# Some applications of the microlocal theory of sheaves to symplectic topology

Pierre Schapira

Université Pierre et Marie Curie
Paris, France

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We shall give (or recall) some examples of applications of the microlocal theory of sheaves to some problems of symplectic topology:

- Arnold non-displaceability conjecture/theorem
  classic: Chaperon, Conley–Zehnder, Hofer, Laudenbach–Sikorav,
  using sheaves: Tamarkin (2008), Guillermou-Kashiwara-Schapira (2012)
- non-displaceability for non-negative Hamiltonian isotopies
  (a notion introduced by Eliashberg-Kim-Polterovich)
  classic: Chernov-Nemirovski, Colin-Ferrand-Pushkar,
  using sheaves: Guillermou-Kashiwara-Schapira (2012)
- the Eliashberg theorem on the $C^0$-limit of symplectomorphisms
  classic: Eliashberg (1987),
  using sheaves: Guillermou (2013)

# References


- S. Guillermou, *The Gromov-Eliashberg theorem by microlocal sheaf theory,* In preparation.
- S. Guillermou, M. Kashiwara and P. Schapira *Sheaf quantization of Hamiltonian isotopies and applications to non displaceability problems.* Duke Math. Journal, (2012). arXiv:math.arXiv:1005.1517
- S. Guillermou and P. Schapira *Microlocal theory of sheaves and Tamarkin's non displaceability theorem,* arXiv:1106.1576 LNM Springer (2014)
- M. Kashiwara and P. Schapira, *Sheaves on Manifolds,* Grundlehren der Math. Wiss. **292** Springer-Verlag (1990).
- D. Tamarkin, *Microlocal conditions for non-displaceability,* arXiv:0809.1584

# Microsupport

Let $M$ be a real $C^1$-manifold. For a commutative unital ring with finite global dimension $\mathbf{k}$, we denote by $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_M)$ the bounded derived category of sheaves of $\mathbf{k}$-modules on $M$.
For a locally closed subset $Z$ of $M$, we denote by $\mathbf{k}_Z$ the constant sheaf with stalk $\mathbf{k}$ on $Z$, extended by 0 on $M \setminus Z$.

**Definition.** (Microsupport or singular support of a sheaf, K-S 81.)
Let $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_M)$. The singular support $\mathrm{SS}(F)$ is the closed conic subset of $T^*M$ defined as follows. An open subset $W$ of $T^*M$ does not intersect $\mathrm{SS}(F)$ if for any $C^1$-function $\varphi\colon M \to \mathbb{R}$ and any $x_0 \in M$ such that $(x_0; d\varphi(x_0)) \in W$, setting $U = \{x; \varphi(x) < \varphi(x_0)\}$, one has for all $j \in \mathbb{Z}$

$$\varinjlim_{V \ni x_0} H^j(U \cup V; F) \simeq H^j(U; F).$$

Therefore, if $(x_0; d\varphi(x_0)) \notin \mathrm{SS}(F)$, then any cohomology class defined on an open subset $U$ as above extends through the boundary in a neighborhood of $x_0$.

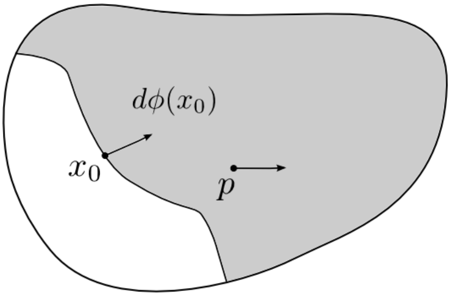


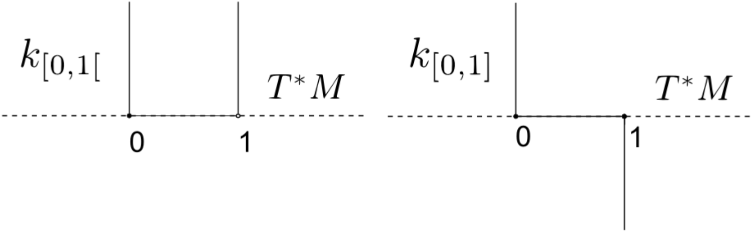

Figure: $p \notin \mathrm{SS}(F)$

- The microsupport is closed and is $\mathbb{R}^+$-conic, that is, invariant by the action of $\mathbb{R}^+$ on $T^*M$.
- $\mathrm{SS}(F) \cap T^*_M M = \pi_M(\mathrm{SS}(F)) = \mathrm{Supp}(F)$, where $\pi_M \colon T^*M \to M$ is the projection.
- The microsupport satisfies the triangular inequality: if $F_1 \to F_2 \to F_3 \xrightarrow{+1}$ is a distinguished triangle in $\mathrm{D^b}(\mathbf{k}_M)$, then $\mathrm{SS}(F_i) \subset \mathrm{SS}(F_j) \cup \mathrm{SS}(F_k)$ for all $i, j, k \in \{1, 2, 3\}$ with $j \neq k$.
- The microsupport is involutive (i.e., co-isotropic). (A precise definition will come later.)

# Examples

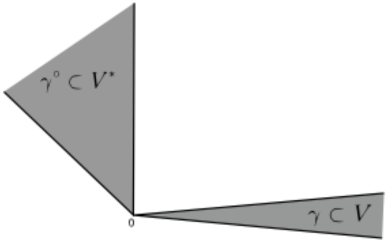
**Examples.**
(i) If $F$ is a non-zero local system on $M$ and $M$ is connected, then $\mathrm{SS}(F) = T^*_M M$, the zero-section.
(ii) If $N$ is a closed submanifold of $M$ and $F = \mathbf{k}_N$, then $\mathrm{SS}(F) = T^*_N M$, the conormal bundle to $N$ in $M$.
(iii) Let $\varphi$ be a $C^1$-function such that $d\varphi(x) \neq 0$ whenever $\varphi(x) = 0$. Let $U = \{x \in M; \varphi(x) > 0\}$ and let $Z = \{x \in M; \varphi(x) \geq 0\}$. Then

$$\mathrm{SS}(\mathbf{k}_U) = U \times_M T^*_M M \cup \{(x; \lambda d\varphi(x)); \varphi(x) = 0, \lambda \leq 0\},$$
$$\mathrm{SS}(\mathbf{k}_Z) = Z \times_M T^*_M M \cup \{(x; \lambda d\varphi(x)); \varphi(x) = 0, \lambda \geq 0\}.$$



(iv) Assume $M = V$ is a vector space and let $\gamma$ be a closed proper convex cone with vertex at 0. Then $\mathrm{SS}(\mathbf{k}_\gamma) \cap \pi_M^{-1}(\{0\}) = \gamma^\circ$ where $\gamma^\circ \subset V^*$ is the polar cone given by

$$\gamma^\circ = \{\theta \in V^*; \langle \theta, v \rangle \geq 0 \text{ for all } v \in \gamma\}.$$




(v) Let $(X, \mathscr{O}_X)$ be a complex manifold and let $\mathscr{M}$ be a coherent module over the ring $\mathscr{D}_X$ of holomorphic differential operators. (Hence, $\mathscr{M}$ represents a system of linear partial differential equations on $X$.) Denote by $F = \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X)$ the complex of holomorphic solutions of $\mathscr{M}$. Then $\mathrm{SS}(F) = \mathrm{char}(\mathscr{M})$, the characteristic variety of $\mathscr{M}$.

# Operations

Let $f\colon M \to N$ be a morphism of real manifolds. To $f$ are associated the diagrams

$$\begin{array}{ccccc} TM & \xrightarrow{f'} & M \times_N TN & \xrightarrow{f_\tau} & TN \\ \downarrow{\scriptstyle \tau_M} & & \downarrow & & \downarrow{\scriptstyle \tau_N} \\ M & = & M & \xrightarrow{f} & N. \end{array} \qquad \begin{array}{ccccc} T^*M & \xleftarrow{f_d} & M \times_N T^*N & \xrightarrow{f_\pi} & T^*N \\ \downarrow{\scriptstyle \pi_M} & & \downarrow & & \downarrow{\scriptstyle \pi_N} \\ M & = & M & \xrightarrow{f} & N. \end{array}$$

Let $\Lambda_M \subset T^*M$ be a closed $\mathbb{R}^+$-conic subset. Then $f_\pi$ is proper on $f_d^{-1}\Lambda_M$ if and only if $f$ is proper on $\Lambda_M \cap T^*_M M$.

Let $\Lambda_N \subset T^*N$ be a closed $\mathbb{R}^+$-conic subset. Then $f_d$ is proper on $f_\pi^{-1}\Lambda_N$ if and only if $f_\pi^{-1}\Lambda_N \cap f_d^{-1} T^*_M M \subset M \times_N T^*_N N$. In this case, one says that $f$ is non-characteristic for $\Lambda_N$.

# Operations

Let $f\colon M \to N$ be a morphism of manifolds, and recall the maps:
$T^*M \xleftarrow{f_d} M \times_N T^*N \xrightarrow{f_\pi} T^*N$.
Let $\Lambda_f \subset T^*(M \times N)$ denotes the conormal to the graph of $f$.

**Theorem.** Let $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_M)$ and let $G \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_N)$.

(i) (The stationary phase lemma.)
Assume that $f$ is proper on $\mathrm{Supp}(F)$. Then
$\mathrm{SS}(\mathrm{R}f_* F) \subset f_\pi f_d^{-1} \mathrm{SS}(F) = \mathrm{SS}(F) \circ \Lambda_f$.

(ii) Assume that $f$ is non-characteristic for $\mathrm{SS}(G)$. Then
$\mathrm{SS}(f^{-1}G) \subset f_d f_\pi^{-1} \mathrm{SS}(G) = \Lambda_f \circ \mathrm{SS}(G)$.

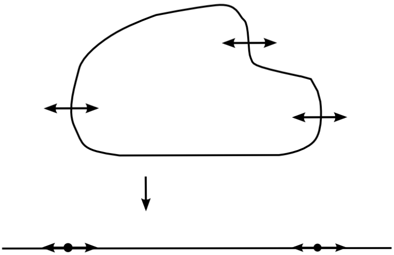


Figure: Direct image

# The Morse lemma

**Theorem.** (The Morse lemma for sheaves.)
Let $F \in \mathrm{D^b}(\mathbf{k}_M)$, let $\psi\colon M \to \mathbb{R}$ be a function of class $C^1$ and assume that $\psi$ is proper on $\mathrm{Supp}(F)$. For $t \in \mathbb{R}$, set $M_t = \psi^{-1}(]-\infty, t[)$. Let $a < b$ in $\mathbb{R}$ and assume that $d\varphi(x) \notin \mathrm{SS}(F)$ for $a \leq \psi(x) < b$. Then the restriction morphism $\mathrm{R}\Gamma(M_b; F) \to \mathrm{R}\Gamma(M_a; F)$ is an isomorphism.

**Proof.** Consider $G = \mathrm{R}\psi_* F \in \mathrm{D^b}(\mathbf{k}_\mathbb{R})$. Then $\mathrm{SS}(G) \cap \{(t; dt); t \in [a, b[\} = \emptyset$ and it follows that $\mathrm{R}\Gamma(]-\infty, b[; G) \to \mathrm{R}\Gamma(]-\infty, a[; G)$ is an isomorphism by the definition of the micro-support.

**Corollary.** Let $F \in \mathrm{D^b}(\mathbf{k}_M)$ and let $\psi\colon M \to \mathbb{R}$ be a function of class $C^1$. Let $\Lambda_\psi = \{(x; d\psi(x))\} \subset T^*M$. Assume that

(i) $\mathrm{Supp}(F)$ is compact,

(ii) $\mathrm{R}\Gamma(M; F) \neq 0$.

Then $\Lambda_\psi \cap \mathrm{SS}(F) \neq \emptyset$.

**Proof.** Otherwise, $\mathrm{R}\Gamma(M; F) \simeq \mathrm{R}\Gamma(\mathbb{R}; \mathrm{R}\psi_* F) \xrightarrow{\sim} \mathrm{R}\Gamma(]t_0, t_1[; \mathrm{R}\psi_* F)$ for all $t_0 < t_1$ and $\mathrm{R}\psi_* F$ has compact support.

# Homogeneous Hamiltonian isotopies

Consider an open interval $I$ with $[0,1] \subset I$. An homogeneous Hamiltonian isotopy $\Phi \colon \dot{T}^*M \times I \to \dot{T}^*M$ is a family of maps $\Phi = \{\varphi_s\}_{s\in I}$ such that

- $\varphi_s$ is a homogeneous symplectic isomorphism for each $s \in I$,
- $\varphi_0 = \mathrm{id}_{\dot{T}^*M}$.

Then setting $f = \langle \alpha_M, \frac{\partial \Phi}{\partial s} \rangle$ we have $\frac{\partial \Phi}{\partial s} = H_{f_s}$ and there exists a conic Lagrangian submanifold $\Lambda$ of $\dot{T}^*M \times \dot{T}^*M \times T^*I$ whose projection is the graph of $\Phi$ in $\dot{T}^*M \times \dot{T}^*M \times I$. Moreover:

- $\Lambda$ is closed in $\dot{T}^*(M \times M \times I)$
- for any $s \in I$, the inclusion $i_s \colon M \times M \to M \times M \times I$ is non-characteristic for $\Lambda$
- the graph of $\varphi_s$ is $\Lambda_s = \Lambda \circ T^*_s I$.

# Quantization of isotopies

For $K \in \mathrm{D}^{\mathrm{lb}}(\mathbf{k}_{M\times M\times I})$ and $s_0 \in I$, we set

$$K_{s_0} := K|_{s=s_0}.$$

**Theorem (GKS12)** We consider $\Phi\colon \dot{T}^*M \times I \to \dot{T}^*M$ as above. Then there exists $K \in \mathrm{D}^{\mathrm{lb}}(\mathbf{k}_{M\times M\times I})$ satisfying

(a) $\mathrm{SS}(K) \subset \Lambda \cup T^*_{M\times M\times I}(M \times M \times I)$,

(b) $K_0 \simeq \mathbf{k}_\Delta$.

Moreover:

(i) both projections $\mathrm{Supp}(K) \rightrightarrows M \times I$ are proper,

(ii) $K_s \circ K_s^{-1} \simeq K_s^{-1} \circ K_s \simeq \mathbf{k}_\Delta$ for all $s \in I$,

(iii) such a $K$ satisfying the conditions (a) and (b) above is unique up to a unique isomorphism,

Here, $K_s^{-1} := v^{-1}\mathrm{R}\mathscr{H}om(K_s, \omega_M \boxtimes \mathbf{k}_M)$.

# Example.

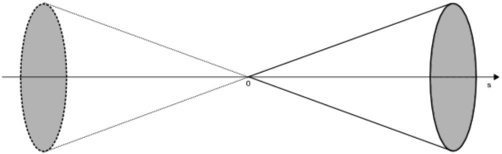
Let $M = \mathbb{R}^n$ and denote by $(x;\xi)$ the homogeneous symplectic coordinates on $T^*\mathbb{R}^n$. Consider the isotopy $\varphi_s(x;\xi) = (x - s\frac{\xi}{|\xi|};\xi)$, $s \in I = \mathbb{R}$. Then

$$\Lambda_s = \{(x,y,\xi,\eta); |x-y| = |s|,\ \xi = -\eta = \lambda(x-y),\ s\lambda < 0\} \quad \text{for } s \neq 0,$$
$$\Lambda_0 = \dot{T}^*_\Delta(M \times M).$$

For $s \in \mathbb{R}$, the morphism $\mathbf{k}_{\{|x-y|\leq s\}} \to \mathbf{k}_{\Delta\times\{s=0\}}$ gives by duality (replacing $s$ with $-s$) $\mathbf{k}_{\Delta\times\{s=0\}} \to \mathbf{k}_{\{|x-y|<-s\}}[n+1]$. We get a morphism $\mathbf{k}_{\{|x-y|\leq s\}} \to \mathbf{k}_{\{|x-y|<-s\}}[n+1]$ and we define $K$ by the distinguished triangle in $\mathrm{D^b}(\mathbf{k}_{M\times M\times I})$:

$$\mathbf{k}_{\{|x-y|<-s\}}[n] \to K \to \mathbf{k}_{\{|x-y|\leq s\}} \xrightarrow{+1}$$

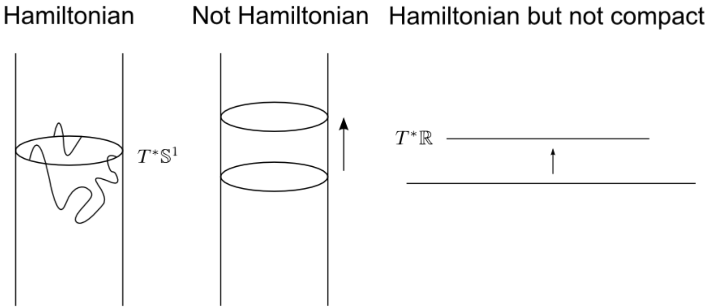
One can show that $K$ is a quantization of the Hamiltonian isotopy $\{\varphi_s\}_s$. We have the isomorphisms in $\mathrm{D^b}(\mathbf{k}_{M\times M})$: $K_s \simeq \mathbf{k}_{\{|x-y|\leq s\}}$ for $s \geq 0$ and $K_s \simeq \mathbf{k}_{\{|x-y|<-s\}}[n]$ for $s < 0$.



# Non displaceability: symplectic case

Consider a compact manifold $N$ and a (no more homogeneous) Hamiltonian isotopy $\Phi = \{\varphi_s\}_{s \in I}$, that is, the $\varphi_s \colon T^*N \to T^*N$ are symplectomorphisms and $\frac{\partial}{\partial_s}\Phi$ is the Hamiltomian vector field of a time dependant function $f$ defined on $T^*N$.

**Theorem** In the above situation, $\varphi_s(T^*_N N) \cap T^*_N N \neq \emptyset$ for all $s \in I$.



# Non displaceability: homogeneous symplectic case

The preceding theorem is deduced from the next one by choosing $M = N \times \mathbb{R}$ and $\psi\colon N \times \mathbb{R} \to \mathbb{R}$ the projection.
Consider a homogeneous Hamiltonian isotopy $\Phi = \{\varphi_s\}_{s\in I}\colon \dot{T}^*M \times I \to \dot{T}^*M$ and a $C^1$-map $\psi\colon M \to \mathbb{R}$ such that the differential $d\psi(x)$ never vanishes. Set

$$\Lambda_\psi := \{(x; d\psi(x));\ x \in M\} \subset \dot{T}^*M.$$

**Theorem.** Let $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_M)$ with compact support and such that $\mathrm{R}\Gamma(M; F) \neq 0$. Then for any $s \in I$, $\varphi_s(\mathrm{SS}(F) \cap \dot{T}^*M) \cap \Lambda_\psi \neq \emptyset$.

**Proof.** Let $K \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{M\times M\times I})$ be the quantization of $\Phi$.
Set:

$$F_s := K_s \circ F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_M) \quad \text{for } s \in I.$$

We have $F_0 = F$ and $\mathrm{R}\Gamma(M; F_s) \simeq \mathrm{R}\Gamma(M; F) \neq 0$. Hence, $\Lambda_\psi \cap \mathrm{SS}(F_s) \neq \emptyset$.
Finally, $\mathrm{SS}(F_s) \cap \dot{T}^*M = \varphi_s(\mathrm{SS}(F) \cap \dot{T}^*M)$.
One could also prove Morse inequalities in this setting.

## Non displaceability: non-negative Hamiltonian isotopies

Consider a manifold $M$ and $\Phi \colon \dot{T}^*M \times I \to \dot{T}^*M$ a homogeneous Hamiltonian isotopy with $[0,1] \subset I$. Define as above $f \colon \dot{T}^*M \times I \to \mathbb{R}$ by $f = \langle \alpha_M, \partial\Phi/\partial s\rangle$ so that $H_f = \partial\Phi/\partial s$.

**Definition** The isotopy $\Phi$ is said to be non-negative if $f \geq 0$ on $[0,1]$.

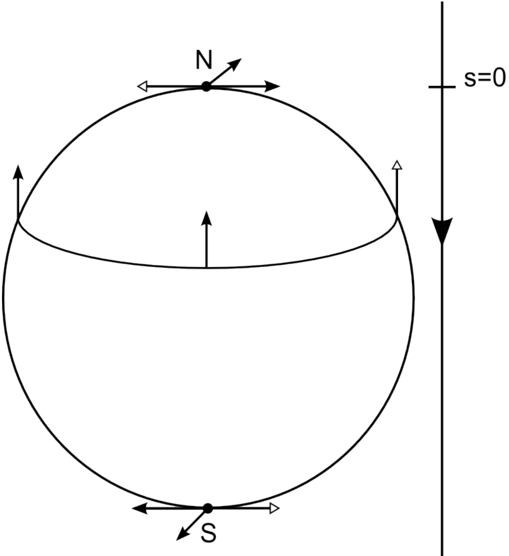
Denoting by $\Lambda \subset \dot{T}^*M \times \dot{T}^*M \times T^*I$ the Lagrangian manifold associated with $\Phi$, this is equivalent to

$$\Lambda \subset \{\tau \leq 0\}.$$

**Theorem** Let $M$ be a connected and non-compact manifold and let $X, Y$ be two compact connected submanifolds of $M$. Let $\Phi = \{\varphi_t\}_{t\in I} \colon \dot{T}^*M \times I \to \dot{T}^*M$ be a non-negative homogeneous Hamiltonian isotopy. Assume that $\varphi_1(\dot{T}^*_X M) = \dot{T}^*_Y M$. Then $X = Y$ and $\varphi_s|_{\dot{T}^*_X M} = \mathrm{id}_{\dot{T}^*_X M}$ for all $s \in [0,1]$.

# Example

On $\dot{T}^*\mathbb{R}^n$ choose $f(s,x,\xi) = \sqrt{\sum_i \xi^2}$, $\Phi(s,x,\xi) = (x + s\xi/|\xi|, \xi)$. A similar example on the $n$-sphere would contradicts the theorem since it interchanges the conormal bundles to the north and south poles.



The main tool of the proof is the following:

**Lemma** Let $N$ be a manifold, $I$ an open interval of $\mathbb{R}$ containing 0. Let $F \in \mathrm{D^b}(\mathbf{k}_{N\times I})$ and, for $t \in I$, set $F_t = F|_{N\times\{t\}} \in \mathrm{D^b}(\mathbf{k}_N)$. Assume that

(a) $\mathrm{SS}(F) \subset \{\tau \leq 0\}$,

(b) $\mathrm{SS}(F) \cap (T^*_N N \times T^*I) \subset T^*_{N\times I}(N\times I)$,

(c) $\mathrm{Supp}(F) \to I$ is proper.

Then for all $a \leq b$ in $I$ there is a natural morphism $r_{b,a}\colon F_a \to F_b$, which induces the isomorphisms

$$\mathrm{R}\Gamma(N\times I; F) \xrightarrow{\sim} \mathrm{R}\Gamma(N; F_a) \xrightarrow[r_{b,a}]{\sim} \mathrm{R}\Gamma(N; F_b) \xleftarrow{\sim} \mathrm{R}\Gamma(N\times I; F).$$

## Statement of the theorem

**Theorem** (Eliashberg)
Let $E$ be a real symplectic finite dimensional vector space. Let $B(R)$ denote the open ball with radius $R > 0$ and $\overline{B}(r)$ the closed ball with radius $r$, $0 < r < R$. Let $\varphi_i \colon B(R) \to E$ be a $C^1$-map, $i \in \mathbb{N} \sqcup \{\infty\}$. Assume that
$\varphi_n$ is a diffeomorphism onto its image for all $n \in \mathbb{N} \sqcup \{\infty\}$,
$\varphi_n$ is a symplectic diffeomorphism onto its image for all $n \in \mathbb{N}$,
$||\varphi_n - \varphi_\infty||_r \xrightarrow{n} 0$ where $||\psi||_r = \sup_{x \in \overline{B}(r)} |\psi(x)|$.
Then $\varphi_\infty|_{B(r)}$ is a symplectic diffeomorphism onto its image.

# Involutivity

**Definition** (KS82)
A locally closed subset $S$ of a symplectic manifold $X$ is co-isotropic (or involutive) at $p \in X$ if

$$C_p(S,S)^{\perp} \subset C_p(S).$$

Here, $C_p(S,Z) \subset T_pX$ is the Whitney normal cone, $C_p(S) = C_p(\{p\},S)$ and $\perp$ is defined through the Hamiltonian isomorphism $TX \xrightarrow{\sim} T^*X$.

**Theorem** (KS82)
Let $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_M)$. Then $\mathrm{SS}(F)$ is co-isotropic in $T^*M$.

(a) Let $S_1 \subset S_2 \subset X$ be locally closed subsets and let $p \in S_1$. If $S_1$ is co-isotropic at $p$ then so is $S_2$.

(b) Let $\rho\colon T^*M \times T^*_{\tau>0}R \to T^*M$ be as above. Let $S \subset T^*M$ be a locally closed subset and let $p \in S$, $q \in \rho^{-1}(p)$. Then $S$ is co-isotropic at $p$ if and only if $\rho^{-1}S$ is co-isotropic at $q$.

## Guillermou's proof

(1) Using an approximation lemma, we reduce to the case where for each $n$ there exists a Hamiltonian isotopy $\Phi_n$ defined on $I \supset [0,1]$ such that $\varphi_n = \Phi^1_n$.

(2) We can lift the isotopies $\Phi_n$ ($n \in \mathbb{N} \sqcup \{\infty\}$) to homogeneous isotopies $\Psi_n$ whose graphs are homogeneous Lagrangian submanifolds $\Lambda_n \subset (T^*E \times T^*E \times T^*\mathbb{R} \times T^*I) \cap \{\tau > 0\}$, where $(t;\tau)$ are the coordinates on $T^*\mathbb{R}$ and denoting by $\rho$ the map

$$\rho\colon T^*E \times T^*E \times T^*\mathbb{R} \times T^*I \cap \{\tau > 0\}, \to T^*E \times T^*E \times T^*I,$$
$$\rho\colon (x, y, t, s; \xi, \eta, \tau, \sigma) \mapsto (x, y, s; \xi/\tau, \eta/\tau, \sigma/\tau)$$

we have $\Gamma_{\Phi_n} = \rho(\Lambda_n)$. We have to show that $\Gamma_{\varphi_\infty}$ is co-isotropic.

(3) For a sequence $K_n \in \mathrm{D}^{[a,b]}(\mathbf{k}_X)$, $n \in \mathbb{N}$, define $K_\infty \in \mathrm{D}^{[a,b]}(\mathbf{k}_X)$ by the d.t.

$$\bigoplus_n K_n \to \prod_{n\in\mathbb{N}} K_n \to K_\infty \xrightarrow{+1} .$$

Note that

$$\mathrm{SS}(K_\infty) \subset \bigcap_{k\in\mathbb{N}} \overline{\bigcup_{n\geq k} \mathrm{SS}(K_n)}.$$

(4) Using the GKS theorem, for each $n$, there exists $K_n \in \mathrm{D}^{[a,b]}(\mathbf{k}_{E\times E\times\mathbb{R}\times I})$ whose microsupport is contained in $\Lambda_n$ and $K_n|_{s=0}$ is the constant sheaf on the diagonal.
(5) Main technical part: prove that given $x \in B(r)$, $K_\infty$ is not constant (in particular, not 0) in a neighborhood of $x$. Since $\mathrm{SS}(K_\infty) \subset \Gamma_{\varphi_\infty}$ and we may assume that $\Gamma_{\varphi_\infty}$ is in generic position, this will completes the proof.