Subanalytic sheaves and exponential $\mathscr{D}$-modules

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Abstract

In this survey talk, we construct the subanalytic site and the six operations on subanalytic sheaves. On a complex manifold, we obtain the (derived) sheaf $\mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}}$ of tempered holomorphic functions and its dual, the sheaf $\mathscr{O}^{\mathrm{w}}_{X_{\mathrm{sa}}}$ of Whitney holomorphic functions. Finally, we recall the role of the sheaf $\mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}}$ in the Riemann-Hilbert correspondence and in the study of exponential $\mathscr{D}$-modules.

Contents

1 Subanalytic sheaves [KS01]

1.1 Sites and sheaves

See [SAGV]. Here, we follow [KS06, Sch25].

We denote by $\mathbf{k}$ a commutative unital ring of finite global dimension. A site X is a small category $\mathscr{C}_X$ endowed with a Grothendieck topology. For $U \in \mathscr{C}_X$, the site U is the presite

$$\mathscr{C}_U := (\mathscr{C}_X)_U$$

endowed with its natural topology induced by that of X.

We assume that $\mathscr{C}_X$, as well as $\mathscr{C}_U$ for any $U \in \mathscr{C}_X$, admits product of two objects.

A morphism of sites $f\colon X \to Y$ is a functor $f^t\colon \mathscr{C}_Y \to \mathscr{C}_X$ which commutes with fiber product of two objects and with coverings of any $V \in \mathscr{C}_Y$.

One has a morphism of sites $i_U\colon U \to X$ which, to $V \in \mathscr{C}_X$, associates $V \times U \in \mathscr{C}_U$.

There is another morphism of sites $j_U\colon X \to U$ which, to $W \to U \in \mathscr{C}_U$, associates $W \in \mathscr{C}_X$. For $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_U)$ one sets:

$$i_{U!}F = j_U^{-1}F.$$

On X, we denote by $\mathrm{Mod}(\mathbf{k}_X)$ the Grothendieck category of sheaves of $\mathbf{k}$-modules and by $\mathrm{D}^*(\mathbf{k}_X)$ its derived category with $* = \mathrm{b}, +, -$. In particular, $\mathrm{Mod}(\mathbf{k}_X)$ admits small limits and colimits and enough injective objects. It admits a small system of generators, namely $\mathrm{Ob}(\mathscr{C}_X)$.

One has the two internal derived operations $\overset{\mathrm{L}}{\otimes}$ and $\mathrm{R}\mathscr{H}om$.

If $f\colon X \to Y$ a morphism of sites, one has the two external derived operations, f^{-1} and $\mathrm{R}f_*$. The functor f^{-1} is exact under the hypothesis that

$$\begin{cases} \text{for any } U \in \mathscr{C}_X, \text{ the category } (\mathscr{C}_Y^U)^{\mathrm{op}} \text{ associated with } f^t\colon \mathscr{C}_Y \to \\ \mathscr{C}_X \text{ is either directed or empty.} \end{cases} \tag{1.1}$$

Using the fact that the homological dimension of $\mathbf{k}$ is finite, we get:

Theorem 1.1. *Let X and Y be two sites and $f\colon X \to Y$ a morphism of sites. Let $U \in \mathscr{C}_X$. The functors below are well defined*

$$\begin{aligned}
&\mathrm{RHom}\,(\,\bullet\,,\,\bullet\,)\colon \mathrm{D}^-(\mathbf{k}_X)^{\mathrm{op}} \times \mathrm{D}^+(\mathbf{k}_X) \to \mathrm{D}^+(\mathbf{k}),\\
&\mathrm{R}\mathscr{H}om\,(\,\bullet\,,\,\bullet\,)\colon \mathrm{D}^-(\mathbf{k}_X)^{\mathrm{op}} \times \mathrm{D}^+(\mathbf{k}_X) \to \mathrm{D}^+(\mathbf{k}_X),\\
&f^{-1}\colon \mathrm{D}^*(\mathbf{k}_Y) \to \mathrm{D}^*(\mathbf{k}_X), (* = \mathrm{b}, +, -) \ (\textit{assuming } (1.1)),\\
&\mathrm{R}f_*\colon \mathrm{D}^+(\mathbf{k}_X) \to \mathrm{D}^+(\mathbf{k}_Y),\\
&\bullet \overset{\mathrm{L}}{\otimes} \bullet\colon \mathrm{D}^*(\mathbf{k}_X) \times \mathrm{D}^*(\mathbf{k}_X) \to \mathrm{D}^*(\mathbf{k}_X) \quad (* = \mathrm{b}, +, -),
\end{aligned}$$

In particular, we have the functors

$$\begin{aligned}
&\mathrm{R}\Gamma(X;\,\bullet)\colon \mathrm{D}^+(\mathbf{k}_X)\to\mathrm{D}^+(\mathbf{k}),\\
&\bullet\overset{\mathrm{L}}{\boxtimes}\bullet\colon \mathrm{D}^*(\mathbf{k}_X)\times\mathrm{D}^*(\mathbf{k}_Y)\to\mathrm{D}^*(\mathbf{k}_{X\times Y})\quad(*=\mathrm{b},+,-),\\
&i_U^{-1}\simeq j_{U*}\colon \mathrm{D}^*(\mathbf{k}_X)\to\mathrm{D}^*(\mathbf{k}_U)\quad(*=\mathrm{b},+,-),\\
&i_{U!}:=j_U^{-1}\colon \mathrm{D}^*(\mathbf{k}_U)\to\mathrm{D}^*(\mathbf{k}_X),\quad(*=\mathrm{b},+,-),\\
&\mathrm{R}i_{U*}\colon \mathrm{D}^+(\mathbf{k}_U)\to\mathrm{D}^+(\mathbf{k}_X).
\end{aligned}$$

1.2 The subanalytic site

We shall freely use the theory of subanalytic sets, due to Gabrielov and Hironaka.

Recall that, for real analytic manifolds X, Y and a closed subanalytic subset S of X, we say that a map $f\colon S\to Y$ is subanalytic if its graph is subanalytic in $X\times Y$. One denotes by $\mathscr{A}_S^{\mathbb{R}}$ the sheaf of continuous $\mathbb{R}$-valued subanalytic maps on S. A *subanalytic space* $(X,\mathscr{A}_X^{\mathbb{R}})$, or simply X for short, is an $\mathbb{R}$-ringed space locally isomorphic to $(S,\mathscr{A}_S^{\mathbb{R}})$ for a closed subanalytic subset S of a real analytic manifold. A subset Z of X is subanalytic in X if it is so locally in X. A morphism of subanalytic spaces is a morphism of $\mathbb{R}$-ringed spaces. Then we obtain the category of subanalytic spaces.

The subanalytic topology was introduced in [KS01].

Definition 1.2. Let X be a subanalytic space.

(a) One denotes by $\mathrm{Op}_{X_{\mathrm{sa}}}$ the category of open subanalytic subsets of X, the morphisms being the inclusions and by $\mathrm{Op}^{\mathrm{c}}_{X_{\mathrm{sa}}}$ the full subcategory of $\mathrm{Op}_{X_{\mathrm{sa}}}$ consisting of relatively compact open subsets.

(b) One endows $\mathrm{Op}_{X_{\mathrm{sa}}}$ with a Grothendieck topology as follows. A family $\{U_i\}_{i\in I}$ of objects of $\mathrm{Op}_{X_{\mathrm{sa}}}$ is a covering of $U\in\mathrm{Op}_{X_{\mathrm{sa}}}$ if $U_i\subset U$ for all $i\in I$ and for any compact subset K of X, there exists $J\subset I$ with J finite such that $U\cap K=\bigcup_{j\in J}U_j\cap K$.

(c) One easily checks that the axioms of a Grothendieck topology are satisfied (see [KS06]) and one denotes by X_{sa} the site so obtained.

(d) One denotes by $\rho_{\mathrm{sa}}\colon X\to X_{\mathrm{sa}}$ the natural morphism of sites.

(e) For $U\in\mathrm{Op}_{X_{\mathrm{sa}}}$, the objects of the site U_{sa} are the open subsets $V\subset U$ which are subanalytic in X. (Hence, V is subanalytic in U but one shall be aware that there are open subanalytic subsets of U which are not subanalytic in X.) In particular, the morphism of sites $j_U\colon X_{\mathrm{sa}}\to U_{\mathrm{sa}}$ is well-defined.

The site X_{sa} admits products of two objects U and V, namely $U\cap V$, and fiber products (again the intersection). It admits a terminal object, namely X and an initial object, $\varnothing$.

If $f\colon X_{\mathrm{sa}}\to Y_{\mathrm{sa}}$ is a morphism of subanalytic spaces, then f^t commutes with products and hypothesis (1.1) is satisfied.

1.3 Subanalytic up to infinity

It is natural to consider spaces which are subanalytic "up to infinity".

Definition 1.3. (a) A b-subanalytic space is a pair $X_\infty = (X, \widehat{X})$ of subanalytic spaces such that X is open, relatively compact and subanalytic in $\widehat{X}$. One denotes by $i_X \colon X \hookrightarrow \widehat{X}$ the embedding.

(b) A morphism $f \colon (X, \widehat{X}) \to (Y, \widehat{Y})$ of subanalytic spaces up to infinity is a morphism of subanalytic spaces $f \colon X \to Y$ such that the graph $\Gamma_f \subset X \times Y$ is subanalytic in $\widehat{X} \times \widehat{Y}$.

When X is the analytification of an algebraic space (over $\mathbb{C}$), such a $\widehat{X}$ exists and moreover the property of being b-subanalytic does not depend on the choice of $\widehat{X}$.

Of course, if X is compact, we may choose $\widehat{X} = X$.

Example 1.4. The set $\mathbb{Z} \subset \mathbb{R}$ is subanalytic in $\mathbb{R}$. However, the subanalytic space $\mathbb{Z}$ is not subanalytic up to infinity since a compact subanalytic space has only finite many connected components.

Definition 1.5. (a) Let $X_\infty = (X, \widehat{X})$ be a b-subanalytic space. One denotes by $\mathrm{Op}_{X_{\infty,\mathrm{sa}}}$ the category whose objects are the open subsets of X subanalytic in $\widehat{X}$, the morphisms being the inclusion morphisms and one still denotes by $X_{\infty,\mathrm{sa}}$ the presite associated with the category $\mathrm{Op}_{X_{\infty,\mathrm{sa}}}$.

(b) The subanalytic site $X_{\infty,\mathrm{sa}}$ is the presite $X_{\infty,\mathrm{sa}}$ endowed with the Grothendieck topology defined as follows. A family $\{U_i\}_{i\in I}$ of objects of $\mathrm{Op}_{X_{\infty,\mathrm{sa}}}$ is a covering of $U \in \mathrm{Op}_{X_{\infty,\mathrm{sa}}}$ if $U_i \subset U$ for all $i \in I$ and there exists a finite subset $J \subset I$ such that $\bigcup_{j\in J} U_j = U$.

(c) We denote by $\rho_{\mathrm{sa}} \colon X \to X_{\infty,\mathrm{sa}}$ the natural morphisms of sites.

Note that if $U \in \mathrm{Op}_{X_{\infty,\mathrm{sa}}}$, $U_\infty = (U, \widehat{X})$ is a b-subanalytic space. and there is a natural morphism of sites $j_U \colon X_{\infty,\mathrm{sa}} \to U_{\infty,\mathrm{sa}}$ given by $\mathrm{Op}_{U_{\infty,\mathrm{sa}}} \ni V \mapsto V \in \mathrm{Op}_{X_{\infty,\mathrm{sa}}}$.

1.4 Subanalytic sheaves

From now on, we assume that $\mathbf{k}$ is a field and X is a subanalytic space. The morphism of sites $\rho_{\mathrm{sa}} \colon X \to X_{\mathrm{sa}}$ gives rise to a pair of adjoint functors $(\rho_{\mathrm{sa}}^{-1}, \rho_{\mathrm{sa}*})$.

There is another pair of adjoint functors $(\rho_{\mathrm{sa}!}, \rho_{\mathrm{sa}}^{-1})$ (see [KS01, Prop. 6.6.3]). For $F \in \mathrm{Mod}(\mathbf{k}_X)$, $\rho_{\mathrm{sa}!}F$ is the sheaf associated with the presheaf $U \mapsto F(\overline{U})$ ($U \in \mathrm{Op}_{X_{\mathrm{sa}}}$) where $\overline{U}$ is the closure of U in X.

$$\begin{tikzcd}
\mathrm{Mod}(\mathbf{k}_X) \arrow[r, shift left=2, "\rho_{\mathrm{sa}*}"] \arrow[r, shift right=2, "\rho_{\mathrm{sa}!}"'] & \mathrm{Mod}(\mathbf{k}_{X_{\mathrm{sa}}}). \arrow[l, "\rho_{\mathrm{sa}}^{-1}" description]
\end{tikzcd}$$

Recall that:

(a) The functors ρ_{sa}^{-1} and $\rho_{\mathrm{sa}!}$ are exact and the functor $\rho_{\mathrm{sa}*}$ is left exact.

(b) The functor $\rho_{\mathrm{sa}*}$ and $\rho_{\mathrm{sa}!}$ are fully faithful or, equivalently, $\rho_{\mathrm{sa}}^{-1} \circ \rho_{\mathrm{sa}*} \simeq \mathrm{id}$ and similarly with $\rho_{\mathrm{sa}!}$.

(c) The functor $\rho_{\mathrm{sa}*}$ is exact when restricted to $\mathrm{Mod}_{\mathbb{R}\text{-c}}(\mathbf{k}_X)$. Hence, in the sequel, we shall identify an object of $\mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbf{k}_X)$ and its image in $\mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbf{k}_{X_{\mathrm{sa}}})$ by $\mathrm{R}\rho_{\mathrm{sa}*}$.

Proposition 1.6 (see [KS01, Pro. 6.4.1] and [GS16, Pro. 2.14])**.** *A presheaf F on X_{sa} is a sheaf if and only if it satisfies:*

(i) $F(\emptyset) = 0$,

(ii) *for any $U_1, U_2 \in \mathrm{Op}^{\mathrm{c}}_{X_{\mathrm{sa}}}$, the sequence $0 \to F(U_1 \cup U_2) \to F(U_1) \oplus F(U_2) \to F(U_1 \cap U_2)$ is exact.*

If moreover, for any $\{U_1, U_2\}$ as above, the sequence $0 \to F(U_1 \cup U_2) \to F(U_1) \oplus F(U_2) \to F(U_1 \cap U_2) \to 0$ is exact, then the sheaf F is Γ-acyclic, that is, $H^k(U; F) \simeq 0$ for all $k > 0$ and all $U \in \mathrm{Op}^{c}_{X_{\mathrm{sa}}}$.

One denotes by $\text{“}\varinjlim\text{”}$ the colimits in $\mathrm{Mod}(\mathbf{k}_{X_{\mathrm{sa}}})$.

Proposition 1.7. *Let $F \in \mathrm{Mod}^{\mathrm{c}}_{\mathbb{R}\text{-}c}(\mathbb{C}_X)$ and let $\{G_i\}_i$ be a small directed inductive system in $\mathrm{Mod}(\mathbf{k}_{X_{\mathrm{sa}}})$. One has the isomorphism:*

$$\text{“}\varinjlim_i\text{”}\, \mathscr{H}om\,(F, G_i) \xrightarrow{\sim} \mathscr{H}om\,(F, \text{“}\varinjlim_i\text{”}\, G_i). \tag{1.2}$$

This follows from the fact that finite coverings are cofinal.

Theorem 1.8 (see [KS01, Prop. 6.3.2 and Th. 6.3.5])**.** *One has an equivalence*

$$\mathrm{Ind}\big(\mathrm{Mod}^{\mathrm{c}}_{\mathbb{R}\text{-}c}(\mathbf{k}_X)\big) \simeq \mathrm{Mod}(\mathbf{k}_{X_{\mathrm{sa}}}).$$

In other words, the subanalytic sheaves are the ind-objects of the $\mathbb{R}$-constructible sheaves with compact support.

Before considering subanalytic sheaves, we have introduced ind-sheaves in [KS01] by setting:

$$\mathrm{I}\mathbf{k}_X := \mathrm{Ind}(\mathrm{Mod}^{\mathrm{c}}(\mathbf{k}_M))$$

and calls an object of this category an *ind-sheaf* on M. The prestack $\mathcal{I}(\mathbf{k}_M)\colon U \mapsto \mathrm{I}\mathbf{k}_U$, U open in M, is a stack and we have natural functors

$$\begin{tikzcd}
\mathrm{Mod}_{\mathbb{R}\text{-c}}(\mathbf{k}_X) \arrow[r] \arrow[d] & \mathrm{Mod}(\mathbf{k}_X) \arrow[d] \arrow[dl, "NC"'] \\
\mathrm{Mod}(\mathbf{k}_{X_{\mathrm{sa}}}) \arrow[r] & \mathrm{I}\mathbf{k}_X.
\end{tikzcd} \tag{1.3}$$

Note that triangle denoted NC does not commute.

The category $\mathrm{I}\mathbf{k}_X$ is an abelian category but it does not admit enough injectives (even if $\dim X = 0$), and thus is not a Grothendieck category. One constructs however the six operations for ind-sheaves.

1.5 The six operations

For $U \in \mathrm{Op}_{X_{\mathrm{sa}}}$, one has morphims of sites

$$U_{\mathrm{sa}} \xrightarrow{i_U} X_{\mathrm{sa}} \xrightarrow{j_U} U_{\mathrm{sa}}.$$

Note that $i_U^{-1} \simeq j_{U*}$. One sets $i_{U!} := j_U^{-1}$ and

$$(\,\bullet\,)_U = i_{U!} \circ i_U^{-1}(\,\bullet\,).$$

Now, let $f\colon X_{\mathrm{sa}} \to Y_{\mathrm{sa}}$ be a morphism of subanalytic spaces. One defines the proper direct image functor $f_{!!}\colon \mathrm{Mod}(\mathbf{k}_{X_{\mathrm{sa}}}) \to \mathrm{Mod}(\mathbf{k}_{Y_{\mathrm{sa}}})$ by the formula

$$f_{!!}F = \text{“}\varinjlim_{U \in \mathrm{Op}^{\mathrm{c}}_{X_{\mathrm{sa}}}}\text{”}\; f_*(F_U). \tag{1.4}$$

To derive this functor one uses the subcategory of quasi-injective sheaves (see below).

One can also define this functor on b-subanalytic spaces by using the open embeddings $i_X\colon X \hookrightarrow \widehat{X}$ and $i_Y\colon Y \hookrightarrow \widehat{Y}$. Denote by $\Gamma_f \subset \widehat{X} \times \widehat{Y}$ the graph of f and denote by q_1 and q_2 the two projections:

$$\begin{tikzcd}
 & & & \widehat{X} \times \widehat{Y} \arrow[ddll, "q_1"'] \arrow[ddrr, "q_2"] & & & \\
 & & & \Gamma_f \arrow[u] \arrow[dl] \arrow[dr] & & & \\
X & \widehat{X} \arrow[l, "j_X"] & X \arrow[l, "i_X"] & & Y \arrow[r, "i_Y"'] & \widehat{Y} \arrow[r, "j_Y"'] & Y
\end{tikzcd}$$

Then one sets for $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{X_{\mathrm{sa}}})$

$$f_{!!}F = i_Y^{-1}\mathrm{R}q_{2*}(q_1^{-1}i_{X!}F \overset{\mathrm{L}}{\otimes} \mathbf{k}_{\Gamma_f}) = i_Y^{-1}(i_{X!}F \circ \mathbf{k}_{\Gamma_f}) \simeq j_{Y*}(j_X^{-1}F \circ \mathbf{k}_{\Gamma_f}).$$

Definition 1.9. Let $F \in \mathrm{Mod}(\mathbf{k}_{X_{\mathrm{sa}}})$. One says that F is quasi-injective if the functor $\mathrm{Hom}\,(\,\bullet\,, F)$ is exact on the category $\mathrm{Mod}_{\mathbb{R}\text{-c}}(\mathbf{k}_X)^{\mathrm{op}}$ and one denotes by $\mathscr{I}_{\mathrm{qinj}}$ the full additive subcategory of $\mathrm{Mod}(\mathbf{k}_{X_{\mathrm{sa}}})$ consisting of quasi-injective sheaves.

Proposition 1.10. (a) *Injective sheaves are quasi-injective.*

(b) *If* $0 \to F' \to F \to F'' \to 0$ *is an exact sequence in* $\mathrm{Mod}(\mathbf{k}_{X_{\mathrm{sa}}})$ *and* F', F *are quasi-injective, then so is* F''*.*

(c) *Directed colimits of quasi-injective sheaves are quasi-injective.*

(d) *The category* $\mathscr{I}_{\mathrm{qinj}}$ *is injective for the functor* $f_{!!}$*.*

(e) *The category* $\mathrm{Mod}(\mathbf{k}_{X_{\mathrm{sa}}})$ *has finite quasi-injective dimension.*

By applying the Brown representability theorem, one gets:

Theorem 1.11. *Let* $f\colon X_{\mathrm{sa}} \to Y_{\mathrm{sa}}$ *be a morphism of subanalytic spaces. Then the functor* $\mathrm{R}f_{!!}\colon \mathrm{D}^+(\mathbf{k}_{X_{\mathrm{sa}}}) \to \mathrm{D}^+(\mathbf{k}_{Y_{\mathrm{sa}}})$ *admits a right adjoint, denoted* $f^!$*.*

2 Tempered and Whitney holomorphic functions

From now on, $\mathbf{k} = \mathbb{C}$ and M is a real analytic manifold. We choose a distance d_M on M such that, for any $x \in M$ and any local chart $(U, \varphi\colon U \hookrightarrow \mathbb{R}^n)$ around x, there exists a neighborhood of x over which d_M is Lipschitz equivalent to the pull-back of the Euclidean distance by φ. If there is no risk of confusion, we write d instead of d_M.

Theorem 2.1 (Lojasiewicz)**.** *Let $U = \bigcup_{j \in J} U_j$ be a finite covering of U with U_j in $\mathrm{Op}^c_{M_{\mathrm{sa}}}$. Then there exist a constant $C > 0$ and a positive integer N such that*

$$d(x, X \setminus U)^N \leq C \cdot (\max_{j \in J} d(x, X \setminus U_j)). \tag{2.1}$$

Remark 2.2. We have endowed M with the subanalytic topology. However, there are other natural Grothendieck topologies on X_{sa}. With S. Guillermou [GS16] we say that the family $\{U_j\}_j$ is 1-regularly situated if one can choose $N = 1$ in (2.1). One then defines the notion of linear coverings and we get the linear subanalytic topology.

By using the linear subanalytic topology, Gilles Lebeau has constructed the Sobolev (derived) sheaves.

2.1 Temperate growth

Notation 2.3. One denotes by

- X a complexification of M with structure sheaf $\mathscr{O}_X$,
- $\mathscr{D}_X$ the sheaf of rings of holomorphic differential operators on X,
- $\mathscr{C}^\infty_M$ the sheaf of C^∞-functions on M,
- $\mathcal{D}b_M$ the sheaf of Schwartz's distributions on M,
- $\mathscr{A}_M = \mathscr{O}_X|_M$,
- $\mathscr{B}_M$ the sheaf of Sato's hyperfunctions on M,
- $\mathscr{D}_M = \mathscr{D}_X|_M$ and $\mathscr{D}_{M_{\mathrm{sa}}} := \rho_{\mathrm{sa}!}\mathscr{D}_M$.
- As far as their is no risk of confusion, we denote by $\mathscr{H}om$ the internal Hom morphism for sheaves on M as well as for sheaves on M_{sa}.

Definition 2.4. Let $U \in \mathrm{Op}^c_{M_{\mathrm{sa}}}$ and let $f \in \mathscr{C}^\infty_M(U)$. One says that f has *polynomial growth* at $p \in M \setminus U$ if it satisfies the following condition. For a local coordinate system $(x_1, \dots, x_n)$ around p, there exist a sufficiently small compact neighborhood K of p a constant $C \geq 0$ and a positive integer N such that

$$\sup_{x \in K \cap U} |f(x)| \leq \big(d(x, K \setminus U)\big)^{-N}. < \infty. \tag{2.2}$$

One says that f is temperate at p if all its derivatives have polynomial growth at p. We say that f is temperate if it is temperate at any point $p \in M \setminus U$.

For $U \in \mathrm{Op}^c_{M_{\mathrm{sa}}}$, one denotes by $\mathscr{C}^{\infty,\mathrm{tp}}_M(U)$ the subspace of $\mathscr{C}^\infty_M(U)$ consisting of temperate functions.

For $U \in \mathrm{Op}^c_{M_{\mathrm{sa}}}$, one denotes by $\mathcal{D}b^{\mathrm{tp}}_M(U)$ the space of temperate distributions on U, defined by the exact sequence

$$0 \to \Gamma_{M\setminus U}(M;\mathcal{D}b_M) \to \Gamma(M;\mathcal{D}b_M) \to \mathcal{D}b^{\mathrm{tp}}_M(U) \to 0.$$

It follows from the work of Lojasiewicz [Loj59] (see (2.1)) that $U \mapsto \mathscr{C}^{\infty,\mathrm{tp}}_M(U)$ and $U \mapsto \mathcal{D}b^{\mathrm{tp}}_M(U)$ are sheaves on M_{sa}.

We denote by $\mathscr{C}^{\infty,\mathrm{tp}}_{M_{\mathrm{sa}}}$ and $\mathcal{D}b^{\mathrm{tp}}_{M_{\mathrm{sa}}}$ these sheaves on M_{sa}. The first one is called the sheaf of $\mathscr{C}^\infty$-functions with temperate growth and the second the sheaf of temperate distributions. Note that both sheaves are sheaves of $\mathscr{D}_{M_{\mathrm{sa}}}$-modules, are Γ-acyclic and the sheaf $\mathcal{D}b^{\mathrm{tp}}_{M_{\mathrm{sa}}}$ is flabby.

2.2 Rapid decay

To S a closed subanalytic subset of M, we associate the sheaf of $\mathscr{D}_M$-modules:

$$V \mapsto \mathscr{I}^\infty_S(V) := \{\varphi \in \mathscr{C}^\infty_M(V); \varphi \text{ vanish up to infinite order on } S \cap V\}.$$

One also sets, following [KS96, § 2]:

$$\mathbb{C}_U \overset{\mathrm{w}}{\otimes} \mathscr{C}^\infty_M \quad := \quad \mathscr{I}^\infty_{M\setminus U}.$$

Hence, $\Gamma(M;\mathbb{C}_U \overset{\mathrm{w}}{\otimes} \mathscr{C}^\infty_M)$ is the space of $\mathscr{C}^\infty$-functions on M which vanish up to infinite order on $M \setminus U$. One calls it the space of functions with rapid decay on U. It naturally contains $\Gamma_c(U;\mathscr{C}^\infty_M)$.

In loc. cit. it is shown that the functor $U \mapsto \mathbb{C}_U \overset{\mathrm{w}}{\otimes} \mathscr{C}^\infty_M$ uniquely extends to an exact functor

$$\bullet \overset{\mathrm{w}}{\otimes} \mathscr{C}^\infty_{M_{\mathrm{sa}}} : \mathrm{Mod}_{\mathbb{R}\text{-c}}(\mathbb{C}_M) \to \mathrm{Mod}(\mathscr{D}_M).$$

One denotes by $\mathscr{C}^{\infty,\mathrm{w}}_{M_{\mathrm{sa}}}$ the sheaf on M_{sa} given by

$$\mathscr{C}^{\infty,\mathrm{w}}_{M_{\mathrm{sa}}}(U) = \Gamma\big(M; H^0(\mathrm{D}'_M\mathbb{C}_U) \overset{\mathrm{w}}{\otimes} \mathscr{C}^\infty_{M_{\mathrm{sa}}}\big),\ U \in \mathrm{Op}_{M_{\mathrm{sa}}}.$$

If $\mathrm{D}'_M\mathbb{C}_U \simeq \mathbb{C}_{\overline{U}}$, then $\mathscr{C}^{\infty,\mathrm{w}}_{M_{\mathrm{sa}}}(U) \simeq \mathscr{C}^\infty(M)/\mathscr{I}^\infty_{M,\overline{U}}$ is the space of Whitney functions on $\overline{U}$. It is thus natural to call $\mathscr{C}^{\infty,\mathrm{w}}_{M_{\mathrm{sa}}}$ the sheaf of Whitney C^∞-functions on M_{sa}.

We have the morphisms in the category $\mathsf{D}^{\mathrm{b}}(\mathbb{C}_{M_{\mathrm{sa}}})$:

$$\rho_{\mathrm{sa}!}\mathscr{A}_M \to \mathscr{C}^{\infty,\mathrm{w}}_{M_{\mathrm{sa}}} \to \mathscr{C}^{\infty,\mathrm{tp}}_{M_{\mathrm{sa}}} \to \mathcal{D}b^{\mathrm{tp}}_{M_{\mathrm{sa}}} \to \rho_{\mathrm{sa}*}\mathscr{B}_M.$$

2.3 Complex setting

Now, let X be a complex manifold of complex dimension d_X, $X_{\mathbb{R}}$ the real underlying manifold (we shall often write X instead of $X_{\mathbb{R}}$) and assume that $X_{\mathbb{R}}$ is endowed with a distance $d(x,y)$ as above. We denote by X_{sa} the site $(X_{\mathbb{R}})_{\mathrm{sa}}$. We denote by X^c the complex manifold conjugate to X. We denote by X_{sa} the site $(X_{\mathbb{R}})_{\mathrm{sa}}$ and similarly with X^c_{sa}. We set for short

$$\mathscr{O}_{X_{\mathrm{sa}}} := \rho_{\mathrm{sa}!}\mathscr{O}_X, \quad \mathscr{D}_{X_{\mathrm{sa}}} := \rho_{\mathrm{sa}!}\mathscr{D}_X.$$

One defines similarly $\mathscr{O}_{X^c_{\mathrm{sa}}}$, $\Omega_{X_{\mathrm{sa}}}$, etc.

One denotes by $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$ the full triangulated subcategory of $\mathsf{D}^{\mathrm{b}}(\mathscr{D}_X)$ consisting of objects with coherent cohomologies. One defines similarly $\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X)$ (holonomic) and $\mathsf{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathscr{D}_X)$ (regular holonomic).

Remark 2.5. One shall be aware that for $n > 1$, $\mathrm{R}\rho_{\mathrm{sa}*}\mathscr{O}_{X_{\mathrm{sa}}}$ is not concentrated in degree 0. Indeed, with the subanalytic topology, only finite coverings are allowed. If one considers for example the open subset $U \subset \mathbb{C}^n$, the difference of an open ball of radius R and a closed ball of radius rR with $0 < r < R$, then the Dolbeault complex will not be exact after any finite covering.

Definition 2.6. (i) The derived sheaf $\mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}}$ of holomorphic functions with temperate growth is the object of $\mathsf{D}^{\mathrm{b}}(\mathbb{C}_{X_{\mathrm{sa}}})$ given by

$$\mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}} \;:=\; \mathrm{R}\mathscr{H}om_{\mathscr{D}_{X^c_{\mathrm{sa}}}}(\mathscr{O}_{X^c_{\mathrm{sa}}}, \mathcal{D}b^{\mathrm{tp}}_{X_{\mathrm{sa}}}) \xleftarrow{\sim} \mathrm{R}\mathscr{H}om_{\mathscr{D}_{X^c_{\mathrm{sa}}}}(\mathscr{O}_{X^c_{\mathrm{sa}}}, \mathscr{C}^{\infty,\mathrm{tp}}_{X}). \tag{2.3}$$

(ii) The derived sheaf $\mathscr{O}^{\mathrm{w}}_{X_{\mathrm{sa}}}$ of Whitney holomorphic functions is the object of $\mathsf{D}^{\mathrm{b}}(\mathbb{C}_{X_{\mathrm{sa}}})$ given by

$$\mathscr{O}^{\mathrm{w}}_{X_{\mathrm{sa}}} \;:=\; \mathrm{R}\mathscr{H}om_{\mathscr{D}_{X^c_{\mathrm{sa}}}}(\mathscr{O}_{X^c_{\mathrm{sa}}}, \mathscr{C}^{\infty,\mathrm{w}}_{X_{\mathrm{sa}}}). \tag{2.4}$$

(iii) The functor of holomorphic functions with rapid decay is the functor defined on $\mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$ with values in $\mathsf{D}^{\mathrm{b}}(\mathbb{C}_{X_{\mathrm{sa}}})$ given by

$$\bullet \overset{\mathrm{w}}{\otimes} \mathscr{O}_{X_{\mathrm{sa}}} := \mathrm{R}\mathscr{H}om_{\mathscr{D}_{X^c_{\mathrm{sa}}}}(\mathscr{O}_{X^c_{\mathrm{sa}}}, \bullet \overset{\mathrm{w}}{\otimes} \mathscr{C}^{\infty}_{M}) \tag{2.5}$$

Note that the isomorphism in (2.3) is far from being obvious.

Hence, $\mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}}$ and $\mathscr{O}^{\mathrm{w}}_{X_{\mathrm{sa}}}$ are represented by the Dolbeault complexes

$$\begin{aligned}
&0 \to \mathcal{D}b^{\mathrm{tp},(0,0)}_{X_{\mathrm{sa}}} \xrightarrow{\bar\partial} \mathcal{D}b^{\mathrm{tp},(0,1)}_{X_{\mathrm{sa}}} \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \mathcal{D}b^{\mathrm{tp},(0,d)}_{X_{\mathrm{sa}}} \to 0,\\
&0 \to \mathscr{C}^{\infty,\mathrm{w},(0,0)}_{X_{\mathrm{sa}}} \xrightarrow{\bar\partial} \mathscr{C}^{\infty,\mathrm{w},(0,1)}_{X_{\mathrm{sa}}} \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \mathscr{C}^{\infty,\mathrm{w},(0,d)}_{X_{\mathrm{sa}}} \to 0.
\end{aligned} \tag{2.6}$$

Note that for $F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$,

$$\mathrm{R}\mathscr{H}om(\mathrm{D}'_X F, \mathscr{O}^{\mathrm{w}}_{X_{\mathrm{sa}}}) \simeq F \overset{\mathrm{w}}{\otimes} \mathscr{O}_X.$$

Example 2.7 ([KS96, Th.5.10 & 5.12]). (i) Let Z be a closed complex analytic subset of the complex manifold X. We have the isomorphisms in $\mathsf{D}^{\mathrm{b}}(\mathscr{D}_X)$:

$$\begin{aligned} &\rho_{\mathrm{sa}}^{-1}(\mathbb{C}_Z \overset{\mathrm{w}}{\otimes} \mathscr{O}_{X_{\mathrm{sa}}}) \simeq \mathscr{O}_X\widehat{|}_Z && \text{(formal completion)},\\ &\rho_{\mathrm{sa}}^{-1}\mathrm{R}\mathscr{H}\!om\,(\mathbb{C}_Z, \mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}}) \simeq \mathrm{R}\Gamma_{[Z]}(\mathscr{O}_X) && \text{(algebraic cohomology)}, \end{aligned}$$

(ii) Let M be a real analytic manifold such that X is a complexification of M. We have the isomorphisms in $\mathsf{D}^{\mathrm{b}}(\mathscr{D}_M)$:

$$\begin{aligned} &\rho_{\mathrm{sa}}^{-1}(\mathbb{C}_M \overset{\mathrm{w}}{\otimes} \mathscr{O}_{X_{\mathrm{sa}}})|_M \simeq \mathscr{C}^\infty_M && (\mathscr{C}^\infty\text{-functions}),\\ &\rho_{\mathrm{sa}}^{-1}\mathrm{R}\mathscr{H}\!om\,(\mathrm{D}'_X\mathbb{C}_M, \mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}})|_M \simeq \mathcal{D}b_M && \text{(distributions)}, \end{aligned}$$

2.4 Duality

This section is extracted from [KS96, § 2.4].

We shall use the theory of topological $\mathbb{C}$-vector spaces of type FN (Fréchet nuclear spaces) or DFN (dual of Fréchet nuclear spaces). The categories of FN spaces and DFN spaces are quasi-abelian and the topological duality functor induces a contravariant equivalence between the category of FN spaces and DFN spaces. It induces therefore an equivalence of triangulated categories

$$\mathrm{D}\colon \mathsf{D}^{\mathrm{b}}(\mathrm{FN})^{\mathrm{op}} \xrightarrow{\sim} \mathsf{D}^{\mathrm{b}}(\mathrm{DFN}).$$

Proposition 2.8 ([KS96, Prop. 2.2]). *Let M be a real analytic manifold and let $F \in \mathbb{R}\text{-}\mathrm{C}(\mathbb{C}_M)$. Then, there exist natural topologies of type* FN *on $\Gamma(M; F \overset{\mathrm{w}}{\otimes} \mathscr{C}^\infty_{M_{\mathrm{sa}}})$ and of type* DFN *on $\Gamma_c(M; \mathscr{H}\!om\,(F, \mathcal{D}b^{\mathrm{tp}\vee}_{M_{\mathrm{sa}}}))$, and they are dual to each other.*

Here, as usual, $\Gamma_c(M; \bullet)$ is the functor of global sections with compact support.

Let X be a complex manifold of complex dimension d_X.

Theorem 2.9 ([Sch21, Sko21]). *Let $U \in \mathrm{Op}^c_{X_{\mathrm{sa}}}$ and assume that U is Stein. Then $\mathrm{R}\Gamma(U; \mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}})$ is concentrated in degree* 0.

It follows that $\mathrm{R}\Gamma(X; \mathbb{C}_U \overset{\mathrm{w}}{\otimes} \mathscr{O}_{X_{\mathrm{sa}}})\,[d_X]$ is concentrated in degree 0 and is dual to the space $\Gamma(U; \mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}})$.

Proof. By Proposition 2.8, $\mathrm{R}\Gamma(U; \mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}})$ and $\mathrm{R}\Gamma(X; \mathbb{C}_U \overset{\mathrm{w}}{\otimes} \mathscr{O}_{X_{\mathrm{sa}}})$ may be represented by two complexes $E^\bullet$ and $F^\bullet$ of type DFN and FN respectively and the cohomology of the DFN complex is concentrated in degree 0 by [Sch21, Sko21].

$$\begin{aligned} E^\bullet: &\qquad 0 \to E^0 \xrightarrow{u_0} \cdots \xrightarrow{u_{d-1}} E^d \to 0,\\ F^\bullet: &\qquad 0 \to F^0 \xrightarrow{v_0} \cdots \xrightarrow{v_{d-1}} F^d \to 0. \end{aligned}$$

Since $E^\bullet$ is exact except in degree 0, the linear maps $u_0, \dots, u_{n-1}$ have closed ranges. It follows that the linear maps $v_0, \dots, v_{n-1}$ also have closed ranges and the complex $F^\bullet$ is concentrated in degree d. Q.E.D.

Theorem 2.10 ([KS96, Th. 6.1],[KS16])**.** *Let* $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X)$ *and let* $F, G \in \mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathbb{C}_X)$*. Then* $\mathscr{H}_1 := \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(G \otimes \mathscr{M}, F \overset{\mathrm{w}}{\otimes} \mathscr{O}_{X_{\mathrm{sa}}})$ *and* $\mathscr{H}_2 := \mathrm{R}\mathscr{H}om(F, \Omega^{\mathrm{tp}}_{X_{\mathrm{sa}}}[d_X]) \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{M} \otimes G$ *belong to* $\mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathbb{C}_X)$ *and are dual to each other, that is,* $\mathscr{H}_1 \simeq \mathrm{D}'_X \mathscr{H}_2$*.*

3 Zariski and subanalytic topologies

In this section, X is a smooth algebraic variety over $\mathbb{C}$. We denote by X_{za} the space X endowed with the Zariski topology. Hence, the open sets are the complementary of closed complex hypersurfaces and their finite unions.

(3.1) In this section, we assume to be chosen $X_\infty = (X, \widehat{X})$ and call for short "subanalytic" what should be called b-subanalytic.

There is a morphism of sites

$$\rho_{\mathrm{za}} \colon X_{\mathrm{sa}} \to X_{za}. \tag{3.2}$$

Therefore, we have a pair of adjoint functors $(\rho_{\mathrm{za}}^{-1}, \rho_{\mathrm{za}*})$:

$$\rho_{\mathrm{za}}^{-1} \colon \mathrm{Mod}(\mathbf{k}_{X_{za}}) \rightleftarrows \mathrm{Mod}(\mathbf{k}_{X_{\mathrm{sa}}}) \colon \rho_{\mathrm{za}*}. \tag{3.3}$$

Theorem 3.1. (a) *The functor* ρ_{za}^{-1} *in* (3.3) *is fully faithful and exact.*

(b) *There is a natural isomorphism* $\mathrm{R}\rho_{\mathrm{za}*}\mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}} \simeq \mathscr{O}_{X_{za}}$*.*

Proof. (a)–(i) The functor $\mathrm{Op}_{X_{za}} \to \mathrm{Op}_{X_{\mathrm{sa}}}$ is fully faithful and the coverings in X_{za} and in X_{sa} are the same. Then the result is well-known (see also [Sch25, Th. 1.8.7]).
(a)–(ii) is classical (or see loc. cit. Th 1.7.4).
(b) The family of Stein Zariski open subsets of X_{za} gives a basis of the Zariski topology. Applying Theorem 2.9, we get that the object $\mathrm{R}\rho_{\mathrm{za}*}\mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}}$ is concentrated in degree 0. Moreover, for Z a closed algebraic hypersurface, one has the isomorphism

$$\Gamma(X \setminus Z; \mathscr{O}_{X_{za}}) \xrightarrow{\sim} \Gamma(X \setminus Z; \mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}}).$$

which follows from the GAGA theorem and the isomorphism $\mathrm{R}\Gamma_{[Z]}\mathscr{O}_{X_{za}} \xrightarrow{\sim} \mathrm{R}\Gamma_Z \mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}}$ of Example 2.7. See also [Pet27] for related results. Q.E.D.

Corollary 3.2 (of Theorem 2.10)**.** *Let* $U \in \mathrm{Op}_{X_{za}}$ *and let* $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_{X_{za}})$*. Then* $\mathrm{RHom}_{\mathscr{D}_X}(\mathscr{M}, \mathbb{C}_U \overset{\mathrm{w}}{\otimes} \mathscr{O}_{X_{\mathrm{sa}}})$ *and* $\mathrm{RHom}(\mathbb{C}_U, \Omega^{\mathrm{tp}}_{X_{\mathrm{sa}}}[d_X] \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{M})$ *are finite dimensional vector spaces dual one to each other.*

Remark 3.3. In [BE04], S. Bloch and H. Esnault proved directly a similar result on an algebraic curve X when assuming that $\mathscr{M}$ is a meromorphic connection with poles on a divisor D. They interpret the duality pairing by considering sections of the type $\gamma \otimes \varepsilon$, where γ is a cycle with boundary on D and ε is a horizontal section of the connection on γ with exponential decay on D. Their work has been extended to higher dimension by M. Hien [Hie09].

4 The Riemann-Hilbert correspondence

The Riemann-Hilbert correspondence for regular holonomic $\mathscr{D}_X$-modules has been stated and proved by Masaki Kashiwara in [Kas80, Kas84].

The use of the subanalytic topology and the sheaf of temperate holomorphic functions makes the proof slightly easier. Lemma 4.2 below, implicitly contained in [Kas84] and extracted from [KS16, Lem. 4.1.11], is quite illuminating.

4.1 A lemma for regular holonomic $\mathscr{D}$-modules

Definition 4.1. Let X be a complex manifold and $D \subset X$ a normal crossing divisor. We say that a holonomic $\mathscr{D}_X$-module $\mathscr{M}$ has *regular normal form* along D if locally on D, for a local coordinate system $(z_1, \dots, z_n)$ on X such that $D = \{z_1 \cdots z_r = 0\}$, $\mathscr{M} \simeq \mathscr{D}_X/\mathscr{I}_\lambda$ for $\lambda = (\lambda_1, \dots, \lambda_r) \in (\mathbb{C} \setminus \mathbb{Z}_{\geq 0})^r$. Here, $\mathscr{I}_\lambda$ is the left ideal generated by the operators $(z_i\partial_i - \lambda_i)$ and ∂_j for $i \in \{1, \dots, r\}$, $j \in \{r+1, \dots, n\}$.

One shall be aware that the property of being of normal form is not stable by duality. Note that, for $\lambda = (\lambda_1, \dots, \lambda_r) \in \mathbb{C}^m$, $\mathscr{D}_X/\mathscr{I}_\lambda \xrightarrow{\sim} (\mathscr{D}_X/\mathscr{I}_\lambda)(*D)$ if and only if $\lambda_i \in \mathbb{C} \setminus \mathbb{Z}_{\geq 0}$ for any $i \in \{1, \dots, r\}$.

Lemma 4.2. *Let $P_X(\mathscr{M})$ be a statement concerning a complex manifold X and a regular holonomic object $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathscr{D}_X)$. Consider the following conditions.*

(a) *Let $X = \bigcup_{i\in I} U_i$ be an open covering. Then $P_X(\mathscr{M})$ is true if and only if $P_{U_i}(\mathscr{M}|_{U_i})$ is true for any $i \in I$.*

(b) *If $P_X(\mathscr{M})$ is true, then $P_X(\mathscr{M}[n])$ is true for any $n \in \mathbb{Z}$.*

(c) *Let $\mathscr{M}' \to \mathscr{M} \to \mathscr{M}'' \xrightarrow{+1}$ be a distinguished triangle in $\mathsf{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathscr{D}_X)$. If $P_X(\mathscr{M}')$ and $P_X(\mathscr{M}'')$ are true, then $P_X(\mathscr{M})$ is true.*

(d) *Let $\mathscr{M}$ and $\mathscr{M}'$ be regular holonomic $\mathscr{D}_X$-modules. If $P_X(\mathscr{M} \oplus \mathscr{M}')$ is true, then $P_X(\mathscr{M})$ is true.*

(e) *Let $f\colon X \to Y$ be a projective morphism and let $\mathscr{M}$ be a good regular holonomic $\mathscr{D}_X$-module. If $P_X(\mathscr{M})$ is true, then $P_Y(\mathrm{D}f_*\mathscr{M})$ is true.*

(f) *If $\mathscr{M}$ is a regular holonomic $\mathscr{D}_X$-module with a regular normal form along a normal crossing divisor of X, then $P_X(\mathscr{M})$ is true.*

If conditions (a)–(f) *are satisfied, then $P_X(\mathscr{M})$ is true for any complex manifold X and any $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathscr{D}_X)$.*

Remark 4.3. There is a result similar to that of Lemma 4.2 for irregular holonomic $\mathscr{D}$-modules, due to Mochizuki [Moc09, Moc11] in the algebraic setting after preliminary results by Sabbah [Sab00] and extended to the analytic setting by Kedlaya [Ked10, Ked11].

It says essentially that one can extend Lemma 4.2 to irregular holonomic $\mathscr{D}$-modules, if, in condition (f), one replaces modules of regular normal form with exponential $\mathscr{D}$-modules, a class of modules that we shall discuss below.

4.2 A lemma for $\mathbb{C}$-constructible sheaves

Of course, one can state an analogue result to Lemma 4.2 for $\mathbb{C}$-constructible sheaves.

Definition 4.4. Let X be a complex manifold and $D \subset X$ a normal crossing divisor. We say that a local system L is *regular along* D if L is locally constant of rank one on $X \setminus D$ and is zero on D.

Lemma 4.5. *Let $P_X(F)$ be a statement concerning a complex manifold X and an object $F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(\mathbb{C}_X)$. Consider the following conditions.*

(a) *Let $X = \bigcup_{i\in I} U_i$ be an open covering. Then $P_X(F)$ is true if and only if $P_{U_i}(F|_{U_i})$ is true for any $i \in I$.*

(b) *If $P_X(F)$ is true, then $P_X(F[n])$ is true for any $n \in \mathbb{Z}$.*

(c) *Let $F' \to F \to F'' \xrightarrow{+1}$ be a distinguished triangle in $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(\mathbb{C}_X)$. If $P_X(F')$ and $P_X(F')$ are true, then $P_X(F)$ is true.*

(d) *Let F and F' be two objects of $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(\mathbb{C}_X)$. If $P_X(F \oplus F')$ is true, then $P_X(F)$ is true.*

(e) *Let $f\colon X \to Y$ be a projective morphism and let $F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(\mathbb{C}_X)$. If $P_X(F)$ is true, then $P_Y(\mathrm{R}f_*F)$ is true.*

(f) *If $F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(\mathbb{C}_X)$ is a local system regular along a normal crossing divisor of X, then $P_X(F)$ is true.*

If conditions (a)–(f) *are satisfied, then $P_X(F)$ is true for any complex manifold X and any $F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(\mathbb{C}_X)$.*

Remark 4.6. Lemma 4.5 has never been explicitly stated but its proof is exactly the same as that of Lemma 4.2. It can also been deduced from this last lemma via the Riemann-Hilbert correspondence.

The regular Riemann-Hilbert correspondence

Consider the two functors

$$\begin{aligned}
&\mathcal{DR}_X\colon \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X) \to \mathsf{D}^{\mathrm{b}}(\mathbb{C}_X), \mathscr{M} \mapsto \Omega_X \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{M},\\
&\mathcal{S}ol_X\colon \mathsf{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathscr{D}_X)^{\mathrm{op}} \to \mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathbb{C}_X), \quad \mathscr{M} \mapsto \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X),\\
&\mathrm{RH}_X\colon \mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathbb{C}_X)^{\mathrm{op}} \to \mathsf{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathscr{O}_X), \quad F \mapsto \rho_{\mathrm{sa}}^{-1}\mathrm{R}\mathscr{H}om(F, \mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}}).
\end{aligned} \tag{4.1}$$

Note that, setting $\mathbb{D}'_X\mathscr{M} = \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}_X \otimes_{\mathscr{O}_X} \Omega_X^{\otimes -1})$, one has $\mathcal{S}ol_X(\mathscr{M}) \simeq \mathcal{DR}_X(\mathbb{D}'_X\mathscr{M})$ for $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$.

By using Lemma 4.2 and 4.5, one proves Theorem 4.7 below (see [Kas84] for the original proof and [KS16] for a detailed proof).

Theorem 4.7 (The R-H correspondence, after [Kas80, Kas84])**.** *The functors $\mathcal{S}ol_X$ and RH_X in* (4.1) *are well defined and quasi-inverse one to each other.*

5 Exponential $\mathscr{D}$-modules

The aim of this section is to show that the use of the sheaf $\mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}}$ plays an essential role in the study of irregular holonomic $\mathscr{D}$-modules. We shall restrict ourselves to exponential $\mathscr{D}$-modules, that is modules of the form $\mathscr{D}_X \cdot \exp\varphi$ with φ meromorphic. We refer to [FJ26] for a detailed study of such modules from an arithmetic point of view.

Setting $\Omega^{\mathrm{tp}}_{X_{\mathrm{sa}}} := \Omega_{X_{\mathrm{sa}}} \otimes_{\mathscr{O}_{X_{\mathrm{sa}}}} \mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}}$, we also define the temperate de Rham and solution functors:

$$\begin{aligned}
&\mathcal{DR}^{\mathrm{tp}}_X \colon \mathsf{D}^{\mathrm{b}}(\mathscr{D}_{X_{\mathrm{sa}}}) \to \mathsf{D}^{\mathrm{b}}(\mathbb{C}_{X_{\mathrm{sa}}}), && \mathscr{M} \mapsto \Omega^{\mathrm{tp}}_{X_{\mathrm{sa}}} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_{X_{\mathrm{sa}}}} \mathscr{M},\\
&\mathcal{S}ol^{\mathrm{tp}}_X \colon \mathsf{D}^{\mathrm{b}}(\mathscr{D}_{X_{\mathrm{sa}}})^{\mathrm{op}} \to \mathsf{D}^{\mathrm{b}}(\mathbb{C}_{X_{\mathrm{sa}}}), && \mathscr{M} \mapsto \mathrm{R}\mathscr{H}om_{\mathscr{D}_{X_{\mathrm{sa}}}}(\mathscr{M}, \mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}}).
\end{aligned}$$

One has $\mathcal{DR}_X \simeq \rho_{\mathrm{sa}}^{-1}\mathcal{DR}^{\mathrm{tp}}_X$ and $\mathcal{S}ol_X \simeq \rho_{\mathrm{sa}}^{-1}\mathcal{S}ol^{\mathrm{tp}}_X$ and

$$\mathcal{S}ol^{\mathrm{tp}}_X(\mathscr{M}) \simeq \mathcal{DR}^{\mathrm{tp}}_X(\mathbb{D}'_X\mathscr{M}). \tag{5.1}$$

Let $Y \subset X$ a complex analytic hypersurface and set $U = X \setminus Y$. For $\varphi \in \mathscr{O}(*Y)$, one sets

$$\begin{aligned}
\mathscr{D}_X \exp\varphi &= \mathscr{D}_X / \{P\,;\, P\exp\varphi = 0 \text{ on } U\},\\
\mathscr{E}^{\varphi}_{U|X} &= (\mathscr{D}_X \exp\varphi)(*Y).
\end{aligned}$$

Hence $\mathscr{D}_X \exp\varphi$ is a $\mathscr{D}_X$-submodule of $\mathscr{E}^{\varphi}_{U|X}$, and $\mathscr{D}_X \exp\varphi$ as well as $\mathscr{E}^{\varphi}_{U|X}$ are holonomic $\mathscr{D}_X$-modules. Note that $\mathscr{E}^{\varphi}_{U|X}$ is isomorphic to $\mathscr{O}(*Y)$ as an $\mathscr{O}$-module, and the connection $\mathscr{O}(*Y) \to \Omega^1_X \otimes_{\mathscr{O}} \mathscr{O}(*Y)$ is given by $u \mapsto du + ud\varphi$. For $c \in \mathbb{R}$, set for short

$$\begin{aligned}
\{\Re\varphi < c\} &:= \{x \in U; \Re\varphi(x) < c\} \subset X,\\
E^{\varphi}_{U|X} &:= \mathrm{R}\mathscr{H}om(\mathbb{C}_U, \text{“}\varinjlim_{c\to+\infty}\text{”}\, \mathbb{C}_{\{\Re\varphi<c\}}) \in \mathsf{D}^{\mathrm{b}}(\mathbb{C}_{X_{\mathrm{sa}}}).
\end{aligned}$$

The next result was first proved in [KS03, Prop. 7.3] in the particular case where $\varphi(z) = 1/z$, then in the general case in [DK14, Prop. 6.2.2].

Theorem 5.1. *Let $Y \subset X$ be a closed complex analytic hypersurface and set $U = X\setminus Y$. For $\varphi \in \mathscr{O}(*Y)$, there is an isomorphism in $\mathsf{D}^{\mathrm{b}}(\mathbb{C}_{X_{\mathrm{sa}}})$*

$$\mathcal{S}ol^{\mathrm{tp}}_X(\mathscr{E}^{\varphi}_{U|X}) \simeq E^{\varphi}_{U|X}.$$

Lemma 5.2. *One has $\mathcal{S}ol^{\mathrm{tp}}_X(\mathscr{E}^{\varphi}_{U|X}) \simeq \mathrm{R}\mathscr{H}om(\mathbb{C}_U, \mathcal{S}ol^{\mathrm{tp}}_X(\mathscr{E}^{\varphi}_{U|X})) \simeq \mathrm{R}\mathscr{H}om(\mathbb{C}_U, \mathcal{S}ol^{\mathrm{tp}}_X(\mathscr{E}^{\varphi}_{U|X}) \otimes \mathbb{C}_U)$.*

Proof. (i) Let us prove the first isomorphism. By (5.1), it is equivalent to prove a similar result for $\mathcal{DR}^{\mathrm{tp}}_X(\mathscr{E}^{\varphi}_{U|X})$. One has

$$\begin{aligned}
\mathcal{DR}^{\mathrm{tp}}_X(\mathscr{E}^{\varphi}_{U|X}) &\simeq \Omega^{\mathrm{tp}}_{X_{\mathrm{sa}}} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_{X_{\mathrm{sa}}}} \mathscr{E}^{\varphi}_{U|X} \simeq \Omega^{\mathrm{tp}}_{X_{\mathrm{sa}}} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_{X_{\mathrm{sa}}}} \mathscr{E}^{\varphi}_{U|X}(*Y)\\
&\simeq \Omega^{\mathrm{tp}}_{X_{\mathrm{sa}}}(*Y) \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_{X_{\mathrm{sa}}}} \mathscr{E}^{\varphi}_{U|X}.
\end{aligned}$$

To conclude, use the isomorphism

$$\Omega^{\mathrm{tp}}_{X_{\mathrm{sa}}}(*Y) \simeq \mathrm{R}\mathscr{H}om\,(\mathbb{C}_U, \Omega^{\mathrm{tp}}_{X_{\mathrm{sa}}})$$

or, equivalently $\mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}}(*Y) \simeq \mathrm{R}\mathscr{H}om\,(\mathbb{C}_U, \mathscr{O}^{\mathrm{tp}}_{X_{\mathrm{sa}}})$, which follows from Theorem 4.7 applied with $F = \mathbb{C}_U$.

(ii) To prove the second isomorphism, note that for any $F \in \mathsf{D}^{\mathrm{b}}(\mathbb{C}_{X_{\mathrm{sa}}})$ one has the isomorphism

$$\mathrm{R}\mathscr{H}om\,(\mathbb{C}_U, F \otimes \mathbb{C}_U) \xrightarrow{\sim} \mathrm{R}\mathscr{H}om\,(\mathbb{C}_U, F). \tag{5.2}$$

Indeed, setting $Z = X \setminus U$, it is enough to prove that $\mathrm{R}\mathscr{H}om\,(\mathbb{C}_U, F \otimes \mathbb{C}_Z) \simeq 0$. Denote by $j\colon Z \hookrightarrow X$ the embedding. Then $F \otimes \mathbb{C}_Z \simeq j_* j^{-1} F$ and the result follows by adjunction since $j^{-1}\mathbb{C}_U \simeq 0$. Q.E.D.

Sketch of the proof of Theorem 5.1. We shall first treat the particular case of $\exp(1/z)$, following [KS03, Prop. 7.3]. Applying this result, we shall treat the general case following [DK14, Prop. 6.2.2]. Q.E.D.

5.1 A particular case

In this subsection, following [KS03, Prop. 7.3], we shall prove Theorem 5.1 in the particular case where $X = \mathbb{C}$ with coordinate z, $U = \mathbb{C} \setminus \{0\}$ and $\varphi(z) = 1/z$. Note that $\mathscr{D}_X \cdot \exp(1/z) = \mathscr{D}_X / \mathscr{D}_X(z^2\partial_z + 1)$. Also note that since $d_X = 1$, $\mathscr{O}^{\mathrm{tp}}_X$ is concentrated in degree 0.

We set for short

$$\mathscr{L} := \mathscr{D}_X \cdot \exp(1/z), \quad \mathscr{S}^{\mathrm{tp}} := H^0(\mathcal{S}ol^{\mathrm{tp}}(\mathscr{L})), \quad \mathscr{S} := H^0(\mathcal{S}ol(\mathscr{L}))$$

The morphism $\mathscr{S}^{\mathrm{tp}} \to \mathscr{S}$ is a monomorphism. Moreover, $\mathscr{S} \simeq \mathbb{C}_{X\setminus\{0\}} \cdot \exp(1/z)$. Let $V \subset X$ be a connected open subanalytic subset. Then the space $\Gamma(V; \mathscr{S})$ has dimension one and is generated by the function $\exp(1/z)$. Hence, the subspace $\Gamma(V; \mathscr{S}^{\mathrm{tp}}) \simeq \Gamma(V; \mathscr{S}) \cap \Gamma(V; \mathscr{O}^{\mathrm{tp}})$ is not zero if and only if $\exp(1/z) \in \Gamma(V; \mathscr{O}^{\mathrm{tp}}_X)$, that is, if and only if $\exp(1/z)|_V$ is tempered.

Let us set $z = x + iy$.

Lemma 5.3. *Let W be an open subanalytic subset of $\mathbb{P}^1(\mathbb{C})$ with $\infty \notin W$. Assume that there exist positive constants C and N such that*

$$\exp(x) \leq C(1 + x^2 + y^2)^N \text{ on } W. \tag{5.3}$$

Then there exists a constant B such that $x \leq B$ on W.

Proof. We shall compactify $\mathbb{C} \subset \mathbb{P}^1(\mathbb{C})$ by $(\mathbb{R} \sqcup \{\infty\})^2$. If x is not bounded on W, then, applying the Curve Selection Lemma, there exists a real analytic curve $\gamma\colon [0, \varepsilon[\to (\mathbb{R} \sqcup \{\infty\})^2$ such that $\Re\gamma(0) = \infty$ and $\gamma(t) \in W$ for $t > 0$. Writing $\gamma(t) = (x(t), y(t))$, one has

$$y(t) = cx(t)^q + \mathrm{O}(x(t)^{q-\varepsilon}).$$

for some $q \in \mathbb{Q}$, $c \in \mathbb{R}$ and $\varepsilon > 0$. Then (5.3) implies that $\exp(x)$ has a polynomial growth when $x \to \infty$, which is a contradiction. Q.E.D.

Let $\bar{B}_\varepsilon$ denote the closed ball with center $(\varepsilon, 0)$ and radius ε and set $U_\varepsilon = X \setminus \bar{B}_\varepsilon$.

Proposition 5.4. *One has the isomorphism*

$$\mathscr{S}^{\mathrm{tp}} \otimes \mathbb{C}_U \simeq \text{“}\varinjlim_{\varepsilon>0}\text{”}\, \mathbb{C}_{U_\varepsilon}. \tag{5.4}$$

Proof. It follows from Lemma 5.3 that $\exp(1/z)$ is temperate (in a neighborhood of 0) on an open subanalytic subset $V \subset X \setminus \{0\}$ if and only if $\mathrm{Re}(1/z)$ is bounded on V, that is, if and only if $V \subset U_\varepsilon$ for some $\varepsilon > 0$.

Let V be a connected relatively compact subanalytic open subset of $X \setminus \{0\}$. Then a morphism $\mathbb{C}_V \to \mathbb{C}_{X\setminus\{0\}} \cdot \exp(1/z)$ factorizes through a morphism $\mathbb{C}_V \to \mathscr{S}^{\mathrm{tp}}$ if and only if it factorizes through $\mathbb{C}_{U_\varepsilon}$. Hence we get the isomorphism (5.4). Q.E.D.

One has $\mathscr{S}^{\mathrm{tp}} \otimes \mathbb{C}_U \simeq \mathcal{S}ol^{\,\mathrm{tp}}(\exp^{1/z}_{U|X} \otimes \mathbb{C}_U)$ and thus it follows from Lemma 5.2 that

$$\mathcal{S}ol^{\,tp}(\exp^{1/z}_{U|X}) \simeq \mathrm{R}\Gamma(U; \text{“}\varinjlim_{\varepsilon>0}\text{”}\, \mathbb{C}_{U_\varepsilon}),$$

which completes the proof of Theorem 5.1 in the particular case of $\varphi = 1/z$. Note that

$$H^1\mathcal{S}ol^{\,tp}(\exp^{1/z}_{U|X}) \simeq \mathbb{C}_{\{0\}} \simeq H^1\mathcal{S}ol(\exp^{1/z}_{U|X}).$$

5.2 General case

Following [DK14, Prop. 6.2.2], one generalizes the preceding example to exponential $\mathscr{D}$-modules.

Sketch of the end of the proof of Theorem 5.1. The proof decomposes into several steps.
(i) Let z be the coordinate on $X = \mathbb{P}(\mathbb{C})$ and let $U = \mathbb{C}$. Then $\mathcal{S}ol^{\,\mathrm{tp}}_X(\mathscr{E}^z_{U|X}) \simeq E^z_{U|X}$. This is equivalent to the case of $\varphi = 1/z$ treated above.
(ii) Let (u, v) be the coordinates on $Y = \mathbb{C}^2$ and let $U = \{v \neq 0\}$. Then $\mathcal{S}ol^{\,tp}_Y(\mathscr{E}^{u/v}_{v\neq 0|Y}) \simeq E^{u/v}_{v\neq 0|Y}$.
(iii) One concludes by writing φ as a quotient a/b with $a, b \in \mathscr{O}_X$ and $Y = b^{-1}(0)$. Consider the map $f = (a, b)\colon X \to Y = \mathbb{C}^2$ and the map $g\colon \{v \neq 0\} \subset \mathbb{C}^2 \to \mathbb{C}$, $g(u, v) = u/v$. Then $\varphi(x) = g \circ f$ and $U = f^{-1}(v \neq 0)$. It remains to check that

$$\begin{aligned} \mathscr{E}^{\varphi}_{U|X} &\simeq f^D \mathscr{E}^{u/v}_{v\neq 0|Y}, \\ \mathcal{S}ol^{\,tp}_X(\mathscr{E}^{\varphi}_{U|X}) &\simeq f^{-1}\mathcal{S}ol^{\,tp}_Y(\mathscr{E}^{u/v}_{v\neq 0|Y}), \\ E^{\varphi}_{U|X} &\simeq f^{-1} E^{u/v}_{v\neq 0|Y}. \end{aligned}$$

Here f^D is the inverse image for $\mathscr{D}$-modules. Q.E.D.

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