An introduction to $\mathscr{D}$-modules

Pierre Schapira
Sorbonne Université
Institut de Mathématiques de Jussieu-Paris Rive Gauche, F-75005 Paris France
http://webusers.imj-prg.fr/~pierre.schapira
e-mail: pierre.schapira@imj-prg.fr

Draft, v7, March 2020

Introduction

The aim of these Notes is to introduce the reader to the theory of $\mathscr{D}$-modules in the analytical setting. This text is a short introduction, not a systematic study. In particular many proofs are skipped and the reader is encouraged to consult the literature. To our opinion, the best reference to $\mathscr{D}$-modules is [Ka03], and, in fact, most of the material of these Notes are extracted from this book.

Indeed, although we do not mention it in the course of the notes, almost all the results and proofs exposed here are due to Masaki Kashiwara.

References for $\mathscr{D}$-modules. Some classical titles are [Ka70, Ka83, Bj93, Ka03] and, in the algebraic setting, [Bo87]. An elementary introduction may also be found in [Co85]. Applications to $\mathscr{D}$-modules to representation theory are studied in [HTT08].

Related theories to $\mathscr{D}$-modules. Microdifferential operators are the natural localization of differential operators. References are made to [SKK73, Ka83, Sc85]. In fact, microdifferential operators may also be considered as an avatar of rings of deformation quantization for which there exists an enormous literature. See [KS12] and the references therein.

References for categories, homological algebra and sheaves. The reader is assumed to be familiar with sheaf theory as well as homological algebra, including derived categories. An exhaustive treatment may be found in [KS06] and a pedagogical treatment is provided in [Sc08]. Among numerous other references, see [GM96], [KS90, Ch. 1, 2] [We94].

History. An outline of $\mathscr{D}$-module theory, including holonomic systems, was proposed by Mikio Sato in the early 60's in a series of lectures at Tokyo University (see [Sc07]). However, it seems that Sato's vision has not been understood until his student, Masaki Kashiwara, wrote his thesis in 1970 (see [Ka70]). Independently and at the same time, J. Bernstein, a student of I. Gelfand at Moscow's University, developed a very similar theory in the algebraic setting (see [Be71]).

Contents

Chapter 1

The ring $\mathscr{D}_X$

In all these Notes, all rings are associative and unital. If R is a ring, an R-module means a left R-module and we denote by $\mathrm{Mod}(R)$ the abelian category of such modules. We denote by R^{op} the opposite ring. Hence, $\mathrm{Mod}(R^{\mathrm{op}})$ denotes the category of right R-modules. If a, b belong to R, their bracket $[a, b]$ is given by $[a, b] = ab - ba$. We use similar conventions and notations for a sheaf of rings $\mathscr{R}$ on a topological space X. In particular, $\mathrm{Mod}(\mathscr{R})$ denotes the category of sheaves of left $\mathscr{R}$-modules on X.

1.1 Construction of $\mathscr{D}_X$

$\mathscr{O}$-modules

Let X denote a complex manifold, $\mathscr{O}_X$ its structural sheaf, that is, the sheaf of holomorphic functions on X. Unless otherwise specified, we denote by d_X the complex dimension of X. We denote by Ω_X^p the sheaf of holomorphic p-forms and one sets $\Omega_X = \Omega_X^{d_X}$. One also sets

$$\Omega^\bullet = \bigoplus_p \Omega_X^p. \tag{1.1.1}$$

We denote by $\mathrm{Mod}(\mathbb{C}_X)$ the abelian category of sheaves of $\mathbb{C}$-vector spaces on X, and we denote by $\mathscr{H}om$ and $\otimes$ the internal Hom and tensor product in this category. For $F \in \mathrm{Mod}(\mathbb{C}_X)$, we set $\mathscr{E}nd(F) = \mathscr{H}om_{\mathbb{C}_X}(F, F)$.

Similarly, we denote by $\mathrm{Mod}(\mathscr{O}_X)$ the abelian category of sheaves of $\mathscr{O}_X$-modules, and we denote by $\mathscr{H}om_{\mathscr{O}}$ and $\otimes_{\mathscr{O}}$ the internal Hom and tensor product in this category. We denote by $\mathrm{Mod}_{\mathrm{c}}(\mathscr{O}_X)$ the full abelian subcategory consisting of coherent sheaves.

One denotes by Θ_X the sheaf of Lie algebras of holomorphic vector fields. Hence, $\Theta_X = \mathscr{H}om_{\mathscr{O}}(\Omega_X^1, \mathscr{O}_X)$.

The sheaf Θ_X has two actions on $\Omega^\bullet$, that we recall. Let $v \in \Theta_X$. The interior derivative $i_v \in \mathscr{E}nd(\Omega_X^\bullet)$ is characterized by the conditions

$$\begin{cases} i_v(a) = 0, \ a \in \mathscr{O}_X \\ i_v(\omega) = \langle v, \omega \rangle, \ \omega \in \Omega^1, \\ i_v(\omega_1 \wedge \omega_2) = (i_v\omega_1) \wedge \omega_2 + (-)^p \omega_1 \wedge (i_v\omega_2), \ \omega_1 \in \Omega_X^p. \end{cases} \tag{1.1.2}$$

Note that $i_v : \Omega_X^p \to \Omega_X^{p-1}$ is of degree -1.

On the other-hand, the Lie derivative $L_v \in \mathscr{E}nd(\Omega_X^\bullet)$ is characterized by the conditions

$$\begin{cases} L_v(a) = v(a) = \langle v, da\rangle, a \in \mathscr{O}_X, \\ d \circ L_v = L_v \circ d, \\ L_v(\omega_1 \wedge \omega_2) = (L_v\omega_1) \wedge \omega_2 + \omega_1 \wedge (L_v\omega_2), \end{cases} \tag{1.1.3}$$

The Lie derivative is of degree 0 and satisfies

$$[L_u, L_v] = L_{[u,v]},\ \ u, v \in \Theta_X. \tag{1.1.4}$$

One has the relations

$$L_v = d \circ i_v + i_v \circ d. \tag{1.1.5}$$

Using $v \mapsto L_v$, one may regard Θ_X as a subsheaf of $\mathscr{E}nd(\mathscr{O}_X)$.

The ring $\mathscr{D}_X$

Definition 1.1.1. One denotes by $\mathscr{D}_X$ the subalgebra of $\mathscr{E}nd(\mathscr{O}_X)$ generated by $\mathscr{O}_X$ and Θ_X.

If $(x_1, \dots, x_n)$ is a local coordinate system on a local chart U of X, then a section P of $\mathscr{D}_X$ on U may be uniquely written as a polynomial

$$P = \sum_{|\alpha| \le m} a_\alpha \partial^\alpha \tag{1.1.6}$$

where $a_\alpha \in \mathscr{O}_X$, $\partial_i = \partial_{x_i} = \frac{\partial}{\partial x_i}$ and we use the classical notations for multi-indices:

$$\begin{cases} \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n, \\ |\alpha| = \alpha_1 + \cdots + \alpha_n, \\ \text{ if } X = (X_1, \dots, X_n), \text{ then } X^\alpha = X^{\alpha_1} \dots X^{\alpha_n}. \end{cases}$$

Proposition 1.1.2. *Let $\mathscr{R}$ be a sheaf of $\mathbb{C}_X$-algebras and let $\iota : \mathscr{O}_X \to \mathscr{R}$ and $\varphi : \Theta_X \to \mathscr{R}$ be $\mathbb{C}_X$-linear morphisms satisfying (here, $a, b \in \mathscr{O}_X$ and $u, v \in \Theta_X$):*

(i) *$\iota\colon \mathscr{O}_X \to \mathscr{R}$ is a ring morphism, that is, $\iota(ab) = \iota(a)\iota(b)$,*

(ii) *$\varphi : \Theta_X \to \mathscr{R}$ is left $\mathscr{O}_X$-linear, that is, $\varphi(av) = \iota(a)\varphi(v)$,*

(iii) *$\varphi : \Theta_X \to \mathscr{R}$ is a morphism of Lie algebras, that is, $[\varphi(u), \varphi(v)] = \varphi([u, v])$,*

(iv) *$[\varphi(v), \iota(a)] = \iota(v(a))$ for any $v \in \Theta_X$ and $a \in \mathscr{O}_X$.*

Then there exists a unique morphism of $\mathbb{C}_X$-algebras $\Psi : \mathscr{D}_X \to \mathscr{R}$ such that the composition $\mathscr{O}_X \to \mathscr{D}_X \to \mathscr{R}$ coincides with ι and the composition $\Theta_X \to \mathscr{D}_X \to \mathscr{R}$ coincides with φ.

The proof is straightforward.

Corollary 1.1.3. *Let $\mathscr{M}$ be an $\mathscr{O}_X$-module and let $\mu\colon \mathscr{O}_X \to \mathscr{E}nd(\mathscr{M})$ be the action of $\mathscr{O}_X$ on $\mathscr{M}$. Let $\psi : \Theta_X \to \mathscr{E}nd(\mathscr{M})$ be a $\mathbb{C}_X$-linear morphism satisfying:*

(i) $\mu(a) \circ \psi(v) = \psi(av)$ (*resp.* $\psi(v) \circ \mu(a) = \psi(av)$).

(ii) $[\psi(v), \psi(w)] = \psi([v, w])$ (*resp.* $[\psi(v), \psi(w)] = -\psi([v, w])$),

(iii) $[\psi(v), \mu(a)] = \mu(v(a))$, (*resp.* $[\psi(v), \mu(a)] = -\mu(v(a))$).

Then there exists one and only one structure of a left (resp. right) $\mathscr{D}_X$-module on $\mathscr{M}$ which extends the action of Θ_X.

Proof. For the structure of a left module, apply Proposition 1.1.2 to $\mathscr{R} = \mathscr{E}nd(\mathscr{M})$. The case of right modules follows since the bracket $[a, b]^{\mathrm{op}}$ in $\mathscr{D}_X^{\mathrm{op}}$ is $-[a, b]$, where $[a, b]$ is the bracket in $\mathscr{D}_X$. □

Examples 1.1.4. (i) The sheaf $\mathscr{O}_X$ is naturally endowed with a structure of a left $\mathscr{D}_X$-module and $1 \in \mathscr{O}_X$ is a generator. Since the anihilator of 1 is the left ideal generated by Θ_X, we find an exact sequence of left $\mathscr{D}_X$-modules

$$\mathscr{D}_X \cdot \Theta_X \to \mathscr{D}_X \to \mathscr{O}_X \to 0.$$

Note that if X is connected and f is a section of $\mathscr{O}_X$, $f \neq 0$ (*i.e.*, f is not identically zero), then f is also a generator of $\mathscr{O}_X$ over $\mathscr{D}_X$. This follows from the Weierstrass Preparation Lemma. Indeed, choosing a local coordinate system $(x_1, \dots, x_n)$, one may write $f = \sum_{j=0}^{m} a_j(x') x_1^j$, with $a_m \equiv 1$. Then $\partial_1^m(f) = m!$.
(ii) The sheaf Ω_X is naturally endowed with a structure of a right $\mathscr{D}_X$-module, by

$$v(\omega) = -L_v(\omega), \quad v \in \Theta_X, \omega \in \Omega_X.$$

(iii) Let $\mathscr{F}$ be an $\mathscr{O}_X$-module. Then $\mathscr{D}_X \otimes_{\mathscr{O}} \mathscr{F}$ is a left $\mathscr{D}_X$-module.
(iv) Let Z be a closed complex submanifold of X of codimension d. Then $H^d_Z(\mathscr{O}_X)$ is a left $\mathscr{D}_X$-module.
(v) Let X be a complex manifold and let P be a differential operator on X. The differential equation $Pu = v$ may be studied via the left $\mathscr{D}_X$-module $\mathscr{D}_X / \mathscr{D}_X \cdot P$. (See below.)
(vi) Let $X = \mathbb{C}^n$ and consider the differential operators $P = \sum_{j=1}^{n} \partial_j^2$, $Q_{ij} = x_i\partial_j - x_j\partial_i$. Consider the left ideal $\mathscr{J}$ of $\mathscr{D}_X$ generated by P and the family $\{Q_{ij}\}_{i<j}$. The left $\mathscr{D}_X$-module $\mathscr{D}_X/\mathscr{J}$ is naturally associated to the operator P and the orthogonal group $\mathbb{O}(n; \mathbb{C})$.

Internal hom and tens

The sheaf $\mathscr{D}_X$ is a sheaf of non commutative rings and $\mathbb{C}_X$ is contained (in fact, is equal, but we have not proved it here) in its center. It follows that we have functors:

$$\begin{aligned} &\mathscr{H}om_{\mathscr{D}} : (\mathrm{Mod}(\mathscr{D}_X))^{\mathrm{op}} \times \mathrm{Mod}(\mathscr{D}_X) \to \mathrm{Mod}(\mathbb{C}_X), \\ &\otimes_{\mathscr{D}} : \mathrm{Mod}(\mathscr{D}_X^{\mathrm{op}}) \times \mathrm{Mod}(\mathscr{D}_X) \to \mathrm{Mod}(\mathbb{C}_X). \end{aligned}$$

We shall now study hom and tens over $\mathscr{O}_X$.
Let $\mathscr{M}, \mathscr{N}$ and $\mathscr{P}$ be left $\mathscr{D}_X$-modules and let $\mathscr{M}'$ and $\mathscr{N}'$ be right $\mathscr{D}_X$-modules.

(a) One endows $\mathscr{M} \otimes_{\mathscr{O}} \mathscr{N}$ with a structure of a left $\mathscr{D}_X$-module by setting

$$v(m \otimes n) = v(m) \otimes n + m \otimes v(n), \quad m \in \mathscr{M}, n \in \mathscr{N}, v \in \Theta_X.$$

(b) One endows $\mathscr{H}om_{\mathscr{O}}(\mathscr{M},\mathscr{N})$ with a structure of a left $\mathscr{D}_X$-module by setting

$$v(f)(m) = v(f(m)) - f(v(m)), \quad m \in \mathscr{M}, f \in \mathscr{H}om_{\mathscr{O}}(\mathscr{M},\mathscr{N}), v \in \Theta_X.$$

(c) One endows $\mathscr{N}' \otimes_{\mathscr{O}} \mathscr{M}$ with a structure of a right $\mathscr{D}_X$-module by setting

$$(n \otimes m)v = nv \otimes m - n \otimes vm, \quad m \in \mathscr{M}, n \in \mathscr{N}', v \in \Theta_X.$$

(d) One endows and $\mathscr{H}om_{\mathscr{O}}(\mathscr{M}',\mathscr{N}')$ with a structure of a left $\mathscr{D}_X$-module by setting

$$v(f)(m) = f(mv) - f(m)v \;\; m \in \mathscr{M}', f \in \mathscr{H}om_{\mathscr{O}}(\mathscr{M}',\mathscr{N}'), v \in \Theta_X.$$

(e) One endows and $\mathscr{H}om_{\mathscr{O}}(\mathscr{M},\mathscr{N}')$ with a structure of a right $\mathscr{D}_X$-module by setting

$$(fv)(m) = f(m)v + f(vm) \;\; m \in \mathscr{M}, f \in \mathscr{H}om_{\mathscr{O}}(\mathscr{M},\mathscr{N}'), v \in \Theta_X.$$

There are isomorphisms of $\mathbb{C}_X$-modules;

$$\begin{aligned}
&\mathscr{H}om_{\mathscr{D}}(\mathscr{M} \otimes_{\mathscr{O}} \mathscr{N}, \mathscr{P}) \simeq \mathscr{H}om_{\mathscr{D}}(\mathscr{M}, \mathscr{H}om_{\mathscr{O}}(\mathscr{N},\mathscr{P})),\\
&\mathscr{H}om_{\mathscr{D}}(\mathscr{M}' \otimes_{\mathscr{O}} \mathscr{M}, \mathscr{N}) \simeq \mathscr{H}om_{\mathscr{D}}(\mathscr{M}, \mathscr{H}om_{\mathscr{O}}(\mathscr{M}',\mathscr{N})),\\
&(\mathscr{M}' \otimes_{\mathscr{O}} \mathscr{M}) \otimes_{\mathscr{D}} \mathscr{N} \simeq \mathscr{M}' \otimes_{\mathscr{D}} (\mathscr{M} \otimes_{\mathscr{O}} \mathscr{N}).
\end{aligned}$$

To summarize, we have functors

$$\begin{aligned}
&\otimes_{\mathscr{O}} \colon \mathrm{Mod}(\mathscr{D}_X) \times \mathrm{Mod}(\mathscr{D}_X) \to \mathrm{Mod}(\mathscr{D}_X),\\
&\otimes_{\mathscr{O}} \colon \mathrm{Mod}(\mathscr{D}_X^{\mathrm{op}}) \times \mathrm{Mod}(\mathscr{D}_X) \to \mathrm{Mod}(\mathscr{D}_X^{\mathrm{op}}),\\
&\mathscr{H}om_{\mathscr{O}} \colon \mathrm{Mod}(\mathscr{D}_X)^{\mathrm{op}} \times \mathrm{Mod}(\mathscr{D}_X) \to \mathrm{Mod}(\mathscr{D}_X),\\
&\mathscr{H}om_{\mathscr{O}} \colon \mathrm{Mod}(\mathscr{D}_X^{\mathrm{op}})^{\mathrm{op}} \times \mathrm{Mod}(\mathscr{D}_X^{\mathrm{op}}) \to \mathrm{Mod}(\mathscr{D}_X),\\
&\mathscr{H}om_{\mathscr{O}} \colon \mathrm{Mod}(\mathscr{D}_X)^{\mathrm{op}} \times \mathrm{Mod}(\mathscr{D}_X^{\mathrm{op}}) \to \mathrm{Mod}(\mathscr{D}_X^{\mathrm{op}}).
\end{aligned}$$

Remark 1.1.5. Following [HTT08] who call it the Oda's rule, one way to memorize the left an right actions is to use the correspondence left $= 0$, right $= 1$, $a \otimes b = a + b$ and $\mathscr{H}om(a,b) = -a + b$.

Twisted $\mathscr{D}_X$-modules

Let $\mathscr{L}$ be a holomorphic line bundle, that is, a locally free $\mathscr{O}_X$-module of rank one. One sets

$$\mathscr{L}^{\otimes -1} = \mathscr{H}om_{\mathscr{O}}(\mathscr{L}, \mathscr{O}_X).$$

There are a natural isomorphisms

$$\mathscr{O}_X \xrightarrow{\sim} \mathscr{H}om_{\mathscr{O}}(\mathscr{L},\mathscr{L}) \xleftarrow{\sim} \mathscr{H}om_{\mathscr{O}}(\mathscr{L},\mathscr{O}_X) \otimes_{\mathscr{O}} \mathscr{L}.$$

If s is a section of $\mathscr{L}^{\otimes -1}$ and t a section of $\mathscr{L}$, their product will be denoted by $\langle s, t\rangle$, a section of $\mathscr{O}_X$.

Let $\mathscr{R}$ be a $\mathscr{O}_X$-ring, that is, a sheaf of rings together with a morphism of rings $\mathscr{O}_X \to \mathscr{R}$. One can define a new $\mathscr{O}_X$-ring $\mathscr{L} \otimes \mathscr{R} \otimes \mathscr{L}^{\otimes -1}$ by setting (with obvious notations)

$$(s \otimes m \otimes t) \cdot (s' \otimes m' \otimes t') = s \otimes m\langle t, s'\rangle m' \otimes t'.$$

If $\mathscr{M}$ is a left $\mathscr{R}$-module, then $\mathscr{L} \otimes_{\mathscr{O}} \mathscr{M}$ is a left $\mathscr{L} \otimes_{\mathscr{O}} \mathscr{R} \otimes_{\mathscr{O}} \mathscr{L}^{\otimes -1}$-module. Clearly:

Proposition 1.1.6. *The functor $\mathscr{M} \mapsto \mathscr{L} \otimes_{\mathscr{O}} \mathscr{M}$ is an equivalence of categories from* $\mathrm{Mod}(\mathscr{R})$ *to* $\mathrm{Mod}(\mathscr{L} \otimes_{\mathscr{O}} \mathscr{R} \otimes_{\mathscr{O}} \mathscr{L}^{\otimes -1})$.

Proposition 1.1.7. *There is an isomorphism of $\mathscr{O}_X$-rings $\mathscr{D}_X^{\mathrm{op}} \simeq \Omega_X \otimes_{\mathscr{O}} \mathscr{D}_X \otimes_{\mathscr{O}} \Omega_X^{\otimes -1}$.*

Proof. The right $\mathscr{D}_X$-module structure of Ω_X defines the morphism of rings

$$\mathscr{D}_X^{\mathrm{op}} \to \mathscr{E}nd(\Omega_X).$$

On the other-hand, the morphism $\mathscr{D}_X \to \mathscr{E}nd(\mathscr{O}_X)$ defines the morphism of rings

$$\Omega_X \otimes_{\mathscr{O}} \mathscr{D}_X \otimes_{\mathscr{O}} \Omega_X^{\otimes -1} \to \mathscr{E}nd(\Omega_X).$$

Both these morphisms are monomorphisms, and to check that their images in $\mathscr{E}nd(\Omega_X)$ are the same, one remark that both rings are generated by $\mathscr{O}_X$ and Θ_X. □

Corollary 1.1.8. *The functor $\mathscr{M} \mapsto \Omega \otimes_{\mathscr{O}} \mathscr{M}$ induces an equivalence of categories* $\mathrm{Mod}(\mathscr{D}_X) \xrightarrow{\sim} \mathrm{Mod}(\mathscr{D}_X^{\mathrm{op}})$

Remark 1.1.9. Suppose to be given a volume form dv on X. Then $f \mapsto f dv$ gives an isomorphism $\mathscr{O}_X \xrightarrow{\sim} \Omega_X$ and we get an isomorphism $\mathscr{D}_X \simeq \mathscr{D}_X^{\mathrm{op}}$. The image of a section $P \in \mathscr{D}_X$ by this isomorphism is called its adjoint with respect to dv and is denoted by P^*. Hence, for a left $\mathscr{D}_X$-module $\mathscr{M}$ and a section u of $\mathscr{M}$, we have

$$P \cdot u = (u \cdot dv) \cdot P^*.$$

Clearly $(Q \circ P)^* = P^* \circ Q^*$. If $(x_1, \dots, x_n)$ is a local coordinate system on X and $dv = dx_1 \wedge \cdots \wedge dx_n$, one checks that $x_i^* = x_i$ and $\partial_{x_i}^* = -\partial_{x_i}$.

1.2 Filtration on $\mathscr{D}_X$

Total symbol of differential operators

Assume X is affine, that is, X is open in a finite dimensional complex vector space E. Let P be a section of $\mathscr{D}_X$. One defines its total symbol

$$\sigma_{\mathrm{tot}}(P)(x;\xi) := \exp\langle -x, \xi\rangle P(\exp\langle x, \xi\rangle) = \sum_{|\alpha| \le m} a_\alpha(x)\xi^\alpha. \tag{1.2.1}$$

Using (1.1.6), one gets that $\sigma_{\mathrm{tot}}(P)$ is a function on $X \times E^*$, polynomial with respect to $\xi \in E^*$. This function highly depends on the affine structure, but its order (a locally constant function on X) does not. It is called the order of P and denoted $\mathrm{ord}(P)$.

If Q is another differential operator with total symbol $\sigma_{\mathrm{tot}}(Q)$, it follows easily from the Leibniz formula that the total symbol $\sigma_{\mathrm{tot}}(R)$ of $R = P \cdot Q$ is given by:

$$\sigma_{\mathrm{tot}}(R) = \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!} \partial_\xi^\alpha(\sigma_{\mathrm{tot}}(P)) \partial_x^\alpha(\sigma_{\mathrm{tot}}(Q)). \tag{1.2.2}$$

By this formula, one gets that

$$\mathrm{ord}(P \cdot Q) = \mathrm{ord}(P) + \mathrm{ord}(Q),$$
$$\mathrm{ord}([P, Q]) \le \mathrm{ord}(P) + \mathrm{ord}(Q) - 1.$$

The ring $\mathscr{D}_X$ is now endowed with the filtration " by the order",

$$\mathrm{Fl}_m(\mathscr{D}_X) = \{P \in \mathscr{D}_X; \mathrm{ord}(P) \le m\}.$$

One can give a more intrinsic definition of the filtration.

Filtration on $\mathscr{D}_X$

Definition 1.2.1. The filtration $\mathrm{Fl}\mathscr{D}_X$ on $\mathscr{D}_X$ is given by

$$\mathrm{Fl}_{-1}\mathscr{D}_X = \{0\}, \quad \mathrm{Fl}_m\mathscr{D}_X = \{P \in \mathscr{D}_X; [P, \mathscr{O}_X] \in \mathrm{Fl}_{m-1}\mathscr{D}_X\}.$$

Note that

$$\begin{cases} \mathrm{Fl}_0\mathscr{D}_X = \mathscr{O}_X, \quad \mathrm{Fl}_1\mathscr{D}_X = \mathscr{O}_X \oplus \Theta_X, \\ \mathrm{Fl}_m\mathscr{D}_X \cdot \mathrm{Fl}_l\mathscr{D}_X \subset \mathrm{Fl}_{m+l}\mathscr{D}_X, \quad [\mathrm{Fl}_m\mathscr{D}_X, \mathrm{Fl}_l\mathscr{D}_X] \subset \mathrm{Fl}_{m+l-1}\mathscr{D}_X. \end{cases} \tag{1.2.3}$$

One denotes by $\mathrm{Gr}\mathscr{D}_X$ the associated graded ring, by $\sigma : \mathrm{Fl}\mathscr{D}_X \to \mathrm{Gr}\mathscr{D}_X$ the "principal symbol map" and by $\sigma_m : \mathrm{Fl}_m\mathscr{D}_X \to \mathrm{Gr}_m\mathscr{D}_X$ the map "symbol of order m".

One shall not confuse the total symbol, which is defined on affine charts, and the principal symbol, which is well defined on manifolds.

It follows from (1.2.2) that $\sigma(P)\sigma(Q) = \sigma(Q)\sigma(P) = \sigma(P \cdot Q)$. Hence, $\mathrm{Gr}(\mathscr{D}_X)$ is a *commutative* graded ring. Moreover, $\mathrm{Gr}_0(\mathscr{D}_X) \simeq \mathscr{O}_X$ and $\mathrm{Gr}_1(\mathscr{D}_X) \simeq \Theta_X$.

Denote by $S_{\mathscr{O}}(\Theta_X)$ the symmetric $\mathscr{O}_X$-algebra associated with the locally free $\mathscr{O}_X$-module Θ_X. By the universal property of symmetric algebras, the morphism $\Theta_X \to \mathrm{Gr}(\mathscr{D}_X)$ may be extended to a morphism of symmetric algebra

$$S_{\mathscr{O}}(\Theta_X) \to \mathrm{Gr}\mathscr{D}_X. \tag{1.2.4}$$

Proposition 1.2.2. *The morphism (1.2.4) is an isomorphism.*

Proof. Choose a local coordinate system $(x_1, \dots, x_n)$ on X. Then $\Theta_X \simeq \bigoplus_{i=1}^n \mathscr{O}_X\partial_i$ and the correspondence $\partial_i \mapsto \xi_i$ gives the isomorphism

$$S_{\mathscr{O}}(\Theta_X) \simeq \bigoplus_\alpha \mathscr{O}_X\partial^\alpha \simeq \mathscr{O}_X[\xi_1, \dots, \xi_n] \simeq \mathrm{Gr}\mathscr{D}_X.$$

□

Denote by $\pi : T^*X \to X$ the projection. There is a natural monomorphism

$$\Theta_X \hookrightarrow \pi_*\mathscr{O}_{T^*X}.$$

Indeed, a vector field on X is a section of the tangent bundle TX, hence defines a linear function on T^*X.

By the universal property of symmetric algebra, we get a monomorphism $S_{\mathscr{O}}(\Theta_X) \hookrightarrow \pi_*\mathscr{O}_{T^*X}$. Applying Proposition 1.2.2, we get an embedding of $\mathbb{C}_X$-algebras:

$$\mathrm{Gr}\mathscr{D}_X \hookrightarrow \pi_*\mathscr{O}_{T^*X}.$$

In the sequel, we shall still denote by

$$\sigma\colon \mathscr{D}_X \to \pi_*\mathscr{O}_{T^*X} \text{ and } \sigma_m\colon \mathrm{Fl}_m\mathscr{D}_X \to \pi_*\mathscr{O}_{T^*X},$$

the maps obtained by applying the inverse of the isomorphism (1.2.4) to σ and σ_m.

Theorem 1.2.3. *The sheaf of rings $\mathscr{D}_X$ is right and left Noetherian.*

Proof. This follows from Proposition 1.2.2 and general results of [Ka03, Th. A.20] on filtered ring with associated commutative graded ring (see Theorem 3.3.5). □

1.3 Characteristic variety

We shall use here the results of § 3.4.

Poisson's structures

The graded ring $\mathrm{Gr}(\mathscr{D}_X)$ is endowed with a natural Poisson bracket induced by the commutator in $\mathscr{D}_X$.

On the other hand, the sheaf $\mathscr{O}_{T^*X}$ (hence, the sheaf $\pi_*\mathscr{O}_{T^*X}$) is endowed with the Poisson bracket induced by the symplectic structure of T^*X. Recall that if $(x_1,\dots,x_n;\xi_1,\dots,\xi_n)$ is a local symplectic coordinate system on T^*X, this Poisson bracket is given by

$$\{f,g\} = \sum_{i=1}^{n} \partial_{\xi_i} f\, \partial_{x_i} g - \partial_{x_i} f\, \partial_{\xi_i} g.$$

Proposition 1.3.1. *The Poisson bracket on $\pi_*\mathscr{O}_{T^*X}$ induces the Poisson bracket on $\mathrm{Gr}(\mathscr{D}_X)$.*

Proof. Let $P \in \mathrm{Fl}_m(\mathscr{D}_X)$ and $Q \in \mathrm{Fl}_l(\mathscr{D}_X)$. Then $[P,Q] \in \mathrm{Fl}_{m+l-1}(\mathscr{D}_X)$ and it follows from (1.2.2) that

$$\sigma_{m+l-1}([P,Q]) = \sum_{i=1}^{n} \big(\partial_{\xi_i}\sigma_m(P)\partial_{x_i}\sigma_l(Q) - \partial_{\xi_i}\sigma_l(Q)\partial_{x_i}\sigma_m(P)\big). \tag{1.3.1}$$

Hence, $\sigma_{m+l-1}([P,Q]) = \{\sigma_m(P), \sigma_l(Q)\}$. □

Good filtration

We shall recall some notions also introduced in § 3.3, 3.4. Recall that a good filtration on a coherent $\mathscr{D}_X$-module $\mathscr{M}$ is a filtration which is locally the image of a finite free filtration. Hence, a filtration $\mathrm{Fl}\mathscr{M}$ on $\mathscr{M}$ is good if and only if,

$$\begin{cases} \text{locally on } X,\ \mathrm{Fl}_j\mathscr{M} = 0 \text{ for } j \ll 0, \\ \mathrm{Fl}_j\mathscr{M} \text{ is } \mathscr{O}_X\text{-coherent}, \\ \text{locally on } X,\ (\mathrm{Fl}_k\mathscr{D}_X)\cdot(\mathrm{Fl}_j\mathscr{M}) = \mathrm{Fl}_{k+j}\mathscr{M} \text{ for } j \gg 0 \text{ and} \\ \text{all } k \geq 0. \end{cases} \tag{1.3.2}$$

Applying Corollary 3.3.6, we get:

Lemma 1.3.2. *Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module, $\mathscr{N} \subset \mathscr{M}$ a coherent submodule. Assume that $\mathscr{M}$ is endowed with a good filtration $\mathrm{Fl}\mathscr{M}$. Then the induced filtration on $\mathscr{N}$ defined by $\mathrm{Fl}_j\mathscr{N} = \mathscr{N} \cap \mathrm{Fl}_j\mathscr{M}$ is good.*

Denote by $\mathrm{Mod}^{\mathrm{gr}}_{\mathrm{coh}}(\mathrm{Gr}\mathscr{D}_X)$ the abelian category of coherent graded $\mathrm{Gr}\mathscr{D}_X$-modules and consider the functor

$$\begin{aligned} \tilde{\ }\colon \mathrm{Mod}^{\mathrm{gr}}_{\mathrm{coh}}(\mathrm{Gr}\mathscr{D}_X) &\to \mathrm{Mod}_{\mathrm{c}}(\pi_*\mathscr{O}_{T^*X}), \\ \mathrm{Gr}\mathscr{M} &\mapsto \pi_*\mathscr{O}_{T^*X} \otimes_{\mathrm{Gr}\mathscr{D}_X} \mathrm{Gr}\mathscr{M}. \end{aligned}$$

This functor is exact and faithful. If $\mathscr{M}$ is a coherent $\mathscr{D}_X$-module endowed with a good filtration, the $\pi_*\mathscr{O}_{T^*X}$-module

$$\widetilde{\mathrm{Gr}\mathscr{M}} = \pi_*\mathscr{O}_{T^*X} \otimes_{\mathrm{Gr}\mathscr{D}_X} \mathrm{Gr}\mathscr{M}$$

is thus coherent and its support satisfies:

$$\mathrm{supp}(\widetilde{\mathrm{Gr}\mathscr{M}}) = \{p \in T^*X; \sigma(P)(p) = 0 \text{ for any } P \in \mathrm{Icar}(\mathscr{M})\}.$$

In the sequel, we shall often confuse $\mathrm{Gr}\mathscr{M}$ and $\widetilde{\mathrm{Gr}\mathscr{M}}$.

Definition 1.3.3. The characteristic variety of $\mathscr{M}$, denoted $\mathrm{char}(\mathscr{M})$, is the closed subset of T^*X characterized as follows: for any open subset U of X such that $\mathscr{M}|_U$ is endowed with a good filtration, $\mathrm{char}(\mathscr{M})|_{T^*U}$ is the support of $\widetilde{\mathrm{Gr}\mathscr{M}|_U}$.

Theorem 1.3.4. (i) *$\mathrm{char}(\mathscr{M})$ is a closed $\mathbb{C}^\times$-conic analytic subset of T^*X.*

(ii) *$\mathrm{char}(\mathscr{M})$ is involutive for the Poisson structure of T^*X, and in particular, $\mathrm{codim}(\mathrm{char}(\mathscr{M})) \leq d_X$.*

(iii) *If $0 \to \mathscr{M}' \to \mathscr{M} \to \mathscr{M}'' \to 0$ is an exact sequence of coherent $\mathscr{D}_X$-modules, then*
$$\mathrm{char}(\mathscr{M}) = \mathrm{char}(\mathscr{M}') \cup \mathrm{char}(\mathscr{M}'').$$

Proof. (i) is obvious, (ii) follows from Gabber's theorem and (iii) follows from Lemma 1.3.2. □

Note that the involutivity theorem has first been proved by Sato, Kashiwara and Kawai [SKK73] using analytical tools, before Gabber gave is purely algebraic proof.

Suppose that a coherent $\mathscr{D}_X$-module $\mathscr{M}$ is generated by a single section u. Then $\mathscr{M} \simeq \mathscr{D}_X/\mathscr{I}$, where $\mathscr{I}$ is the anihilator of u. There is a natural filtration on $\mathscr{M}$, the image of $\mathrm{Fl}\mathscr{D}_X$. Put $\mathrm{Fl}_j\mathscr{I} = \mathscr{I} \cap \mathrm{Fl}_j\mathscr{D}_X$. It follows from Corollary 3.3.6 that the graded ideal $\mathrm{Gr}\mathscr{I}$ is coherent. Moreover, since $\mathrm{Gr}\mathscr{M} = \mathrm{Gr}\mathscr{D}_X/\mathrm{Gr}\mathscr{I}$, we get

$$\mathrm{char}(\mathscr{M}) = \{p \in T^*X; \sigma_j(P)(p) = 0 \text{ for all } P \in \mathrm{Fl}_j(\mathscr{I})\}. \tag{1.3.3}$$

If $\{P_0, \dots, P_N\}$ generates $\mathscr{I}$ it follows that

$$\mathrm{char}(\mathscr{M}) \subset \bigcap_j \sigma(P_j)^{-1}(0).$$

In general the equality does not hold, since the family of the P_j's may generate $\mathscr{I}$ although the family of the $\sigma_{m_j}(P_j)$'s does not generate $\mathrm{Gr}\mathscr{I}$.

Example 1.3.5. If $X = \mathbb{A}^1(\mathbb{C})$, the affine line, the ideal generated by ∂ and x is $\mathscr{D}_X$, but the ideal generated by their principal symbols is not $\mathscr{O}_{T^*X}$.

Corollary 1.3.6. *Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module, let $p \in T^*X$ and assume that $p \notin \mathrm{char}(\mathscr{M})$. Let $u \in \mathscr{M}$. Then there exists a section $P \in \mathscr{D}_X$ defined in a neighborhood of $\pi(p)$ with $Pu = 0$ and $\sigma(P)(p) \neq 0$.*

Proof. Consider the sub-$\mathscr{D}_X$-module $\mathscr{D}_X u$ generated by u. It is coherent and its characteristic variety is contained in that of $\mathscr{M}$. Let $\mathscr{I}$ denotes the anihilator ideal of u in $\mathscr{D}_X$ and let $P_1, \dots, P_N$ denotes sections of this ideal such that $\sigma(P_1), \dots, \sigma(P_N)$ generate the graded ideal $\mathrm{Gr}\mathscr{I}$. Such a finite family exists since $\mathrm{Gr}\mathscr{I}$ is coherent. Since $p \notin \mathrm{char}(\mathscr{D}_X u)$, there exists j with $\sigma(P_j)(p) \neq 0$. □

Example 1.3.7. (i) $\mathrm{char}(\mathscr{O}_X) = T^*_X X$, the zero-section of T^*X.
(ii) $\mathrm{char}(\mathscr{D}_X/\mathscr{D}_X \cdot P) = \{p \in T^*X; \sigma(P)(p) = 0\}$.

Multiplicities

By the result of Proposition 3.5.2, one sees that if $\mathscr{M}$ is a coherent $\mathscr{D}_X$-module and V is an irreducible component of $\mathrm{char}(\mathscr{M}) \cup V$, then $\mathrm{mult}_V(\widetilde{\mathrm{Gr}\mathscr{M}})$ depends only on $\mathscr{M}$.

Definition 1.3.8. Let V be a closed analytic subset of T^*X and let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module such that V is an irreducible component of $\mathrm{char}(\mathscr{M}) \cup V$. The number $\mathrm{mult}_V(\widetilde{\mathrm{Gr}\mathscr{M}})$ is called the multiplicity of $\mathscr{M}$ along V and denoted $\mathrm{mult}_V(\mathscr{M})$.

If $0 \to \mathscr{M}' \to \mathscr{M} \to \mathscr{M}'' \to 0$ is an exact sequence of cherent $\mathscr{D}_X$-modules with V irreducible in $\mathrm{char}(\mathscr{M}) \cup V$, then

$$\mathrm{mult}_V(\mathscr{M}) = \mathrm{mult}_V(\mathscr{M}') + \mathrm{mult}_V(\mathscr{M}'').$$

Involutive basis

Definition 1.3.9. Let $\mathscr{I}$ be a coherent ideal of $\mathscr{D}_X$ and let $\{P_1, \dots, P_N\}$ be a family of sections of $\mathscr{I}$, with P_j of order m_j. One says that this family is an involutive basis of $\mathscr{I}$ if the family $\{\sigma(P_1), \dots, \sigma(P_N)\}$ generates $\mathrm{Gr}\mathscr{I}$.

Proposition 1.3.10. *Assume*

(i) $\cap_{j=1}^{N} \sigma_{m_j}(P_j)^{-1}(0)$ *is of codimension* N,

(ii) *there exist* $Q_{jkl} \in \mathrm{Fl}_{m_j+m_k-m_l-1}\mathscr{D}_X$ *such that for all* j, k

$$[P_j, P_k] = \sum_l Q_{jkl} P_l$$

Then $\{P_1, \dots, P_N\}$ *is an involutive basis.*

Proof. Set $p_j = \sigma(P_j)$. Let $a_j \in \mathrm{Gr}_{l-m_j}\mathscr{D}_X$ with

$$\sum_j a_j p_j = 0.$$

By Proposition 3.4.9, it is enough to find $A_j \in \mathscr{D}_X$ with $\sigma(A_j) = a_j$ and such that

$$\sum_j A_j P_j = 0.$$

By the hypothesis, the sequence $\{p_1, \dots, p_N\}$ is a regular sequence. Hence, we may find $r_{ij} \in \mathrm{Gr}_{l-m_i-m_j}\mathscr{D}_X$ satisfying

$$a_j = \sum_i r_{ij} p_i, \quad r_{ij} = -r_{ji}.$$

Next we choose $R_{ij} \in \mathrm{Fl}_{l-m_i-m_j}\mathscr{D}_X$ with $\sigma(R_{ij}) = r_{ij}$ and $R_{ij} = -R_{ji}$. Set $A_j = \sum_i R_{ij} P_i$. Then $\sigma_{l-m_j}(A_j) = a_j$ and

$$\begin{aligned}\sum_j A_j P_j &= \sum_{i,j} R_{ij} P_i P_j = \sum_{i<j} R_{ij}[P_i, P_j] \\ &= \sum_{i<j} \sum_k R_{ij} Q_{ijk} P_k.\end{aligned}$$

Set $S_k = \sum_{i<j} R_{ij} Q_{ijk}$. Then S_k has order $\leq l - m_k - 1$, $\sum_j (A_j - S_j) P_j = 0$ and $\sigma_l(A_j - S_j) = a_j$. ☐

1.4 De Rham and Spencer complexes

If A is a ring, M is an A-module, and $\varphi := (\varphi_1, \dots, \varphi_n)$ are n-commuting endomorphisms of M, one can define the Koszul complex $K^\bullet(M;\varphi)$ and the co-Koszul complex $K_\bullet(M;\varphi)$. We refer to [Sc08] for an exposition.

Also recall the De Rham complex

$$\mathrm{DR}_X(\mathscr{O}_X) := 0 \to \Omega_X^0 \xrightarrow{d} \Omega_X^1 \xrightarrow{d} \cdots \to \Omega_X^{d_X} \to 0, \tag{1.4.1}$$

where d is the differential.

Let $\mathscr{M}$ be a left $\mathscr{D}_X$-module. One defines the differential $d\colon \mathscr{M} \to \Omega_X^1 \otimes_{\mathscr{O}} \mathscr{M}$ as follows. In a local coordinate system $(x_1, \dots, x_{d_X})$ on X, the differential d is given by

$$\mathscr{M} \to \Omega_X^1 \otimes_{\mathscr{O}} \mathscr{M}, \quad m \mapsto \sum_i dx_i \otimes \partial_i m$$

and one checks easily that this does not depend on the choice of the local coordinate system.

One defines the De Rham complex of $\mathscr{M}$, denoted $\mathrm{DR}_X(\mathscr{M})$, as the complex

$$\mathrm{DR}_X(\mathscr{M}) := \quad 0 \to \Omega_X^0 \otimes_{\mathscr{O}} \mathscr{M} \xrightarrow{d} \cdots \to \Omega_X^{d_X} \otimes_{\mathscr{O}} \mathscr{M} \to 0, \tag{1.4.2}$$

where $\Omega_X^0 \otimes_{\mathscr{O}} \mathscr{M}$ is in degree 0 and the differential d is characterized by:

$$d(\omega \otimes m) = d\omega \otimes m + (-)^p \omega \wedge dm, \quad \omega \in \Omega_X^p, m \in \mathscr{M}.$$

Note that $\mathrm{DR}_X(\mathscr{D}_X) \in \mathrm{C^b}(\mathrm{Mod}(\mathscr{D}_X^{\mathrm{op}}))$, the category of bounded complexes of right $\mathscr{D}_X$-modules, and

$$\mathrm{DR}_X(\mathscr{M}) \simeq \mathrm{DR}_X(\mathscr{D}_X) \otimes_{\mathscr{D}} \mathscr{M}. \tag{1.4.3}$$

Recall that there is a natural right $\mathscr{D}$-linear morphism $\Omega_X \otimes_{\mathscr{O}} \mathscr{D}_X \to \Omega_X$. Moreover, one checks easily that the composition

$$\Omega_X^{d_X - 1} \otimes_{\mathscr{O}} \mathscr{D}_X \to \Omega_X^{d_X} \otimes_{\mathscr{O}} \mathscr{D}_X \to \Omega_X$$

is zero. Hence, we get a morphism in the derived category $\mathrm{D}^b(\mathscr{D}_X^{\mathrm{op}})$

$$\mathrm{DR}_X(\mathscr{D}_X) \to \Omega_X[-d_X]. \tag{1.4.4}$$

Proposition 1.4.1. *The morphism (1.4.4) induces an isomorphism in* $\mathrm{D}^b(\mathscr{D}_X^{\mathrm{op}})$.

Proof. Since the morphism is well defined on X, we may argue locally and choose a local coordinate system. In this case, there is an isomorphism of complexes

$$\mathrm{DR}_X(\mathscr{D}_X) \simeq K^\bullet(\mathscr{D}_X; \partial_1 \cdot, \dots, \partial_{d_X} \cdot) \tag{1.4.5}$$

where the right hand side is the Koszul complex of the the sequence $\partial_1 \cdot, \dots, \partial_n \cdot$ acting on the left on $\mathscr{D}_X$. Since this sequence is clearly regular, the result follows. □

Applying Proposition 1.4.1 and isomorphism (1.4.3), we get:

Corollary 1.4.2. *Let $\mathscr{M}$ be a left $\mathscr{D}_X$-module. Then*

$$\mathrm{DR}_X(\mathscr{M}) \simeq \Omega_X \overset{\mathrm{L}}{\otimes}_{\mathscr{D}} \mathscr{M}\,[-d_X].$$

Let us apply the contravariant functor $\mathscr{H}om_{\mathscr{D}^{\mathrm{op}}}(\,\bullet\,, \mathscr{D}_X)$ to the complex $\mathrm{DR}_X(\mathscr{D}_X)$. One sets

$$\mathrm{SP}_{\mathrm{X}}(\mathscr{D}_{\mathrm{X}}) := \mathscr{H}om_{\mathscr{D}}(\mathrm{DR}_X(\mathscr{D}_X), \mathscr{D}_X), \tag{1.4.6}$$

and calls $\mathrm{SP}_{\mathrm{X}}(\mathscr{D}_{\mathrm{X}})$ the Spencer complex.

$$\mathrm{SP}_{\mathrm{X}}(\mathscr{D}_{\mathrm{X}}) := \quad 0 \to \mathscr{D}_{\mathrm{X}} \otimes_{\mathscr{O}} \bigwedge^{\mathrm{d_X}} \Theta_{\mathrm{X}} \xrightarrow{\mathrm{d}} \cdots \to \mathscr{D}_{\mathrm{X}} \otimes_{\mathscr{O}} \Theta_{\mathrm{X}} \to \mathscr{D}_{\mathrm{X}} \to 0, \tag{1.4.7}$$

One deduces from (1.4.5) the isomorphism of complexes

$$\mathrm{SP}_{\mathrm{X}}(\mathscr{D}_{\mathrm{X}}) \simeq \mathrm{K}_\bullet(\mathscr{D}_{\mathrm{X}}; \cdot\partial_1, \dots, \cdot\partial_{\mathrm{d_X}}) \tag{1.4.8}$$

where the right hand side is the co-Koszul complex of the sequence $\cdot\partial_1, \dots, \cdot\partial_{d_X}$ acting on the right on $\mathscr{D}_X$. Since this sequence is clearly regular, we obtain:

Proposition 1.4.3. *The left $\mathscr{D}$-linear morphism $\mathscr{D}_X \to \mathscr{O}_X$ induces an isomorphism $\mathrm{SP}_{\mathrm{X}}(\mathscr{D}_{\mathrm{X}}) \xrightarrow{\sim} \mathscr{O}_{\mathrm{X}}$ in $\mathrm{D^b}(\mathscr{D}_X)$.*

Corollary 1.4.4. *Let $\mathscr{M}$ be a left $\mathscr{D}_X$-module. There is an isomorphism in $\mathrm{D}^b(\mathbb{C}_X)$*

$$\mathrm{R}\mathscr{H}om_{\mathscr{D}}(\mathscr{O}_X, \mathscr{M}) \simeq \mathrm{DR}_X(\mathscr{M}).$$

Proof. Since $\mathrm{SP}_{\mathrm{X}}(\mathscr{D}_{\mathrm{X}})$ is a complex of locally free $\mathscr{D}_X$-modules of finite rank, one has

$$\begin{aligned}
\mathrm{R}\mathscr{H}om_{\mathscr{D}}(\mathscr{O}_X, \mathscr{M}) &\simeq \mathscr{H}om_{\mathscr{D}}(\mathrm{SP}_{\mathrm{X}}(\mathscr{D}_{\mathrm{X}}), \mathscr{M}) \\
&\simeq \mathscr{H}om_{\mathscr{D}}(\mathrm{SP}_{\mathrm{X}}(\mathscr{D}_{\mathrm{X}}), \mathscr{D}_{\mathrm{X}}) \otimes_{\mathscr{D}} \mathscr{M} \\
&\simeq \mathrm{DR}_X(\mathscr{D}_X) \otimes_{\mathscr{D}} \mathscr{M} \\
&\simeq \mathrm{DR}_X(\mathscr{M}).
\end{aligned}$$

□

Proposition 1.4.5. *One has the isomorphism*

$$\begin{aligned}
&\mathrm{R}\mathscr{H}om_{\mathscr{D}}(\mathscr{O}_X, \mathscr{D}_X)[d_X] \simeq \Omega_X \\
&\mathrm{R}\mathscr{H}om_{\mathscr{D}^{\mathrm{op}}}(\Omega_X, \mathscr{D}_X)[d_X] \simeq \mathscr{O}_X \\
&\mathrm{R}\mathscr{H}om_{\mathscr{D}}(\mathscr{O}_X, \mathscr{O}_X) \simeq \mathbb{C}_X.
\end{aligned}$$

Proof. (i) One has the chain of isomorphisms

$$\begin{aligned}
\mathrm{R}\mathscr{H}om_{\mathscr{D}}(\mathscr{O}_X, \mathscr{D}_X)[d_X] &\simeq \mathrm{R}\mathscr{H}om_{\mathscr{D}}(\mathrm{SP}_{\mathrm{X}}(\mathscr{D}_{\mathrm{X}}), \mathscr{D}_{\mathrm{X}})[-\mathrm{d_X}] \\
&\simeq \mathscr{H}om_{\mathscr{D}}(\mathrm{SP}_{\mathrm{X}}(\mathscr{D}_{\mathrm{X}}), \mathscr{D}_{\mathrm{X}})[-\mathrm{d_X}] \\
&\simeq \mathrm{DR}(\mathscr{D}_X)[-d_X] \simeq \Omega_X.
\end{aligned}$$

(ii) The proof is similar.
(iii) The canonical morphism $\mathbb{C}_X \to \mathscr{H}om_{\mathscr{D}}(\mathscr{O}_X, \mathscr{O}_X)$ induces the morphism

$$\begin{aligned}
\mathbb{C}_X &\to \mathrm{R}\mathscr{H}om_{\mathscr{D}}(\mathscr{O}_X, \mathscr{O}_X) \\
&\simeq \mathscr{H}om_{\mathscr{D}}(\mathrm{SP}_{\mathrm{X}}(\mathscr{D}_{\mathrm{X}}), \mathscr{O}_{\mathrm{X}}) \\
&\simeq \Omega_X^\bullet.
\end{aligned}$$

The isomorphism $\mathbb{C}_X \xrightarrow{\sim} \Omega_X^\bullet$ is the classical Poincaré lemma. □

1.5 Homological properties of $\mathscr{D}_X$

Vanishing theorems and dimension

There is a corresponding theorem to Theorem 3.5.6 for $\mathscr{D}$-modules.

Theorem 1.5.1. *Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module. Then*

(i) $\mathscr{E}xt^k_{\mathscr{D}}(\mathscr{M}, \mathscr{D}_X)$ *is coherent for all k and is 0 for $k < \operatorname{codim}(\operatorname{char}(\mathscr{M}))$,*

(ii) $\operatorname{codim}(\operatorname{char}(\mathscr{E}xt^k_{\mathscr{D}}(\mathscr{M}, \mathscr{D}_X))) \geq k$,

(iii) $\operatorname{char}(\mathscr{E}xt^k_{\mathscr{D}}(\mathscr{M}, \mathscr{D}_X)) \subset \operatorname{char}(\mathscr{M})$,

(iv) $\mathscr{E}xt^k_{\mathscr{D}}(\mathscr{M}, \mathscr{D}_X) = 0$ *for $k > d_X$.*

Corollary 1.5.2. *Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module. Then the support of $\mathscr{E}xt^{d_X}_{\mathscr{D}}(\mathscr{M}, \mathscr{D}_X)$ has pure dimension d_X.*

Proof. First we construct by induction a finite free filtered resolution of $\mathrm{Fl}\mathscr{M}$, that is, a filtered exact sequence of $\mathrm{Fl}\mathscr{D}_X$-modules

$$\cdots \to \mathrm{Fl}\mathscr{L}_1 \to \mathrm{Fl}\mathscr{L}_0 \to \mathrm{Fl}\mathscr{M} \to 0$$

where the $\mathrm{Fl}\mathscr{L}_j$'s are filtered finite free. We denote by d^j the differential. Set:

$$\mathrm{Fl}\mathscr{L}_\bullet := \cdots \to \mathrm{Fl}\mathscr{L}_1 \to \mathrm{Fl}\mathscr{L}_0 \to 0,$$
$$\mathrm{Gr}\mathscr{L}_\bullet := \cdots \to \mathrm{Gr}\mathscr{L}_1 \to \mathrm{Gr}\mathscr{L}_0 \to 0.$$

Then

$$\cdots \to \mathrm{Gr}\mathscr{L}_1 \to \mathrm{Gr}\mathscr{L}_0 \to \mathrm{Gr}\mathscr{M} \to 0$$

is exact. Put

$$\mathscr{L}_j^* = \mathscr{H}om_{\mathscr{D}}(\mathscr{L}_j, \mathscr{D}_X),$$
$$\mathscr{L}_\bullet^* = \mathscr{H}om_{\mathscr{D}}(\mathscr{L}_\bullet, \mathscr{D}_X) = 0 \to \mathscr{L}_0^* \to \mathscr{L}_1^* \to \cdots$$

One defines a filtration $\mathrm{Fl}\mathscr{L}_j^*$ on $\mathscr{L}_j^*$ by setting

$$\mathrm{Fl}_m\mathscr{L}_j^* = \{\varphi \in \mathscr{H}om_{\mathscr{D}}(\mathscr{L}_j, \mathscr{D}_X); \varphi(\mathrm{Fl}_k\mathscr{L}_j) \subset \mathrm{Fl}_{k+m}\mathscr{D}_X \text{ for all } k\}.$$

Clearly, this filtration on $\mathscr{L}_j^*$ is good and moreover $\mathscr{H}om_{\mathrm{Gr}\mathscr{D}}(\mathrm{Gr}\mathscr{L}_j, \mathrm{Gr}\mathscr{D}) \simeq \mathrm{Gr}\mathscr{L}_j^*$. In other words,

$$\mathscr{H}om_{\mathrm{Gr}\mathscr{D}}(\mathrm{Gr}\mathscr{L}_\bullet, \mathrm{Gr}\mathscr{D}) \simeq \mathrm{Gr}\mathscr{L}_\bullet^*.$$

Put

$$\mathscr{Z}^k = \ker(\mathscr{L}_k \xrightarrow{d^k} \mathscr{L}_{k+1}), \quad \mathscr{I}^k = \mathrm{Im}(\mathscr{L}_{k-1} \to \mathscr{L}_k) \quad H^k(\mathscr{L}_\bullet^*) = \mathscr{Z}^k/\mathscr{I}^k.$$

We endow $\mathscr{Z}^k$ with the induced filtration and $H^k(\mathscr{L}_\bullet^*)$ with the filtration image of $\mathrm{Fl}\mathscr{Z}^k$. Since $\mathscr{E}xt^k_{\mathscr{D}}(\mathscr{M}, \mathscr{D}_X) \simeq H^k(\mathscr{L}_\bullet^*)$, we get a filtration $\mathrm{Fl}\mathscr{E}xt^k_{\mathscr{D}}(\mathscr{M}, \mathscr{D}_X)$ on this module. Moreover $\mathscr{E}xt^k_{\mathrm{Gr}\mathscr{D}}(\mathrm{Gr}\mathscr{M}, \mathrm{Gr}\mathscr{D}_X)) \simeq H^k(\mathrm{Gr}\mathscr{L}_\bullet^*)$.

In order to complete the proof, we need a lemma. ∎

Lemma 1.5.3. $\mathrm{Gr}H^k(\mathscr{L}^*_\bullet)$ *is a subquotient of* $H^k(\mathrm{Gr}\mathscr{L}^*_\bullet)$.

Proof of Lemma 1.5.3.

$$H^k(\mathrm{Gr}_m\mathscr{L}^*_\bullet) = \frac{\mathrm{Fl}_m(\mathscr{L}^*_k) \cap (d^k)^{-1}\mathrm{Fl}_{m-1}\mathscr{L}^*_{k+1}}{\mathrm{Fl}_{m-1}(\mathscr{L}^*_k) + d^{k-1}\mathrm{Fl}_m\mathscr{L}^*_{k-1}} \supset \frac{\mathrm{Fl}_m(\mathscr{Z}^k)}{\mathrm{Fl}_{m-1}(\mathscr{Z}_k) + d^{k-1}\mathrm{Fl}_m\mathscr{L}^*_{k-1}}.$$

On the other-hand,

$$\mathrm{Gr}_m H^k(\mathscr{L}^*_\bullet) = \frac{\mathrm{Fl}_m(\mathscr{Z}^k)}{\mathrm{Fl}_{m-1}(\mathscr{Z}_k) + \mathscr{I}^k \cap \mathrm{Fl}_m(\mathscr{Z}_k)}.$$

The result then follows from

$$\mathrm{Fl}_{m-1}(\mathscr{Z}_k) + d^{k-1}\mathrm{Fl}_m\mathscr{L}^*_{k-1} \subset \mathrm{Fl}_{m-1}(\mathscr{Z}_k) + \mathscr{I}^k \cap \mathrm{Fl}_m(\mathscr{Z}_k).$$

□

End of proof of Theorem 1.5.1. It follows that

(1.5.1) $$\mathrm{char}(\mathscr{E}xt^k_{\mathscr{D}}(\mathscr{M}, \mathscr{D}_X)) \subset \mathrm{supp}(\mathscr{E}xt^k_{\mathrm{Gr}\mathscr{D}}(\mathrm{Gr}\mathscr{M}, \mathrm{Gr}\mathscr{D}_X))).$$

(i) By Theorem 3.5.6, $\mathscr{E}xt^k_{\mathscr{O}}(\widetilde{\mathrm{Gr}\mathscr{M}}, \mathscr{O}_{T^*X})) = 0$ for $k < \mathrm{codim}(\mathrm{char}(\mathscr{M}))$. By (1.5.1), we get that $\mathscr{E}xt^k_{\mathscr{D}}(\mathscr{M}, \mathscr{D}_X) = 0$ for $k < \mathrm{codim}(\mathrm{char}(\mathscr{M}))$.

(ii) By Theorem 3.5.6, $\mathrm{codim}(\mathrm{supp}(\mathscr{E}xt^k_{\mathrm{Gr}\mathscr{D}}(\mathrm{Gr}\mathscr{M}, \mathrm{Gr}\mathscr{D}_X))) \geq k$. By (1.5.1), we get that $\mathrm{codim}(\mathrm{char}(\mathscr{E}xt^k_{\mathscr{D}}(\mathscr{M}, \mathscr{D}_X))) \geq k$.

(iii) follows from the inclusion

$$\mathrm{supp}(\mathscr{E}xt^k_{\mathrm{Gr}\mathscr{D}}(\mathrm{Gr}\mathscr{M}, \mathrm{Gr}\mathscr{D}_X)) \subset \mathrm{supp}(\mathrm{Gr}\mathscr{M}).$$

(iv) follows from (ii) and the involutivity of the characteristic variety of $\mathscr{E}xt^k_{\mathscr{D}}(\mathscr{M}, \mathscr{D}_X)$. □

Example 1.5.4. Let $d_X = 1$. Then any coherent ideal $\mathscr{I}$ of $\mathscr{D}_X$ is projective since $\mathscr{E}xt^j_{\mathscr{D}}(\mathscr{D}_X/\mathscr{I}, \mathscr{D}_X) = 0$ for $j > 1$.

Let t denote a local holomorphic coordinate. The left ideal of $\mathscr{D}_X$ generated by t^2 and $t\partial_t - 1$ is projective. By Theorem 1.3.4, its characteristic is T^*X. Since it is contained in $\mathscr{D}_X$, its multiplicity on T^*X is 1. This module does not admits a single generator, and it follows that it is not free.

Free resolutions

Theorem 1.5.5. *Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module. Then, locally on X, $\mathscr{M}$ admits a finite free resolution of length $\leq d_X$. In other words, there locally exists an exact sequence*

$$0 \to \mathscr{L}^{d_X} \to \cdots \to \mathscr{L}^0 \to \mathscr{M} \to 0,$$

where the $\mathscr{L}^i$'s are free of finite rank over $\mathscr{D}_X$.

Proof. Set $n = d_X$. Since we argue locally, we may endow $\mathscr{M}$ with a good filtration $\mathrm{Fl}\mathscr{M}$. We may locally find a finite free filtered resolution

$$\cdots \to \mathrm{Fl}\mathscr{L}^n \to \cdots \to \mathrm{Fl}\mathscr{L}^0 \to \mathrm{Fl}\mathscr{M} \to 0.$$

On the other-hand, we know that $\mathscr{E}xt^j_{\mathrm{Gr}\mathscr{D}}(\mathrm{Gr}\mathscr{M}, \mathrm{Gr}\mathscr{D}_X) = 0$ for $j > n$. Set $\mathscr{K}^n = \ker(\mathscr{L}^{n-1} \to \mathscr{L}^{n-2})$ and let us endow $\mathscr{K}_n$ with the induced filtration. Then the sequence

$$0 \to \mathrm{Gr}\mathscr{K}^n \to \mathrm{Gr}\mathscr{L}^{n-1} \to \cdots \to \mathrm{Gr}\mathscr{L}^0 \to \mathrm{Gr}\mathscr{M} \to 0$$

is exact and it follows that $\mathrm{Gr}\mathscr{K}^n$ is projective. Since projective modules over $\mathrm{Gr}\mathscr{D}_X$ are stably free, there exists a finite free $\mathscr{D}_X$ module $\mathscr{L}$ such that $\mathrm{Gr}\mathscr{K}^n \oplus \mathrm{Gr}\mathscr{L}$ is free and this implies that $\mathscr{K}^n \oplus \mathscr{L}$ is a free $\mathscr{D}_X$-module. The sequence

$$0 \to \mathscr{K}^n \oplus \mathscr{L} \to \mathscr{L}^{n-1} \oplus \mathscr{L} \to \cdots \to \mathscr{L}^0 \to \mathscr{M} \to 0$$

is a finite free resolution of $\mathscr{M}$. □

Homological dimension

Let R be a ring. Recall that the global homological dimension of R, $\mathrm{gld}(R)$, is the biggest $d \in \mathbb{N} \cup \{\infty\}$ such that there exist left R-modules M and N with $\mathrm{Ext}^d_R(M, N) \neq 0$.

For a sheaf of rings $\mathscr{R}$ on a topological space X, the global homological dimension of R, $\mathrm{gld}(R)$, is the biggest $d \in \mathbb{N} \cup \{\infty\}$ such that there exist sheaves of $\mathscr{R}$-modules $\mathscr{M}$ and $\mathscr{N}$ with $\mathrm{Ext}^d_{\mathscr{R}}(\mathscr{N}, \mathscr{M}) \neq 0$.

The weak global homological dimension of R, $\mathrm{wgld}(R)$, also called the Tor-dimension of R, is the biggest $d \in \mathbb{N} \cup \{\infty\}$ such that there exists a right R-module N and a left R-module M with $\mathscr{T}or[R]d(N, M) \neq 0$.

For a sheaf of rings $\mathscr{R}$, $\mathrm{wgld}(\mathscr{R})$ is the maximum of $\mathrm{wgld}(\mathscr{R}_x)$, for $x \in X$.

Lemma 1.5.6. (i) *The $\mathscr{O}_X$-module $\mathscr{D}_X$ is flat.*

(ii) *If a $\mathscr{D}_X$-module $\mathscr{I}$ is injective in the category* $\mathrm{Mod}(\mathscr{D}_X)$*, then it is injective in the category* $\mathrm{Mod}(\mathscr{O}_X)$*.*

Proof. (i) Locally, $\mathscr{D}_X$ is isomorphic to ${\mathscr{O}_X}^{(\mathbb{N})}$.

(ii) follows from (i). Indeed, if $\mathscr{N}$ is a $\mathscr{D}_X$-module, then

$$\mathscr{H}om_{\mathscr{O}}(\mathscr{N}, \mathscr{I}) \simeq \mathrm{Hom}_{\mathscr{D}}(\mathscr{D}_X \otimes_{\mathscr{O}} \mathscr{N}, \mathscr{I}).$$

□

Recall that if $\mathscr{M}$ and $\mathscr{N}$ are two left $\mathscr{D}_X$-modules, $\mathscr{H}om_{\mathscr{O}}(\mathscr{M}, \mathscr{N})$ has a natural structure of a left $\mathscr{D}_X$-modules. By Lemma 1.5.6 we get that the natural forgetful functor $D^b(\mathscr{D}_X) \to D^b(\mathscr{O}_X)$ commutes with $\mathrm{R}\mathscr{H}om_{\mathscr{O}}$.

Lemma 1.5.7. *Let $\mathscr{M}, \mathscr{N} \in \mathrm{Mod}(\mathscr{D}_X)$. Then*

$$\mathrm{R}\mathscr{H}om_{\mathscr{D}}(\mathscr{M}, \mathscr{N}) \simeq \mathrm{R}\mathscr{H}om_{\mathscr{D}}(\mathscr{O}_X, \mathrm{R}\mathscr{H}om_{\mathscr{O}}(\mathscr{M}, \mathscr{N})).$$

Proof. Since this formula is true when replacing $\mathrm{R}\mathscr{H}om$ with $\mathscr{H}om$, it is enough to show that if $\mathscr{N}$ is an injective $\mathscr{D}_X$-module, then

$$H^j(\mathrm{R}\mathscr{H}om_{\mathscr{D}}(\mathscr{O}_X, \mathscr{H}om_{\mathscr{O}}(\mathscr{M}, \mathscr{N}))) = 0 \text{ for } j > 0.$$

Choose a finite free $\mathscr{D}_X$-resolution $\mathscr{L}^\bullet$ of $\mathscr{O}_X$ (for example, take $\mathscr{L}^\bullet = \mathrm{SP_X}(\mathscr{D}_\mathrm{X})$). Notice that $\mathscr{L}^\bullet \otimes_{\mathscr{O}} \mathscr{M} \to \mathscr{M}$ is a quasi-isomorphism of left $\mathscr{D}_X$-modules. Using the fact that $\mathscr{N}$ is $\mathscr{O}_X$ and $\mathscr{D}_X$-injective, we get:

$$\begin{aligned}\mathrm{R}\mathscr{H}om_{\mathscr{D}}(\mathscr{O}_X, \mathscr{H}om_{\mathscr{O}}(\mathscr{M}, \mathscr{N})) &\simeq \mathrm{R}\mathscr{H}om_{\mathscr{D}}(\mathscr{O}_X, \mathrm{R}\mathscr{H}om_{\mathscr{O}}(\mathscr{M}, \mathscr{N}))\\ &\simeq \mathrm{R}\mathscr{H}om_{\mathscr{D}}(\mathscr{L}^\bullet, \mathscr{H}om_{\mathscr{O}}(\mathscr{M}, \mathscr{N}))\\ &\simeq \mathrm{R}\mathscr{H}om_{\mathscr{D}}(\mathscr{L}^\bullet \otimes_{\mathscr{O}} \mathscr{M}, \mathscr{N})\\ &\simeq \mathrm{R}\mathscr{H}om_{\mathscr{D}}(\mathscr{M}, \mathscr{N}) \simeq \mathscr{H}om_{\mathscr{D}}(\mathscr{M}, \mathscr{N}).\end{aligned}$$

□

Theorem 1.5.8. *Let $x \in X$. The global homological dimension $\mathrm{gld}(\mathscr{D}_{X,x})$ is d_X. In other words, the conditions* (i)–(ii) *below are satisfied:*

(i) *let M and N be two $\mathscr{D}_{X,x}$-modules. Then $\mathrm{Ext}^j_{\mathscr{D}_{X,x}}(M, N) = 0$ for $j > d_X$,*

(ii) *there exist two $\mathscr{D}_{X,x}$-modules M and N such that $\mathrm{Ext}^j_{\mathscr{D}_{X,x}}(M, N) \neq 0$, with $j = d_X$.*

Proof. (i) By classical results (see [We94, Th. 4.1.2]), it is enough to prove the result when assuming that M is finitely generated. Since $\mathscr{D}_{X,x}$ is noetherian, there exists a coherent $\mathscr{D}_X$ module $\mathscr{M}$ defined in a neighborhood of x such that $M = \mathscr{M}_x$. Then the result follows from Theorem 1.5.5 in this case.

(ii) Choose $M = \mathscr{O}_{X,x}$ and $N = \mathscr{D}_{X,x}$. □

Theorem 1.5.9. *The weak global dimension $\mathrm{wgld}(\mathscr{D}_{X,x})$ of $\mathscr{D}_X$ is equal to d_X. In other words, the conditions* (i)–(ii) *below are satisfied:*

(i) *for any left (resp. right) $\mathscr{D}_X$-module $\mathscr{M}$ (resp. $\mathscr{N}$), one has $Tor^{\mathscr{D}}_j(\mathscr{N}, \mathscr{M}) = 0$ for $j > d_X$,*

(ii) *there exist a left $\mathscr{D}_X$-module $\mathscr{M}$ and a right $\mathscr{D}_X$-module $\mathscr{N}$, such that $\mathscr{T}or[\mathscr{D}]d_X(\mathscr{N}, \mathscr{M}) \neq 0$.*

Proof. (i) It is well known that if R is a ring, $\mathrm{wgld}(R)$ is less or equal to $\mathrm{gld}(R)$ (see [We94, Ch. 4]). Therefore, $\mathrm{wgld}(\mathscr{D}_X)$ is bounded by $\mathrm{gld}(\mathscr{D}_{X,x})$, that is, by d_X.

(ii) Choose $\mathscr{N} = \Omega_X$ and $\mathscr{M} = \mathscr{O}_X$. □

Theorem 1.5.10. *The global dimension of $\mathscr{D}_X$ is $2d_X + 1$. In other words, the conditions* (i)–(ii) *below are satisfied:*

(i) *let $\mathscr{M}$ and $\mathscr{N}$ be two $\mathscr{D}_X$-modules. Then $\mathrm{Ext}^j_{\mathscr{D}}(\mathscr{M}, \mathscr{N}) = 0$ for $j > 2d_X + 1$,*

(ii) *there exist two $\mathscr{D}_X$-modules $\mathscr{M}$ and $\mathscr{N}$ such that $\mathrm{Ext}^{2d_X+1}_{\mathscr{D}}(\mathscr{M}, \mathscr{N}) \neq 0$.*

Proof. Let $n = \dim X$.

(i) By Lemma 1.5.7 one has

$$\mathrm{RHom}_{\mathscr{D}}(\mathscr{M}, \mathscr{N}) \simeq \mathrm{RHom}_{\mathscr{D}}(\mathscr{O}_X, \mathrm{R}\mathscr{H}om_{\mathscr{O}}(\mathscr{M}, \mathscr{N})).$$

Let $\mathrm{SP_X}(\mathscr{D}_\mathrm{X})$ be the Spencer complex of $\mathscr{D}_X$. This complex has length n, is locally free and is qis to $\mathscr{O}_X$.

On the other hand, consider a resolution in the category $\mathrm{Mod}(\mathscr{D}_X)$:

$$0 \to \mathscr{N}^{n+1} \to \mathscr{N}^n \to \cdots \to \mathscr{N}^0 \to \mathscr{N} \to 0$$

such that $\mathscr{N}^0, \dots, \mathscr{N}^n$ are $\mathscr{D}_X$-injective. Then these modules will be $\mathscr{O}_X$-injective and it follows from Theorem 3.5.7 that $\mathscr{N}^{n+1}$ is $\mathscr{O}_X$-injective. Set $\mathscr{L}^i = \mathscr{H}om_{\mathscr{O}_X}(\mathscr{M}, \mathscr{N}^i)$. This is a left $\mathscr{D}_X$-module, and a flabby sheaf. Consider the complex

$$\mathscr{L}^\bullet := 0 \to \mathscr{L}^0 \to \cdots \to \mathscr{L}^{n+1} \to 0.$$

Then $\mathrm{R}\mathscr{H}om_{\mathscr{D}}(\mathscr{M}, \mathscr{N})$ is represented by the complex $\mathscr{H}om_{\mathscr{D}}(\mathrm{SP_X}(\mathscr{D}_\mathrm{X}), \mathscr{L}^\bullet)$. This complex has length $2n+1$ and its components are flabby sheaves. Therefore

$$\mathrm{RHom}_{\mathscr{D}}(\mathscr{M}, \mathscr{N}) \simeq \mathrm{R}\Gamma(X; \mathscr{H}om_{\mathscr{D}}(\mathrm{SP_X}(\mathscr{D}_\mathrm{X}), \mathscr{L}^\bullet))$$

is concentrated in degree $[0, 2n+1]$.

(ii) Let $x \in X$. One has

$$\mathrm{Ext}^j_{\mathscr{D}}(\mathscr{O}_{X,x}, \mathscr{D}_X^{(\mathbb{N})}) \neq 0 \text{ for } j = 2n+1.$$

Indeed, $\mathrm{R}\mathscr{H}om_{\mathscr{D}}(\mathscr{O}_{X,x}, \mathscr{D}_X) \simeq \Omega_X[-n]$, we get

$$\mathrm{Ext}^{j+n}_{\mathscr{D}}(\mathscr{O}_{X,x}, (\mathscr{D}_X)^{(\mathbb{N})}) \simeq H^j(\mathrm{R}\Gamma_{\{x\}}(X; \Omega_X^{(\mathbb{N})})).$$

Then the result follows from Proposition 3.5.8. □

1.6 Derived category and duality

Recall that $\mathrm{Mod}(\mathscr{D}_X)$ is a Grothendieck category (see for example [KS06, Th. 18.1.6]) and thus has enough injectives. One denotes by $\mathrm{Mod_c}(\mathscr{D}_X)$ the thick abelian subcategory of $\mathrm{Mod}(\mathscr{D}_X)$ consisting of coherent modules and by $\mathrm{D^b_{coh}}(\mathscr{D}_X)$ the full triangulated category of the bounded derived category $\mathrm{D^b}(\mathscr{D}_X)$ consisting of objects with coherent cohomology.

If $\mathscr{M} \in \mathrm{D^b_{coh}}(\mathscr{D}_X)$, we set

$$\mathrm{char}(\mathscr{M}) = \bigcup_j \mathrm{char}(H^j(\mathscr{M})). \tag{1.6.1}$$

Internal operations

We denote by $\mathrm{R}\mathscr{H}om_{\mathscr{O}}$ the right derived functor of $\mathscr{H}om_{\mathscr{O}}$ and by $\overset{\mathrm{L}}{\underline{\otimes}}$ the left derived functor of $\otimes_{\mathscr{O}}$ acting on $\mathscr{D}$-modules. Hence, we get the functors

$$\begin{aligned}
\bullet\overset{\mathrm{L}}{\underline{\otimes}}\bullet &: \mathrm{D}^{\mathrm{b}}(\mathscr{D}_X)\times\mathrm{D}^{\mathrm{b}}(\mathscr{D}_X)\to\mathrm{D}^{\mathrm{b}}(\mathscr{D}_X),\\
\bullet\overset{\mathrm{L}}{\underline{\otimes}}\bullet &: \mathrm{D}^{\mathrm{b}}(\mathscr{D}_X^{\mathrm{op}})\times\mathrm{D}^{\mathrm{b}}(\mathscr{D}_X)\to\mathrm{D}^{\mathrm{b}}(\mathscr{D}_X^{\mathrm{op}}),\\
\mathrm{R}\mathscr{H}om_{\mathscr{O}}(\bullet,\bullet) &: \mathrm{D}^{\mathrm{b}}(\mathscr{D}_X)^{\mathrm{op}}\times\mathrm{D}^{\mathrm{b}}(\mathscr{D}_X)\to\mathrm{D}^{\mathrm{b}}(\mathscr{D}_X),\\
\mathrm{R}\mathscr{H}om_{\mathscr{O}}(\bullet,\bullet) &: \mathrm{D}^{\mathrm{b}}(\mathscr{D}_X^{\mathrm{op}})^{\mathrm{op}}\times\mathrm{D}^{\mathrm{b}}(\mathscr{D}_X^{\mathrm{op}})\to\mathrm{D}^{\mathrm{b}}(\mathscr{D}_X).
\end{aligned}$$

The tensor product is commutative and associative, that is, for $\mathscr{L},\mathscr{M},\mathscr{N}$ in $\mathrm{D}^{\mathrm{b}}(\mathscr{D}_X)$ there are natural isomorphisms $\mathscr{M}\overset{\mathrm{L}}{\underline{\otimes}}\mathscr{N}\simeq\mathscr{N}\overset{\mathrm{L}}{\underline{\otimes}}\mathscr{M}$ and $(\mathscr{M}\overset{\mathrm{L}}{\underline{\otimes}}\mathscr{N})\overset{\mathrm{L}}{\underline{\otimes}}\mathscr{L}\simeq\mathscr{M}\overset{\mathrm{L}}{\underline{\otimes}}(\mathscr{N}\overset{\mathrm{L}}{\underline{\otimes}}\mathscr{L})$. Moreover $\mathscr{O}_X\overset{\mathrm{L}}{\underline{\otimes}}\mathscr{M}\simeq\mathscr{M}$.

There are also natural functors

$$\begin{aligned}
\mathrm{R}\mathscr{H}om_{\mathscr{D}}(\bullet,\bullet) &: \mathrm{D}^{\mathrm{b}}(\mathscr{D}_X)^{\mathrm{op}}\times\mathrm{D}^{\mathrm{b}}(\mathscr{D}_X)\to\mathrm{D}^{\mathrm{b}}(\mathbb{C}_X),\\
\bullet\overset{\mathrm{L}}{\otimes}_{\mathscr{D}}\bullet &: \mathrm{D}^{\mathrm{b}}(\mathscr{D}_X^{\mathrm{op}})\times\mathrm{D}^{\mathrm{b}}(\mathscr{D}_X)\to\mathrm{D}^{\mathrm{b}}(\mathbb{C}_X).
\end{aligned}$$

These functors are related by the formulas (1.6.2) and (1.6.3) below.

Proposition 1.6.1. *For $\mathscr{L},\mathscr{M},\mathscr{N}$ in $\mathrm{D}^{\mathrm{b}}(\mathscr{D}_X)$ and $\mathscr{K}$ in $\mathrm{D}^{\mathrm{b}}(\mathscr{D}_X^{\mathrm{op}})$ there are natural isomorphisms*

$$\mathscr{K}\overset{\mathrm{L}}{\otimes}_{\mathscr{D}}(\mathscr{M}\overset{\mathrm{L}}{\underline{\otimes}}\mathscr{N})\simeq(\mathscr{K}\overset{\mathrm{L}}{\underline{\otimes}}\mathscr{M})\overset{\mathrm{L}}{\otimes}_{\mathscr{D}}\mathscr{N},\tag{1.6.2}$$

$$\mathrm{R}\mathscr{H}om_{\mathscr{D}}(\mathscr{L},\mathrm{R}\mathscr{H}om_{\mathscr{O}}(\mathscr{M},\mathscr{N}))\simeq\mathrm{R}\mathscr{H}om_{\mathscr{D}}(\mathscr{L}\overset{\mathrm{L}}{\underline{\otimes}}\mathscr{M},\mathscr{N}).\tag{1.6.3}$$

Duality

We define the duality functors on $\mathrm{D}^{\mathrm{b}}(\mathscr{D}_X)$ or $\mathrm{D}^{\mathrm{b}}(\mathscr{D}_X^{\mathrm{op}})$, all denoted by $\mathbb{D}'_{\mathscr{D}}$ and $\mathbb{D}_{\mathscr{D}}$, by setting

$$\mathbb{D}'_{\mathscr{D}}(\mathscr{M}):=\mathrm{R}\mathscr{H}om_{\mathscr{D}}(\mathscr{M},\mathscr{D}_X)\ \ (\mathscr{M}\in\mathrm{D}^{\mathrm{b}}(\mathscr{D}_X)\text{ or }\mathscr{M}\in\mathrm{D}^{\mathrm{b}}(\mathscr{D}_X^{\mathrm{op}})),\tag{1.6.4}$$

$$\mathbb{D}_{\mathscr{D}}(\mathscr{M}):=\mathrm{R}\mathscr{H}om_{\mathscr{D}}(\mathscr{M},\mathscr{D}_X\otimes_{\mathscr{O}}\Omega_X^{\otimes-1}[d_X])\ (\mathscr{M}\in\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)),\tag{1.6.5}$$

$$\mathbb{D}_{\mathscr{D}}(\mathscr{M}):=\mathrm{R}\mathscr{H}om_{\mathscr{D}}(\mathscr{M},\Omega_X[d_X]\otimes_{\mathscr{O}}\mathscr{D}_X)\ (\mathscr{M}\in\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X^{\mathrm{op}})).\tag{1.6.6}$$

Proposition 1.6.2. *For $\mathscr{M},\mathscr{N}$ in $\mathrm{D}^{\mathrm{b}}(\mathscr{D}_X)$, we have a natural morphism*

$$\mathrm{R}\mathscr{H}om_{\mathscr{D}}(\mathscr{O}_X,\mathbb{D}_{\mathscr{D}}\mathscr{M}\overset{\mathrm{L}}{\underline{\otimes}}\mathscr{N})\to\mathrm{R}\mathscr{H}om_{\mathscr{D}}(\mathscr{M},\mathscr{N})\tag{1.6.7}$$

and if $\mathscr{M}$ of $\mathscr{N}$ belongs to $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$, this morphism is an isomorphism.

Proof. We have the isomorphism

$$\begin{aligned}
\mathrm{R}\mathscr{H}om_{\mathscr{D}}(\mathscr{O}_X,\mathbb{D}_{\mathscr{D}}\mathscr{M}\overset{\mathrm{L}}{\underline{\otimes}}\mathscr{N}) &\simeq\mathrm{R}\mathscr{H}om_{\mathscr{D}}(\mathscr{O}_X,\mathscr{D}_X)\overset{\mathrm{L}}{\otimes}_{\mathscr{D}}(\mathbb{D}_{\mathscr{D}}\mathscr{M}\overset{\mathrm{L}}{\underline{\otimes}}\mathscr{N})\\
&\simeq\Omega_X\overset{\mathrm{L}}{\otimes}_{\mathscr{D}}(\mathbb{D}_{\mathscr{D}}\mathscr{M}\overset{\mathrm{L}}{\underline{\otimes}}\mathscr{N})[-d_X]\\
&\simeq(\Omega_X\overset{\mathrm{L}}{\underline{\otimes}}\mathbb{D}_{\mathscr{D}}\mathscr{M})\overset{\mathrm{L}}{\otimes}_{\mathscr{D}}\mathscr{N}[-d_X]\\
&\simeq\mathbb{D}'_{\mathscr{D}}\mathscr{M}\overset{\mathrm{L}}{\otimes}_{\mathscr{D}}\mathscr{N}\to\mathrm{R}\mathscr{H}om_{\mathscr{D}}(\mathscr{M},\mathscr{N}).
\end{aligned}$$

Cleary, if $\mathscr{M}$ of $\mathscr{N}$ belongs to $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$, the last morphism is an isomorphism. □

Proposition 1.6.3. (i) *The functor* $\mathbb{D}'_{\mathscr{D}}\colon \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)^{\mathrm{op}} \to \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X^{\mathrm{op}})$ *is well-defined and satisfies* $\mathbb{D}'_{\mathscr{D}} \circ \mathbb{D}'_{\mathscr{D}} \simeq \mathrm{id}$ *and similarly with* $\mathbb{D}_{\mathscr{D}}$.

(ii) *If* $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$, *then* $\mathrm{char}(\mathbb{D}'_{\mathscr{D}}(\mathscr{M})) = \mathrm{char}(\mathscr{M})$.

Proof. (i) There is a natural morphism $\mathrm{id} \to \mathbb{D}'_{\mathscr{D}}\circ\mathbb{D}'_{\mathscr{D}}$. To prove it is an isomorphism, we argue by induction on the amplitude of $\mathscr{M}$ and reduce to the case where $\mathscr{M}$ is a coherent $\mathscr{D}_X$-module. More precisely, assume $H^j(\mathscr{M}) = 0$ for $j \notin [j_0, j_1]$ and the result has been proved for modules with amplitude $j_1 - j_0 - 1$. Consider the distinguished triangle (d.t. for short)

$$H^{j_0}(\mathscr{M})[-j_0] \to \mathscr{M} \to \tau^{>j_0}(\mathscr{M}) \xrightarrow{+1} \tag{1.6.8}$$

and apply the functor $\mathbb{D}'_{\mathscr{D}} \circ \mathbb{D}'_{\mathscr{D}}$. We get a new d.t. with two objects isomorphic to two objects of the d.t. (1.6.8). hence the third objects of these d.t. will be isomorphic.

Hence, we are reduced to treat the case of $\mathscr{M} \in \mathrm{Mod}_{\mathrm{c}}(\mathscr{D}_X)$. We may argue locally and replace $\mathscr{M}$ with a bounded complex of finite free $\mathscr{D}_X$-modules. It reduces to the case where $\mathscr{M} = \mathscr{D}_X$.

(ii) It is enough to prove the inclusion $\mathrm{char}(\mathbb{D}'_{\mathscr{D}}(\mathscr{M})) \subset \mathrm{char}(\mathscr{M})$. We argue by induction on the amplitude of $\mathscr{M}$. Assume $H^j(\mathscr{M}) = 0$ for $j \notin [j_0, j_1]$. Consider the distinguished triangle (1.6.8) Applying the functor $\mathbb{D}'_{\mathscr{D}}$ we find the d.t.

$$\mathbb{D}(\tau^{>j_0}\mathscr{M}) \to \mathbb{D}'_{\mathscr{D}}\mathscr{M} \to \mathbb{D}'_{\mathscr{D}}(H^{j_0}(\mathscr{M}))[j_0] \xrightarrow{+1}$$

Since $\mathrm{char}(\mathscr{M}) = \mathrm{char}(H^{j_0}(\mathscr{M})) \cup \mathrm{char}(\tau^{>j_0}(\mathscr{M}))$, the induction proceeds, and we are reduced to the case where $\mathscr{M}$ is a coherent $\mathscr{D}_X$-module. Then the result follows from Theorem 1.5.1 (iii). □

Chapter 2

Operations on $\mathscr{D}$-modules

2.1 External product

Let X and Y be two manifolds. For a $\mathscr{D}_X$-module $\mathscr{M}$ and a $\mathscr{D}_Y$-module $\mathscr{N}$, we define their external product, denoted $\mathscr{M}\underline{\boxtimes}\mathscr{N}$, by

$$\mathscr{M}\underline{\boxtimes}\mathscr{N} := \mathscr{D}_{X\times Y} \otimes_{\mathscr{D}_X \boxtimes \mathscr{D}_Y} (\mathscr{M} \boxtimes \mathscr{N}).$$

Note that the functor $\mathscr{M} \mapsto \mathscr{M}\underline{\boxtimes}\mathscr{N}$ is exact.

Theorem 2.1.1. *Let $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$ and $\mathscr{N} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_Y)$. Then $\mathscr{M}\underline{\boxtimes}\mathscr{N} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_{X\times Y})$ and $\mathrm{char}(\mathscr{M}\underline{\boxtimes}\mathscr{N}) = \mathrm{char}(\mathscr{M}) \times \mathrm{char}(\mathscr{N})$.*

Proof. (i) By dévissage, one reduces to the case where $\mathscr{M} \in \mathrm{Mod_c}(\mathscr{D}_X)$ and $\mathscr{N} \in \mathrm{Mod_c}(\mathscr{D}_Y)$.

(ii) Let us show that $\mathscr{M}\underline{\boxtimes}\mathscr{N}$ is coherent. Consider finite free presentations of $\mathscr{M}$ and $\mathscr{N}$:

$$\mathscr{D}_X^{M_1} \xrightarrow{\cdot P} \mathscr{D}_X^{M_0} \to \mathscr{M} \to 0,\ \mathscr{D}_Y^{N_1} \xrightarrow{\cdot Q} \mathscr{D}_Y^{N_0} \to \mathscr{N} \to 0.$$

Then

$$(\mathscr{D}_X \boxtimes \mathscr{D}_Y)^{N_1+M_1} \xrightarrow{\left(\begin{smallmatrix} P & 0 \\ 0 & Q \end{smallmatrix}\right)} (\mathscr{D}_X \boxtimes \mathscr{D}_Y)^{N_0+M_0} \to \mathscr{M} \boxtimes \mathscr{N} \to 0$$

is a finite free presentation of $\mathscr{M} \boxtimes \mathscr{N}$ over $\mathscr{D}_X \boxtimes \mathscr{D}_Y$. To conclude, apply the exact functor $\mathscr{D}_{X\times Y} \otimes_{\mathscr{D}_X \boxtimes \mathscr{D}_Y} \bullet$ to this sequence.

(iii) Let us endow $\mathscr{M}$ and $\mathscr{N}$ with good filtrations $\mathrm{Fl}\mathscr{M}$ and $\mathrm{Fl}\mathscr{N}$. Set

$$\mathrm{Fl}_k(\mathscr{M}\underline{\boxtimes}\mathscr{N}) = \sum_{i+j=k} \mathrm{Fl}_i(\mathscr{M})\underline{\boxtimes}\mathrm{Fl}_j(\mathscr{N}).$$

Then $\{\mathrm{Fl}_k(\mathscr{M}\underline{\boxtimes}\mathscr{N})\}_k$ is a good filtration on $\mathscr{M}\underline{\boxtimes}\mathscr{N}$ and the result follows from

$$\mathrm{Gr}(\mathscr{M}\underline{\boxtimes}\mathscr{N}) \simeq \mathrm{Gr}(\mathscr{M}) \overset{\mathrm{GrD}}{\boxtimes} \mathrm{Gr}(\mathscr{N})$$

where $\overset{\mathrm{GrD}}{\boxtimes}$ is defined similarly as $\underline{\boxtimes}$. □

2.2 Transfert bimodule

Let $f\colon X \to Y$ be a morphism of complex manifolds. Recall (see (3.1.14)) that to f are associated the maps

$$TX \xrightarrow{f'} X \times_Y TY \xrightarrow{f_\tau} TY. \tag{2.2.1}$$

We shall construct a $(\mathscr{D}_X, f^{-1}\mathscr{D}_Y)$-bimodule denoted $\mathscr{D}_{X\to Y}$ which shall allow one to pass from left $\mathscr{D}_Y$-modules to left $\mathscr{D}_X$-modules and from right $\mathscr{D}_X$-modules to right $\mathscr{D}_Y$-modules.

Set

$$\mathscr{D}_{X\to Y} = \mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\mathscr{D}_Y.$$

This sheaf on X is naturally endowed with a structure of an $(\mathscr{O}_X, f^{-1}\mathscr{D}_Y)$-bimodule. We shall endow it of a structure of a left $\mathscr{D}_X$-module by defining the action Θ_X and verifying that this action satisfies the hypothesis of Corollary 1.1.3. Let $v \in \Theta_X$. Then $f'_* v \in \mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\Theta_Y$. Hence

$$f'_* v = \sum_j a_j \otimes w_j,$$

with $a_j \in \mathscr{O}_X$ and $w_j \in f^{-1}\Theta_Y$. Define the action of v on $a \otimes P \in \mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\mathscr{D}_Y$ by setting

$$v(a \otimes P) = v(a) \otimes P + \sum_j a a_j \otimes w_j \circ P. \tag{2.2.2}$$

If one chooses a local coordinate system $(y_1, \dots, y_m)$ on Y and writes $f = (f_1, \dots, f_m)$, then

$$v(f^*\varphi) = \sum_{j=1}^m v(f_j) \frac{\partial \varphi}{\partial y_j},$$

which implies

$$f'_* v = \sum_{j=1}^m v(f_j) \otimes \partial_{y_j}.$$

A section P of $\mathscr{D}_{X\to Y}$ may formally be written as $P = \sum_\alpha a_\alpha(x)\partial_y^\alpha$.

By composing the monomorphism $\mathscr{D}_Y \hookrightarrow \mathscr{H}om_{\mathbb{C}_Y}(\mathscr{O}_Y, \mathscr{O}_Y)$ with $\mathscr{D}_{X\to Y} = \mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\mathscr{D}_Y$ we get the monomorphisms

$$\begin{aligned} \mathscr{D}_{X\to Y} &\hookrightarrow \mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\mathscr{H}om_{\mathbb{C}_Y}(\mathscr{O}_Y, \mathscr{O}_Y) \\ &\hookrightarrow \mathscr{H}om_{\mathbb{C}_X}(f^{-1}\mathscr{O}_Y, \mathscr{O}_X) \end{aligned}$$

and the section $1_{X\to Y} := 1 \otimes 1 \in \mathscr{D}_{X\to Y}$ corresponds to the canonical morphism

$$\begin{aligned} f^{-1}\mathscr{O}_Y &\to \mathscr{O}_X \\ \varphi &\mapsto \varphi \circ f. \end{aligned}$$

Note that $\mathscr{D}_Y$ being flat over $\mathscr{O}_Y$,

$$\mathscr{D}_{X\to Y} \simeq \mathscr{O}_X \overset{\mathrm{L}}{\otimes}_{f^{-1}\mathscr{O}_Y} f^{-1}\mathscr{D}_Y.$$

One also introduces the $(f^{-1}\mathscr{D}_Y, \mathscr{D}_X)$-bimodule $\mathscr{D}_{Y\leftarrow X}$ by setting

$$\mathscr{D}_{Y\leftarrow X} = \Omega_X \otimes_{\mathscr{O}_X} \mathscr{D}_{X\to Y} \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\Omega_Y^{\otimes -1}.$$

Proposition 2.2.1. *Let $f\colon X \to Y$, $g\colon Y \to Z$ be morphisms of manifolds and set $h = g \circ f\colon X \to Z$. Then there is an isomorphism of $(\mathscr{D}_X, h^{-1}\mathscr{D}_Z)$-bimodules*

$$\mathscr{D}_{X\to Y} \overset{\mathrm{L}}{\otimes}_{f^{-1}\mathscr{D}_Y} f^{-1}\mathscr{D}_{Y\to Z} \simeq \mathscr{D}_{X\to Z}. \tag{2.2.3}$$

In particular, the left hand side is concentrated in degree zero.

Proof. One has the isomorphisms of $(\mathscr{O}_X, h^{-1}\mathscr{D}_Z)$-bimodules:

$$\begin{aligned}\mathscr{D}_{X\to Y} \overset{\mathrm{L}}{\otimes}_{f^{-1}\mathscr{D}_Y} f^{-1}\mathscr{D}_{Y\to Z} &= (\mathscr{O}_X \overset{\mathrm{L}}{\otimes}_{f^{-1}\mathscr{O}_Y} f^{-1}\mathscr{D}_Y) \overset{\mathrm{L}}{\otimes}_{f^{-1}\mathscr{D}_Y} f^{-1}(\mathscr{O}_Y \overset{\mathrm{L}}{\otimes}_{g^{-1}\mathscr{O}_Z} g^{-1}\mathscr{D}_Z)\\ &\simeq \mathscr{O}_X \overset{\mathrm{L}}{\otimes}_{f^{-1}\mathscr{O}_Y} (f^{-1}\mathscr{D}_Y \overset{\mathrm{L}}{\otimes}_{f^{-1}\mathscr{D}_Y} f^{-1}\mathscr{O}_Y \overset{\mathrm{L}}{\otimes}_{h^{-1}\mathscr{O}_Z} h^{-1}\mathscr{D}_Z)\\ &\simeq \mathscr{O}_X \overset{\mathrm{L}}{\otimes}_{h^{-1}\mathscr{O}_Z} h^{-1}\mathscr{D}_Z \simeq \mathscr{O}_X \otimes_{h^{-1}\mathscr{O}_Z} h^{-1}\mathscr{D}_Z.\end{aligned}$$

(Recall that $\mathscr{D}_Z$ is flat over $\mathscr{O}_Z$.) Then, one checks that these isomorphisms extend as isomorphisms of $(\mathscr{D}_X, h^{-1}\mathscr{D}_Z)$-bimodules. □

Proposition 2.2.2. (i) *Assume f is submersive. Then $\mathscr{D}_{X\to Y}$ is $\mathscr{D}_X$-coherent and $f^{-1}\mathscr{D}_Y$-flat.*

(ii) *Assume f is a closed embedding. Then $\mathscr{D}_{X\to Y}$ is $\mathscr{D}_Y$-coherent and $\mathscr{D}_X$-flat.*

Proof. (i) Since the problem is local on X, we may assume that $X = Z \times Y$ and f is the second projection. In this case, $\mathscr{D}_{X\to Y} \simeq \mathscr{O}_Z \boxtimes \mathscr{D}_Y$. Note that if $x = (t, y)$ is a local coordinate system on $Z \times Y$ with $t = (t_1, \dots, t_m)$, then

$$\mathscr{D}_{X\to Y} \simeq \mathscr{D}_X / \mathscr{D}_X \cdot \partial_t$$

where $\mathscr{D}_X \cdot \partial_t$ denotes the left ideal generated by $(\partial_{t_1}, \dots, \partial_{t_m})$.
(ii) For a local coordinate system $y = (t, x)$ on Y such that $X = \{t = 0\}$, we have

$$\mathscr{D}_{X\to Y} \simeq \mathscr{D}_Y / t \cdot \mathscr{D}_Y$$

where $t \cdot \mathscr{D}_Y$ denotes the right ideal generated by $(t_1, \dots, t_m)$. □

If f is submersive, one has

$$\mathscr{D}_{X\to Y} \simeq \mathscr{D}_X / \mathscr{D}_X \cdot \Theta_f$$

where $\mathscr{D}_X \cdot \Theta_f$ denotes the left ideal generated by the vector fields tangent to the leaves of f.

If f is a closed embedding, one has

$$\mathscr{D}_{X\to Y} \simeq \mathscr{D}_Y / \mathscr{I}_X \cdot \mathscr{D}_Y$$

where $\mathscr{I}_X \cdot \mathscr{D}_Y$ denotes the right ideal generated sections of $\mathscr{O}_Y$ vanishing on X.

Notice that any morphism $f\colon X \to Y$ may be decomposed as

$$f\colon X \hookrightarrow X \times Y \to Y$$

where the first map is the graph (closed) embedding and the second map is the projection.

Example 2.2.3. One has $\mathscr{D}_{X\to \mathrm{pt}} \simeq \mathscr{O}_X$ and $\mathscr{D}_{\mathrm{pt}\leftarrow X} \simeq \Omega_X$.

Inverse and direct images of $\mathscr{D}$-modules

Definition 2.2.4. Let $f\colon X\to Y$ be a morphism of complex manifolds.

(i) One defines the inverse image functor $f_{\mathscr{D}}^{-1}\colon \mathrm{D}^{\mathrm{b}}(\mathscr{D}_Y)\to\mathrm{D}^{\mathrm{b}}(\mathscr{D}_X)$ by setting for $\mathscr{N}\in\mathrm{D}^{\mathrm{b}}(\mathscr{D}_Y)$:

$$f_{\mathscr{D}}^{-1}\mathscr{N} := \mathscr{D}_{X\to Y}\overset{\mathrm{L}}{\otimes}_{f^{-1}\mathscr{D}_Y} f^{-1}\mathscr{N}.$$

(ii) One defines the direct image functors $f_*^{\mathscr{D}}, f_!^{\mathscr{D}}\colon \mathrm{D}^{\mathrm{b}}(\mathscr{D}_X)\to\mathrm{D}^{\mathrm{b}}(\mathscr{D}_Y)$ by setting for $\mathscr{M}\in\mathrm{D}^{\mathrm{b}}(\mathscr{D}_X^{\mathrm{op}})$:

$$f_*^{\mathscr{D}}\mathscr{M} := \mathrm{R}f_*(\mathscr{M}\overset{\mathrm{L}}{\otimes}_{\mathscr{D}}\mathscr{D}_{X\to Y}),\ f_!^{\mathscr{D}}\mathscr{M} := \mathrm{R}f_!(\mathscr{M}\overset{\mathrm{L}}{\otimes}_{\mathscr{D}}\mathscr{D}_{X\to Y}).$$

Using the bimodule $\mathscr{D}_{Y\leftarrow X}$, one defines similarly the inverse image of a right $\mathscr{D}_Y$-module or the direct images of a left $\mathscr{D}_X$-module. Note that, if $g\colon Y\to Z$ is another morphism of complex manifolds, we have

$$(g\circ f)_{\mathscr{D}}^{-1}\simeq f_{\mathscr{D}}^{-1}\circ g_{\mathscr{D}}^{-1}, \tag{2.2.4}$$

$$(g\circ f)_*^{\mathscr{D}}\simeq g_*^{\mathscr{D}}\circ f_*^{\mathscr{D}}, \tag{2.2.5}$$

$$(g\circ f)_!^{\mathscr{D}}\simeq g_!^{\mathscr{D}}\circ f_!^{\mathscr{D}}. \tag{2.2.6}$$

2.3 Inverse images

Definition 2.3.1. Let $\mathscr{N}$ be a coherent $\mathscr{D}_Y$-module. One says that f is non characteristic for $\mathscr{N}$ (or $\mathscr{N}$ is non characteristic for f) if f is non characteristic for $\mathrm{char}(\mathscr{N})$. (See Definition 3.1.10.)

Example 2.3.2. (i) Since $\mathrm{char}(\mathscr{O}_Y)=T^*_YY$, the $\mathscr{D}_Y$-module $\mathscr{O}_Y$ is non characteristic for any morphism $f\colon X\to Y$. Note that $f_{\mathscr{D}}^{-1}\mathscr{O}_Y\simeq\mathscr{O}_X$.
(ii) See Exercise 2.2.

Example 2.3.3. Assume to be given a coordinate system $(y)=(x_1,\dots,x_n,t)=(x,t)$ on Y such that $X=\{t=0\}$. Let P be a differential operator of order m. Then X is non-characteristic with respect to P (*i.e.*, for the $\mathscr{D}_Y$-module $\mathscr{D}_Y/\mathscr{D}_Y\cdot P$) in a neighborhood of $(x_0,0)\in X$ if and only if P is written as

$$P(x,t;\partial_x,\partial_t)=\sum_{0\le j\le m}a_j(x,t,\partial_x)\partial_t^j \tag{2.3.1}$$

where $a_j(x,t,\partial_x)$ is a differential operator not depending on ∂_t of order $\le m-j$ and $a_m(x,t)$ (which is a holomorphic function on Y) satisfies: $a_m(x_0,0)\neq 0$.

Lemma 2.3.4. *Let X,Y and P be as in Example 2.3.3. Let $\mathscr{N}=\mathscr{D}_Y/\mathscr{D}_Y\cdot P$. Then $\mathscr{D}_{X\to Y}\otimes_{\mathscr{D}_Y}\mathscr{N}\simeq\mathscr{D}_X^m$.*

Proof. Notice that

$$\mathscr{D}_{X\to Y}\otimes_{\mathscr{D}}\mathscr{N}\simeq\mathscr{D}_Y/(t\cdot\mathscr{D}_Y+\mathscr{D}_Y\cdot P).$$

By the Weierstrass preparation theorem, any $Q(x,t,\partial_x,\partial_t) \in \mathscr{D}_Y$ may be written uniquely as

$$Q(x,t,\partial_x,\partial_t) = S(x,t,\partial_x,\partial_t)\cdot P(x,t,\partial_x,\partial_t) + \sum_{j=0}^{m-1} R_j(x,t,\partial_x)\partial_t^j.$$

Hence, $Q(x,t,\partial_x,\partial_t) \in \mathscr{D}_Y$ may be written uniquely as

$$\begin{aligned} &Q(x,t,\partial_x,\partial_t) = \\ &\qquad S(x,t,\partial_x,\partial_t)\cdot P(x,t,\partial_x,\partial_t) + t\cdot T(x,t,\partial_x) + \sum_{j=0}^{m-1} P_j(x,\partial_x)\partial_t^j. \end{aligned}$$

□

Proposition 2.3.5. . *For $\mathscr{M},\mathscr{N} \in \mathrm{D^b}(\mathscr{D}_X)$, one has*

$$\mathscr{M}\underline{\overset{\mathrm{L}}{\otimes}}\mathscr{N} \simeq \delta_{\mathscr{D}}^{-1}(\mathscr{M}\underline{\boxtimes}\mathscr{N}),$$

where $\delta\colon X \to X\times X$ is the diagonal embedding.

Proof. Let us identify X with Δ, the diagonal of $X\times X$. One has the chain of isomorphisms

$$\begin{aligned} \delta_{\mathscr{D}}^{-1}(\mathscr{M}\underline{\boxtimes}\mathscr{N}) &\simeq \mathscr{O}_\Delta\overset{\mathrm{L}}{\otimes}_{\mathscr{O}}\mathscr{D}_{X\times X}\overset{\mathrm{L}}{\otimes}_{\mathscr{D}}(\mathscr{M}\underline{\boxtimes}\mathscr{N}) \\ &\simeq \mathscr{O}_\Delta\overset{\mathrm{L}}{\otimes}_{\mathscr{O}}(\mathscr{M}\underline{\boxtimes}\mathscr{N}) \simeq \mathscr{M}\underline{\overset{\mathrm{L}}{\otimes}}\mathscr{N}. \end{aligned}$$

□

Corollary 2.3.6. *Let $f\colon X\to Y$ be a morphism of complex manifolds. For $\mathscr{N}_1,\mathscr{N}_2 \in \mathrm{D^b}(\mathscr{D}_Y)$, one has*

$$f_{\mathscr{D}}^{-1}(\mathscr{N}_1\underline{\overset{\mathrm{L}}{\otimes}}\mathscr{N}_2) \simeq f_{\mathscr{D}}^{-1}\mathscr{N}_1\underline{\overset{\mathrm{L}}{\otimes}}f_{\mathscr{D}}^{-1}\mathscr{N}_2.$$

Proof. Denote by δ_X the diagonal embedding $X\to X\times X$ and similarly with δ_Y, and denote by $\widetilde{f}\colon X\times X\to Y\times Y$ the map associated with f. One has the chain of isomorphisms

$$\begin{aligned} f_{\mathscr{D}}^{-1}(\mathscr{N}_1\underline{\overset{\mathrm{L}}{\otimes}}\mathscr{N}_2) &\simeq f_{\mathscr{D}}^{-1}{\delta_Y}_{\mathscr{D}}^{-1}(\mathscr{N}_1\underline{\boxtimes}\mathscr{N}_2) \simeq {\delta_X}_{\mathscr{D}}^{-1}\widetilde{f}_{\mathscr{D}}^{-1}(\mathscr{N}_1\underline{\boxtimes}\mathscr{N}_2) \\ &\simeq {\delta_X}_{\mathscr{D}}^{-1}(f_{\mathscr{D}}^{-1}\mathscr{N}_1\underline{\boxtimes}f_{\mathscr{D}}^{-1}\mathscr{N}_2) \simeq f_{\mathscr{D}}^{-1}\mathscr{N}_1\underline{\overset{\mathrm{L}}{\otimes}}f_{\mathscr{D}}^{-1}\mathscr{N}_2. \end{aligned}$$

□

Theorem 2.3.7. *Let $\mathscr{N} \in \mathrm{Mod_c}(\mathscr{D}_Y)$ and assume that f is non characteristic for $\mathscr{N}$. Then*

(a) *$f_{\mathscr{D}}^{-1}\mathscr{N}$ is concentrated in degree 0,*

(b) *$f_{\mathscr{D}}^{-1}\mathscr{N}$ is $\mathscr{D}_X$-coherent,*

(c) $\mathrm{char}(f_{\mathscr{D}}^{-1}\mathscr{N}) \subset f_d f_\pi^{-1}\,\mathrm{char}(\mathscr{N})$.

Remark 2.3.8. In fact, there is a better result, namely $\operatorname{char}(f_{\mathscr{D}}^{-1}\mathscr{N}) = f_d f_\pi^{-1}\operatorname{char}(\mathscr{N})$, and the characteristic cycle of $f_{\mathscr{D}}^{-1}\mathscr{N}$ is the image by $f_d f_\pi^{-1}$ of the characteristic cycle of $\mathscr{N}$ (see [Ka83]).

Proof. The map $f\colon X \to Y$ decomposes as

$$X \xrightarrow{h} X \times Y \xrightarrow{p} Y$$

where h is the graph embedding and p is the projection. Using (2.2.4) and Lemma 3.1.13, it is enough to prove the result for p and for h. Hence, we shall treat separately the case where f is submersive and the case where f is a closed embedding.

(i) Assume $f\colon X \to Y$ is submersive. The problem is local on X. Hence, we may assume $X = Y \times Z$ and f is the projection. In this case, $f_{\mathscr{D}}^{-1}(\bullet) \simeq \mathscr{O}_X \boxtimes \bullet$. Hence, this functor is exact and the result follows from Theorem 2.1.1.

(ii) Assume $f\colon X \to Y$ is a closed embedding. Let d denote the codimension of X in Y. Since our problem is local, we may assume that there are submanifolds $X = X_0 \subset X_1 \subset \cdots \subset X_d = Y$. Using (2.2.4) and Lemma 3.1.13 again, we are reduced to treat the case $d = 1$. Since the problem is local we may assume to be given a local coordinate system in a neighborhood of $x_0 \in X$, $(y) = (x_1, \dots, x_n, t) = (x, t)$ on Y such that $X = \{t = 0\}$. Let $(x, t; \xi, \tau)$ denote the associated coordinate system on T^*Y. Set $\Lambda = \operatorname{char}(\mathscr{N})$. By the hypothesis, $(x_0, 0; 0, 1) \notin \Lambda$. By Corollary 1.3.6, for each section u of $\mathscr{N}$ defined in a neighborhood of $(x_0, 0)$, there exists a differential operator P, say of order m, such that

$$Pu = 0, \qquad \sigma_m(P)(x_0, 0; 0, 1) \neq 0. \tag{2.3.2}$$

(iii) Let us prove that $f_{\mathscr{D}}^{-1}\mathscr{N}$ is concentrated in degree 0. Since $\mathscr{D}_{X\to Y} \simeq \mathscr{D}_Y / t\cdot\mathscr{D}_Y$, $f_{\mathscr{D}}^{-1}\mathscr{N}$ is isomorphic to the complex $\mathscr{N} \xrightarrow{t\cdot} \mathscr{N}$. Hence, we have to show that $t\cdot$ acting on $\mathscr{N}$ is injective. Let $u \in \mathscr{N}$ with $tu = 0$. Let P satisfying (2.3.2). Set $\operatorname{Ad}(P) = [P, \bullet]$. We obtain

$$\operatorname{Ad}^m(P)(t)u = m!u = 0.$$

Hence, $u = 0$.

(iv) Let us prove that $f_{\mathscr{D}}^{-1}\mathscr{N}$ is $\mathscr{D}_X$-coherent. Let $(u_1, \dots, u_N)$ be a system of generators of $\mathscr{N}$ in a neighborhood of $(x_0, 0)$. For each $j, 1 \leq j \leq N$, there exists a differential operator P_j of order m_j, such that $P_j u_j = 0$ and $\sigma_{m_j}(P_j)(x_0, 0; 0, 1) \neq 0$. Set

$$\mathscr{M} = \oplus_{j=1}^N \mathscr{D}_Y / \mathscr{D}_Y \cdot P_j.$$

It follows from (iii) and Lemma 2.3.4 that $f_{\mathscr{D}}^{-1}\mathscr{M}$ is concentrated in degree 0 and is $\mathscr{D}_X$-coherent.

Denote by v_j the canonical generator of $\mathscr{D}_Y / \mathscr{D}_Y \cdot P_j$, the image of $1 \in \mathscr{D}_Y$. There is a well-defined $\mathscr{D}_Y$-linear epimorphism $\psi\colon \mathscr{M} \twoheadrightarrow \mathscr{N}$ which associates u_j to v_j. The functor $f_{\mathscr{D}}^{-1}$ being right exact, the epimorphism ψ defines the epimorphism $f_{\mathscr{D}}^{-1}\mathscr{M} \twoheadrightarrow f_{\mathscr{D}}^{-1}\mathscr{N}$. Therefore, $f_{\mathscr{D}}^{-1}\mathscr{N}$ is locally finitely generated.

Define the coherent $\mathscr{D}_Y$-module $\mathscr{L}$ by the exact sequence

$$0 \to \mathscr{L} \to \mathscr{M} \to \mathscr{N} \to 0. \tag{2.3.3}$$

It follows from (iii) that the sequence

$$0 \to f_{\mathscr{D}}^{-1}\mathscr{L} \to f_{\mathscr{D}}^{-1}\mathscr{M} \to f_{\mathscr{D}}^{-1}\mathscr{N} \to 0 \tag{2.3.4}$$

is exact. Since X is non-characteristic for $\mathscr{M}$, it is non-characteristic for its sub-module $\mathscr{L}$. Therefore, $f_{\mathscr{D}}^{-1}\mathscr{L}$ is locally finitely generated and $f_{\mathscr{D}}^{-1}\mathscr{M}$ being coherent, this implies that $f_{\mathscr{D}}^{-1}\mathscr{N}$ is coherent.

(v) Let us prove (c).
(v)–(a) Let us choose a local coordinate system (x,t) on Y such that $X = \{(x,t); t = 0\}$. Then $f_{\mathscr{D}}^{-1}\mathscr{N} \simeq \mathscr{N}/t\cdot\mathscr{N}$. Set

$$\mathscr{M} := f_{\mathscr{D}}^{-1}\mathscr{N}.$$

Let $\mathrm{Fl}\mathscr{N} = \{\mathscr{N}_j\}_{j\in\mathbb{Z}}$ be a good filtration on $\mathscr{N}$. We define a filtration on $\mathrm{Fl}\mathscr{M} = \{\mathscr{M}_j\}_{j\in\mathbb{Z}}$ by setting

$$\mathscr{M}_j = \mathscr{N}_j/(t\cdot\mathscr{N}\cap\mathscr{N}_j). \tag{2.3.5}$$

(v)–(b) Let us show that $\mathrm{Fl}\mathscr{M}$ is a good filtration. It is enough to check that the $\mathscr{M}_j$'s are $\mathscr{O}_X$-coherent. Since

$$t\cdot\mathscr{N}\cap\mathscr{N}_j = \bigcup_k (t\cdot\mathscr{N}_k\cap\mathscr{N}_j),$$

and $\mathscr{N}_j$ is $\mathscr{O}_Y$-coherent, this sequence is locally stationary. It follows that $\mathscr{M}_j$ is $\mathscr{O}_Y$-coherent. Being supported by X, $\mathscr{M}_j$ is $\mathscr{O}_X$-coherent.

(v)–(c) The exact sequence $0 \to \mathscr{N}_{j-1} \to \mathscr{N}_j \to \mathrm{Gr}_j\mathscr{N} \to 0$ gives rise to the exact sequence

$$\mathscr{N}_{j-1}/t\cdot\mathscr{N}_{j-1} \to \mathscr{N}_j/t\cdot\mathscr{N}_j \to \mathrm{Gr}_j\mathscr{N}/t\cdot\mathrm{Gr}_j\mathscr{N} \to 0. \tag{2.3.6}$$

We deduce from (2.3.5) and (2.3.6) an epimorphism $\mathrm{Gr}_j\mathscr{N}/t\cdot\mathrm{Gr}_j\mathscr{N} \twoheadrightarrow \mathrm{Gr}_j\mathscr{M}$, hence, an epimorphism

$$\mathrm{Gr}\mathscr{N}/t\cdot\mathrm{Gr}\mathscr{N} \twoheadrightarrow \mathrm{Gr}\mathscr{M}. \tag{2.3.7}$$

It follows that the support of $\mathrm{Gr}\mathscr{M}$ in $X\times_Y T^*Y$ (*i.e.*, as an $\mathscr{O}_X\otimes_{\mathscr{O}_Y}\mathrm{Gr}\mathscr{D}_Y$-module) is contained in $f_\pi^{-1}\mathrm{char}(\mathscr{N})$. Since f_d is finite over $f_\pi^{-1}\mathrm{char}(\mathscr{N})$, the support of $\mathrm{Gr}\mathscr{M}$ as a $\mathrm{Gr}\mathscr{D}_Y$-module is contained in $f_d f_\pi^{-1}\mathrm{char}(\mathscr{N})$. □

Corollary 2.3.9. *Let* $\mathscr{M},\mathscr{N}\in\mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X)$ *and assume that* $\mathrm{char}(\mathscr{M})\cap\mathrm{char}(\mathscr{N})\subset T^*_XX$. *Then* $\mathscr{M}\overset{\mathrm{L}}{\underline{\otimes}}\mathscr{N}$ *is* $\mathscr{D}_X$*-coherent and*

$$\mathrm{char}(\mathscr{M}\overset{\mathrm{L}}{\underline{\otimes}}\mathscr{N}) \subset \mathrm{char}(\mathscr{M}) \underset{X}{+} \mathrm{char}(\mathscr{N}).$$

Recall that for two conic subsets Λ_1 and Λ_2 of T^*X ,

$$\Lambda_1 \underset{X}{+} \Lambda_2 := \{(x;\xi_1+\xi_2); (x;\xi_j\in\Lambda_j, j=1,2\}.$$

Proof. Apply Proposition 2.3.5 and Theorem 2.3.7. □

Duality and inverse images

Let $\mathscr{N} \in \mathrm{D}^{\mathrm{b}}(\mathscr{D}_Y)$. Recall that its dual, $\mathbb{D}_{\mathscr{D}}\mathscr{N} \in \mathrm{D}^{\mathrm{b}}(\mathscr{D}_Y)$ has been constructed in (1.6.6)

Theorem 2.3.10. *Let $f\colon X \to Y$ be a morphism of complex manifolds and let $\mathscr{N} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_Y)$. Assume that f is non characteristic for $\mathscr{N}$. Then there exists a natural isomorphism :*

$$\psi\colon \mathbb{D}_{\mathscr{D}} f_{\mathscr{D}}^{-1}\mathscr{N} \xrightarrow{\sim} f_{\mathscr{D}}^{-1}\mathbb{D}_{\mathscr{D}}\mathscr{N}.$$

Proof. First, we shall construct the morphism ψ. By Proposition 1.6.2, we have an isomorphism

$$\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(\mathscr{D}_Y)}(\mathscr{N},\mathscr{N}) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(\mathscr{D}_Y)}(\mathscr{O}_Y, \mathbb{D}_{\mathscr{D}}\mathscr{N} \overset{\mathrm{L}}{\underline{\otimes}} \mathscr{N}).$$

It defines the morphism $\mathscr{O}_Y \to \mathbb{D}_{\mathscr{D}}\mathscr{N} \overset{\mathrm{L}}{\underline{\otimes}} \mathscr{N}$. Applying the functor $f_{\mathscr{D}}^{-1}$ we get the morphisms

$$\begin{aligned} f_{\mathscr{D}}^{-1}\mathscr{O}_Y \simeq \mathscr{O}_X &\to f_{\mathscr{D}}^{-1}\mathbb{D}_{\mathscr{D}}\mathscr{N} \overset{\mathrm{L}}{\underline{\otimes}} f_{\mathscr{D}}^{-1}\mathscr{N} \\ &\to f_{\mathscr{D}}^{-1}\mathbb{D}_{\mathscr{D}}\mathscr{N} \overset{\mathrm{L}}{\underline{\otimes}} \mathbb{D}_{\mathscr{D}}\mathbb{D}_{\mathscr{D}} f_{\mathscr{D}}^{-1}\mathscr{N}. \end{aligned}$$

Hence, we have obtained a morphism

$$\begin{aligned} \psi &\in \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(\mathscr{D}_X)}(\mathscr{O}_X, f_{\mathscr{D}}^{-1}\mathbb{D}_{\mathscr{D}}\mathscr{N} \overset{\mathrm{L}}{\underline{\otimes}} \mathbb{D}_{\mathscr{D}}\mathbb{D}_{\mathscr{D}} f_{\mathscr{D}}^{-1}\mathscr{N}) \\ &\simeq \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(\mathscr{D}_X)}(\mathbb{D}_{\mathscr{D}} f_{\mathscr{D}}^{-1}\mathscr{N}, f_{\mathscr{D}}^{-1}\mathbb{D}_{\mathscr{D}}\mathscr{N}). \end{aligned}$$

To prove that ψ is an isomorphism, we proceed as in the proof of Theorem 2.3.7 and reduce to the case where X is a closed hypersurface of Y and $\mathscr{N} = \mathscr{D}_Y/\mathscr{D}_Y \cdot P$ for a differential operator P of order m. In this case, $f_{\mathscr{D}}^{-1}\mathscr{N} \simeq \mathscr{D}_X^m$ and $\mathbb{D}_{\mathscr{D}} f_{\mathscr{D}}^{-1}\mathscr{N} \simeq \mathscr{D}_X^m[d_X]$. On the other hand, $\mathscr{N}$ is represented by the complex $0 \to \mathscr{D}_Y \xrightarrow{\cdot P} \mathscr{D}_Y \to 0$ and it follows that

$$\mathbb{D}_{\mathscr{D}}\mathscr{N} \simeq \mathscr{N}[d_Y - 1].$$

Therefore, $f_{\mathscr{D}}^{-1}\mathbb{D}_{\mathscr{D}}\mathscr{N} \simeq \mathscr{D}_X^m[d_Y - 1]$. □

2.4 Holomorphic solutions of inverse images

Let $f\colon X \to Y$ be a morphism of complex manifolds and let $\mathscr{N}_1, \mathscr{N}_2 \in \mathrm{Mod}(\mathscr{D}_Y)$. There is a natural morphism

$$f^{-1}\mathrm{R}\mathscr{H}om_{\mathscr{D}_Y}(\mathscr{N}_1,\mathscr{N}_2) \to \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(f_{\mathscr{D}}^{-1}\mathscr{N}_1, f_{\mathscr{D}}^{-1}\mathscr{N}_2). \tag{2.4.1}$$

obtained as the composition

$$\begin{aligned} f^{-1}\mathrm{R}\mathscr{H}om_{\mathscr{D}_Y}(\mathscr{N}_1,\mathscr{N}_2) &\to \mathrm{R}\mathscr{H}om_{f^{-1}\mathscr{D}_Y}(f^{-1}\mathscr{N}_1, f^{-1}\mathscr{N}_2) \\ &\to \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{D}_{X\to Y} \overset{\mathrm{L}}{\otimes}_{f^{-1}\mathscr{D}} f^{-1}\mathscr{N}_1, \mathscr{D}_{X\to Y} \overset{\mathrm{L}}{\otimes}_{f^{-1}\mathscr{D}} f^{-1}\mathscr{N}_2). \end{aligned}$$

Also recall the natural isomorphism

$$f_{\mathscr{D}}^{-1}\mathscr{O}_Y \simeq \mathscr{O}_X. \tag{2.4.2}$$

Theorem 2.4.1. (Cauchy-Kowalevski-Kashiwara) *Let $f\colon X \to Y$ be a morphism of complex manifolds and let $\mathscr{N} \in \mathrm{Mod}(\mathscr{D}_Y)$. Assume that f is non characteristic for $\mathscr{N}$. Then there exists a natural isomorphism :*

$$f^{-1}\mathrm{R}\mathscr{H}om_{\mathscr{D}_Y}(\mathscr{N},\mathscr{O}_Y) \xrightarrow{\sim} \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(f_{\mathscr{D}}^{-1}\mathscr{N},\mathscr{O}_X). \tag{2.4.3}$$

Proof. As in the proof of Theorem 2.3.7, we may check separately the case of a projection and a closed embedding.

(a) If f is submersive, the morphism (2.4.1) is an isomorphism. Indeed, we may reduce to the case where $\mathscr{N}_1 = \mathscr{N}_2 = \mathscr{D}_Y$. In such a case, the isomorphism reduces to:

$$f^{-1}\mathscr{D}_Y \simeq \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{D}_{X\to Y},\mathscr{D}_{X\to Y}).$$

We may assume f is the projection $X = Y\times Z \to Y$, and the result is a relative version of the De Rham isomorphism $\mathbb{C}_Z \simeq \mathrm{R}\mathscr{H}om_{\mathscr{D}_Z}(\mathscr{O}_Z,\mathscr{O}_Z)$.

(b) Now assume f is a closed embedding. Again, we reduce to the case where X is a hypersurface. First we treat the case where $\mathscr{N} = \mathscr{D}_Y/\mathscr{D}_Y\cdot P$. We may assume that we have a local coordinate system (x,t) such that $X = \{(x,t); t=0\}$ and P is a differential operator of order m as in Lemma 2.3.3. The complex $\mathrm{R}\mathscr{H}om_{\mathscr{D}_Y}(\mathscr{N},\mathscr{O}_Y)$ is represented by the complex $0 \to \mathscr{O}_Y|_X \xrightarrow{P\cdot} \mathscr{O}_Y|_X \to 0$, where $\mathscr{O}_Y|_X$ on the left is in degree 0. Since $\mathscr{N}_{\mathscr{D}}^{-1} \simeq \mathscr{D}_X^m$, the complex $\mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{N}_{\mathscr{D}}^{-1},\mathscr{O}_X)$ is represented by the complex $\mathscr{O}_X^m$ in degree 0. The morphism (2.4.3) reduces to the morphism

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathscr{O}_Y|_X & \xrightarrow{P} & \mathscr{O}_Y|_X & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle\gamma} & & \downarrow & & \\
0 & \longrightarrow & \mathscr{O}_X^m & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

Here, the vertical arrow γ is the morphism which, to $f\in\mathscr{O}_Y|_X$ associates the first m traces of f

$$\gamma(f) = f|_X, \partial_t f|_X, \dots, \partial_t^{m-1} f|_X.$$

Then the theorem asserts that P acting on $\mathscr{O}_Y|_X$ is an epimorphism and $\ker P$ acting on this sheaf is isomorphic by γ to $\mathscr{O}_X^m$. This is the Cauchy-Kovalevski theorem.

(c) As in the proof of Theorem 2.3.7, we construct an exact sequence (2.3.3) $0\to \mathscr{L}\to\mathscr{M}\to\mathscr{N}\to 0$ where $\mathscr{M}$ is a finite direct sum of modules of the type $\mathscr{D}_Y/\mathscr{D}_Y\cdot P$. let us apply the functor $\mathrm{R}\mathscr{H}om_{\mathscr{D}_Y}(\,\bullet\,,\mathscr{O}_Y)$ to the sequence (2.3.3) and the functor $\mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\,\bullet\,,\mathscr{O}_X)$ to the image by $(\,_{\mathscr{D}}^{-1}\bullet\,)$ of the sequence (2.3.3). Let us set for short

$$\mathrm{Sol}_Y(\,\bullet\,) := \mathrm{R}\mathscr{H}om_{\mathscr{D}_Y}(\,\bullet\,,\mathscr{O}_Y)$$

and similarly with $\mathrm{Sol}_X(\,\bullet\,)$. We find the morphism of distinguished triangles

$$\begin{array}{ccccccc}
f^{-1}\mathrm{Sol}_Y(\mathscr{N}) & \longrightarrow & f^{-1}\mathrm{Sol}_Y(\mathscr{M}) & \longrightarrow & f^{-1}\mathrm{Sol}_Y(\mathscr{L}) & \xrightarrow{+1} & \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathrm{Sol}_Y(f_{\mathscr{D}}^{-1}\mathscr{N}) & \longrightarrow & \mathrm{Sol}_Y(f_{\mathscr{D}}^{-1}\mathscr{M}) & \longrightarrow & \mathrm{Sol}_Y(f_{\mathscr{D}}^{-1}\mathscr{L}) & \xrightarrow{+1} &
\end{array}$$

Let us apply the cohomology functor H^0 to this morphism of distinguished triangles. We find a morphism of long exact sequences

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & H^0(A_1) & \longrightarrow & H^0(A_2) & \longrightarrow & H^0(A_3) & \longrightarrow & H^1(A_1) & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle u_1^0} & & \downarrow{\scriptstyle u_2^0} & & \downarrow{\scriptstyle u_3^0} & & \downarrow{\scriptstyle u_1^1} & & \\ 0 & \longrightarrow & H^0(B_1) & \longrightarrow & H^0(B_2) & \longrightarrow & H^0(B_3) & \longrightarrow & H^1(B_1) & \longrightarrow & \cdots. \end{array}$$

By (b), all morphisms $u_2^n, n \geq 0$ are isomorphisms. It follows that u_1^0 is a monomorphism, and the module $\mathscr{M}$ satisfying the non characteristicity hypothesis, the morphism u_3^0 is also a monomorphism. Therefore, u_1^0 is an isomorphism, hence u_3^0 is also an isomorphism. By induction, we get that all u_1^n are isomorphism. □

2.5 Direct images

Good $\mathscr{D}$-modules

Definition 2.5.1. (i) Let $\mathscr{F} \in \mathrm{Mod}(\mathscr{O}_X)$. One says that $\mathscr{F}$ is good if for any relatively compact open subset $U \subset\subset X$, there exists a small and filtrant category I, an inductive system $\{\mathscr{F}_i\}_{i\in I}$ of coherent $\mathscr{O}_U$-modules and an isomorphism $\operatorname{colim}_i \mathscr{F}_i \xrightarrow{\sim} \mathscr{F}|_U$.

(ii) One denotes by $\mathrm{Mod}_{\gamma\circ a}(\mathscr{O}_X)$ the full subcategory of $\mathrm{Mod}(\mathscr{O}_X)$ consisting of good $\mathscr{O}_X$-modules.

(iii) A coherent $\mathscr{D}_X$-module $\mathscr{M}$ is good if it is good as an $\mathscr{O}_X$-module.

(iv) One denotes by $\mathrm{Mod}_{\gamma\circ a}(\mathscr{D}_X)$ the full subcategory of $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X)$ consisting of good $\mathscr{O}_X$-modules.

Note that $\mathscr{D}_X$ is good. For generally, if a coherent $\mathscr{D}_X$-module may be endowed with a good filtration, then it is good. However, there exist coherent $\mathscr{D}_X$-modules which are not good.

Lemma 2.5.2. *The category $\mathrm{Mod}_{\gamma\circ a}(\mathscr{O}_X)$ is a thick abelian subcategory of the category $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X)$. In particular, the full subcategory $\mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{D}_X)$ of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$ consisting of objects $\mathscr{M}$ such that $H^j(\mathscr{M})$ is good for all j is triangulated.*

Proof. For the proof, we refer to [Ka03]. □

Lemma 2.5.3. *Let $\mathscr{M} \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X)$. Then $\mathscr{M}$ is good if and only if, for any relatively compact open subset $U \subset\subset X$, there exists $\mathscr{F} \subset \mathscr{M}|_U$ with $\mathscr{F} \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{O}_U)$ and an epimorphism of $\mathscr{D}_U$-modules $\mathscr{F} \otimes_{\mathscr{O}_U} \mathscr{D}_U \twoheadrightarrow \mathscr{M}|_U$.*

Proof. After replacing X with a relatively compact open subset of X containing the closure of U, we may assume that $\mathscr{M} = \operatorname{colim}_i \mathscr{F}_i$ where I is small and filtrant and $\mathscr{F}_i$ is $\mathscr{O}_X$-coherent. Set

$$\mathscr{L}_i = \mathrm{Im}(\mathscr{F}_i \otimes_{\mathscr{O}_X} \mathscr{D}_X \to \mathscr{M}).$$

Since $\mathscr{M}$ is $\mathscr{D}_X$-coherent, the family $\{\mathscr{L}_i\}_{i\in I}$ of coherent $\mathscr{D}_X$-modules is locally stationary hence is stationary on the closure of U. □

Coherency

Theorem 2.5.4. *Let* $f\colon X \to Y$ *be a morphism of complex manifolds and let* $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{D}_X^{\mathrm{op}})$*. Assume that* f *is proper on* $\mathrm{supp}(\mathscr{M})$*. Then*

(i) $f_*^{\mathscr{D}}\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{D}_Y^{\mathrm{op}})$,

(ii) $\mathrm{char}(f_*^{\mathscr{D}}\mathscr{M}) \subset f_\pi f_d(\mathrm{char}(\mathscr{M}))$.

(iii) *Moreover, if* f *is finite on* $\mathrm{supp}(\mathscr{M})$*, the above inclusion is an equality.*

Proof. (i)–(a) By "dévissage", we reduce to the case where $\mathscr{M}$ is a good $\mathscr{D}_X$-module. More precisely, assume $H^j(\mathscr{M}) = 0$ for $j \notin [j_0, j_1]$ and the result has been proved for modules with amplitude $j_1 - j_0 - 1$. Consider the distinguished triangle

$$H^{j_0}(\mathscr{M})[-j_0] \to \mathscr{M} \to \tau^{>j_0}(\mathscr{M}) \xrightarrow{+1}$$

Applying the functor $f_!^{\mathscr{D}}$ to this d.t., we get the d.t.:

$$f_!^{\mathscr{D}}(H^{j_0}(\mathscr{M}))[-j_0] \to f_!^{\mathscr{D}}\mathscr{M} \to f_!^{\mathscr{D}}(\tau^{>j_0}(\mathscr{M})) \xrightarrow{+1}$$

It follows from the induction hypothesis and Lemma 2.5.2 that $f_!^{\mathscr{D}}\mathscr{M}$ belongs to $\mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{D}_Y^{\mathrm{op}})$.

(i)–(b) First, assume that $\mathscr{M} \simeq \mathscr{F} \otimes_{\mathscr{O}} \mathscr{D}_X$ for a coherent $\mathscr{O}_X$-module $\mathscr{F}$ and f is proper on $\mathrm{supp}(\mathscr{F})$. Then

$$\begin{aligned} f_*^{\mathscr{D}}\mathscr{M} &\simeq \mathrm{R}f_!(\mathscr{F} \otimes_{\mathscr{O}} \mathscr{D}_X \otimes_{\mathscr{D}_X} \mathscr{D}_{X\to Y}) \\ &\simeq \mathrm{R}f_!(\mathscr{F} \otimes_{\mathscr{O}} \mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\mathscr{D}_Y) \\ &\simeq \mathrm{R}f_!\mathscr{F} \otimes_{\mathscr{O}} \mathscr{D}_Y. \end{aligned}$$

The coherence of $\mathrm{R}f_!\mathscr{F}$ follows from Grauert's theorem.

(i)–(c) Since the problem is local on Y and f is proper on $\mathrm{supp}(\mathscr{M})$, we may assume by Lemma 2.5.3 that there exists an exact sequence in $\mathrm{Mod}(\mathscr{D}_X^{\mathrm{op}})$:

$$0 \to \mathscr{M}' \to \mathscr{F} \otimes_{\mathscr{O}} \mathscr{D}_X \to \mathscr{M} \to 0$$

and f is proper on $\mathrm{supp}(\mathscr{F})$. We apply the functor $f_!^{\mathscr{D}}$ to this sequence and take the cohomology. Setting $\mathscr{L} = \mathscr{F} \otimes_{\mathscr{O}} \mathscr{D}_X$ we find a long exact sequence

$$\cdots \to H^j(f_!^{\mathscr{D}}\mathscr{M}') \to H^j(f_!^{\mathscr{D}}\mathscr{L}) \to H^j(f_!^{\mathscr{D}}\mathscr{M}) \to H^{j+1}(f_!^{\mathscr{D}}\mathscr{M}') \to \cdots.$$

Assume $H^j(f_!^{\mathscr{D}}\mathscr{M})$ is good for all $\mathscr{M}$ and all $j > j_0$. Set

$$\mathscr{K}^j := \ker(H^{j_0+1}(f_!^{\mathscr{D}}\mathscr{M}') \to H^{j_0+1}(f_!^{\mathscr{D}}\mathscr{L})).$$

Then $\mathscr{K}^j$ is good. Moreover, we have an exact sequence

$$H^j(f_!^{\mathscr{D}}\mathscr{L}) \to H^j(f_!^{\mathscr{D}}\mathscr{M}) \to \mathscr{K}^j \to 0$$

from which we deduce that $H^j(f_!^{\mathscr{D}}\mathscr{M})$ is locally finitely generated over $\mathscr{D}_Y^{\mathrm{op}}$. Set

$$\mathscr{R}^j := \mathrm{Coker}\, H^j(f_!^{\mathscr{D}}\mathscr{M}') \to H^j(f_!^{\mathscr{D}}\mathscr{L}).$$

Being a quotient of a good $\mathscr{D}_Y^{\mathrm{op}}$-module by a finitely generated module, it is a good $\mathscr{D}_Y^{\mathrm{op}}$-module. By the exact sequence

$$0 \to \mathscr{R}^j \to H^j(f_!^{\mathscr{D}}\mathscr{M}) \to \mathscr{K}^j \to 0$$

we conclude that $H^{j_0}(f_!^{\mathscr{D}}\mathscr{M})$ is a good $\mathscr{D}_X$-module and the induction proceeds.

(ii)–(iii) The proof is similar to that of Theorem 2.3.7 and left to the reader. □

Example 2.5.5. (i) Assume X is compact and let $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{D}_X^{\mathrm{op}})$. Denote by a_X the projection $X \to \{\mathrm{pt}\}$. Then $a_{X*}^{\mathscr{D}}\mathscr{M} \simeq \mathrm{R}\Gamma(X; \mathscr{M}\overset{\mathrm{L}}{\otimes}_{\mathscr{D}}\mathscr{O}_X)$ and for all $j \in \mathbb{Z}$, $H^j(\mathrm{R}\Gamma(X; \mathscr{M}\overset{\mathrm{L}}{\otimes}_{\mathscr{D}}\mathscr{O}_X)$ is a finite-dimensional $\mathbb{C}$-vector space.
(ii) Let $f\colon X \to Y$ be a proper map and assume that Y is a curve (*i.e.*, $d_Y = 1$). The object $f_!^{\mathscr{D}}\mathscr{O}_X$ is called the Gauss-Manin connection on Y associated with f. It is of particular importance when f is finite (hence, X is again a curve). Note that the characteristic variety of the Gauss-Manin connection satisfies

$$\begin{aligned}\mathrm{char}(f_!^{\mathscr{D}}\mathscr{O}_X) &\subset f_\pi f_d^{-1}(T_X^*X)\\ &= \{(y;\eta) \in T^*Y;\ \text{there exist } x \in X \text{ with } f_d(x)\eta = 0\}.\end{aligned}$$

In other words, this characteristic variety is contained in the union of the zero-section of T^*Y and the conormal bundles to the points $y \in Y$ which are critical values of f.

We state without proof an important result due to Kashiwara.

Theorem 2.5.6. *Let $j\colon Z \hookrightarrow X$ be a closed embedding of a smooth manifold. Then the functor $j_*^{\mathscr{D}}$ induces an equivalence of categories $\mathrm{Mod}(\mathscr{D}_Z) \xrightarrow{\sim} \mathrm{Mod}_Z(\mathscr{D}_X)$, where $\mathrm{Mod}_Z(\mathscr{D}_X)$ denotes the full abelian subcategory of $\mathrm{Mod}(\mathscr{D}_X)$ consisting of objects with support contained in Z. Moreover, this equivalence induces an equivalence of the subcategories consisting of coherent modules.*

A quasi-inverse functor to $j_*^{\mathscr{D}}$ is given by $j^{-1}\mathscr{H}om_{\mathscr{D}}(\mathscr{D}_{X\leftarrow Z}, \bullet)$.

Although we do not give the proof here and refer to [Ka03, Th. 4.28], the next result will be used in the sequel.

Theorem 2.5.7. Projection formula for $\mathscr{D}$-modules *Let $f\colon X \to Y$ be a morphism of complex manifolds. Let $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}(\mathscr{D}_X^{\mathrm{op}})$ and let $\mathscr{N} \in \mathrm{D}^{\mathrm{b}}(\mathscr{D}_Y)$. There is a natural isomorphism in $\mathrm{D}^{\mathrm{b}}(\mathscr{D}_Y)$*

$$f_!^{\mathscr{D}}(\mathscr{M}\overset{\mathrm{L}}{\underline{\otimes}} f_{\mathscr{D}}^{-1}\mathscr{N}) \simeq f_!^{\mathscr{D}}\mathscr{M}\overset{\mathrm{L}}{\underline{\otimes}}\mathscr{N}. \tag{2.5.1}$$

Proof. □

2.6 Trace morphism

Theorem 2.6.1. *For each morphism of complex manifolds $f\colon X \to Y$, there exists a "trace morphism" in $\mathrm{D}^{\mathrm{b}}(\mathscr{D}_Y^{\mathrm{op}})$*

$$\mathrm{tr}_f\colon f_!^{\mathscr{D}}\Omega_X\,[d_X] \to \Omega_Y\,[d_Y] \tag{2.6.1}$$

with the following properties:

(i) tr_f *is functorial in* f*, that is,* $\mathrm{tr}_{\mathrm{id}_X} = \mathrm{id}$ *and* $\mathrm{tr}_{g\circ f} = \mathrm{tr}_g \circ \mathrm{tr}_f$ *for morphisms* $X \xrightarrow{f} Y \xrightarrow{g} Z$*,*

(ii) *when* X *is a curve and* $Y = \{\mathrm{pt}\}$*,* tr_f *induces the residues morphism on* $H^1_c(X;\Omega_X)$*.*

Using the direct images functor for left $\mathscr{D}$-modules, (2.6.1) gives the functorial morphism

$$\mathrm{tr}_f\colon f^{\mathscr{D}}_! \mathscr{O}_X\,[d_X] \to \mathscr{O}_Y\,[d_Y]. \tag{2.6.2}$$

Proof. Recall that $\Omega_X[-d_X]$ is quasi-isomorphic in $\mathrm{D}^{\mathrm{b}}(\mathscr{D}_X^{\mathrm{op}})$ to the De Rham complex $\mathrm{DR}_X(\mathscr{D}_X)$ (see (1.4.4)):

$$\mathrm{DR}_X(\mathscr{D}_X) := \quad 0 \to \Omega^0_X \otimes_{\mathscr{O}} \mathscr{D}_X \xrightarrow{d} \cdots \to \Omega_X \otimes_{\mathscr{O}} \mathscr{D}_X \to 0,$$

where the differential d is characterized by:

$$d(\omega\otimes m) = d\omega\otimes P + (-)^p\omega\wedge dP, \quad \omega\in\Omega^p_X, P\in\mathscr{D}_X$$

and $dP = \sum_i dx_i \otimes \partial_i \circ P$ in a local coordinate system.

Let us identify $X_{\mathbb{R}}$, the real analytic manifold underlying the complex manifold X with the diagonal of $X\times\overline{X}$. Hence, the real tangent bundle $TX_{\mathbb{R}}$ is isomorphic to $TX\times_{X_{\mathbb{R}}} T\overline{X}$ and the differential $d_{X_{\mathbb{R}}}$ splits as

$$d_{X_{\mathbb{R}}} = \partial \oplus \overline{\partial}.$$

Denote by $\mathscr{D}b_{X_{\mathbb{R}}}$ the sheaf of distributions on the real analytic manifold $X_{\mathbb{R}}$. The sheaf Ω^p_X is quasi-isomorphic to the Dolbeault complex

$$0\to \mathscr{D}b^{(p,0)}_{X_{\mathbb{R}}} \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \mathscr{D}b^{(p,d_X)}_{X_{\mathbb{R}}} \to 0,$$

where $\mathscr{D}b^{(p,q)}_{X_{\mathbb{R}}}$ is the sheaf of forms of type (p,q) with coefficients in $\mathscr{D}b_{X_{\mathbb{R}}}$. It follows that there is a qis

$$\Omega_X[-d_X] \to \mathscr{D}b^{\bullet,\bullet}_{X_{\mathbb{R}}} \otimes_{\mathscr{O}} \mathscr{D}_X, (\partial,\overline{\partial}) \tag{2.6.3}$$

where the bidifferential $(\partial,\overline{\partial})$ satisfies

$$\partial(u\otimes P) = \partial u\otimes P + (-)^p u\wedge dP, \tag{2.6.4}$$
$$\overline{\partial}(u\otimes P) = \overline{\partial} u\otimes P. \tag{2.6.5}$$

Denote by $\mathscr{C}^{\infty(p,q)}_{X_{\mathbb{R}}}$ the sheaf of forms of type (p,q) with coefficients in the sheaf $\mathscr{C}^{\infty}_{X_{\mathbb{R}}}$ of complex valued $\mathscr{C}^\infty$-functions on $X_{\mathbb{R}}$. There is a natural morphism

$$f^*\colon f^{-1}\mathscr{C}^{\infty(p,q)}_{Y_{\mathbb{R}}} \to \mathscr{C}^{\infty(p,q)}_{X_{\mathbb{R}}}. \tag{2.6.6}$$

Since $\Gamma_c(X;\mathscr{D}b^{(p+d_X,q+d_X)}_{X_{\mathbb{R}}})$ is the dual of the space $\Gamma_c(X;\mathscr{C}^{\infty(p,q)}_{X_{\mathbb{R}}})$, the morphism (2.6.6) defines the morphism

$$\int_f\colon f_!\mathscr{D}b^{(p+d_X,q+d_X)}_{X_{\mathbb{R}}} \to \mathscr{D}b^{(p+d_Y,q+d_Y)}_{Y_{\mathbb{R}}}. \tag{2.6.7}$$

Moreover, $\int_f$ commutes with $\overline{\partial}$ and ∂.

The object $\Omega_X[d_X]\overset{\mathrm{L}}{\otimes}_{\mathscr{D}}\mathscr{D}_{X\to Y}$ of $\mathrm{D^b}(\mathscr{D}_X^{\mathrm{op}})$ is isomorphic to the complex $\mathscr{D}b_{X_{\mathbb{R}}}^{\bullet,\bullet}\otimes_{\mathscr{O}} f^{-1}\mathscr{D}_Y\,[2d_X]$ where $\overline{\partial}(u\otimes P)=\overline{\partial}u\otimes P$ and the action of ∂ is given by (2.6.4) and (2.2.2). Noticing that the sheaves $\mathscr{D}b_{X_{\mathbb{R}}}^{(p,q)}$ are soft, we get the chain of morphisms and isomorphisms

$$\begin{aligned} f_!^{\mathscr{D}}\Omega_X\,[d_X] &\simeq f_!(\mathscr{D}b_{X_{\mathbb{R}}}^{\bullet,\bullet}\otimes_{\mathscr{O}}\mathscr{D}_X\otimes_{\mathscr{D}}\mathscr{O}_X\otimes_{f^{-1}\mathscr{O}_Y}f^{-1}\mathscr{D}_Y)[2d_X]\\ &\simeq f_!(\mathscr{D}b_{X_{\mathbb{R}}}^{\bullet,\bullet}\otimes_{f^{-1}\mathscr{O}_Y}f^{-1}\mathscr{D}_Y)[2d_X]\\ &\xrightarrow{\int_f}\mathscr{D}b_{Y_{\mathbb{R}}}^{\bullet,\bullet}\otimes_{\mathscr{O}}\mathscr{D}_Y[2d_Y]\\ &\simeq \Omega_Y\,[d_Y]. \end{aligned}$$

The properties (i) and (ii) of the morphism tr_f are easily checked. □

Corollary 2.6.2. *Let $\mathscr{N}\in\mathrm{D^b}(\mathscr{D}_Y)$. There exists a canonical morphism in $\mathrm{D^b}(\mathscr{D}_Y)$:*

$$f_!^{\mathscr{D}}(f_{\mathscr{D}}^{-1}\mathscr{N}\otimes_{\mathscr{O}}\Omega_X\,[d_X])\to\mathscr{N}\otimes_{\mathscr{O}}\Omega_Y\,[d_Y]. \tag{2.6.8}$$

Proof. By Theorem 2.5.7, we have an isomorphism

$$\begin{aligned} f_!^{\mathscr{D}}(f_{\mathscr{D}}^{-1}\mathscr{N}\otimes_{\mathscr{O}}\Omega_X\,[d_X]) &= f_!^{\mathscr{D}}(f_{\mathscr{D}}^{-1}\mathscr{N}\overset{\mathrm{L}}{\underline{\otimes}}\Omega_X\,[d_X])\\ &\simeq\mathscr{N}\overset{\mathrm{L}}{\underline{\otimes}}f_!^{\mathscr{D}}\Omega_X\,[d_X]. \end{aligned}$$

To conclude, apply the trace morphism $f_!^{\mathscr{D}}\Omega_X\,[d_X]\to\Omega_Y\,[d_Y]$. □

Corollary 2.6.3. *Let $\mathscr{M}\in\mathrm{D^b}(\mathscr{D}_X)$ and let $\mathscr{N}\in\mathrm{D^b}(\mathscr{D}_Y)$. There is a canonical morphism*

$$\mathrm{R}f_*\mathrm{R}\mathscr{H}om_{\mathscr{D}}(\mathscr{M},f_{\mathscr{D}}^{-1}\mathscr{N})\,[d_X]\to\mathrm{R}\mathscr{H}om_{\mathscr{D}}(f_!^{\mathscr{D}}\mathscr{M},\mathscr{N})\,[d_Y]. \tag{2.6.9}$$

Proof. Consider the chain of morphisms

$$\begin{aligned} &\mathrm{R}f_*\mathrm{R}\mathscr{H}om_{\mathscr{D}}(\mathscr{M},f_{\mathscr{D}}^{-1}\mathscr{N})\,[d_X]\\ &\quad\to\mathrm{R}f_*\mathrm{R}\mathscr{H}om_{\mathscr{D}}(\mathscr{D}_{Y\leftarrow X}\otimes_{\mathscr{D}}\mathscr{M},\mathscr{D}_{Y\leftarrow X}\otimes_{\mathscr{D}}f_{\mathscr{D}}^{-1}\mathscr{N})\,[d_X]\\ &\quad\to\mathrm{R}\mathscr{H}om_{\mathscr{D}}(\mathrm{R}f_!(\mathscr{D}_{Y\leftarrow X}\otimes_{\mathscr{D}}\mathscr{M}),\mathrm{R}f_*(\mathscr{D}_{Y\leftarrow X}\otimes_{\mathscr{D}}f_{\mathscr{D}}^{-1}\mathscr{N}))\,[d_X]\\ &\quad\simeq\mathrm{R}\mathscr{H}om_{\mathscr{D}}(f_!^{\mathscr{D}}\mathscr{M},f_!^{\mathscr{D}}f_{\mathscr{D}}^{-1}\mathscr{N})\,[d_X]\\ &\quad\to\mathrm{R}\mathscr{H}om_{\mathscr{D}}(f_!^{\mathscr{D}}\mathscr{M},\mathscr{N})\,[d_Y] \end{aligned}$$

where the last morphism follows from (2.6.8). □

Duality and direct images

Let again $f\colon X\to Y$ be a morphism of complex manifolds.

Lemma 2.6.4. *Let $\mathscr{M}\in\mathrm{D^b}(\mathscr{D}_X^{\mathrm{op}})$. There is a canonical morphism in $\mathrm{D^b}(\mathscr{D}_Y^{\mathrm{op}})$:*

$$f_!^{\mathscr{D}}\mathbb{D}_{\mathscr{D}}\mathscr{M}\to\mathbb{D}_{\mathscr{D}}f_!^{\mathscr{D}}\mathscr{M}. \tag{2.6.10}$$

Proof. By choosing $\mathscr{N} = \mathscr{D}_Y$ in Corollary 2.6.3, we get the chain of morphisms

$$\begin{aligned} f_!^{\mathscr{D}} \mathbb{D}_{\mathscr{D}} \mathscr{M} &= \mathrm{R}f_!(\mathrm{R}\mathscr{H}om_{\mathscr{D}}(\mathscr{M}, \mathscr{D}_X \otimes_{\mathscr{O}} \Omega_X\,[d_X]) \otimes_{\mathscr{D}} \mathscr{D}_{X\to Y}) \\ &\to \mathrm{R}f_!(\mathrm{R}\mathscr{H}om_{\mathscr{D}}(\mathscr{M}, \Omega_X \otimes_{\mathscr{O}} \mathscr{D}_{X\to Y})\,[d_X] \\ &\simeq \mathrm{R}f_!(\mathrm{R}\mathscr{H}om_{\mathscr{D}}(\mathscr{M}, f_{\mathscr{D}}^{-1}\Omega_Y)\,[d_X]) \\ &\to \mathrm{R}\mathscr{H}om_{\mathscr{D}}(f_!^{\mathscr{D}}\mathscr{M}, \mathscr{D}_Y \otimes_{\mathscr{O}} \Omega_Y)\,[d_Y] \\ &= \mathbb{D}_{\mathscr{D}} f_!^{\mathscr{D}} \mathscr{M}. \end{aligned}$$

□

Theorem 2.6.5. *Let $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{D}_X^{\mathrm{op}})$ and assume that f is proper on* $\mathrm{supp}(\mathscr{M})$. *Then the morphism* (2.6.10) *is an isomorphism.*

Proof. We may reduce to the case where $\mathscr{M} \in \mathrm{Mod}_{\gamma^{\circ a}}(\mathscr{D}_X^{\mathrm{op}})$ and, as in the proof of Theorem 2.5.4, that $\mathscr{M} = \mathscr{F} \otimes_{\mathscr{O}} \mathscr{D}_X$ for a coherent $\mathscr{O}_X$-module $\mathscr{F}$. In this case,

$$\begin{aligned} f_!^{\mathscr{D}} \mathbb{D}_{\mathscr{D}} \mathscr{M} &\simeq \mathrm{R}f_!(\mathrm{R}\mathscr{H}om_{\mathscr{D}}(\mathscr{F} \otimes_{\mathscr{O}} \mathscr{D}_X, \mathscr{D}_{Y\leftarrow X} \otimes_{\mathscr{O}} f^{-1}\Omega_Y)\,[d_X] \\ &\simeq \mathrm{R}f_! \mathrm{R}\mathscr{H}om_{\mathscr{O}}(\mathscr{F}, \mathscr{O}_X) \otimes_{\mathscr{O}} \mathscr{D}_Y\,[d_X] \otimes_{\mathscr{O}} \Omega_Y \\ &\simeq \mathrm{R}\mathscr{H}om_{\mathscr{O}}(\mathrm{R}f_!\mathscr{F}, \mathscr{O}_Y) \otimes_{\mathscr{O}} \mathscr{D}_Y \otimes_{\mathscr{O}} \Omega_Y\,[d_Y] \\ &\simeq \mathrm{R}\mathscr{H}om_{\mathscr{D}}(\mathrm{R}f_!\mathscr{F} \otimes_{\mathscr{O}} \mathscr{D}_Y, \mathscr{D}_Y) \otimes_{\mathscr{O}} \Omega_Y\,[d_Y] \\ &\simeq \mathbb{D}_{\mathscr{D}} f_!^{\mathscr{D}} \mathscr{M}. \end{aligned}$$

Here, we have used the fact that proper direct images commute with duality for $\mathscr{O}$-modules (Theorem 3.5.11). □

Theorem 2.6.6. *Let $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{D}_X^{\mathrm{op}})$ and assume that f is proper on* $\mathrm{supp}(\mathscr{M})$. *Then the morphism* (2.6.8) *is an isomorphism.*

Proof. Since $\mathscr{M}$ and $f_!^{\mathscr{D}}\mathscr{M}$ have coherent cohomologies, we have the isomorphisms

$$\begin{aligned} \mathrm{R}\mathscr{H}om_{\mathscr{D}}(\mathscr{M}, f_{\mathscr{D}}^{-1}\mathscr{N}) &\simeq \mathrm{R}\mathscr{H}om_{\mathscr{D}}(\mathscr{M}, \mathscr{D}_{X\leftarrow Y}) \overset{\mathrm{L}}{\otimes}_{f^{-1}\mathscr{D}_Y} f^{-1}\mathscr{N}, \\ \mathrm{R}\mathscr{H}om_{\mathscr{D}}(f_!^{\mathscr{D}}\mathscr{M}, \mathscr{N}) &\simeq \mathrm{R}\mathscr{H}om_{\mathscr{D}}(f_!^{\mathscr{D}}\mathscr{M}, \mathscr{D}_Y) \overset{\mathrm{L}}{\otimes}_{\mathscr{D}} \mathscr{N}. \end{aligned}$$

Hence, we are reduced to prove the result when $\mathscr{N} = \mathscr{D}_Y$, and it follows immediately from Theorem 2.6.5. □

Corollary 2.6.7. *Let $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{D}_X)$ and assume f is proper on* $\mathrm{supp}(\mathscr{M})$. *There is a canonical isomorphism*

$$\mathrm{R}f_* \mathrm{R}\mathscr{H}om_{\mathscr{D}}(\mathscr{M}, \mathscr{O}_X)\,[d_X] \xrightarrow{\sim} \mathrm{R}\mathscr{H}om_{\mathscr{D}}(f_!^{\mathscr{D}}\mathscr{M}, \mathscr{O}_Y)\,[d_Y].$$

2.7 $\mathscr{D}$-modules associated with a submanifold

Let Z be a hypersurface of X. One denotes by $\mathscr{O}_X(*Z)$ the sheaf of meromorphic functions on X with poles in Z. Hence, if $\{f = 0\}$ is a local equation of Z, a section u of $\mathscr{O}_X(*Z)$ is locally written as a quotient $u = g/f^m$, for some $m \in \mathbb{N}$ and g a section of $\mathscr{O}_X$. Clearly, $\mathscr{O}_X(*Z)$ is a left $\mathscr{D}_X$-module.

One also introduces the left $\mathscr{D}_X$-module $\mathscr{B}_{Z|X}$ by the exact sequence

$$0 \to \mathscr{O}_X \to \mathscr{O}_X(*Z) \to \mathscr{B}_{Z|X} \to 0.$$

If $\{f = 0\}$ is a local equation of Z, then

$$\mathscr{B}_{Z|X} \simeq (\mathscr{O}_X[1/f])/\mathscr{O}_X.$$

More generally, let $Z = \{f_j = 0; j = 1, \dots, d\}$ be a complete intersection. One sets

$$\mathscr{B}_{Z|X} \simeq \mathscr{O}_X[1/f_1 \dots f_d]/\sum_i \mathscr{O}_X[1/f_1 \dots \widehat{f_i} \dots f_d]. \tag{2.7.1}$$

We shall see that this does not depend on the choice of the $f_j's$. For that purpose, we recall the construction of the functor $\Gamma_{[Z]}$ and its derived functors.

The functor $\Gamma_{[Z]}$ for $\mathscr{O}$-modules

Let X be a complex manifold, Z a closed analytic subset, $\mathscr{I}_Z$ its defining ideal. Let $\mathscr{F}$ be an $\mathscr{O}_X$-module. Recall that $\Gamma_Z\mathscr{F}$ denotes the subsheaf of sections supported by Z.

Definition 2.7.1. One sets

$$\Gamma_{[Z]}\mathscr{F} \simeq \operatorname{colim}_j \mathscr{H}om_{\mathscr{O}}(\mathscr{O}_X/\mathscr{I}_Z^j, \mathscr{F}),$$
$$\Gamma_{[X\setminus Z]}\mathscr{F} = \operatorname{colim}_j \mathscr{H}om_{\mathscr{O}}(\mathscr{I}_Z^j, \mathscr{F}).$$

Notice that

- $\Gamma_{[Z]}\mathscr{F}$ is the subsheaf of $\Gamma_Z\mathscr{F}$ consisting of sections s such that, locally on X, there exists $j \geq 0$ such that $\mathscr{I}_Z^j s = 0$,
- there is a monomorphism $\Gamma_{[Z]}\mathscr{F} \rightarrowtail \Gamma_Z\mathscr{F}$,
- in Definition 2.7.1, one may replace the defining ideal $\mathscr{I}_Z$ with any coherent ideal $\mathscr{I}$ such that $\operatorname{supp}(\mathscr{O}_X/\mathscr{I}) = Z$. Indeed, for such an ideal, there exists locally an integer k such that $\mathscr{I}_Z^k \subset \mathscr{I} \subset \mathscr{I}_Z$,
- the functors $\Gamma_{[Z]}(\,\bullet\,)$ and $\Gamma_{[X\setminus Z]}(\,\bullet\,)$ are left exact,
- there is an exact sequence of sheaves

$$0 \to \Gamma_{[Z]}\mathscr{F} \to \mathscr{F} \to \Gamma_{[X\setminus Z]}\mathscr{F}. \tag{2.7.2}$$

We shall concentrate our study on the functor $\Gamma_{[Z]}$.

Proposition 2.7.2. *Let Z_1 and Z_2 be two closed subsets of X. There is a natural isomorphism*

$$\Gamma_{[Z_1]}\Gamma_{[Z_2]}\mathscr{F} \simeq \Gamma_{[Z_1\cap Z_2]}\mathscr{F}.$$

Proof. One has the chain of isomorphisms

$$\begin{aligned}\Gamma_{[Z_1]}\Gamma_{[Z_2]}\mathscr{F} &= \operatorname*{colim}_{j_1} \mathscr{H}om_{\mathscr{O}}(\mathscr{O}_X/\mathscr{I}_{Z_1}^{j_1}, \operatorname*{colim}_{j_2} \mathscr{H}om_{\mathscr{O}}(\mathscr{O}_X/\mathscr{I}_{Z_2}^{j_2}, \mathscr{F}))\\ &\simeq \operatorname*{colim}_{j_1} \operatorname*{colim}_{j_2} \mathscr{H}om_{\mathscr{O}}(\mathscr{O}_X/\mathscr{I}_{Z_1}^{j_1}, \mathscr{H}om_{\mathscr{O}}(\mathscr{O}_X/\mathscr{I}_{Z_2}^{j_2}, \mathscr{F}))\\ &\simeq \operatorname*{colim}_{j_1} \operatorname*{colim}_{j_2} \mathscr{H}om_{\mathscr{O}}(\mathscr{O}_X/(\mathscr{I}_{Z_1}^{j_1} + \mathscr{I}_{Z_2}^{j_2}), \mathscr{F})\\ &\simeq \operatorname*{colim}_{j} \mathscr{H}om_{\mathscr{O}}(\mathscr{O}_X/(\mathscr{I}_{Z_1} + \mathscr{I}_{Z_2})^j, \mathscr{F}).\end{aligned}$$

Here, we have used

$$(\mathscr{I}_{Z_1} + \mathscr{I}_{Z_2})^{2j} \subset \mathscr{I}_{Z_1}^j + \mathscr{I}_{Z_2}^j \subset (\mathscr{I}_{Z_1} + \mathscr{I}_{Z_2})^j$$

Since $\operatorname{supp}(\mathscr{I}_{Z_1} + \mathscr{I}_{Z_2}) = Z_1 \cap Z_2$, the result follows from Lemma **??**. □

Let $x \in X$ and let $\mathscr{F} \in \operatorname{Mod}(\mathscr{O}_X)$. Denote by $j_x\colon \{x\} \hookrightarrow X$ the inclusion. One shall be aware that one uses the notation $\mathscr{F}_x$ for both the stalk of $\mathscr{F}$ at x, an object of $\operatorname{Mod}(\mathscr{O}_{X,x})$ and for the sheaf $j_{x*}j_x^{-1}\mathscr{F}$, an object of $\operatorname{Mod}(\mathscr{O}_X)$.

Proposition 2.7.3. *Let $\mathscr{F}$ be an $\mathscr{O}_X$-module and let $x \in X$. Then there is a natural isomorphism $(\Gamma_{[Z]}\mathscr{F})_x \simeq \Gamma_{[Z]}\mathscr{F}_x$.*

Proof. By the the coherence of $\mathscr{O}_X/\mathscr{I}_Z^m$ we have the isomorphisms

$$\begin{aligned}(\mathscr{H}om_{\mathscr{O}}(\mathscr{O}_X/\mathscr{I}_Z^j, \mathscr{F}))_x &\simeq \mathscr{H}om_{\mathscr{O}_{X,x}}((\mathscr{O}_X/\mathscr{I}_Z^j)_x, \mathscr{F}_x)\\ &\simeq \mathscr{H}om_{\mathscr{O}}(\mathscr{O}_X/\mathscr{I}_Z^j, \mathscr{F}_x).\end{aligned}$$

□

Proposition 2.7.4. *Let $\mathscr{G}$ be a coherent $\mathscr{O}_X$-module and let $\mathscr{F}$ be an $\mathscr{O}_X$-module. There are natural isomorphisms*

$$\begin{aligned}\mathscr{H}om_{\mathscr{O}}(\mathscr{G}, \Gamma_{[Z]}\mathscr{F}) &\simeq \Gamma_{[Z]}\mathscr{H}om_{\mathscr{O}}(\mathscr{G}, \mathscr{F})\\ &\simeq \operatorname*{colim}_{j} \mathscr{H}om_{\mathscr{O}}(\mathscr{G}/\mathscr{I}_Z^j\mathscr{G}, \mathscr{F}).\end{aligned} \tag{2.7.3}$$

Proof. (i) Since $\mathscr{G}$ is coherent, the functor $\mathscr{H}om_{\mathscr{O}}(\mathscr{G}, \bullet)$ commutes with filtrant inductive limits. Hence

$$\begin{aligned}\mathscr{H}om_{\mathscr{O}}(\mathscr{G}, \Gamma_{[Z]}\mathscr{F}) &\simeq \mathscr{H}om_{\mathscr{O}}(\mathscr{G}, \operatorname*{colim}_{j} \mathscr{H}om_{\mathscr{O}}(\mathscr{O}_X/\mathscr{I}_Z^j, \mathscr{F}))\\ &\simeq \operatorname*{colim}_{j} \mathscr{H}om_{\mathscr{O}}(\mathscr{G}, \mathscr{H}om_{\mathscr{O}}(\mathscr{O}_X/\mathscr{I}_Z^j, \mathscr{F}))\\ &\simeq \operatorname*{colim}_{j} \mathscr{H}om_{\mathscr{O}}(\mathscr{O}_X/\mathscr{I}_Z^j, \mathscr{H}om_{\mathscr{O}}(\mathscr{G}, \mathscr{F})).\end{aligned}$$

(ii) The second isomorphism follows from

$$\mathscr{G} \otimes_{\mathscr{O}} \mathscr{O}_X/\mathscr{I}_Z^j \simeq \mathscr{G}/\mathscr{I}_Z^j\mathscr{G}.$$

□

The functor $\Gamma_{[Z]}$ for $\mathscr{D}$-modules

Note that $\mathscr{D}_X$ being flat over $\mathscr{O}_X$,

$$\mathscr{D}_X \mathscr{I}_Z^j \simeq \mathscr{D}_X \otimes_{\mathscr{O}} \mathscr{I}_Z^j,$$
$$\mathscr{D}_X / \mathscr{D}_X \mathscr{I}_Z^j \simeq \mathscr{D}_X \otimes_{\mathscr{O}} \mathscr{O}_X / \mathscr{I}_Z^j.$$

Hence, if $\mathscr{M}$ is a $\mathscr{D}_X$-module:

$$\Gamma_{[Z]}\mathscr{M} \simeq \operatorname*{colim}_j \mathscr{H}om_{\mathscr{D}}(\mathscr{D}_X/\mathscr{D}_X\mathscr{I}_Z^j, \mathscr{M}). \tag{2.7.4}$$

Proposition 2.7.5. *Let $\mathscr{M}$ be a left $\mathscr{D}_X$-module. Then $\Gamma_{[Z]}\mathscr{M}$ is naturally endowed with a strucure of a left $\mathscr{D}_X$-module.*

Proof. The proof decomposes into several steps.
(i) Let $\mathscr{I}$ be an ideal of $\mathscr{O}_X$. Then

$$\mathscr{I}^{n+m}\mathrm{Fl}_m\mathscr{D}_X \subset \mathrm{Fl}_m\mathscr{D}_X\mathscr{I}^n. \tag{2.7.5}$$

First, we treat the case $m = 1$. Let $v \in \mathrm{Fl}_1\mathscr{D}_X$ and let $a_1, \dots, a_n \in \mathscr{I}$. Then

$$a_0 \cdots a_n v = v a_0 \cdots a_n - \sum_{i=0}^{n} [v, a_i] a_0 \cdots \widehat{a_i} \cdots a_n \in F_1\mathscr{I}^n.$$

The inclusion (2.7.5) follows by induction. Indeed, $\mathrm{Fl}_m\mathscr{D}_X = \mathrm{Fl}_1\mathscr{D}_X\mathrm{Fl}_{m-1}\mathscr{D}_X$, and we get

$$\begin{aligned}\mathscr{I}^{n+m}\mathrm{Fl}_m\mathscr{D}_X &\simeq \mathscr{I}^{n+m}\mathrm{Fl}_1\mathscr{D}_X\mathrm{Fl}_{m-1}\mathscr{D}_X \\ &\subset \mathrm{Fl}_1\mathscr{D}_X\mathscr{I}^{n+m-1}\mathrm{Fl}_{m-1}\mathscr{D}_X \\ &\subset \mathrm{Fl}_1\mathscr{D}_X\mathrm{Fl}_{m-1}\mathscr{D}_X\mathscr{I}^n.\end{aligned}$$

(ii) Let Z be a closed analytic subset. It follows that if $P \in F_m\mathscr{D}_X$, then $\cdot P$ defines a morphism $\mathscr{D}_X\mathscr{I}_Z^{n+m} \xrightarrow{\cdot P} \mathscr{D}_X\mathscr{I}_Z^n$, hence a morphism

$$P\cdot\colon \mathscr{H}om_{\mathscr{D}}(\mathscr{D}_X/\mathscr{D}_X\mathscr{I}_Z^{j+m}, \mathscr{M}) \to \mathscr{H}om_{\mathscr{D}}(\mathscr{D}_X/\mathscr{D}_X\mathscr{I}_Z^j, \mathscr{M}).$$

It follows from (2.7.4) that P acts on $\Gamma_{[Z]}\mathscr{M}$.. □

Definition 2.7.6. We denote by $\mathscr{J}$ the full additive subcategory of $\mathscr{D}_X$ consisting of objects $\mathscr{M}$ such that $\mathscr{M}_x$ is $\mathscr{O}_{X,x}$-injective for all $x \in X$.

Lemma 2.7.7. *Let Z be an closed analytic subset. The category $\mathscr{J}$ satisfies:*

(i) *for any $\mathscr{M} \in \mathrm{Mod}(\mathscr{D}_X)$, there exists $\mathscr{N} \in \mathscr{J}$ and a monomorphism $\mathscr{M} \rightarrowtail \mathscr{N}$,*

(ii) *for any exact sequence $0 \to \mathscr{M}' \to \mathscr{M} \to \mathscr{M}'' \to 0$ with $\mathscr{M}$ and $\mathscr{M}'$ in $\mathscr{J}$, then $\mathscr{M}'' \in \mathscr{J}$,*

(iii) *for any exact sequence as above with $\mathscr{M}'$ in $\mathscr{J}$, the sequence $0 \to \Gamma_{[Z]}\mathscr{M}' \to \Gamma_{[Z]}\mathscr{M} \to \Gamma_{[Z]}\mathscr{M}'' \to 0$ is exact,*

(iv) *for any $\mathscr{M} \in \mathscr{J}$, $\Gamma_{[Z]}\mathscr{M} \in \mathscr{J}$.*

Proof. (i)–(ii) are easy and left to the reader.
(iii) It is enough to check that this sequence is exact after applying the functor $(\bullet)_x$ for $x \in X$. Indeed, the sequence $0 \to \mathscr{M}'_x \to \mathscr{M}_x \to \mathscr{M}''_x \to 0$ is exact and the sequence obtained by applying the functor $\mathscr{H}om_{\mathscr{O}}(\mathscr{O}/\mathscr{I}_Z^j, \bullet)$ will remain exact since $\mathscr{M}'_x$ is $\mathscr{O}_{X,x}$-injective. Then the result follows from Proposition 2.7.3.
(iv) By Proposition 2.7.3, it is enough to check that $\Gamma_{[Z]}\mathscr{M}_x$ is $\mathscr{O}_{X,x}$-injective. By classical results (see [We94, Ch. 2 § 3]) we are thus reduced to show that if $\mathscr{G}' \subset \mathscr{G}$ are coherent $\mathscr{O}_X$-modules, then $\mathscr{H}om_{\mathscr{O}}(\mathscr{G}, \Gamma_{[Z]}\mathscr{M}) \to \mathscr{H}om_{\mathscr{O}}(\mathscr{G}', \Gamma_{[Z]}\mathscr{M})$ is an epimorphism. Since $\mathscr{M}_x$ is injective for all $x \in X$ and $\mathscr{G}$, $\mathscr{G}'$ are coherent, the sequence

$$\mathscr{H}om_{\mathscr{O}}(\mathscr{G}/\mathscr{I}_Z^j\mathscr{G}, \mathscr{M}) \to \mathscr{H}om_{\mathscr{O}}(\mathscr{G}'/(\mathscr{G}' \cap \mathscr{I}_Z^j\mathscr{G}), \mathscr{M}) \to 0$$

is exact. Hence, it is enough to prove the isomorphism

$$\operatorname*{colim}_j \mathscr{H}om_{\mathscr{O}}(\mathscr{G}'/\mathscr{I}_Z^j\mathscr{G}', \mathscr{M}) \xrightarrow{\sim} \operatorname*{colim}_j \mathscr{H}om_{\mathscr{O}}(\mathscr{G}'/(\mathscr{G}' \cap \mathscr{I}_Z^j\mathscr{G}), \mathscr{M}).$$

This follows from the Artin-Rees theorem (see Theorem 3.5.10) which asserts that there locally exists $r \gg 0$ such that $\mathscr{G}' \cap (\mathscr{I}_Z^{j+r}\mathscr{G}) \subset \mathscr{I}_Z^j\mathscr{G}'$. □

We can define the right derived functor $\mathrm{R}\Gamma_{[Z]}\colon \mathrm{D^b}(\mathscr{D}_X) \to \mathrm{D^b}(\mathscr{D}_X)$. Using the category $\mathscr{J}$, we obtain

Proposition 2.7.8. *Let $\mathscr{F} \in \mathrm{D^b}(\mathscr{O}_X)$.*

(i) $\mathrm{R}\Gamma_{[Z_1]} \circ \mathrm{R}\Gamma_{[Z_2]} \simeq \mathrm{R}\Gamma_{[Z_1 \cap Z_2]}$,

(ii) *if $\mathscr{G}$ is $\mathscr{O}$-coherent,* $\mathrm{R}\mathscr{H}om_{\mathscr{O}}(\mathscr{G}, \mathrm{R}\Gamma_{[Z]}\mathscr{F}) \simeq \mathrm{R}\Gamma_{[Z]}\mathrm{R}\mathscr{H}om_{\mathscr{O}}(\mathscr{G}, \mathscr{F})$.

Proof. Remark first that it follows from Lemma 2.7.7 that if $\mathscr{F} \in \mathrm{Mod}(\mathscr{D}_X)$ and $\mathscr{F} \to \mathscr{F}^\bullet$ is a qis with $\mathscr{F}^\bullet \in \mathrm{C}^+(\mathscr{J})$, then $\mathrm{R}\Gamma_{[Z]}\mathscr{F} \simeq \Gamma_{[Z]}\mathscr{F}^\bullet$ in $\mathrm{D^b}(\mathscr{D}_X)$.
(i) By Proposition 2.7.2, it is enough to prove that the derived functor of $\Gamma_{[Z_1]} \circ \Gamma_{[Z_2]}$ is the composition $\mathrm{R}\Gamma_{[Z_1]} \circ \mathrm{R}\Gamma_{[Z_2]}$. This follows from Lemma 2.7.7 (iv).
(ii) We may assume that $\mathscr{F} \in \mathscr{J}$. In this case the formula reduce to the first isomorphism in Proposition 2.7.4. □

Proposition 2.7.9. *Let $\mathscr{N} \in \mathrm{D^b}(\mathscr{D}_X)$ and $\mathscr{M} \in \mathrm{D^b_{gd}}(\mathscr{D}_X)$. Then there is a natural isomorphism* $\mathrm{R}\Gamma_{[Z]}(\mathscr{N} \overset{\mathrm{L}}{\underline{\otimes}} \mathscr{M}) \simeq (\mathrm{R}\Gamma_{[Z]}\mathscr{N}) \overset{\mathrm{L}}{\otimes}_{\mathscr{O}} \mathscr{M}$ *in* $\mathrm{D^b}(\mathscr{D}_X)$.

Proof. (i) First, we construct the morphism. One proves the isomorphism

$$\mathrm{R}\Gamma_{[Z]}(\mathrm{R}\Gamma_{[Z]}\mathscr{N} \overset{\mathrm{L}}{\underline{\otimes}} \mathscr{M}) \simeq \mathrm{R}\Gamma_{[Z]}\mathscr{N} \overset{\mathrm{L}}{\underline{\otimes}} \mathscr{M}.$$

(We shall not give the proof here.) Hence, the morphism $\mathrm{R}\Gamma_{[Z]}\mathscr{N} \overset{\mathrm{L}}{\underline{\otimes}} \mathscr{M} \to \mathscr{N} \overset{\mathrm{L}}{\underline{\otimes}} \mathscr{M}$ factorizes uniquely through $\mathrm{R}\Gamma_{[Z]}\mathscr{N} \overset{\mathrm{L}}{\underline{\otimes}} \mathscr{M} \to \mathrm{R}\Gamma_{[Z]}(\mathscr{N} \overset{\mathrm{L}}{\underline{\otimes}} \mathscr{M})$.

(ii) Then, we prove the isomorphism in $\mathrm{D^b}(\mathscr{O}_X)$, that is, for the functor $\overset{\mathrm{L}}{\otimes}_{\mathscr{O}}$. By dévissage, we reduce to the case where $\mathscr{N}$ and $\mathscr{M}$ belong to $\mathrm{Mod}(\mathscr{O}_X)$. Then, we may reduce to the case where $\mathscr{M}$ is coherent. Set $\mathscr{M}^* = \mathrm{R}\mathscr{H}om_{\mathscr{O}}(\mathscr{M},\mathscr{O})$. In this case,

$$\begin{aligned}(\mathrm{R}\Gamma_{[Z]}\mathscr{N})\overset{\mathrm{L}}{\otimes}_{\mathscr{O}}\mathscr{M} &\simeq \mathrm{R}\mathscr{H}om_{\mathscr{O}}(\mathscr{M}^*,\mathrm{R}\Gamma_{[Z]}\mathscr{N})\\ &\simeq \mathrm{R}\Gamma_{[Z]}\mathrm{R}\mathscr{H}om_{\mathscr{O}}(\mathscr{M}^*,\mathscr{N})\\ &\simeq \mathrm{R}\Gamma_{[Z]}(\mathscr{N}\overset{\mathrm{L}}{\otimes}_{\mathscr{O}}\mathscr{M}).\end{aligned}$$

(iii) The morphism in (i) is an isomorphism by (ii). □

The $\mathscr{D}_X$-module $\mathscr{B}_{Z|X}$

Lemma 2.7.10. *Let Z be a closed analytic subset of X. Then*

$$H^k(\mathrm{R}\Gamma_{[Z]}\mathscr{O}_X) \simeq \operatorname*{colim}_j \mathscr{E}xt^k_{\mathscr{O}}(\mathscr{O}_X/\mathscr{I}_Z^j,\mathscr{O}_X). \tag{2.7.6}$$

Proof. Let $\mathscr{F}^\bullet$ be a resolution of $\mathscr{O}_X$ with $\mathscr{F}^j \in \mathscr{J}$. Then the left hand side of (2.7.6) is the k-th cohomology object of $\operatorname*{colim}_j \mathscr{H}om_{\mathscr{O}}(\mathscr{O}_X,\mathscr{F}^\bullet)$. Sincve the inductive limit is filtrant, it commutes with H^k. Moreover,

$$H^k(\mathscr{H}om_{\mathscr{O}}(\mathscr{O}_X/\mathscr{I}_Z^j,\mathscr{F}^\bullet)) \simeq \mathscr{E}xt^k_{\mathscr{O}}(\mathscr{O}_X/\mathscr{I}_Z^j,\mathscr{O}_X),$$

since the germs of the $\mathscr{F}^j$'s are $\mathscr{O}_{X,x}$-injective and $\mathscr{O}_X/\mathscr{I}_Z^j$ is $\mathscr{O}_X$-coherent. □

Recall that if Z is a closed complex analytic hypersurface of X and $j\colon (X\setminus Z)\hookrightarrow X$ is the open embedding, the sheaf $j_*j^{-1}\mathscr{O}_X$ describes the sheaf of holomorphic functions on $X\setminus Z$, with essential singularities on Z. It contains the subsheaf $\mathscr{O}_X[*Z]$ of meromorphic functions with poles in Z. If $\{f=0\}$ is an equation of Z (such an f exists locally), then $\mathscr{O}_X[*Z]\simeq\mathscr{O}_X[1/f]$.

Proposition 2.7.11. (i) *Let Z be a closed analytic subset of codimension $\geq l$. Then $H^j(\mathrm{R}\Gamma_{[Z]}\mathscr{O}_X)=0$ for $j<l$.*

(ii) *If Z is a hypersurface, then $H^j(\mathrm{R}\Gamma_{[Z]}\mathscr{O}_X)=0$ for $j\neq 1$ and if $\{f=0\}$ is an equation of Z then $H^1(\mathrm{R}\Gamma_{[Z]}\mathscr{O}_X)\simeq\mathscr{O}_X[1/f]/\mathscr{O}_X$.*

Proof. (i) using (2.7.6), this is a particular case of Theorem 3.5.6.
(ii) For $j>0$, let us apply the left exact functor $\mathscr{H}om_{\mathscr{O}}(\,\bullet\,,\mathscr{O}_X)$ to the exact sequence $0\to\mathscr{O}_X\xrightarrow{f^j}\mathscr{O}_X\to\mathscr{O}_X/\mathscr{I}_Z^j\to 0$. We get the sequence

$$0\to\mathscr{O}_X\xrightarrow{f^j}\mathscr{O}_X\to\mathscr{E}xt^1_{\mathscr{O}}(\mathscr{O}_X/\mathscr{I}_Z^j,\mathscr{O}_X)\to 0.$$

Hence, $H^1(\mathrm{R}\Gamma_{[Z]}\mathscr{O}_X)\simeq\operatorname*{colim}_j\mathscr{O}_X/f^j\mathscr{O}_X$. The isomorphism $\operatorname*{colim}_j\mathscr{O}_X/f^j\mathscr{O}_X\xrightarrow{\sim}\mathscr{O}_X[1/f]/\mathscr{O}_X$ associates $1/f^j\in\mathscr{O}_X[1/f]/\mathscr{O}_X$ to the image of $1\in\mathscr{O}_X$ in $\mathscr{O}_X/f^j\mathscr{O}_X$. □

Recall that $H^d_Z(\,\bullet\,)$ is the d-th derived functor of the functor $\Gamma_Z(\,\bullet\,)\colon \mathrm{Mod}(\mathbb{C}_X)\to\mathrm{Mod}(\mathbb{C}_X)$.

Definition 2.7.12. When Z is a closed subset of pure codimension d, one sets

$$\mathscr{B}_{Z|X}=H^d(\mathrm{R}\Gamma_{[Z]}\mathscr{O}_X),\ \mathscr{B}^\infty_{Z|X}=H^d_Z(\mathscr{O}_X).$$

Note that

$$\mathscr{B}_{X|X} = \mathscr{O}_X.$$

Also note that the morphism of functors $\mathrm{R}\Gamma_{[Z]}(\bullet) \to \Gamma_Z(\bullet)$ defines the morphism $\mathrm{R}\Gamma_{[Z]}(\bullet) \to \mathrm{R}\Gamma_Z(\bullet)$ and in particular, the morphism

$$\mathscr{B}_{Z|X} \to \mathscr{B}^{\infty}_{Z|X}.$$

Recall that a closed analytic subvariety of codimension d is called a local complete intersection if locally on X there exists d holomorphic functions $f_1, \dots, f_d$ such that, setting $Z_j = \{x \in X; f_j = 0\}$, $Z = \bigcap_j^d Z_j$.

Proposition 2.7.13. *Assume $Z = \bigcap_j^d Z_j$ is a local complete intersection of codimension d. Then $H^j(\mathrm{R}\Gamma_{[Z]}\mathscr{O}_X) = 0$ for $j \neq d$ and*

$$\mathscr{B}_{Z|X} \simeq \mathscr{B}_{Z_1|X} \overset{\mathrm{L}}{\underline{\otimes}} \cdots \overset{\mathrm{L}}{\underline{\otimes}} \mathscr{B}_{Z_d|X}. \tag{2.7.7}$$

Proof. Since $\mathscr{B}_{Z|X}$ is concentrated in degree ≥ 0 and the right-hand side of (2.7.7) is concentrated in degree ≤ 0, it is enough to prove this formula. One has

$$\mathrm{R}\Gamma_{[Z]}\mathscr{O}_X[d] \simeq \mathrm{R}\Gamma_{[Z_1]}\mathscr{O}_X[1] \overset{\mathrm{L}}{\underline{\otimes}} \cdots \overset{\mathrm{L}}{\underline{\otimes}} \mathrm{R}\Gamma_{[Z_d]}\mathscr{O}_X[1]. \tag{2.7.8}$$

Since each $\mathrm{R}\Gamma_{[Z_i]}\mathscr{O}_X[1]$ is concentrated in degree 0, the result follows. □

Corollary 2.7.14. *Let $Z = \{f_j = 0; j = 1, \dots, d\}$ be a complete intersection. Then*

$$\mathscr{B}_{Z|X} \simeq \mathscr{O}_X[1/f_1 \dots f_d] / \sum_i \mathscr{O}_X[1/f_1 \dots \widehat{f_i} \dots f_d]. \tag{2.7.9}$$

Corollary 2.7.15. *Let $x = (x', x'')$ be a local coordinate system on X, with $x' = (x_1, \dots, x_d)$. Assume $Z = \{x' = 0\}$. Then*

$$\mathscr{B}_{Z|X} \simeq \mathscr{D}_X / \mathscr{D}_X(x', \partial_{x''}).$$

Corollary 2.7.16. *Let Z be a closed smooth submanifold of X. Then $\mathscr{B}_{Z|X}$ is a coherent $\mathscr{D}_X$-module and its characteristic variety is $T^*_Z X$, the conormal bundle to Z in X.*

Notation 2.7.17. Let f be a non zero section of $\mathscr{O}_X$ (on a connected open set) and let $Z = \{f = 0\}$. One denotes by $\delta(f)$ the generator of $\mathscr{B}_{Z|X} \simeq \mathscr{O}_X[*Z]/\mathscr{O}_X$ associated with $1/f$.

Let Z_1 and Z_2 be hypersurfaces and assume $Z_1 \cap Z_2$ has codimension 2. Consider the diagram below in which all morphisms are isomorphisms:

$$\begin{array}{ccc}
\mathscr{B}_{Z_1|X} \otimes_{\mathscr{O}} \mathscr{B}_{Z_2|X} & \longrightarrow & \mathscr{B}_{Z_1 \cap Z_2|X} \\
\downarrow & & \| \\
\mathscr{B}_{Z_2|X} \otimes_{\mathscr{O}} \mathscr{B}_{Z_1|X} & \longrightarrow & \mathscr{B}_{Z_1 \cap Z_2|X}
\end{array}$$

Note that $\delta(f_1) \otimes \delta(f_2)$ is a generator of $\mathscr{B}_{Z_1 \cap Z_2|X}$ and

$$\delta(f_1) \otimes \delta(f_2) = -\delta(f_2) \otimes \delta(f_1). \tag{2.7.10}$$

Remark 2.7.18. One proves similarly that $Z = \bigcap_j^d Z_j$ being a local complete intersection of codimension d, then $H^j_Z(\mathscr{O}_X) = 0$ for $j \neq d$ and

$$\mathscr{B}^{\infty}_{Z|X} \simeq \mathscr{B}^{\infty}_{Z_1|X} \overset{\mathrm{L}}{\underline{\otimes}} \cdots \overset{\mathrm{L}}{\underline{\otimes}} \mathscr{B}^{\infty}_{Z_d|X}. \tag{2.7.11}$$

Proposition 2.7.19. *Let Z be a complete intersection of codimension d and assume $\mathscr{I}_Z = \mathscr{O}_X f_1 + \cdots + \mathscr{O}_X f_d$. Then the section*

$$\delta(f_1) \otimes \cdots \otimes \delta(f_d) \otimes df_1 \wedge \cdots \wedge df_d \in \mathscr{B}_{Z|X} \otimes_{\mathscr{O}} \Omega^d_X$$

does not depend on the choice of the sequence $(f_1, \ldots, f_d)$.

Proof. Let $(f'_1, \ldots, f'_d)$ be another sequence defining the ideal $\mathscr{I}_Z$. There exists a section $A \in Gl(\mathscr{O}_X, d)$ which interchanges these two sequences. The group $Gl(\mathscr{O}_X, d)$ is generated by the transformations

(i) $(f_1, \ldots, f_d) \mapsto (af_1, \ldots, f_d)$, with $a \in \mathscr{O}^{\times}_X$,

(ii) $(f_1, \ldots, f_i, f_{i+1}, \ldots f_d) \mapsto (f_1, \ldots, f_{i+1}, f_i, \ldots, f_d)$

(iii) $(f_1, \ldots, f_d) \mapsto (f_1, f_2 + bf_1, \ldots, f_d)$

Then, it is enough to notice that

$$\begin{aligned}
&1/af_1 \cdot 1/f_2 d(af_1) \wedge df_2 = 1/f_1 \cdot 1/f_2 df_1 \wedge df_2,\\
&1/f_2 \cdot 1/f_1 df_2 \wedge df_1 = 1/f_1 \cdot 1/f_2 df_1 \wedge df_2\\
&1/f_1 \cdot 1/(f_2 + bf_1) df_1 \wedge d(f_2 + bf_1) = 1/f_1 \cdot 1/f_2 df_1 \wedge df_2.
\end{aligned}$$

□

Definition 2.7.20. Assume that Z is smooth of codimension d. We shall denote by $\delta(Z)dx$ the canonical section of $\mathscr{B}_{Z|X} \otimes_{\mathscr{O}} \Omega^d_X$ constructed in Proposition 2.7.19. One calls it the fundamental class of Z in X.

Note that $\delta(Z)dx$ belongs to $\bigwedge^d \mathscr{L}_Z$ where $\mathscr{L}_Z$ denotes the subsheaf of Ω^1_X consisting of sections with values in the conormal bundle $T^*_Z X$.

Denote by Δ the diagonal in $X \times X$ and by q_1 and q_2 the first and second projections $X \times X \to X$. The projection q_2 allows us to identify $T^*_\Delta X \times X$ with T^*X. There is a natural $\mathscr{D}_X \otimes \mathscr{D}^{\mathrm{op}}_X$-linear morphism

$$\mathscr{D}_X \to \mathscr{B}_{\Delta|X \times X} \otimes_{q_2^{-1}\mathscr{O}} q_2^{-1}\Omega_X, \tag{2.7.12}$$

given by $1 \mapsto \delta(\Delta)dx$

Proposition 2.7.21. *The morphism (2.7.12) is an isomorphism.*

Proof. We may choose a local coordinate system (x) on X and denote by (y) a copy of this system. Then (x, y) is a local coordinate system on $X \times X$. Replace this coordinate system by the new system $(u, v) = (x + y, x - y)$. Then $\mathscr{B}_{\Delta|X \times X}$ is isomorphic to $\mathscr{D}_{X \times X}/(v, \partial_u)$ and the result follows. □

Exercises to Chapter 2

Exercise 2.1. Let Z_1 and Z_2 be two smooth submanifolds of X and assume they are transversal. Calculate
(i) $\mathrm{R}\mathscr{H}om_{\mathscr{D}}(\mathscr{B}_{Z_1|X}, \mathscr{B}_{Z_2|X})$,
(ii) $\mathscr{B}_{Z_1|X} \underset{=}{\overset{\mathrm{L}}{\otimes}} \mathscr{B}_{Z_2|X}$.

Exercise 2.2. Let $f\colon X \to Y$ be a morphism of complex manifolds and let Z be a smooth closed submanifold of Y. Assume that f is transversal to Z, that is, f is non characteristic for $T^*_Z Y$, or, equivalently, for $\mathscr{B}_{Z|Y}$. Prove that $S := f^{-1}Z$ is a smooth closed submanifold of X and that $f_{\mathscr{D}}^{-1}\mathscr{B}_{Z|Y} \simeq \mathscr{B}_{S|X}$.

Exercise 2.3. Denote by $j\colon Z \hookrightarrow X$ the closed embedding of a smooth submanifold Z of X.
(i) Prove that $\mathscr{B}_{Z|X} \simeq j_!^{\mathscr{D}} \mathscr{O}_Z$.
(ii) Calculate $\mathrm{R}\mathscr{H}om_{\mathscr{D}}(\mathscr{B}_{Z|X}, \mathscr{D}_X)$ for a smooth submanifold Z of X.

Exercise 2.4. Let $\mathscr{M} \in \mathrm{Mod_c}(\mathscr{D}_X)$ and assume that $\mathrm{char}(\mathscr{M}) \subset T^*_X X$. Prove that locally on X, there is an isomorphism of $\mathscr{D}_X$-modules $\mathscr{M} \simeq \mathscr{O}_X^N$ for some integer N. (Hint: see [Ka03, Prop. 4.43]).

Exercise 2.5. Let $f\colon X \to Y$ be a morphism of complex manifolds. Let $\mathscr{M} \in \mathrm{D^b}(\mathscr{D}_X^{\mathrm{op}})$ and let $\mathscr{N} \in \mathrm{D^b}(\mathscr{D}_Y)$. Prove that there is a natural isomorphism in $\mathrm{D^b}(\mathbb{C}_Y)$

$$\mathrm{R}f_!(\mathscr{M} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}} f_{\mathscr{D}}^{-1}\mathscr{N}) \simeq f_!^{\mathscr{D}} \mathscr{M} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}} \mathscr{N}.$$

Exercise 2.6. Let X and Y be two complex manifolds and denote by q_i the i-th projection defined on $X \times Y$ and by p_i the i-th projection defined on $T^*X \times T^*Y$ $(i = 1, 2)$. Let $\mathscr{M} \in \mathrm{D^b}(\mathscr{D}_X)$ and $\mathscr{L} \in \mathrm{D^b}(\mathscr{D}_{X\times Y}^{\mathrm{op}})$.
(i) Prove the isomorphism

$$\mathscr{L} \circ \mathscr{M} := q_{2!}^{\mathscr{D}}(\mathscr{L} \underset{=}{\overset{\mathrm{L}}{\otimes}} q_1{}_{\mathscr{D}}^{-1}\mathscr{M}) \simeq \mathrm{R}q_{2!}(\mathscr{L} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}} q_1^{-1}\mathscr{M}).$$

(ii) Assume now that $\mathscr{M} \in \mathrm{D^b_{gd}}(\mathscr{D}_X)$, $\mathscr{L} \in \mathrm{D^b_{gd}}(\mathscr{D}_{X\times Y}^{\mathrm{op}})$ and that p_2 is proper on $p_1^{-1}\,\mathrm{char}(\mathscr{M}) \cap \mathrm{char}(\mathscr{L})$. Prove that $p_1^{-1}\,\mathrm{char}(\mathscr{M}) \cap \mathrm{char}(\mathscr{L}) \subset T^*_{X\times Y} X \times Y$ and that $\mathscr{L} \circ \mathscr{M}\mathscr{M} \in \mathrm{D^b_{gd}}(\mathscr{D}_Y^{\mathrm{op}})$.
(iii) Show that the construction of the inverse or direct image of a $\mathscr{D}$-module can be obtained by this procedure.

Chapter 3

Appendix

In this Appendix, we collect basic and classical results of various fields of Mathematics which are of constant use in $\mathscr{D}$-modules theory.

We give a few proofs of results that, although elementary, are not always well-known. Here, $\mathbf{k}$ denotes a commutative ring.

3.1 Symplectic geometry

The theory developed in this section works for real vector spaces and real manifolds, as well as for complex vector spaces and complex manifolds.

Linear symplectic geometry

A finite dimensional symplectic vector space (E, θ) is a finite dimensional vector space E endowed with a non degenerate skew symmetric 2-form θ. In such a case E has even dimension.

Definition 3.1.1. A symplectic basis on a symplectic vector space (E, θ) is a basis $(e; f) = (e_1, \ldots, e_n; f_1, \ldots, f_n)$ such that denoting by $(e^*; f^*) = (e_1^*, \ldots, e_n^*; f_1^*, \ldots, f_n^*)$ the dual basis, on E^*, one has

$$\theta = \sum_{i=1}^{n} f_i^* \wedge e_i^*.$$

One proves easily that any finite dimensional symplectic vector space (E, θ) admits a symplectic basis.

Example 3.1.2. Let V be a finite dimensional vector space. The space $E = V \oplus V^*$ is endowed with a symplectic structure, by setting for $(x; \xi) \in V \oplus V^*$:

$$\theta((x; \xi)(x'; \xi')) = \langle x', \xi \rangle - \langle x, \xi' \rangle.$$

Since θ is non degenerate, it defines an isomorphism

$$\begin{aligned} &H : E^* \xrightarrow{\sim} E \\ &\langle \xi, v \rangle = \theta(v, H(\xi)), \quad v \in E, \xi \in E^*. \end{aligned}$$

The isomorphism H is called the Hamiltonian isomorphism. If $\xi \in E^*$, one also writes H_ξ instead of $H(\xi)$ and calls H_ξ the Hamiltonian vector of ξ.

The Poisson bracket. denoted $\{\cdot,\cdot\}$, is the symplectic form on E^*, the image of θ by H. It is thus given by:

$$\{\xi,\eta\} = \theta(H^{-1}(\xi), H^{-1}(\eta)).$$

If E is endowed with a symplectic basis and one calculates the image by H of the dual symplectic basis, one finds

$$H(e_i^*) = -f_i, \quad H(f_i^*) = e_i. \tag{3.1.1}$$

Let ρ be a linear subspace of E. One sets

$$\rho^\perp = \{v \in E; \theta(v,\rho) = 0.$$

Note that

$$\rho^{\perp\perp} = \rho, \quad (\rho_1+\rho_2)^\perp = \rho_1^\perp \cap \rho_2^\perp, (\rho_1\cap\rho_2)^\perp = \rho_1^\perp + \rho_2^\perp.$$

Definition 3.1.3. A linear subspace ρ of E is called

(i) isotropic if $\rho \subset \rho^\perp$,

(ii) involutive (or else, co-isotropic) if $\rho^\perp \subset \rho$,

(iii) Lagrangian if $\rho = \rho^\perp$.

Note that if $\dim E = 2n$ and ρ is isotropic (resp. involutive, resp. Lagrangian), then $\dim\rho \leq n$ (resp. $\dim\rho \geq n$, resp. $\dim\rho = n$). A line is always isotropic and a hyperplane is always involutive.

Symplectic manifolds

A real or complex symplectic manifold $(\mathfrak{X},\theta)$ is a manifold $\mathfrak{X}$ endowed with a closed 2-form θ such that θ^n never vanishes.

At each $p \in T^*X$, the 2-form $\theta_X(p)$ is a bilinear skew symmetric non degenerate form on T_pT^*X, hence induces a linear isomorphism $H(p) : T_p^*\mathfrak{X} \simeq T_p\mathfrak{X}$. Hence θ defines an isomorphism of vector bundles

$$H : T^*\mathfrak{X} \simeq T\mathfrak{X},$$

or, equivalenly, a sheaf isomorphism

$$H : \Theta_{\mathfrak{X}} \simeq \Omega_{\mathfrak{X}}. \tag{3.1.2}$$

Definition 3.1.4. (i) Let f be a section of the sheaf $\mathscr{O}_{\mathfrak{X}}$, one sets

$$H_f = H(df),$$

the section of $\Theta_{\mathfrak{X}}$ associated with df by the isomorphism (3.1.2). One calls H_f the Hamiltonian vector field of f.

(ii) Given two sections f and g of $\mathscr{O}_{\mathfrak{X}}$, one defines their Poisson bracket $\{f,g\}$ as

$$\{f,g\} = H_f(g). \tag{3.1.3}$$

The Poisson bracket satisfies the Jacobi identities:

$$\begin{cases} \{f,g\} = -\{g,f\} \\ \{f,hg\} = h\{f,g\} + g\{f,h\} \\ \{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} = 0 \end{cases} \tag{3.1.4}$$

Moreover,

$$[H_f, H_g] = H_{\{f,g\}}. \tag{3.1.5}$$

Definition 3.1.5. A symplectic local coordinate system $(x;\xi)$ is a local coordinate system $(x_1,\dots,x_n;\xi_1\dots,\xi_n)$ on $\mathfrak{X}$ such that

$$\theta = \sum_i d\xi_i \wedge dx_i. \tag{3.1.6}$$

The Darboux Theorem asserts that a symplectic local coordinate system always locally exists.

In a symplectic local coordinate system, one finds, using (3.1.1):

$$H_{x_i} = -\partial_{\xi_i}, \quad H_{\xi_i} = \partial_{x_i} \tag{3.1.7}$$

$$H_f = \sum_{i=1}^{n} \big(\frac{\partial f}{\partial_{\xi_i}}\frac{\partial}{\partial_{x_i}} - \frac{\partial f}{\partial_{x_i}}\frac{\partial}{\partial_{\xi_i}}\big) \tag{3.1.8}$$

$$\{f,g\} = \sum_{i=1}^{n} \big(\frac{\partial f}{\partial_{\xi_i}}\frac{\partial g}{\partial_{x_i}} - \frac{\partial f}{\partial_{x_i}}\frac{\partial g}{\partial_{\xi_i}}\big). \tag{3.1.9}$$

If S is a locally closed analytic subvariety of a smooth complex manifold X, one denotes by S_{reg} the manifold given by the non singular points of S, and by $\mathscr{I}_S$ the defining sheaf of ideals of S.

Definition 3.1.6. Let V be a locally closed analytic subset of $\mathfrak{X}$. One says that V isotropic (resp. involutive, resp. Lagrangian) if for each $p \in V_{reg}$, the vector space T_pV_{reg} is isotropic (resp. involutive, resp. Lagrangian) in $T_p\mathfrak{X}$.

One can prove that V is involutive if and only if its symbol ideal $\mathscr{I}_V$ is stable by the Poisson product, that is, if for any f, g vanishing on V, the function $\{f,g\}$ also vanishes on V. If V is involutive, then all irreductible components of V have dimension at least n.

If V is smooth, then V is involutive if and only if for any function f which vanishes on V, then H_f is tangent to V. Indeed, $TV^\perp$ is generated by the vector fields H_f, with $f|_V = 0$. By (3.1.5), it follows that the sub-bundle $TV^\perp$ of TV is table by brackets, that is, satisfies the Frobenius integrability conditions. Therefore there exists a foliation of V, and the leaves of this foliations are called the "bicharacteristic leaves" of V.

An involutive manifold has dimension $\geq n$. A hypersurface is always involutive.

One proves that V is isotropic if and only if, for any manifold S and any morphism $f : S \to V$, the 2-form $f^*\theta_X$ vanishes. If V is isotropic, then all irreductible components of V have dimension at most n. A curve is always isotropic.

If V is Lagrangian, then it is pure dimensional.

Realification of complex cotangent bundles

For a complex manifold X we denote by $X_{\mathbb{R}}$ the real underlying submanifold to X. When there is no risk of confusion, we simply write X instead of $X_{\mathbb{R}}$.

We denote by $\overline{X}$ the complex conjugate manifold to X. (Recall that $\overline{X} = X$ as a topological space, but the sheaf of holomorphic functions on $\overline{X}$ is the sheaf of anti-holomorphic functions on X.) Then, identifying X with the diagonal of $X \times \overline{X}$, the complex manifold $X \times \overline{X}$ is a complexification of $X_{\mathbb{R}}$.

Denote by $d\alpha_X$ the symplectic form on T^*X and by $d\alpha_{X_{\mathbb{R}}}$ the symplectic form on $T^*X_{\mathbb{R}}$. Then

$$d\alpha_{X_{\mathbb{R}}} = 2\Re\, d\alpha_X.$$

Homogeneous symplectic manifolds

A homogeneous symplectic manifold is the data of a symplectic manifold $(\mathfrak{X}, \theta)$ together with a vector field v on $\mathfrak{X}$ such that

$$L_v\theta = \theta. \tag{3.1.10}$$

Define the 1-form ω by

$$\omega = i_v\theta. \tag{3.1.11}$$

Since $L_v = d \circ i_v + i_v \circ d$, we get

$$d\omega = \theta, \quad H(\omega) = -v. \tag{3.1.12}$$

In such a case one calls $\mathfrak{X}$ a homogeneous symplectic manifold.

Definition 3.1.7. A homogeneous symplectic local coordinate system $(x;\xi)$ is a local coordinate system $(x_1, \dots, x_n; \xi_1 \dots, \xi_n)$ such that

$$\omega = \sum_i \xi_i dx_i. \tag{3.1.13}$$

It follows from Darboux's theorem that such a local coordinate system always locally exists.

Cotangent bundle

Let X be a manifold and let $E \to X$ be a real vector bundle over X. Then E is endowed with an action of $\mathbb{R}^\times$ and in particular, an action of $\mathbb{R}^+$. One says that a subset $\Lambda \subset E$ is R^+ conic if it is invariant by this action. One defines similarly the $\mathbb{C}^\times$-conic subsets of a complex vector bundle.

If X is a manifold, we denote by $\tau\colon TX \to X$ and $\pi\colon T^*X \to X$ the tangent and cotangent bundles, respectively.

Let $f : X \to Y$ be a morphism of manifolds. To f are associated the tangent morphisms

$$TX \xrightarrow{f'} X \times_Y TY \xrightarrow{f_\tau} TY. \tag{3.1.14}$$

Taking the dual bundles, we find the canonical morphisms

$$\begin{array}{ccccc} T^*X & \xleftarrow{f_d} & X \times_Y T^*Y & \xrightarrow{f_\pi} & T^*Y \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ X & = & X & \xrightarrow{f} & Y. \end{array} \tag{3.1.15}$$

The projection $\pi : T^*X \to X$ defines $\pi_\pi : T^*X \times_X T^*X \to T^*T^*X$. By composing with the diagonal embedding $T^*X \hookrightarrow T^*X \times_X T^*X$, we find the map

$$T^*X \to T^*T^*X,$$

which is a section of the projection $T^*T^*X \to T^*X$. We have thus constructed a canonical 1-form ω_X on T^*X.

Let $x = (x_1, \ldots, x_n)$ a local coordinate system on X. It defines canonically a local coordinate system on T^*X,

$$(x;\xi) = (x_1, \ldots, x_n; \xi_1, \ldots, \xi_n) \tag{3.1.16}$$

and the 1-form ω_X associates $(x;\xi;;\xi;0)$ to $(x;\xi) \in T^*X$. Therefore

$$\omega_X = \sum_{i=1}^{n} \xi_i dx_i,$$
$$H(\omega) = \sum_{i=1}^{n} -\xi_i \frac{\partial}{\partial_{\xi_i}}.$$

The vector field $H(\omega)$ is called the Euler vector field and denoted $\mathrm{eu}_{\mathrm{T^*X}}$. It is the vector field associated with the action of $\mathbb{C}^\times$ (in case of complex manifolds, $\mathbb{R}^\times$ in case of real manifolds) on the vector bundle T^*X.

Set $\theta_X = d\omega_X$. In local coordinates,

$$\theta_X = \sum_{i=1}^{n} d\xi_i \wedge dx_i.$$

Hence, $(T^*X, \theta, \mathrm{eu}_{\mathrm{T^*X}})$ is a homogeneous symplectic manifold.

Definition 3.1.8. (i) One denotes by $T^*_X X$ the zero-section of the vector bundle T^*X.

(ii) Consider a morphism $f\colon X \to Y$ of manifolds. The conormal bundle to X in Y is the sub-vector bundle of $X \times_Y T^*Y$ given by $f_d^{-1}(T^*_X X)$.

When Z is a smooth submanifold of X, the conormal bundle $T^*_Z X$ is identified with a sub-bundle of $T * X$. Note that the zero-section $T *_X X$ is also the conormal bundle to X in X.

Let Z be a smooth submanifold to X. Then $T^*_Z X$ is a Lagrangian submanifold of T^*X and $Z \times_X T^*X$ is an involutive submanifold.

Example 3.1.9. Assume we have a local coordinate system $(x) = (x', x'')$ on X, with $(x') = (x_1, \ldots, x_p)$ and $(x'') = (x_{p+1}, \ldots, x_n)$. Let $(x;\xi) = (x', x''; \xi', \xi'')$ denote the associate coordinates on T^*X and let $Z = \{x \in X; x' = 0\}$. Then

$$T^*_Z X = \{(x;\xi) \in T^*X; x' = 0, \xi'' = 0\},$$
$$Z \times_X T^*X = \{(x;\xi) \in T^*X; x' = 0\}.$$

Non characteristic morphisms

Definition 3.1.10. Consider a morphism $f\colon X \to Y$ of real manifolds and let $\Lambda \subset T^*Y$ be a closed $\mathbb{R}^+$-conic subset. One says that f is non-caracteristic for Λ (or else, Λ is non-caracteristic for f, or f and Λ are transversal) if, with the notations in (3.1.15),

$$f_\pi^{-1}(\Lambda) \cap T^*_X Y \subset X \times_Y T^*_Y Y.$$

Lemma 3.1.11. (i) *Let Λ be a closed $\mathbb{R}^+$-conic subset of T^*Y. Then a morphism $f\colon X \to Y$ is non characteristic for Λ if and only if $f_d\colon X \times_Y T^*Y \to T^*X$ is proper on $f_\pi^{-1}(\Lambda)$.*

(ii) *In particular, if f is non characteristic for Λ, then $f_d f_\pi^{-1}(\Lambda)$ is closed and $\mathbb{R}^+$-conic in T^*X.*

(iii) *If f is a morphism of complex manifolds and Λ is a complex analytic $\mathbb{C}^\times$-conic subset, then f_d is finite on $f_\pi^{-1}(\Lambda)$ and $f_d f_\pi^{-1}(\Lambda)$ is a complex analytic $\mathbb{C}^\times$-conic subset of T^*X.*

Proof. The first assertion follows from the fact that if λ is a closed cone in a vector space E and $u\colon E \to F$ is a linear map, then $u|_\lambda$ is proper if and only if $\lambda \cap u^{-1}(0) \subset \{0\}$, and the others are easily deduced. $\square$

Example 3.1.12. Let Z be a closed and smooth submanifold of Y. Then f is non-characteristic for $T^*_Z Y$ if and only if f is transversal to Z.

Lemma 3.1.13. *Consider morphisms of real manifolds $X \xrightarrow{f} Y \xrightarrow{g} Z$ and set $h = g \circ f$. Let Λ be a closed $\mathbb{R}^+$-conic subset of T^*Z.*

(i) *Assume that g is non characteristic for Λ and f is non characteristic for $g_d g_\pi^{-1}(\Lambda)$. Then h is non characteristic for Λ.*

(ii) *Assume that h is non characteristic for Λ. Then g is non characteristic for Λ on a neighborhhod of $f(X)$ and f is non characteristic for $g_d g_\pi^{-1}\Lambda$.*

Proof. Set $\Lambda_0 = g_\pi^{-1}\Lambda$. Consider the diagram in which the square labelled $\square$ is Cartesian:

(3.1.17)
$$\begin{array}{ccccc}
T^*X & \xleftarrow{f_d} & X \times_Y T^*Y & \xleftarrow{\varphi} & X \times_Z T^*Z \\
 & & \downarrow{\scriptstyle f_\pi} & \square & \downarrow{\scriptstyle \psi} \\
 & & T^*Y & \xleftarrow{g_d} & Y \times_Z T^*Z \supset \Lambda_0 \\
 & & & & \downarrow{\scriptstyle g_\pi} \\
 & & & & T^*Z \supset \Lambda.
\end{array}$$

Note that

$$(g \circ f)_d = f_d \circ \varphi,\ (g \circ f)_\pi = g_\pi \circ \psi,$$
$$\psi^{-1}(T^*_Y Z) \subset T^*_X Z,\ \varphi^{-1}(T^*_X Y) = T^*_X Z,$$
$$\Lambda_0 := g_\pi^{-1}(\Lambda)\ f_\pi^{-1} g_d(\Lambda_0) = \varphi\psi^{-1}(\Lambda_0).$$

It follows that

$$(g\circ f)_d(g\circ f)_\pi^{-1}(\Lambda) = f_d f_\pi^{-1} g_d g_\pi^{-1}(\Lambda).$$

(i) Since f is non characteristic for $g_d g_\pi^{-1}(\Lambda)$ and $f_\pi^{-1} g_d g_\pi^{-1}(\Lambda) = \varphi\psi^{-1}g_\pi^{-1}(\Lambda)$, we get

$$f_d^{-1}(T^*_X X)\cap \varphi\psi^{-1}g_\pi^{-1}(\Lambda) \subset X\times_Y T^*_Y Y.$$

Hence

$$\begin{aligned}\varphi^{-1}f_d^{-1}(T^*_X X)\cap\psi^{-1}g_\pi^{-1}(\Lambda) &= (g\circ f)_d^{-1}(T^*_X X)\cap (g\circ f)_\pi^{-1}(\Lambda)\\ &\subset X\times_Z T^*_Z Z.\end{aligned}$$

(ii)–(a) By the hypothesis,

$$\psi^{-1}(g_\pi^{-1}(\Lambda)\cap T^*_Y Z)\subset (g\circ f)_\pi^{-1}(\Lambda)\cap T^*_X Z\subset X\times_Z T^*_Z Z.$$

Therefore, g is non characteristic for Λ on a neighborhood of $f(X)$.
(ii)–(b) We have

$$\begin{aligned}f_\pi^{-1}(g_d g_\pi^{-1}(\Lambda))\cap T^*_X Y &= \varphi\psi^{-1}g_\pi^{-1}(\Lambda)\cap T^*_X Y\\ &= \varphi((g\circ f)_\pi^{-1}(\Lambda)\cap T^*_X Z)\\ &\subset \varphi(X\times_Z T^*_Z Z)\subset X\times_Y T^*_Y Y.\end{aligned}$$

(Note that we have used the equality $\varphi(A)\cap B = \varphi(A\cap\varphi^{-1}B)$.) □

3.2 Coherent sheaves

Let X be a topological space and let $\mathscr{R}$ be a $\mathbf{k}$-algebra (*i.e.*, a sheaf of $\mathbf{k}$-algebras) on X. Let us recall a few classical definitions.

- An $\mathscr{R}$-module $\mathscr{M}$ is locally finitely generated if there locally exists an exact sequence

  (3.2.1) $$\mathscr{L}_0\to\mathscr{M}\to 0$$

  such that $\mathscr{L}_0$ is locally free of finite rank over $\mathscr{R}$.

- An $\mathscr{R}$-module $\mathscr{M}$ is locally of finite presentation if there locally exists an exact sequence

  (3.2.2) $$\mathscr{L}_1\to\mathscr{L}_0\to\mathscr{M}\to 0$$

  such that $\mathscr{L}_1$ and $\mathscr{L}_0$ are locally free of finite rank over $\mathscr{R}$. This is equivalent to saying that there locally exists an exact sequence

  (3.2.3) $$0\to\mathscr{K}\xrightarrow{u}\mathscr{N}\to\mathscr{M}\to 0$$

  where $\mathscr{N}$ is locally free of finite rank and $\mathscr{K}$ is locally finitely generated. This is also equivalent to saying that there locally exists an exact sequence

  (3.2.4) $$\mathscr{K}\to\mathscr{N}\to\mathscr{M}\to 0$$

  where $\mathscr{N}$ is locally of finite presentation and $\mathscr{K}$ is locally finitely generated.

- An $\mathscr{R}$-module $\mathscr{M}$ is pseudo-coherent if for any locally defined morphism $u\colon \mathscr{N} \to \mathscr{M}$ with $\mathscr{N}$ of finite presentation, $\ker u$ is locally finitely generated. This is also equivalent to saying that any locally defined $\mathscr{R}$-submodule of $\mathscr{M}$ is locally of finite presentation as soon as it is locally finitely generated.

- An $\mathscr{R}$-module $\mathscr{M}$ is coherent if it is locally finitely generated and pseudo-coherent. A ring is a coherent ring if it is so as a module over itself. One denotes by $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{R})$ the full additive subcategory of $\mathrm{Mod}(\mathscr{R})$ consisting of coherent modules. Note that $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{R})$ is a full abelian subcategory of $\mathrm{Mod}(\mathscr{R})$, stable by extension, and the natural functor $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{R}) \to \mathrm{Mod}(\mathscr{R})$ is exact (see [?K-S3, Exe. 8.23]).

- An $\mathscr{R}$-module $\mathscr{M}$ is Noetherian (see [?Ka2, Def. A.7]) if it is coherent, $\mathscr{M}_x$ is a Noetherian $\mathscr{R}_x$-module for any $x \in X$, and for any open subset $U \subset X$, any filtrant family of coherent submodules of $\mathscr{M}|_U$ is locally stationary. (This means that given a family $\{\mathscr{M}_i\}_{i\in I}$ of coherent submodules of $\mathscr{M}|_U$ indexed by a filtrant ordered set I, with $\mathscr{M}_i \subset \mathscr{M}_j$ for $i \leq j$, there locally exists $i_0 \in I$ such that $\mathscr{M}_{i_0} \xrightarrow{\sim} \mathscr{M}_j$ for any $j \geq i_0$.) A ring is a Noetherian ring if it is so as a left module over itself.

Let $\mathscr{M}$ and $\mathscr{N}$ be two $\mathscr{R}$-modules. Consider the natural morphism

$$\varphi_x : (\mathscr{H}om_{\mathscr{R}}(\mathscr{M}, \mathscr{N}))_x \to \mathrm{Hom}_{\mathscr{R}_x}(\mathscr{M}_x, \mathscr{N}_x).$$

If $\mathscr{M}$ is locally finitely generated (resp. of finite presentation), then φ_x is injective (resp. bijective). By choosing $\mathscr{N} = \mathscr{M}$, one gets that if $\mathscr{M}$ is locally of finite presentation and $\mathscr{M}_x = 0$, then there exists an open neighborhood U of x such that $\mathscr{M}|_U = 0$.

Example 3.2.1. Let U be an open subset of X with $\overline{U} \neq U$. Then the sheaf $\mathscr{R}_U$ is not of finite presentation since, choosing $x \in \overline{U} \setminus U$, $(\mathscr{R}_U)_x \simeq 0$.

Proposition 3.2.2. *If $\mathscr{R}$ is Noetherian, then all coherent $\mathscr{R}$-modules are Noetherian.*

Proposition 3.2.3. *Let $X = Y \times Z$ be a product of topological spaces and let $f\colon X \to Y$ be the projection. Let $\mathscr{R}$ be a sheaf of $\mathbf{k}_Y$-algebras on Y.*

(i) *If $\mathscr{R}$ is coherent, then $f^{-1}\mathscr{R}$ is coherent.*

(ii) *If $\mathscr{R}$ is Noetherian and moreover Z is a topological manifold, then $f^{-1}\mathscr{R}$ is Noetherian.*

3.3 Filtered sheaves

As above, $\mathbf{k}$ denotes a commutative unitary ring and X a topological space.

Definition 3.3.1. (i) A graded sheaf $\mathrm{Gr}\mathscr{M}$ on X is a sheaf of $\mathbf{k}$-modules together with a familly $\mathrm{Gr}_j\mathscr{M}$, $j \in \mathbb{Z}$ of subsheaves satisfying :

$$\mathrm{Gr}\mathscr{M} \simeq \bigoplus_j \mathrm{Gr}_j\mathscr{M}.$$

(ii) The shifted graduation $\mathrm{Gr}^{[p]}\mathscr{M}$ is given by $\mathrm{Gr}^{[p]}_j\mathscr{M} = \mathrm{Gr}_{p+j}\mathscr{M}$.

(iii) A morphism of graded sheaves $\mathrm{Gr}f : \mathrm{Gr}\mathscr{M} \to \mathrm{Gr}\mathscr{N}$ is a morphism of sheaves such that $\mathrm{Gr}f(\mathrm{Gr}_j\mathscr{M}) \subset \mathrm{Gr}_j\mathscr{N}$ for all $j \in \mathbb{Z}$.

(iv) A graded ring $\mathrm{Gr}\mathscr{R}$ on X is a graded sheaf of rings satisfying: $1 \in \mathrm{Gr}_0\mathscr{R}$ and $\mathrm{Gr}_i\mathscr{R} \cdot \mathrm{Gr}_j\mathscr{R} \subset \mathrm{Gr}_{i+j}\mathscr{R}$ for all i, j.

(v) A graded $\mathrm{Gr}\mathscr{R}$-module $\mathrm{Gr}\mathscr{M}$ is a graded sheaf of $\mathrm{Gr}\mathscr{R}$-modules satisfying:

$$\mathrm{Gr}_i\mathscr{R} \cdot \mathrm{Gr}_j\mathscr{M} \subset \mathrm{Gr}_{i+j}\mathscr{M} \text{ for all } i, j.$$

(vi) We denote by $\mathrm{Mod}^{\mathrm{gr}}(\mathrm{Gr}\mathscr{R})$ the abelian category of graded $\mathrm{Gr}\mathscr{R}$-modules.

Definition 3.3.2. (i) A filtered sheaf $\mathrm{Fl}\mathscr{M}$ on X is a sheaf $\mathscr{M}$ of $\mathbf{k}$-modules together with a familly $\mathrm{Fl}_j\mathscr{M}, j \in \mathbb{Z}$ of subsheaves satisfying :

$$\mathrm{Fl}_j\mathscr{M} \subset \mathrm{Fl}_{j+1}\mathscr{M}, \quad \operatorname*{colim}_j \mathrm{Fl}_j\mathscr{M} = \mathscr{M}.$$

One calls $\mathscr{M}$ the underlying sheaf.

(ii) The shifted filtration $\mathrm{Fl}^{[p]}\mathscr{M}$ is given by $\mathrm{Fl}^{[p]}_j\mathscr{M} = \mathrm{Fl}_{p+j}\mathscr{M}$.

(iii) A morphism of filtered sheaves $\mathrm{Fl}f : \mathrm{Fl}\mathscr{M} \to \mathrm{Fl}\mathscr{N}$ is a morphism of sheaves $f : \mathscr{M} \to \mathscr{N}$ such that $f(\mathrm{Fl}_j\mathscr{M}) \subset \mathrm{Fl}_j\mathscr{N}$ for all m.

(iv) The graded sheaf $\mathrm{Gr}\mathscr{M}$ associated to $\mathrm{Fl}\mathscr{M}$ is the sheaf $\bigoplus_j \mathrm{Gr}_j\mathscr{M}$, where $\mathrm{Gr}_j\mathscr{M} = \mathrm{Fl}_j\mathscr{M}/\mathrm{Fl}_{j-1}\mathscr{M}$. If $\mathrm{Fl}f : \mathrm{Fl}\mathscr{M} \to \mathrm{Fl}\mathscr{N}$ is a filtered morphism, one denotes by $\mathrm{Gr}f : \mathrm{Gr}\mathscr{M} \to \mathrm{Gr}\mathscr{N}$ the associated morphism of graded sheaves.

(v) One denote by $\sigma_j : \mathrm{Fl}_j\mathscr{M} \to \mathrm{Gr}_j\mathscr{M}$ the canonical morphism and calls it the"symbol of order j" morphism. One denotes by $\sigma : \mathrm{Fl}\mathscr{M} \to \mathrm{Gr}\mathscr{M}$ the morphism deduced from the σ_j and calls it the"principal symbol" morphism. (One shall be aware that σ_j is an additive morphism, contrarily to σ.)

(vi) A filtered ring $\mathrm{Fl}\mathscr{R}$ on X is a filtered sheaf of rings satisfying: $1 \in \mathrm{Fl}_0\mathscr{R}$ and $\mathrm{Fl}_i\mathscr{R} \cdot \mathrm{Fl}_j\mathscr{R} \subset \mathrm{Fl}_{i+j}\mathscr{R}$ for all i, j.

(vii) A filtered $\mathscr{R}$-module $\mathrm{Fl}\mathscr{M}$, or equivalently an $\mathrm{Fl}\mathscr{R}$-module, is an $\mathscr{R}$-module endowed with a filtration satisfying: $\mathrm{Fl}_i\mathscr{R} \cdot \mathrm{Fl}_j\mathscr{M} \subset \mathrm{Fl}_{i+j}\mathscr{M}$.

Consider an exact sequence of sheaves

$$0 \to \mathscr{M}' \xrightarrow{f} \mathscr{M} \xrightarrow{g} \mathscr{M}'' \to 0$$

and assume that $\mathscr{M}$ is endowed with a filtration $\mathrm{Fl}\mathscr{M}$. The induced filtration on $\mathscr{M}'$ is given by $\mathrm{Fl}_j\mathscr{M}' = f^{-1}(\mathrm{Fl}_j\mathscr{M})$. The image filtration on $\mathscr{M}''$ is given by $\mathrm{Fl}_j\mathscr{M}'' = g(\mathrm{Fl}_j\mathscr{M})$.

Let us denote by $\mathrm{Mod}^{\mathrm{fil}}(k_X)$ the category of filtered sheaves. Clearly, the category $\mathrm{Mod}^{\mathrm{fil}}(k_X)$ is additive and admits kernels and cokernels.

Remark 3.3.3. One shall be aware that the category $\mathrm{Mod}^{\mathrm{fil}}(k_X)$ is not abelian, even when $X = \mathrm{pt}$. Indeed, consider a filtered $\mathbf{k}$-module $\mathrm{Fl}M$ and the identity morhism $u : \mathrm{Fl}M \to \mathrm{Fl}^{[1]}M$. Its kernel and cokernel are zero, although this morphism is not an isomorphism in general.

Here, we shall assume that the filtration is positive, that is,

$$\mathrm{Fl}_m\mathscr{R} = 0 \text{ for } m \ll 0. \tag{3.3.1}$$

Definition 3.3.4. Let $\mathrm{Fl}\mathscr{R}$ be a filtered ring and $\mathscr{M}$ an $\mathscr{R}$-module.

(i) A filtration $\mathrm{Fl}\mathscr{M}$ on $\mathscr{M}$ is locally finite free if it is locally isomorphic to a finite direct sum of $\mathrm{Fl}^{[i]}\mathscr{R}$.

(ii) A filtration $\mathrm{Fl}\mathscr{M}$ on $\mathscr{M}$ is locally finitely generated if it is locally the image of a finite free filtration.

(iii) One defines similarly the notion of a filtration locally of finite presentation.

(iv) A locally finitely generated filtration is called a good filtration.

If $\mathscr{M} \twoheadrightarrow \mathscr{N}$ is an epimorphism and $\mathscr{M}$ is endowed with a good filtration, then the image filtration on $\mathscr{N}$ is good. Note that if $\mathscr{M}$ is a finitely generated $\mathscr{R}$-module, then $\mathscr{M}$ may be endowed with a good filtration. Namely, if $\mathscr{R}^m \twoheadrightarrow \mathscr{M}$ is an epimorphism, one endows $\mathscr{M}$ with the image filtration.

We shall give conditions in order that the induced filtration on a submodule is good.

Recall that if $\mathscr{R}$ is a sheaf of rings, then $\mathscr{R}[T]$ is the sheaf of rings associated with the presheaf $\mathscr{R} \otimes_k k[T]$.

Theorem 3.3.5. *Let $\mathscr{R}$ be a filtered ring. Assume*

(i) *$\mathrm{Gr}_0\mathscr{R}$ and $\mathrm{Gr}\mathscr{R}$ are Noetherian sheaves of rings,*

(ii) *all $\mathrm{Gr}_i\mathscr{R}$ are locally finitely generated over $\mathrm{Gr}_0\mathscr{R}$.*

Then the sheaves $\mathscr{R}$ and $\mathscr{R}[T]$ are Noetherian.

Corollary 3.3.6. *We make the hypotheses of Theorem 3.3.5.*

(i) *Let $\mathrm{Fl}\mathscr{M}$ be an $\mathrm{Fl}\mathscr{R}$-module with $\mathrm{Fl}_m\mathscr{M} = 0$ for $m << 0$ and assume that $\mathrm{Gr}\mathscr{M}$ is locally finitely generated (resp. coherent). Then $\mathscr{M}$ is locally finitely generated (resp. coherent).*

(ii) *Let $\mathscr{M}$ be a coherent $\mathscr{R}$-module endowed with a good filtration $\mathrm{Fl}\mathscr{M}$ and let $\mathscr{N}$ be a coherent submodule. Then the induced filtration $\mathrm{Fl}\mathscr{N}$ on $\mathscr{N}$ is good.*

(iii) *Let $\mathscr{M}$ be a coherent $\mathscr{R}$-module endowed with a good filtration $\mathrm{Fl}\mathscr{M}$. Then $\mathrm{Gr}\mathscr{M}$ is a coherent $\mathrm{Gr}\mathscr{R}$-module.*

3.4 Almost commutative filtered rings

In this section, for simplicity, we shall not consider sheaves of filtered rings, but simply filtered rings.

If $\mathrm{Fl}A$ is a filtered ring with $\mathrm{Fl}_iA = 0$ for $i << 0$ and $a \in A$, the order of a, denoted $\mathrm{ord}(a)$ is the smallest integer m such that $a \in \mathrm{Fl}_mA$.

Poisson bracket

From now on and until the end of this section, we shall assume that

$$[\mathrm{Fl}_i A, \mathrm{Fl}_j A] \subset \mathrm{Fl}_{i+j-1} A. \tag{3.4.1}$$

Hence, for any $a, b \in \mathrm{Fl}A$ one has:

$$\mathrm{ord}[a,b] \leq \mathrm{ord}(a) + \mathrm{ord}(b) - 1. \tag{3.4.2}$$

Clearly, condition (3.4.1) is equivalent to the fact that $\mathrm{Gr}A$ is commutative. One defines a Poisson bracket on $\mathrm{Gr}A$ by setting for homogeneous element $\bar{a}_i$ and $\bar{a}_j$ of order i and j, respectively:

$$\{\bar{a}_i, \bar{a}_j\} = \sigma_{i+j-1}([a_i, a_j]), \tag{3.4.3}$$

where $a_i \in \mathrm{Fl}_i A$, $a_j \in \mathrm{Fl}_j A$, $\sigma_i(a_i) = \bar{a}_i$ and $\sigma_j(a_j) = \bar{a}_j$. Clearly, the right hand side of (3.4.3) does not depend on the choice of a_i and a_j. The relations

$$\begin{cases} [f,g] = -[g,f] \\ [f,hg] = h[f,g] + g[f,h] \\ [[f,g],h] + [[g,h],f] + [[h,f],g] = 0 \end{cases} \tag{3.4.4}$$

tell us that the bracket $\{\bullet, \bullet\}$ satisfies the Jacobi identities (3.1.4).

Definition 3.4.1. A graded ideal $\mathrm{Gr}I$ of $\mathrm{Gr}A$ is involutive if it is stable by the Poisson bracket, that is, $a, b \in \mathrm{Gr}I$ implies $\{a,b\} \in \mathrm{Gr}I$.

Additivity

Recall that an additive semi-group $\mathscr{S}$ is a set endowed with an associative, commutative law $\mathscr{S} \times \mathscr{S} \to \mathscr{S}$, $(a,b) \mapsto a+b$ and a zero element 0, such that $0 + a = a$ for all a.

Examples 3.4.2. (i) If S is a set and $\mathscr{S} = \mathscr{P}(S)$ is the set of subsets of S, then the map $\mathscr{S} \times \mathscr{S} \to \mathscr{S}$, $(a,b) \mapsto a \cup b$ makes $\mathscr{S}$ an additive semi-group. The zero element is the empty set.
(ii) Let B be a commutative ring and let $\mathscr{S}$ denote the family of its ideals. Then the map $\mathscr{S} \times \mathscr{S} \to \mathscr{S}$, $(I,J) \mapsto I \cdot J$ makes $\mathscr{S}$ an additive semi-group. The zero element is B.

Definition 3.4.3. Let $\mathscr{C}$ be an abelian category

(i) The Grothendieck group $K(\mathscr{C})$ of $\mathscr{C}$ is the additive group generated by the isomorphy classes $[X]$ of objects of $\mathscr{C}$ wih relations $[X] = [X'] + [X'']$ if there exists an exact sequence $0 \to X' \to X \to X'' \to 0$.

(ii) Let $\mathscr{C}'$ a full additive subcategory of $\mathscr{C}$ stable by isomorphisms in $\mathscr{C}$. One denotes by $K(\mathscr{C}')$ the semigroup of $K(\mathscr{C})$ of elements $[X]$ with $X \in \mathscr{C}'$.

(iii) Let $\chi : \mathrm{Ob}(\mathscr{C}') \to \mathscr{S}$ a function. One says that χ is additive if for any exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathscr{C}$, with X', X, X'' in $\mathscr{C}'$, one has

$$\chi(X) = \chi(X') + \chi(X''). \tag{3.4.5}$$

Clearly, an additive function χ as above defines an additive function $\chi : K(\mathscr{C}') \to \mathscr{S}$.

Let FlA be a filtered ring with GrA commutative. We denote by $\mathrm{Mod}^{\mathrm{gr}}_f(\mathrm{Gr}A)$ the full additive subcategory of the abelian category $\mathrm{Mod}^{\mathrm{gr}}(\mathrm{Gr}A)$ consisting of finitely generated graded modules.

Theorem 3.4.4. *Let $\chi : \mathrm{Mod}^{\mathrm{gr}}_f(\mathrm{Gr}A) \to \mathscr{S}$ be an additive function. We assume that χ is invariant by the shift functors $[i]$. Let M be an A-module of finite type. Let us endow M with a finite filtration $\mathrm{Fl}M$. Then $\chi(\mathrm{Gr}M)$ does not depend on the choice of the finite filtration.*

Proof. (i) Let $\mathrm{Fl}M$ and $\mathrm{Fl}'M$ be two finite filtrations on M. There exists an $n_0 \in \mathbb{N}$ such that $\mathrm{Fl}'_iM \subset \mathrm{Fl}_{i+n_0}M$ for all i. Replacing $\mathrm{Fl}M$ by $\mathrm{Fl}^{[n_0]}M$, we may assume from the beginning that

$$\mathrm{Fl}'_iM \subset \mathrm{Fl}_iM \subset \mathrm{Fl}'_{i+n_0}M \text{ for all } i. \tag{3.4.6}$$

We shall argue by induction on n_0. If $n_0 = 0$ the result is clear.
(ii) Assume $n_0 = 1$. Consider the exact sequences

$$0 \to \oplus_i\mathrm{Fl}'_iM/\mathrm{Fl}_{i-1}M \to \oplus_i\mathrm{Fl}_iM/\mathrm{Fl}_{i-1}M \to \oplus_i\mathrm{Fl}_iM/\mathrm{Fl}'_iM \to 0,$$
$$0 \to \oplus_i\mathrm{Fl}_{i-1}M/\mathrm{Fl}'_{i-1}M \to \oplus_i\mathrm{Fl}'_iM/\mathrm{Fl}'_{i-1}M \to \oplus_i\mathrm{Fl}'_iM/\mathrm{Fl}_{i-1}M \to 0.$$

Set $\mathrm{Gr}L' = \oplus_i\mathrm{Fl}'_iM/\mathrm{Fl}_{i-1}M$, $\mathrm{Gr}L'' = \oplus_i\mathrm{Fl}_iM/\mathrm{Fl}'_iM$. We get exact sequences

$$0 \to \mathrm{Gr}L' \to \mathrm{Gr}M \to \mathrm{Gr}L'' \to 0,$$
$$0 \to \mathrm{Gr}^{[-1]}L' \to \mathrm{Gr}'M \to \mathrm{Gr}L'' \to 0.$$

(iii) Assume $n_0 > 1$. Set $\mathrm{Fl}''_iM = \mathrm{Fl}_{i-1}M + \mathrm{Fl}'_iM$. Then

$$\mathrm{Fl}''_iM \subset \mathrm{Fl}_iM \subset \mathrm{Fl}''_{i+1}M,$$
$$\mathrm{Fl}'_iM \subset \mathrm{Fl}''_iM \subset \mathrm{Fl}'_{i+n_0-1}M.$$

Since $\chi(\mathrm{Gr}M) = \chi(\mathrm{Gr}''M)$ by (ii), the induction proceeds. □

Corollary 3.4.5. *We make the hypotheses of Theorem 3.3.5. Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of finitely generated A-modules. Then*

$$\chi(M) = \chi(M') + \chi(M''). \tag{3.4.7}$$

Gabber's theorem

Recall that if B is a commutative ring and I an ideal, the radical $\sqrt{I}$ of I is the ideal

$$x \in \sqrt{I} \Leftrightarrow \text{ there exists } k \geq 0 \text{ with } x^k \in I.$$

If N is a B-module, the annihilator I_N of N is the ideal given by

$$x \in I_N \Leftrightarrow xu = 0 \text{ for all } u \in N.$$

If $0 \to M' \to M \to M'' \to 0$ is an exact sequence in $\mathrm{Mod}(B)$, then clearly

$$\sqrt{I_M} = \sqrt{I_{M'}} \cap \sqrt{I_{M''}} = \sqrt{I_{M'} \cdot I_{M''}}. \tag{3.4.8}$$

In other words, the map $M \mapsto \sqrt{I_M}$ is additive. If $\mathrm{Gr}M$ is a graded $\mathrm{Gr}A$-module, then $\sqrt{I_{\mathrm{Gr}M}}$ is a graded ideal.

Let $\mathrm{Fl}A$ be a filtered ring with $\mathrm{Gr}A$ commutative. Let M be a finitely generated A-module. Let us endow M with a finite filtration $\mathrm{Fl}M$. Applying Theorem 3.4.4, we can state:

Definition 3.4.6. Let M is a finitely generated A-module. One sets

$$\mathrm{Icar}(M) = \sqrt{I_{\mathrm{Gr}M}}, \tag{3.4.9}$$

where $\mathrm{Gr}M$ is the graded module associated with a finite filtration $\mathrm{Fl}M$ on M.

Let us give a direct proof of the fact that $\mathrm{Icar}(M)$ does not depend on the choice of the filtration. Let $\bar{a} \in \sqrt{I_{\mathrm{Gr}M}}$ of order p. There exists q such that $\bar{a}^q \in I_{\mathrm{Gr}M}$ and there exists $a \in \mathrm{Fl}_p M$ such that $\sigma(a) = \bar{a}$. Then

$$a^q \mathrm{Fl}_k M \subset \mathrm{Fl}_{k+pq-1} \text{ for all } k,$$
$$a^{lq} \mathrm{Fl}_k M \subset \mathrm{Fl}_{k+lpq-l} \text{ for all } k.$$

If $\mathrm{Fl}'M$ is another filtration, there exists r such that $\mathrm{Fl}'_{k-r}M \subset \mathrm{Fl}_k M \subset \mathrm{Fl}'_{k+r}M$. Hence, $a^{lq}\mathrm{Fl}'_k \subset \mathrm{Fl}'_{k+lpq-1}$ for $l >> 0$.

As an application, assume moreover that $\mathrm{Gr}A$ has no zero divisors. let $a \neq 0, b \neq 0$ in A. Then

$$A \cdot a \cap A \cdot b \neq \{0\}.$$

Indeed, the sequence below of left A-modules is exact.

$$0 \to A/(A \cdot a \cap A \cdot b) \to A/A \cdot a \oplus A/A \cdot b \to A/(A \cdot a + A \cdot b) \to 0$$

It follows that

$$0 \neq \mathrm{Icar}(A/A \cdot a) \cap \mathrm{Icar}(A/A \cdot b) \subset \mathrm{Icar}(A/(A \cdot a \cap A \cdot b)).$$

Theorem 3.4.7. (Gabber's Theorem.) *Assume that* $\mathrm{Gr}A$ *is a commutative Noetherian* $\mathbb{Q}$*-algebra. Let* M *be a finitely generated* A*-module. Then* $\mathrm{Icar}(M)$ *is involutive.*

Note that if $\bar{a}$ and $\bar{b}$ belong to $I_{\mathrm{Gr}M}$, then $\{\bar{a}, \bar{b}\}$ obviously belongs to $I_{\mathrm{Gr}M}$. The difficulty is that one assumes that $\bar{a}$ and $\bar{b}$ belong to the radical of $I_{\mathrm{Gr}M}$.

Involutive basis

Let $\mathrm{Fl}A$ and $\mathrm{Gr}A$ be as in Theorem 3.3.5 with $\mathrm{Gr}A$ commutative.

Definition 3.4.8. Let I be an ideal of A and $\{u_1, \ldots, u_{N_0}\}$ a system of generators. One says that this system is an involutive basis if $\{\sigma(u_1), \ldots, \sigma(u_{N_0})\}$ is a system of generators of $\mathrm{Gr}I$.

Let m_j denote the order of u_j. We endow I with the induced filtration by $\mathrm{Fl}A$. Consider the sequences

$$\oplus_{j=1}^{N_0}\mathrm{Fl}^{[-m_j]}A \xrightarrow{\mathrm{Fl}f} \mathrm{Fl}I \to 0, \quad \text{where } \mathrm{Fl}f(\oplus_j b_j) = \sum_j b_j u_j, \tag{3.4.10}$$

$$\oplus_{j=1}^{N_0}\mathrm{Gr}^{[-m_j]}A \xrightarrow{\mathrm{Gr}f} \mathrm{Gr}I \to 0, \quad \text{where } \mathrm{Gr}f(\oplus_j \bar{b}_j) = \sum_j \bar{b}_j \bar{u}_j. \tag{3.4.11}$$

Proposition 3.4.9. *The following conditions are equivalent.*

(i) $(u_1, \dots, u_{N_0})$ *is an involutive basis of* I*,*

(ii) *the sequence* (3.4.10) *is filtered exact,*

(iii) *the sequence* (3.4.11) *is exact,*

(iv) *for any* $l \in \mathbb{Z}$ *and* $\bar{b}_j \in \mathrm{Gr}_{l-m_j}A$ *such that* $\sum_j \bar{b}_j\sigma(u_j) = 0$*, there exists* $b_j \in \mathrm{Fl}_{l-m_j}A$ *such that* $\sum_j b_j u_j = 0$*.*

Proof. (i) $\Leftrightarrow$ (iii) by definition and (ii) $\Leftrightarrow$ (iii) by Proposition **??**.
Let us prove that (iv)$\Leftrightarrow$ (iii). Let $\mathrm{Gr}I'$ denote the ideal generated by $\{\sigma(u_1), \dots, \sigma(u_{N_0})\}$. Consider the exact sequences

$$0 \to \mathrm{Gr}K' \to \oplus_{j=1}^{N_0}\mathrm{Gr}^{[-m_j]}A \xrightarrow{\mathrm{Gr}f} \mathrm{Gr}I' \to 0$$
$$0 \to \mathrm{Fl}\ker(f) \to \oplus_{j=1}^{N_0}\mathrm{Fl}^{[-m_j]}A \xrightarrow{\mathrm{Fl}f} \mathrm{Fl}I \to 0,$$

where $\ker f$ is endowed with the induced filtration. Then $\mathrm{Gr}I' \xrightarrow{\sim} \mathrm{Gr}I$ if and only if $\mathrm{Gr}K' \xleftarrow{\sim} \mathrm{Gr}\ker(f)$. □

Note that since $\mathrm{Gr}A$ is Noetherian, there always exist involutive basis.

3.5 $\mathscr{O}$-modules

Coherency

Let X be a complex manifold of complex dimension d_X, $\mathscr{O}_X$ it structural sheaf.

Theorem 3.5.1. *The sheaf* $\mathscr{O}_X$ *is Noetherian.*

If Z is a closed complex analytic subset, we shall denote by $\mathscr{I}_Z$ its defining ideal. Note that $\mathscr{I}_Z$ is coherent.

One denotes by $\mathrm{Mod}_c(\mathscr{O}_X)$ the abelian category of coherent sheaves of $\mathscr{O}_X$-modules. If S is a closed analytic subset of X, we shall denote by $\mathrm{Mod}_c(\mathscr{O}_X)_S$ the abelian category of coherent sheaves with support in S.

Cycles

Let Z be a closed analytic irreducible component of S. To each $\mathscr{F} \in \mathrm{Mod}_{\mathrm{c}}(\mathscr{O}_X)_S$ one can associates a number $\mathrm{mult}_Z(\mathscr{F})$, the multiplicity of $\mathscr{F}$ along Z, as follows. First, assume that $\mathscr{I}_Z\mathscr{F} = 0$. Then $\mathscr{F}$ is an $\mathscr{O}_Z$-module, and generically, Z is smooth and $\mathscr{F}$ is free of finite rank r over $\mathscr{O}_Z$. Then we set $\mathrm{mult}_Z(\mathscr{F}) = r$. In the general case, since locally at generic points of Z, $\mathrm{supp}\,\mathscr{F} \subset Z$, there locally exists an integer N such that $\mathscr{I}_Z^N\mathscr{F} = 0$ and one sets

$$\mathrm{mult}_Z(\mathscr{F}) = \sum_{j\geq 0} \mathrm{mult}_Z(\mathscr{I}_Z^j\mathscr{F}/\mathscr{I}_Z^{j+1}\mathscr{F}).$$

Proposition 3.5.2. *Let S a closed analytic subset of X and let Z be an irreducible component of S. The function $\mathrm{mult}_Z(\,\bullet\,)$ on $\mathrm{Mod}_{\mathrm{c}}(\mathscr{O}_X)_S$ is additive.*

Let us introduce the group $\mathfrak{Z}_X^d$ of cycles of codimension d as the free abelian group generated by the symbols $[S]$ where S is a closed irreducible subset of X of codimension d. One sets

$$\mathfrak{Z}_X = \bigoplus_d \mathfrak{Z}_X^d$$

and calls this graded group the group of cycles of X.

If $\mathscr{F}$ is a coherent sheaf, S its support, $\{Z_j\}_j$ the (locally finite) family of closed irreducible components of S, the cycle associated with $\mathscr{F}$ is defined by

$$[\mathscr{F}] = \sum_j \mathrm{mult}_{Z_j}(\mathscr{F})[Z_j].$$

One shall be aware that $[\,\bullet\,]$ is not additive on the category $\mathrm{Mod}_{\mathrm{c}}(\mathscr{O}_X)$.

Example 3.5.3. Let $X = \mathbb{C}$ with holomorphic coordinate x and let $\mathscr{F} = \mathscr{O}_X/x^2\mathscr{O}_X$. Let $Z = \{0\}$. Then $\mathrm{mult}_Z(\mathscr{F}) = 2$ and $[\mathscr{F}] = 2[\{0\}]$. On the other hand $[\mathscr{O}_X \oplus \mathscr{F}] = [\mathscr{O}_X] = [X]$.

One denotes by $[\mathscr{F}]^d$ the homogeneous part of degree d of the cycle $[\mathscr{F}]$. Then the function $[\,\bullet\,]^d$ is additive on the full category of $\mathrm{Mod}_{\mathrm{c}}(\mathscr{O}_X)$ consisting of sheaves $\mathscr{F}$ such that $\mathrm{codim}(\mathrm{supp}(\mathscr{F})) \geq d$.

Operations on $\mathscr{O}$-modules

For a complex manifold X, one denotes by $\mathrm{Mod}_{\mathrm{c}}(\mathscr{O}_X)$ the thick abelian subcategory of $\mathrm{Mod}(\mathscr{O}_X)$ consisting of coherent modules. One denotes by $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_X)$ the full triangulated category of the bounded derived category $\mathrm{D}^{\mathrm{b}}(\mathscr{O}_X)$ consisting of objects with coherent cohomology.

We shall also encounter the duality functors for $\mathscr{O}$-modules:

$$\begin{aligned}
\mathbb{D}'_{\mathscr{O}}\mathscr{F} &:= \mathrm{R}\mathscr{H}om_{\mathscr{O}}(\mathscr{F}, \mathscr{O}_X),\\
\mathbb{D}_{\mathscr{O}}\mathscr{F} &:= \mathrm{R}\mathscr{H}om_{\mathscr{O}}(\mathscr{F}, \Omega_X\,[d_X]).
\end{aligned}$$

Recall that d_X is the complex dimension of X and $\Omega_X = \Omega_X^{d_X}$.

Let X and Y be two manifolds. For an $\mathscr{O}_X$-module $\mathscr{F}$ and an $\mathscr{O}_Y$-module $\mathscr{G}$, we define their external product, denoted $\mathscr{F}\underline{\boxtimes}\mathscr{G}$, by

$$\mathscr{F}\underline{\boxtimes}\mathscr{G} = \mathscr{O}_{X\times Y} \otimes_{\mathscr{O}_X\boxtimes\mathscr{O}_Y} (\mathscr{F}\boxtimes\mathscr{G}).$$

Note that the functor $\mathscr{F} \mapsto \mathscr{F}\underline{\boxtimes}\mathscr{G}$ is exact. Clearly, if $\mathscr{F} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_X)$ and $\mathscr{G} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_Y)$, then $\mathscr{F}\underline{\boxtimes}\mathscr{G} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_{X\times Y})$.

Let $f\colon X \to Y$ be a morphism of complex manifolds. There is a natural morphism of rings $f^{-1}\mathscr{O}_Y \to \mathscr{O}_X$. Using this morphism, the direct images $f_*\mathscr{F}$ and $f_!\mathscr{F}$ of an $\mathscr{O}_X$-module are well defined as $\mathscr{O}_Y$-modules. One denotes as usual by $\mathrm{R}f_*$ and $\mathrm{R}f_!$ their derived functors. The inverse image of an $\mathscr{O}_Y$-module $\mathscr{G}$ is defined by $f^* := \mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\mathscr{G}$. Its right derived functor is denoted Lf^*. The following result is left as an exercise.

Proposition 3.5.4. *Let $\mathscr{G} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_Y)$. Then $Lf^*\mathscr{G} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_X)$ and there is a natural isomorphism*

$$Lf^*\mathbb{D}'_{\mathscr{O}}\mathscr{G} \simeq \mathbb{D}'_{\mathscr{O}}Lf^*\mathscr{G}.$$

There is a similar result for direct images:

Theorem 3.5.5. Grauert's theorem. *Let $\mathscr{F} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_X)$ and assume that f is proper on* $\mathrm{supp}(\mathscr{F})$. *Then $\mathrm{R}f_!\mathscr{F} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_Y)$ and there is a natural isomorphism*

$$\mathrm{R}f_!\mathbb{D}_{\mathscr{O}}\mathscr{F} \simeq \mathbb{D}_{\mathscr{O}}\mathrm{R}f_!\mathscr{F}.$$

Note that Grauert's theorem is a relative version of the Cartan-Serre's finiteness theorem and the Serre's duality theorem.

Homological properties

Recall the well known results.

Theorem 3.5.6. *Let X be a smooth manifold and let $\mathscr{F}$ be a coherent $\mathscr{O}_X$-module. Then*

(i) $\mathscr{E}xt^k_{\mathscr{O}}(\mathscr{F},\mathscr{O}_X) = 0$ *for* $k < \operatorname{codim}\operatorname{supp}(\mathscr{F})$,

(ii) $\operatorname{codim}(\operatorname{supp}(\mathscr{E}xt^k_{\mathscr{O}}(\mathscr{F},\mathscr{O}_X))) \geq k$.

Theorem 3.5.7. (*Golovin*) *The global homological dimension of $\mathscr{O}_X$ is $d_X + 1$.*

In other words, any $\mathscr{O}_X$-module $\mathscr{F}$ admits an injectve resolution of length $\leq \dim X + 1$, or equivalently, for any $\mathscr{O}_X$-modules $\mathscr{F}$ and $\mathscr{G}$, one has $\mathrm{Ext}^j_{\mathscr{O}}(\mathscr{F},\mathscr{G}) = 0$ for $j > d_X + 1$.

Let us only show that this dimension is at least $d_X + 1$.

Proposition 3.5.8. *Let $x \in X$. Then $H^j(\mathrm{R}\Gamma_{\{x\}}(X;\mathscr{O}_X^{(\mathbb{N})})) \neq 0$ for $j = d_X + 1$.*

We may assume $X = \mathbb{C}^n$. Let $Y = \mathbb{C}^{n-1}$ and let $f\colon X \to Y$ be the projection. We have a short exact sequence $0 \to f^{-1}\mathscr{O}_Y \to \mathscr{O}_X \xrightarrow{\partial_n} \mathscr{O}_X \to 0$ form wich we deduce the exact sequence

$$\cdots \to H^{n+1}(\mathrm{R}\Gamma_{\{x\}}(X;\mathscr{O}_X^{(\mathbb{N})})) \to H^{n+2}(\mathrm{R}\Gamma_{\{x\}}(X;f^{-1}\mathscr{O}_Y^{(\mathbb{N})})) \\ \to H^{n+2}(\mathrm{R}\Gamma_{\{x\}}(X;\mathscr{O}_X^{(\mathbb{N})})) = 0.$$

Since for any sheaf $\mathscr{F}$ on Y

$$H^{j+2}(\mathrm{R}\Gamma_{\{0\}}(X; f^{-1}\mathscr{F})) \simeq H^j(\mathrm{R}\Gamma_{\{0\}}(Y;\mathscr{F})),$$

we are reduced to prove the result for $n = 1$. Let $X = \mathbb{P}^1(\mathbb{C})$ denote the Riemann sphere. Since X is compact, $H^j(X;\mathscr{O}_X[T]) \simeq H^j(X;\mathscr{O}_X)[T]$ and this group is zero for $j > 0$. Therefore, $H^2_{\{x\}}(X;\mathscr{O}_X^{(\mathbb{N})}) \simeq H^1(X \setminus \{0\};\mathscr{O}_X[T])$.

Lemma 3.5.9. *Set $X = \mathbb{A}(\mathbb{C})$, the affine line. Then $H^1(X;\mathscr{O}_X[T]) \neq 0$.*

Proof. Let $\delta(n)$ denote the Dirac mass at $n \in X$ and set $u = \Sigma_n \delta(n)T^n \in \Gamma(X;\mathrm{D}^{\mathrm{b}}_X[T])$. The equation $\overline{\partial} v = u$ has no solution in $\Gamma(X;\mathrm{D}^{\mathrm{b}}_X[T])$. The exact sequence of sheaves

$$0 \to \mathscr{O}_X[T] \to \mathrm{D}^{\mathrm{b}}_X[T] \xrightarrow{\overline{\partial}} \mathrm{D}^{\mathrm{b}}_X[T] \to 0$$

and the vanishing of $H^1(X;\mathrm{D}^{\mathrm{b}}_X[T])$ give the result. □

The Artin-Rees theorem

Theorem 3.5.10. *Let $\mathscr{I}$ be a coherent ideal of $\mathscr{O}_X$ and let $\mathscr{F}$ be a coherent $\mathscr{O}_X$-module. Then, locally, there exists $m_0 \geq 0$ such that for any $m \geq m_0$, the morphism*

$$\mathscr{I}^m \otimes_{\mathscr{O}} \mathscr{F} \to \mathscr{I}^{m-m_0} \otimes_{\mathscr{O}} \mathscr{F}$$

factorizes uniquely through

$$\mathscr{I}^m \otimes_{\mathscr{O}} \mathscr{F} \to \mathscr{I}^m \mathscr{F} \to \mathscr{I}^{m-m_0} \otimes_{\mathscr{O}} \mathscr{F}.$$

In fact, there is a similar theorem in the more general setting of commutative Noetherian rings.

The Grauert theorem

Theorem 3.5.11. *Let $f\colon X \to Y$ be a morphism of complex manifolds and let $\mathscr{F} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_X)$. Assume that f is proper on* $\mathrm{supp}(\mathscr{F})$. *Then $\mathrm{R}f_!\mathscr{F}$ belongs to $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_Y)$.*

Theorem 3.5.12. *Let $f\colon X \to Y$ be a morphism of complex manifolds and let $\mathscr{F} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_X)$. Assume that f is proper on* $\mathrm{supp}(\mathscr{F})$. *Then there is a canonical isomorphism*

$$\mathrm{R}f_!\mathrm{R}\mathscr{H}om_{\mathscr{O}}(\mathscr{F},\mathscr{O}_X)\,[d_X] \simeq \mathrm{R}\mathscr{H}om_{\mathscr{O}}(\mathrm{R}f_!\mathscr{F},\mathscr{O}_Y)\,[d_Y].$$

Bibliography

[Be71] I. Bernstein, *Modules over a ring of differential operators,* Funct. Analysis and Appl. **5** pp 89–101 (1971).

[Bj93] J-E. Bjork, *Analytic $\mathscr{D}$-modules and applications,* Kluwer **247** (1993).

[Bo87] A. Borel et al, *Algebraic $\mathscr{D}$-modules,* Perspective in Mathematics, Academic Press, (1987).

[Co85] Coutinho, *A primer of algebraic $\mathscr{D}$-modules,* London Math. Society **33** Cambridge University Press (1995)

[Ga81] O. Gabber, *The integrability of the characteristic variety.* Amer. Journ. Math. **103** pp 445–468 (1981),

[GM96] S.I. Gelfand and Yu.I. Manin, *Methods of homological algebra,* Springer (1996)

[HTT08] R. Hotta, K. Takeuchi and T. Tanisaki, *D-modules, perverse sheaves, and representation theory,* Progress in Math. **236**, Birkhauser (2008)

[Ka70] M. Kashiwara, *Algebraic study of systems of partial diffential equations,* Thesis, Tokyo Univ. (1970), Mémoires Soc. Math. France **63** (1995)

[Ka83] ______, *Systems of microdiffential equations,* Progress in Math **34** Birkhauser (1983)

[Ka03] ______, *D-modules and Microlocal Calculus,* Translations of Mathematical Monographs, **217** American Math. Soc. (2003)

[KS90] M. Kashiwara and P. Schapira, *Sheaves on manifolds,* Grundlehren der Math. Wiss. **292** Springer (1990)

[KS06] ______, *Categories and sheaves,* Grundlehren der Math. Wiss. **332** Springer (2006)

[KS12] ______, *Deformation quantization modules,* Astérisque Soc. Math. France **345** (2012). `arXiv:math.arXiv:1003.3304`

[La83] G. Laumon, *Sur la catégorie dérivée des $\mathscr{D}$-modules filtrés,* Algebraic Geometry (M. Raynaud and T. Shioda eds) Lecture Notes in Math. Springer-Verlag **1016** pp. 151–237 (1983).

[Sc85] P. Schapira, *Microdifferential systems in the complex domain,* Grundlehren der Math. Wiss. **269** Springer (1985)

[Sc08] ______, *Categories and homological algebra,*
An introduction to Sheaves,
Course at Paris VI University,
http://www.math.jussieu.fr/~schapira/lectnotes

[Sc07] ______, *Mikio Sato, a visionary of mathematics,* Notices AMS, Feb. 2007

[ScSn94] P. Schapira and J-P. Schneiders, *Index theorem for elliptic pairs II. Euler class and relative index theorem,* Astérisque Soc. Math. France 224 (1994).

[Sn95] J-P. Schneiders, *An introduction to $\mathscr{D}$-modules,* Bull. Soc. Royale Sci. Liège, pp 223–295 (1995)

[SKK73] M. Sato, T. Kawai and M. Kashiwara, *Hyperfunctions and pseudodifferential equations,* Lecture Notes in Math., Springer **287**, 265–529 (1973)

[We94] C. Weibel, *An introduction to homological algebra,* Cambridge Studies in Advanced Math. **38** (1994)