if and only if it has shift 0 at generic points of $\mathrm{SS}(F)$.

Example 3.8. Denote by $x = (x_1, x_2)$ the coordinates on $\mathbb{R}^2$, by $(x;\xi) = (x_1, x_2; \xi_1, \xi_2)$ the associated coordinates on $T^*\mathbb{R}^2$ and consider the "locally closed cusp":

$$Z = \{(x_1, x_2); x_1 > 0, -x_1^{\frac{3}{2}} \leq x_2 < x_1^{\frac{3}{2}}\}.$$

Then the microsupport of $\mathbf{k}_Z$ outside of the zero-section is the smooth Lagrangian manifold

$$\begin{aligned}\Lambda &= \{(x;\xi); \xi_2 > 0, x_1 = (2\xi_1/3\xi_2)^2, x_2 = -(2\xi_1/3\xi_2)^3\},\\ &= \{(t^2, t^3; -3tu, 2u)\}; t \in \mathbb{R}, u \in \mathbb{R}_{>0}\}.\end{aligned}$$

It follows from Example 3.6 that $\mathbf{k}_Z$ is simple with shift $1/2$ on $\Lambda \cap \{\xi_1 > 0\}$ and simple with shift $-1/2$ on $\Lambda \cap \{\xi_1 < 0\}$. Let us calculate $\mathbf{k}_Z$ at $p = (0; dx_2)$. One has $(\mathrm{R}\Gamma_{\varphi \geq 0}\mathbf{k}_Z)_0 \simeq \mathbf{k}[-1]$. On the other hand, the function $\varphi(x) = x_2$ is transversal to Λ at $(0; dx_2)$ and $\tau_\varphi(p) = 0$. Therefore, $\mathbf{k}_Z$ is simple with shift 0 at p.

Remark 3.9. Let Λ be a conic smooth Lagrangian locally closed submanifold of $\dot{T}^*M$. We assume that the projection $\Lambda/\mathbb{R}^+ \to M$ is finite and we let $\Lambda_0 \subset \Lambda$ be the open subset where $\Lambda \to M$ is of maximal rank. We assume that $\mathrm{Pure}(\Lambda, \mathbf{k})$ is non empty and we let $F \in \mathrm{Pure}(\Lambda, \mathbf{k})$. The fonction $m\colon \Lambda \to \frac{1}{2}\mathbb{Z}$, $p \mapsto$ "shift of F at p", is constant on the connected components of Λ_0 and changes by 1 when p goes over a cusp, as follows from Example 3.8. Hence m is a Maslov potential for Λ in the sense of [PC05].

4 Legendrian knots, cusps and zigzags

Legendrian knots

In [STZ14], the authors study the special situation in which $M = \mathbb{R} \times \mathbb{R}$ and Λ is a connected smooth Lagrangian (equivalently, Legendrian) closed submanifold of $T^*\mathbb{R}\times\mathbb{R}$. More precisely, they consider the category $\mathrm{Simple}(\Lambda, \mathbf{k})$ (that they call $\mathscr{M}_1(\Lambda, \mathbf{k})$) of constructible sheaves of $\mathbf{k}$-modules on $M = \mathbb{R} \times \mathbb{R}$, whose microsupport is contained in the union of the zero-section and the set $\{(x, t; \xi, \tau); \tau < 0\}$ with $((x; \xi/\tau), t) \in \Lambda$. That is, the "downward pointing" co-vectors of M are in Λ.

It follows from Corollary 2.13 that $\mathscr{M}_1(\Lambda, \mathbf{k})$ only depends on the Legendrian isotopy class of Λ. Hence we can deform Λ and assume that its front $\pi(\Lambda)$ is a curve in M with ordinary double points and cusps as its only singularities. The objects of $\mathscr{M}_1(\Lambda, \mathbf{k})$ are in particular constructible with respect to the stratification of M given by the singularities of $\pi(\Lambda)$, the smooth arcs of $\pi(\Lambda)$ and the components of the complement of $\pi(\Lambda)$.

These constructible sheaves have a combinatorial description: such a sheaf is given by a complex associated with each stratum and some gluing condition.

We fix some Maslov potential m along Λ and let $\mathscr{M}_{1,m}(\Lambda, \mathbf{k})$ be the subcategory of $\mathscr{M}_1(\Lambda, \mathbf{k})$ whose objects have their shift given by p (see Remark 3.9). Let $|\mathscr{M}_{1,m}(\Lambda, \mathbf{k})|$ be the number of isomorphism classes of objects in $\mathscr{M}_{1,m}(\Lambda, \mathbf{k})$. We can associate a Legendrian knot Λ_B with a braid B (draw the braid horizontally in the plane, join the left and right ends by upper arcs, with cusps to avoid vertical tangents – then Λ_B is the conormal bundle of the resulting curve). The main result of [STZ14] relates the count function $p \mapsto |\mathscr{M}_{1,m}(\Lambda_B, \mathbb{Z}/p\mathbb{Z})|$, p prime, with the HOMFLY polynomial of B.

Cusps

Here is a partial converse to Example 1.2 (iii). For an open subset $U \subset M$ with smooth boundary we denote by $T^{*,in}_{\partial U}M$ the inner conormal bundle of ∂U (that is, $T^{*,in}_{\partial U}M = \dot{\mathrm{SS}}(\mathbf{k}_{\overline{U}})$ in Example 1.2 (iii)).

Lemma 4.1. *Let M be a manifold and let U be an open subset. Let $F \in \mathrm{D^b}(\mathbf{k}_M)$. We assume that $\mathrm{supp}(F) \subset \overline{U}$ and that, for any $x \in \partial U$, there exists a C^1-function $\varphi\colon M \to \mathbb{R}$ such that $U \subset \varphi^{-1}(]0, +\infty[)$ near x, $d\varphi_x \neq 0$, and $(x; d\varphi_x) \notin \mathrm{SS}(F)$. Then the morphism $F_U \to F$ is an isomorphism.*

In particular, if U has a smooth boundary, $\mathrm{supp}(F) \subset \overline{U}$ and $\dot{\mathrm{SS}}(F) \cap T^{*,in}_{\partial U}M = \varnothing$, then $F_U \xrightarrow{\sim} F$.

Proof. Since $\mathrm{supp}(F) \subset \overline{U}$ it is enough to see that $F_x \simeq 0$ for all $x \in \partial U$. Let $\varphi\colon M \to \mathbb{R}$ be as in the lemma. We have $(x; d\varphi_x) \notin \mathrm{SS}(F)$. The definition of the microsupport gives $(\mathrm{R}\Gamma_{\varphi^{-1}([0,+\infty[)}(F))_x \simeq 0$. Since $\mathrm{supp}(F) \subset \overline{U} \subset \varphi^{-1}([0, +\infty[)$ we have $\mathrm{R}\Gamma_{\varphi^{-1}([0,+\infty[)}(F) \simeq F$. Hence $F_x \simeq 0$, as required.

If U has a smooth boundary, we can choose for φ any C^1-function such that $U = \varphi^{-1}(]0, +\infty[)$ near x and $d\varphi_x \neq 0$. Then the second part of the lemma follows from the first. $\square$

Here is the dual statement of Lemma 4.1.

Lemma 4.2. *Let M be a manifold and let U be an open subset. Let $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_M)$. We assume that* $\mathrm{supp}(F) \subset \overline{U}$ *and that, for any $x \in \partial U$, there exists a* C^1*-function $\varphi\colon M \to \mathbb{R}$ such that $U \subset \varphi^{-1}(]0,+\infty[)$ near x, $d\varphi_x \neq 0$, and $(x; -d\varphi_x) \notin \mathrm{SS}(F)$. Then the morphism $F \to \mathrm{R}\Gamma_U(F)$ is an isomorphism.*

In particular, if U has a smooth boundary, $\mathrm{supp}(F) \subset \overline{U}$ *and* $\dot{\mathrm{SS}}(F) \cap (T^{*,in}_{\partial U}M)^a = \varnothing$, *then $F \xrightarrow{\sim} \mathrm{R}\Gamma_U(F)$.*

Example 4.3. As a special case of Lemma 4.1 we consider an open subset $U \subset M$ which is contractible and has smooth boundary. Then the sheaves $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_M)$ such that $\mathrm{supp}(F) \subset \overline{U}$ and $\dot{\mathrm{SS}}(F) \cap T^{*,in}_{\partial U}M = \varnothing$ are the sheaves of the form $F \simeq L_U$ for some $L \in \mathrm{D}^{\mathrm{b}}(\mathbf{k})$.

Example 4.4. In $\mathbb{R}^2$ with coordinates (x, y) we define the following locally closed subset bounded by the cusp

$$W = \{(x, y);\ x > 0,\ -x^{3/2} \leq y < x^{3/2}\}. \tag{4.1}$$

It follows from [KS90, Ex. 5.3.4] that, outside the zero section, $\mathrm{SS}(\mathbf{k}_W)$ is the smooth Lagrangian submanifold

$$\Lambda_{cusp} = \{(t^2, t^3; -3tu, 2u);\ t \in \mathbb{R},\ u > 0\}. \tag{4.2}$$

We will now describe all objects in $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{R}^2})$ with a microsupport contained in Λ_{cusp} outside of the zero-section. Let $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{R}^2})$ with $\dot{\mathrm{SS}}(F) \subset \Lambda_{cusp}$. In particular F is locally constant on the open subsets $U_0 = \mathrm{Int}(W)$ and $U_1 = \mathbb{R}^2 \setminus \overline{W}$. Since U_0 and U_1 are contractible, $F|_{U_0}$ and $F|_{U_1}$ are in fact constant.

We first assume that $F|_{U_1} \simeq 0$.

Lemma 4.5. *Let $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{R}^2})$ be such that $\dot{\mathrm{SS}}(F) \subset \Lambda_{cusp}$ and $F|_{U_1} \simeq 0$. Then there exists $L \in \mathrm{D}^{\mathrm{b}}(\mathbf{k})$ such that $F \simeq L_W$.*

Proof. (i) We define $U = \{(x, y) \in \mathbb{R}^2;\ x > 0,\ y < x^{3/2}\}$. Let us prove that $F_U \xrightarrow{\sim} F$ by checking the hypothesis of Lemma 4.1. We clearly have $\mathrm{supp}(F) \subset \overline{U}$. For a given $z \in \partial U$ the existence of φ as in the lemma is easy when z is smooth. It remains to consider $z = (0, 0)$. In this case we see that $\varphi(x, y) = x - y$ satisfies the hypothesis.

(ii) Now we consider $F|_U$. By Lemma 4.2 applied to $M = U$ and $U = \mathrm{Int}(W)$ we have $F|_U \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{Int}W}(F)$. Since $\mathrm{Int}(W)$ is contractible and $\dot{\mathrm{SS}}(F|_{\mathrm{Int}(W)})$ is empty we have $\mathrm{R}\Gamma_{\mathrm{Int}W}(F) \simeq \mathrm{R}j_*(L_{\mathrm{Int}(W)})$ for some $L \in \mathrm{D}^{\mathrm{b}}(\mathbf{k})$, where $j\colon \mathrm{Int}(W) \to U$ is the inclusion. Since $\partial W \cap U$ is smooth we have $\mathrm{R}j_*(L_{\mathrm{Int}(W)}) \simeq L_W$ (this is an isomorphism in $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_U)$) and the lemma follows. □

Lemma 4.6. *Let $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{R}^2})$ be such that $\dot{\mathrm{SS}}(F) \subset \Lambda_{cusp}$. We recall that $U_1 = \mathbb{R}^2 \setminus \overline{W}$ and we choose $z \in U_1$. Then the restriction morphisms $\mathrm{R}\Gamma(\mathbb{R}^2; F) \to \mathrm{R}\Gamma(U_1; F) \to F_z$ are isomorphisms.*

Proof. Since $F|_{U_1}$ is constant the second morphism is an isomorphism. Hence it is enough to check that $\mathrm{R}\Gamma(\mathbb{R}^2; F) \to F_z$ is also an isomorphism and we can even assume that $z = (-1, 0)$. We define $\varphi\colon \mathbb{R}^2 \to \mathbb{R}$ by $\varphi(z') = d(z, z')^2$. Then, for $z' \neq z$, we have $(z'; d\varphi_{z'}) \notin \mathrm{SS}(F)$. By the microlocal Morse lemma (see [KS90, Ex. 5.4.19]) it follows that $\mathrm{R}\Gamma(B_r; F) \to \mathrm{R}\Gamma(B_{1/2}; F)$ is an isomorphism for all $r \geq 1/2$, where B_r is the open ball of radius r centered at z. We have $\mathrm{R}\Gamma(B_{1/2}; F) \xrightarrow{\sim} F_z$ and $\varprojlim_r H^i(B_r; F) \simeq H^i(\mathbb{R}^2; F)$ for all $i \in \mathbb{Z}$ by [KS90, Prop. 2.7.1]. The lemma follows. □

The following result could be deduced from [STZ14, Thm. 3.12].

Proposition 4.7. *Let $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{R}^2})$ be such that $\dot{\mathrm{SS}}(F) \subset \Lambda_{cusp}$. Then there exist $L, L' \in \mathrm{D}^{\mathrm{b}}(\mathbf{k})$ such that $F \simeq L_W \oplus L'_{\mathbb{R}^2}$.*

Proof. (i) We set $L' = \mathrm{R}\Gamma(\mathbb{R}^2; F)$. Let a be the map from $\mathbb{R}^2$ to a point. The adjunction $(a^{-1}, \mathrm{R}a_*)$ gives a morphism $L'_{\mathbb{R}^2} \to F$ which induces an isomorphism on the global sections. We define G by the distinguished triangle $L'_{\mathbb{R}^2} \to F \to G \xrightarrow{+1}$. Then $\mathrm{R}\Gamma(\mathbb{R}^2; G) \simeq 0$. By Lemma 4.6 it follows that $G_z \simeq 0$ for all $z \in U_1$. Hence, by Lemma 4.5, there exists $L \in \mathrm{D}^{\mathrm{b}}(\mathbf{k})$ such that $G \simeq L_W$.

(ii) By (i) we have a distinguished triangle $L'_{\mathbb{R}^2} \to F \to L_W \xrightarrow{u} L'_{\mathbb{R}^2}[1]$. By the adjunction $(\mathrm{R}a_!, a^{-1})$ we have $\mathrm{Hom}\,(L_W, L'_{\mathbb{R}^2}[1]) \simeq \mathrm{Hom}\,(\mathrm{R}a_!(L_W), L'[1])$. We see that $\mathrm{R}a_!(L_W) \simeq 0$ and it follows that $u = 0$. We deduce that F is the direct sum given in the lemma. □

Corollary 4.8. *Let $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{R}^2})$ be such that $\dot{\mathrm{SS}}(F) \subset \Lambda_{cusp}$. For a given $x_0 > 0$ we define $i\colon \mathbb{R} \to \mathbb{R}^2$, $y \mapsto (x_0, y)$ and we let $a < b \in \mathbb{R}$ be the inverse images of the cusp $\{x^3 = y^2\}$ by i. Then there exist $L, L' \in \mathrm{D}^{\mathrm{b}}(\mathbf{k})$ such that $i^{-1}F \simeq L_{[a,b[} \oplus L'_{\mathbb{R}}$.*

Zigzags

Now we consider the following double cusp. We let C be the cusp $C = \{x^3 = y^2\}$ and we set $z = (1, -1) \in C$. We let $C^+ = C \cap (\mathbb{R} \times [-1, +\infty[)$ be the portion of C "above z". We let $C' = C^+ - z$ be the translation of C^+ which ends at $(0, 0)$ and we define C_2 as the union of C' and its image by $(x, y) \mapsto (-x, -y)$. Then C_2 has two cusps, at $c_0 = (-1, 1)$ and $c_1 = (1, -1)$, and is a smooth curve of class C^1 outside the cusps. The closure of $\dot{T}^*_{C_2}(\mathbb{R}^2 \setminus \{c_0, c_1\})$ in $\dot{T}^*\mathbb{R}^2$ is a smooth Lagrangian submanifold with two connected components. We let Λ_{C_2} be one of these components. It is well-known that Λ_{C_2} cannot be described by a generating function. We see that it cannot be the microsupport of a sheaf. This is done for example in [STZ14, Prop. 5.8]. Here is a slightly different proof.

Proposition 4.9. *Let $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{R}^2})$ be such that $\dot{\mathrm{SS}}(F) \subset \Lambda_{C_2}$. Then F is a constant sheaf and $\dot{\mathrm{SS}}(F) = \varnothing$.*

Proof. We define $i\colon \mathbb{R} \to \mathbb{R}^2$, $y \mapsto (0, y)$ and we let $a = -2$, $b = 0$, $c = 2$ be the points in $i^{-1}(C_2)$. We set $U_+ = \mathbb{R}\times]-1/2, +\infty[$ and $U_- = \mathbb{R}\times]-\infty, 1/2[$. Then $C_2 \cap U_\pm$ is diffeomorphic to the usual cusp (by a diffeomorphism of class C^1). By Corollary 4.8 we deduce that there exist $L, L', M, M' \in \mathrm{D}^{\mathrm{b}}(\mathbf{k})$ such that $(i^{-1}F)|_{]-\infty,1/2[} \simeq L_{]-\infty,1/2[} \oplus$

$L'_{[-2,0[}$ and $(i^{-1}F)|_{]-1/2,+\infty[} \simeq M_{]-1/2,+\infty[} \oplus M'_{[0,2[}$. Restricting to $I =]-1/2, 1/2[$ we obtain $L_I \oplus L'_{]-1/2,0[} \simeq M_I \oplus M'_{[0,1/2[}$. This implies $L' = M' = 0$ and the lemma follows. □

Problem 4.10. Consider the situation of (2.8). It is natural to ask the question of the quantization of the Lagrangian manifold Λ or, equivalently, of the Legendrian manifold Λ_0. More precisely, one asks the question:

(4.3) to give necessary and sufficient geometrical conditions in order that there exists a globally defined simple sheaf along Λ.

(i) The zigzag example (Proposition 4.9) shows that such a quantization does not always exist.

(ii) In [Gui19, Th. 13.5.1] it is proved that if Λ comes from a compact exact Lagrangian submanifold of T^*N, then a quantization exists (and we can choose one in a canonical way).

(iii) It follows from Remark 3.9 that a necessary condition is that the Maslov potential of Λ vanishes.

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