Sheaves for spacetime
Alan Weinstein 80th Birthday Conference

Pierre Schapira

June 20, 2023

I was 20 years old. I won't let anyone say that it's the most beautiful age in life.
Paul Nizan, Aden Arabie, 1931.

Introduction

I will discuss two applications of microlocal sheaf theory to spacetime, one by solving the global Cauchy problem on globally hyperbolic manifolds, another one by interpreting the "Big Bang" as an Hamiltonian isotopy that one can quantify, obtaining a "shifted spacetime". This talk is based on various papers in collaboration with Benoît Jubin, Stéphane Guillermou and Masaki Kashiwara.

History: Mikio Sato 1970 introduced microlocal analysis giving rise to [SKK73], soon followed by Lärs Hörmander and many others in the period 70–80 with a totally different approach (analytical methods vs Fourier analysis). Then Masaki Kashiwara and myself [KS82] extend Sato's idea to sheaves giving rise to [KS90].

The natural framework for microlocal sheaf theory is the cotangent bundle T^*M to a real manifold M, whence the use of symplectic geometry. Conversely, 25 years later, David Nadler and Eric Zaslow [NZ09] made a link with the Fukaya category and Dmitry Tamarkin [Tam12] initiated the use of microlocal sheaf theory to prove results of symplectic topology, a strategy now systematically used by many authors (in particular, Stéphane Guillermou [Gui23], Vivek Shende, etc.).

Linear PDE (systems, that is, $\mathscr{D}$-modules) theory becomes sheaf theory.

Plan of my talk:

Microlocal sheaf theory (with Kashiwara)
Holomorphic solutions of $\mathscr{D}$-modules (with Kashiwara)
Applications to causal manifolds (with Benoît Jubin [JS16])
Sheaves for the Big Bang (extracted from [GKS12], with and Kashiwara)

1 Microlocal sheaf theory

Idea of microlocal in Analysis: Mikio Sato, around 1970, developed in [SKK73]. Microlocal sheaves: initiated in KS82, developed in [KS90].

Let X be a real manifold.

- $\tau\colon TX \to X$ the tangent bundle, $\pi\colon T^*X \to X$ the cotangent bundle.
- for M a submanifold of X, $T_M X$ the normal bundle, $T^*_M X$ the conormal bundle.
- X identified with $T^*_X X$, the zero-section of T^*X.

Let $\mathbf{k}$ be a field (more generally, a commutative unital ring of finite global dimension). $\mathrm{D^b}(\mathbf{k}_X)$: he bounded derived category of sheaves of $\mathbf{k}$-modules. Let $F \in \mathrm{D^b}(\mathbf{k}_X)$.

Definition 1.1. The microsupport $\mathrm{SS}(F)$ is the closed $\mathbb{R}^+$-conic subset of T^*X defined as follows: for an open subset $W \subset T^*X$ one has $W \cap \mathrm{SS}(F) = \emptyset$ if and only if for any $x_0 \in X$ and any real $\mathscr{C}^1$-function φ on X defined in a neighborhood of x_0 with $(x_0; d\varphi(x_0)) \in W$, one has $(\mathrm{R}\Gamma_{\{x;\varphi(x)\geq\varphi(x_0)\}}F)_{x_0} \simeq 0$.

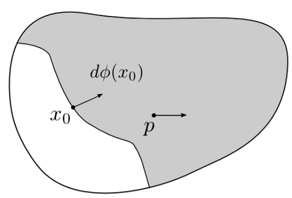


Figure 1: Propagation

In other words, $p \notin \mathrm{SS}(F)$ if the sheaf F has no cohomology supported by "half-spaces" whose conormals are contained in a neighborhood of p.
Set $U = \{x \in X; \varphi(x) < \varphi(x_0)\}$. then

$$(\mathrm{R}\Gamma_{\{x;\varphi(x)\geq\varphi(x_0)\}}F)_{x_0} \simeq 0 \Leftrightarrow \varinjlim_{V \ni x_0} H^j(U \cup V; F) \xrightarrow{\sim} H^j(U; F) \text{ for all } j \in \mathbb{Z}.$$

- By its construction, the microsupport is $\mathbb{R}^+$-conic, that is, invariant by the action of $\mathbb{R}^+$ on T^*X.

- $\mathrm{SS}(F) \cap T^*_X X = \pi(\mathrm{SS}(F)) = \mathrm{supp}(F)$.

- The microsupport satisfies the triangular inequality: if $F_1 \to F_2 \to F_3 \xrightarrow{+1}$ is a distinguished triangle in $\mathrm{D^b}(\mathbf{k}_X)$, then $\mathrm{SS}(F_i) \subset \mathrm{SS}(F_j) \cup \mathrm{SS}(F_k)$ for all $i, j, k \in \{1, 2, 3\}$ with $j \neq k$.

- The microsupport $\mathrm{SS}(F)$ is **co-isotropic**, (one also says involutive) ([KS90, Def. 6.5.1]). Using Whitney's normal cone, for any $p \in T^*X$:

$$C_p(\mathrm{SS}(F), \mathrm{SS}(F))^{\perp} \subset C_p(\mathrm{SS}(F)).$$

In the sequel, for a locally closed subset $A \subset X$, we denote by $\mathbf{k}_A$ the sheaf on X which is the constant sheaf with stalk $\mathbf{k}$ on A and is zero on $X \setminus A$.

Example 1.2. (i) If F is a non-zero local system on X and X is connected, then $\mathrm{SS}(F) = T^*_X X$.
(ii) If M is a closed submanifold of X and $F = \mathbf{k}_M$, then $\mathrm{SS}(F) = T^*_M X$, the conormal bundle to M in X.
(iii) When $X = \mathbb{R}$, $F = \mathbf{k}_I$, I an interval:

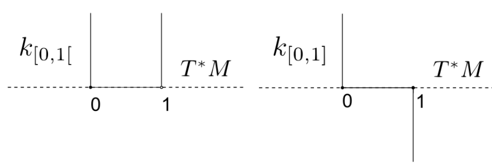


Figure 2: Examples

Note that one can now define the "conormal bundle" to any locally closed subset A by setting $T^*_A X := \mathrm{SS}(\mathbf{k}_A)$.

Let $f\colon X \to Y$ a morphism of manifolds. We get the maps

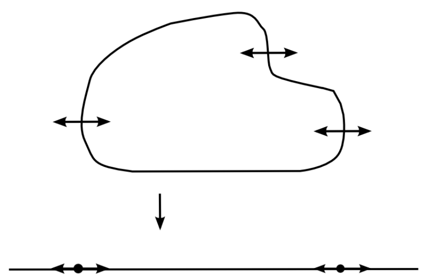
$$\begin{array}{ccccc} TX & \xrightarrow{f'} & X \times_Y TY & \xrightarrow{f_\tau} & TY \\ & {\scriptstyle\tau_X}\searrow & \downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle\tau_Y} \\ & & X & \xrightarrow{f} & Y \end{array} \qquad \begin{array}{ccccc} T^*X & \xleftarrow{f_d} & X \times_Y T^*Y & \xrightarrow{f_\pi} & T^*Y \\ & {\scriptstyle\pi_X}\searrow & \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi_Y} \\ & & X & \xrightarrow{f} & Y. \end{array} \tag{1.1}$$

Theorem 1.3. *Let $f\colon X \to Y$ be a morphism of manifolds, let $F \in \mathrm{D^b}(\mathbf{k}_X)$ and $G \in \mathrm{D^b}(\mathbf{k}_Y)$.*

(a) *Assume that f is proper on* $\mathrm{supp}(F)$*. Then $\mathrm{SS}(\mathrm{R}f_!F) \subset f_\pi f_d^{-1}\,\mathrm{SS}(F)$.*

(b) *Assume that f is non characteristic for G, that is, f_d is proper on $f_\pi^{-1}\,\mathrm{SS}(G)$. Then $\mathrm{SS}(f^{-1}G) \subset f_d(f_\pi^{-1}\,\mathrm{SS}(G))$. Moreover $f^{-1}G \otimes \omega_{X/Y} \xrightarrow{\sim} f^!F$.*

Note that these inclusions may be strict (see the picture below for the direct image).



2 D-modules

Let $(X, \mathscr{O}_X)$ be a complex manifold, $\mathscr{D}_X$ the sheaf of rings of differential operators (see [Kas03]). An object of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$ is locally isomorphgic to a bounded complex where $\cdot P_0$ operates on the right.

$$\mathscr{M} \simeq 0 \to \mathscr{D}_X^{N_r} \to \cdots \to \mathscr{D}_X^{N_1} \xrightarrow{\cdot P_0} \mathscr{D}_X^{N_0} \to 0.$$

Then $\mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X)$ is given by

$$\mathrm{Sol}(\mathscr{M}) \simeq 0 \to \mathscr{O}_X^{N_0} \xrightarrow{P_0 \cdot} \mathscr{O}_X^{N_1} \to \cdots \mathscr{O}_X^{N_r} \to 0, \tag{2.1}$$

where now $P_0 \cdot$ operates on the left.

One can define the characteristic variety of $\mathscr{M}$, denoted $\mathrm{char}(\mathscr{M})$, a closed complex analytic subset of T^*X, conic with respect to the action of $\mathbb{C}^\times$ on T^*X. For example, if $\mathscr{M} = \mathscr{D}_X/\mathscr{D}_X \cdot P$, then

$$\mathrm{char}(\mathscr{M}) = \{(z;\zeta) \in T^*X; \sigma(P)(z;\zeta) = 0\}.$$

where $\sigma(P)$ denotes the principal symbol of P.

The set $\mathrm{char}(\mathscr{M})$ is **co-isotropic** thanks to [SKK73]. Later, Gabber [Gab81] gave a purely algebraic proof of this fundamental result

Let Y be a complex submanifold of the complex manifold X and let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module. One defines $\mathscr{M}_Y \in \mathrm{D}^{\mathrm{b}}(\mathscr{D}_Y)$ which is neither concentrated in degree zero nor coherent in general. One says that Y is non-characteristic for $\mathscr{M}$ if $\mathrm{char}(\mathscr{M}) \cap T^*_Y X \subset T^*_X X$.

Theorem 2.1. (a) *Kashiwara* [Kas70]*. Assume Y is non-characteristic for $\mathscr{M}$. Then $\mathscr{M}_Y$ is a coherent $\mathscr{D}_Y$-module and*

$$\mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X)|_Y \xrightarrow{\sim} \mathrm{R}\mathscr{H}om_{\mathscr{D}_Y}(\mathscr{M}_Y, \mathscr{O}_Y). \tag{2.2}$$

(b) *Kashiwara-Schapira* [KS82]*. One has*

$$\mathrm{SS}(\mathrm{Sol}(\mathscr{M})) = \mathrm{char}(\mathscr{M}). \tag{2.3}$$

The proof only uses the classical Cauchy-Kowalevsky theorem (in its precised form, see [Hör83, Th. 9.4.5–9.4.7]).

3 Applications to LPDE

Elliptic systems

We denote by $\mathrm{D}^{\mathrm{b}}_{\mathrm{cc}}(\mathbf{k}_X)$ the full triangulated subcategory of $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$ consisting of cohomologically constructible sheaves.

Theorem 3.1 (Petrowsky theorem for sheaves)**.** *Let* $G \in \mathrm{D}^{\mathrm{b}}_{\mathrm{cc}}(\mathbf{k}_X)$, $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$ *and assume* $\mathrm{SS}(G) \cap \mathrm{SS}(F) \subset T^*_X X$*. Then*

$$\mathrm{R}\mathscr{H}om\,(G, \mathbf{k}_X) \otimes F \xrightarrow{\sim} \mathrm{R}\mathscr{H}om\,(G, F).$$

Now consider a real analytic manifold M and a complexification X. We choose $\mathbf{k} = \mathbb{C}$. The sheaf of real analytic function $\mathscr{A}_M$ is defined by

$$\mathscr{A}_M = \mathscr{O}_X \otimes \mathbb{C}_M \simeq \mathrm{R}\mathscr{H}om\,(\mathrm{D}'_X \mathbb{C}_M, \mathbb{C}_X) \otimes \mathscr{O}_X.$$

The sheaf of Sato's hyperfunctions is thus naturally defined by

$$\mathscr{B}_M = \mathrm{R}\mathscr{H}om\,(\mathrm{D}'_X \mathbb{C}_M, \mathscr{O}_X) \simeq \mathrm{R}\Gamma_M(\mathscr{O}_X) \otimes \omega_{X/M},$$

where ω_X is the dualizing complex, $\omega_{X/M} = \omega_X \otimes \omega_M^{\otimes -1} \simeq \mathrm{or}_M\,[\dim M]$. Recall that the complex $\mathscr{B}_M$ is proved to be concentrated in degree 0.

Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module, $F = \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X)$, $G = \mathrm{D}'\mathbb{C}_M$. Then $\mathrm{SS}(F) = \mathrm{char}(\mathscr{M})$, $\mathrm{SS}(G) = T^*_M X$.

Classically, one says that $\mathscr{M}$ is elliptic if $\mathrm{char}(\mathscr{M}) \cap T^*_M X \subset T^*_X X$. The Petrowsky theorem thus gives the isomorphism

$$\mathrm{R}\mathscr{H}om\,(\mathscr{M}, \mathscr{A}_M) \xrightarrow{\sim} \mathrm{R}\mathscr{H}om\,(\mathscr{M}, \mathscr{B}_M).$$

Hyperbolic systems

Let M be a closed submanifold of the real manifold X

$$T^*_MX \to M \text{ defines } T^*M \times_M T^*_MX \hookrightarrow T^*T^*_MX \text{ hence } T^*M \hookrightarrow T^*T^*_MX.$$

In local coordinates $(x, y; \xi, \eta) \in T^*X$, $M = \{y = 0\}$, $T^*M \hookrightarrow T^*T^*_MX$ is given by

$$(x; \xi) \mapsto (x, 0; \xi, 0),$$

Let $A \subset T^*X$, conic. The Whitney normal cone $C_{T^*_MX}(A) \subset T_{T^*_MX}T^*X \simeq T^*T^*_MX$

$$(x_0; \xi_0) \in T^*M \cap C_{T^*_MX}(A) \Leftrightarrow \quad \begin{array}{l}\text{there exists } (x_n, y_n; \xi_n, \eta_n)_n \subset A,\\ (x_n; \xi_n) \xrightarrow{n} (x_0; \xi_0), |y_n| \xrightarrow{n} 0, |y_n||\eta_n| \xrightarrow{n} 0.\end{array}$$

Notation 3.2. For $F \in \mathrm{D^b}(\mathbf{k}_X)$, $T^*M \cap C_{T^*_MX}(\mathrm{SS}(F)) \subset T^*M$ is intrinsically well defined. (or $\mathscr{M} \in \mathrm{D^b_{coh}}(\mathscr{D}_X)$ in case X is complex):

$$\mathrm{SS}_M(F) = T^*M \cap C_{T^*_MX}(\mathrm{SS}(F)), \quad \mathrm{char}_M(\mathscr{M}) = T^*M \cap C_{T^*_MX}(\mathrm{char}(\mathscr{M})).$$

Theorem 3.3 ([KS90, Cor. 6.4.4])**.** *One has*

$$\begin{aligned}&\mathrm{SS}(\mathrm{R}\Gamma_M F) \subset \mathrm{SS}_M(F),\\ &\mathrm{SS}(F|_M) \subset \mathrm{SS}_M(F).\end{aligned}$$

Corollary 3.4. *Let M be a real analytic manifold, X a complexification, Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module. Then*

$$\begin{aligned}&\mathrm{SS}(\mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{B}_M)) \subset \mathrm{char}_M(\mathscr{M}),\\ &\mathrm{SS}(\mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{A}_M)) \subset \mathrm{char}_M(\mathscr{M}).\end{aligned}$$

Let $N \hookrightarrow M$ is a real analytic smooth closed submanifold of M of and $Y \hookrightarrow X$ is a complexification of N in X. We assume that Y is non characteristic for $\mathscr{M}$ in a neighborhood of N and

$$T^*_NM \cap \mathrm{char}_M(\mathscr{M}) \subset T^*_MM. \tag{3.1}$$

Then the isomorphism (2.2) *induces the isomorphism*

$$\mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{B}_M)|_N \xrightarrow{\sim} \mathrm{R}\mathscr{H}om_{\mathscr{D}_Y}(\mathscr{M}_Y, \mathscr{B}_N). \tag{3.2}$$

4 Causal manifolds

Definition 4.1. A *causal manifold* (M,γ) is a *connected* manifold M equipped with an open proper convex cone $\gamma \subset TM$ such that $\gamma_x \neq \varnothing$ for all $x \in M$.

When $M \subset \mathbb{V}$ is open, an open set U of M is γ-open if $U = U + \gamma_0$ for any closed convex proper cone γ_0 such that $M \times \gamma_0 \subset \gamma$. Similar definition on a manifold, by using local chart.

Definition 4.2. Consider a preorder $\preceq$ and its graph $\Delta_{\preceq} \subset M \times M$.

(a) The preorder is closed if $\Delta_{\preceq}$ is closed in $M \times M$.

(b) The preorder is proper if q_{13} is proper on $\Delta_{\preceq} \times_M \Delta_{\preceq}$, that is, **diamonds are compact**, for any $x, y \in M$, $J^+_{\preceq}(x) \cap J^-_{\preceq}(y)$ is compact.

(c) The preorder $\preceq$ is causal if for any $y \in M$, the set $\{x; x \preceq y\}$ contains the closure of y for the γ-topology.

Example 4.3. A causal path (resp. strictly causal path) is a piecewise smooth curve $c\colon [0,1] \to M$, with $c(0) = x$, $c(1) = y$ and $c'_l(t), c'_r(t)$ in $\overline{\gamma}$ (resp. in γ). We get a causal preorder: $x \leq_{ps} y$ if there exists a causal path from x to y.

Definition 4.4. A causal manifold (M,γ) is globally hyperbolic if the causal preorder $\leq_{ps}$ is proper and there are no causal loops.

Definition 4.5 (Jubin-S [JS16])**.** (a) Let $(M,\gamma,\preceq)$ be a causal manifold endowed with a closed causal preorder $\preceq$. A Cauchy time function $q\colon (M,\gamma) \to (\mathbb{R},+)$ is a submersive morphism of causal manifolds such that for any compact set K, the map q is proper both on $J^+_{\preceq}(K)$ and on $J^-_{\preceq}(K)$.

(b) A G-causal (G for Geroch) manifold $(M,\gamma,\preceq,q)$ is a causal manifold endowed with a proper causal preorder $\preceq$ and a Cauchy time function q.

Note that q is increasing from $(M,\preceq)$ to $(\mathbb{R},\leq)$. It is strictly increasing on strictly causal paths.

Theorem 4.6. *If (M,γ) is globally hyperbolic then there exists a Cauchy time function q and $(M,\gamma,\leq_{ps},q)$ is G-causal.*

See Geroch [Ger70], Bernal-Sanchez [BS05], the survey paper Minguzi-Sanchez [MS08]. See also Fathi-Siconolli [FS11] for a more general version.

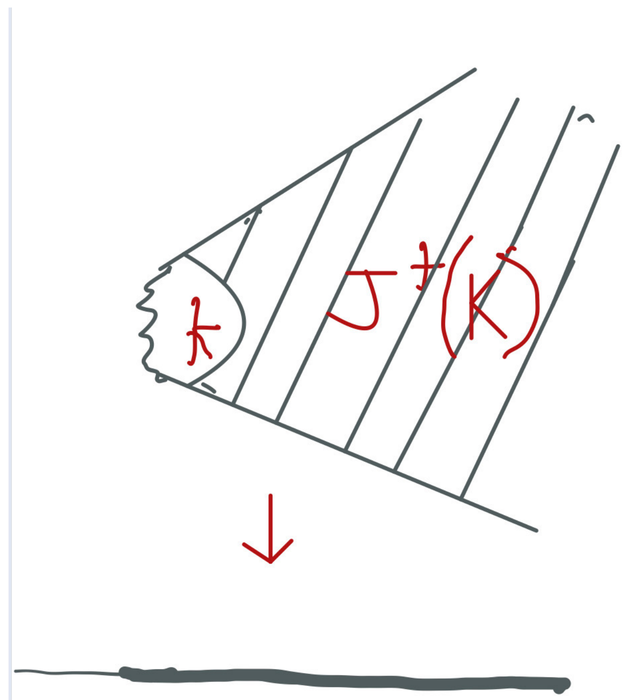


Figure 3: Time function

The link with the micro-support of sheaves is given by:

Lemma 4.7. *Let $(M, \gamma, \preceq)$ be a causal manifold endowed with a closed causal preorder. Set $\lambda = \gamma^{\circ}$. For K compact, set $Z = J^{-}_{\preceq}(K)$. Then $\mathrm{SS}(\mathbf{k}_Z) \subset \lambda^a$.*

Theorem 4.8 (Jubin-S)**.** *Let $(M, \gamma, \preceq, q)$ be a G-causal manifold, $\lambda = \gamma^{\circ}$ and let $F \in \mathrm{D^b}(\mathbf{k}_M)$. Assume that $\mathrm{SS}(F) \cap \lambda \subset T^*_M M$. Then*

$$\mathrm{SS}(\mathrm{R}q_* F) \cap \{\tau \leq 0\} \subset T^*_{\mathbb{R}}\mathbb{R}.$$

Proof. Set $Z = J^{-}_{\preceq}(K)$. Then $\mathrm{SS}(\mathbf{k}_Z) \subset \lambda^a$ and thus $\mathrm{SS}(F_Z) \cap \lambda \subset T^*_M M$. Since q is proper on Z, $\mathrm{SS}(\mathrm{R}q_* F_Z) \cap \{\tau \leq 0\} \subset T^*_{\mathbb{R}}\mathbb{R}$. Then recover (locally on $\mathbb{R}$) the space M with an increasing family of such Z. Q.E.D.

Corollary 4.9. *Let $(M, \gamma, \preceq, q)$ be a G-causal manifold. Let $F \in \mathrm{D^b}(\mathbf{k}_M)$ satisfying $\mathrm{SS}(F) \cap (\lambda \cup \lambda^a) \subset T^*_M M$. Assume $0 \in q(M)$ and set $M_0 = q^{-1}(0)$. Then the natural restriction morphism below is an isomorphism:*

$$\mathrm{R}\Gamma(M; F) \xrightarrow{\sim} \mathrm{R}\Gamma(M_0; F|_{M_0}). \tag{4.1}$$

Now consider a G-causal manifold $(M, \lambda, \preceq, q)$ and we assume moreover that M and q are real analytic. Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module. Applying the preceding result with $F = \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{B}_M)$, we get:

Theorem 4.10 ([JS16])**.** *Let $(M, \gamma, \preceq, q)$ be an analytic G-causal manifold. Let $N = q^{-1}(0)$ and let Y be a complexification of N in X. Assume $\mathrm{char}_M(\mathscr{M}) \cap \lambda \subset T^*_M M$.*

(a) *One has the natural isomorphism*

$$\mathrm{RHom}_{\mathscr{D}_X}(\mathscr{M}, \mathscr{B}_M) \xrightarrow{\sim} \mathrm{RHom}_{\mathscr{D}_Y}(\mathscr{M}_Y, \mathscr{B}_N).$$

(b) *Let K be a compact subset of M and let A be either $J^+_{\preceq}(K)$ or $J^-_{\preceq}(K)$. Then* $\mathrm{RHom}_{\mathscr{D}_X}(\mathscr{M}, \Gamma_A \mathscr{B}_M) \simeq 0$.

In other words, the Cauchy problem for hyperfunctions with data on N is *globally* well-posed.

The theorem applies when $\mathscr{M} = \mathscr{D}_X / \mathscr{D}_X \cdot P$ for P a wave operator associated with a quadratic form of signature $(+, -, \cdots, -)$

5 Before the Big Bang

Let us represent the universe as a closed ball in $\mathbb{R}^n$ whose radius grows linearly with the time t. We represent spacetime as a closed cone in $\mathbb{R}^{n+1}$ with vertex at $t = 0$, similarly as a light cone in a Minkowski space.

What happens for $t < 0$? If one replaces the spacetime with the constant sheaf supported by it, the sheaf $\mathbf{k}_{\{|x| \le t\}}$ defined on $t \ge 0$, we need to extend it naturally for $t < 0$. The micro-support of this sheaf at the boundary is the interior conormal. If we extend it naturally for $t < 0$ we get the exterior conormal which is the micro-support of the constant sheaf on the open cone.

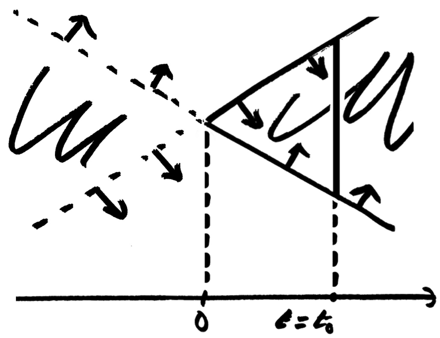


Figure 4: Before the Big Bang

With Guillermou and Kashiwara [GKS12], we have constructed a "distinguished triangle" as follows. Set $X = \mathbb{R}^n_x \times \mathbb{R}_t$. The morphism $\mathbf{k}_{\{|x| \le -t\}} \to \mathbf{k}_{\{0\}}$ and the isomorphisms

$$\mathrm{D}'_X \mathbf{k}_{\{0\}} \simeq \mathbf{k}_{\{0\}}[-n-1], \quad \mathrm{D}'_X \mathbf{k}_{\{|x| \le -t\}} \simeq \mathbf{k}_{\{|x| < -t\}}$$

induce the morphism

$$\mathbf{k}_{\{0\}}[-n-1] \to \mathbf{k}_{\{|x| < -t\}}.$$

Composing with $\mathbf{k}_{\{|x| \le t\}} \to \mathbf{k}_{\{0\}}$, we get the morphism $\mathbf{k}_{\{|x| \le t\}} \xrightarrow{\psi} \mathbf{k}_{\{|x| < -t\}}[n+1]$ hence a distinguished triangle

$$\mathbf{k}_{\{|x| < -t\}}[n] \to K \to \mathbf{k}_{\{|x| \le t\}} \xrightarrow[\psi]{+1}$$

The micro-support of K outside the zero-section is the smooth Lagrangian manifold, the image of $T^*_{\{0\}}\mathbb{R}^n$ by the Hamiltonian isotopy

$$(x; \xi) \mapsto (x - t\xi/|\xi|; \xi).$$

One can modify the Lorentzian case encountered above and replace $\mathbb{R}^n_x$ with a Riemannian manifold (with convexity radius and injectivity radius > 0) using the Hamiltonian isotopy associated with $||\xi||_x$.

In particular, one can consider the n-dimensional unit sphere $M = \mathbb{S}^n$ $(n \geq 2)$ endowed with the canonical Riemannian metric. In this case, the sheaf obtained has a shift which jumps by the dimension minus one when $t \in \pi\mathbb{Z}$.

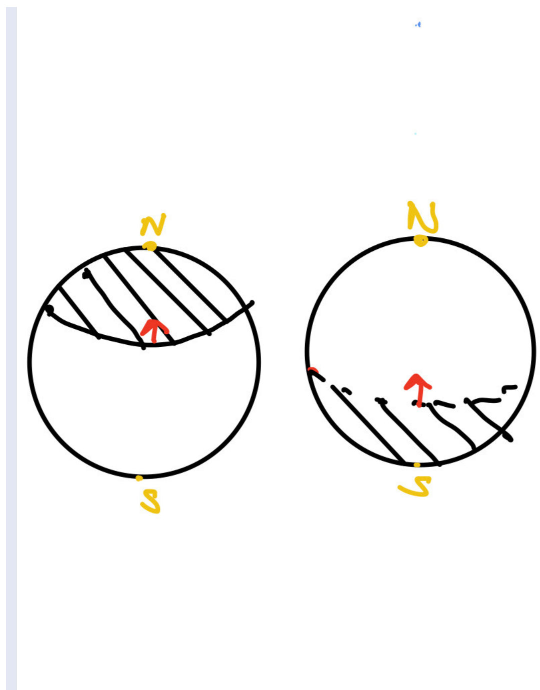


Figure 5: Periodic Big Bang

One also constructs a (non-zero) morphism $\mathbf{k}_{\{|x|=t\}}[-n] \to \mathbf{k}_{\{|x|=-t\}}$, hence a distinguished triangle

$$\mathbf{k}_{\{|x|=-t\}}[n-1] \to K' \to \mathbf{k}_{\{|x|=t\}} \xrightarrow[\psi]{+1}$$

In this case, the micro-support of K' is no more smooth, hence K is not associated with an Hamiltonian isotopy.

It would be natural to extend these constructions to the case of a family of compact Riemannian manifolds depending on $t > 0$ and whose diameter goes to 0 for $t \to 0$.

References

[BGP07] Christian Bär, Nicolas Ginoux, and Frank Pfäffle, *Wave equations on Lorentzian manifolds and quantization*, ESI Lectures in Mathematics and Physics, European Mathematical Society (EMS), Zürich, 2007.

[BS05] Antonio N. Bernal and Miguel Sánchez, *Smoothness of time functions and the metric splitting of globally hyperbolic spacetimes*, Comm. Math. Phys. **257** (2005), no. 1, 43–50.

[FS11] Albert Fathi and Antonio Siconolfi, *On smooth time functions*, Mathematical Proceedings, CUP (2011), available at `hal-00660452`.

[Gab81] Ofer Gabber, *The integrability of the characteristic variety*, Amer. Journ. Math. **103** (1981), 445–468.

[Ger70] Robert Geroch, *Domain of dependence*, J. Mathematical Phys. **11** (1970), 437–449.

[Gui23] Stéphane Guillermou, *Sheaves and symplectic geometry of cotangent bundles*, Astérisque **440** (2023), available at `arXiv:1905.07341`.

[GKS12] Stéphane Guillermou, Masaki Kashiwara, and Pierre Schapira, *Sheaf quantization of Hamiltonian isotopies and applications to nondisplaceability problems*, Duke Math Journal **161** (2012).

[Hör83] Lars Hörmander, *The analysis of linear partial differential operators I, II, III, IV*, Grundlehren der Math. Wiss, vol. 256, 257, 274, 275, Springer-Verlag, 1983, 1985.

[JS16] Benoît Jubin and Pierre Schapira, *Sheaves and D-modules on causal manifolds,* Letters in Mathematical Physics **16** (2016), 607-648.

[Kas70] Masaki Kashiwara, *Algebraic study of systems of partial differential equations, Master's thesis, Tokyo university 1970*, Mémoires SMF, vol. 63, Soc. Math. France, 1995. xiii+72 pp.

[Kas03] ———, *D-modules and microlocal calculus*, Translations of Mathematical Monographs, vol. 217, 2003.

[KS82] Masaki Kashiwara and Pierre Schapira, *Microsupport des faisceaux: applications aux modules différentiels*, C. R. Acad. Sci. Paris **295, 8** (1982), 487–490.

[KS90] ———, *Sheaves on manifolds*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 292, Springer-Verlag, Berlin, 1990.

[MS08] Ettore Minguzzi and Miguel Sánchez, *The causal hierarchy of spacetimes*, Recent developments in pseudo-Riemannian geometry, ESI Lect. Math. Phys., Eur. Math. Soc., Zürich, 2008, pp. 299–358.

[NZ09] David Nadler and Eric Zaslow, *Constructible sheaves and the Fukaya category*, J. Amer. Math. Soc. **22** (2009), 233–286.

[SKK73] Mikio Sato, Takahiro Kawai, and Masaki Kashiwara, *Microfunctions and pseudo-differential equations*, Hyperfunctions and pseudo-differential equations (Proc. Conf., Katata, 1971; dedicated to the memory of André Martineau), Springer, Berlin, 1973, pp. 265–529. Lecture Notes in Math., Vol. 287.

[Tam12] Dmitry Tamarkin, *Algebraic and analytic microlocal analysis*, Microlocal conditions for non-displaceability, Vol. 269, Springer, proceedings in Mathematics & Statistics, 2012&2013, pp. 99–223, available at `arXiv:0809.1584`.