An Introduction to Categories and Homological Algebra

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Comments are welcome.

Preface

Since more than half a century, the set theoretical point of view in mathematics has been supplanted by the categorical perspective. Category theory was introduced by Samuel Eilenberg and Saunders McLane at more or less at the same time as sheaf theory was by Jean Leray. Both theories, categories and sheaves, were incredibly developed by Alexander Grothendieck who made them the natural language for algebraic geometry, two cornerstones being first his famous Tohoku paper, second the introduction of the so-called "6 operations". These new techniques are now basic in many fields, not only algebraic geometry, but also algebraic topology, analytic geometry, algebraic analysis and D-module theory, singularity theory, representation theory, etc. and, more recently, computational geometry.

The underlying idea of category theory is that mathematical objects only take their full force in relation with other objects of the same type. As we shall see, category theory is a very nice and natural language, not difficult to assimilate for any one having a bit of experience in mathematics, someone familiar with linear algebra and general topology. It opens new horizons in mathematics, a new way, a "functorial way", of doing mathematics. A typical example is that of real finite dimensional vector spaces. One can look at this category as well as at the category whose objects are the integers and the morphisms are matrices with real coefficients. A good part of the first year of study at University consists in fact in proving that these two categories are equivalent. Category theory reveals fundamental concepts and notions which across all mathematics, such as adjunction formulas, limits and colimits, or the difference between equalities and isomorphisms. And many theorems of today's mathematics are simply expressed as equivalences of categories. The famous homological mirror symmetry, as formulated by Maxim Kontsevich, is a good illustration of this trend.

However, a difficulty soon appears: one should be careful with the size of the objects one manipulates and one is led to work in a given universe, changing of universe when necessary. An easy and classical illustration of this fact is given in Remark 2.7.5. We shall not introduce cardinals, preferring to work with universes and remaining rather sketchy with this notion.

There is a class of categories which plays a central role: these are the additive categories and among them the abelian categories, in which one can perform homological algebra. Homological algebra is essentially linear algebra, no more over a field but over a ring and by extension, in abelian categories. When replacing a field with a ring, a submodule has in general no supplementary and the classical functors of tensor product and internal hom are no more exact and one has to consider their derived functors. Derived functors are of fundamental importance and many phenomena only appear in their light. Two classical examples are local cohomology of sheaves and duality. The calculation of the derived functor of a composition leads to technical difficulties, known as “spectral sequences”. Fortunately, the use of derived categories makes things much more elementary as we shall see in this book which never uses spectral sequences.

These Notes are a preparation for the reading of [KS24a,KS24b] and may also be considered as a development of [KS90, Ch. 1] as well as an introduction to [KS06].

Contents

Introduction

In these Notes, we introduce the reader to the language of categories and we present the main notions of homological algebra, including triangulated and derived categories.

After having presented the basic concepts and results of category theory, in particular the Yoneda lemma and the notion of representable functors, we define limits and colimits (also called projective and inductive limits). We start with the particular cases of kernels and products (and the dual notions of cokernels and co-products) and study directed colimits.

As a fundamental application, one has the so-called Kan extension of functors, which allows one to construct the natural operations of direct and inverse images of presheaves, a presheaf being nothing but a contravariant functor, a terminology due to Grothendieck. (Sheaves on Grothendieck topologies will be studied in forthcoming notes.)

We also introduce, as a preparation for derived categories, the notion of localization of categories. Then we treat additive categories, complexes, shifted complexes, mapping cones and the homotopy category. In the course of the exposition, we introduce the truncation functors, an essential tool in practice and a substitute to the famous "spectral sequences" which shall not appear here. We also have a glance to simplicial constructions. Then we define abelian categories and study complexes in this framework. We admit without proof the Grothendieck theorem which asserts that abelian categories satisfying suitable properties admit enough injectives. We treat in some details Koszul complexes, giving many examples.

We introduce the basic notions on triangulated categories, study their localization and state, without proof, the Brown representability theorem. Finally we define and study derived categories and derived functors and bifunctors.

As it is well-known, it not possible to develop category theory without some caution about the size of the objects one considers. An easy and classical illustration of this fact is given in Remark 2.7.5. We shall not introduce cardinals, preferring to work with universes and referring to [KS06] for details. We shall mention when necessary (perhaps not always!) that a category is "small" or "big" with respect to a given universe $\mathscr{U}$, passing to a bigger universe if necessary. Notice that Grothendieck's theorem about the existence of injectives is a typical example (and historically, the first one) where universes play an essential role.

Some historical comments and references.[1] As already mentioned, category theory was introduced by Samuel Eilenberg and Saunders McLane [EML45]. At the prehistory of homological algebra is the book [CE56] by Henri Cartan and Samuel

[1]These few lines are not written by an historian of mathematics and should be read with caution.

Eilenberg including the Appendix by David A. Buchsbaum in which abelian categories are introduced for the first time, before being considerably developed and systematically studied by Gothendieck in [Gro57, SGA4]. The natural framework of homological algebra is that of derived categories, whose idea is, once more, due to Grothendieck and which was written done by his student, Jean-Louis Verdier who realized the importance of triangulated categories, a notion already used at that time in algebraic topology. Derived categories are constructed by "localizing" the homotopy category and the basic reference for localization is the book [GZ67] by Gabriel and Zisman. Category theory would not exist without the axiom of universes (or anything equivalent) and this is Grothendieck who introduced this axiom (see [SGA4]). We refer to [Krö07] for interesting thoughts on this topic.

Category theory is the natural language to develop sheaf theory, a point of view introduced in the foundational paper by Grothendieck [Gro57].

These Notes are largely inspired by [KS06], a book itself inspired by [SGA4]. Other references for category theory are [Mac98, Bor94] for the general theory, [MM92] for links with logic, [GM96, Wei94] and [KS90, Ch. 1] for homological algebra, including derived categories, as well as [Nee01, Yek20] for a more exhaustive study of triangulated categories and derived categories, the last reference developing the DG (differential graded) point of view.

Prerequisites. The reader is supposed to have basic knowledges in algebra, essentially with the notions of modules over a ring.

Conventions. In these Notes, all rings are unital and associative but not necessarily commutative. The operations, the zero element, and the unit are denoted by $+, \cdot, 0, 1$, respectively. However, we shall often write for short ab instead of $a \cdot b$. All along these Notes, $\mathbf{k}$ will denote a *commutative* ring of finite global dimension (see [Wei94, 4.1.2] or [KS90, Exe. I.28]). This means that there exists $n \in \mathbb{N}$ such that any $\mathbf{k}$-module M admits an injective (equivalently, a projective) resolution of length $\leq n$. (Sometimes, $\mathbf{k}$ will be a field.) We denote by $\varnothing$ the empty set and by $\{\mathrm{pt}\}$ a set with one element. We denote by $\mathbb{N}$ the set of non-negative integers, $\mathbb{N} = \{0, 1, \dots\}$, by $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ the fields of rational numbers, real numbers and complex numbers, respectively.

A comment: ∞-categories. These Notes are written in the "classical" language of category theory, that is, 1- or 2-categories. However, some specialists of algebraic geometry or algebraic topology are now using the language of ∞-categories, the homotopical approach replacing the homological one. The difficulty of this last theory is for the moment of another order of magnitude than that of the classical theory, this last one being perfectly suited for the applications we have in mind. References are made to [Cis19, Lan21, Lur09, Lur17, RV22, Toë14].

Chapter 1

The language of categories

Summary

In this chapter we start with some reminders on the categories **Set** of sets and $\mathrm{Mod}(A)$ of modules over a (not necessarily commutative) ring A. Then we explain the basic language of categories and functors. A key point is the Yoneda lemma, which asserts that a category $\mathscr{C}$ may be embedded in the category $\mathscr{C}^\wedge$ of contravariant functors from $\mathscr{C}$ to the category **Set**. This naturally leads to the concept of representable functor and adjoint functors. Many examples are treated, in particular in the categories **Set** and $\mathrm{Mod}(A)$.

Caution. All along this book, we shall be rather sketchy with the notion of universes, mentioning when necessary (perhaps not always!) that a category is "small" or "big" with respect to a universe $\mathscr{U}$. Indeed, it not possible to develop category theory without some caution about the size of the objects we consider. An easy and classical illustration of this fact is given in Remark 2.7.5.

Some references. As already mentioned, Category Theory was invented by Samuel Eilenberg and Saunders McLane [EML45] and one certainly should quote the seminal book [CE56] by Henri Cartan and Samuel Eilenberg as well as the fundamental contribution of Alexander Grothendieck in [Gro57, SGA4]. This is in particular in this seminar that he introduced the notion of Universes. For a modern treatment, see [KS06, Wei94], among many others. For more historical comments and other references, see the introduction.

1.1 Sets and maps

The aim of this section is to fix some notations and to recall some elementary constructions on sets.

If $f\colon X \to Y$ is a map from a set X to a set Y, we shall often say that f is a morphism (of sets) from X to Y. We shall denote by $\mathrm{Hom}_{\mathbf{Set}}(X,Y)$, or simply $\mathrm{Hom}\,(X,Y)$ or also Y^X, the set of all maps from X to Y. If $g\colon Y \to Z$ is another map, we can define the composition $g\circ f\colon X \to Z$. Hence, we get two maps:

$$\begin{aligned} g\circ\colon \mathrm{Hom}\,(X,Y) &\to \mathrm{Hom}\,(X,Z),\\ \circ f\colon \mathrm{Hom}\,(Y,Z) &\to \mathrm{Hom}\,(X,Z). \end{aligned}$$

If f is bijective we shall say that f is an isomorphism and write $f\colon X \xrightarrow{\sim} Y$. This is equivalent to saying that there exists $g\colon Y \to X$ such that $g\circ f$ is the identity of

X and $f \circ g$ is the identity of Y. If there exists an isomorphism $f\colon X \xrightarrow{\sim} Y$, we say that X and Y are isomorphic and write $X \simeq Y$.

Notice that if $X = \{x\}$ and $Y = \{y\}$ are two sets with one element each, then there exists a unique isomorphism $X \xrightarrow{\sim} Y$. Of course, if X and Y are finite sets with the same cardinal $\pi > 1$, X and Y are still isomorphic, but the isomorphism is no more unique.

Notation 1.1.1. We shall denote by $\varnothing$ the empty set and by $\{\mathrm{pt}\}$ a set with one element. Note that for any set X, there is a unique map $\varnothing \to X$ and a unique map $X \to \{\mathrm{pt}\}$.

Let $\{X_i\}_{i\in I}$ be a family of sets indexed by a set I. Their union is denoted by $\bigcup_i X_i$. The product of the X_i's, denoted $\prod_{i\in I} X_i$, or simply $\prod_i X_i$, is defined as

$$\prod_{i\in I} X_i = \{f \in \mathrm{Hom}\,(I, \bigcup_i X_i); f(i) \in X_i \text{ for all } i \in I\}. \tag{1.1.1}$$

Hence, if $X_i = X$ for all $i \in I$, we get

$$\prod_{i\in I} X_i = \mathrm{Hom}\,(I, X) = X^I.$$

If I is the ordered set $\{1, 2\}$, one sets

$$X_1 \times X_2 = \{(x_1, x_2); x_i \in X_i, i = 1, 2\}, \tag{1.1.2}$$

and there are natural isomorphisms

$$X_1 \times X_2 \simeq \prod_{i\in I} X_i \simeq X_2 \times X_1.$$

This notation and these isomorphisms extend to the case of a finite ordered set I.

If $\{X_i\}_{i\in I}$ is a family of sets indexed by a set I as above, one also considers their disjoint union, also called their coproduct. The coproduct of the X_i's is denoted $\bigsqcup_{i\in I} X_i$ or simply $\bigsqcup_i X_i$. If $X_i = X$ for all $i \in I$, one uses the notation $X^{\sqcup I}$. If $I = \{1, 2\}$, one often writes $X_1 \sqcup X_2$ instead of $\bigsqcup_{i\in\{1,2\}} X_i$.

For three sets I, X, Y, there is a natural isomorphism

$$\mathrm{Hom}\,(I, \mathrm{Hom}\,(X, Y)) = \mathrm{Hom}\,(X, Y)^I \simeq \mathrm{Hom}\,(I \times X, Y). \tag{1.1.3}$$

For a set Y, there is a natural isomorphism

$$\mathrm{Hom}\,(Y, \prod_i X_i) \simeq \prod_i \mathrm{Hom}\,(Y, X_i). \tag{1.1.4}$$

Note that

$$X \times I \simeq X^{\sqcup I}. \tag{1.1.5}$$

For a set Y, there is a natural isomorphism

$$\mathrm{Hom}\,(\bigsqcup_i X_i, Y) \simeq \prod_i \mathrm{Hom}\,(X_i, Y). \tag{1.1.6}$$

In particular,

$$\mathrm{Hom}\,(X^{\sqcup I}, Y) \simeq \mathrm{Hom}\,(X, Y^I) \simeq \mathrm{Hom}\,(X, Y)^I.$$

Consider two sets X and Y and two maps f, g from X to Y. We write for short $f, g\colon X \rightrightarrows Y$. The kernel (or equalizer) of (f, g), denoted $\mathrm{Ker}(f, g)$, is defined as

$$\mathrm{Ker}(f, g) = \{x \in X; f(x) = g(x)\}. \tag{1.1.7}$$

Note that for a set Z, one has

$$\mathrm{Hom}\,(Z, \mathrm{Ker}(f, g)) \simeq \mathrm{Ker}(\mathrm{Hom}\,(Z, X) \rightrightarrows \mathrm{Hom}\,(Z, Y)). \tag{1.1.8}$$

Let us recall a few elementary definitions.

- A relation $\mathscr{R}$ on a set X is a subset of $X \times X$. One writes $x\mathscr{R}y$ if $(x, y) \in \mathscr{R}$.
- The opposite relation $\mathscr{R}^{\mathrm{op}}$ is defined by: $x\mathscr{R}^{\mathrm{op}}y$ if and only if $y\mathscr{R}x$.
- A relation $\mathscr{R}$ is reflexive if it contains the diagonal, that is, $x\mathscr{R}x$ for all $x \in X$.
- A relation $\mathscr{R}$ is symmetric if $x\mathscr{R}y$ implies $y\mathscr{R}x$.
- A relation $\mathscr{R}$ is transitive if $x\mathscr{R}y$ and $y\mathscr{R}z$, implies $x\mathscr{R}z$.
- A relation $\mathscr{R}$ is an equivalence relation if it is reflexive, symmetric and transitive.
- A relation $\mathscr{R}$ is a preorder if it is reflexive and transitive. A preorder is often denoted by $\leq$. A set endowed with a preorder is called a preordered set.
- Let $(I, \leq)$ be a preordered set. One says that $(I, \leq)$ is directed if I is non empty and for any $i, j \in I$ there exists k with $i \leq k$ and $j \leq k$.
- Assume $(I, \leq)$ is a directed preordered set and let $J \subset I$ be a subset. One says that J is cofinal to I if for any $i \in I$ there exists $j \in J$ with $i \leq j$.

If $\mathscr{R}$ is a relation on a set X, there is a smallest equivalence relation which contains $\mathscr{R}$. (Take the intersection of all subsets of $X \times X$ which contain $\mathscr{R}$ and which are equivalence relations.)

Let $\mathscr{R}$ be an equivalence relation on a set X. A subset S of X is saturated if $x \in S$ and $x\mathscr{R}y$ implies $y \in S$. For $x \in X$, the smallest saturated subset $\widehat{x}$ of X containing x is called the equivalence class of x. One then defines a new set $X/\mathscr{R}$ and a canonical map $f\colon X \to X/\mathscr{R}$ as follows: the elements of $X/\mathscr{R}$ are the sets $\widehat{x}$ and the map f associates the set $\widehat{x}$ to $x \in X$.

1.2 Modules and linear maps

All along this book, a ring A means a unital associative ring, but A is not necessarily commutative. We shall denote by $\mathbf{k}$ a commutative ring. Recall that a $\mathbf{k}$-algebra A is a ring endowed with a morphism of rings $\varphi\colon \mathbf{k} \to A$ such that the image of $\mathbf{k}$ is contained in the center of A (*i.e.,* $\varphi(x)a = a\varphi(x)$ for any $x \in \mathbf{k}$ and $a \in A$). Notice that a ring A is always a $\mathbb{Z}$-algebra. If A is commutative, then A is an A-algebra.

Since we do not assume that A is commutative, we have to distinguish between left and right structures. Unless otherwise specified, a module M over A means a left A-module.

Recall that an A-module M is an additive group (whose operations and zero element are denoted by $+, 0$) endowed with an external law $A \times M \to M$ (denoted by $(a, m) \mapsto a \cdot m$ or simply $(a, m) \mapsto am$) satisfying:

$$\begin{cases} (ab)m = a(bm), \\ (a+b)m = am + bm, \\ a(m+m') = am + am,' \\ 1 \cdot m = m, \end{cases}$$

where $a, b \in A$ and $m, m' \in M$.

Note that, when A is a $\mathbf{k}$-algebra, M inherits a structure of a $\mathbf{k}$-module via φ. In the sequel, if there is no risk of confusion, we shall not write φ.

We denote by A^{op} the ring A with the opposite structure. Hence the product ab in A^{op} is the product ba in A and an A^{op}-module is a right A-module.

Note that if the ring A is a field (here, a field is always commutative), then an A-module is nothing but a vector space.

Also note that an abelian group is nothing but a $\mathbb{Z}$-module.

Examples 1.2.1. (i) The first example of a ring is $\mathbb{Z}$, the ring of integers. Since a field is a ring, $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ are rings. If A is a commutative ring, then $A[x_1, \dots, x_n]$, the ring of polynomials in n variables with coefficients in A, is also a commutative ring. It is a sub-ring of $A[[x_1, \dots, x_n]]$, the ring of formal powers series with coefficients in A.

(ii) Let $\mathbf{k}$ be a field. For $n > 1$, the ring $M_n(\mathbf{k})$ of square $(n \times n)$ matrices with entries in $\mathbf{k}$ is non-commutative.

(iii) Let $\mathbf{k}$ be a field. The *Weyl algebra* in n variables, denoted by $W_n(\mathbf{k})$, is the non commutative ring of polynomials in the variables x_i, ∂_j $(1 \le i, j \le n)$ with coefficients in $\mathbf{k}$ and relations :

$$[x_i, x_j] = 0,\ [\partial_i, \partial_j] = 0,\ [\partial_j, x_i] = \delta_j^i$$

where $[p, q] = pq - qp$ and δ_j^i is the Kronecker symbol.

The Weyl algebra $W_n(\mathbf{k})$ may be regarded as the ring of differential operators with coefficients in $\mathbf{k}[x_1, \dots, x_n]$, and $\mathbf{k}[x_1, \dots, x_n]$ becomes a left $W_n(\mathbf{k})$-module: x_i acts by multiplication and ∂_i is the derivation with respect to x_i.

A morphism $f\colon M \to N$ of A-modules is an A-linear map, *i.e.,* f satisfies:

$$\begin{cases} f(m+m') = f(m) + f(m') \quad m, m' \in M, \\ f(am) = af(m) \quad m \in M, a \in A. \end{cases}$$

A morphism f is an isomorphism if there exists a morphism $g : N \to M$ with $f \circ g = \mathrm{id}_N, g \circ f = \mathrm{id}_M$.

If f is bijective, it is easily checked that the inverse map $f^{-1} : N \to M$ is itself A-linear. Hence f is an isomorphism if and only if f is A-linear and bijective.

A submodule N of M is a nonempty subset N of M such that if $n, n' \in N$, then $n + n' \in N$ and if $n \in N, a \in A$, then $an \in N$. A submodule of the A-module A is called an ideal of A. Note that if A is a field, it has no non-trivial ideal, *i.e.,* its only ideals are $\{0\}$ and A. If $A = \mathbb{C}[x]$, then $I = \{P \in \mathbb{C}[x]; P(0) = 0\}$ is a non trivial ideal.

If N is a submodule of M, it defines an equivalence relation: $m\mathscr{R}m'$ if and only if $m - m' \in N$. One easily checks that the quotient set $M/\mathscr{R}$ is naturally endowed with a structure of a left A-module. This module is called the quotient module and is denoted by M/N.

Let $f\colon M \to N$ be a morphism of A-modules. One sets:

$$\begin{aligned}\operatorname{Ker} f &= \{m \in M;\ f(m) = 0\},\\ \operatorname{Im} f &= \{n \in N;\ \text{there exists } m \in M, \quad f(m) = n\}.\end{aligned}$$

These are submodules of M and N respectively, called the kernel and the image of f, respectively. One also introduces the cokernel and the coimage of f:

$$\operatorname{Coker} f = N/\operatorname{Im} f, \quad \operatorname{Coim} f = M/\operatorname{Ker} f.$$

Note that the natural morphism $\operatorname{Coim} f \to \operatorname{Im} f$ is an isomorphism.

Example 1.2.2. Let $W_n(\mathbf{k})$ denote as above the Weyl algebra. Consider the left $W_n(\mathbf{k})$-linear map $W_n(\mathbf{k}) \to \mathbf{k}[x_1, \dots, x_n]$, $W_n(\mathbf{k}) \ni P \mapsto P(1) \in \mathbf{k}[x_1, \dots, x_n]$. This map is clearly surjective and its kernel is the left ideal generated by $(\partial_1, \cdots, \partial_n)$. Hence, one has the isomorphism of left $W_n(\mathbf{k})$-modules:

$$W_n(\mathbf{k})/\sum_j W_n(\mathbf{k})\partial_j \xrightarrow{\sim} \mathbf{k}[x_1, \dots, x_n]. \tag{1.2.1}$$

Products and direct sums

Let I be a set and let $\{M_i\}_{i\in I}$ be a family of A-modules indexed by I. The set $\prod_i M_i$ is naturally endowed with a structure of a left A-module by setting

$$\begin{aligned}(m_i)_i + (m'_i)_i &= (m_i + m'_i)_i,\\ a \cdot (m_i)_i &= (a \cdot m_i)_i.\end{aligned}$$

The direct sum $\bigoplus_i M_i$ is the submodule of $\prod_i M_i$ whose elements are the $(m_i)_i$'s such that $m_i = 0$ for all but a finite number of $i \in I$. In particular, if the set I is finite, we have $\bigoplus_i M_i = \prod_i M_i$. If $M_i = M$ for all i, one writes $M^{\oplus I}$ or $M^{(I)}$ instead of $\bigoplus_i M$.

Linear maps

Let M and N be two A-modules. Recall that an A-linear map $f\colon M \to N$ is also called a morphism of A-modules. One denotes by $\operatorname{Hom}_A(M, N)$ the set of A-linear maps $f\colon M \to N$. When A is a $\mathbf{k}$-algebra, $\mathrm{Hom}_A(M,N)$ is a $\mathbf{k}$-module. In fact one defines the action of $\mathbf{k}$ on $\mathrm{Hom}_A(M,N)$ by setting: $(\lambda f)(m) = \lambda(f(m))$. Hence $(\lambda f)(am) = \lambda f(am) = \lambda a f(m) = a\lambda f(m) = a(\lambda f)(m)$, and $\lambda f \in \mathrm{Hom}_A(M,N)$.

There is a natural isomorphism $\mathrm{Hom}_A(A,M) \simeq M$: to $u \in \mathrm{Hom}_A(A,M)$ one associates $u(1)$ and to $m \in M$ one associates the linear map $A \to M, a \mapsto am$. More generally, if I is an ideal of A then $\mathrm{Hom}_A(A/I, M) \simeq \{m \in M; Im = 0\}$.

Note that if A is a $\mathbf{k}$-algebra and $L \in \mathrm{Mod}(\mathbf{k})$, $M \in \mathrm{Mod}(A)$, the $\mathbf{k}$-module $\mathrm{Hom}_{\mathbf{k}}(L,M)$ is naturally endowed with a structure of a left A-module. If N is a right A-module, then $\mathrm{Hom}_{\mathbf{k}}(N,L)$ is naturally endowed with a structure of a left A-module.

Tensor product

Consider a right A-module N, a left A-module M and a $\mathbf{k}$-module L. Let us say that a map $f\colon N \times M \to L$ is $(A,\mathbf{k})$-bilinear if f is additive with respect to each of its arguments and satisfies $f(na,m) = f(n,am)$ and $f(n\lambda,m) = \lambda(f(n,m))$ for all $(n,m) \in N \times M$ and $a \in A, \lambda \in \mathbf{k}$.

One naturally identifies a set I with a subset of $\mathbf{k}^{(I)}$. Then the tensor product $N \otimes_A M$ is the $\mathbf{k}$-module defined as the quotient of $\mathbf{k}^{(N\times M)}$ by the submodule generated by the following elements (where $n, n' \in N, m, m' \in M, a \in A, \lambda \in \mathbf{k}$ and $N \times M$ is identified with a subset of $\mathbf{k}^{(N\times M)}$):

$$\begin{cases} (n+n',m) - (n,m) - (n',m), \\ (n,m+m') - (n,m) - (n,m'), \\ (na,m) - (n,am), \\ \lambda(n,m) - (n\lambda,m). \end{cases}$$

The image of (n,m) in $N \otimes_A M$ is denoted by $n \otimes m$. Hence an element of $N \otimes_A M$ may be written (not uniquely!) as a finite sum $\sum_j n_j \otimes m_j$, $n_j \in N$, $m_j \in M$ and:

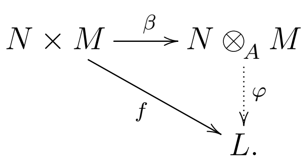
$$\begin{cases} (n+n') \otimes m = n \otimes m + n' \otimes m, \\ n \otimes (m+m') = n \otimes m + n \otimes m', \\ na \otimes m = n \otimes am, \\ \lambda(n \otimes m) = n\lambda \otimes m = n \otimes \lambda m. \end{cases}$$

Denote by $\beta\colon N \times M \to N \otimes_A M$ the natural map which associates $n \otimes m$ to (n,m).

Proposition 1.2.3. *The map β is $(A,\mathbf{k})$-bilinear and for any $\mathbf{k}$-module L and any $(A,\mathbf{k})$-bilinear map $f\colon N \times M \to L$, the map f factorizes uniquely through a $\mathbf{k}$-linear map $\varphi\colon N \otimes_A M \to L$.*

The proof is left to the reader.

Proposition 1.2.3 is visualized by the diagram:



Consider three A-modules L, M, N and an A-linear map $f\colon M \to L$. It defines a linear map $\mathrm{id}_N \times f\colon N \times M \to N \times L$, hence a $(A,\mathbf{k})$-bilinear map $N \times M \to N \otimes_A L$, and finally a **k**-linear map

$$\mathrm{id}_N \otimes f \colon N \otimes_A M \to N \otimes_A L.$$

One constructs similarly $g \otimes \mathrm{id}_M$ associated to $g \colon N \to L$.

There are natural isomorphisms $A \otimes_A M \simeq M$ and $N \otimes_A A \simeq N$.

Denote by $\mathrm{Bil(N \times M, L)}$ the **k**-module of $(A, \mathbf{k})$-bilinear maps from $N \times M$ to L. One has the isomorphisms

$$\begin{aligned} \mathrm{Bil(N \times M, L)} &\simeq \mathrm{Hom}_{\mathbf{k}}(\mathrm{N} \otimes_{\mathrm{A}} \mathrm{M, L}) \\ &\simeq \mathrm{Hom}_A(M, \mathrm{Hom}_{\mathbf{k}}(N, L)) \\ &\simeq \mathrm{Hom}_A(N, \mathrm{Hom}_{\mathbf{k}}(M, L)). \end{aligned} \tag{1.2.2}$$

For $L \in \mathrm{Mod}(\mathbf{k})$ and $M \in \mathrm{Mod}(A)$, the **k**-module $L \otimes_{\mathbf{k}} M$ is naturally endowed with a structure of a left A-module. For $M, N \in \mathrm{Mod}(A)$ and $L \in \mathrm{Mod}(\mathbf{k})$, we have the isomorphisms (whose verification is left to the reader):

$$\begin{aligned} \mathrm{Hom}_A(L \otimes_{\mathbf{k}} N, M) &\simeq \mathrm{Hom}_A(N, \mathrm{Hom}_{\mathbf{k}}(L, M)) \\ &\simeq \mathrm{Hom}_{\mathbf{k}}(L, \mathrm{Hom}_A(N, M)). \end{aligned} \tag{1.2.3}$$

If A is commutative, $N \otimes_A M$ is naturally an A-module and there is an isomorphism: $N \otimes_A M \simeq M \otimes_A N$ given by $n \otimes m \mapsto m \otimes n$. Moreover, the tensor product is associative, that is, if L, M, N are A-modules, there are natural isomorphisms $L \otimes_A (M \otimes_A N) \simeq (L \otimes_A M) \otimes_A N$. One simply writes $L \otimes_A M \otimes_A N$.

1.3 Categories and functors

Definition 1.3.1. A category $\mathscr{C}$ consists of:

(i) a set $\mathrm{Ob}(\mathscr{C})$ whose elements are called the objects of $\mathscr{C}$,

(ii) for each $X, Y \in \mathrm{Ob}(\mathscr{C})$, a set $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ whose elements are called the morphisms from X to Y,

(iii) for any $X, Y, Z \in \mathrm{Ob}(\mathscr{C})$, a map, called the composition, $\mathrm{Hom}_{\mathscr{C}}(X, Y) \times \mathrm{Hom}_{\mathscr{C}}(Y, Z) \to \mathrm{Hom}_{\mathscr{C}}(X, Z)$, and denoted by $(f, g) \mapsto g \circ f$,

these data satisfying:

(a) $\circ$ is associative,

(b) for each $X \in \mathrm{Ob}(\mathscr{C})$, there exists $\mathrm{id}_X \in \mathrm{Hom}\,(X, X)$ such that for all $f \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$ and $g \in \mathrm{Hom}_{\mathscr{C}}(Y, X)$, $f \circ \mathrm{id}_X = f$, $\mathrm{id}_X \circ g = g$.

Note that $\mathrm{id}_X \in \mathrm{Hom}\,(X, X)$ is characterized by the condition in (b).

Universes

With such a definition of a category, there is no category of sets, since there is no set of "all" sets. The set-theoretical dangers encountered in category theory will be illustrated in Remark 2.7.5.

To overcome this difficulty, one has to be more precise when using the word "set". One way is to use the notion of *universe*. We do not give in this book the precise definition of a universe, only recalling that a universe $\mathscr{U}$ is a set (a very big one) stable by many operations. In particular, $\varnothing \in \mathscr{U}$, $\mathbb{N} \in \mathscr{U}$, $x \in \mathscr{U}$ and $y \in x$ implies $y \in \mathscr{U}$, $x \in \mathscr{U}$ and $y \subset x$ implies $y \in \mathscr{U}$, if $I \in \mathscr{U}$ and $u_i \in \mathscr{U}$ for all $i \in I$, then $\bigcup_{i\in I} u_i \in \mathscr{U}$ and $\prod_{i\in I} u_i \in \mathscr{U}$. See for example [KS06, Def. 1.1.1].

Definition 1.3.2. Let $\mathscr{U}$ be a universe.

(a) A set E is a $\mathscr{U}$-set if it belongs to $\mathscr{U}$.

(b) A set E is $\mathscr{U}$-small if it is isomorphic to a $\mathscr{U}$-set.

(c) A $\mathscr{U}$-category $\mathscr{C}$ is a category such that for any $X, Y \in \mathscr{C}$, the set $\mathrm{Hom}_{\mathscr{C}}(X,Y)$ is $\mathscr{U}$-small.

(d) A $\mathscr{U}$-category $\mathscr{C}$ is $\mathscr{U}$-small if moreover the set $\mathrm{Ob}(\mathscr{C})$ is $\mathscr{U}$-small.

Note that the set $\mathscr{U}$ is not $\mathscr{U}$-small.

The crucial point is Grothendieck's axiom which says that any set belongs to some universe.

By a "big" category, we mean a category in a bigger universe. Note that, by Grothendieck's axiom, any category is an $\mathscr{V}$-category for a suitable universe $\mathscr{V}$ and one even can choose $\mathscr{V}$ so that $\mathscr{C}$ is $\mathscr{V}$-small.

As far as it has no implication, we shall not always be precise on this matter and the reader may skip the words "small" and "big".

Notation 1.3.3. One often writes $X \in \mathscr{C}$ instead of $X \in \mathrm{Ob}(\mathscr{C})$ and $f \colon X \to Y$ (or else $f \colon Y \leftarrow X$) instead of $f \in \mathrm{Hom}_{\mathscr{C}}(X,Y)$. One calls X the source and Y the target of f.

- A morphism $f \colon X \to Y$ is an *isomorphism* if there exists $g \colon X \leftarrow Y$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$. In such a case, one writes $f \colon X \xrightarrow{\sim} Y$ or simply $X \simeq Y$. Of course g is unique, and one also denotes it by f^{-1}.

- A morphism $f \colon X \to Y$ is a *monomorphism* (resp. an *epimorphism*) if for any morphisms g_1 and g_2, $f \circ g_1 = f \circ g_2$ (resp. $g_1 \circ f = g_2 \circ f$) implies $g_1 = g_2$. One sometimes writes $f \colon X \rightarrowtail Y$ or else $X \hookrightarrow Y$ (resp. $f \colon X \twoheadrightarrow Y$) to denote a monomorphism (resp. an epimorphism).

- Two morphisms f and g are parallel if they have the same sources and targets, visualized by $f, g \colon X \rightrightarrows Y$.

- A category is *discrete* if the only morphisms are the identity morphisms. Note that a set is naturally identified with a discrete category (and conversely).

- A category $\mathscr{C}$ is *finite* if the family of all morphisms in $\mathscr{C}$ (hence, in particular, the family of objects) is a finite set.

- A category $\mathscr{C}$ is a *groupoid* if all morphisms are isomorphisms.

One introduces the *opposite category* $\mathscr{C}^{\mathrm{op}}$:

$$\mathrm{Ob}(\mathscr{C}^{\mathrm{op}}) = \mathrm{Ob}(\mathscr{C}), \quad \mathrm{Hom}_{\mathscr{C}^{\mathrm{op}}}(X,Y) = \mathrm{Hom}_{\mathscr{C}}(Y,X),$$

the identity morphisms and the composition of morphisms being the obvious ones.

A category $\mathscr{C}'$ is a *subcategory* of $\mathscr{C}$, denoted by $\mathscr{C}' \subset \mathscr{C}$, if:

(a) $\mathrm{Ob}(\mathscr{C}') \subset \mathrm{Ob}(\mathscr{C})$,

(b) $\mathrm{Hom}_{\mathscr{C}'}(X,Y) \subset \mathrm{Hom}_{\mathscr{C}}(X,Y)$ for any $X, Y \in \mathscr{C}'$, the composition $\circ$ in $\mathscr{C}'$ is induced by the composition in $\mathscr{C}$ and the identity morphisms in $\mathscr{C}'$ are induced by those in $\mathscr{C}$.

- One says that $\mathscr{C}'$ is a *full* subcategory if for all $X, Y \in \mathscr{C}'$, $\mathrm{Hom}_{\mathscr{C}'}(X,Y) = \mathrm{Hom}_{\mathscr{C}}(X,Y)$.
- One says that a full subcategory $\mathscr{C}'$ of $\mathscr{C}$ is *saturated* if $X \in \mathscr{C}$ belongs to $\mathscr{C}'$ as soon as it is isomorphic to an object of $\mathscr{C}'$.

Examples 1.3.4. (i) **Set** is the category of sets and maps (in a given universe $\mathscr{U}$). If necessary, one calls this category $\mathscr{U}$-**Set**. Then $\mathbf{Set}^f$ is the full subcategory consisting of finite sets.

(ii) **Rel** is defined by: $\mathrm{Ob}(\mathbf{Rel}) = \mathrm{Ob}(\mathbf{Set})$ and $\mathrm{Hom}_{\mathbf{Rel}}(X,Y) = \mathscr{P}(X \times Y)$, the set of subsets of $X \times Y$. The composition law is defined as follows. For $f\colon X \to Y$ and $g\colon Y \to Z$, $g \circ f$ is the set

$$\{(x,z) \in X \times Z;\ \text{there exists } y \in Y \text{ with } (x,y) \in f, (y,z) \in g\}.$$

Of course, $\mathrm{id}_X = \Delta_X \subset X \times X$, the diagonal of $X \times X$.

(iii) Let A be a ring. The category of left A-modules and A-linear maps is denoted by $\mathrm{Mod}(A)$. In particular $\mathrm{Mod}(\mathbb{Z})$ is the category of abelian groups.

We shall use the notation $\mathrm{Hom}_A(\,\bullet\,,\,\bullet\,)$ instead of $\mathrm{Hom}_{\mathrm{Mod}(A)}(\,\bullet\,,\,\bullet\,)$.

One denotes by $\mathrm{Mod}^{\mathrm{f}}(A)$ the full subcategory of $\mathrm{Mod}(A)$ consisting of finitely generated A-modules.

(iv) One associates to a preordered set $(I, \leq)$ a category, still denoted by I for short, as follows. $\mathrm{Ob}(I) = I$, and the set of morphisms from i to j has a single element if $i \leq j$, and is empty otherwise. Note that I^{op} is the category associated with I endowed with the opposite preorder.

(v) We denote by **Top** the category of topological spaces and continuous maps.

(vi) We denote by **Arr** the category which consists of two objects, say $\{a, b\}$, and one morphism $a \to b$ other than id_a and id_b. One represents it by the diagram $\bullet \to \bullet$.

(vii) We represent by $\bullet \rightrightarrows \bullet$ the category with two objects, say $\{a, b\}$, and two parallel morphisms $a \rightrightarrows b$ other than id_a and id_b.

(viii) Let G be a group. We may attach to it the groupoid $\mathscr{G}$ with one object, say $\{a\}$, and morphisms $\mathrm{Hom}_{\mathscr{G}}(a,a) = G$.

(ix) Let X be a topological space locally arcwise connected. We attach to it a category $\widetilde{X}$ as follows: $\mathrm{Ob}(\widetilde{X}) = X$ and for $x, y \in X$, a morphism $f\colon x \to y$ is a path form x to y. (Precise definitions are left to the reader.)

Definition 1.3.5. (i) An object $P \in \mathscr{C}$ is called initial if $\mathrm{Hom}_{\mathscr{C}}(P, X) \simeq \{\mathrm{pt}\}$ for all $X \in \mathscr{C}$. One often denotes by $\varnothing_{\mathscr{C}}$ an initial object in $\mathscr{C}$.

(ii) One says that P is terminal if P is initial in $\mathscr{C}^{\mathrm{op}}$, *i.e.,* for all $X \in \mathscr{C}$, $\mathrm{Hom}_{\mathscr{C}}(X, P) \simeq \{\mathrm{pt}\}$. One often denotes by $\mathrm{pt}_{\mathscr{C}}$ a terminal object in $\mathscr{C}$.

(iii) One says that P is a zero-object if it is both initial and terminal. In such a case, one often denotes it by 0. If $\mathscr{C}$ has a zero-object, for any objects $X, Y \in \mathscr{C}$, the morphism obtained as the composition $X \to 0 \to Y$ is still denoted by $0 \colon X \to Y$.

Note that initial (resp. terminal) objects are unique up to unique isomorphisms.

Examples 1.3.6. (i) In the category **Set**, $\varnothing$ is initial and $\{\mathrm{pt}\}$ is terminal.

(ii) The zero module 0 is a zero-object in $\mathrm{Mod}(A)$.

(iii) The category associated with the ordered set $(\mathbb{Z}, \leq)$ has neither initial nor terminal object.

Definition 1.3.7. Let $\mathscr{C}$ and $\mathscr{C}'$ be two categories. A functor $F \colon \mathscr{C} \to \mathscr{C}'$ consists of a map $F \colon \mathrm{Ob}(\mathscr{C}) \to \mathrm{Ob}(\mathscr{C}')$ and for all $X, Y \in \mathscr{C}$, of a map still denoted by F: $\mathrm{Hom}_{\mathscr{C}}(X, Y) \to \mathrm{Hom}_{\mathscr{C}'}(F(X), F(Y))$ such that

$$F(\mathrm{id}_X) = \mathrm{id}_{F(X)}, \quad F(f \circ g) = F(f) \circ F(g).$$

A contravariant functor from $\mathscr{C}$ to $\mathscr{C}'$ is a functor from $\mathscr{C}^{\mathrm{op}}$ to $\mathscr{C}'$. In other words, it satisfies $F(g \circ f) = F(f) \circ F(g)$. If one wishes to put the emphasis on the fact that a functor is not contravariant, one says it is covariant.

One denotes by $\mathrm{id}_{\mathscr{C}}$ (or simply id) the identity functor on $\mathscr{C}$. One denotes by $\mathrm{op} \colon \mathscr{C} \to \mathscr{C}^{\mathrm{op}}$ the natural contravariant functor.

Example 1.3.8. Let $\mathscr{C}$ be a category and let $X \in \mathscr{C}$.

(i) $\mathrm{Hom}_{\mathscr{C}}(X, \bullet)$ is a functor from $\mathscr{C}$ to **Set**. To $Y \in \mathscr{C}$, it associates the set $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ and to a morphism $f \colon Y \to Z$ in $\mathscr{C}$, it associates the map

$$\mathrm{Hom}_{\mathscr{C}}(X, f) \colon \mathrm{Hom}_{\mathscr{C}}(X, Y) \xrightarrow{f\circ} \mathrm{Hom}_{\mathscr{C}}(X, Z).$$

(ii) $\mathrm{Hom}_{\mathscr{C}}(\bullet, X)$ is a functor from $\mathscr{C}^{\mathrm{op}}$ to **Set**. To $Y \in \mathscr{C}$, it associates the set $\mathrm{Hom}_{\mathscr{C}}(Y, X)$ and to a morphism $f \colon Y \to Z$ in $\mathscr{C}$, it associates the map

$$\mathrm{Hom}_{\mathscr{C}}(f, X) \colon \mathrm{Hom}_{\mathscr{C}}(Z, X) \xrightarrow{\circ f} \mathrm{Hom}_{\mathscr{C}}(Y, X).$$

Example 1.3.9. Let A be a $\mathbf{k}$-algebra and let $M \in \mathrm{Mod}(A)$. Similarly as in Example 1.3.8, we have the functors

$$\mathrm{Hom}_A(M, \bullet) \colon \mathrm{Mod}(A) \to \mathrm{Mod}(\mathbf{k}),$$
$$\mathrm{Hom}_A(\bullet, M) \colon \mathrm{Mod}(A)^{\mathrm{op}} \to \mathrm{Mod}(\mathbf{k})$$

Clearly, the functor $\mathrm{Hom}_A(M, \bullet)$ commutes with products in $\mathrm{Mod}(A)$, that is,

$$\mathrm{Hom}_A(M, \prod_i N_i) \simeq \prod_i \mathrm{Hom}_A(M, N_i)$$

and the functor $\mathrm{Hom}_A(\,\bullet\,, N)$ commutes with direct sums in $\mathrm{Mod}(A)$, that is,

$$\mathrm{Hom}_A(\bigoplus_i M_i, N) \simeq \prod_i \mathrm{Hom}_A(M_i, N).$$

(ii) Let N be a right A-module. Then $N \otimes_A \bullet \colon \mathrm{Mod}(A) \to \mathrm{Mod}(\mathbf{k})$ is a functor. Clearly, the functor $N \otimes_A \bullet$ commutes with direct sums, that is,

$$N \otimes_A (\bigoplus_i M_i) \simeq \bigoplus_i (N \otimes_A M_i),$$

and similarly with the functor $\bullet \otimes_A M$.

Definition 1.3.10. Let $F\colon \mathscr{C} \to \mathscr{C}'$ be a functor.

(i) One says that F is faithful (resp. full, resp. fully faithful) if for $X, Y \in \mathscr{C}$ $\mathrm{Hom}_{\mathscr{C}}(X, Y) \to \mathrm{Hom}_{\mathscr{C}'}(F(X), F(Y))$ is injective (resp. surjective, resp. bijective).

(ii) One says that F is essentially surjective if for each $Y \in \mathscr{C}'$ there exist $X \in \mathscr{C}$ and an isomorphism $F(X) \simeq Y$.

(iii) One says that F is conservative if a morphism $f\colon X \to Y$ in $\mathscr{C}$ is an isomorphism as soon as $F(f)$ is an isomorphism.

Examples 1.3.11. (i) The forgetful functor $\mathit{for}\colon \mathrm{Mod}(A) \to \mathbf{Set}$ associates to an A-module M the set M, and to a linear map f the map f. The functor *for* is faithful and conservative but not fully faithful.
(ii) The forgetful functor $\mathit{for}\colon \mathbf{Top} \to \mathbf{Set}$ (defined similarly as in (i)) is faithful. It is neither fully faithful nor conservative.
(iii) Consider the functor $\mathit{for}\colon \mathbf{Set} \to \mathbf{Rel}$ which is the identity on the objects of these categories and which, to a morphism $f\colon X \to Y$ in $\mathbf{Set}$, associates its graph $\Gamma_f \subset X \times Y$. This forgetful functor is faithful but not fully faithful. It is conservative (this is left as an exercise).

Definition 1.3.12. Let $\mathscr{C}$ be a category. The category $\mathrm{Mor}(\mathscr{C})$ of morphisms in $\mathscr{C}$ is defined as follows.

$$\begin{aligned}
&\mathrm{Ob}(\mathrm{Mor}(\mathscr{C})) = \{(U, V, s); U, V \in \mathscr{C}, s \in \mathrm{Hom}_{\mathscr{C}}(U, V),\\
&\mathrm{Hom}_{\mathrm{Mor}(\mathscr{C})}((s\colon U \to V), (s'\colon U' \to V')\\
&\qquad = \{u\colon U \to U', v\colon V \to V'; v \circ s = s' \circ u\}.
\end{aligned}$$

If $(u, v)\colon (U, V, s) \to (U', V', s')$ is as above and $(u', v')\colon (U', V', s') \to (U'', V'', s'')$ is another morphism in $\mathrm{Mor}(\mathscr{C})$, the composition $(u', v') \circ (u, v)$ is given by $(u' \circ u, v' \circ v)$.

The category $\mathrm{Mor}_0(\mathscr{C})$ is defined as follows.

$$\begin{aligned}
&\mathrm{Ob}(\mathrm{Mor}_0(\mathscr{C})) = \{(U, V, s); U, V \in \mathscr{C}_X, s \in \mathrm{Hom}_{\mathscr{C}}(U, V),\\
&\mathrm{Hom}_{\mathrm{Mor}_0(\mathscr{C})}((s\colon U \to V), (s'\colon U' \to V')\\
&\qquad = \{u\colon U \to U', w\colon V' \to V; s = w \circ s' \circ u\}.
\end{aligned}$$

A morphism $(s\colon U\to V)\to(s'\colon U'\to V')$ in $\mathrm{Mor}(\mathscr{C})$ (resp. $\mathrm{Mor}_0(\mathscr{C})$) is visualized by the commutative diagram on the left (resp. on the right) below:

$$\begin{array}{ccc} U & \xrightarrow{s} & V \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ U' & \xrightarrow{s'} & V', \end{array} \qquad \begin{array}{ccc} U & \xrightarrow{s} & V \\ {\scriptstyle u}\downarrow & & \uparrow{\scriptstyle w} \\ U' & \xrightarrow{s'} & V'. \end{array}$$

If $(u,w)\colon(U,V,s)\to(U',V',s')$ is as above and $(u',w')\colon(U',V',s')\to(U'',V'',s'')$ is another morphism in $\mathrm{Mor}_0(\mathscr{C})$, the composition $(u',w')\circ(u,w)$ is given by $(u'\circ u),(w\circ w')$.

Product of categories

One defines the product of two categories $\mathscr{C}$ and $\mathscr{C}'$ by :

$$\mathrm{Ob}(\mathscr{C}\times\mathscr{C}')=\mathrm{Ob}(\mathscr{C})\times\mathrm{Ob}(\mathscr{C}')$$
$$\mathrm{Hom}_{\mathscr{C}\times\mathscr{C}'}((X,X'),(Y,Y'))=\mathrm{Hom}_{\mathscr{C}}(X,Y)\times\mathrm{Hom}_{\mathscr{C}'}(X',Y').$$

A bifunctor $F\colon\mathscr{C}\times\mathscr{C}'\to\mathscr{C}''$ is a functor on the product category. This means that for $X\in\mathscr{C}$ and $X'\in\mathscr{C}'$, $F(X,\bullet)\colon\mathscr{C}'\to\mathscr{C}''$ and $F(\bullet,X'):\mathscr{C}\to\mathscr{C}''$ are functors, and moreover for any morphisms $f:X\to Y$ in $\mathscr{C}$, $g:X'\to Y'$ in $\mathscr{C}'$, the diagram below commutes:

$$\begin{array}{ccc} F(X,X') & \xrightarrow{F(X,g)} & F(X,Y') \\ {\scriptstyle F(f,X')}\downarrow & \searrow^{F(f,g)} & \downarrow{\scriptstyle F(f,Y')} \\ F(Y,X') & \xrightarrow[F(Y,g)]{} & F(Y,Y'). \end{array}$$

In fact, $(f,g)=(\mathrm{id}_Y,g)\circ(f,\mathrm{id}_{X'})=(f,\mathrm{id}_{Y'})\circ(\mathrm{id}_X,g)$.

Examples 1.3.13. (i) $\mathrm{Hom}_{\mathscr{C}}(\bullet,\bullet):\mathscr{C}^{\mathrm{op}}\times\mathscr{C}\to\mathbf{Set}$ is a bifunctor.
(ii) If A is a $\mathbf{k}$-algebra, we have met the bifunctors

$$\mathrm{Hom}_A(\bullet,\bullet):\ \mathrm{Mod}(A)^{\mathrm{op}}\times\mathrm{Mod}(A)\to\mathrm{Mod}(\mathbf{k}),$$
$$\bullet\otimes_A\bullet:\ \mathrm{Mod}(A^{\mathrm{op}})\times\mathrm{Mod}(A)\to\mathrm{Mod}(\mathbf{k}).$$

Morphisms of functors

Definition 1.3.14. Let F_1,F_2 be two functors from $\mathscr{C}$ to $\mathscr{C}'$. A morphism of functors $\theta\colon F_1\to F_2$ is the data for all $X\in\mathscr{C}$ of a morphism $\theta(X):F_1(X)\to F_2(X)$ such that for all $f\colon X\to Y$, the diagram below commutes:

$$\begin{array}{ccc} F_1(X) & \xrightarrow{\theta(X)} & F_2(X) \\ {\scriptstyle F_1(f)}\downarrow & & \downarrow{\scriptstyle F_2(f)} \\ F_1(Y) & \xrightarrow{\theta(Y)} & F_2(Y). \end{array} \tag{1.3.1}$$

A morphism of functors is visualized by a diagram:

$$\mathscr{C} \;\underset{F_2}{\overset{F_1}{\rightrightarrows}}\; \mathscr{C}' \qquad (\Downarrow \theta)$$

Hence, by considering the family of functors from $\mathscr{C}$ to $\mathscr{C}'$ and the morphisms of such functors, we get a new category.

Notation 1.3.15. (i) We denote by $\mathrm{Fct}(\mathscr{C},\mathscr{C}')$ the category of functors from $\mathscr{C}$ to $\mathscr{C}'$. One may also use the shorter notation $(\mathscr{C}')^{\mathscr{C}}$.

Examples 1.3.16. Let $\mathbf{k}$ be a field and consider the functor

$$\begin{aligned} &{}^*\colon \mathrm{Mod}(\mathbf{k})^{\mathrm{op}} \to \mathrm{Mod}(\mathbf{k}),\\ &V \mapsto V^* = \mathrm{Hom}_{\mathbf{k}}(V,\mathbf{k}), \quad u\colon V \to W \mapsto u^*\colon W^* \to V^*. \end{aligned}$$

Then there is a morphism of functors $\mathrm{id}_{\mathrm{Mod}(\mathbf{k})} \to {}^* \circ {}^*$ in $\mathrm{Fct}(\mathrm{Mod}(\mathbf{k}), \mathrm{Mod}(\mathbf{k}))$. Indeed, for any $V \in \mathrm{Mod}(\mathbf{k})$, there is a natural morphism $V \to V^{**}$ and for $u\colon V \to W$ a linear map, the diagram below commutes:

$$\begin{array}{ccc} V & \longrightarrow & V^{**} \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle u^{**}} \\ W & \longrightarrow & W^{**}. \end{array} \tag{1.3.2}$$

(ii) We shall encounter morphisms of functors when considering pairs of adjoint functors (see (1.5.2)).

In particular we have the notion of an isomorphism of categories. A functor $F\colon \mathscr{C} \to \mathscr{C}'$ is an isomorphism of categories if there exists $G : \mathscr{C}' \to \mathscr{C}$ such that: $G \circ F = \mathrm{id}_{\mathscr{C}}$ and $F \circ G = \mathrm{id}_{\mathscr{C}'}$. This implies that for all $X \in \mathscr{C}$, $G \circ F(X) = X$. In practice, such a situation rarely occurs and is not really interesting. There is a weaker notion that we introduce below.

Definition 1.3.17. A functor $F\colon \mathscr{C} \to \mathscr{C}'$ is an equivalence of categories if there exists $G\colon \mathscr{C}' \to \mathscr{C}$ such that: $G \circ F$ is isomorphic to $\mathrm{id}_{\mathscr{C}}$ and $F \circ G$ is isomorphic to $\mathrm{id}_{\mathscr{C}'}$.

We shall not give the proof of the following important result below.

Theorem 1.3.18. *The functor $F\colon \mathscr{C} \to \mathscr{C}'$ is an equivalence of categories if and only if F is fully faithful and essentially surjective.*

If two categories are equivalent, all results and concepts in one of them have their counterparts in the other one. This is why this notion of equivalence of categories plays an important role in Mathematics.

Examples 1.3.19. (i) Let $\mathbf{k}$ be a field and let $\mathscr{C}$ denote the category defined by $\mathrm{Ob}(\mathscr{C}) = \mathbb{N}$ and $\mathrm{Hom}_{\mathscr{C}}(n,m) = M_{m,n}(\mathbf{k})$, the space of matrices of type (m,n) with entries in a field $\mathbf{k}$ (the composition being the usual composition of matrices). Define the functor $F\colon \mathscr{C} \to \mathrm{Mod}^f(\mathbf{k})$ as follows. To $n \in \mathbb{N}$, $F(n)$ associates $\mathbf{k}^n \in \mathrm{Mod}^f(\mathbf{k})$ and to a matrix of type (m,n), F associates the induced linear map from $\mathbf{k}^n$ to $\mathbf{k}^m$.

Clearly F is fully faithful. Since any finite dimensional vector space admits a basis, it is isomorphic to $\mathbf{k}^n$ for some n, hence F is essentially surjective. In conclusion, F is an equivalence of categories.

(ii) Let $\mathscr{C}$ and $\mathscr{C}'$ be two categories. There is an equivalence

$$\mathrm{Fct}(\mathscr{C}, \mathscr{C}')^{\mathrm{op}} \simeq \mathrm{Fct}(\mathscr{C}^{\mathrm{op}}, \mathscr{C}'^{\mathrm{op}}). \tag{1.3.3}$$

(iii) Let I, J and $\mathscr{C}$ be categories. There are equivalences

$$\mathrm{Fct}(I \times J, \mathscr{C}) \simeq \mathrm{Fct}(J, \mathrm{Fct}(I, \mathscr{C})) \simeq \mathrm{Fct}(I, \mathrm{Fct}(J, \mathscr{C})). \tag{1.3.4}$$

1.4 The Yoneda Lemma

Definition 1.4.1. Let $\mathscr{C}$ be a category. One defines the big categories

$$\mathscr{C}^\wedge = \mathrm{Fct}(\mathscr{C}^{\mathrm{op}}, \mathbf{Set}),\ \mathscr{C}^\vee = \mathrm{Fct}(\mathscr{C}^{\mathrm{op}}, \mathbf{Set}^{\mathrm{op}}),$$

and the functors

$$\begin{aligned} \mathrm{h}_{\mathscr{C}} : \mathscr{C} \to \mathscr{C}^\wedge, &\quad X \mapsto \mathrm{Hom}_{\mathscr{C}}(\,\bullet\,, X) \\ \mathrm{k}_{\mathscr{C}} : \mathscr{C} \to \mathscr{C}^\vee, &\quad X \mapsto \mathrm{Hom}_{\mathscr{C}}(X, \,\bullet\,). \end{aligned}$$

Since there is a natural equivalence of categories

$$\mathscr{C}^\vee \simeq \mathscr{C}^{\mathrm{op},\wedge,\mathrm{op}}, \tag{1.4.1}$$

we shall concentrate our study on $\mathscr{C}^\wedge$.

Theorem 1.4.2 (The Yoneda lemma)**.** *For $A \in \mathscr{C}^\wedge$ and $X \in \mathscr{C}$, there is an isomorphism $\mathrm{Hom}_{\mathscr{C}^\wedge}(\mathrm{h}_{\mathscr{C}}(X), A) \simeq A(X)$, functorial with respect to X and A.*

Proof. One constructs the morphism $\varphi\colon \mathrm{Hom}_{\mathscr{C}^\wedge}(\mathrm{h}_{\mathscr{C}}(X), A) \to A(X)$ by the chain of morphisms: $\mathrm{Hom}_{\mathscr{C}^\wedge}(\mathrm{h}_{\mathscr{C}}(X), A) \to \mathrm{Hom}_{\mathbf{Set}}(\mathrm{Hom}_{\mathscr{C}}(X, X), A(X)) \to A(X)$, where the last map is associated with id_X.

To construct $\psi\colon A(X) \to \mathrm{Hom}_{\mathscr{C}^\wedge}(\mathrm{h}_{\mathscr{C}}(X), A)$, it is enough to associate with $s \in A(X)$ and $Y \in \mathscr{C}$ a map from $\mathrm{Hom}_{\mathscr{C}}(Y, X)$ to $A(Y)$. It is defined by the chain of maps $\mathrm{Hom}_{\mathscr{C}}(Y, X) \to \mathrm{Hom}_{\mathbf{Set}}(A(X), A(Y)) \to A(Y)$ where the last map is associated with $s \in A(X)$.

One checks that φ and ψ are inverse to each other. □

Corollary 1.4.3. *The functors $\mathrm{h}_{\mathscr{C}}$ and $\mathrm{k}_{\mathscr{C}}$ are fully faithful.*

Proof. For $X, Y \in \mathscr{C}$, one has $\mathrm{Hom}_{\mathscr{C}^\wedge}(\mathrm{h}_{\mathscr{C}}(X), \mathrm{h}_{\mathscr{C}}(Y)) \simeq \mathrm{h}_{\mathscr{C}}(Y)(X) = \mathrm{Hom}_{\mathscr{C}}(X, Y)$. □

One calls $\mathrm{h}_{\mathscr{C}}$ and $\mathrm{k}_{\mathscr{C}}$ the Yoneda embeddings.

Hence, one may consider $\mathscr{C}$ as a full subcategory of $\mathscr{C}^\wedge$. In particular, for $X \in \mathscr{C}$, $\mathrm{h}_{\mathscr{C}}(X)$ determines X up to unique isomorphism, that is, an isomorphism $\mathrm{h}_{\mathscr{C}}(X) \simeq \mathrm{h}_{\mathscr{C}}(Y)$ determines a unique isomorphism $X \simeq Y$.

Corollary 1.4.4. *Let $\mathscr{C}$ be a category and let $f\colon X \to Y$ be a morphism in $\mathscr{C}$.*

(i) *Assume that for any* $Z \in \mathscr{C}$*, the map* $\mathrm{Hom}_{\mathscr{C}}(Z,X) \xrightarrow{f\circ} \mathrm{Hom}_{\mathscr{C}}(Z,Y)$ *is bijective. Then* f *is an isomorphism.*

(ii) *Assume that for any* $Z \in \mathscr{C}$*, the map* $\mathrm{Hom}_{\mathscr{C}}(Y,Z) \xrightarrow{\circ f} \mathrm{Hom}_{\mathscr{C}}(X,Z)$ *is bijective. Then* f *is an isomorphism.*

Proof. (i) By the hypothesis, $\mathrm{h}_{\mathscr{C}}(f) : \mathrm{h}_{\mathscr{C}}(X) \to \mathrm{h}_{\mathscr{C}}(Y)$ is an isomorphism in $\mathscr{C}^\wedge$. Since $\mathrm{h}_{\mathscr{C}}$ is fully faithful, this implies that f is an isomorphism (see Exercise 1.3 (ii)).
(ii) follows by replacing $\mathscr{C}$ with $\mathscr{C}^{\mathrm{op}}$. □

Definition 1.4.5. Let $\mathscr{C}$ and $\mathscr{C}'$ be categories, $F\colon \mathscr{C} \to \mathscr{C}'$ a functor and let $Z \in \mathscr{C}'$.

(i) The category $\mathscr{C}_Z$ is defined as follows:

$$\mathrm{Ob}(\mathscr{C}_Z) = \{(X,u); X \in \mathscr{C}, u\colon F(X) \to Z\},$$
$$\mathrm{Hom}_{\mathscr{C}_Z}((X_1,u_1),(X_2,u_2)) = \{v\colon X_1 \to X_2; u_1 = u_2 \circ F(v)\}.$$

(ii) The category $\mathscr{C}^Z$ is defined as follows:

$$\mathrm{Ob}(\mathscr{C}^Z) = \{(X,u); X \in \mathscr{C}, u\colon Z \to F(X))\},$$
$$\mathrm{Hom}_{\mathscr{C}^Z}((X_1,u_1),(X_2,u_2)) = \{v\colon X_1 \to X_2; u_2 = F(v) \circ u_1\}.$$

Note that the natural functors $(X,u) \mapsto X$ from $\mathscr{C}_Z$ and $\mathscr{C}^Z$ to $\mathscr{C}$ are faithful.

The morphisms in $\mathscr{C}_Z$ (resp. $\mathscr{C}^Z$) are visualized by the commutative diagram on the left (resp. on the right) below:

$$\begin{array}{ccc} F(X_1) & \xrightarrow{u_1} & Z, \\ {\scriptstyle F(v)}\downarrow & \nearrow_{u_2} & \\ F(X_2) & & \end{array} \qquad \begin{array}{ccc} Z & \xrightarrow{u_1} & F(X_1) \\ & {}_{u_2}\searrow & \downarrow{\scriptstyle F(v)} \\ & & F(X_2). \end{array}$$

1.5 Representable functors, adjoint functors

Representable functors

Definition 1.5.1. (i) One says that a functor $F\colon \mathscr{C}^{\mathrm{op}} \to \mathbf{Set}$ is representable if there exists $X \in \mathscr{C}$ such that $F(Y) \simeq \mathrm{Hom}_{\mathscr{C}}(Y,X)$ functorially in $Y \in \mathscr{C}$. In other words, $F \simeq \mathrm{h}_{\mathscr{C}}(X)$ in $\mathscr{C}^\wedge$. Such an object X is called a representative of F.

(ii) Similarly, a functor $G\colon \mathscr{C} \to \mathbf{Set}$ is representable if there exists $X \in \mathscr{C}$ such that $G(Y) \simeq \mathrm{Hom}_{\mathscr{C}}(X,Y)$ functorially in $Y \in \mathscr{C}$.

It is important to notice that the isomorphisms above determine X up to unique isomorphism. More precisely, given two isomorphisms $F \xrightarrow{\sim} \mathrm{h}_{\mathscr{C}}(X)$ and $F \xrightarrow{\sim} \mathrm{h}_{\mathscr{C}}(X')$ there exists a unique isomorphism $\theta\colon X \xrightarrow{\sim} X'$ making the diagram below commutative:

$$\begin{array}{ccc} & F & \\ {\scriptstyle\sim}\swarrow & & \searrow{\scriptstyle\sim} \\ \mathrm{h}_{\mathscr{C}}(X) & \xrightarrow[\sim]{\mathrm{h}_{\mathscr{C}}(\theta)} & \mathrm{h}_{\mathscr{C}}(X'). \end{array}$$

Representable functors provides a categorical language to deal with universal problems. Let us illustrate this by an example.

Example 1.5.2. Let A be a $\mathbf{k}$-algebra. Let N be a right A-module, M a left A-module and L a $\mathbf{k}$-module. Denote by $B(N \times M, L)$ the set of $(A, \mathbf{k})$-bilinear maps from $N \times M$ to L. Then the functor $F \colon L \mapsto B(N \times M, L)$ is representable by $N \otimes_A M$ by (1.2.2).

Adjoint functors

Definition 1.5.3. Let $F \colon \mathscr{C} \to \mathscr{C}'$ and $G \colon \mathscr{C}' \to \mathscr{C}$ be two functors. One says that (F, G) is a pair of adjoint functors or that F is a left adjoint to G, or that G is a right adjoint to F if there exists an isomorphism of bifunctors:

$$\mathrm{Hom}_{\mathscr{C}'}(F(\,\bullet\,), \,\bullet\,) \simeq \mathrm{Hom}_{\mathscr{C}}(\,\bullet\,, G(\,\bullet\,)). \tag{1.5.1}$$

If G is an adjoint to F, then G is unique up to isomorphism. In fact, $G(Y)$ is a representative of the functor $X \mapsto \mathrm{Hom}_{\mathscr{C}'}(F(X), Y)$.

The isomorphism (1.5.1) gives the isomorphisms

$$\begin{aligned} &\mathrm{Hom}_{\mathscr{C}'}(F \circ G(\,\bullet\,), \,\bullet\,) \simeq \mathrm{Hom}_{\mathscr{C}}(G(\,\bullet\,), G(\,\bullet\,)), \\ &\mathrm{Hom}_{\mathscr{C}'}(F(\,\bullet\,), F(\,\bullet\,)) \simeq \mathrm{Hom}_{\mathscr{C}}(\,\bullet\,, G \circ F(\,\bullet\,)). \end{aligned}$$

In particular, we have morphisms $X \to G \circ F(X)$, functorial in $X \in \mathscr{C}$, and morphisms $F \circ G(Y) \to Y$, functorial in $Y \in \mathscr{C}'$. In other words, we have morphisms of functors

$$\varepsilon \colon F \circ G \to \mathrm{id}_{\mathscr{C}'}, \qquad \iota \colon \mathrm{id}_{\mathscr{C}} \to G \circ F. \tag{1.5.2}$$

Moreover,

$$\begin{aligned} &\mathrm{id}_G \text{ is the composition } G \xrightarrow{\iota \circ G} G \circ F \circ G \xrightarrow{G \circ \varepsilon} G, \\ &\mathrm{id}_F \text{ is the composition } F \xrightarrow{F \circ \iota} F \circ G \circ F \xrightarrow{\varepsilon \circ F} F. \end{aligned} \tag{1.5.3}$$

Conversely, if $F \colon \mathscr{C} \to \mathscr{C}'$ and $G \colon \mathscr{C}' \to \mathscr{C}$ are two functors and ε and ι are morphisms of functors as in (1.5.2) satisfying (1.5.3), then (F, G) is a pair of adjoint functors.

Examples 1.5.4. (i) Let $X \in \mathbf{Set}$. Using the bijection (1.1.3), we get that the functor $\mathrm{Hom}_{\mathbf{Set}}(X, \,\bullet\,) \colon \mathbf{Set} \to \mathbf{Set}$ is right adjoint to the functor $\bullet \times X$.

(ii) Let A be a $\mathbf{k}$-algebra and let $L \in \mathrm{Mod}(\mathbf{k})$. Using the first isomorphism in (1.2.3), we get that the functor $\mathrm{Hom}_{\mathbf{k}}(L, \,\bullet\,) \colon \mathrm{Mod}(A)$ to $\mathrm{Mod}(A)$ is right adjoint to the functor $\bullet \otimes_{\mathbf{k}} L$.

(iii) Let A be a $\mathbf{k}$-algebra. Using the isomorphisms in (1.2.3) with $N = A$, we get that the "forgetful functor" $\mathit{for} \colon \mathrm{Mod}(A) \to \mathrm{Mod}(\mathbf{k})$ which, to an A-module associates the underlying $\mathbf{k}$-module, is right adjoint to the "extension of scalars functor" $A \otimes_{\mathbf{k}} \bullet \colon \mathrm{Mod}(\mathbf{k}) \to \mathrm{Mod}(A)$.

Exercises to Chapter 1

Exercise 1.1. In a category $\mathscr{C}$, consider three morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$. Prove that if $g \circ f$ and $h \circ g$ are isomorphisms, then f is an isomorphism.

Exercise 1.2. Prove that the categories **Set** and $\mathbf{Set}^{\mathrm{op}}$ are not equivalent and similarly with the categories $\mathbf{Set}^f$ and $(\mathbf{Set}^f)^{\mathrm{op}}$.
(Hint: if $F\colon \mathbf{Set} \to \mathbf{Set}^{\mathrm{op}}$ were such an equivalence, then $F(\varnothing) \simeq \{\mathrm{pt}\}$ and $F(\{\mathrm{pt}\}) \simeq \varnothing$. Now compare $\mathrm{Hom}_{\mathbf{Set}}(\{\mathrm{pt}\}, X)$ and $\mathrm{Hom}_{\mathbf{Set}^{\mathrm{op}}}(F(\{\mathrm{pt}\}), F(X))$ when X is a set with two elements.)

Exercise 1.3. (i) Let $F\colon \mathscr{C} \to \mathscr{C}'$ be a faithful functor and let f be a morphism in $\mathscr{C}$. Prove that if $F(f)$ is a monomorphism (resp. an epimorphism), then f is a monomorphism (resp. an epimorphism).
(ii) Assume now that F is fully faithful. Prove that if $F(f)$ is an isomorphism, then f is an isomorphism. In other words, fully faithful functors are conservative.

Exercise 1.4. Is the natural functor $\mathbf{Set} \to \mathbf{Rel}$ full, faithful, fully faithful, conservative?

Exercise 1.5. Prove that the category $\mathscr{C}$ is equivalent to the opposite category $\mathscr{C}^{\mathrm{op}}$ in the following cases:
(i) $\mathscr{C}$ denotes the category of finite abelian groups,
(ii) $\mathscr{C}$ is the category **Rel** of relations.

Exercise 1.6. (i) Prove that in the category **Set**, a morphism f is a monomorphism (resp. an epimorphism) if and only if it is injective (resp. surjective).
(ii) Prove that in the category of rings, the morphism $\mathbb{Z} \to \mathbb{Q}$ is an epimorphism. (Hint: if $f\colon \mathbb{Q} \to A$ is a morphism of rings, then $f(p/q) = f(p) \times f(q)^{-1}$.)
(iii) In the category **Top**, give an example of a morphism which is both a monomorphism and an epimorphism and which is not an isomorphism.

Exercise 1.7. Let $F\colon \mathscr{C} \to \mathbf{Set}$ be a functor and let $u\colon X \to Y$ be a morphisms in $\mathscr{C}$. Assume that F is faithful. Prove that u is an epimorphism (resp. a monomorphism) as soon as $F(u)$ is surjective (resp. injective).

Exercise 1.8. Let $\mathscr{C}$ be a category. We denote by $\mathrm{id}_{\mathscr{C}}\colon \mathscr{C} \to \mathscr{C}$ the identity functor of $\mathscr{C}$ and by $\mathrm{End}\,(\mathrm{id}_{\mathscr{C}})$ the set of endomorphisms of the identity functor $\mathrm{id}_{\mathscr{C}} : \mathscr{C} \to \mathscr{C}$, that is,

$$\mathrm{End}\,(\mathrm{id}_{\mathscr{C}}) = \mathrm{Hom}_{\,\mathrm{Fct}(\mathscr{C},\mathscr{C})}(\mathrm{id}_{\mathscr{C}}, \mathrm{id}_{\mathscr{C}}).$$

Prove that the composition law on $\mathrm{End}\,(\mathrm{id}_{\mathscr{C}})$ is commutative.

Chapter 2

Limits

Summary

After treating the particular cases of kernels and cokernels, products and coproducts, we shall construct limits and colimits, starting with limits in the category **Set**. We show that limits may be obtained as a combination of products and kernels, hence that colimits may be obtained as a combination of coproducts and cokernels. In particular the category **Set** of sets (in a given universe) admits small limits and colimits, as well as the category $\mathrm{Mod}(A)$ of modules over a ring A. As a particular case of the notions of limits and colimits we get those of fiber product and fiber coproduct. Then we introduce the fundamental notion of directed colimits and cofinal functors. We show that in the category **Set**, directed colimits commute with finite limits. Finally we have a glance to the theory of ind-objects, following [SGA4] (see also [KS06, KS01]).

Caution. We may sometimes use the terms "projective limit" or "inductive limits" instead of "limit" or "colimit".

References for this chapter already appeared at the beginning of Chapter 1.

2.1 Products and coproducts

Let $\mathscr{C}$ be a category (in a given universe $\mathscr{U}$) and consider a family $\{X_i\}_{i\in I}$ of objects of $\mathscr{C}$ indexed by a small set I. Consider the two functors

$$\mathscr{C}^{\mathrm{op}} \to \mathbf{Set}, Y \mapsto \prod_{i\in I} \mathrm{Hom}_{\mathscr{C}}(Y, X_i), \tag{2.1.1}$$

$$\mathscr{C} \to \mathbf{Set}, Y \mapsto \prod_{i\in I} \mathrm{Hom}_{\mathscr{C}}(X_i, Y). \tag{2.1.2}$$

Definition 2.1.1. (i) Assume that the functor in (2.1.1) is representable. In this case one denotes by $\prod_{i\in I} X_i$ a representative and calls this object the product of the X_i's. In case $I = \{1, 2\}$, one denotes this object by $X_1 \times X_2$.

(ii) Assume that the functor in (2.1.2) is representable. In this case one denotes by $\coprod_{i\in I} X_i$ a representative and calls this object the coproduct of the X_i's. In case $I = \{1, 2\}$, one denotes this object by $X_1 \coprod X_2$ or even $X_1 \sqcup X_2$.

(iii) If for any family of objects $\{X_i\}_{i\in I}$, the product (resp. coproduct) exists, one says that the category $\mathscr{C}$ admits products (resp. coproducts) indexed by I.

(iv) If $X_i = X$ for all $i \in I$, one writes:

$$X^I := \prod_{i\in I} X_i,\ X^{\coprod I} := \coprod_{i\in I} X_i.$$

(v) One often write $\prod_i X_i$ instead of $\prod_{i\in I} X_i$ and similarly with coproducts.

In case of additive categories (see § 4.1 below), one writes $\bigoplus_i X_i$ instead of $\coprod_i X_i$ and $X^{(I)}$ or $X^{\oplus I}$ instead of $X^{\coprod I}$. If $\mathscr{C} = \mathbf{Set}$, one often writes $\bigsqcup_i X_i$ instead of $\coprod_i X_i$. and $X^{\sqcup I}$ instead of $X^{\coprod I}$.

Note that the coproduct in $\mathscr{C}$ is the product in $\mathscr{C}^{\rm op}$.

By this definition, if the product or the coproduct exists, then one has the isomorphisms, functorial with respect to $Y \in \mathscr{C}$:

$$\operatorname{Hom}_{\mathscr{C}}(Y, \prod_i X_i) \simeq \prod_i \operatorname{Hom}_{\mathscr{C}}(Y, X_i), \tag{2.1.3}$$

$$\operatorname{Hom}_{\mathscr{C}}(\coprod_i X_i, Y) \simeq \prod_i \operatorname{Hom}_{\mathscr{C}}(X_i, Y). \tag{2.1.4}$$

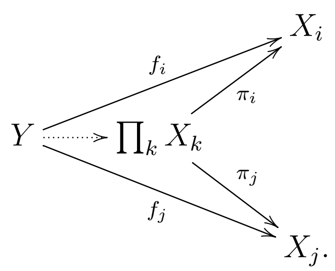
Assume that $\prod_i X_i$ exists. By choosing $Y = \prod_i X_i$ in (2.1.3), we get the morphisms

$$\pi_i \colon \prod_j X_j \to X_i. \tag{2.1.5}$$

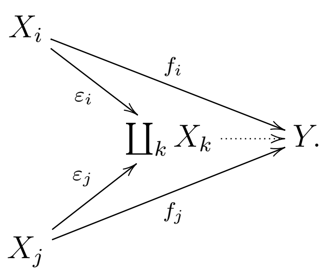
Similarly, assume that $\coprod_i X_i$ exists. By choosing $Y = \coprod_i X_i$ in (2.1.4), we get the morphisms

$$\varepsilon_i \colon X_i \to \coprod_j X_j. \tag{2.1.6}$$

The isomorphism (2.1.3) may be translated as follows. Given an object Y and a family of morphisms $f_i \colon Y \to X_i$, this family factorizes uniquely through $\prod_i X_i$. This is visualized by the diagram



The isomorphism (2.1.4) may be translated as follows. Given an object Y and a family of morphisms $f_i \colon X_i \to Y$, this family factorizes uniquely through $\coprod_i X_i$. This is visualized by the diagram



Example 2.1.2. (i) The category **Set** (in a given universe) admits small products (that is, products indexed by small sets) and the two definitions, that given in (1.1.1) and that given in Definition 2.1.1, coincide.

(ii) The category **Set** admits coproducts indexed by small sets, namely, the disjoint union.

(iii) Let A be a ring. The category $\mathrm{Mod}(A)$ admits products, as defined in § 1.2. The category $\mathrm{Mod}(A)$ also admits coproducts, which are the direct sums defined in § 1.2 and are denoted by $\bigoplus$.

(iv) Let X be a set and denote by $\mathfrak{X}$ the category of subsets of X. (The set $\mathfrak{X}$ is ordered by inclusion, hence defines a category.) For $S_1, S_2 \in \mathfrak{X}$, their product in the category $\mathfrak{X}$ is their intersection and their coproduct is their union.

Remark 2.1.3. The forgetful functor $\mathit{for}\colon \mathrm{Mod}(A) \to \mathbf{Set}$ commutes with products but does not commute with coproducts. The coproduct of two modules is not their disjoint union. That is the reason why the coproduct in the category $\mathrm{Mod}(A)$ is called the direct sum and is denoted differently, namely by $\oplus$.

2.2 Kernels and cokernels

Let $\mathscr{C}$ be a category and consider two parallel arrows $f, g\colon X_0 \rightrightarrows X_1$ in $\mathscr{C}$. Consider the two functors (recall 1.1.7)

$$\mathscr{C}^{\mathrm{op}} \to \mathbf{Set}, Y \mapsto \mathrm{Ker}(\mathrm{Hom}_{\mathscr{C}}(Y, X_0) \rightrightarrows \mathrm{Hom}_{\mathscr{C}}(Y, X_1)), \tag{2.2.1}$$

$$\mathscr{C} \to \mathbf{Set}, Y \mapsto \mathrm{Ker}(\mathrm{Hom}_{\mathscr{C}}(X_1, Y) \rightrightarrows \mathrm{Hom}_{\mathscr{C}}(X_0, Y)). \tag{2.2.2}$$

Definition 2.2.1. (i) Assume that the functor in (2.2.1) is representable. In this case one denotes by $\mathrm{Ker}(f, g)$ a representative and calls this object a kernel (one also says an equalizer) of (f, g).

(ii) Assume that the functor in (2.2.2) is representable. In this case one denotes by $\mathrm{Coker}(f, g)$ a representative and calls this object a cokernel (one also says a co-equalizer) of (f, g).

(iii) A sequence $Z \to X_0 \rightrightarrows X_1$ (resp. $X_0 \rightrightarrows X_1 \to Z$) is exact if Z is isomorphic to the kernel (resp. cokernel) of $X_0 \rightrightarrows X_1$.

(iv) Assume that the category $\mathscr{C}$ admits a zero-object 0. Let $f\colon X \to Y$ be a morphism in $\mathscr{C}$. A kernel (resp. a cokernel) of f, if it exists, is a kernel (resp. a cokernel) of $f, 0\colon X \rightrightarrows Y$. It is denoted by $\mathrm{Ker}(f)$ (resp. $\mathrm{Coker}(f)$).

Note that the cokernel in $\mathscr{C}$ is the kernel in $\mathscr{C}^{\mathrm{op}}$.

By this definition, the kernel or the cokernel of $f, g\colon X_0 \rightrightarrows X_1$ exists if and only if one has the isomorphisms, functorial in $Y \in \mathscr{C}$:

$$\mathrm{Hom}_{\mathscr{C}}(Y, \mathrm{Ker}(f, g)) \simeq \mathrm{Ker}(\mathrm{Hom}_{\mathscr{C}}(Y, X_0) \rightrightarrows \mathrm{Hom}_{\mathscr{C}}(Y, X_1)), \tag{2.2.3}$$

$$\mathrm{Hom}_{\mathscr{C}}(\mathrm{Coker}(f, g), Y) \simeq \mathrm{Ker}(\mathrm{Hom}_{\mathscr{C}}(X_1, Y) \rightrightarrows \mathrm{Hom}_{\mathscr{C}}(X_0, Y)). \tag{2.2.4}$$

Assume that $\mathrm{Ker}(f, g)$ exists. By choosing $Y = \mathrm{Ker}(f, g)$ in (2.2.3), we get the morphism

$$h\colon \mathrm{Ker}(X_0 \rightrightarrows X_1) \to X_0.$$

Similarly, assume that $\mathrm{Coker}(f,g)$ exists. By choosing $Y = \mathrm{Coker}(f,g)$ in (2.2.4), we get the morphism

$$k\colon X_1 \to \mathrm{Coker}(X_0 \rightrightarrows X_1).$$

Proposition 2.2.2. *The morphism* $h\colon \mathrm{Ker}(X_0 \rightrightarrows X_1) \to X_0$ *is a monomorphism and the morphism* $k\colon X_1 \to \mathrm{Coker}(X_0 \rightrightarrows X_1)$ *is an epimorphism.*

Proof. (i) Let us write Hom instead of $\mathrm{Hom}_{\mathscr{C}}$. Note that for $Y \in \mathscr{C}$, one has $\mathrm{Hom}\,(Y, \mathrm{Ker}(X_0 \rightrightarrows X_1)) \simeq \mathrm{Ker}(\mathrm{Hom}_{\mathscr{C}}(Y, X_0) \rightrightarrows \mathrm{Hom}\,(Y, X_1))$. Hence, the map

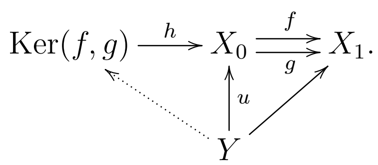
$$\mathrm{Hom}\,(h, Y)\colon \mathrm{Hom}\,(Y, \mathrm{Ker}(X_0 \rightrightarrows X_1)) \to \mathrm{Hom}\,(Y, X_0)$$

is injective.

(ii) The case of cokernels follows, by reversing the arrows. □

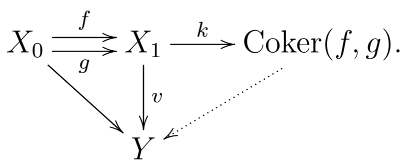
The isomorphism (2.2.3) may be translated as follows. Given an objet Y and a morphism $u\colon Y \to X_0$ such that $f \circ u = g \circ u$, the morphism u factors uniquely through $\mathrm{Ker}(f,g)$. This is visualized by the diagram

(2.2.5)



The isomorphism (2.2.4) may be translated as follows. Given an objet Y and a morphism $v\colon X_1 \to Y$ such that $v \circ f = v \circ g$, the morphism v factors uniquely through $\mathrm{Coker}(f,g)$. This is visualized by diagram:

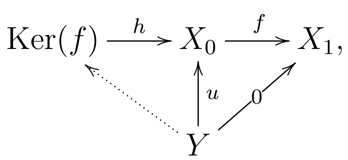
(2.2.6)

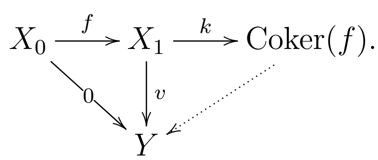


Example 2.2.3. (i) The category **Set** admits kernels and the two definitions (that given in (1.1.7) and that given in Definition 2.2.1) coincide.

(ii) The category **Set** admits cokernels. If $f, g\colon Z_0 \rightrightarrows Z_1$ are two maps, the cokernel of (f,g) is the quotient set $Z_1/\mathscr{R}$ where $\mathscr{R}$ is the equivalence relation generated by the relation $x \sim y$ if there exists $z \in Z_0$ with $f(z) = x$ and $g(z) = y$.

(iii) Let A be a ring. The category $\mathrm{Mod}(A)$ admits a zero object. Hence, the kernel or the cokernel of a morphism f means the kernel or the cokernel of $(f, 0)$. As already mentioned, the kernel of a linear map $f\colon M \to N$ is the A-module $f^{-1}(0)$ and the cokernel is the quotient module $M/\operatorname{Im} f$. The kernel and cokernel are visualized by the diagrams





2.3 Limits and colimits

Let us generalize and unify the preceding constructions.

Definition 2.3.1. Let I and $\mathscr{C}$ categories with I small. A projective system (resp. an inductive system) in $\mathscr{C}$ indexed by I is nothing but a functor $\beta\colon I^{\mathrm{op}} \to \mathscr{C}$ (resp. $\alpha\colon I \to \mathscr{C}$).

For example, if $(I, \leq)$ is a pre-ordered set, I the associated category, an inductive system indexed by I is the data of a family $(X_i)_{i\in I}$ of objects of $\mathscr{C}$ and for all $i \leq j$, a morphism $X_i \to X_j$ with the natural compatibility conditions.

Projective limits in Set

Assume first that $\mathscr{C}$ is the category **Set** and let us consider a projective system $\beta\colon I^{\mathrm{op}} \to \mathbf{Set}$. One sets

$$\lim \beta = \{(x_i)_i \in \prod_i \beta(i); \beta(s)(x_j) = x_i \text{ for all } s \in \mathrm{Hom}_I(i,j)\}. \tag{2.3.1}$$

The next result is obvious.

Lemma 2.3.2. *Let $\beta\colon I^{\mathrm{op}} \to \mathbf{Set}$ be a functor and let $X \in \mathbf{Set}$. There is a natural isomorphism*

$$\mathrm{Hom}_{\mathbf{Set}}(X, \lim \beta) \xrightarrow{\sim} \lim \mathrm{Hom}_{\mathbf{Set}}(X, \beta),$$

where $\mathrm{Hom}_{\mathbf{Set}}(X,\beta)$ denotes the functor $I^{\mathrm{op}} \to \mathbf{Set}$, $i \mapsto \mathrm{Hom}_{\mathbf{Set}}(X, \beta(i))$.

Limits and colimits

Consider now two functors $\beta\colon I^{\mathrm{op}} \to \mathscr{C}$ and $\alpha\colon I \to \mathscr{C}$. For $X \in \mathscr{C}$, we get functors from I^{op} to **Set**:

$$\begin{aligned}
&\mathrm{Hom}_{\mathscr{C}}(X,\beta)\colon I^{\mathrm{op}} \ni i \mapsto \mathrm{Hom}_{\mathscr{C}}(X,\beta(i)) \in \mathbf{Set},\\
&\mathrm{Hom}_{\mathscr{C}}(\alpha,X)\colon I^{\mathrm{op}} \ni i \mapsto \mathrm{Hom}_{\mathscr{C}}(\alpha,X) \in \mathbf{Set}.
\end{aligned}$$

Definition 2.3.3. (i) Assume that the functor $X \mapsto \lim \mathrm{Hom}_{\mathscr{C}}(X,\beta)$ is representable. We denote by $\lim \beta$ its representative and say that the functor β admits a limit (or "a projective limit") in $\mathscr{C}$. In other words, we have the isomorphism, functorial in $X \in \mathscr{C}$:

$$\mathrm{Hom}_{\mathscr{C}}(X, \lim \beta) \simeq \lim \mathrm{Hom}_{\mathscr{C}}(X,\beta). \tag{2.3.2}$$

(ii) Assume that the functor $X \mapsto \lim \mathrm{Hom}_{\mathscr{C}}(\alpha,X)$ is representable. We denote by $\mathrm{colim}\, \alpha$ its representative and say that the functor α admits a colimit (or "an inductive limit") in $\mathscr{C}$. In other words, we have the isomorphism, functorial in $X \in \mathscr{C}$:

$$\mathrm{Hom}_{\mathscr{C}}(\mathrm{colim}\, \alpha, X) \simeq \lim \mathrm{Hom}_{\mathscr{C}}(\alpha,X), \tag{2.3.3}$$

Remark 2.3.4. The limit of the functor β is not only the object $\lim \beta$ but also the isomorphism of functors given in (2.3.2), and similarly with colimits.

When $\mathscr{C} = \mathbf{Set}$ this definition of $\lim \beta$ coincides with the former one, in view of Lemma 2.3.2.

Notice that both limits and colimits are defined using limits in **Set**.

Assume that $\lim \beta$ exists in $\mathscr{C}$. One gets:

$$\lim \operatorname{Hom}_{\mathscr{C}}(\lim \beta, \beta) \simeq \operatorname{Hom}_{\mathscr{C}}(\lim \beta, \lim \beta)$$

and the identity of $\lim \beta$ defines a family of morphisms

$$\pi_i \colon \lim \beta \to \beta(i).$$

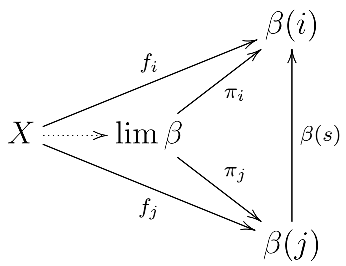
Consider a family of morphisms $\{f_i \colon X \to \beta(i)\}_{i \in I}$ in $\mathscr{C}$ satisfying the compatibility conditions

$$f_i = \beta(s) \circ f_j \text{ for all } s \in \operatorname{Hom}_I(i, j). \tag{2.3.4}$$

This family of morphisms is nothing but an element of $\lim\limits_{i} \operatorname{Hom}(X, \beta(i))$, hence by (2.3.2), an element of $\operatorname{Hom}(X, \lim \beta)$. Therefore, $\lim \beta$ is characterized by the "universal property":

$$\begin{cases} \text{for all } X \in \mathscr{C} \text{ and all family of morphisms } \{f_i \colon X \to \beta(i)\}_{i \in I} \\ \text{in } \mathscr{C} \text{ satisfying (2.3.4), the morphisms } f_i\text{'s factorize uniquely} \\ \text{through } \lim \beta. \end{cases} \tag{2.3.5}$$

This is visualized by the diagram:



Similarly, assume that $\operatorname{colim} \alpha$ exists in $\mathscr{C}$. One gets:

$$\lim \operatorname{Hom}_{\mathscr{C}}(\alpha, \operatorname{colim} \alpha) \simeq \operatorname{Hom}_{\mathscr{C}}(\operatorname{colim} \alpha, \operatorname{colim} \alpha)$$

and the identity of $\operatorname{colim} \alpha$ defines a family of morphisms

$$\varepsilon_i \colon \alpha(i) \to \operatorname{colim} \alpha.$$

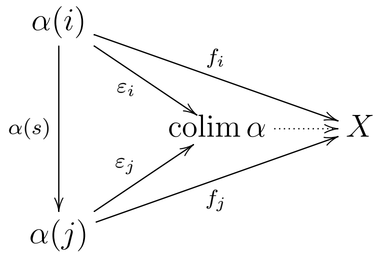
Consider a family of morphisms $\{f_i \colon \alpha(i) \to X\}_{i \in I}$ in $\mathscr{C}$ satisfying the compatibility conditions

$$f_i = f_j \circ \alpha(s) \text{ for all } s \in \operatorname{Hom}_I(i, j). \tag{2.3.6}$$

This family of morphisms is nothing but an element of $\lim\limits_{i} \operatorname{Hom}(\alpha(i), X)$, hence by (2.3.3), an element of $\operatorname{Hom}(\operatorname{colim} \alpha, X)$. Therefore, $\operatorname{colim} \alpha$ is characterized by the "universal property":

$$\begin{cases} \text{for all } X \in \mathscr{C} \text{ and all family of morphisms } \{f_i \colon \alpha(i) \to X\}_{i \in I} \\ \text{in } \mathscr{C} \text{ satisfying (2.3.6), the morphisms } f_i\text{'s factorize uniquely} \\ \text{through } \operatorname{colim} \alpha. \end{cases} \tag{2.3.7}$$

This is visualized by the diagram:



Example 2.3.5. Let X be a set and let $\mathfrak{X}$ be the category of subsets of X (see Example 2.1.2 (iv)). Let $\beta\colon I^{\mathrm{op}} \to \mathfrak{X}$ and $\alpha\colon I \to \mathfrak{X}$ be two functors. Then

$$\lim \beta \simeq \bigcap_i \beta(i), \ \operatorname{colim} \alpha \simeq \bigcup_i \alpha(i).$$

Examples 2.3.6. (i) When the category I is discrete, limits and colimits indexed by I are nothing but products and coproducts indexed by I.

(ii) Consider the category I with two objects and two parallel morphisms other than identities, visualized by $\bullet \rightrightarrows \bullet$. A functor $\alpha\colon I \to \mathscr{C}$ is characterized by two parallel arrows in $\mathscr{C}$:

$$f, g\colon X_0 \rightrightarrows X_1 \tag{2.3.8}$$

In the sequel we shall identify such a functor with the diagram (2.3.8). Then, the kernel (resp. cokernel) of (f, g) is nothing but the limit (resp. colimit) of the functor α.

(iii) If I is the empty category and $\alpha\colon I \to \mathscr{C}$ is a functor, then $\lim \alpha$ exists in $\mathscr{C}$ if and only if $\mathscr{C}$ has a terminal object $\mathrm{pt}_{\mathscr{C}}$, and in this case $\lim \alpha \simeq \mathrm{pt}_{\mathscr{C}}$. Similarly, $\operatorname{colim} \alpha$ exists in $\mathscr{C}$ if and only if $\mathscr{C}$ has an initial object $\varnothing_{\mathscr{C}}$, and in this case $\operatorname{colim} \alpha \simeq \varnothing_{\mathscr{C}}$.

(iv) If I admits a terminal object, say i_o and if $\beta\colon I^{\mathrm{op}} \to \mathscr{C}$ and $\alpha\colon I \to \mathscr{C}$ are functors, then

$$\lim \beta \simeq \beta(i_o), \ \operatorname{colim} \alpha \simeq \alpha(i_o).$$

This follows immediately of (2.3.5) and (2.3.7).

If every functor from I^{op} to $\mathscr{C}$ admits a limit, one says that $\mathscr{C}$ admits limits indexed by I.

Remark 2.3.7. Assume that $\mathscr{C}$ admits limits (resp. colimits) indexed by I. Then $\lim\colon \mathrm{Fct}(I^{\mathrm{op}}, \mathscr{C}) \to \mathscr{C}$ (resp. $\operatorname{colim}\colon \mathrm{Fct}(I, \mathscr{C}) \to \mathscr{C}$) is a functor.

Definition 2.3.8. One says that a category $\mathscr{C}$ admits small limits (resp. small colimits) if for any small category I and any functor $\beta\colon I^{\mathrm{op}} \to \mathscr{C}$ (resp. $\alpha\colon I \to \mathscr{C}$) $\lim \beta$ (resp. $\operatorname{colim} \alpha$) exists in $\mathscr{C}$.

Similarly one says that $\mathscr{C}$ admits finite limits or colimits if the preceding conditions hold when assuming that I is finite.

Caution We shall often neglect the adjective “small” before the words “limit” and “colimit”.

Limits as kernels and products

We have seen that products and kernels (resp. coproducts and cokernels) are particular cases of limits (resp. colimits). One can show that conversely, limits can be obtained as kernels of products and colimits can be obtained as cokernels of coproducts.

Recall that for a category I, $\mathrm{Mor}(I)$ denote the set of morphisms in I. There are two natural maps (source and target) from $\mathrm{Mor}(I)$ to $\mathrm{Ob}(I)$:

$$\sigma: \mathrm{Mor}(I) \to \mathrm{Ob}(I), \quad (s\colon i \to j) \mapsto i,$$
$$\tau: \mathrm{Mor}(I) \to \mathrm{Ob}(I), \quad (s\colon i \to j) \mapsto j.$$

Let $\mathscr{C}$ be a category which admits limits and let $\beta\colon I^{\mathrm{op}} \to \mathscr{C}$ be a functor. For $s\colon i \to j$, we get two morphisms in $\mathscr{C}$:

$$\beta(i) \times \beta(j) \underset{\beta(s)}{\overset{\mathrm{id}_{\beta(i)}}{\rightrightarrows}} \beta(i)$$

from which we deduce the morphisms in $\mathscr{C}$: $\prod_{k\in I} \beta(k) \rightrightarrows \beta(\sigma(s)) \times \beta(\tau(s)) \rightrightarrows \beta(\sigma(s))$. These morphisms define the two morphisms in $\mathscr{C}$:

$$\prod_{k\in I} \beta(k) \underset{b}{\overset{a}{\rightrightarrows}} \prod_{s\in \mathrm{Mor}(I)} \beta(\sigma(s)). \tag{2.3.9}$$

Similarly, assume that $\mathscr{C}$ admits colimits and let $\alpha\colon I \to \mathscr{C}$ be a functor. By reversing the arrows, one gets the two morphisms in $\mathscr{C}$:

$$\coprod_{s\in \mathrm{Mor}(I)} \alpha(\sigma(s)) \underset{b}{\overset{a}{\rightrightarrows}} \coprod_{k\in I} \alpha(k). \tag{2.3.10}$$

Proposition 2.3.9. (i) $\lim \beta$ *is the kernel of* (a, b) *in* (2.3.9),

(ii) $\mathrm{colim}\, \alpha$ *is the cokernel of* (a, b) *in* (2.3.10).

Sketch of proof. By the definition of limits and colimits we are reduced to check (i) when $\mathscr{C} = \mathbf{Set}$ and in this case this is obvious. □

In particular, a category $\mathscr{C}$ admits finite limits if and only if it satisfies:

(i) $\mathscr{C}$ admits a terminal object,

(ii) for any $X, Y \in \mathrm{Ob}(\mathscr{C})$, the product $X \times Y$ exists in $\mathscr{C}$,

(iii) for any parallel arrows in $\mathscr{C}$, $f, g\colon X \rightrightarrows Y$, the kernel exists in $\mathscr{C}$.

There is a similar result for finite colimits, replacing a terminal object by an initial object, products by coproducts and kernels by cokernels.

Theorem 2.3.10. (a) *The category* **Set** *admits small limits and colimits.*

(b) *Let A be a ring. The category* $\mathrm{Mod}(A)$ *admits small limits and colimits and the forgetful functor* $\mathit{for}\colon \mathrm{Mod}(A) \to \mathbf{Set}$ *commutes with limits.*

Proof. (i) Both categories admit small products and coproducts as well as kernels and cokernels (see Example 2.2.3).

(ii) The forgetful functor *for* commutes with products and kernels. □

Recall that the forgetful functor *for* does not commute with coproducts (see Remark 2.1.3).

2.4 Fiber products and coproducts

Consider the category I with three objects $\{a, b, c\}$ and two morphisms other than the identities, visualized by the diagram

$$I\colon a \leftarrow c \rightarrow b.$$

Let $\mathscr{C}$ be a category. A functor $\beta\colon I^{\mathrm{op}} \to \mathscr{C}$ (resp. $\alpha\colon I \to \mathscr{C}$) is nothing but the data of three objects X_0, X_1, Y and two morphisms (f, g) (resp. (k, l)) visualized by the arrows on the left (resp. on the right)

$$X_0 \xrightarrow{f} Y \xleftarrow{g} X_1, \quad X_0 \xleftarrow{k} W \xrightarrow{l} X_1.$$

The fiber product $X_0 \times_Y X_1$ of X_0 and X_1 over Y, if it exists, is the limit of β.

The fiber coproduct $X_0 \sqcup_W X_1$ of X_0 and X_1 over W, if it exists, is the colimit of α.

Consider a commutative diagram in $\mathscr{C}$:

(2.4.1)
$$\begin{array}{ccc} W & \longrightarrow & X_1 \\ \downarrow & & \downarrow \\ X_0 & \longrightarrow & Y \end{array}$$

Definition 2.4.1. The square (2.4.1) is Cartesian if $W \simeq X_0 \times_Y X_1$. It is co-Cartesian if $Y \simeq X_0 \sqcup_W X_1$

Proposition 2.4.2. (a) *Assume that $\mathscr{C}$ admits products of two objects and kernels. Then $X_0 \times_Y X_1 \simeq \mathrm{Ker}(X_0 \times X_1 \rightrightarrows Y)$.*

(b) *Assume that $\mathscr{C}$ admits coproducts of two objects and cokernels. Then $X_0 \sqcup_W X_1 \simeq \mathrm{Coker}(W \rightrightarrows X_0 \coprod X_1)$.*

Proof. It follows from the characterizations of limits and colimits given in (2.3.5) and (2.3.7). □

Proposition 2.4.3. (a) *The category $\mathscr{C}$ admits finite limits if and only if it admits fiber products and a terminal object.*

(b) *The category $\mathscr{C}$ admits finite colimits if and only if it admits fiber coproducts and an initial object.*

Proof. (a) If $\mathscr{C}$ admits finite limits, then it admits fiber products by Proposition 2.4.2 (a). Conversely, if $\mathscr{C}$ admits a terminal object $\mathrm{pt}_{\mathscr{C}}$ and fiber products, then it admits product of two objects (X_0, X_1), namely $X_0 \times_{\mathrm{pt}_{\mathscr{C}}} X_1$. It admits kernels since given $(f, g)\colon X \rightrightarrows Y$, then $\mathrm{Ker}(f, g) \simeq X \times_Y X$ again by Proposition 2.4.2 (a).
(b) is deduced from (a) by reversing the arrows. □

Note that

(2.4.2)
$$X_0 \times X_1 \simeq X_0 \times_{\mathrm{pt}_{\mathscr{C}}} X_1, \quad X_0 \sqcup X_1 \simeq X_0 \sqcup_{\varnothing_{\mathscr{C}}} X_1$$

Definition 2.4.4. Let $\mathscr{C}$ be a category which admits finite limits and colimits and let $f\colon X \to Y$ be a morphism. One sets

$$\operatorname{Coim} f := \operatorname{Coker}(X \times_Y X \rightrightarrows X),\ \operatorname{Im} f := \operatorname{Ker}(Y \rightrightarrows Y \sqcup_X Y). \tag{2.4.3}$$

Here, the fiber product $X \times_Y X$ as well as the fiber coproduct $Y \sqcup_X Y$ are associated with two copies of the map f.

One calls $\operatorname{Coim}(f)$ and $\operatorname{Im}(f)$ the co-image and the image of f, respectively.

One has a natural epimorphism $s\colon X \to \operatorname{Coim} f$ and a natural monomorphism $t\colon \operatorname{Im} f \to Y$. Moreover, one can construct a natural morphism $u\colon \operatorname{Coim}(f) \to \operatorname{Im}(f)$ such that the composition $X \to \operatorname{Coim}(f) \to \operatorname{Im}(f) \to Y$ is f (see [KS06, Prop. 5.1.2] and Section 5.1 for a similar construction in the abelian setting).

2.5 Properties of limits

Double limits

For two categories I and $\mathscr{C}$, recall the notation $\mathscr{C}^I := \operatorname{Fct}(I, \mathscr{C})$ and for a third category J, recall the equivalence (1.3.4);

$$\operatorname{Fct}(I \times J, \mathscr{C}) \simeq \operatorname{Fct}(I, \operatorname{Fct}(J, \mathscr{C})).$$

Consider a bifunctor $\beta\colon I^{\mathrm{op}} \times J^{\mathrm{op}} \to \mathscr{C}$ with I and J small. It defines a functor $\beta_J\colon I^{\mathrm{op}} \to \mathscr{C}^{J^{\mathrm{op}}}$ as well as a functor $\beta_I\colon J^{\mathrm{op}} \to \mathscr{C}^{I^{\mathrm{op}}}$. One easily checks that

$$\lim \beta \simeq \lim \lim \beta_J \simeq \lim \lim \beta_I. \tag{2.5.1}$$

Similarly, if $\alpha\colon I \times J \to \mathscr{C}$ is a bifunctor, it defines a functor $\alpha_J\colon I \to \mathscr{C}^J$ as well as a functor $\alpha_I\colon J \to \mathscr{C}^I$ and one has the isomorphisms

$$\operatorname{colim} \alpha \simeq \operatorname{colim}(\operatorname{colim} \alpha_J) \simeq \operatorname{colim}(\operatorname{colim} \alpha_I). \tag{2.5.2}$$

In other words:

$$\lim_{i,j} \beta(i,j) \simeq \lim_j \lim_i (\beta(i,j)) \simeq \lim_i \lim_j (\beta(i,j)), \tag{2.5.3}$$

$$\operatorname*{colim}_{i,j} \alpha(i,j) \simeq \operatorname*{colim}_j (\operatorname*{colim}_i (\alpha(i,j)) \simeq \operatorname*{colim}_i \operatorname*{colim}_j (\alpha(i,j)). \tag{2.5.4}$$

Limits with values in a category of functors

Consider another category $\mathscr{A}$ and a functor $\beta\colon I^{\mathrm{op}} \to \operatorname{Fct}(\mathscr{A}, \mathscr{C})$. It defines a functor $\widetilde{\beta}\colon I^{\mathrm{op}} \times \mathscr{A} \to \mathscr{C}$, hence for each $A \in \mathscr{A}$, a functor $\widetilde{\beta}(A)\colon I^{\mathrm{op}} \to \mathscr{C}$. Assuming that $\mathscr{C}$ admits limits indexed by I, one checks easily that $A \mapsto \lim \widetilde{\beta}(A)$ is a functor, that is, an object of $\operatorname{Fct}(\mathscr{A}, \mathscr{C})$, and is a limit of β. There is a similar result for colimits. Hence:

Proposition 2.5.1. *Let I be a small category and assume that $\mathscr{C}$ admits limits indexed by I. Then for any category $\mathscr{A}$, the category $\operatorname{Fct}(\mathscr{A}, \mathscr{C})$ admits limits indexed* *by I. Moreover, if $\beta\colon I^{\mathrm{op}} \to \mathrm{Fct}(\mathscr{A},\mathscr{C})$ is a functor, then $\lim\beta \in \mathrm{Fct}(\mathscr{A},\mathscr{C})$ is given by*

$$(\lim\beta)(A) = \lim(\beta(A)), \quad A \in \mathscr{A}.$$

Similarly, assume that $\mathscr{C}$ admits colimits indexed by I. Then for any category $\mathscr{A}$, the category $\mathrm{Fct}(\mathscr{A},\mathscr{C})$ admits colimits indexed by I. Moreover, if $\alpha\colon I \to \mathrm{Fct}(\mathscr{A},\mathscr{C})$ is a functor, then $\operatorname{colim}\alpha \in \mathrm{Fct}(\mathscr{A},\mathscr{C})$ is given by

$$(\operatorname{colim}\alpha)(A) = \operatorname{colim}(\alpha(A)), \quad A \in \mathscr{A}.$$

Corollary 2.5.2. *Let $\mathscr{C}$ be a category. Then the categories $\mathscr{C}^\wedge$ and $\mathscr{C}^\vee$ admit small limits and colimits.*

Composition of limits

Let $I, \mathscr{C}$ and $\mathscr{C}'$ be categories with I small and let $\alpha\colon I \to \mathscr{C}$, $\beta\colon I^{\mathrm{op}} \to \mathscr{C}$ and $F\colon \mathscr{C} \to \mathscr{C}'$ be functors. When $\mathscr{C}$ and $\mathscr{C}'$ admit limits or colimits indexed by I, there are natural morphisms

$$F(\lim\beta) \to \lim(F \circ \beta), \tag{2.5.5}$$

$$\operatorname{colim}(F \circ \alpha) \to F(\operatorname{colim}\alpha). \tag{2.5.6}$$

This follows immediately from (2.3.7) and (2.3.5).

Definition 2.5.3. Let I be a small category and let $F\colon \mathscr{C} \to \mathscr{C}'$ be a functor.

(i) Assume that $\mathscr{C}$ and $\mathscr{C}'$ admit limits indexed by I. One says that F commutes with such limits if (2.5.5) is an isomorphism.

(ii) Similarly, assume that $\mathscr{C}$ and $\mathscr{C}'$ admit colimits indexed by I. One says that F commutes with such colimits if (2.5.6) is an isomorphism.

Examples 2.5.4. (i) Let $\mathscr{C}$ be a category which admits limits indexed by I and let $X \in \mathscr{C}$. By (2.3.2), the functor $\mathrm{Hom}_{\mathscr{C}}(X, \bullet)\colon \mathscr{C} \to \mathbf{Set}$ commutes with limits indexed by I. Similarly, if $\mathscr{C}$ admits colimits indexed by I, then the functor $\mathrm{Hom}_{\mathscr{C}}(\bullet, X)\colon \mathscr{C}^{\mathrm{op}} \to \mathbf{Set}$ commutes with limits indexed by I, by (2.3.3).
(ii) Let I and J be two small categories and assume that $\mathscr{C}$ admits limits (resp. colimits) indexed by $I \times J$. Then the functor $\lim : \mathrm{Fct}(J^{\mathrm{op}},\mathscr{C}) \to \mathscr{C}$ (resp. the functor $\operatorname{colim} : \mathrm{Fct}(J,\mathscr{C}) \to \mathscr{C}$) commutes with limits (resp. colimits) indexed by I. This follows from the isomorphisms (2.5.1) and (2.5.2).
(iii) Let $\mathbf{k}$ be a field, $\mathscr{C} = \mathscr{C}' = \mathrm{Mod}(\mathbf{k})$, and let $X \in \mathscr{C}$. Then the functor $\mathrm{Hom}_{\mathbf{k}}(X, \bullet)$ does not commute with colimit if X is infinite dimensional.

Proposition 2.5.5. *Let $F\colon \mathscr{C} \to \mathscr{C}'$ be a functor and let I be a small category.*

(i) *Assume that $\mathscr{C}$ and $\mathscr{C}'$ admit limits indexed by I and F admits a left adjoint $G\colon \mathscr{C}' \to \mathscr{C}$. Then F commutes with limits indexed by I, that is, $F(\lim_i \beta(i)) \simeq \lim_i F(\beta(i))$.*

(ii) *Similarly, if $\mathscr{C}$ and $\mathscr{C}'$ admit colimits indexed by I and F admits a right adjoint, then F commutes with such colimits.*

Proof. It is enough to prove the first assertion. To check that (2.5.5) is an isomorphism, we apply Corollary 1.4.4. Let $Y \in \mathscr{C}'$. One has the chain of isomorphisms

$$\begin{aligned}\mathrm{Hom}_{\mathscr{C}'}(Y, F(\lim_i \beta(i))) &\simeq \mathrm{Hom}_{\mathscr{C}}(G(Y), \lim_i \beta(i))\\ &\simeq \lim_i \mathrm{Hom}_{\mathscr{C}}(G(Y), \beta(i))\\ &\simeq \lim_i \mathrm{Hom}_{\mathscr{C}'}(Y, F(\beta(i)))\\ &\simeq \mathrm{Hom}_{\mathscr{C}'^\wedge}(Y, \lim_i F(\beta(i))).\end{aligned}$$

□

2.6 Directed colimits

As already seen in Theorem 2.3.10, the category **Set** admits small colimits. In the category **Set** one uses the notation $\bigsqcup$ rather than $\coprod$.

We shall construct colimits more explicitly.

Let $\alpha\colon I \to$ **Set** be a functor (with I small) and consider the relation on $\bigsqcup_{i\in I}\alpha(i))$:

(2.6.1) $$\begin{cases}\alpha(i) \ni x\mathscr{R}y \in \alpha(j) \text{ if there exists } k \in I,\ s\colon i \to k \text{ and } t\colon j \to k\\ \text{with } \alpha(s)(x) = \alpha(t)(y).\end{cases}$$

The relation $\mathscr{R}$ is reflexive and symmetric but is not transitive in general.

Proposition 2.6.1. *With the notations above, denote by $\sim$ the equivalence relation generated by $\mathscr{R}$. Then*

$$\mathrm{colim}\,\alpha \simeq (\bigsqcup_{i\in I}\alpha(i))/\sim.$$

Proof. Apply Proposition 2.3.9 and Example 2.2.3 (ii). □

For a ring A, the category $\mathrm{Mod}(A)$ admits coproducts and cokernels. Hence, the category $\mathrm{Mod}(A)$ admits colimits. One shall be aware that the functor *for*: $\mathrm{Mod}(A) \to$ **Set** does not commute with colimits. For example, if I is empty and $\alpha\colon I \to \mathrm{Mod}(A)$ is a functor, then $\alpha(I) = \{0\}$ and *for*$(\{0\})$ is not an initial object in **Set**.

Definition 2.6.2. A category I is called directed if it satisfies the conditions (i)–(iii) below.

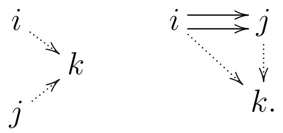
(i) I is non empty,

(ii) for any i and j in I, there exist $k \in I$ and morphisms $i \to k, j \to k$,

(iii) for any parallel morphisms $f, g\colon i \rightrightarrows j$, there exists a morphism $h\colon j \to k$ such that $h \circ f = h \circ g$.

One says that I is codirected if I^{op} is directed.

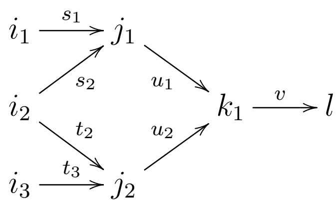
The conditions (ii)–(iii) of being directed are visualized by the diagrams:



Of course, if $(I, \leq)$ is a directed ordered set, then the associated category I is directed.

Proposition 2.6.3. *Let $\alpha \colon I \to \mathbf{Set}$ be a functor, with I directed. The relation $\mathscr{R}$ on $\coprod_i \alpha(i)$ given by (2.6.1) is an equivalence relation.*

Proof. Let $x_j \in \alpha(i_j)$, $j = 1, 2, 3$ with $x_1 \sim x_2$ and $x_2 \sim x_3$. There exist morphisms visualized by the diagram:



such that $\alpha(s_1)x_1 = \alpha(s_2)x_2$, $\alpha(t_2)x_2 = \alpha(t_3)x_3$, and $v \circ u_1 \circ s_2 = v \circ u_2 \circ t_2$. Set $w_1 = v \circ u_1 \circ s_1$, $w_2 = v \circ u_1 \circ s_2 = v \circ u_2 \circ t_2$ and $w_3 = v \circ u_2 \circ t_3$. Then $\alpha(w_1)x_1 = \alpha(w_2)x_2 = \alpha(w_3)x_3$. Hence $x_1 \sim x_3$. □

Corollary 2.6.4. *Let $\alpha \colon I \to \mathbf{Set}$ be a functor, with I small and directed.*

(i) *Let S be a finite subset in $\operatorname{colim} \alpha$. Then there exists $i \in I$ such that S is contained in the image of $\alpha(i)$ by the natural map $\alpha(i) \to \operatorname{colim} \alpha$.*

(ii) *Let $i \in I$ and let x and y be elements of $\alpha(i)$ with the same image in $\operatorname{colim} \alpha$. Then there exists $s \colon i \to j$ such that $\alpha(s)(x) = \alpha(s)(y)$ in $\alpha(j)$.*

Proof. (i) Denote by $\lambda \colon \bigsqcup_{i \in I} \alpha(i) \to \operatorname{colim} \alpha$ the quotient map. Let $S = \{x_1, \ldots, x_n\}$. For $j = 1, \ldots, n$, there exists $y_j \in \alpha(i_j)$ such that $x_j = \lambda(y_j)$. Choose $k \in I$ such that there exist morphisms $s_j \colon \alpha(i_j) \to \alpha(k)$. Then $x_j = \lambda(\alpha(s_j(y_j)))$.
(ii) For $x, y \in \alpha(i)$, $x \mathscr{R} y$ if and only if there exists $s \colon i \to j$ with $\alpha(s)(x) = \alpha(s)(y)$ in $\alpha(j)$. □

Corollary 2.6.5. *Let A be a ring and denote by for the forgetful functor* $\operatorname{Mod}(A) \to \mathbf{Set}$. *Then the functor for commutes with directed colimits. In other words, if I is small and directed and $\alpha \colon I \to \operatorname{Mod}(A)$ is a functor, then*

$$\mathit{for} \circ (\operatorname*{colim}_i \alpha(i)) = \operatorname*{colim}_i (\mathit{for} \circ \alpha(i)).$$

The proof is left as an exercise (see Exercise 2.8).

Colimits with values in **Set** indexed by small directed categories commute with finite limits. More precisely:

Theorem 2.6.6. *For a small directed category I, a finite category J and a functor $\alpha\colon I \times J^{\rm op} \to \mathbf{Set}$, one has $\operatorname*{colim}_i \lim_j \alpha(i,j) \xrightarrow{\sim} \lim_j \operatorname*{colim}_i \alpha(i,j)$. In other words, the functor*

$$\operatorname{colim} : \operatorname{Fct}(I, \mathbf{Set}) \to \mathbf{Set}$$

commutes with finite limits.

Proof. It is enough to prove that colim commutes with kernels and with finite products.
(i) colim commutes with kernels. Let $\alpha, \beta\colon I \to \mathbf{Set}$ be two functors and let $f, g\colon \alpha \rightrightarrows \beta$ be two morphisms of functors. We denote by $f_i, g_i\colon \alpha(i) \rightrightarrows \beta(i)$ the morphisms associated with f, g and $i \in I$.

Define γ as the kernel of (f, g), that is, we have exact sequences

$$\gamma(i) \to \alpha(i) \rightrightarrows \beta(i).$$

Let Z denote the kernel of $\operatorname*{colim}_i \alpha(i) \rightrightarrows \operatorname*{colim}_i \beta(i)$. We have to prove that the natural map $\lambda\colon \operatorname*{colim}_i \gamma(i) \to Z$ is bijective.

(i) (a) The map λ is surjective. Indeed for $x \in Z$, represent x by some $x_i \in \alpha(i)$. Then $f_i(x_i)$ and $g_i(x_i)$ in $\beta(i)$ having the same image in $\operatorname{colim}\beta$, there exists $s\colon i \to j$ such that $\beta(s)f_i(x_i) = \beta(s)g_i(x_i)$. Set $x_j = \alpha(s)x_i$. Then $f_j(x_j) = g_j(x_j)$, which means that $x_j \in \gamma(j)$. Clearly, $\lambda(x_j) = x$.

(i) (b) The map λ is injective. Indeed, let $x, y \in \operatorname{colim}\gamma$ with $\lambda(x) = \lambda(y)$. We may represent x and y by elements x_i and y_i of $\gamma(i)$ for some $i \in I$. Since x_i and y_i have the same image in $\operatorname{colim}\alpha$, there exists $i \to j$ such that they have the same image in $\alpha(j)$. Therefore their images in $\gamma(j)$ will be the same.
(ii) colim commutes with finite products. The proof is similar to the preceding one and left to the reader. □

Corollary 2.6.7. *Let A be a ring and let I be a small directed category. Then the functor* $\operatorname{colim} : \operatorname{Fct}(I, \operatorname{Mod}(A)) \to \operatorname{Mod}(A)$ *commutes with finite limits.*

Exact functors

Definition 2.6.8. Let $F\colon \mathscr{C} \to \mathscr{C}'$ be a functor.

(a) Assume that both $\mathscr{C}$ and $\mathscr{C}'$ admit finite limits. If F commutes with such limits, one says that F is left exact.

(b) Assume that both $\mathscr{C}$ and $\mathscr{C}'$ admit finite colimits. If F commutes with such limits, one says that F is right exact.

(c) Assume that both $\mathscr{C}$ and $\mathscr{C}'$ admit finite limits and colimits. If F commutes with such limits, one says that F is exact.

If $\mathscr{C}$ admits limits indexed by a category I, the functor $\lim : \operatorname{Fct}(I, \mathscr{C}^{\rm op}) \to \mathscr{C}$ is left exact and similarly for the functor colim . Moreover, Theorem 2.6.6 and Corollary 2.6.7 may be translated by saying that in these situations, the functor colim is left exact.

Cofinal functors

Let $\varphi\colon J \to I$ be a functor. If there are no risk of confusion, we still denote by φ the associated functor $\varphi\colon J^{\rm op} \to I^{\rm op}$. For two functors $\alpha\colon I \to \mathscr{C}$ and $\beta\colon I^{\rm op} \to \mathscr{C}$, we have natural morphisms:

$$\lim\,(\beta\circ\varphi) \leftarrow \lim\beta, \tag{2.6.2}$$

$$\operatorname{colim}\,(\alpha\circ\varphi) \to \operatorname{colim}\alpha. \tag{2.6.3}$$

This follows immediately of (2.3.7) and (2.3.5).

Definition 2.6.9. (a) Let $\varphi\colon J \to I$ be a functor. Assume that I and J are directed. One says that φ is cofinal if for any $i \in I$ there exists $j \in J$ and a morphism $s\colon i \to \varphi(j)$.

(b) Let I be a directed category. One says that I is cofinally small if there exists a fully faithful functor $\varphi\colon J \to I$ such that J is small and φ is cofinal.

Example 2.6.10. A subset $J \subset \mathbb{N}$ defines a cofinal subcategory of $(\mathbb{N}, \leq)$ if and only if it is infinite.

Proposition 2.6.11. *Let $\varphi\colon J \to I$ be a fully faithful functor. Assume that I is directed and φ is cofinal. Then*

(i) *for any category $\mathscr{C}$ and any functor $\beta\colon I^{\rm op} \to \mathscr{C}$, the morphism (2.6.2) is an isomorphism,*

(ii) *for any category $\mathscr{C}$ and any functor $\alpha\colon I \to \mathscr{C}$, the morphism (2.6.3) is an isomorphism.*

Proof. Let us prove (ii), the other proof being similar. By the hypothesis, for each $i \in I$ we get a morphism $\alpha(i) \to \operatorname*{colim}_{j\in J}(\alpha\circ\varphi(j))$ from which one deduce a morphism

$$\operatorname*{colim}_{i\in I}\alpha(i) \to \operatorname*{colim}_{j\in J}(\alpha\circ\varphi(j)).$$

One checks easily that this morphism is inverse to the morphism in (2.5.6). □

Example 2.6.12. Let X be a topological space, $x \in X$ and denote by I_x the set of open neighborhoods of x in X. We endow I_x with the order: $U \leq V$ if $V \subset U$. Given U and V in I_x, and setting $W = U \cap V$, we have $U \leq W$ and $V \leq W$. Therefore, I_x is directed.

Denote by $\mathscr{C}^0(U)$ the $\mathbb{C}$-vector space of complex valued continuous functions on U. The restriction maps $\mathscr{C}^0(U) \to \mathscr{C}^0(V), V \subset U$ define an inductive system of $\mathbb{C}$-vector spaces indexed by I_x. One sets

$$\mathscr{C}^0_{X,x} = \operatorname*{colim}_{U\in I_x}\mathscr{C}^0(U). \tag{2.6.4}$$

An element φ of $\mathscr{C}^0_{X,x}$ is called a germ of continuous function at 0. Such a germ is an equivalence class $(U, \varphi_U)/\sim$ with U a neighborhood of x, φ_U a continuous function on U, and $(U, \varphi_U) \sim 0$ if there exists a neighborhood V of x with $V \subset U$ such that the restriction of φ_U to V is the zero function. Hence, a germ of function is zero at x if this function is identically zeroon a neighborhood of x.

2.7 Ind-objects

The aim of this section is to have a glance to the notion of ind-objects. Since we shall almost not use this theory in these notes, we shall be rather sketchy.

By Theorem 2.3.10, the category **Set** admits small limits and colimits. It follows from Proposition 2.5.1 that for any category $\mathscr{C}$, the big category $\mathscr{C}^\wedge$ also admits small limits and colimits. One denotes by "colim" the colimit in $\mathscr{C}^\wedge$.

One could also define "lim" in $\mathscr{C}^\vee$ but we shall concentrate here on colimits.

In the sequel we identify $\mathscr{C}$ to a full subcategory of $\mathscr{C}^\wedge$ by the Yoneda functor $\mathrm{h}_\mathscr{C}$ and when there is no risk of confusion, we shall write X instead of $\mathrm{h}_\mathscr{C}(X)$. Hence, for a small a category I and a functors $\alpha\colon I \to \mathscr{C}$, we have:

$$\mathrm{Hom}_{\mathscr{C}^\wedge}(X, \text{“colim”}\,\alpha) \simeq \mathrm{colim}\,\mathrm{Hom}_\mathscr{C}(X,\alpha).$$

Assume that the category $\mathscr{C}$ admits small colimits. Then the natural morphism

$$\mathrm{colim}\,\mathrm{Hom}_\mathscr{C}(X,\alpha) \to \mathrm{Hom}_\mathscr{C}(X, \mathrm{colim}\,\alpha)$$

defines the morphism

$$\text{“colim”}\,\alpha \to \mathrm{colim}\,\alpha. \tag{2.7.1}$$

This morphism is not an isomorphism in general (see Exercise 2.8). In other words, the Yoneda functor $\mathrm{h}_\mathscr{C}$ does not commute with colimits.

On the other hand, assuming that $\mathscr{C}$ admits limits, if $\beta\colon I^{\mathrm{op}} \to \mathscr{C}$ is a functor, then

$$\mathrm{Hom}_{\mathscr{C}^\wedge}(X, \lim\beta) \simeq \lim \mathrm{Hom}_\mathscr{C}(X,\beta).$$

Hence, the Yoneda functor $\mathrm{h}_\mathscr{C}$ commutes with limits in this case.

Let $A \in \mathscr{C}^\wedge$. Applying Definition 1.4.5 to the Yoneda functor, we get the category $\mathscr{C}_A$.

Lemma 2.7.1. *Let $A \in \mathscr{C}^\wedge$. Then $A \simeq \underset{(X\to A)\in\mathscr{C}_A}{\text{“colim”}} X$.*

Proof. Let $B \in \mathscr{C}^\wedge$. One has the chain of isomorphisms

$$\begin{aligned}\mathrm{Hom}_{\mathscr{C}^\wedge}(A,B) &\simeq \lim_{(X\to A)\in\mathscr{C}_A} B(X)\\ &\simeq \lim_{(X\to A)\in\mathscr{C}_A} \mathrm{Hom}_{\mathscr{C}^\wedge}(X,B) \simeq \mathrm{Hom}_{\mathscr{C}^\wedge}(\underset{(X\to A)}{\text{“colim”}}\, X, B),\end{aligned}$$

where the first isomorphism follows from the definition of a morphism of functors. □

Consider a functor $F\colon \mathscr{C} \to \mathscr{C}'$. One defines the functor

$$IF\colon \mathscr{C}^\wedge \to \mathscr{C}'^\wedge, \quad IF(A) = \underset{(X\to A)\in\mathscr{C}_A}{\text{“colim”}} F(X). \tag{2.7.2}$$

We shall not prove here that IF is well defined.

Definition 2.7.2. One denotes by $\mathrm{Ind}(\mathscr{C})$ the full subcategory of $\mathscr{C}^\wedge$ consisting of objects isomorphic to "colim" α for some functor $\alpha\colon I \to \mathscr{C}$ with I small and directed. One calls an object of $\mathrm{Ind}(\mathscr{C})$ an ind-object.

Proposition 2.7.3. (a) *Let* $F\colon \mathscr{C} \to \mathscr{C}'$ *be a functor. Then* IF *induces a functor* (*we keep the same notation*) $IF\colon \mathrm{Ind}(\mathscr{C}) \to \mathrm{Ind}(\mathscr{C}')$.

(b) *Let* I *be small and directed and let* $\alpha\colon I \to \mathscr{C}$ *be a functor. Then* "colim" $(F \circ \alpha) \xrightarrow{\sim} IF(\text{"colim"}\,\alpha)$.

(c) *Let* I *be small and directed and let* $\alpha\colon I \to \mathscr{C}$ *be a functor. If* "colim" α *is representable by* $X \in \mathscr{C}$, *then* $\mathrm{colim}\,\alpha$ *exists in* $\mathscr{C}$ *and is isomorphic to* X.

Proof. (a)–(b) follow from (2.7.2).
(c) For $Y \in \mathscr{C}$, one has

$$\begin{aligned}\mathrm{Hom}_{\mathscr{C}}(X,Y) &\simeq \mathrm{Hom}_{\mathscr{C}^\wedge}(\text{"colim"}\,\alpha, Y)\\ &\simeq \lim \mathrm{Hom}_{\mathscr{C}}(\alpha, Y) \simeq \mathrm{Hom}_{\mathscr{C}}(\mathrm{colim}\,\alpha, Y).\end{aligned}$$

□

Since we shall not use the next result, we skip its proof, referring to [KS06, Prop. 6.1.5, Th. 6.1.8]. Note that the "if"part of the first statement follows immediately from Lemma 2.7.1.

Proposition 2.7.4. (a) *Let* $A \in \mathscr{C}^\wedge$. *Then* $A \in \mathrm{Ind}(\mathscr{C})$ *if and only if the category* $\mathscr{C}_A$ *is directed and cofinally small.*

(b) *The category* $\mathrm{Ind}(\mathscr{C})$ *admits small directed colimits and the embedding* $\mathrm{Ind}(\mathscr{C}) \hookrightarrow \mathscr{C}^\wedge$ *commutes with colimits* (*which will still be denoted by* "colim").

A set-theoretical remark

Remark 2.7.5. As already mentioned, all categories $\mathscr{C}$, $\mathscr{C}'$, etc. are $\mathscr{U}$-categories for some universe $\mathscr{U}$ and all limits or colimits are indexed by $\mathscr{U}$-small categories I, J, etc. Let us give an example which shows that without some care, we may have troubles.

Let $\mathscr{C}$ be a category which admits products and assume there exist $X, Y \in \mathscr{C}$ such that $\mathrm{Hom}_{\mathscr{C}}(X,Y)$ has more than one element. Set $M = \mathrm{Mor}(\mathscr{C})$, where $\mathrm{Mor}(\mathscr{C})$ denotes the big set of all morphisms in $\mathscr{C}$. Let $\pi = \mathrm{card}(M)$, the cardinal of the set M. We have

$$\mathrm{Hom}_{\mathscr{C}}(X, Y^M) \simeq \mathrm{Hom}_{\mathscr{C}}(X,Y)^M$$

and therefore $\mathrm{card}(\mathrm{Hom}_{\mathscr{C}}(X, Y^M)) \geq 2^\pi$. On the other hand, $\mathrm{Hom}_{\mathscr{C}}(X, Y^M) \subset \mathrm{Mor}(\mathscr{C})$ which implies $\mathrm{card}(\mathrm{Hom}_{\mathscr{C}}(X, Y^M)) \leq \pi$.

The "contradiction" comes from the fact that $\mathscr{C}$ does not admit products indexed by such a big set as $\mathrm{Mor}(\mathscr{C})$. (This remark is extracted from [Fre64].)

Exercises to Chapter 2

Exercise 2.1. (i) Let I be a small set and $\{X_i\}_{i\in I}$ a family of sets indexed by I. Show that $\coprod_i X_i = \bigsqcup_i X_i$, the disjoint union of the sets X_i.
(ii) Construct the natural map $\bigsqcup_i \mathrm{Hom}_{\mathbf{Set}}(Y, X_i) \to \mathrm{Hom}_{\mathbf{Set}}(Y, \bigsqcup_i X_i)$ and prove it is injective and not surjective in general.

Exercise 2.2. Let $X, Y \in \mathscr{C}$ and consider the category $\mathscr{D}$ whose objects are triplets $Z \in \mathscr{C}, f\colon Z \to X, g\colon Z \to Y$, the morphisms being the natural ones. Prove that this category admits a terminal object if and only if the product $X \times Y$ exists in $\mathscr{C}$, and that in such a case this terminal object is isomorphic to $X \times Y, X \times Y \to X, X \times Y \to Y$. Deduce that if $X \times Y$ exists, it is unique up to unique isomorphism.

Exercise 2.3. Let I and $\mathscr{C}$ be two categories with I small and denote by Δ the functor from $\mathscr{C}$ to $\mathscr{C}^I$ which, to $X \in \mathscr{C}$, associates the constant functor $\Delta(X)\colon I \ni i \mapsto X \in \mathscr{C}$, $(i \to j) \in \mathrm{Mor}(I) \mapsto \mathrm{id}_X$.

(i) Assume that colimits indexed by I exist. Prove the formula, for $\alpha\colon I \to \mathscr{C}$ and $Y \in \mathscr{C}$:

$$\mathrm{Hom}_{\mathscr{C}}(\operatorname*{colim}_i \alpha(i), Y) \simeq \mathrm{Hom}_{\,\mathrm{Fct}(I,\mathscr{C})}(\alpha, \Delta(Y)).$$

(ii) Assuming that limits exist, deduce the formula for $\beta\colon I^{\mathrm{op}} \to \mathscr{C}$ and $X \in \mathscr{C}$:

$$\mathrm{Hom}_{\mathscr{C}}(X, \lim_i \beta(i)) \simeq \mathrm{Hom}_{\,\mathrm{Fct}(I^{\mathrm{op}},\mathscr{C})}(\Delta(X), \beta).$$

Exercise 2.4. Let $\mathscr{C}$ be a category which admits small directed colimits. One says that an object X of $\mathscr{C}$ is of finite type if for any functor $\alpha\colon I \to \mathscr{C}$ with I directed, the natural map $\operatorname{colim} \mathrm{Hom}_{\mathscr{C}}(X, \alpha) \to \mathrm{Hom}_{\mathscr{C}}(X, \operatorname{colim} \alpha)$ is injective. Show that this definition coincides with the classical one when $\mathscr{C} = \mathrm{Mod}(A)$, for a ring A.

(Hint: let $X \in \mathrm{Mod}(A)$. To prove that if X is of finite type in the categorical sense then it is of finite type in the usual sense, use the fact that, denoting by $\mathscr{S}$ be the family of submodules of finite type of X ordered by inclusion, we have $\operatorname*{colim}_{V \in \mathscr{S}} X/V \simeq 0$.)

Exercise 2.5. Let $\mathscr{C}$ be a category which admits small directed colimits. One says that an object X of $\mathscr{C}$ is of finite presentation if for any functor $\alpha\colon I \to \mathscr{C}$ with I small and directed, the natural map $\operatorname{colim} \mathrm{Hom}_{\mathscr{C}}(X, \alpha) \to \mathrm{Hom}_{\mathscr{C}}(X, \operatorname{colim} \alpha)$ is bijective. Show that this definition coincides with the classical one when $\mathscr{C} = \mathrm{Mod}(A)$, for a ring A.

Exercise 2.6. In the situation of Definition 2.4.4, construct the natural morphism $u\colon \mathrm{Coim}(f) \to \mathrm{Im}(f)$ such that the composition $X \to \mathrm{Coim}(f) \to \mathrm{Im}(f) \to Y$ is f. (See [KS06, Prop. 5.1.2].)

Exercise 2.7. Let I be a directed ordered set and let $\{A_i\}_{i \in I}$ be an inductive system of rings indexed by I.

(i) Prove that $A := \operatorname*{colim}_i A_i$ is naturally endowed with a ring structure.

(ii) Define the notion of an inductive system M_i of A_i-modules, and define the A-module $\operatorname*{colim}_i M_i$.

(iii) Let N_i (resp. M_i) be an inductive system of right (resp. left) A_i modules. Prove the isomorphism

$$\operatorname*{colim}_i (N_i \otimes_{A_i} M_i) \xrightarrow{\sim} \operatorname*{colim}_i N_i \otimes_A \operatorname*{colim}_i M_i.$$

Exercise 2.8. Prove Corollary 2.6.5.

Exercise 2.9. (i) Let $\mathscr{C}$ be a category which admits colimits indexed by a category I. Let $\alpha\colon I \to \mathscr{C}$ be a functor and let $X \in \mathscr{C}$. Construct the natural morhism

$$\varinjlim_i \operatorname{Hom}_{\mathscr{C}}(X, \alpha(i)) \to \operatorname{Hom}_{\mathscr{C}}(X, \varinjlim_i \alpha(i)). \tag{2.7.3}$$

(ii) Let $\mathbf{k}$ be a field and denote by $\mathbf{k}[x]^{\le n}$ the $\mathbf{k}$-vector space consisting of polynomials of degree $\le n$. Prove the isomorphism $\mathbf{k}[x] \simeq \operatorname{colim}_n \mathbf{k}[x]^{\le n}$ in $\operatorname{Mod}(\mathbf{k})$ and, noticing that $\mathrm{id}_{\mathbf{k}[x]} \notin \operatorname{colim}_n \operatorname{Hom}_{\mathbf{k}}(\mathbf{k}[x], \mathbf{k}[x]^{\le n})$, deduce that the morphism (2.7.3) is not an isomorphism in general.

Exercise 2.10. Let I be a small set and let $\mathscr{J}$ be the set of finite subsets of I, ordered by inclusion. We consider both I and $\mathscr{J}$ as categories. Let $\mathscr{C}$ be a category which admits small colimits and let $\alpha\colon I \to \mathscr{C}$ be a functor. For $J \in \mathscr{J}$ we denote by $\alpha_J\colon J \to \mathscr{C}$ the restriction of α to J.

(i) Prove that the category $\mathscr{J}$ is directed.

(ii) Prove the isomorphism $\operatorname{colim}_{J\in\mathscr{J}} \operatorname{colim}_{j\in J} \alpha_J \xrightarrow{\sim} \operatorname{colim} \alpha$.

Exercise 2.11. Let $\mathscr{C}$ be a category which admits a zero-object and kernels. Prove that if a morphism $f\colon X \to Y$ is a monomorphism then $\operatorname{Ker} f \simeq 0$. Prove the converse when assuming that $\mathscr{C}$ is additive (see Chapter 4).

Exercise 2.12. We consider the ordered set $\mathbb{N}$ as a category. Hence, for a category $\mathscr{C}$, a functor $\alpha\colon \mathbb{N} \to \mathscr{C}$ is defined by the data of the objects $\alpha(n) \in \mathscr{C}$, $n \in \mathbb{N}$, and the morphisms $\alpha(n < n+1)\colon \alpha(n) \to \alpha(n+1)$.

(i) Consider the functor $\alpha\colon \mathbb{N} \to \operatorname{Mod}(\mathbb{Z})$ given by $\alpha(n) = \mathbb{Z}$ and $\alpha(n < n+1) = 2\cdot\colon \mathbb{Z} \to \mathbb{Z}$. Calculate $\operatorname{colim} \alpha$.
(Hint: one can represent this colimit as a subgroup of $\mathbb{Q}$.)

(ii) Give an example of a functor $\alpha\colon \mathbb{N} \to \operatorname{Mod}(\mathbb{Z})$ in which all $\alpha(n)$ are not 0 and all morphisms $\alpha(n < n+1)$ are not 0 but $\operatorname{colim} \alpha \simeq 0$.

Exercise 2.13. Let $\mathbf{k}$ be a field and denote as usual by $\operatorname{Mod}(\mathbf{k})$ the category of $\mathbf{k}$-vector spaces (in a given universe $\mathscr{U}$). Denote by $\operatorname{Mod}^{\mathrm{f}}(\mathbf{k})$ the full subcategory consisting of finite dimensional vector spaces and set for short $\mathbf{Ik} = \operatorname{Ind}(\operatorname{Mod}(\mathbf{k}))$.

Let $\mathbb{V}$ denote an infinite dimensional vector space and denote by $\mathscr{V}^f$ the category consisting of finite dimensional vector subspaces of $\mathbb{V}$ and linear maps.

(i) Prove that the category $\mathscr{V}^f$ is small and directed and set $\widetilde{\mathbb{V}} = \text{“}\operatorname{colim}\text{”}_{W\in\mathscr{V}^f} W \in \mathbf{Ik}$.

(ii) Construct the morphism $\widetilde{\mathbb{V}} \to \mathbb{V}$ in $\mathbf{Ik}$ and prove it is a monomorphism.

(iii) Let $\mathbb{L} \in \operatorname{Mod}(\mathbf{k})$. Prove that the morphism $\operatorname{Hom}_{\mathbf{Ik}}(\mathbb{L}, \widetilde{\mathbb{V}}) \to \operatorname{Hom}_{\mathbf{Ik}}(\mathbb{L}, \mathbb{V})$ is an isomorphism if and only if $\mathbb{L} \in \operatorname{Mod}^{\mathrm{f}}(\mathbf{k})$.

(iv) Set $\mathbb{W} = \mathbb{V}/\widetilde{\mathbb{V}}$. Prove that $\operatorname{Hom}_{\mathbf{Ik}}(\mathbf{k}, \mathbb{W}) \simeq 0$ although $\mathbb{W} \neq 0$.

(v) Consider the functor $\alpha\colon \operatorname{Ind}(\operatorname{Mod}^{\mathrm{f}}(\mathbf{k})) \to \operatorname{Mod}(\mathbf{k})$ which, to $\text{“}\operatorname{colim}\text{”}_{i\in I} V_i$ (I small and directed), associates $\operatorname{colim}_{i\in I} V_i$. Prove that α is an equivalence of categories.

Exercise 2.14. Let I be a small and directed category and let $\alpha\colon I \to \mathscr{C}$ be a functor. Assume that for any morphism $s\colon i \to j$ in I, $\alpha(s)$ is an isomorphism. Prove that “colim” α exists in $\mathscr{C}$ and moreover, for any $X \in \mathscr{C}$, one has the isomorphism

$$\operatorname{colim}_i \operatorname{Hom}_{\mathscr{C}}(X, \alpha(i)) \xrightarrow{\sim} \operatorname{Hom}_{\mathscr{C}}(X, \operatorname{colim}_i \alpha(i)).$$

Exercise 2.15. Recall Definition 2.7.2.

(i) Prove that the Yoneda functor induces a fully faithful functor $\mathscr{C} \hookrightarrow \mathrm{Ind}(\mathscr{C})$, that $\mathrm{Ind}(\mathscr{C})$ admits small directed colimits and that the functor $\mathrm{Ind}(\mathscr{C}) \hookrightarrow \mathscr{C}^\wedge$ commutes with such colimits.

(ii) Let $\mathbf{k}$ be a field and let $\mathscr{C} = \mathrm{Mod}(\mathbf{k})$. Prove that the Yoneda functor $\mathrm{h}_\mathscr{C} \colon \mathscr{C} \to \mathscr{C}^\wedge$ does not commute with colimits.

Exercise 2.16. Recall that **Set** denotes the category of sets in a given universe $\mathscr{U}$. Denote by $\mathbf{Set}^f$ the full subcategory of the category **Set** consisting of finite sets. Prove the equivalence $\mathrm{Ind}(\mathbf{Set}^f) \simeq \mathbf{Set}$. (See [KS06, Exa. 6.3.6].)

Chapter 3

Localization

Summary

Consider a category $\mathscr{C}$ and a family $\mathscr{S}$ of morphisms in $\mathscr{C}$. The aim of localization is to find a new category $\mathscr{C}_{\mathscr{S}}$ and a functor $Q\colon \mathscr{C} \to \mathscr{C}_{\mathscr{S}}$ which sends the morphisms belonging to $\mathscr{S}$ to isomorphisms in $\mathscr{C}_{\mathscr{S}}$, $(Q, \mathscr{C}_{\mathscr{S}})$ being "universal" for such a property.

In this chapter, we shall construct the localization of a category when $\mathscr{S}$ satisfies suitable conditions and we shall construct the localization of functors.

Note that we shall only use the localization of categories in the situation of triangulated categories, essentially in order to define derived categories. Hence, the reading of this chapter may be skipped until § 6.4.

References. A classical reference for the localization of categories is the book [GZ67]. Here, we follow the presentation of [KS06]. We shall skip some proofs, referring to this last item in this case.

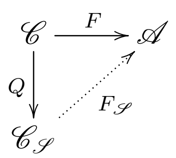
3.1 Localization of categories

Let $\mathscr{C}$ be a category and let $\mathscr{S}$ be a family of morphisms in $\mathscr{C}$.

Definition 3.1.1. A localizaton of $\mathscr{C}$ by $\mathscr{S}$ is the data of a category $\mathscr{C}_{\mathscr{S}}$ and a functor $Q\colon \mathscr{C} \to \mathscr{C}_{\mathscr{S}}$ satisfying:

(a) for all $s \in \mathscr{S}$, $Q(s)$ is an isomorphism,

(b) for any functor $F\colon \mathscr{C} \to \mathscr{A}$ such that $F(s)$ is an isomorphism for all $s \in \mathscr{S}$, there exists a functor $F_{\mathscr{S}}\colon \mathscr{C}_{\mathscr{S}} \to \mathscr{A}$ and an isomorphism $F \simeq F_{\mathscr{S}} \circ Q$,



(c) if G_1 and G_2 are two objects of $\mathrm{Fct}(\mathscr{C}_{\mathscr{S}}, \mathscr{A})$, then the natural map

$$\text{(3.1.1)} \qquad \mathrm{Hom}_{\,\mathrm{Fct}(\mathscr{C}_{\mathscr{S}},\mathscr{A})}(G_1, G_2) \to \mathrm{Hom}_{\,\mathrm{Fct}(\mathscr{C},\mathscr{A})}(G_1 \circ Q, G_2 \circ Q)$$

is bijective.

Note that (c) means that the functor $\circ Q\colon \mathrm{Fct}(\mathscr{C}_\mathscr{S}, \mathscr{A}) \to \mathrm{Fct}(\mathscr{C}, \mathscr{A})$ is fully faithful. This implies that $F_\mathscr{S}$ in (b) is unique up to unique isomorphism.

Proposition 3.1.2. (i) *If $\mathscr{C}_\mathscr{S}$ exists, it is unique up to equivalence of categories.*

(ii) *If $\mathscr{C}_\mathscr{S}$ exists, then, denoting by $\mathscr{S}^{\mathrm{op}}$ the image of $\mathscr{S}$ in $\mathscr{C}^{\mathrm{op}}$ by the functor* op, *$(\mathscr{C}^{\mathrm{op}})_{\mathscr{S}^{\mathrm{op}}}$ exists and there is an equivalence of categories:*

$$(\mathscr{C}_\mathscr{S})^{\mathrm{op}} \simeq (\mathscr{C}^{\mathrm{op}})_{\mathscr{S}^{\mathrm{op}}}.$$

Proof. (i) is clear.
(ii) Assume $\mathscr{C}_\mathscr{S}$ exists. Set $(\mathscr{C}^{\mathrm{op}})_{\mathscr{S}^{\mathrm{op}}} := (\mathscr{C}_\mathscr{S})^{\mathrm{op}}$ and define $Q^{\mathrm{op}}\colon \mathscr{C}^{\mathrm{op}} \to (\mathscr{C}^{\mathrm{op}})_{\mathscr{S}^{\mathrm{op}}}$ by $Q^{\mathrm{op}} = \mathrm{op} \circ Q \circ \mathrm{op}$. Then properties (a), (b) and (c) of Definition 3.1.1 are clearly satisfied. ☐

Definition 3.1.3. One says that $\mathscr{S}$ is a right multiplicative system if it satisfies the axioms S1-S4 below.

S1 For all $X \in \mathscr{C}$, $\mathrm{id}_X \in \mathscr{S}$.

S2 For all $f \in \mathscr{S}, g \in \mathscr{S}$, if $g \circ f$ exists then $g \circ f \in \mathscr{S}$.

S3 Given two morphisms, $f\colon X \to Y$ and $s\colon X \to X'$ with $s \in \mathscr{S}$, there exist $t\colon Y \to Y'$ and $g\colon X' \to Y'$ with $t \in \mathscr{S}$ and $g \circ s = t \circ f$. This can be visualized by the diagram:

$$\begin{array}{ccc} X' & \overset{g}{\dashrightarrow} & Y' \\ {\scriptstyle s}\uparrow & & \uparrow{\scriptstyle t} \\ X & \xrightarrow[f]{} & Y \end{array}$$

meaning that the dotted arrows may bwe completed, making the diagram commutative.

S4 Let $f, g\colon X \to Y$ be two parallel morphisms. If there exists $s \in \mathscr{S}\colon W \to X$ such that $f \circ s = g \circ s$ then there exists $t \in \mathscr{S}\colon Y \to Z$ such that $t \circ f = t \circ g$. This can be visualized by the diagram:

$$W \xrightarrow{s} X \overset{f}{\underset{g}{\rightrightarrows}} Y \overset{t}{\dashrightarrow} Z$$

Notice that these axioms are quite natural if one wants to invert the elements of $\mathscr{S}$. In other words, if the element of $\mathscr{S}$ would be invertible, then these axioms would clearly be satisfied.

Remark 3.1.4. Axioms S1-S2 asserts that $\mathscr{S}$ is the family of morphisms of a subcategory $\widetilde{\mathscr{S}}$ of $\mathscr{C}$ with $\mathrm{Ob}(\widetilde{\mathscr{S}}) = \mathrm{Ob}(\mathscr{C})$.

Remark 3.1.5. One defines the notion of a left multiplicative system $\mathscr{S}$ by reversing the arrows. This means that the condition S3 is replaced by: given two morphisms,

$f\colon X \to Y$ and $t\colon Y' \to Y$, with $t \in \mathscr{S}$, there exist $s\colon X' \to X$ and $g\colon X' \to Y'$ with $s \in \mathscr{S}$ and $t \circ g = f \circ s$. This can be visualized by the diagram:

$$\begin{array}{ccc} X' & \overset{g}{\dashrightarrow} & Y' \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle t} \\ X & \xrightarrow{f} & Y \end{array}$$

meaning that the dotted arrows may bwe completed, making the diagram commutative.

Condition S4 is replaced by: if there exists $t \in \mathscr{S}\colon Y \to Z$ such that $t \circ f = t \circ g$ then there exists $s \in \mathscr{S}\colon W \to X$ such that $f \circ s = g \circ s$. This is visualized by the diagram

$$W \overset{s}{\dashrightarrow} X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{t} Z$$

In the literature, one often calls a multiplicative system a system which is both right and left multiplicative.

Definition 3.1.6. Assume that $\mathscr{S}$ satisfies the axioms S1-S2 and let $X \in \mathscr{C}$. One defines the categories $\mathscr{S}_X$ and $\mathscr{S}^X$ as follows.

$$\begin{aligned} \mathrm{Ob}(\mathscr{S}^X) &= \{s\colon X \to X'; s \in \mathscr{S}\} \\ \mathrm{Hom}_{\mathscr{S}^X}((s\colon X \to X'), (s'\colon X \to X'')) &= \{h \in \mathrm{Hom}_{\mathscr{C}}(X', X''); h \circ s = s'\} \\ \mathrm{Ob}(\mathscr{S}_X) &= \{s\colon X' \to X; s \in \mathscr{S}\} \\ \mathrm{Hom}_{\mathscr{S}_X}((s\colon X' \to X), (s'\colon X'' \to X)) &= \{h \in \mathrm{Hom}_{\mathscr{C}}(X', X''); s' \circ h = s\}. \end{aligned}$$

Note that $\mathscr{S}^X$ and $\mathscr{S}_X$ are full subcategories of $\mathscr{C}^X$ and $\mathscr{C}_X$ (see Definition 1.4.5), respectively.

Recall the definition of the category $\widetilde{\mathscr{S}}$ of Remark 3.1.4. Then one shall be aware that $\mathscr{S}^X \neq \widetilde{\mathscr{S}}^X$ and $\mathscr{S}_X \neq \widetilde{\mathscr{S}}_X$ since we do not ask $h \in \mathscr{S}$ in the preceding definition.

Proposition 3.1.7. *Assume that $\mathscr{S}$ is a right (resp. left) multiplicative system. Then the category $\mathscr{S}^X$ (resp. $\mathscr{S}_X^{\mathrm{op}}$) is directed.*

Proof. By reversing the arrows, both results are equivalent. We treat the case of $\mathscr{S}^X$.

(a) The category $\mathscr{S}^X$ is non empty since it contains id_X.

(b) Let $s\colon X \to X'$ and $s'\colon X \to X''$ belong to $\mathscr{S}$. By S3, there exists $t\colon X' \to X'''$ and $t'\colon X'' \to X'''$ such that $t' \circ s' = t \circ s$, and $t \in \mathscr{S}$. Hence, $t \circ s \in \mathscr{S}$ by S2 and $(X \to X''')$ belongs to $\mathscr{S}^X$.

(c) Let $s\colon X \to X'$ and $s'\colon X \to X''$ belong to $\mathscr{S}$, and consider two morphisms $f, g\colon X' \to X''$, with $f \circ s = g \circ s = s'$. By S4 there exists $t\colon X'' \to W, t \in \mathscr{S}$ such that $t \circ f = t \circ g$. Hence $t \circ s'\colon X \to W$ belongs to $\mathscr{S}^X$. □

One defines the functors:

$$\begin{aligned} \alpha_X\colon \mathscr{S}^X \to \mathscr{C} \qquad & (s\colon X \to X') \mapsto X', \\ \beta_X\colon \mathscr{S}_X^{\mathrm{op}} \to \mathscr{C} \qquad & (s\colon X' \to X) \mapsto X'. \end{aligned}$$

We shall concentrate on right multiplicative system.

Definition 3.1.8. Let $\mathscr{S}$ be a right multiplicative system and let $X, Y \in \mathrm{Ob}(\mathscr{C})$. We set

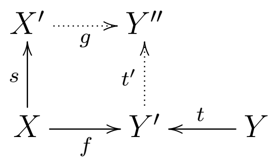
$$\mathrm{Hom}_{\mathscr{C}^r_{\mathscr{S}}}(X,Y) = \operatorname*{colim}_{(Y \to Y') \in \mathscr{S}^Y} \mathrm{Hom}_{\mathscr{C}}(X,Y'). \tag{3.1.2}$$

Roughly speaking, a morphism in $\mathscr{C}^r_{\mathscr{S}}$ is represented by morphisms $X \to Y' \xleftarrow{t} Y$ with $t \in \mathscr{S}$.

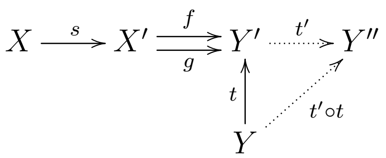
Lemma 3.1.9. *Assume that $\mathscr{S}$ is a right multiplicative system. Let $Y \in \mathscr{C}$ and let $s\colon X \to X' \in \mathscr{S}$. Then s induces an isomorphism*

$$\mathrm{Hom}_{\mathscr{C}^r_{\mathscr{S}}}(X',Y) \xrightarrow[\circ s]{\sim} \mathrm{Hom}_{\mathscr{C}^r_{\mathscr{S}}}(X,Y).$$

Proof. (i) The map $\circ s$ is surjective. This follows from S3, as visualized by the diagram in which $s, t, t' \in \mathscr{S}$:



Indeed, the map (f,t) is the image by $\circ s$ of the map $(g, t' \circ t)$.
(ii) The map $\circ s$ is injective. Since the category $\mathscr{S}^Y$ is directed, we may represent two morphisms in $\mathrm{Hom}_{\mathscr{C}^r_{\mathscr{S}}}(X',Y)$ by a diagram $f, g\colon X' \rightrightarrows Y' \xleftarrow{t} Y$. If $f \circ s = g \circ s$, there exists by S4 a morphism $t'\colon Y' \to Y$ with $t' \circ f = t' \circ g$. We get the diagram in which $s, t, t' \in \mathscr{S}$:



This shows that (f,t) and (g,t) have the same image in $\mathrm{Hom}_{\mathscr{C}^r_{\mathscr{S}}}(X',Y)$. □

Using Lemma 3.1.9, we define the composition

$$\mathrm{Hom}_{\mathscr{C}^r_{\mathscr{S}}}(X,Y) \times \mathrm{Hom}_{\mathscr{C}^r_{\mathscr{S}}}(Y,Z) \to \mathrm{Hom}_{\mathscr{C}^r_{\mathscr{S}}}(X,Z) \tag{3.1.3}$$

as

$$\begin{aligned}
&\operatorname*{colim}_{Y \to Y'} \mathrm{Hom}_{\mathscr{C}}(X,Y') \times \operatorname*{colim}_{Z \to Z'} \mathrm{Hom}_{\mathscr{C}}(Y,Z') \\
&\qquad \simeq \operatorname*{colim}_{Y \to Y'} \big(\mathrm{Hom}_{\mathscr{C}}(X,Y') \times \operatorname*{colim}_{Z \to Z'} \mathrm{Hom}_{\mathscr{C}}(Y,Z')\big) \\
&\qquad \xleftarrow{\sim} \operatorname*{colim}_{Y \to Y'} \big(\mathrm{Hom}_{\mathscr{C}}(X,Y') \times \operatorname*{colim}_{Z \to Z'} \mathrm{Hom}_{\mathscr{C}}(Y',Z')\big) \\
&\qquad \to \operatorname*{colim}_{Y \to Y'} \operatorname*{colim}_{Z \to Z'} \mathrm{Hom}_{\mathscr{C}}(X,Z') \\
&\qquad \simeq \operatorname*{colim}_{Z \to Z'} \mathrm{Hom}_{\mathscr{C}}(X,Z')
\end{aligned}$$

Lemma 3.1.10. *The composition (3.1.3) is associative.*

The verification is left to the reader.

Definition 3.1.11. (a) We denote by $\mathscr{C}^r_{\mathscr{S}}$ the category whose objects are those of $\mathscr{C}$ and morphisms are given by (3.1.2).

(b) We denote by $Q_{\mathscr{S}}\colon \mathscr{C} \to \mathscr{C}^r_{\mathscr{S}}$ the natural functor. If there is no risk of confusion, we denote this functor simply by Q.

Note that Q is associated with the natural map

$$\mathrm{Hom}_{\mathscr{C}}(X,Y) \to \operatorname*{colim}_{(Y\to Y')\in\mathscr{S}^Y} \mathrm{Hom}_{\mathscr{C}}(X,Y').$$

Lemma 3.1.12. *If $s\colon X \to Y$ belongs to $\mathscr{S}$, then $Q(s)$ is invertible.*

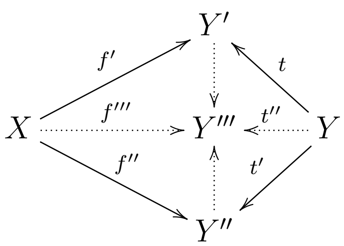
Proof. For any $Z \in \mathscr{C}^r_{\mathscr{S}}$, the map $\mathrm{Hom}_{\mathscr{C}^r_{\mathscr{S}}}(Y,Z) \to \mathrm{Hom}_{\mathscr{C}^r_{\mathscr{S}}}(X,Z)$ is bijective by Lemma 3.1.9. □

A morphism $f\colon X \to Y$ in $\mathscr{C}^r_{\mathscr{S}}$ is thus given by an equivalence class of triplets (Y', t, f') with $t\colon Y \to Y', t \in \mathscr{S}$ and $f'\colon X \to Y'$, that is:

$$X \xrightarrow[f']{} Y' \xleftarrow[t]{} Y,$$

the equivalence relation being defined as follows: $(Y', t, f') \sim (Y'', t', f'')$ if there exists (Y''', t'', f''') $(t, t', t'' \in S)$ and a commutative diagram:

(3.1.4)



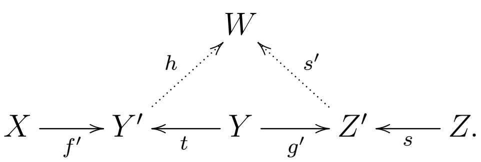
Note that the morphism (Y', t, f') in $\mathscr{C}^r_{\mathscr{S}}$ is $Q(t)^{-1} \circ Q(f')$, that is,

$$f = Q(t)^{-1} \circ Q(f'). \tag{3.1.5}$$

For two parallel arrows $f, g\colon X \rightrightarrows Y$ in $\mathscr{C}$ we have the equivalence

(3.1.6) $Q(f) = Q(g) \in \mathscr{C}^r_{\mathscr{S}} \Leftrightarrow$ there exits $s\colon Y \to Y', s \in \mathscr{S}$ with $s \circ f = s \circ g$.

The composition of two morphisms $(Y', t, f')\colon X \to Y$ and $(Z', s, g')\colon Y \to Z$ is defined by the diagram below in which $t, s, s' \in \mathscr{S}$:



In other words, this composition is given by $(W, s' \circ s, h \circ f')$.

Theorem 3.1.13. *Assume that $\mathscr{S}$ is a right multiplicative system. Then the category $\mathscr{C}^r_{\mathscr{S}}$ and the functor Q define a localization of $\mathscr{C}$ by $\mathscr{S}$.*

We refer to [KS06, Th. 7.1.16] for a proof.

Notation 3.1.14. From now on, we shall write $\mathscr{C}_{\mathscr{S}}$ instead of $\mathscr{C}^r_{\mathscr{S}}$. This is justified by Theorem 3.1.13.

Remark 3.1.15. (i) In the above construction, we have used the property of $\mathscr{S}$ of being a right multiplicative system. If $\mathscr{S}$ is a left multiplicative system, one sets

$$\mathrm{Hom}_{\mathscr{C}^l_{\mathscr{S}}}(X,Y) = \operatorname*{colim}_{(X'\to X)\in\mathscr{S}_X} \mathrm{Hom}_{\mathscr{C}}(X',Y).$$

By Proposition 3.1.2 (i), the two constructions give equivalent categories.
(ii) If $\mathscr{S}$ is both a right and left multiplicative system,

$$\mathrm{Hom}_{\mathscr{C}_{\mathscr{S}}}(X,Y) \simeq \operatorname*{colim}_{(X'\to X)\in\mathscr{S}_X,(Y\to Y')\in\mathscr{S}^Y} \mathrm{Hom}_{\mathscr{C}}(X',Y').$$

Remark 3.1.16. In general, $\mathscr{C}_{\mathscr{S}}$ is no more a $\mathscr{U}$-category. However, if one assumes that for any $X \in \mathscr{C}$ the category $\mathscr{S}^X$ is small (or more generally, cofinally small, which means that there exists a small category cofinal to it), then $\mathscr{C}_{\mathscr{S}}$ is a $\mathscr{U}$-category, and there is a similar result with the $\mathscr{S}_X$'s.

Saturated multiplicative systems

In this subsection, $\mathscr{C}$ is a category, $\mathscr{S}$ is a right multiplicative system and $Q\colon \mathscr{C} \to \mathscr{C}_{\mathscr{S}}$ is the localization functor.

Proposition 3.1.17 (see [KS06, Prop. 7.1.20])**.** *For a morphism $f\colon X \to Y$, $Q(f)$ is an isomorphism in $\mathscr{C}^r_{\mathscr{S}}$ if and only if there exist $g\colon Y \to Z$ and $h\colon Z \to W$ such that $g \circ f \in \mathscr{S}$ and $h \circ g \in \mathscr{S}$.*

Proof. (i) Assume that $Q(f)$ is an isomorphism. Let us represent the inverse of $Q(f)$ by morphisms (g, s) as on the diagram below, with $s \in \mathscr{S}$:

$$X \xrightarrow{f} Y \xrightarrow{g} X' \xleftarrow{s} X.$$

Then $Q(s)^{-1} \circ Q(g)$ is the inverse of $Q(f)$ and $Q(g) \circ Q(f) = Q(f \circ g) = Q(s)$. By (3.1.6), there exists $t\colon X' \to X''$ in $\mathscr{S}$ such that $t \circ g \circ f = t \circ s$. Changing our notations and replacing g with $t \circ g$, we have found $g\colon Y \to Z$ such that $g \circ f \in \mathscr{S}$. Then $Q(g) \circ Q(f)$ is an isomorphism, hence, by the hypothesis, $Q(g)$ is an isomorphism. By the preceding argument applied to g instead of f, there exists $h\colon Z \to W$ such that $h \circ g \in \mathscr{S}$.
(ii) The converse assertion follows from the result of Exercise 1.1 applied to $Q(f)$, $Q(g)$, $Q(h)$. □

Definition 3.1.18. One says that $\mathscr{S}$ is saturated[1] if it satisfies

S5 for any morphisms $f\colon X \to Y$, $g\colon Y \to Z$ and $h\colon Z \to W$ such that $g \circ f$ and $h \circ g$ belong to $\mathscr{S}$, the morphism f belongs to $\mathscr{S}$.

[1]One shall note confuse the notion of a saturated multiplicative system with that of a saturated subcategory, defined in § 1.3

Corollary 3.1.19. *The two conditions below are equivalent.*

(a) *The multiplicative system $\mathscr{S}$ is saturated.*

(b) *A morphism f in $\mathscr{C}$ belongs to $\mathscr{S}$ if and only if $Q(f)$ is an isomorphism.*

Proof. (a)⇒(b). If $f \in \mathscr{S}$, then $Q(f)$ is an isomorphism by Lemma 3.1.12. Conversely, assume that $Q(f)$ is an isomorphism. Then $f \in \mathscr{S}$ by Proposition 3.1.17 and the definition of being saturated.

(b)⇒(a). Consider morphisms f, g, h as in Definition 3.1.18, with $g \circ f$ and $h \circ g$ in $\mathscr{S}$. Then $Q(f)$ is an isomorphism by Proposition 3.1.17 and this implies that f belongs to $\mathscr{S}$ by the hypothesis (b). Therefore, S5 is satisfied. □

Proposition 3.1.20. *Let $\mathscr{C}$ and $\mathscr{S}$ be as above. Let $\mathscr{T}$ be the set of morphisms $f\colon X \to Y$ in $\mathscr{C}$ such that there exist $g\colon Y \to Z$ and $h\colon Z \to W$, with $h \circ g$ and $g \circ f$ in $\mathscr{S}$. Then $\mathscr{T}$ is a right saturated multiplicative system and the natural functor $\mathscr{C}_{\mathscr{S}} \to \mathscr{C}_{\mathscr{T}}$ is an equivalence.*

The proof is left as an exercise.

3.2 Localization of subcategories

Proposition 3.2.1. *Let $\mathscr{C}$ be a category, $\mathscr{I}$ a full subcategory, $\mathscr{S}$ a right multiplicative system in $\mathscr{C}$, $\mathscr{T}$ the family of morphisms in $\mathscr{I}$ which belong to $\mathscr{S}$.*

(i) *Assume that $\mathscr{T}$ is a right multiplicative system in $\mathscr{I}$. Then the functor $\mathscr{I}_{\mathscr{T}} \to \mathscr{C}_{\mathscr{S}}$ is well-defined.*

(ii) *Assume that for every $f\colon Y \to X$, $f \in \mathscr{S}$, $Y \in \mathscr{I}$, there exist $W \in \mathscr{I}$ and $g\colon X \to W$ with $g \circ f \in \mathscr{S}$. Then $\mathscr{T}$ is a right multilplicative system and the functor $\mathscr{I}_{\mathscr{T}} \to \mathscr{C}_{\mathscr{S}}$ is fully faithful.*

Proof. (i) A morphism $X \to Y$ in $\mathscr{I}_{\mathscr{T}}$ is represented by morphisms $X \xrightarrow{f'} Y' \xleftarrow{t} Y$ in $\mathscr{I}$ with $t \in \mathscr{T}$. Since $t \in \mathscr{S}$, we get a morpism in $\mathscr{C}_{\mathscr{S}}$.

(ii) It is left to the reader to check that $\mathscr{T}$ is a right multpiplicative system. For $X \in \mathscr{I}$, $\mathscr{T}^X$ is the full subcategory of $\mathscr{S}^X$ whose objects are the morphisms $s\colon X \to Y$ with $Y \in \mathscr{I}$. By Proposition 3.1.7 and the hypothesis, the functor $\mathscr{T}^X \to \mathscr{S}^X$ is cofinal, and the result follows from Definition 3.1.8. □

Corollary 3.2.2. *Let $\mathscr{C}$ be a category, $\mathscr{I}$ a full subcategory, $\mathscr{S}$ a right multiplicative system in $\mathscr{C}$, $\mathscr{T}$ the family of morphisms in $\mathscr{I}$ which belong to $\mathscr{S}$. Assume that for any $X \in \mathscr{C}$ there exists $s\colon X \to W$ with $W \in \mathscr{I}$ and $s \in \mathscr{S}$.*

Then $\mathscr{T}$ is a right multpiplicative system and $\mathscr{I}_{\mathscr{T}}$ is equivalent to $\mathscr{C}_{\mathscr{S}}$.

Proof. It follows from Proposition 3.2.1 that $\mathscr{T}$ is a right multpiplicative system, the natural functor $\mathscr{I}_{\mathscr{T}} \to \mathscr{C}_{\mathscr{S}}$ is fully faithful by the same proposition Proposition and is essentially surjective by the assumption. □

3.3 Localization of functors

Let $\mathscr{C}$ be a category, $\mathscr{S}$ a right multiplicative system in $\mathscr{C}$ and $F\colon \mathscr{C} \to \mathscr{A}$ a functor. In general, F does not send morphisms in $\mathscr{S}$ to isomorphisms in $\mathscr{A}$. In other words, F does not factorize through $\mathscr{C}_{\mathscr{S}}$. It is however possible in some cases to define a localization of F as follows.

Definition 3.3.1. A right localization of F (if it exists) is a functor $F_{\mathscr{S}}\colon \mathscr{C}_{\mathscr{S}} \to \mathscr{A}$ and a morphism of functors $\tau\colon F \to F_{\mathscr{S}} \circ Q$ such that for any functor $G\colon \mathscr{C}_{\mathscr{S}} \to \mathscr{A}$ the map

$$\text{(3.3.1)} \qquad \operatorname{Hom}_{\operatorname{Fct}(\mathscr{C}_{\mathscr{S}},\mathscr{A})}(F_{\mathscr{S}}, G) \to \operatorname{Hom}_{\operatorname{Fct}(\mathscr{C},\mathscr{A})}(F, G \circ Q)$$

is bijective. (This map is obtained as the composition $\operatorname{Hom}_{\operatorname{Fct}(\mathscr{C}_{\mathscr{S}},\mathscr{A})}(F_{\mathscr{S}}, G) \to \operatorname{Hom}_{\operatorname{Fct}(\mathscr{C},\mathscr{A})}(F_{\mathscr{S}} \circ Q, G \circ Q) \xrightarrow{\tau} \operatorname{Hom}_{\operatorname{Fct}(\mathscr{C},\mathscr{A})}(F, G \circ Q)$.)

We shall say that F is right localizable if it admits a right localization.

One defines similarly the left localization. Since we mainly consider right localization, we shall sometimes omit the word "right" as far as there is no risk of confusion.

If $(\tau, F_{\mathscr{S}})$ exists, it is unique up to unique isomorphism. Indeed, $F_{\mathscr{S}}$ is a representative of the functor

$$G \mapsto \operatorname{Hom}_{\operatorname{Fct}(\mathscr{C},\mathscr{A})}(F, G \circ Q).$$

(This last functor is defined on the category $\operatorname{Fct}(\mathscr{C}_{\mathscr{S}}, \mathscr{A})$ with values in **Set**.)

Proposition 3.3.2. *Let $\mathscr{C}$ be a category, $\mathscr{I}$ a full subcategory, $\mathscr{S}$ a right multiplicative system in $\mathscr{C}$, $\mathscr{T}$ the family of morphisms in $\mathscr{I}$ which belong to $\mathscr{S}$. Let $F\colon \mathscr{C} \to \mathscr{A}$ be a functor. Assume that*

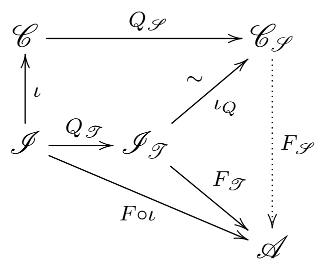
(i) *for any $X \in \mathscr{C}$ there exists $s\colon X \to W$ with $W \in \mathscr{I}$ and $s \in \mathscr{S}$,*

(ii) *for any $t \in \mathscr{T}$, $F(t)$ is an isomorphism.*

Then F is right localizable.

Proof. We shall apply Corollary 3.2.2.

Denote by $\iota\colon \mathscr{I} \to \mathscr{C}$ the natural functor. By the hypothesis, the localization $F_{\mathscr{T}}$ of $F \circ \iota$ exists. Consider the diagram:



Denote by ι_Q^{-1} a quasi-inverse of ι_Q and set $F_{\mathscr{S}} := F_{\mathscr{T}} \circ \iota_Q^{-1}$. Let us show that $F_{\mathscr{S}}$ is the localization of F. Let $G\colon \mathscr{C}_{\mathscr{S}} \to \mathscr{A}$ be a functor. We have the chain of morphisms:

$$\begin{aligned}\operatorname{Hom}_{\mathrm{Fct}(\mathscr{C},\mathscr{A})}(F, G\circ Q_{\mathscr{S}}) &\xrightarrow{\lambda} \operatorname{Hom}_{\mathrm{Fct}(\mathscr{I},\mathscr{A})}(F\circ\iota, G\circ Q_{\mathscr{S}}\circ\iota)\\ &\simeq \operatorname{Hom}_{\mathrm{Fct}(\mathscr{I},\mathscr{A})}(F_{\mathscr{T}}\circ Q_{\mathscr{T}}, G\circ\iota_Q\circ Q_{\mathscr{T}})\\ &\simeq \operatorname{Hom}_{\mathrm{Fct}(\mathscr{I}_{\mathscr{T}},\mathscr{A})}(F_{\mathscr{T}}, G\circ\iota_Q)\\ &\simeq \operatorname{Hom}_{\mathrm{Fct}(\mathscr{C}_{\mathscr{S}},\mathscr{A})}(F_{\mathscr{T}}\circ\iota_Q^{-1}, G)\\ &\simeq \operatorname{Hom}_{\mathrm{Fct}(\mathscr{C}_{\mathscr{S}},\mathscr{A})}(F_{\mathscr{S}}, G).\end{aligned}$$

We shall not prove here that λ is an isomorphism referring to [KS06, Prop. 7.3.2]. The first isomomorphism above (after λ) follows from the fact that $Q_{\mathscr{T}}$ is a localization functor (see Definition 3.1.1 (c)). The other isomorphisms are obvious. □

Remark 3.3.3. Let $\mathscr{C}$ (resp. $\mathscr{C}'$) be a category and $\mathscr{S}$ (resp. $\mathscr{S}'$) a right multiplicative system in $\mathscr{C}$ (resp. $\mathscr{C}'$). One checks immediately that $\mathscr{S}\times\mathscr{S}'$ is a right multiplicative system in the category $\mathscr{C}\times\mathscr{C}'$ and $(\mathscr{C}\times\mathscr{C}')_{\mathscr{S}\times\mathscr{S}'}$ is equivalent to $\mathscr{C}_{\mathscr{S}}\times\mathscr{C}'_{\mathscr{S}'}$. Since a bifunctor is a functor on the product $\mathscr{C}\times\mathscr{C}'$, we may apply the preceding results to the case of bifunctors. In the sequel, we shall write $F_{\mathscr{S}\mathscr{S}'}$ instead of $F_{\mathscr{S}\times\mathscr{S}'}$.

Exercises to Chapter 3

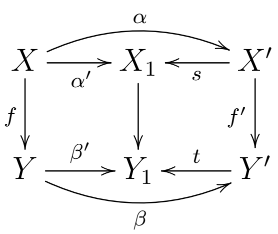
Exercise 3.1. Let $\mathscr{C}$ be a category, $\mathscr{S}$ a right and left multiplicative system. Prove that $\mathscr{S}$ is saturated if and only if for any $f\colon X\to Y$, $g\colon Y\to Z$, $h\colon Z\to W$, $h\circ g\in\mathscr{S}$ and $g\circ f\in\mathscr{S}$ imply $g\in\mathscr{S}$.

Exercise 3.2. Let $\mathscr{C}$ be a category with a zero object 0, $\mathscr{S}$ a right and left saturated multiplicative system.

(i) Show that $\mathscr{C}_{\mathscr{S}}$ has a zero object (still denoted by 0).

(ii) Prove that $Q(X)\simeq 0$ if and only if the zero morphism $0\colon X\to X$ belongs to $\mathscr{S}$.

Exercise 3.3. Let $\mathscr{C}$ be a category, $\mathscr{S}$ a right multiplicative system. Consider morphisms $f\colon X\to Y$ and $f'\colon X'\to Y'$ in $\mathscr{C}$ and morphisms $\alpha\colon X\to X'$ and $\beta\colon Y\to Y'$ in $\mathscr{C}_{\mathscr{S}}$, and assume that $f'\circ\alpha=\beta\circ f$ in $\mathscr{C}_{\mathscr{S}}$. Prove that there exists a commutative diagram in $\mathscr{C}$



with s and t in $\mathscr{S}$, $\alpha=Q(s)^{-1}\circ Q(\alpha')$ and $\beta=Q(t)^{-1}\circ Q(\beta')$.

Exercise 3.4 (See [KS06, Exe. 7.5])**.** Let $F\colon\mathscr{C}\to\mathscr{A}$ be a functor and assume that $\mathscr{C}$ admits finite colimits and F commutes with such colimits. Let $\mathscr{S}$ denote the set of morphisms s in $\mathscr{C}$ such that $F(s)$ is an isomorphism.

(i) Prove that $\mathscr{S}$ is a right saturated multiplicative system.

(ii) Prove that the localized functor $F_{\mathscr{S}}\colon\mathscr{C}_{\mathscr{S}}\to\mathscr{A}$ is faithful.

Chapter 4

Additive categories

Summary

Many results or constructions in the category Mod(A) of modules over a ring A are naturally adapted to other contexts, such as finitely generated A-modules, or graded modules over a graded ring, or sheaves of A-modules, etc. Hence, it is natural to look for a common language which avoids to repeat the same arguments. This is the language of additive and abelian categories.

In this chapter we introduce additive categories and study the category of complexes in such categories. We introduce the shifted complex, the mapping cone of a morphism, the homotopy category and the simple complex associated with a double complex. We apply this last construction to the study of bifunctors, particularly the bifunctor Hom We also briefly study the simplicial category and explain how to associate complexes to simplicial objects.

References for this chapter already appeared at the beginning of Chapter 1.

4.1 Additive categories

Definition 4.1.1. A category $\mathscr{C}$ is additive if it satisfies conditions (i)-(v) below:

(i) for any $X, Y \in \mathscr{C}$, $\quad \mathrm{Hom}_{\mathscr{C}}(X,Y) \in \mathrm{Mod}(\mathbb{Z})$,

(ii) the composition law $\circ$ is bilinear,

(iii) there exists a zero object in $\mathscr{C}$,

(iv) the category $\mathscr{C}$ admits finite coproducts,

(v) the category $\mathscr{C}$ admits finite products.

Note that $\mathrm{Hom}_{\mathscr{C}}(X,Y) \neq \varnothing$ since it is a group and for all $X \in \mathscr{C}$, $\mathrm{Hom}_{\mathscr{C}}(X,0) = \mathrm{Hom}_{\mathscr{C}}(0,X) = 0$. (The morphism 0 should not be confused with the object 0.)

Notation 4.1.2. If X and Y are two objects of $\mathscr{C}$, one denotes by $X \oplus Y$ (instead of $X \coprod Y$) their coproduct, and calls it their direct sum. One denotes as usual by $X \times Y$ their product. This change of notations is motivated by the fact that if A is a ring, the forgetful functor $\mathit{for}\colon \mathrm{Mod}(A) \to \mathbf{Set}$ does not commute with coproducts.

Similarly, if $\mathscr{C}$ admits coproducts indexed by a category I and $\{X_i\}_{i\in I}$ is a family of objects of $\mathscr{C}$, one denotes by $\bigoplus_{i\in I} X_i$ their coproduct.

Lemma 4.1.3. *Let $\mathscr{C}$ be a category satisfying conditions* (i)–(iii) *in* Definition 4.1.1. *Consider the condition*

(vi) *for any two objects X and Y in $\mathscr{C}$, there exists $Z \in \mathscr{C}$ and morphisms $i_1 \colon X \to Z$, $i_2 \colon Y \to Z$, $p_1 \colon Z \to X$ and $p_2 \colon Z \to Y$ satisfying*

$$p_1 \circ i_1 = \mathrm{id}_X, \quad p_1 \circ i_2 = 0 \tag{4.1.1}$$
$$p_2 \circ i_2 = \mathrm{id}_Y, \quad p_2 \circ i_1 = 0, \tag{4.1.2}$$
$$i_1 \circ p_1 + i_2 \circ p_2 = \mathrm{id}_Z\,. \tag{4.1.3}$$

Then the conditions (iv), (v) *and* (vi) *are equivalent and the objects $X \oplus Y$, $X \times Y$ and Z are naturally isomorphic.*

Proof. (a) Let us assume condition (iv). The identity of X and the zero morphism $Y \to X$ define the morphism $p_1 \colon X \oplus Y \to X$ satisfying (4.1.1). We construct similarly the morphism $p_2 \colon X \oplus Y \to Y$ satisfying (4.1.2). To check (4.1.3), we use the fact that if $f \colon X \oplus Y \to X \oplus Y$ satisfies $f \circ i_1 = i_1$ and $f \circ i_2 = i_2$, then $f = \mathrm{id}_{X \oplus Y}$.
(b) Let us assume condition (vi). Let $W \in \mathscr{C}$ and consider morphisms $f \colon X \to W$ and $g \colon Y \to W$. Set $h := f \circ p_1 \oplus g \circ p_2$. Then $h \colon Z \to W$ satisfies $h \circ i_1 = f$ and $h \circ i_2 = g$ and such an h is unique. Hence $Z \simeq X \oplus Y$.
(c) We have proved that conditions (iv) and (vi) are equivalent and moreover that if they are satisfied, then $Z \simeq X \oplus Y$. Replacing $\mathscr{C}$ with $\mathscr{C}^{\mathrm{op}}$, we get that these conditions are equivalent to (v) and $Z \simeq X \times Y$. □

Example 4.1.4. (i) If A is a ring, $\mathrm{Mod}(A)$ and $\mathrm{Mod}^{\mathrm{f}}(A)$ (see Example 1.3.4) are additive categories.
(ii) **Ban**, the category of $\mathbb{C}$-Banach spaces and linear continuous maps is additive.
(iii) If $\mathscr{C}$ is additive, then $\mathscr{C}^{\mathrm{op}}$ is additive.
(iv) Let I be a small category. If $\mathscr{C}$ is additive, the category $\mathrm{Fct}(I, \mathscr{C})$ of functors from I to $\mathscr{C}$ is additive.

Let $F \colon \mathscr{C} \to \mathscr{C}'$ be a functor of additive categories. One says that F is additive if for $X, Y \in \mathscr{C}$, $\mathrm{Hom}_{\mathscr{C}}(X, Y) \to \mathrm{Hom}_{\mathscr{C}'}(F(X), F(Y))$ is a morphism of groups. We shall not prove here the following result.

Proposition 4.1.5. *Let $F \colon \mathscr{C} \to \mathscr{C}'$ be a functor of additive categories. Then F is additive if and only if it commutes with direct sum, that is, for X and Y in $\mathscr{C}$:*

$$F(0) \simeq 0$$
$$F(X \oplus Y) \simeq F(X) \oplus F(Y).$$

Unless otherwise specified, functors between additive categories will be assumed to be additive.

Generalization. Let $\mathbf{k}$ be a *commutative* unital ring. One defines the notion of a $\mathbf{k}$-additive category by assuming that for X and Y in $\mathscr{C}$, $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ is a $\mathbf{k}$-module and the composition is $\mathbf{k}$-bilinear.

4.2 Complexes in additive categories

Let $\mathscr{C}$ denote an additive category.

A differential object $(X^\bullet, d_X^\bullet)$ in $\mathscr{C}$ is a sequence of objects X^k and morphisms d_X^k $(k \in \mathbb{Z})$:

$$\cdots \to X^{k-1} \xrightarrow{d_X^{k-1}} X^k \xrightarrow{d_X^k} X^{k+1} \to \cdots . \tag{4.2.1}$$

A morphism of differential objects $f^\bullet \colon X^\bullet \to Y^\bullet$ is visualized by a commutative diagram:

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & X^n & \xrightarrow{d_X^n} & X^{n+1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f^n} & & \downarrow{\scriptstyle f^{n+1}} & & \\
\cdots & \longrightarrow & Y^n & \xrightarrow{d_Y^n} & X^{n+1} & \longrightarrow & \cdots
\end{array}$$

Hence, the category $\mathrm{Diff}(\mathscr{C})$ of differential objects in $\mathscr{C}$ is nothing but the category $\mathrm{Fct}(\mathbb{Z}, \mathscr{C})$. In particular, it is an additive category.

Definition 4.2.1. (i) A complex in $\mathscr{C}$ is a differential object $(X^\bullet, d_X^\bullet)$ such that $d_X^n \circ d_X^{n-1} = 0$ for all $n \in \mathbb{Z}$.

(ii) One denotes by $\mathrm{C}(\mathscr{C})$ the full additive subcategory of $\mathrm{Diff}(\mathscr{C})$ consisting of complexes in $\mathscr{C}$.

From now on, we shall concentrate our study on the category $\mathrm{C}(\mathscr{C})$.

A complex is bounded (resp. bounded below, bounded above) if $X^n = 0$ for $|n| >> 0$ (resp. $n << 0$, $n >> 0$). One denotes by $\mathrm{C}^*(\mathscr{C})(* = \mathrm{b}, +, -)$ the full additive subcategory of $\mathrm{C}(\mathscr{C})$ consisting of bounded complexes (resp. bounded below, bounded above). We also use the notation $\mathrm{C}^{\mathrm{ub}}(\mathscr{C}) = \mathrm{C}(\mathscr{C})$ (ub for "unbounded"). For $a \in \mathbb{Z}$ we shall denote by $\mathrm{C}^{\geq a}(\mathscr{C})$ the full additive subcategory of $\mathrm{C}(\mathscr{C})$ consisting of objects $X^\bullet$ such that $X^j \simeq 0$ for $j < a$. One defines similarly the categories $\mathrm{C}^{\leq a}(\mathscr{C})$ and, for $a \leq b$, $\mathrm{C}^{[a,b]}(\mathscr{C})$.

One considers $\mathscr{C}$ as a full subcategory of $\mathrm{C}^{\mathrm{b}}(\mathscr{C})$ by identifying an object $X \in \mathscr{C}$ with the complex $X^\bullet$ "concentrated in degree 0":

$$X^\bullet := \cdots \to 0 \to X \to 0 \to \cdots$$

where X stands in degree 0. In other words, one identifies $\mathscr{C}$ and $\mathrm{C}^{[0,0]}(\mathscr{C})$.

Notation 4.2.2. In the definitions above of a differential object or a complex, we assumed that X^k is defined for $k \in \mathbb{Z}$. If X^k is only defined for $k \in I$, I being an interval of Z, we consider again $X^\bullet$ as a differential object or a complex by setting $X^k = 0$ for $k \notin I$.

From now on, we shall often simply denote by X an object of $\mathrm{C}(\mathscr{C})$.

Shift functor

Let $\mathscr{C}$ be an additive category, let $X \in \mathrm{C}(\mathscr{C})$ and let $p \in \mathbb{Z}$. One defines the shifted complex $X[p]$ by[1]:

$$(X[p])^n = X^{n+p}, \quad d^n_{X[p]} = (-)^p d^{n+p}_X$$

If $f\colon X \to Y$ is a morphism in $\mathrm{C}(\mathscr{C})$ one defines $f[p]\colon X[p] \to Y[p]$ by $(f[p])^n = f^{n+p}$.

The shift functor $[1]\colon X \mapsto X[1]$ is an automorphism (*i.e.* an invertible functor) of $\mathrm{C}(\mathscr{C})$.

Mapping cone

Definition 4.2.3. Let $f\colon X \to Y$ be a morphism in $\mathrm{C}(\mathscr{C})$. The mapping cone of f, denoted $\mathrm{Mc}(f)$, is the object of $\mathrm{C}(\mathscr{C})$ defined by:

$$\mathrm{Mc}(f)^n = (X[1])^n \oplus Y^n, \quad d^n_{\mathrm{Mc}(f)} = \begin{pmatrix} d^n_{X[1]} & 0 \\ f^{n+1} & d^n_Y \end{pmatrix}.$$

Of course, before to state this definition, one should check that $d^{n+1}_{\mathrm{Mc}(f)} \circ d^n_{\mathrm{Mc}(f)} = 0$. Indeed:

$$\begin{pmatrix} -d^{n+2}_X & 0 \\ f^{n+2} & d^{n+1}_Y \end{pmatrix} \circ \begin{pmatrix} -d^{n+1}_X & 0 \\ f^{n+1} & d^n_Y \end{pmatrix} = 0.$$

Notice that although $\mathrm{Mc}(f)^n = (X[1])^n \oplus Y^n$, $\mathrm{Mc}(f)$ is not isomorphic to $X[1] \oplus Y$ in $\mathrm{C}(\mathscr{C})$ unless f is the zero morphism.

There are natural morphisms of complexes

$$\alpha(f)\colon Y \to \mathrm{Mc}(f), \quad \beta(f)\colon \mathrm{Mc}(f) \to X[1]. \tag{4.2.2}$$

and $\beta(f) \circ \alpha(f) = 0$.

Example 4.2.4. Let $f\colon X \to Y$ be a morphism in $\mathscr{C}$ and let us identify $\mathscr{C}$ with a full subcategory of $\mathrm{C}(\mathscr{C})$. Then X and Y are complexes concentrated in degree 0 and f is a morphism of complexes. One checks immediately that $\mathrm{Mc}(f)$ is the complex $\cdots 0 \to X \xrightarrow{f} Y \to 0 \to \cdots$ where Y stands in degree 0.

If $F\colon \mathscr{C} \to \mathscr{C}'$ is an additive functor, then $F(\mathrm{Mc}(f)) \simeq \mathrm{Mc}(F(f))$.

4.3 Double complexes

Let $\mathscr{C}$ be an additive category as above. A double complex $(X^{\bullet,\bullet}, d_X)$ in $\mathscr{C}$ is the data of

$$\{X^{n,m}, d'^{n,m}_X, d''^{n,m}_X; (n,m) \in \mathbb{Z} \times \mathbb{Z}\}$$

where $X^{n,m} \in \mathscr{C}$ and the "differentials" $d'^{n,m}_X\colon X^{n,m} \to X^{n+1,m}$, $d''^{n,m}_X : X^{n,m} \to X^{n,m+1}$ satisfy:

$$d'^2_X = d''^2_X = 0, \; d'_X \circ d''_X = d''_X \circ d'_X. \tag{4.3.1}$$

[1]In these notes, we shall sometimes write $(-)^p$ instead of $(-1)^p$

One can represent a double complex by a commutative diagram:

$$(4.3.2)\qquad \begin{array}{ccccccc} & & \downarrow & & \downarrow & & \\ \longrightarrow & X^{n,m} & \xrightarrow{d''^{\,n,m}_X} & X^{n,m+1} & \longrightarrow \\ & {\scriptstyle d'^{\,n,m}_X}\downarrow & & {\scriptstyle d'^{\,n,m+1}_X}\downarrow & \\ \longrightarrow & X^{n+1,m} & \xrightarrow[d''^{\,n+1,m}_X]{} & X^{n+1,m+1} & \longrightarrow \\ & \downarrow & & \downarrow & \end{array}$$

One defines naturally the notion of a morphism of double complexes and one obtains the additive category $\mathrm{C}^2(\mathscr{C})$ of double complexes.

There is a functor $F_I\colon \mathrm{C}^2(\mathscr{C}) \to \mathrm{C}(\mathrm{C}(\mathscr{C}))$ which, to a double complex X, associates the complex whose objects are the rows of X. More precisely, for $n \in \mathbb{Z}$, consider the simple complex

$$X_I^n = \{X^{n,m}, d''^{n,m}\}_{m\in\mathbb{Z}}$$

The family of morphisms $\{d'^{n,m}\}_{m\in\mathbb{Z}}$ defines a morphism $d_I^n\colon X_I^n \to X_I^{n+1}$ and one checks that $d_I^{n+1}\circ d_I^n = 0$. Therefore, $\{X_I^n, d_I^n\}_{n\in\mathbb{Z}}$ is a complex in $\mathrm{C}(\mathscr{C})$ and we have constructed the functor

$$F_I\colon \mathrm{C}^2(\mathscr{C}) \to \mathrm{C}(\mathrm{C}(\mathscr{C})).$$

By reversing the role of the rows and the columns, one constructs similarly the functor F_{II}. Clearly, the two functors F_I and F_{II} are isomorphisms of categories.

Assume

$$(4.3.3)\qquad \mathscr{C} \text{ admits countable direct sums.}$$

One can then associate to the double complex X a simple complex $\mathrm{tot}_\oplus(X)$ by setting:

$$(4.3.4)\qquad \mathrm{tot}_\oplus(X)^p = \oplus_{n+m=p}X^{n,m}, \quad d^p_{\mathrm{tot}_\oplus(X)}\circ\varepsilon_{n,m} = \varepsilon_{n+1,m}\circ d'^{n,m} + \varepsilon_{n,m+1}\circ(-)^n d''^{n,m}.$$

(See (2.1.6) for the notation $\varepsilon_{n,m}$.) This is visualized by the diagram:

$$X^{n,m} \xrightarrow{(d'^{n,m},(-)^n d''^{n,m})} X^{n+1,m}\oplus X^{n,m+1} \to \mathrm{tot}_\oplus(X)^{p+1}.$$

Similarly, assume

$$(4.3.5)\qquad \mathscr{C} \text{ admits countable products.}$$

One can then associate to the double complex X a simple complex $\mathrm{tot}_\pi(X)$ by setting:

$$(4.3.6)\qquad \mathrm{tot}_\pi(X)^p = \prod_{m+n=p} X^{n,m}, \quad \pi_{n,m}\circ d^{p-1}_{\mathrm{tot}_\pi(X)} = d'^{n-1,m}\circ\pi_{n-1,m} + (-)^n d''^{n,m-1}\circ\pi_{n,m-1}.$$

This is visualized by the diagram:

$$\mathrm{tot}_\pi(X)^{p-1} \to X^{n-1,m} \oplus X^{n,m-1} \to \begin{pmatrix} d'^{n-1,m} \\ (-)^n d''^{n,m-1} \end{pmatrix} X^{n,m}.$$

One also encounters the finiteness condition:

$$\text{for all } p \in \mathbb{Z}, \quad \{(m,n) \in \mathbb{Z} \times \mathbb{Z}; X^{n,m} \neq 0, m+n=p\} \text{ is finite.} \tag{4.3.7}$$

To such an X one associates its "total complex" $\mathrm{tot}(X) = \mathrm{tot}_\oplus(X) \simeq \mathrm{tot}_\pi(X)$. In the sequel, we denote by $C^2_f(\mathscr{C})$ the full subcategory of $\mathrm{C}^2(\mathscr{C})$ consisting of objects X satisfying (4.3.7).

Proposition 4.3.1. *Assume* (4.3.3). *Then the differential object* $\{\mathrm{tot}_\oplus(X)^p, d^p_{\mathrm{tot}_\oplus(X)}\}_{p\in\mathbb{Z}}$ *is a complex* (i.e., $d^{p+1}_{\mathrm{tot}_\oplus(X)} \circ d^p_{\mathrm{tot}_\oplus(X)} = 0$) *and* $\mathrm{tot}_\oplus \colon \mathrm{C}^2(\mathscr{C}) \to \mathrm{C}(\mathscr{C})$ *is a functor of additive categories.*

There is a similar result assuming (4.3.5) *or assuming that* $X \in C^2_f(\mathscr{C})$.

Proof. For short, we write simply d_{tot} or even d instead of $d_{\mathrm{tot}_\oplus(X)}$. We also write $d|_{X^{n,m}}$ instead of $\varepsilon_{n,m} \circ d$.

For $(n,m) \in \mathbb{Z} \times \mathbb{Z}$, one has

$$\begin{aligned} d \circ d|_{X^{n,m}} &= d'' \circ d''|_{X^{n,m}} + d' \circ d'|_{X^{n,m}} \\ &\quad + (-)^{n+1} d'' \circ d'|_{X^{n,m}} + (-)^n d' \circ d''|_{X^{n,m}} \\ &= 0. \end{aligned}$$

The fact that $\mathrm{tot}_\oplus$ is an additive functor is obvious. □

Example 4.3.2. Let $f^\bullet : X^\bullet \to Y^\bullet$ be a morphism in $\mathrm{C}(\mathscr{C})$. Consider the double complex $Z^{\bullet,\bullet}$ such that $Z^{-1,\bullet} = X^\bullet$, $Z^{0,\bullet} = Y^\bullet$, $Z^{i,\bullet} = 0$ for $i \neq -1, 0$, with differentials $f^j \colon Z^{-1,j} \to Z^{0,j}$. Then

$$\mathrm{tot}(Z^{\bullet,\bullet}) \simeq \mathrm{Mc}(f^\bullet). \tag{4.3.8}$$

Bifunctor

Let $\mathscr{C}, \mathscr{C}'$ and $\mathscr{C}''$ be additive categories and let $F \colon \mathscr{C} \times \mathscr{C}' \to \mathscr{C}''$ be an additive bifunctor (*i.e.*, $F(\,\bullet\,,\,\bullet\,)$ is additive with respect to each argument). It defines an additive bifunctor $\mathrm{C}^2(F) \colon \mathrm{C}(\mathscr{C}) \times \mathrm{C}(\mathscr{C}') \to \mathrm{C}^2(\mathscr{C}'')$. In other words, if $X \in \mathrm{C}(\mathscr{C})$ and $X' \in \mathrm{C}(\mathscr{C}')$ are complexes, then $\mathrm{C}^2(F)(X, X')$ is a double complex.

Example 4.3.3. Consider the bifunctor $\bullet \otimes \bullet : \mathrm{Mod}(A^{\mathrm{op}}) \times \mathrm{Mod}(A) \to \mathrm{Mod}(\mathbb{Z})$. In the sequel, we shall simply write $\otimes$ instead of $\mathrm{C}^2(\otimes)$. Then, for $X \in \mathrm{C}(\mathrm{Mod}(A^{\mathrm{op}}))$ and $Y \in \mathrm{C}(\mathrm{Mod}(A))$, one has

$$\begin{aligned} &(X \otimes Y)^{n,m} = X^n \otimes Y^m, \quad d'^{n,m} = d^n_X \otimes \mathrm{id}_{Y^m}, d''^{n,m} = \mathrm{id}_{X^n} \otimes d^m_Y, \\ &(\mathrm{tot}_\oplus(X,Y))^k = \bigoplus_{n+m=k} X^n \otimes Y^m, \quad d_{\mathrm{tot}(X\otimes Y)}|_{X^n \otimes Y^m} = d'^{n,m} + (-)^n d''^{n,m}. \end{aligned}$$

The complex $\mathrm{Hom}^\bullet$

Consider the bifunctor $\mathrm{Hom}_{\mathscr{C}}\colon \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathrm{Mod}(\mathbb{Z})$. In the sequel, we shall write $\mathrm{Hom}_{\mathscr{C}}^{\bullet,\bullet}$ instead of $\mathrm{C}^2(\mathrm{Hom}_{\mathscr{C}})$. If X and Y are two objects of $\mathrm{C}(\mathscr{C})$, one has

$$\begin{aligned}&\mathrm{Hom}_{\mathscr{C}}^{\bullet,\bullet}(X,Y)^{n,m} = \mathrm{Hom}_{\mathscr{C}}(X^{-m}, Y^n),\\ &d'^{n,m} = \mathrm{Hom}_{\mathscr{C}}(X^{-m}, d_Y^n), \quad d''^{n,m} = \mathrm{Hom}_{\mathscr{C}}((-)^{m+1}d_X^{-m-1}, Y^n).\end{aligned}$$

Note that $\mathrm{Hom}_{\mathscr{C}}^{\bullet,\bullet}(X,Y)$ is a double complex in the category $\mathrm{Mod}(\mathbb{Z})$ and should not be confused with the group $\mathrm{Hom}_{\mathrm{C}(\mathscr{C})}(X,Y)$.

Let $X, Y \in \mathrm{C}(\mathscr{C})$. Using the fact that $\mathrm{Mod}(\mathbb{Z})$ admits countable products, one sets

$$\mathrm{Hom}_{\mathscr{C}}^{\bullet}(X,Y) = \mathrm{tot}_\pi \mathrm{Hom}_{\mathscr{C}}^{\bullet,\bullet}(X,Y), \text{ an object of } \mathrm{C}(\mathrm{Mod}(\mathbb{Z})). \tag{4.3.9}$$

Hence, $\mathrm{Hom}_{\mathscr{C}}(X,Y)^n = \prod_j \mathrm{Hom}_{\mathscr{C}}(X^j, Y^{n+j})$ and $d^n\colon \mathrm{Hom}_{\mathscr{C}}(X,Y)^n \to \mathrm{Hom}_{\mathscr{C}}(X,Y)^{n+1}$ is defined as follows. To $f = \{f^j\}_j \in \prod_{j\in\mathbb{Z}} \mathrm{Hom}_{\mathscr{C}}(X^j, Y^{n+j})$ one associates

$$d^n f = \{g^j\}_j \in \prod_{j\in\mathbb{Z}} \mathrm{Hom}_{\mathscr{C}}(X^j, Y^{n+j+1}), \quad g^j = d'^{n+j,-j} f^j + (-)^{j+n+1} d''^{j+n+1,-j-1} f^{j+1}.$$

In other words, the components of df in $\mathrm{Hom}_{\mathscr{C}}(X,Y)^{n+1}$ will be given by

$$(d^n f)^j = d_Y^{j+n} \circ f^j + (-)^{n+1} f^{j+1} \circ d_X^j. \tag{4.3.10}$$

Note that for $X, Y, Z \in \mathrm{C}(\mathscr{C})$, there is a natural composition map

$$\mathrm{Hom}_{\mathscr{C}}^{\bullet}(X,Y) \times \mathrm{Hom}_{\mathscr{C}}^{\bullet}(Y,Z) \xrightarrow{\circ} \mathrm{Hom}_{\mathscr{C}}^{\bullet}(X,Z) \tag{4.3.11}$$

associated with the map

$$\begin{aligned}&\mathrm{Hom}_{\mathscr{C}}(X,Y)^m \times \mathrm{Hom}_{\mathscr{C}}(Y,Z)^n \to \mathrm{Hom}_{\mathscr{C}}(X,Z)^{m+n},\\ &\prod_i \mathrm{Hom}_{\mathscr{C}}(X^i, Y^{i+m}) \times \prod_i \mathrm{Hom}_{\mathscr{C}}(Y^{i+m}, Z^{i+m+n}) \to \prod_i \mathrm{Hom}_{\mathscr{C}}(X^i, Z^{i+m+n}).\end{aligned}$$

4.4 The homotopy category

Let $\mathscr{C}$ be an additive category.

Definition 4.4.1. (i) A morphism $f\colon X \to Y$ in $\mathrm{C}(\mathscr{C})$ is homotopic to zero if for all p there exists a morphism $s^p\colon X^p \to Y^{p-1}$ such that:

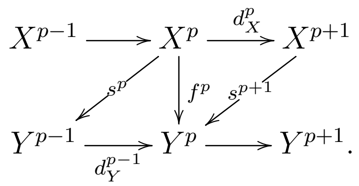
$$f^p = s^{p+1} \circ d_X^p + d_Y^{p-1} \circ s^p.$$

Two morphisms $f, g : X \to Y$ are homotopic if $f - g$ is homotopic to zero.

(ii) An object X in $\mathrm{C}(\mathscr{C})$ is homotopic to 0 if id_X is homotopic to zero.

(iii) A morphism $f\colon X \to Y$ in $\mathrm{C}(\mathscr{C})$ is a homotopy equivalence if there exists $g\colon Y \to X$ such that $g \circ f$ is homotopic to id_X and $f \circ g$ is homotopic to id_Y.

A morphism homotopic to zero is visualized by the diagram (which is not commutative):



Note that an additive functor sends a morphism homotopic to zero to a morphism homotopic to zero.

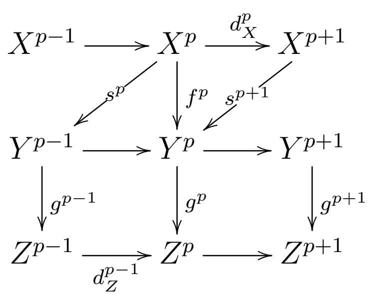
Example 4.4.2. (i) Let $X, Y \in \mathrm{C}(\mathscr{C})$. If both X and Y are homotopic to zero, then so is $X \oplus Y$.

(ii) Let $X \in \mathscr{C}$. Then the complex $0 \to X \xrightarrow{\mathrm{id}_X} X \to 0$ is homotopic to zero.

(iii) In particular, for $X', X'' \in \mathscr{C}$, the complex $0 \to X' \to X' \oplus X'' \to X'' \to 0$ is homotopic to zero.

Lemma 4.4.3. *Let $f\colon X \to Y$ and $g\colon Y \to Z$ be two morphisms in $\mathrm{C}(\mathscr{C})$. If f or g is homotopic to zero, then $g \circ f$ is homotopic to zero.*

Proof. Assume for example that f is homotopic to zero. In this case the proof is visualized by the diagram below.



Indeed, the equality $f^p = s^{p+1} \circ d_X^p + d_Y^{p-1} \circ s^p$ implies

$$g^p \circ f^p = g^p \circ s^{p+1} \circ d_X^p + d_Z^{p-1} \circ g^{p-1} \circ s^p.$$

□

We shall construct a new category by deciding that a morphism in $\mathrm{C}(\mathscr{C})$ homotopic to zero is isomorphic to the zero morphism. Set:

$$Ht(X, Y) = \{f\colon X \to Y; f \text{ is homotopic to } 0\}.$$

Lemma 4.4.3 allows us to state:

Definition 4.4.4. The homotopy category $\mathrm{K}(\mathscr{C})$ is defined by:

$$\begin{aligned} \mathrm{Ob}(\mathrm{K}(\mathscr{C})) &= \mathrm{Ob}(\mathrm{C}(\mathscr{C})) \\ \mathrm{Hom}_{\mathrm{K}(\mathscr{C})}(X, Y) &= \mathrm{Hom}_{\mathrm{C}(\mathscr{C})}(X, Y)/Ht(X, Y). \end{aligned}$$

In other words, a morphism homotopic to zero in $\mathrm{C}(\mathscr{C})$ becomes the zero morphism in $\mathrm{K}(\mathscr{C})$ and a homotopy equivalence becomes an isomorphism.

One defines similarly $\mathrm{K}^*(\mathscr{C})$, $(* = \mathrm{b}, +, -)$. They are clearly additive categories endowed with an automorphism, the shift functor $[1]\colon X \mapsto X[1]$.

Recall (4.3.9).

Proposition 4.4.5. *Let $\mathscr{C}$ be an additive category and let $X, Y \in \mathrm{C}(\mathscr{C})$. There are isomorphisms:*

$$\begin{aligned} Z^0(\mathrm{Hom}^\bullet_{\mathscr{C}}(X,Y)) &:= \mathrm{Ker}\, d^0 \simeq \mathrm{Hom}_{\mathrm{C}(\mathscr{C})}(X,Y), \\ B^0(\mathrm{Hom}^\bullet_{\mathscr{C}}(X,Y)) &:= \mathrm{Im}\, d^{-1} \simeq Ht(X,Y), \\ H^0(\mathrm{Hom}^\bullet_{\mathscr{C}}(X,Y)) &:= \mathrm{Ker}\, d^0 / \mathrm{Im}\, d^{-1} \simeq \mathrm{Hom}_{\mathrm{K}(\mathscr{C})}(X,Y). \end{aligned}$$

Proof. (i) Let us calculate $Z^0(\mathrm{Hom}^\bullet_{\mathscr{C}}(X,Y))$. By (4.3.10), the component of $d^0\{f^j\}_j$ in $\mathrm{Hom}_{\mathscr{C}}(X^j, Y^{j+1})$ will be zero if and only if $d_Y^{j+1} \circ f^j = f^{j+1} \circ d_X^j$, that is, if the family $\{f^j\}_j$ defines a morphism of complexes.

(ii) Let us calculate $B^0(\mathrm{Hom}^\bullet_{\mathscr{C}}(X,Y))$. An element $f^j \in \mathrm{Hom}_{\mathscr{C}}(X^j, Y^j)$ will be in the image of d^{-1} if it is in the sum of the image of $\mathrm{Hom}_{\mathscr{C}}(X^j, Y^{j-1})$ by d_Y^{j-1} and the image of $\mathrm{Hom}_{\mathscr{C}}(X^{j+1}, Y^j)$ by d_X^j. Hence, if it can be written as $f^j = d_Y^{j-1} \circ s^j + s^{j+1} \circ d_X^j$.

(iii) The third isomorphism follows. □

Remark 4.4.6. The preceding constructions could be developed in the general setting of DG-categories. Roughly speaking, a DG-category is an additive category in which the morphisms are no more additive groups but are complexes of such groups.

The category $\mathrm{C}(\mathscr{C})$ endowed for each $X, Y \in \mathrm{C}(\mathscr{C})$ with the complex $\mathrm{Hom}^\bullet_{\mathscr{C}}(X,Y))$ and the composition being given by (4.3.11) is an example of such a DG-category. More details on this subject, see for example [Kel06, Yek20].

We shall come back to the category $\mathrm{K}(\mathscr{C})$ in § 6.3.

4.5 Simplicial constructions

We shall define the simplicial category and use it to construct complexes and homotopies in additive categories.

Definition 4.5.1. (a) The simplicial category, denoted by $\mathbf{\Delta}$, is the category whose objects are the finite totally ordered sets and the morphisms are the order-preserving maps.

(b) We denote by $\mathbf{\Delta}_{inj}$ the subcategory of $\mathbf{\Delta}$ such that $\mathrm{Ob}(\mathbf{\Delta}_{inj}) = \mathrm{Ob}(\mathbf{\Delta})$, the morphisms being the injective order-preserving maps.

For integers n, m denote by $[n, m]$ the totally ordered set $\{k \in \mathbb{Z};\ n \leq k \leq m\}$.

Proposition 4.5.2. (i) *the natural functor $\mathbf{\Delta} \to \mathbf{Set}^f$ is faithful,*

[1]§ 4.5 may be skipped.

(ii) *the full subcategory of* $\mathbf{\Delta}$ *consisting of objects* $\{[0,n]\}_{n\geq -1}$ *is equivalent to* $\mathbf{\Delta}$,

(iii) $\mathbf{\Delta}$ *admits an initial object, namely* $\varnothing$, *and a terminal object, namely* $\{0\}$.

The proof is obvious.

Let us denote by

$$d_i^n\colon [0,n] \to [0,n+1] \qquad (0\leq i\leq n+1)$$

the injective order-preserving map which does not take the value i. In other words

$$d_i^n(k) = \begin{cases} k & \text{for } k<i, \\ k+1 & \text{for } k\geq i. \end{cases}$$

One checks immediately that

$$d_j^{n+1}\circ d_i^n = d_i^{n+1}\circ d_{j-1}^n \text{ for } 0\leq i<j\leq n+2. \tag{4.5.1}$$

Indeed, both morphisms are the unique injective order-preserving map which does not take the values i and j.

The category $\mathbf{\Delta}_{inj}$ is visualized by

$$\varnothing \xrightarrow{d_0^{-1}} [0] \underset{d_1^0}{\overset{d_0^0}{\rightrightarrows}} [0,1] \overset{d_0^1,\ d_1^1,\ d_2^1}{\rightrightarrows} [0,1,2] \cdots \tag{4.5.2}$$

Let $\mathscr{C}$ be an additive category and $F\colon \mathbf{\Delta}_{inj}\to\mathscr{C}$ a functor. We set for $n\in\mathbb{Z}$:

$$F^n = \begin{cases} F([0,n]) & \text{for } n\geq -1, \\ 0 & \text{otherwise}, \end{cases}$$

$$d_F^n\colon F^n\to F^{n+1}, \quad d_F^n = \sum_{i=0}^{n+1}(-)^i F(d_i^n).$$

Consider the differential object

$$F^\bullet := \cdots \to 0\to F^{-1}\xrightarrow{d_F^{-1}} F^0 \xrightarrow{d_F^0} F^1\to\cdots\to F^n \xrightarrow{d_F^n} \cdots . \tag{4.5.3}$$

Theorem 4.5.3. (i) *The differential object* $F^\bullet$ *is a complex.*

(ii) *Assume that there exist morphisms* $s_F^n\colon F^n\to F^{n-1}$ $(n\geq 0)$*satisfying:*

$$\begin{cases} s_F^{n+1}\circ F(d_0^n) = \mathrm{id}_{F^n} & \textit{for } n\geq -1, \\ s_F^{n+1}\circ F(d_{i+1}^n) = F(d_i^{n-1})\circ s_F^n & \textit{for } i>0, n\geq 0. \end{cases}$$

Then $F^\bullet$ *is homotopic to zero.*

Proof. (i) By (4.5.1), we have

$$\begin{aligned} d_F^{n+1}\circ d_F^n &= \sum_{j=0}^{n+2}\sum_{i=0}^{n+1}(-)^{i+j}F(d_j^{n+1}\circ d_i^n) \\ &= \sum_{0\leq j\leq i\leq n+1}(-)^{i+j}F(d_j^{n+1}\circ d_i^n) + \sum_{0\leq i<j\leq n+2}(-)^{i+j}F(d_j^{n+1}\circ d_i^n) \\ &= \sum_{0\leq j\leq i\leq n+1}(-)^{i+j}F(d_j^{n+1}\circ d_i^n) + \sum_{0\leq i<j\leq n+2}(-)^{i+j}F(d_i^{n+1}\circ d_{j-1}^n) \\ &= 0\,. \end{aligned}$$

Here, we have used

$$\sum_{0\le i<j\le n+2}(-)^{i+j}F(d_i^{n+1}\circ d_{j-1}^n)=\sum_{0\le i\le j\le n+1}(-)^{i+j+1}F(d_i^{n+1}\circ d_j^n)$$
$$=\sum_{0\le j\le i\le n+1}(-)^{i+j+1}F(d_j^{n+1}\circ d_i^n).$$

(ii) We have

$$\begin{aligned}
&s_F^{n+1}\circ d_F^n+d_F^{n-1}\circ s^n\\
&\quad=\sum_{i=0}^{n+1}(-1)^i s_F^{n+1}\circ F(d_i^n)+\sum_{i=0}^{n}(-1)^i F(d_i^{n-1}\circ s_F^n)\\
&\quad=s_F^{n+1}\circ F(d_0^n)+\sum_{i=0}^{n}(-1)^{i+1}s_F^{n+1}\circ F(d_{i+1}^n)+\sum_{i=0}^{n}(-1)^i F(d_i^{n-1}\circ s_F^n)\\
&\quad=\mathrm{id}_{F^n}+\sum_{i=0}^{n}(-1)^{i+1}F(d_i^{n-1}\circ s_F^n)+\sum_{i=0}^{n}(-1)^i F(d_i^{n-1}\circ s_F^n)\\
&\quad=\mathrm{id}_{F^n}.
\end{aligned}$$

□

Exercises to Chapter 4

Exercise 4.1. Let $\mathscr{C}$ be an additive category and let $X\in \mathrm{C}(\mathscr{C})$ with differential d_X. Let $\{a_n\}_{n\in\mathbb{Z}}$ be a sequence in $\mathbb{Z}$. Define the morphism $\delta_X\colon X\to X[1]$ by setting $\delta_X^n=a_nd_X^n$. Prove that δ_X is a morphism in $\mathrm{C}(\mathscr{C})$ and is homotopic to zero.

Exercise 4.2 (See [KS06, Exe. 11.4]). Let $\mathscr{C}$ be an additive category, $f,g\colon X\rightrightarrows Y$ two morphisms in $\mathrm{C}(\mathscr{C})$. Prove that f and g are homotopic if and only if there exists a commutative diagram in $\mathrm{C}(\mathscr{C})$

$$\begin{array}{ccccc}
Y & \xrightarrow[\alpha(f)]{} & \mathrm{Mc}(f) & \xrightarrow[\beta(f)]{} & X[1]\\
\| & & \downarrow{\scriptstyle u} & & \|\\
Y & \xrightarrow[\alpha(g)]{} & \mathrm{Mc}(g) & \xrightarrow[\beta(g)]{} & X[1].
\end{array}$$

In such a case, prove that u is an isomorphism in $\mathrm{C}(\mathscr{C})$.

Exercise 4.3 (See [KS06, Exe. 11.6]). Let $\mathscr{C}$ be an additive category and let $f\colon X\to Y$ be a morphism in $\mathrm{C}(\mathscr{C})$.
Prove that the following conditions are equivalent:

(a) f is homotopic to zero,

(b) f factors through $\alpha(\mathrm{id}_X)\colon X\to\mathrm{Mc}(\mathrm{id}_X)$,

(c) f factors through $\beta(\mathrm{id}_Y)[-1]\colon \mathrm{Mc}(\mathrm{id}_Y)[-1]\to Y$,

(d) f decomposes as $X\to Z\to Y$ with Z a complex homotopic to zero.

Exercise 4.4. (See [KS06, § 10.1].) A category with translation $(\mathscr{A}, T)$ is a category $\mathscr{A}$ together with an equivalence $T\colon \mathscr{A} \to \mathscr{A}$. A differential object (X, d_X) in a category with translation $(\mathscr{A}, T)$ is an object $X \in \mathscr{A}$ together with a morphism $d_X\colon X \to T(X)$. A morphism $f\colon (X, d_X) \to (Y, d_Y)$ of differential objects is a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{d_X} & TX \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle T(f)} \\ Y & \xrightarrow{d_Y} & TY. \end{array}$$

One denotes by $\mathscr{A}_d$ the category consisting of differential objects and morphisms of such objects. If $\mathscr{A}$ is additive, one says that a differential object (X, d_X) in $(\mathscr{A}, T)$ is a complex if the composition $X \xrightarrow{d_X} T(X) \xrightarrow{T(d_X)} T^2(X)$ is zero. One denotes by $\mathscr{A}_c$ the full subcategory of $\mathscr{A}_d$ consisting of complexes.
(i) Let $\mathscr{C}$ be a category. Denote by $\mathbb{Z}_d$ the set $\mathbb{Z}$ considered as a discrete category and still denote by $\mathbb{Z}$ the ordered set $(\mathbb{Z}, \leq)$ considered as a category. Prove that $\mathrm{Fct}(\mathbb{Z}_d, \mathscr{C})$ is a category with translation.
(ii) Show that the category $\mathrm{Fct}(\mathbb{Z}, \mathscr{C})$ may be identified to the category of differential objects in $\mathrm{Fct}(\mathbb{Z}_d, \mathscr{C})$.
(iii) Let $\mathscr{C}$ be an additive category. Show that the notions of differential objects and complexes given above coincide with those in Definition 4.2.1 when choosing $\mathscr{A} = \mathrm{C}(\mathscr{C})$ and $T = [1]$.

Exercise 4.5. Consider the category $\mathbf{\Delta}$ and for $n > 0$, denote by

$$s_i^n\colon [0, n] \to [0, n-1] \qquad (0 \leq i \leq n-1)$$

the surjective order-preserving map which takes the same value at i and $i+1$. In other words

$$s_i^n(k) = \begin{cases} k & \text{for } k \leq i, \\ k-1 & \text{for } k > i. \end{cases}$$

Check the relations:

$$\begin{cases} s_j^n \circ s_i^{n+1} = s_{i-1}^n \circ s_j^{n+1} & \text{for } 0 \leq j < i \leq n, \\ s_j^{n+1} \circ d_i^n = d_i^{n-1} \circ s_{j-1}^n & \text{for } 0 \leq i < j \leq n, \\ s_j^{n+1} \circ d_i^n = \mathrm{id}_{[0,n]} & \text{for } 0 \leq i \leq n+1, i = j, j+1, \\ s_j^{n+1} \circ d_i^n = d_{i-1}^{n-1} \circ s_j^n & \text{for } 1 \leq j+1 < i \leq n+1. \end{cases}$$

Chapter 5

Abelian categories

Summary

The toy model of abelian categories is the category Mod(A) of modules over a ring A and for sake of simplicity, we shall argue most of the time as if we were working in a full abelian subcategory of a category Mod(A). This is not restrictive in view of a famous theorem of Fred and Mitchell [Mit60, Fre64].

We introduce injective and projective objects and state without proof the famous Grothendieck theorem which asserts that what is now called a Grothendieck category admits enough injectives.

We explain the notions of exact sequences and right or left exact functors, we give some basic lemmas such as “the five lemma” and “the snake lemma”, we construct the long exact sequence associated with an exact sequence of complexes and we also study double complexes. We also study the so-called Mittag-Leffler condition introduced first in [EGA3], an efficient tool to treat projective limits of modules.

Finally, we study with some details Koszul complexes and show how they naturally appear in Algebra and Analysis.

Some references. See [CE56, Gro57] for historical references and [Wei94, KS06] for a more modern exposition. Here we shall often follow this last reference.

5.1 Abelian categories

Let $\mathscr{C}$ be an additive category which admits kernels and cokernels (recall Definition 2.2.1). Equivalently, $\mathscr{C}$ admits finite limits and colimits.

Let $f\colon X \to Y$ be a morphism in $\mathscr{C}$. We have already defined the image and co-image of f in Definition 2.4.4. Denote by $h\colon \operatorname{Ker} f \to X$ and $k\colon Y \to \operatorname{Coker} f$ the natural morphisms.

Lemma 5.1.1. *One has the isomorphisms*

$$\operatorname{Coim} f \simeq \operatorname{Coker} h, \ \operatorname{Im} f \simeq \operatorname{Ker} k.$$

Proof. Of course, it is enough to prove the first isomorphism. For $Z \in \mathscr{C}$, one has (see Diagram 2.2.6)

$$\operatorname{Hom}_{\mathscr{C}}(\operatorname{Coim} f, Z) = \{u\colon X \to Z; u \circ p_1 = u \circ p_2\},$$

where $p_1, p_2 \colon X \times_Y X \to X$ are the two projections. Since $X \times_Y X$ is the kernel of $(f \circ p_1, f \circ p_2) \colon X \times X \rightrightarrows Y$, one also have

$$\operatorname{Hom}_{\mathscr{C}}(\operatorname{Coim} f, Z) = \{u \colon X \to Z; u \circ v_1 = u \circ v_2 \text{ for any } W \text{ and } (v_1, v_2) \colon W \rightrightarrows X \text{ such that } f \circ v_1 = f \circ v_2.\}$$

Equivalently,

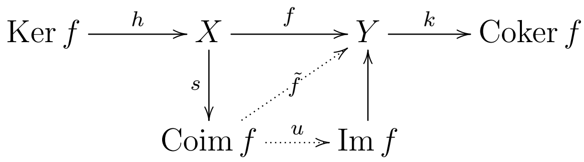
$$\operatorname{Hom}_{\mathscr{C}}(\operatorname{Coim} f, Z) = \{u \colon X \to Z; u \circ v = 0 \text{ for any } W \text{ and } v \colon W \to X \text{ such that} f \circ v = 0.\}$$

Since such a v factorizes uniquely through h, we get

$$\begin{aligned}\operatorname{Hom}_{\mathscr{C}}(\operatorname{Coim} f, Z) &= \{u \colon X \to Z; u \circ h = 0\} \\ &\simeq \operatorname{Hom}_{\mathscr{C}}(\operatorname{Coker} h, Z).\end{aligned}$$

Since this isomorphism is functorial in Z (this point being left to the reader), we get the result by the Yoneda lemma. □

Consider the diagram:



Since $f \circ h = 0$, f factors uniquely through $\operatorname{Coim} f$, which defines $\tilde{f}$ (see Diagram 2.2.6) and thus $k \circ f$ factors through $k \circ \tilde{f}$. Since $k \circ f = k \circ \tilde{f} \circ s = 0$ and s is an epimorphism, we get that $k \circ \tilde{f} = 0$. Hence $\tilde{f}$ factors through $\operatorname{Ker} k = \operatorname{Im} f$, which defines u (see Diagram 2.2.5). We have thus constructed a canonical morphism:

$$\operatorname{Coim} f \xrightarrow{u} \operatorname{Im} f. \tag{5.1.1}$$

Examples 5.1.2. (i) For a ring A and a morphism f in $\operatorname{Mod}(A)$, (5.1.1) is an isomorphism.
(ii) The category **Ban** admits kernels and cokernels. If $f \colon X \to Y$ is a morphism of Banach spaces, define $\operatorname{Ker} f = f^{-1}(0)$ and $\operatorname{Coker} f = Y/\overline{\operatorname{Im} f}$ where $\overline{\operatorname{Im} f}$ denotes the closure of the space $\operatorname{Im} f$. It is well-known that there exist continuous linear maps $f \colon X \to Y$ which are injective, with dense and non closed image. For such an f, $\operatorname{Ker} f = \operatorname{Coker} f = 0$ although f is not an isomorphism. Thus $\operatorname{Coim} f \simeq X$ and $\operatorname{Im} f \simeq Y$. Hence, the morphism (5.1.1) is not an isomorphism.
(iii) Let A be a ring, I an ideal which is not finitely generated and let $M = A/I$. Then the natural morphism $A \to M$ in $\operatorname{Mod}^{\mathrm{f}}(A)$ has no kernel.

Definition 5.1.3. Let $\mathscr{C}$ be an additive category. One says that $\mathscr{C}$ is abelian if:

(i) any morphism in $\mathscr{C}$ admits a kernel and a cokernel,

(ii) for any morphism f in $\mathscr{C}$, the natural morphism $\operatorname{Coim} f \to \operatorname{Im} f$ is an isomorphism.

Examples 5.1.4. (i) If A is a ring, $\mathrm{Mod}(A)$ is an abelian category. If A is noetherian, then $\mathrm{Mod}^{\mathrm{f}}(A)$ is abelian.
(ii) The category **Ban** admits kernels and cokernels but is not abelian. (See Examples 5.1.2 (ii).)
(iii) If $\mathscr{C}$ is abelian, then $\mathscr{C}^{\mathrm{op}}$ is abelian.

Proposition 5.1.5. *Let I be category and let $\mathscr{C}$ be an abelian category. Then the category $\mathrm{Fct}(I, \mathscr{C})$ of functors from I to $\mathscr{C}$ is abelian.*

Proof. (i) Let $F, G \colon I \to \mathscr{C}$ be two functors and $\varphi \colon F \to G$ a morphism of functors. Let us define a new functor H as follows. For $i \in I$, set $H(i) = \mathrm{Ker}(F(i) \to G(i))$. Let $s \colon i \to j$ be a morphism in I. In order to define the morphism $H(s) \colon H(i) \to H(j)$, consider the diagram

$$\begin{array}{ccccc} H(i) & \xrightarrow{h_i} & F(i) & \xrightarrow{\varphi(i)} & G(i) \\ {\scriptstyle H(s)}\downarrow & & {\scriptstyle F(s)}\downarrow & & \downarrow{\scriptstyle G(s)} \\ H(j) & \xrightarrow{h_j} & F(j) & \xrightarrow{\varphi(i)} & G(j). \end{array}$$

Since $\varphi(j) \circ F(s) \circ h_i = 0$, the morphism $F(s) \circ h_i$ factorizes uniquely through $H(j)$. This gives $H(s)$. One checks immediately that for a morphism $t \colon j \to k$ in I, one has $H(t) \circ H(s) = H(t \circ s)$. Therefore H is a functor and one also easily cheks that H is a kernel of the morphism of functors φ.
(ii) One defines similarly the functor $\mathrm{Coim}\,\varphi$. Since, for each $i \in I$, the natural morphism $\mathrm{Coim}\,\varphi(i) \to \mathrm{Im}\,\varphi(i)$ is an isomorphism, one deduces that the natural morphism of functors $\mathrm{Coim}\,\varphi \to \mathrm{Im}\,\varphi$ is an isomorphism. □

Corollary 5.1.6. *If $\mathscr{C}$ is abelian, then the categories of complexes $\mathrm{C}^*(\mathscr{C})$ $(* = \mathrm{ub}, \mathrm{b}, +, -)$ are abelian.*

Proof. It follows from Proposition 5.1.5 that the category $\mathrm{Diff}(\mathscr{C})$ of differential objects of $\mathscr{C}$ is abelian. One checks immediately that if $f^\bullet \colon X^\bullet \to Y^\bullet$ is a morphism of complexes, its kernel in the category $\mathrm{Diff}(\mathscr{C})$ is a complex and is a kernel in the category $\mathrm{C}(\mathscr{C})$, and similarly with cokernels. □

For example, if $f \colon X \to Y$ is a morphism in $\mathrm{C}(\mathscr{C})$, the complex Z defined by $Z^n = \mathrm{Ker}(f^n \colon X^n \to Y^n)$, with differential induced by those of X, will be a kernel for f, and similarly for $\mathrm{Coker}\, f$.

Note the following results.

- An abelian category admits finite limits and finite colimits. (Indeed, an abelian category admits an initial object, a terminal object, finite products and finite coproducts and kernels and cokernels.)

- In an abelian category, a morphism f is a monomorphism (resp. an epimorphism) if and only if $\mathrm{Ker}\, f \simeq 0$ (resp. $\mathrm{Coker}\, f \simeq 0$) (see Exercise 2.11). Moreover, a morphism $f \colon X \to Y$ is an isomorphism as soon as $\mathrm{Ker}\, f \simeq 0$ and $\mathrm{Coker}\, f \simeq 0$. Indeed, in such a case, $X \xrightarrow{\sim} \mathrm{Coim}\, f$ and $\mathrm{Im}\, f \xrightarrow{\sim} Y$.

Unless otherwise specified, we assume until the end of this chapter that $\mathscr{C}$ is abelian.

Consider a complex $X' \xrightarrow{f} X \xrightarrow{g} X''$ (hence, $g \circ f = 0$). It defines a morphism $\mathrm{Coim}\, f \to \mathrm{Ker}\, g$, hence, $\mathscr{C}$ being abelian, a morphism $\mathrm{Im}\, f \to \mathrm{Ker}\, g$.

Definition 5.1.7. (i) One says that a complex $X' \xrightarrow{f} X \xrightarrow{g} X''$ is exact if $\operatorname{Im} f \xrightarrow{\sim} \operatorname{Ker} g$. (Note that this condition is equivalent to $\operatorname{Coker} f \xrightarrow{\sim} \operatorname{Im} g$.)

(ii) More generally, a sequence of morphisms $X^p \xrightarrow{d^p} \cdots \to X^n$ with $d^{i+1} \circ d^i = 0$ for all $i \in [p, n-1]$ is exact if $\operatorname{Im} d^i \xrightarrow{\sim} \operatorname{Ker} d^{i+1}$ for all $i \in [p, n-1]$.

(iii) A short exact sequence is an exact sequence $0 \to X' \to X \to X'' \to 0$

Any morphism $f\colon X \to Y$ may be decomposed into short exact sequences:

$$0 \to \operatorname{Ker} f \to X \to \operatorname{Coim} f \to 0,$$
$$0 \to \operatorname{Im} f \to Y \to \operatorname{Coker} f \to 0,$$

with $\operatorname{Coim} f \simeq \operatorname{Im} f$.

Proposition 5.1.8. *Let*

$$0 \to X' \xrightarrow{f} X \xrightarrow{g} X'' \to 0 \tag{5.1.2}$$

be a short exact sequence in $\mathscr{C}$. Then the conditions (a) *to* (e) *are equivalent.*

(a) *there exists $h\colon X'' \to X$ such that $g \circ h = \mathrm{id}_{X''}$.*

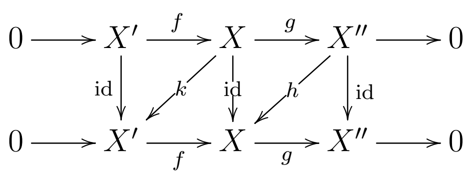
(b) *there exists $k\colon X \to X'$ such that $k \circ f = \mathrm{id}_{X'}$.*

(c) *there exists $\varphi = (k, g)$ and $\psi = \begin{pmatrix} f \\ h \end{pmatrix}$ such that $X \xrightarrow{\varphi} X' \oplus X''$ and $X' \oplus X'' \xrightarrow{\psi} X$ are isomorphisms inverse to each other.*

(d) *The complex* (5.1.2) *is homotopic to* 0.

(e) *The complex* (5.1.2) *is isomorphic to the complex* $0 \to X' \to X' \oplus X'' \to X'' \to 0$.

Proof. (a) ⇒ (c). Since $g = g \circ h \circ g$, we get $g \circ (\mathrm{id}_X - h \circ g) = 0$, which implies that $\mathrm{id}_X - h \circ g$ factors through $\operatorname{Ker} g$, that is, through X'. Hence, there exists $k\colon X \to X'$ such that $\mathrm{id}_X - h \circ g = f \circ k$.
(b) ⇒ (c) follows by reversing the arrows.
(c) ⇒ (a). Since $g \circ f = 0$, we find $g = g \circ h \circ g$, that is $(g \circ h - \mathrm{id}_{X''}) \circ g = 0$. Since g is an epimorphism, this implies $g \circ h - \mathrm{id}_{X''} = 0$.
(c) ⇒ (b) follows by reversing the arrows.
(d) By definition, the complex (5.1.2) is homotopic to zero if and only if there exists a diagram



such that $\mathrm{id}_{X'} = k \circ f$, $\mathrm{id}_{X''} = g \circ h$ and $\mathrm{id}_X = h \circ g + f \circ k$.
(e) is obvious by (c). ∎

Definition 5.1.9. In the above situation, one says that the exact sequence splits.

Note that an additive functor of abelian categories sends split exact sequences to split exact sequences.

If $\mathscr{C} = \mathrm{Mod}(\mathbf{k})$ and $\mathbf{k}$ is a field, then all exact sequences split, but this is not the case in general.

Example 5.1.10. The exact sequence of $\mathbb{Z}$-modules

$$0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$$

does not split.

Definition 5.1.11. Let $\mathscr{C}$ be an abelian category and $\mathscr{J}$ a full additive subcategory. Denote by $\mathscr{J}'$ the full subcategory of $\mathscr{C}$ consisting of objects isomorphic to some object of $\mathscr{J}$.

(a) One says that $\mathscr{J}$ is closed (one also says "stable") by kernels if for any morphism $u\colon X \to Y$ in $\mathscr{J}$ the kernel of u in $\mathscr{C}$ belongs to $\mathscr{J}'$. One defines similarly the notions of being closed by cokernels.

(b) One says that $\mathscr{J}$ is closed by extension if for any exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathscr{C}$, with X', X'' in $\mathscr{J}$, we have $X \in \mathscr{J}'$.

(c) One says that $\mathscr{J}$ is thick in $\mathscr{C}$ if it is closed by kernels, cokernels and extensions.

5.2 Exact functors

Recall Definition 2.6.8. Hence, an additive functor of abelian categories $F\colon \mathscr{C} \to \mathscr{C}'$ is left exact if it commutes with finite limits, right exact if it commutes with finite colimits and exact if it is both left and right exact.

Lemma 5.2.1. *Consider an additive functor $F\colon \mathscr{C} \to \mathscr{C}'$.*

(a) *The conditions below are equivalent:*

  (i) *F is left exact,*

  (ii) *F commutes with kernels, that is, for any morphism $f\colon X \to Y$, $F(\mathrm{Ker}(f)) \xrightarrow{\sim} \mathrm{Ker}(F(f))$,*

  (iii) *for any exact sequence $0 \to X' \to X \to X''$ in $\mathscr{C}$, the sequence $0 \to F(X') \to F(X) \to F(X'')$ is exact in $\mathscr{C}'$,*

  (iv) *for any exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathscr{C}$, the sequence $0 \to F(X') \to F(X) \to F(X'')$ is exact in $\mathscr{C}'$.*

(b) *The conditions below are equivalent:*

  (i) *F is exact,*

  (ii) *for any exact sequence $X' \to X \to X''$ in $\mathscr{C}$, the sequence $F(X') \to F(X) \to F(X'')$ is exact in $\mathscr{C}'$,*

  (iii) *for any exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathscr{C}$, the sequence $0 \to F(X') \to F(X) \to F(X'') \to 0$ is exact in $\mathscr{C}'$.*

There is a similar result to (a) for right exact functors.

Proof. Since F is additive, it commutes with terminal objects and products of two objects. Hence, by Proposition 2.3.9, F is left exact if and only if it commutes with kernels.
The proof of the other assertions are left as an exercise. □

Proposition 5.2.2. (i) *The functor* $\mathrm{Hom}_{\mathscr{C}}\colon \mathscr{C}^{\mathrm{op}}\times\mathscr{C} \to \mathrm{Mod}(\mathbb{Z})$ *is left exact with respect to each of its arguments.*

(ii) *If a functor* $F\colon \mathscr{C} \to \mathscr{C}'$ *admits a left (resp. right) adjoint then* F *is left (resp. right) exact.*

(iii) *Let* I *be a small category. If* $\mathscr{C}$ *admits limits indexed by* I*, then the functor* $\lim\colon \mathrm{Fct}(I^{\mathrm{op}},\mathscr{C}) \to \mathscr{C}$ *is left exact. Similarly, if* $\mathscr{C}$ *admits colimits indexed by* I*, then the functor* $\mathrm{colim}\colon \mathrm{Fct}(I,\mathscr{C}) \to \mathscr{C}$ *is right exact.*

(iv) *Let* A *be a ring and let* I *be a small set. The two functors* $\prod_{i\in I}$ *and* $\bigoplus_{i\in I}$ *from* $\mathrm{Fct}(I,\mathrm{Mod}(A))$ *to* $\mathrm{Mod}(A)$ *are exact.*

(v) *Let* A *be a ring and let* I *be a small directed category. The functor* colim *from* $\mathrm{Fct}(I,\mathrm{Mod}(A))$ *to* $\mathrm{Mod}(A)$ *is exact.*

Proof. (i) follows from (2.3.2) and (2.3.3).
(ii) Apply Proposition 2.5.5.
(iii) Apply Proposition 2.5.1.
(iv) is left as an exercise (see Exercise 5.1).
(v) follows from Corollary 2.6.7. □

Example 5.2.3. Let A be a ring and let N be a right A-module. Since the functor $N\otimes_A \bullet$ admits a right adjoint, it is right exact. Let us show that the functors $\mathrm{Hom}_A(\,\bullet\,,\,\bullet\,)$ and $N\otimes_A\bullet$ are not exact in general. In the sequel, we choose $A=\mathbf{k}[x]$, with $\mathbf{k}$ a field, and we consider the exact sequence of A-modules:

$$0 \to A \xrightarrow{\cdot x} A \to A/Ax \to 0, \tag{5.2.1}$$

where $\cdot x$ means multiplication by x.
(i) Apply the functor $\mathrm{Hom}_A(\,\bullet\,,A)$ to the exact sequence (5.2.1). We get the sequence:

$$0 \to \mathrm{Hom}_A(A/Ax,A) \to A \xrightarrow{x\cdot} A \to 0$$

which is not exact since $x\cdot$ is not surjective. On the other hand, since $x\cdot$ is injective and $\mathrm{Hom}_A(\,\bullet\,,A)$ is left exact, we find that $\mathrm{Hom}_A(A/Ax,A)=0$.
(ii) Apply $\mathrm{Hom}_A(A/Ax,\,\bullet\,)$ to the exact sequence (5.2.1). We get the sequence:

$$0 \to \mathrm{Hom}_A(A/Ax,A) \to \mathrm{Hom}_A(A/Ax,A) \to \mathrm{Hom}_A(A/Ax,A/Ax) \to 0.$$

Since $\mathrm{Hom}_A(A/Ax,A)=0$ and $\mathrm{Hom}_A(A/Ax,A/Ax)\neq 0$, this sequence is not exact.
(iii) Apply $\bullet\otimes_A A/Ax$ to the exact sequence (5.2.1). We get the sequence:

$$0 \to A/Ax \xrightarrow{x\cdot} A/Ax \to A/xA\otimes_A A/Ax \to 0.$$

Multiplication by x is 0 on A/Ax. Hence this sequence is the same as:

$$0 \to A/Ax \xrightarrow{0} A/Ax \to (A/Ax) \otimes_A (A/Ax) \to 0$$

which shows that $(A/Ax) \otimes_A (A/Ax) \simeq A/Ax$ and moreover that this sequence is not exact.

(iv) Notice that the functor $\mathrm{Hom}_A(\,\bullet\,, A)$ being additive, it sends split exact sequences to split exact sequences. This shows that (5.2.1) does not split.

Example 5.2.4. We shall show that the functor $\lim : \mathrm{Fct}(I^{\mathrm{op}}, \mathrm{Mod}(\mathbf{k})) \to \mathrm{Mod}(\mathbf{k})$ is not right exact in general, even if $\mathbf{k}$ is a field.

Consider as above the $\mathbf{k}$-algebra $A := \mathbf{k}[x]$ over a field $\mathbf{k}$. Denote by $I = A \cdot x$ the ideal generated by x. Notice that $A/I^{n+1} \simeq \mathbf{k}[x]^{\leq n}$, where $\mathbf{k}[x]^{\leq n}$ denotes the $\mathbf{k}$-vector space consisting of polynomials of degree $\leq n$. For $p \leq n$ denote by $v_{pn} \colon A/I^n \twoheadrightarrow A/I^p$ the natural epimorphisms. They define a projective system of A-modules. One checks easily that

$$\lim_n A/I^n \simeq \mathbf{k}[[x]],$$

the ring of formal series with coefficients in $\mathbf{k}$. On the other hand, for $p \leq n$ the monomorphisms $I^n \rightarrowtail I^p$ define a projective system of A-modules and one has

$$\lim_n I^n \simeq 0.$$

Now consider the projective system of exact sequences of A-modules

$$0 \to I^n \to A \to A/I^n \to 0.$$

By taking the limit of these exact sequences one gets the sequence $0 \to 0 \to \mathbf{k}[x] \to \mathbf{k}[[x]] \to 0$ which is no more exact, neither in the category $\mathrm{Mod}(A)$ nor in the category $\mathrm{Mod}(\mathbf{k})$.

5.3 Injective and projective objects

Definition 5.3.1. Let $\mathscr{C}$ be an abelian category.

(i) An object I of $\mathscr{C}$ is injective if the functor $\mathrm{Hom}_{\mathscr{C}}(\,\bullet\,, I)$ is exact.

(ii) One says that $\mathscr{C}$ has enough injectives if for any $X \in \mathscr{C}$ there exists a monomorphism $X \rightarrowtail I$ with I injective.

(iii) An object P is projective in $\mathscr{C}$ if it is injective in $\mathscr{C}^{\mathrm{op}}$, *i.e.,* if the functor $\mathrm{Hom}_{\mathscr{C}}(P, \bullet\,)$ is exact.

(iv) One says that $\mathscr{C}$ has enough projectives if for any $X \in \mathscr{C}$ there exists an epimorphism $P \twoheadrightarrow X$ with P projective.

Proposition 5.3.2. *The object $I \in \mathscr{C}$ is injective if and only if, for any diagram in $\mathscr{C}$ in which the row is exact:*

$$\begin{array}{ccccc} 0 & \longrightarrow & X' & \xrightarrow{f} & X \\ & & {\scriptstyle k}\downarrow & \swarrow{\scriptstyle h} & \\ & & I & & \end{array}$$

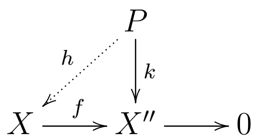
the dotted arrow may be completed, making the diagram commutative.

Proof. (i) Assume that I is injective and let X'' denote the cokernel of the morphism $X' \to X$. Applying the functor $\mathrm{Hom}_{\mathscr{C}}(\,\bullet\,, I)$ to the sequence $0 \to X' \to X \to X''$, one gets the exact sequence:

$$\mathrm{Hom}_{\mathscr{C}}(X'', I) \to \mathrm{Hom}_{\mathscr{C}}(X, I) \xrightarrow{\circ f} \mathrm{Hom}_{\mathscr{C}}(X', I) \to 0.$$

Thus there exists $h\colon X \to I$ such that $h \circ f = k$.
(ii) Conversely, consider an exact sequence $0 \to X' \xrightarrow{f} X \xrightarrow{g} X'' \to 0$. Then the sequence $0 \to \mathrm{Hom}_{\mathscr{C}}(X'', I) \xrightarrow{\circ h} \mathrm{Hom}_{\mathscr{C}}(X, I) \xrightarrow{\circ f} \mathrm{Hom}_{\mathscr{C}}(X', I) \to 0$ is exact by the hypothesis. Therefore, the functor $\mathrm{Hom}_{\mathscr{C}}(\,\bullet\,, I)$ is exact by Lemma 5.2.1. □

By reversing the arrows, we get that P is projective if and only if for any diagram in which the row is exact:



the dotted arrow may be completed, making the diagram commutative.

Lemma 5.3.3. *Let $0 \to X' \xrightarrow{f} X \xrightarrow{g} X'' \to 0$ be an exact sequence in $\mathscr{C}$, and assume that X' is injective. Then the sequence splits.*

Proof. Applying the preceding result with $k = \mathrm{id}_{X'}$, we find $h\colon X \to X'$ such that $k \circ f = \mathrm{id}_{X'}$. Then apply Proposition 5.1.8. □

It follows that if $F\colon \mathscr{C} \to \mathscr{C}'$ is an additive functor of abelian categories, and the hypotheses of the lemma are satisfied, then the sequence $0 \to F(X') \to F(X) \to F(X'') \to 0$ splits and in particular is exact.

Lemma 5.3.4. *Let X', X'' belong to $\mathscr{C}$. Then $X' \oplus X''$ is injective if and only if X' and X'' are injective.*

Proof. It is enough to remark that for two additive functors of abelian categories F and G, the functor $F \oplus G\colon X \mapsto F(X) \oplus G(X)$ is exact if and only if the functors F and G are exact. □

Applying Lemmas 5.3.3 and 5.3.4, we get:

Proposition 5.3.5. *Let $0 \to X' \to X \to X'' \to 0$ be an exact sequence in $\mathscr{C}$ and assume X' and X are injective. Then X'' is injective.*

Example 5.3.6. (i) Let A be a ring. An A-module M is free if it is isomorphic to a direct sum of copies of A, that is, $M \simeq A^{\oplus I}$ for some small set I. It follows from (2.1.4) and Proposition 5.2.2 (iv) that free modules are projective.

Let $M \in \mathrm{Mod}(A)$. For $m \in M$, denote by A_m a copy of A and denote by $1_m \in A_m$ the unit. Define the linear map

$$\psi\colon \bigoplus_{m \in M} A_m \to M$$

by setting $\psi(1_m) = m$ and extending by linearity. This map is clearly surjective. Since the left A-module $\bigoplus_{m\in M} A_m$ is free, it is projective. This shows that the category $\mathrm{Mod}(A)$ has enough projectives.

More generally, if there exists an A-module N such that $M \oplus N$ is free then M is projective (see Exercise 5.3).

One can prove that $\mathrm{Mod}(A)$ has enough injectives (see Exercise 5.4).

(ii) If $\mathbf{k}$ is a field, then any object of $\mathrm{Mod}(\mathbf{k})$ is both injective and projective.

(iii) Let A be a $\mathbf{k}$-algebra and let $M \in \mathrm{Mod}(A^{\mathrm{op}})$. One says that M is flat if the functor $M \otimes_A \bullet \colon \mathrm{Mod}(A) \to \mathrm{Mod}(\mathbf{k})$ is exact. Projective modules are flat (see Exercise 5.3).

Although Proposition 5.3.7 below is a particular case of Theorem 7.2.2, we include it for pedagogical reasons.

For a category $\mathscr{C}$, denote by $\mathscr{I}_{\mathscr{C}}$ the full additive subcategory of injective objects.

Proposition 5.3.7. *Let $\mathscr{C}$ be an abelian category which admits enough injectives. Then, for any $X \in \mathscr{C}$, there exists an exact sequence*

$$0 \to X \to I_X^0 \to \cdots \to I_X^n \to \cdots \tag{5.3.1}$$

with $I_X^n \in \mathscr{I}_{\mathscr{C}}$ for all $n \geq 0$.

Proof. We proceed by induction. Assume to have constructed:

$$0 \to X \to I_X^0 \to \cdots \to I_X^n.$$

For $n = 0$ this is the hypothesis. Set $B^n = \mathrm{Coker}(I_X^{n-1} \to I_X^n)$ (with $I_X^{-1} = X$). Then $I_X^{n-1} \to I_X^n \to B^n \to 0$ is exact. Embed B^n in an injective object: $0 \to B^n \to I_X^{n+1}$. Then $I_X^{n-1} \to I_X^n \to I_X^{n+1}$ is exact, and the induction proceeds. □

The sequence

$$I_X^\bullet := 0 \to I_X^0 \to \cdots \to I_X^n \to \cdots \tag{5.3.2}$$

is called an injective resolution of X.

Remark 5.3.8. Note that, identifying X and $I_X^\bullet$ with objects of $\mathrm{C}^+(\mathscr{C})$, the morphism $X \to I_X^\bullet$ in $\mathrm{C}^+(\mathscr{C})$ induces an isomorphism in the cohomology object, that is, is a quasi-isomorphism, following the terminology of Definition 5.5.4 below.

Of course, there is a similar result for projective resolutions. If for any $X \in \mathscr{C}$ there is an exact sequence $Y \to X \to 0$ with Y projective, then one can construct a projective resolution of X, that is, a quasi-isomorphism $P_X^\bullet \to X$, where the P_X^n's are projective.

5.4 Generators and Grothendieck categories

In this section it is essential to fix a universe $\mathscr{U}$. Hence, a category means a $\mathscr{U}$-category and small means $\mathscr{U}$-small.

Definition 5.4.1. Let $\mathscr{C}$ be a category. A system of generators in $\mathscr{C}$ is a family of objects $\{G_i\}_{i\in I}$ of $\mathscr{C}$ such that I is small and a morphism $f\colon X \to Y$ in $\mathscr{C}$ is an isomorphism as soon as $\mathrm{Hom}_{\mathscr{C}}(G_i, X) \to \mathrm{Hom}_{\mathscr{C}}(G_i, Y)$ is an isomorphism for all $i \in I$.

If the family contains a single element, say G, one says that G is a generator.

If $\{G_i\}_{i\in I}$ is a system of generators, then the functor $\prod_{i\in I} \mathrm{Hom}_{\mathscr{C}}(G_i, \bullet)\colon \mathscr{C} \to$ **Set** is conservative. If $\mathscr{C}$ is additive, these two conditions are equivalent [1]. Moreover, if $\mathscr{C}$ is additive, admits small coproducts and a system of generators as above, then it admits a generator, namely the object $\bigoplus_{i\in I} G_i$.

Lemma 5.4.2. *Let $\mathscr{C}$ be an abelian category which admits small coproducts and a generator G.*

(a) *The functor* $\mathrm{Hom}_{\mathscr{C}}(G, \bullet)$ *is faithful.*

(b) *For any $X \in \mathscr{C}$, there exist a small set I and an epimorphism $G^{\oplus I} \twoheadrightarrow X$.*

Proof. In this proof, we write $\mathrm{Hom}\,(Y, Z)$ instead of $\mathrm{Hom}_{\mathscr{C}}(Y, Z)$.
(a) The functor $\mathrm{Hom}\,(G, \bullet)$ is left exact and conservative by the hypothesis. Then use Exercise 5.13.
(b) There is a natural isomorphism (see Exercise 5.12):

$$\mathrm{Hom}_{\mathbf{Set}}(\mathrm{Hom}\,(G, X), \mathrm{Hom}\,(G, X)) \simeq \mathrm{Hom}\,(G^{\oplus \mathrm{Hom}\,(G,X)}, X).$$

The identity morphism on the left-hand side defines the morphism $G^{\oplus \mathrm{Hom}\,(G,X)} \to X$. This morphism defines the morphism

$$\mathrm{Hom}\,(G, G^{\oplus \mathrm{Hom}\,(G,X)}) \to \mathrm{Hom}\,(G, X).$$

This last morphism being obviously surjective, the result follows from Exercise 5.14. □

Definition 5.4.3. A Grothendieck category is an abelian category which admits small limits and small colimits, a generator and such that directed small colimits are exact.

We shall not give the proof of the important Grothendieck's theorem below, referring to [KS06, Th. 9.6.2]. See [Gro57] for the original proof.

Theorem 5.4.4. *Let $\mathscr{C}$ be an abelian Grothendieck category. Then $\mathscr{C}$ admits enough injectives.*

5.5 Complexes in abelian categories

One still denotes by $\mathscr{C}$ an abelian category.

Solving linear equations

The aim of this subsection is to illustrate and motivate the constructions which will appear further. In this subsection, we work in the category $\mathrm{Mod}(A)$ for a $\mathbf{k}$-algebra A. Recall that the category $\mathrm{Mod}(A)$ admits enough projectives.

[1] There was a mistake in [KS06, Def. 5.2.1], see the Errata on the webpage of the author PS.

Suppose that one is interested in studying a system of linear equations

$$\sum_{j=1}^{N_0} p_{ij}u_j = v_i, \quad (i = 1, \dots, N_1) \tag{5.5.1}$$

where the p_{ij}'s belong to the ring A and u_j, v_i belong to some left A-module S. Using matrix notations, one can write equations (5.5.1) as

$$P_0 u = v \tag{5.5.2}$$

where P_0 is the matrix (p_{ij}) with N_1 rows and N_0 columns, defining the A-linear map $P_0\cdot : S^{N_0} \to S^{N_1}$. Now consider the right A-linear map

$$\cdot P_0 : A^{N_1} \to A^{N_0}, \tag{5.5.3}$$

where $\cdot P_0$ operates on the right and the elements of A^{N_0} and A^{N_1} are written as rows. Let $(e_1, \dots, e_{N_0})$ and $(f_1, \dots, f_{N_1})$ denote the canonical basis of A^{N_0} and A^{N_1}, respectively. One gets:

$$f_i \cdot P_0 = \sum_{j=1}^{N_0} p_{ij}e_j, \quad (i = 1, \dots, N_1). \tag{5.5.4}$$

Hence $\operatorname{Im} P_0$ is generated by the elements $\sum_{j=1}^{N_0} p_{ij}e_j$ for $i = 1, \dots, N_1$. Denote by M the quotient module $A^{N_0}/A^{N_1} \cdot P_0$ and by $\psi : A^{N_0} \to M$ the natural A-linear map. Let $(u_1, \dots, u_{N_0})$ denote the images by ψ of $(e_1, \dots, e_{N_0})$. Then M is a left A-module with generators $(u_1, \dots, u_{N_0})$ and relations $\sum_{j=1}^{N_0} p_{ij}u_j = 0$ for $i = 1, \dots, N_1$. By construction, we have an exact sequence of left A-modules:

$$A^{N_1} \xrightarrow{\cdot P_0} A^{N_0} \xrightarrow{\psi} M \to 0. \tag{5.5.5}$$

Applying the left exact functor $\operatorname{Hom}_A(\bullet, S)$ to this sequence, we find the exact sequence of $\mathbf{k}$-modules:

$$0 \to \operatorname{Hom}_A(M, S) \to S^{N_0} \xrightarrow{P_0\cdot} S^{N_1} \tag{5.5.6}$$

(where $P_0\cdot$ operates on the left). Hence, the $\mathbf{k}$-module of solutions of the homogeneous equation associated to (5.5.1) is described by $\operatorname{Hom}_A(M, S)$.

Assume now that A is left Noetherian, that is, any submodule of a free A-module of finite rank is of finite type. In this case, arguing as in the proof of Proposition 5.3.7, we construct an exact sequence

$$\cdots \to A^{N_2} \xrightarrow{\cdot P_1} A^{N_1} \xrightarrow{\cdot P_0} A^{N_0} \xrightarrow{\psi} M \to 0.$$

In other words, we have a projective resolution $L^\bullet \to M$ of M by finite free left A-modules:

$$L^\bullet : \cdots \to L^n \to L^{n-1} \to \cdots \to L^0 \to 0.$$

Applying the left exact functor $\operatorname{Hom}_A(\bullet, S)$ to $L^\bullet$, we find the complex of $\mathbf{k}$-modules:

$$0 \to S^{N_0} \xrightarrow{P_0\cdot} S^{N_1} \xrightarrow{P_1\cdot} S^{N_2} \to \cdots . \tag{5.5.7}$$

Then (see (5.5.8) below for the definition of H^0 and H^1):

$$\begin{cases} H^0(\mathrm{Hom}_A(L^\bullet, S)) \simeq \mathrm{Ker}\, P_0, \\ H^1(\mathrm{Hom}_A(L^\bullet, S)) \simeq \mathrm{Ker}(P_1)/\mathrm{Im}(P_0). \end{cases}$$

Hence, a necessary condition to solve the equation $P_0 u = v$ is that $P_1 v = 0$ and this necessary condition is sufficient if $H^1(\mathrm{Hom}_A(L^\bullet, S)) \simeq 0$. As we shall see in § 7.3, the cohomology groups $H^j(\mathrm{Hom}_A(L^\bullet, S))$ do not depend, up to isomorphisms, of the choice of the projective resolution $L^\bullet$ of M and are denoted by $\mathrm{Ext}^j_A(M, S)$.

Cohomology

Recall that the categories $\mathrm{C}^*(\mathscr{C})$ are abelian for $* = \mathrm{ub}, +, -, \mathrm{b}$.

Let $X \in \mathrm{C}(\mathscr{C})$:

$$X := \cdots \to X^{n-1} \to X^n \to X^{n+1} \to \cdots .$$

One defines the following objects of $\mathscr{C}$:

(5.5.8)
$$\begin{aligned} Z^n(X) &:= \mathrm{Ker}\, d_X^n, \\ B^n(X) &:= \mathrm{Im}\, d_X^{n-1}, \\ H^n(X) &:= Z^n(X)/B^n(X) \quad (:= \mathrm{Coker}(B^n(X) \to Z^n(X))). \end{aligned}$$

One calls $H^n(X)$ the n-th cohomology object of X. If $f\colon X \to Y$ is a morphism in $\mathrm{C}(\mathscr{C})$, then it induces morphisms $Z^n(X) \to Z^n(Y)$ and $B^n(X) \to B^n(Y)$, thus a morphism $H^n(f)\colon H^n(X) \to H^n(Y)$. Clearly, $H^n(X \oplus Y) \simeq H^n(X) \oplus H^n(Y)$. Hence we have obtained an additive functor:

$$H^n(\,\bullet\,) : \mathrm{C}(\mathscr{C}) \to \mathscr{C}.$$

Notice that $H^n(X) = H^0(X[n])$.

There are exact sequences

$$X^{n-1} \xrightarrow{d^{n-1}} \mathrm{Ker}\, d_X^n \to H^n(X) \to 0,$$
$$0 \to H^n(X) \to \mathrm{Coker}\, d_X^{n-1} \xrightarrow{d^n} X^{n+1}.$$

The next result is easily checked.

Lemma 5.5.1. *For $n \in \mathbb{Z}$, the sequence below is exact:*

(5.5.9)
$$0 \to H^n(X) \to \mathrm{Coker}(d_X^{n-1}) \xrightarrow{d_X^n} \mathrm{Ker}\, d_X^{n+1} \to H^{n+1}(X) \to 0.$$

One defines the truncation functors:

(5.5.10)
$$\begin{aligned} \tau^{\leq n}, \widetilde{\tau}^{\leq n} &\colon C(\mathscr{C}) \to C^-(\mathscr{C}) \\ \tau^{\geq n}, \widetilde{\tau}^{\geq n} &\colon C(\mathscr{C}) \to C^+(\mathscr{C}) \end{aligned}$$

as follows. Let $X := \cdots \to X^{n-1} \to X^n \to X^{n+1} \to \cdots$. One sets:

$$\begin{aligned} \tau^{\leq n} X &:= \cdots \to X^{n-2} \to X^{n-1} \to \mathrm{Ker}\, d_X^n \to 0 \to \cdots \\ \widetilde{\tau}^{\leq n} X &:= \cdots \to X^{n-1} \to X^n \to \mathrm{Im}\, d_X^n \to 0 \to \cdots \\ \tau^{\geq n} X &:= \cdots \to 0 \to \mathrm{Coker}\, d_X^{n-1} \to X^{n+1} \to X^{n+2} \to \cdots \\ \widetilde{\tau}^{\geq n} X &:= \cdots \to 0 \to \mathrm{Im}\, d_X^{n-1} \to X^n \to X^{n+1} \to \cdots \end{aligned}$$

There is a chain of morphisms in $\mathrm{C}(\mathscr{C})$:

$$\tau^{\leq n}X \to \widetilde{\tau}^{\leq n}X \to X \to \widetilde{\tau}^{\geq n}X \to \tau^{\geq n}X,$$

and there are exact sequences in $\mathrm{C}(\mathscr{C})$:

$$\begin{cases} 0 \to \widetilde{\tau}^{\leq n-1}X \to \tau^{\leq n}X \to H^n(X)[-n] \to 0, \\ 0 \to H^n(X)[-n] \to \tau^{\geq n}X \to \widetilde{\tau}^{\geq n+1}X \to 0, \\ 0 \to \tau^{\leq n}X \to X \to \widetilde{\tau}^{\geq n+1}X \to 0, \\ 0 \to \widetilde{\tau}^{\leq n-1}X \to X \to \tau^{\geq n}X \to 0. \end{cases} \tag{5.5.11}$$

We have the isomorphisms

$$\begin{aligned} H^j(\tau^{\leq n}X) &\xrightarrow{\sim} H^j(\widetilde{\tau}^{\leq n}X) \simeq \begin{cases} H^j(X) & j \leq n, \\ 0 & j > n. \end{cases} \\ H^j(\widetilde{\tau}^{\geq n}X) &\xrightarrow{\sim} H^j(\tau^{\geq n}X) \simeq \begin{cases} H^j(X) & j \geq n, \\ 0 & j < n. \end{cases} \end{aligned} \tag{5.5.12}$$

The verification is straightforward.

Remark 5.5.2. Let $X \in \mathrm{C}(\mathscr{C})$ be as above. One also defines the *stupid trunctated* complexes at $n \in \mathbb{Z}$ as

$$\begin{aligned} \sigma^{\leq n}X &:= \cdots \to X^{n-2} \to X^{n-1} \to X^n \to 0 \to \cdots \\ \sigma^{\geq n+1}X &:= \qquad\qquad\qquad\qquad \cdots \to 0 \to X^{n+1} \to X^{n+2} \to \cdots . \end{aligned}$$

Note that there is an exact sequence in $\mathrm{C}(\mathscr{C})$

$$0 \to \sigma^{\geq n+1}X \to X \to \sigma^{\leq n}X \to 0. \tag{5.5.13}$$

Lemma 5.5.3. *Let $\mathscr{C}$ be an abelian category and let $f\colon X \to Y$ be a morphism in $\mathrm{C}(\mathscr{C})$ homotopic to zero. Then $H^n(f)\colon H^n(X) \to H^n(Y)$ is the 0 morphism.*

Proof. Let $f^n = s^{n+1} \circ d_X^n + d_Y^{n-1} \circ s^n$. Then $d_X^n = 0$ on $\operatorname{Ker} d_X^n$ and the image of d_Y^{n-1} is 0 on $\operatorname{Ker} d_Y^n / \operatorname{Im} d_Y^{n-1}$. Hence $H^n(f)\colon \operatorname{Ker} d_X^n / \operatorname{Im} d_X^{n-1} \to \operatorname{Ker} d_Y^n / \operatorname{Im} d_Y^{n-1}$ is the zero morphism. □

In view of Lemma 5.5.3, the functor $H^0\colon \mathrm{C}(\mathscr{C}) \to \mathscr{C}$ extends as a functor

$$H^0\colon \mathrm{K}(\mathscr{C}) \to \mathscr{C}.$$

One shall be aware that the additive category $\mathrm{K}(\mathscr{C})$ is not abelian in general.

Definition 5.5.4. One says that a morphism $f\colon X \to Y$ in $\mathrm{C}(\mathscr{C})$ is a quasi-isomorphism (a qis, for short) if $H^k(f)$ is an isomorphism for all $k \in \mathbb{Z}$. In such a case, one says that X and Y are quasi-isomorphic. In particular, $X \in \mathrm{C}(\mathscr{C})$ is qis to 0 if and only if the complex X is exact.

Remark 5.5.5. By Lemma 5.5.3, a complex homotopic to 0 is qis to 0, but the converse is false. In particular, the property for a complex of being homotopic to 0 is preserved when applying an additive functor, contrarily to the property of being qis to 0.

Remark 5.5.6. Consider a bounded complex $X^\bullet$ and denote by $Y^\bullet$ the complex given by $Y^j = H^j(X^\bullet), d_Y^j \equiv 0$. One has:

$$Y^\bullet = \bigoplus_i H^i(X^\bullet)[-i]. \tag{5.5.14}$$

The complexes $X^\bullet$ and $Y^\bullet$ have the same cohomology objects. In other words, $H^j(Y^\bullet) \simeq H^j(X^\bullet)$. However, in general these isomorphisms are neither induced by a morphism from $X^\bullet \to Y^\bullet$, nor by a morphism from $Y^\bullet \to X^\bullet$, and the two complexes $X^\bullet$ and $Y^\bullet$ are not quasi-isomorphic.

Long exact sequence

Lemma 5.5.7 (The "five lemma"). *Consider a commutative diagram:*

$$\begin{array}{ccccccc} X^0 & \xrightarrow{\alpha_0} & X^1 & \xrightarrow{\alpha_1} & X^2 & \xrightarrow{\alpha_2} & X^3 \\ {\scriptstyle f^0}\downarrow & & {\scriptstyle f^1}\downarrow & & {\scriptstyle f^2}\downarrow & & {\scriptstyle f^3}\downarrow \\ Y^0 & \xrightarrow[\beta_0]{} & Y^1 & \xrightarrow[\beta_1]{} & Y^2 & \xrightarrow[\beta_2]{} & Y^3 \end{array}$$

and assume that the rows are exact.

(i) *If f^0 is an epimorphism and f^1, f^3 are monomorphisms, then f^2 is a monomorphism.*

(ii) *If f^3 is a monomorphism and f^0, f^2 are epimorphisms, then f^1 is an epimorphism.*

As already mentioned in the introduction of this Chapter, there is a theorem of Fred and Mitchell [Mit60, Fre64] which asserts that we may assume that $\mathscr{C}$ is a full abelian subcategory of $\mathrm{Mod}(A)$ for some ring A, what we will do here. Hence we may choose elements in the objects of $\mathscr{C}$.

Proof. (i) Let $x_2 \in X_2$ and assume that $f^2(x_2) = 0$. Then $f^3 \circ \alpha_2(x_2) = 0$ and f^3 being a monomorphism, this implies $\alpha_2(x_2) = 0$. Since the first row is exact, there exists $x_1 \in X_1$ such that $\alpha_1(x_1) = x_2$. Set $y_1 = f^1(x_1)$. Since $\beta_1 \circ f^1(x_1) = 0$ and the second row is exact, there exists $y_0 \in Y^0$ such that $\beta_0(y_0) = f^1(x_1)$. Since f^0 is an epimorphism, there exists $x_0 \in X^0$ such that $y_0 = f^0(x_0)$. Since $f^1 \circ \alpha_0(x_0) = f^1(x_1)$ and f^1 is a monomorphism, $\alpha_0(x_0) = x_1$. Therefore, $x_2 = \alpha_1(x_1) = 0$.
(ii) is nothing but (i) in $\mathscr{C}^{\mathrm{op}}$. □

Lemma 5.5.8 (The snake lemma). *Consider the commutative diagram in $\mathscr{C}$ below with exact rows:*

$$\begin{array}{ccccccccc} & & X' & \xrightarrow{f} & X & \xrightarrow{g} & X'' & \longrightarrow & 0 \\ & & {\scriptstyle \alpha}\downarrow & & {\scriptstyle \beta}\downarrow & & {\scriptstyle \gamma}\downarrow & & \\ 0 & \longrightarrow & Y' & \xrightarrow{f'} & Y & \xrightarrow{g'} & Y'' & & \end{array}$$

Then there exists a morphism $\delta\colon \mathrm{Ker}\,\gamma \to \mathrm{Coker}\,\alpha$ giving rise to an exact sequence:

$$\mathrm{Ker}\,\alpha \to \mathrm{Ker}\,\beta \to \mathrm{Ker}\,\gamma \xrightarrow{\delta} \mathrm{Coker}\,\alpha \to \mathrm{Coker}\,\beta \to \mathrm{Coker}\,\gamma. \tag{5.5.15}$$

Proof. Here again, we shall assume that $\mathscr{C}$ is a full abelian subcategory of $\mathrm{Mod}(A)$ for some ring A.

(i) Let us first prove that the sequence $\mathrm{Ker}\,\alpha \to \mathrm{Ker}\,\beta \to \mathrm{Ker}\,\gamma$ is exact. Let $x \in \mathrm{Ker}\,\beta$ with $g(x) = 0$. Using the fact that the first row is exact, there exists $x' \in X'$ with $f(x') = x$. Then $f' \circ \alpha(x') = \beta \circ f(x') = 0$. Since f' is a monomorphism, $\alpha(x') = 0$ and $x' \in \mathrm{Ker}\,\alpha$.

(ii) The sequence $\mathrm{Coker}\,\alpha \to \mathrm{Coker}\,\beta \to \mathrm{Coker}\,\gamma$ is exact. If one works in the abstract setting of abelian categories, this follows from (i) by reversing the arrows. Otherwise, if one wishes to remain in the setting of A-modules, one can adapt the proof of (i)[2].

(iii) Let us construct the map δ making the sequence exact. Let $x'' \in \mathrm{Ker}\,\gamma$ and choose $x \in X$ with $g(x) = x''$. Set $y = \beta(y)$. Since $g'(y) = 0$, there exists $y' \in Y'$ with $f'(y') = y$. One defines $\delta(x'')$ as the image of y' in $\mathrm{Coker}\,\alpha$, that is, in $Y'/\operatorname{Im}\alpha$.

The reader will check that the map δ is well-defined (i.e., the construction does not depend on the various choices) and that the sequence (5.5.15) is exact. □

One shall be aware that the morphism δ is not unique. Replacing δ with $-\delta$ does not change the conclusion.

Theorem 5.5.9. *Let* $0 \to X' \xrightarrow{f} X \xrightarrow{g} X'' \to 0$ *be an exact sequence in* $\mathrm{C}(\mathscr{C})$.

(i) *For each* $k \in \mathbb{Z}$, *the sequence* $H^k(X') \to H^k(X) \to H^k(X'')$ *is exact.*

(ii) *For each* $k \in \mathbb{Z}$, *there exists* $\delta^k : H^k(X'') \to H^{k+1}(X')$ *making the long sequence*

$$\cdots \to H^k(X) \to H^k(X'') \xrightarrow{\delta^k} H^{k+1}(X') \to H^{k+1}(X) \to \cdots \tag{5.5.16}$$

exact. Moreover, one can construct δ^k *functorial with respect to short exact sequences of* $\mathrm{C}(\mathscr{C})$.

Proof. Consider the commutative diagrams:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & H^k(X') & & H^k(X) & & H^k(X'') \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \mathrm{Coker}\, d_{X'}^{k-1} & \xrightarrow{f} & \mathrm{Coker}\, d_X^{k-1} & \xrightarrow{g} & \mathrm{Coker}\, d_{X''}^{k-1} \longrightarrow 0 \\
 & & \downarrow{\scriptstyle d_{X'}^k} & & \downarrow{\scriptstyle d_X^k} & & \downarrow{\scriptstyle d_{X''}^k} \\
0 & \longrightarrow & \mathrm{Ker}\, d_{X'}^{k+1} & \xrightarrow{f} & \mathrm{Ker}\, d_X^{k+1} & \xrightarrow{g} & \mathrm{Ker}\, d_{X''}^{k+1} \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & H^{k+1}(X') & & H^{k+1}(X) & & H^{k+1}(X'') \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
\end{array}$$

The columns are exact by Lemma 5.5.1 and the rows are exact by the hypothesis. Hence, the result follows from Lemma 5.5.8. □

[2] The reader shall be aware that the opposite of an abelian category is still abelian, but in general, the category $\mathrm{Mod}(A)^{\mathrm{op}}$ is not equivalent to a category $\mathrm{Mod}(B)$ for some ring B.

Corollary 5.5.10. *Consider a morphism $f\colon X \to Y$ in $\mathrm{C}(\mathscr{C})$ and recall that $\mathrm{Mc}(f)$ denotes the mapping cone of f. There is a long exact sequence:*

$$(5.5.17)\qquad \cdots \to H^{k-1}(\mathrm{Mc}(f)) \to H^k(X) \xrightarrow{f} H^k(Y) \to H^k(\mathrm{Mc}(f)) \to \cdots .$$

Proof. Using (4.2.2), we get a complex:

$$(5.5.18)\qquad 0 \to Y \to \mathrm{Mc}(f) \to X[1] \to 0.$$

Clearly, this complex is exact. Indeed, in degree n, it gives the split exact sequence $0 \to Y^n \to Y^n \oplus X^{n+1} \to X^{n+1} \to 0$. Applying Theorem 5.5.9, we find a long exact sequence

$$(5.5.19)\qquad \cdots \to H^{k-1}(\mathrm{Mc}(f)) \to H^{k-1}(X[1]) \xrightarrow{\delta^{k-1}} H^k(Y) \to H^k(\mathrm{Mc}(f)) \to \cdots .$$

It remains to check that, up to a sign, the morphism $\delta^{k-1}\colon H^k(X) \to H^k(Y)$ is $H^k(f)$. We shall not give the proof here. □

One shall be aware that although the exact sequences $0 \to Y^n \to Y^n \oplus X^{n+1} \to X^{n+1} \to 0$ split, the exact sequence of complexes (5.5.18) does not split in general.

5.6 Double complexes in abelian categories

In this subsection we shall illustrate the fact that the use of truncation functors is an alternative to that of spectral sequences (and is much easier). We follow [KS06, § 12.5].

Let $\mathscr{C}$ denote an abelian category.

Recall that, for a double complex $X = X^{\bullet,\bullet}$, the finiteness condition (4.3.7) says that for all $p \in \mathbb{Z}$, the set $\{(m,n) \in \mathbb{Z}\times\mathbb{Z}; m+n = p \text{ such that } X^{n,m} \neq 0\}$ is finite. From now on,

(5.6.1) We assume that X satisfies (4.3.7).

Note that

(5.6.2) The functor $\mathrm{tot}\colon \mathrm{C}^2_f(\mathscr{C}) \to \mathrm{C}(\mathscr{C})$ is exact.

In Section 4.3, we have constructed the functors $F_I\colon \mathrm{C}^2(\mathscr{C}) \to \mathrm{C}(\mathrm{C}(\mathscr{C}))$. Since now $\mathscr{C}$ is abelian, we can consider the truncation functors $\tau_I^{\leq n}$, $\tilde\tau_I^{\leq n}$, $\tau_I^{\geq n}$, etc. For example, one defines $\tau_I^{\leq n} := F_I^{-1} \circ \tau^{\leq n} \circ F_I$, that is, setting $X_I = F_I(X)$:

$$\tau_I^{\leq n}(X) = \cdots \to X_I^{n-1} \xrightarrow{d_I^{n-1}} X_I^n \to \mathrm{Ker}\, d_I^n \to 0.$$

For $n \in \mathbb{Z}$, we also introduce the simple complex

$$H_I^n(X) = H^n(F_I(X)).$$

Of course, the same constructions hold with F_{II} instead of F_I.

It follows from (5.5.12) and (5.6.2) that

(5.6.3) the natural morphism $\mathrm{tot}(\tau_I^{\leq n}(X)) \to \mathrm{tot}(\tilde\tau_I^{\leq n}(X))$ is a qis for all n.

We deduce from (5.5.11) the exact sequence in $\mathrm{C}(\mathscr{C})$, functorial with respect to X:

$$(5.6.4)\qquad 0 \to \mathrm{tot}(\tilde\tau_I^{\leq n-1}(X)) \to \mathrm{tot}(\tau_I^{\leq n}(X)) \to H_I^n(X)\,[-n] \to 0.$$

Theorem 5.6.1. *Let $f\colon X \to Y$ be a morphism of double complexes in $\mathscr{C}$, both satisfying (4.3.7). Assume that f induces an isomorphism $H_{II}H_I(X) \xrightarrow{\sim} H_{II}H_I(Y)$. Then $\mathrm{tot}(f)\colon \mathrm{tot}(X) \to \mathrm{tot}(Y)$ is a quasi-isomorphism.*

Proof. The hypothesis is equivalent to

(5.6.5) $H_I^n(f)\colon H_I^n(X) \to H_I^n(Y)$ is a qis for all n.

Since $H_I^n(\tau_I^{\geq p}X)$ is isomorphic to $H_I^n(X)$ for $n \geq p$ or to 0 otherwise, we get the isomorphisms

$$H_{II}H_I(\tau_I^{\geq p}X) \xrightarrow{\sim} H_{II}H_I(\tau_I^{\geq p}Y) \text{ for all } p.$$

For n fixed, we have

$$H^n(\mathrm{tot}(X)) \simeq H^n(\mathrm{tot}(\tau_I^{\geq p}X)) \text{ for } p \ll 0, \tag{5.6.6}$$

and similarly with Y instead of X. Hence, replacing X and Y with $\tau_I^{\geq p}X$ and $\tau_I^{\geq p}Y$, we may assume from the beginning that

$$X_I^n = 0 \text{ and } Y_I^n = 0 \text{ for } n \ll 0. \tag{5.6.7}$$

Using (5.6.4), we get a commutative diagram of exact sequences:

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{tot}(\tilde\tau_I^{\leq n-1}(X)) & \to & \mathrm{tot}(\tau_I^{\leq n}(X)) & \to & H_I^n(X)\,[-n] & \to & 0 \\ & & \downarrow{\scriptstyle \mathrm{tot}(\tilde\tau_I^{\leq n-1}(f))} & & \downarrow{\scriptstyle \mathrm{tot}(\tau_I^{\leq n}(f))} & & \downarrow{\scriptstyle H_I^n(f)[-n]} & & \\ 0 & \to & \mathrm{tot}(\tilde\tau_I^{\leq n-1}(Y)) & \to & \mathrm{tot}(\tau_I^{\leq n}(Y)) & \to & H_I^n(Y)\,[-n] & \to & 0 \end{array} \tag{5.6.8}$$

By (5.6.5), the vertical arrow on the right is a qis for all $n \in \mathbb{Z}$. Thanks to (5.6.7), the vertical arrow on the left is a qis for $n \ll 0$. It follows by induction, using (5.6.3), that all vertical arrows are qis. Then the result follows from (5.6.6). □

Corollary 5.6.2. *Let X be a double complex in $\mathscr{C}$ satisfying (4.3.7). If $H_I(X) \simeq 0$, then $\mathrm{tot}(X)$ is qis to 0.*

Proof. Apply Theorem 5.6.1 with $Y = 0$ and use (5.6.5). □

Corollary 5.6.3. *Let X be a double complex in $\mathscr{C}$ satisfying (4.3.7). Assume that all rows $X^{j,\bullet}$ are exact for $j \neq n$. Then $\mathrm{tot}(X)$ is qis to $X^{n,\bullet}\,[-n]$.*

Proof. Denote by $\sigma_I^{\geq n}$ the "stupid" truncation functor which to a double complex X associates the double complex whose rows are those of X for $j \geq n$ and are 0 for $j < n$. Define similarly $\sigma_I^{\leq n}$. Now apply Theorem 5.6.1 to the morphism $\sigma_I^{\geq n}(X) \to X$, next to the morphism $\sigma_I^{\geq n}(X) \to \sigma_I^{\leq n}\sigma_I^{\geq n}(X) \simeq X^{n,\bullet}\,[-n]$. □

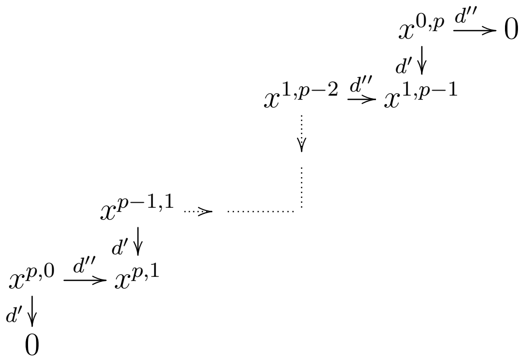
Corollary 5.6.4. *Let $X^{\bullet,\bullet}$ be a double complex. Assume that all rows $X^{j,\bullet}$ and columns $X^{\bullet,j}$ are 0 for $j < 0$ and are exact for $j > 0$. Then $H^p(X^{0,\bullet}) \simeq H^p(X^{\bullet,0})$ for all p.*

Proof. Both $X^{0,\bullet}$ and $X^{\bullet,0}$ are qis to $\mathrm{tot}(X)$. □

Let us describe the isomorphism $H^p(X^{0,\bullet}) \simeq H^p(X^{\bullet,0})$ in the case where $\mathscr{C} = \mathrm{Mod}(A)$ by the so-called "Weil procedure".

Let $x^{p,0} \in X^{p,0}$, with $d'x^{p,0} = 0$ which represents $y \in H^p(X^{\bullet,0})$. Define $x^{p,1} = d''x^{p,0}$. Then $d'x^{p,1} = 0$, and the first column being exact, there exists $x^{p-1,1} \in X^{p-1,1}$ with $d'x^{p-1,1} = x^{p,1}$. One can iterate this procedure until getting $x^{0,p} \in X^{0,p}$. Since $d'd''x^{0,p} = 0$, and d' is injective on $X^{0,p}$ for $p > 0$ by the hypothesis, we get $d''x^{0,p} = 0$. The class of $x^{0,p}$ in $H^p(X^{0,\bullet})$ will be the image of y by the Weil procedure. Of course, one has to check that this image does not depend of the various choices we have made, and that it induces an isomorphism.

This can be visualized by the diagram:



5.7 The Mittag-Leffler condition

References are made to [EGA3] (see [KS90, § 1.12]). Consider a projective system of abelian groups indexed by $\mathbb{N}$, $\{M_n, \rho_{n,p}\}_{n\in\mathbb{N}}$, with $\rho_{n,p}\colon M_p \to M_n$ $(p \geq n)$. (In the sequel we shall simply denote such a system by $\{M_n\}_n$.)

Definition 5.7.1. One says that the system $\{M_n\}_n$ satisfies the Mittag-Leffler condition (ML for short) if for any $n \in \mathbb{N}$ the decreasing sequence $\{\rho_{n,p}M_p\}$ of subgroups of M_n is stationary.

Of course, this condition is in particular satisfied if all maps $\rho_{n,p}$ are surjective.

Notation 5.7.2. For a projective system of abelian groups $\{M_n\}_n$, we set $M_\infty = \varprojlim_n M_n$.

Consider a projective system of exact sequences of abelian groups indexed by $\mathbb{N}$. Hence, for each $n \in \mathbb{N}$ we have an exact sequence

$$E_n\colon 0 \to M'_n \xrightarrow{f_n} M_n \xrightarrow{g_n} M''_n \to 0, \tag{5.7.1}$$

and we have morphisms $\rho_{n,p}\colon E_p \to E_n$ satisfying the compatibility conditions.

Lemma 5.7.3. *If the projective system $\{M'_n\}_n$ satisfies the* ML *condition, then the sequence*

$$E_\infty\colon 0 \to M'_\infty \xrightarrow{f} M_\infty \xrightarrow{g} M''_\infty \to 0 \tag{5.7.2}$$

is exact.

Proof. Since the functor $\lim$ is left exact by Proposition 5.2.2, it remains to show that g is surjective. For simplicity, we shall assume that for each n, the map $M'_{n+1} \to M'_n$ is surjective, leaving to the reader the proof in the general situation.

Let us denote by v_p the morphisms $M_p \to M_{p-1}$ which define the projective system $\{M_n\}_n$, and similarly for v'_p, v''_p. Let $(x''_p)_p \in M''_\infty$. Hence $x''_p \in M''_p$, and $v''_p(x''_p) = x''_{p-1}$.

We shall first show that $v_n \colon g_n^{-1}(x''_n) \to g_{n-1}^{-1}(x''_{n-1})$ is surjective. Let $x_{n-1} \in g_{n-1}^{-1}(x''_{n-1})$. Take $x_n \in g_n^{-1}(x''_n)$. Then $g_{n-1}(v_n(x_n) - x_{n-1})) = 0$. Hence $v_n(x_n) - x_{n-1} = f_{n-1}(x'_{n-1})$. By the hypothesis $f_{n-1}(x'_{n-1}) = f_{n-1}(v'_n(x'_n))$ for some x'_n and thus $v_n(x_n - f_n(x'_n)) = x_{n-1}$.

Then we can choose $x_n \in g_n^{-1}(x''_n)$ inductively such that $v_n(x_n) = x_{n-1}$. □

Lemma 5.7.4. *Consider the projective system of exact sequences (5.7.1).*

(a) *If $\{M'_n\}_n$ and $\{M''_n\}_n$ satisfy the ML condition, then so does $\{M_n\}_n$*

(b) *If $\{M_n\}_n$ satisfies the ML condition, then so does $\{M''\}_n$.*

The proof is left as an exercise (or see [KS90, Prop. 1.12.2]).

Mittag-Leffler theorem for complexes

Now, instead of considering an exact sequence of projective systems, we consider a complex of projective systems:

$$\{M_n^\bullet\}_n \colon \cdots \to \{M_n^{k-1}\}_n \xrightarrow{d^{k-1}} \{M_n^k\}_n \xrightarrow{d^k} \{M_n^{k+1}\}_n \to \cdots , \tag{5.7.3}$$

and its projective limit

$$M_\infty^\bullet \colon \cdots \to M_\infty^{k-1} \to M_\infty^k \to M_\infty^{k+1} \to \cdots . \tag{5.7.4}$$

Hence, we have commutative diagrams for $p \geq n$

$$\begin{array}{ccccccc}
\cdots \longrightarrow & M_p^k & \xrightarrow{d_p^k} & M_p^{k+1} & \longrightarrow \cdots \\
& {\scriptstyle \rho_{n,p}^k}\downarrow & & \downarrow{\scriptstyle \rho_{n,p}^{k+1}} & \\
\cdots \longrightarrow & M_n^k & \xrightarrow{d_n^k} & M_n^{k+1} & \longrightarrow \cdots
\end{array}$$

Denote by

$$\Phi_k \colon H^k(M_\infty^\bullet) \to \lim_n H^k(M_n^\bullet) \tag{5.7.5}$$

the natural morphism.

Proposition 5.7.5 (See [KS90, Prop. 1.12.4])**.** *Assume that for all $k \in \mathbb{Z}$, the system $\{M_n^k\}_n$ satisfies the* ML *condition. Then*

(a) *for each $k \in \mathbb{Z}$, the map Φ_k in (5.7.5) is surjective,*

(b) *if moreover, for a given i the system $\{H^{i-1}(M_n^\bullet)\}_n$ satisfies the* ML *condition, then Φ_i is bijective.*

Proof. (i) Set $Z_n^k = \operatorname{Ker} d_n^k \colon M_n^k \to M_n^{k+1}$ and $B_n^k = \operatorname{Im} d_n^{k-1} \colon M_n^{k-1} \to M_n^k$. There are sequences

$$
\begin{aligned}
&0 \to Z_n^k \to M_n^k \to B_n^{k+1} \to 0,\\
&0 \to B_n^k \to Z_n^k \to H^k(M_n^\bullet) \to 0,\\
&0 \to B_\infty^k \to Z_\infty^k \to \lim_n H^k(M_n^\bullet) \to 0.
\end{aligned}
\tag{5.7.6}
$$

The two first sequences are clearly exact. The third one is also exact thanks to Lemma 5.7.3 since the projective system $\{B_n^k\}_n$ satisfies the ML condition by Lemma 5.7.4,
(ii) The functor lim being left exact, we have:

$$Z_\infty^k \simeq \operatorname{Ker}(M_\infty^k \to M_\infty^{k+1}).$$

Consider the diagram

$$
\begin{array}{ccccccccc}
 & & M_\infty^{k-1} & \longrightarrow & \operatorname{Ker}(M_\infty^k \to M_\infty^{k+1}) & \longrightarrow & H^k(M_n^\bullet) & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle \Psi_k} & & \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \Phi_k} & & \\
0 & \longrightarrow & B_\infty^k & \longrightarrow & Z_\infty^k & \longrightarrow & \lim_n H^k(M_n^\bullet) & \longrightarrow & 0.
\end{array}
\tag{5.7.7}
$$

Since the rows are exact, we get that Φ_k is surjective.
(iii) Assume now that for i given, the projective system $\{H^{i-1}(M_n^\bullet)\}_n$ satisfies the ML condition. It follows from the second exact sequence in (5.7.6) and Lemma 5.7.4 that the projective system $\{Z_n^{i-1}\}_n$ satisfies the ML condition. Applying Lemma 5.7.3 to the first exact sequence in (5.7.6), we get the exact sequence

$$0 \to Z_\infty^{i-1} \to M_\infty^{i-1} \to B_\infty^i \to 0.$$

□

A basic lemma

The next lemma, although elementary, is extremely useful. It is due to M. Kashiwara [Kas83].

Let $\{X_s, \rho_{s,t}\}_{s\in\mathbb{R}}$ be a projective system of sets indexed by $\mathbb{R}$. Hence, the X_s are sets and $\rho_{s,t} \colon X_t \to X_s$ are maps defined for $s \leq t$, satisfying the natural compatibility conditions. Set

$$\lambda_s \colon X_s \to \lim_{r<s} X_r, \quad \mu_s \colon \operatorname{colim}_{t>s} X_t \to X_s.$$

Lemma 5.7.6 (See [KS90, Prop. 1.12.6])**.** *Assume that for each $s \in \mathbb{R}$, both maps λ_s and μ_s are injective (resp. surjective). Then all maps ρ_{s_0,s_1} $(s_0 \leq s_1)$ are injective (resp. surjective).*

Proof. (i) The map ρ_{s_0,s_1} is injective. Let $x, y \in X_{s_1}$ be such that $\rho_{s_0,s_1}(x) = \rho_{s_0,s_1}(y)$. Set

$$I = \{s \in \mathbb{R}; s \leq s_1, \rho_{s,s_1}(x) = \rho_{s,s_1}(y)\}.$$

Then $s_0 \in I$ and $s \in I$, $r < s$ implies $r \in I$. Let $s_2 = \sup I$. Then $\rho_{s,s_1}(x) = \rho_{s,s_1}(y)$ for all $s < s_2$ which implies $\lambda_{s_2}(x) = \lambda_{s_2}(y)$. Since λ_{s_2} is injective, we get that $s_2 \in I$. If $s_2 < s_1$, the map μ_{s_2} being injective, we find again that there exists some $t > s$ such that $\rho_{t,s_1}(x) = \rho_{t,s_1}(y)$ which is a contradiction. Therefore, $s_2 = s_1$. Hence, $x = y$.

(ii) The map ρ_{s_0,s_1} is surjective. Let $x_0 \in X_{s_0}$ and let A be the set

$$A = \{(s, x); s_0 \le s \le s_1, x \in X_s \text{ and } \rho_{s_0,s}(x) = x_0\}.$$

We order A as follows.

$$(s, x) \le (s', x') \Leftrightarrow s \le s' \text{ and } \rho_{s,s'}(x') = x.$$

Let us show that A is inductively ordered. Let $B \subset A$ be totally ordered and let us show that A contains an upper bound of B. Let

$$I = \{s \in \mathbb{R}; s_0 \le s \le s_1, \text{ there exists } x \in X_s \text{ with } (s, x) \in B\}.$$

Let $s_2 = \sup I$. If $s_2 \in I$, then B has a maximal element. If $s_2 \notin I$, then there exists $(s_2, x_2) \in A$ greater than any element of B by the surjectivity of λ_{s_2}. By Zorn's lemma, we get that A admits a maximal element (s, x). If $s = s_1$, the proof is complete. Assume $s < s_1$. Since μ_s is surjective, there exists s' with $s < s' \le s_1$ and $x' \in X_{s'}$ with $\rho_{s,s'}(x') = x$. This is a contradiction. Hence, $s = s_1$. □

5.8 Koszul complexes

Recall that $\mathbf{k}$ denotes a commutative unital ring. In this section, we do not work in abstract abelian categories but in the category $\mathrm{Mod}(\mathbf{k})$.

If L is a finitely generated free $\mathbf{k}$-module of rank n, one denotes by $\bigwedge^j L$ the j-th exterior power of L. Recall that $L^* = \mathrm{Hom}_{\mathbf{k}}(L, \mathbf{k})$.

Note that $\bigwedge^1 L \simeq L$ and $\bigwedge^n L \simeq \mathbf{k}$. One sets $\bigwedge^0 L = \mathbf{k}$.

If $(e_1, \dots, e_n)$ is a basis of L and $I = \{i_1 < \cdots < i_j\} \subset \{1, \dots, n\}$, one sets

$$e_I = e_{i_1} \wedge \cdots \wedge e_{i_j}.$$

For a subset $I \subset \{1, \dots, n\}$, one denotes by $|I|$ its cardinal. Recall that:

$$\bigwedge^j L \text{ is free with basis } \{e_I; I \subset \{1, \dots, n\}, |I| = j\}.$$

If $i_1, \dots, i_m$ belong to the set $(1, \dots, n)$, one defines $e_{i_1} \wedge \cdots \wedge e_{i_m}$ by reducing to the case where $i_1 < \cdots < i_j$, using the convention $e_i \wedge e_j = -e_j \wedge e_i$.

Let M be a $\mathbf{k}$-module and let $\varphi = (\varphi_1, \dots, \varphi_n)$ be n $\mathbf{k}$-linear endomorphisms of M which commute with one another:

$$[\varphi_i, \varphi_j] = 0, \ 1 \le i, j \le n.$$

(Recall the notation $[a, b] := ab - ba$.) Set

$$M^{(j)} = M \otimes \bigwedge^j \mathbf{k}^n.$$

Hence $M^{(0)} = M$ and $M^{(n)} \simeq M$. Denote by $(e_1, \dots, e_n)$ the canonical basis of $\mathbf{k}^n$. Hence, any element of $M^{(j)}$ may be written uniquely as a sum

$$m = \sum_{|I|=j} m_I \otimes e_I.$$

One defines $d \in \mathrm{Hom}_{\mathbf{k}}(M^{(j)}, M^{(j+1)})$ by:

$$d(m \otimes e_I) = \sum_{i=1}^{n} \varphi_i(m) \otimes e_i \wedge e_I$$

and extending d by $\mathbf{k}$-linearity. Using the commutativity of the φ_i's one checks easily that $d \circ d = 0$. Hence we get a complex, called a Koszul complex and denoted by $K^\bullet(M, \varphi)$:

$$0 \to M^{(0)} \xrightarrow{d} \cdots \to M^{(n)} \to 0.$$

When $n = 1$, the cohomology of this complex gives the kernel and cokernel of φ_1. More generally,

$$H^0(K^\bullet(M,\varphi)) \simeq \operatorname{Ker}\varphi_1 \cap \ldots \cap \operatorname{Ker}\varphi_n,$$
$$H^n(K^\bullet(M,\varphi)) \simeq M/(\varphi_1(M) + \cdots + \varphi_n(M)).$$

Set $\varphi' = \{\varphi_1, \dots, \varphi_{n-1}\}$ and denote by d' the differential in $K^\bullet(M, \varphi')$. Then φ_n defines a morphism

(5.8.1) $$\widetilde{\varphi}_n : K^\bullet(M, \varphi') \to K^\bullet(M, \varphi')$$

Lemma 5.8.1. *The complex $K^\bullet(M, \varphi)[1]$ is isomorphic to the mapping cone of $-\widetilde{\varphi}_n$.*

Proof. [3] Consider the diagram

$$\begin{array}{ccc} \mathrm{Mc}(\widetilde{\varphi}_n)^p & \xrightarrow[d_M^p]{} & \mathrm{Mc}(\widetilde{\varphi}_n)^{p+1} \\ {\scriptstyle\lambda^p}\downarrow & & \downarrow{\scriptstyle\lambda^{p+1}} \\ K^{p+1}(M,\varphi) & \xrightarrow[d_K^{p+1}]{} & K^{p+2}(M,\varphi) \end{array}$$

given explicitly by:

$$\begin{array}{ccc} (M \otimes \bigwedge^{p+1} \mathbf{k}^{n-1}) \oplus (M \otimes \bigwedge^{p} \mathbf{k}^{n-1}) & \xrightarrow[\begin{pmatrix} -d' & 0 \\ -\varphi_n & d' \end{pmatrix}]{} & (M \otimes \bigwedge^{p+2} \mathbf{k}^{n-1}) \oplus (M \otimes \bigwedge^{p+1} \mathbf{k}^{n-1}) \\ {\scriptstyle \mathrm{id} \oplus (\mathrm{id} \otimes e_n \wedge)}\downarrow & & \downarrow{\scriptstyle \mathrm{id} \oplus (\mathrm{id} \otimes e_n \wedge)} \\ M \otimes \bigwedge^{p+1} \mathbf{k}^n & \xrightarrow[-d]{} & M \otimes \bigwedge^{p+2} \mathbf{k}^n \end{array}$$

[3] The proof may be skipped

Then

$$\begin{aligned} &d^p_M(a\otimes e_J + b\otimes e_K) = -d'(a\otimes e_J) + (d'(b\otimes e_K) - \varphi_n(a)\otimes e_J),\\ &\lambda^p(a\otimes e_J + b\otimes e_K) = a\otimes e_J + b\otimes e_n\wedge e_K.\end{aligned}$$

(i) The vertical arrows are isomorphisms. Indeed, let us treat the first one. It is described by:

$$\sum_J a_J\otimes e_J + \sum_K b_K\otimes e_K \mapsto \sum_J a_J\otimes e_J + \sum_K b_K\otimes e_n\wedge e_K \tag{5.8.2}$$

with $|J| = p+1$ and $|K| = p$. Any element of $M\otimes\bigwedge^{p+1}\mathbf{k}^n$ may uniquely be written as in the right hand side of (5.8.2).

(ii) The diagram commutes. Indeed,

$$\begin{aligned} \lambda^{p+1}\circ d^p_M(a\otimes e_J + b\otimes e_K) &= -d'(a\otimes e_J) + e_n\wedge d'(b\otimes e_K) - \varphi_n(a)\otimes e_n\wedge e_J\\ &= -d'(a\otimes e_J) - d'(b\otimes e_n\wedge e_K) - \varphi_n(a)\otimes e_n\wedge e_J,\\ d^{p+1}_K\circ\lambda^p(a\otimes e_J + b\otimes e_K) &= -d(a\otimes e_J + b\otimes e_n\wedge e_K)\\ &= -d'(a\otimes e_J) - \varphi_n(a)\otimes e_n\wedge e_J - d'(b\otimes e_n\wedge e_K).\end{aligned}$$

□

Theorem 5.8.2. *There exists a* $\mathbf{k}$*-linear long exact sequence*

$$\cdots\to H^j(K^\bullet(M,\varphi'))\xrightarrow{\varphi_n} H^j(K^\bullet(M,\varphi'))\to H^{j+1}(K^\bullet(M,\varphi))\to\cdots \tag{5.8.3}$$

Proof. Apply Lemma 5.8.1 and the long exact sequence (5.5.17). □

Definition 5.8.3. (i) If for each j, $1\le j\le n$, φ_j is injective as an endomorphism of $M/(\varphi_1(M)+\cdots+\varphi_{j-1}(M))$, one says that $(\varphi_1,\dots,\varphi_n)$ is a regular sequence.

(ii) If for each j, $1\le j\le n$, φ_j is surjective as an endomorphism of $\operatorname{Ker}\varphi_1\cap\ldots\cap\operatorname{Ker}\varphi_{j-1}$, one says that $(\varphi_1,\dots,\varphi_n)$ is a coregular sequence.

Corollary 5.8.4. (i) *If $(\varphi_1,\dots,\varphi_n)$ is a regular sequence, then $H^j(K^\bullet(M,\varphi))\simeq 0$ for $j\neq n$.*

(ii) *If $(\varphi_1,\dots,\varphi_n)$ is a coregular sequence, then $H^j(K^\bullet(M,\varphi))\simeq 0$ for $j\neq 0$.*

Proof. Assume for example that $(\varphi_1,\dots,\varphi_n)$ is a regular sequence and let us argue by induction on n. The cohomology of $K^\bullet(M,\varphi')$ is thus concentrated in degree $n-1$ and is isomorphic to $M/(\varphi_1(M)+\cdots+\varphi_{n-1}(M))$. By the hypothesis, φ_n is injective on this group, and Corollary 5.8.4 follows. □

Second proof in case $n=2$. Let us give a direct proof of the Corollary in case $n=2$ for coregular sequences. Hence we consider the complex:

$$0\to M\xrightarrow{d} M\oplus M\xrightarrow{d} M\to 0$$

where $d(x) = (\varphi_1(x),\varphi_2(x))$, $d(y,z) = \varphi_2(y)-\varphi_1(z)$ and we assume φ_1 is surjective on M, φ_2 is surjective on $\operatorname{Ker}\varphi_1$.

Let $(y,z)\in M\oplus M$ with $\varphi_2(y) = \varphi_1(z)$. We look for $x\in M$ solution of $\varphi_1(x) = y,\quad \varphi_2(x) = z$. First choose $x'\in M$ with $\varphi_1(x') = y$. Then $\varphi_2\circ\varphi_1(x') = \varphi_2(y) = \varphi_1(z) = \varphi_1\circ\varphi_2(x')$. Thus $\varphi_1(z-\varphi_2(x')) = 0$ and there exists $t\in M$ with $\varphi_1(t) = 0,\quad \varphi_2(t) = z-\varphi_2(x')$. Hence $y = \varphi_1(t+x'),\quad z = \varphi_2(t+x')$ and $x = t+x'$ is a solution to our problem. □

Example 5.8.5. Let $\mathbf{k}$ be a field of characteristic 0 and let $A = \mathbf{k}[x_1, \dots, x_n]$.
(i) Denote by $x_i\cdot$ the multiplication by x_i in A. We get the complex:

$$0 \to A^{(0)} \xrightarrow{d} \cdots \xrightarrow{d} A^{(n)} \to 0$$

where:

$$d(\sum_I a_I \otimes e_I) = \sum_{j=1}^n \sum_I x_j \cdot a_I \otimes e_j \wedge e_I.$$

The sequence $(x_1\cdot, \dots, x_n\cdot)$ is a regular sequence. Hence the Koszul complex is exact except in degree n where its cohomology is isomorphic to $\mathbf{k}$.
(ii) Denote by ∂_i the partial derivation with respect to x_i. This is a $\mathbf{k}$-linear map on the $\mathbf{k}$-vector space A. Hence we get a Koszul complex

$$0 \to A^{(0)} \xrightarrow{d} \cdots \xrightarrow{d} A^{(n)} \to 0$$

where:

$$d(\sum_I a_I \otimes e_I) = \sum_{j=1}^n \sum_I \partial_j(a_I) \otimes e_j \wedge e_I.$$

The sequence $(\partial_1\cdot, \dots, \partial_n\cdot)$ is a coregular sequence and the above complex is exact except in degree 0 where its cohomology is isomorphic to $\mathbf{k}$. Writing dx_j instead of e_j, we recognize the "de Rham complex".

Example 5.8.6. Let $\mathbf{k}$ be a field and let $A = \mathbf{k}[x, y]$, $M = \mathbf{k} \simeq A/xA + yA$. Let us calculate a free (hence, projective) resolution of M. Since (x, y) is a regular sequence of endomorphisms of A (viewed as a $\mathbf{k}$-module), M is quasi-isomorphic to the complex:

$$M^\bullet : 0 \to A \xrightarrow{u} A^2 \xrightarrow{v} A \to 0,$$

where $u(a) = (ya, -xa)$, $v(b, c) = xb + yc$ and the module A on the right stands in degree 0. Therefore, for N an A-module, $\mathrm{Hom}_A(M^\bullet, N)$ is represented by the complex:

$$0 \to N \xrightarrow{v'} N^2 \xrightarrow{u'} N \to 0,$$

where $v' = \mathrm{Hom}(v, N)$, $u' = \mathrm{Hom}(u, N)$ and the module N on the left stands in degree 0. Since $v'(n) = (xn, yn)$ and $u'(m, l) = ym - xl$, we find again a Koszul complex. Choosing $N = A$, its cohomology is concentrated in degree 2 and isomorphic to $\mathbf{k}$.

Example 5.8.7. Let $W = W_n(\mathbf{k})$ be the Weyl algebra introduced in Example 1.2.2, and denote by $\cdot\partial_i$ the multiplication on the right by ∂_i. Then $(\cdot\partial_1, \dots, \cdot\partial_n)$ is a regular sequence on W and we get the Koszul complex:

$$0 \to W^{(0)} \xrightarrow{\delta} \cdots \to W^{(n)} \to 0$$

where:

$$\delta(\sum_I a_I \otimes e_I) = \sum_{j=1}^n \sum_I a_I \cdot \partial_j \otimes e_j \wedge e_I.$$

This complex is exact except in degree n where its cohomology is isomorphic to $\mathbf{k}[x]$ (see Exercise 5.10).

Remark 5.8.8. One may also encounter co-Koszul complexes. For $I = (i_1, \dots, i_k)$, introduce

$$e_j \lfloor e_I = \begin{cases} 0 & \text{if } j \notin \{i_1, \dots, i_k\}, \\ (-1)^{l+1} e_{I_{\hat{l}}} := (-1)^{l+1} e_{i_1} \wedge \dots \wedge \widehat{e_{i_l}} \wedge \dots \wedge e_{i_k}, & \text{if } e_{i_l} = e_j, \end{cases}$$

where $e_{i_1} \wedge \dots \wedge \widehat{e_{i_l}} \wedge \dots \wedge e_{i_k}$ means that e_{i_l} should be omitted in $e_{i_1} \wedge \dots \wedge e_{i_k}$. Define δ by:

$$\delta(m \otimes e_I) = \sum_{j=1}^{n} \varphi_j(m) e_j \lfloor e_I.$$

Here again one checks easily that $\delta \circ \delta = 0$, and we get the complex:

$$K_\bullet(M, \varphi) : 0 \to M^{(n)} \xrightarrow{\delta} \cdots \to M^{(0)} \to 0,$$

This complex is in fact isomorphic to a Koszul complex. Consider the isomorphism

$$*\colon \bigwedge^{j} \mathbf{k}^n \xrightarrow{\sim} \bigwedge^{n-j} \mathbf{k}^n$$

which associates $\varepsilon_I m \otimes e_{\hat{I}}$ to $m \otimes e_I$, where $\hat{I} = (1, \dots, n) \setminus I$ and ε_I is the signature of the permutation which sends $(1, \dots, n)$ to $I \sqcup \hat{I}$ (any $i \in I$ is smaller than any $j \in \hat{I}$). Then, up to a sign, $*$ interchanges d and δ.

De Rham complexes

Let E be a real vector space of dimension n and let U be an open subset of E. Denote as usual by $\mathscr{C}^\infty(U)$ the $\mathbb{C}$-algebra of $\mathbb{C}$-valued functions on U of class C^∞. Recall that $\Omega^1(U)$ denotes the $\mathscr{C}^\infty(U)$-module of C^∞-functions on U with values in $E^* \otimes_\mathbb{R} \mathbb{C} \simeq \mathrm{Hom}_\mathbb{R}(E, \mathbb{C})$. Hence

$$\Omega^1(U) \simeq E^* \otimes_\mathbb{R} \mathscr{C}^\infty(U).$$

For $p \in \mathbb{N}$, one sets

$$\begin{aligned} \Omega^p(U) &:= \bigwedge^{p} \Omega^1(U) \\ &\simeq (\bigwedge^{p} E^*) \otimes_\mathbb{R} \mathscr{C}^\infty(U). \end{aligned}$$

(The first exterior product is taken over the commutative ring $\mathscr{C}^\infty(U)$ and the second one over $\mathbb{R}$.) Hence, $\Omega^0(U) = \mathscr{C}^\infty(U)$, $\Omega^p(U) = 0$ for $p > n$ and $\Omega^n(U)$ is free of rank 1 over $\mathscr{C}^\infty(U)$. The differential is a $\mathbb{C}$-linear map

$$d\colon \mathscr{C}^\infty(U) \to \Omega^1(U).$$

The differential extends by multilinearity as a $\mathbb{C}$-linear map $d\colon \Omega^p(U) \to \Omega^{p+1}(U)$ satisfying

$$\text{(5.8.4)} \qquad \begin{cases} d^2 = 0, \\ d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-)^p \omega_1 \wedge d\omega_2 \text{ for any } \omega_1 \in \Omega^p(U). \end{cases}$$

We get a complex, called the De Rham complex, that we denote by $\mathrm{DR}(U)$:

$$\mathrm{DR}(U) := 0 \to \Omega^0(U) \xrightarrow{d} \cdots \to \Omega^n(U) \to 0. \tag{5.8.5}$$

Let us choose a basis $(e_1, \dots, e_n)$ of E and denote by x_i the function which, to $x = \sum_{i=1}^n x_i \cdot e_i \in E$, associates its i-th cordinate x_i. Then $(dx_1, \dots, dx_n)$ is the dual basis on E^* and the differential of a function φ is given by

$$d\varphi = \sum_{i=1}^n \partial_i \varphi \, dx_i.$$

where $\partial_i \varphi := \dfrac{\partial \varphi}{\partial x_i}$. By its construction, the Koszul complex of $(\partial_1, \dots, \partial_n)$ acting on $\mathscr{C}^\infty(U)$ is nothing but the De Rham complex:

$$K^\bullet(\mathscr{C}^\infty(U), (\partial_1, \dots, \partial_n)) = \mathrm{DR}(U).$$

Note that $H^0(\mathrm{DR}(U))$ is the space of locally constant functions on U, and therefore is isomorphic to $\mathbb{C}^{\# cc(U)}$ where $\# cc(U)$ denotes the cardinal of the set of connected components of U. Using sheaf theory, one proves that all cohomology groups $H^j(\mathrm{DR}(U))$ are topological invariants of U.

Holomorphic De Rham complexes

Replacing $\mathbb{R}^n$ with $\mathbb{C}^n$, $\mathscr{C}^\infty(U)$ with $\mathscr{O}(U)$, the space of holomorphic functions on U and the real derivation with the holomorphic derivation, one constructs similarly the holomorphic De Rham complex.

Example 5.8.9. Let $n = 1$ and let $U = \mathbb{C} \setminus \{0\}$. The holomorphic De Rham complex reduces to

$$0 \to \mathscr{O}(U) \xrightarrow{\partial_z} \mathscr{O}(U) \to 0.$$

Its cohomology is isomorphic to $\mathbb{C}$ in degree 0 and 1.

Exercises to Chapter 5

Exercise 5.1. Prove assertion (iv) in Proposition 5.2.2, that is, prove that for a ring A and a small set I, the two functors $\prod$ and $\bigoplus$ from $\mathrm{Fct}(I, \mathrm{Mod}(A))$ to $\mathrm{Mod}(A)$ are exact.

Exercise 5.2. Consider two complexes in an abelian category $\mathscr{C}$: $X_1' \to X_1 \to X_1''$ and $X_2' \to X_2 \to X_2''$. Prove that the two sequences are exact if and only if the sequence $X_1' \oplus X_2' \to X_1 \oplus X_2 \to X_1'' \oplus X_2''$ is exact.

Exercise 5.3. Let A be a ring.

(i) Prove that a free module is projective.

(ii) Prove that a module P is projective if and only if it is a direct summand of a free module (*i.e.,* there exists a module K such that $P \oplus K$ is free).

(iii) An A-module M is flat if the functor $\bullet \otimes_A M$ is exact. (One defines similarly flat right A-modules.) Deduce from (ii) that projective modules are flat.

Exercise 5.4 (See [God58, Th. 1.2.2]). If M is a $\mathbb{Z}$-module, set $M^\vee = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$.

(i) Prove that $\mathbb{Q}/\mathbb{Z}$ is injective in $\mathrm{Mod}(\mathbb{Z})$.

(ii) Prove that the map $\mathrm{Hom}_{\mathbb{Z}}(M, N) \to \mathrm{Hom}_{\mathbb{Z}}(N^\vee, M^\vee)$ is injective for any $M, N \in \mathrm{Mod}(\mathbb{Z})$.

(iii) Prove that if P is a right projective A-module, then $P^\vee$ is left A-injective.

(iv) Let M be an A-module. Prove that there exists an injective A-module I and a monomorphism $M \to I$.

(Hint: for (iii) Use formula (1.2.3), for (iv) prove that $M \mapsto M^{\vee\vee}$ is an injective map using (ii), and replace M with $M^{\vee\vee}$.)

Exercise 5.5. Let $\mathscr{C}$ be an additive category which admits small colimits. Let $\{X_i\}_{i \in I}$ be a family of objects of $\mathscr{C}$ indexed by a small set I and let $i_0 \in I$. Prove that the natural morphism $X_{i_0} \to \bigoplus_{i \in I} X_i$ is a monomorphism.

Exercise 5.6. Let $\mathscr{C}$ be an abelian category.

(i) Prove that a complex $0 \to X \to Y \to Z$ is exact iff and only if for any object $W \in \mathscr{C}$ the complex of abelian groups $0 \to \mathrm{Hom}_{\mathscr{C}}(W, X) \to \mathrm{Hom}_{\mathscr{C}}(W, Y) \to \mathrm{Hom}_{\mathscr{C}}(W, Z)$ is exact.

(ii) By reversing the arrows, state and prove a similar statement for a complex $X \to Y \to Z \to 0$.

Exercise 5.7. Let $\mathscr{C}$ be an abelian category, $\mathscr{J}$ a full additive subcategory.

(a) Assume that $\mathscr{J}$ is closed by kernels and cokernels. Prove that $\mathscr{J}$ is abelian and the embedding functor $\mathscr{J} \to \mathscr{C}$ is exact.

(b) Prove that $\mathscr{J}$ is thick in $\mathscr{C}$ if and only if for any exact sequence $X_0 \to X_1 \to X_2 \to X_3 \to X_4$ in $\mathscr{C}$ with $X_i \in \mathscr{J}$ for $j = 0, 1, 3, 4$, X_2 is isomorphic to an object of $\mathscr{J}$. (See [KS06, Rem. 8.3.22].)

Exercise 5.8. Recall Diagram 2.4.1 and Definition 2.4.1. Let $\mathscr{C}$ be an abelian category and consider a commutative diagram:

$$\begin{array}{ccc} V & \xrightarrow{g'} & X \\ \big\downarrow{\scriptstyle f'} & & \big\downarrow{\scriptstyle f} \\ Y & \xrightarrow{g} & Z. \end{array}$$

The square is Cartesian if the sequence $0 \to V \to X \times Y \to Z$ is exact, that is, if $V \simeq X \times_Z Y$ (recall that $X \times_Z Y = \mathrm{Ker}(f - g)$, where $f - g \colon X \oplus Y \to Z$). The square is co-Cartesian if the sequence $V \to X \oplus Y \to Z \to 0$ is exact, that is, if $Z \simeq X \oplus_V Y$ (recall that $X \oplus_Z Y = \mathrm{Coker}(f' - g')$, where $f' - g' \colon V \to X \oplus Y$).

(i) Assume the square is Cartesian and f is an epimorphism. Prove that f' is an epimorphism.

(ii) Assume the square is co-Cartesian and f' is a monomorphism. Prove that f is a monomorphism.

Exercise 5.9. Let $\mathscr{C}$ be an abelian category and consider a double complexe

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \to & X'_0 & \to & X_0 & \to & X''_0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \to & X'_1 & \to & X_1 & \to & X''_1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \to & X'_2 & \to & X_2 & \to & X''_2 \end{array}$$

Assume that all rows are exact as well as the middle and right column. Prove that all columns are exact.

Exercise 5.10. Let $\mathbf{k}$ be a field of characteristic 0, $W := W_n(\mathbf{k})$ the Weyl algebra in n variables.

(i) Denote by $x_i\cdot\colon W \to W$ the left multiplication by x_i on W (hence, the $x_i\cdot$'s are morphisms of right W-modules). Prove that $\varphi = (x_1\cdot, \dots, x_n\cdot)$ is a regular sequence and calculate $H^j(K^\bullet(W, \varphi))$.

(ii) Denote $\cdot\partial_i$ the right multiplication by ∂_i on W. Prove that $\psi = (\cdot\partial_1, \dots, \cdot\partial_n)$ is a regular sequence and calculate $H^j(K^\bullet(W, \psi))$.

(iii) Now consider the left $W_n(\mathbf{k})$-module $\mathscr{O} := \mathbf{k}[x_1, \dots, x_n]$ and the $\mathbf{k}$-linear map $\partial_i : \mathscr{O} \to \mathscr{O}$ (derivation with respect to x_i). Prove that $\lambda = (\partial_1, \dots, \partial_n)$ is a coregular sequence and calculate $H^j(K^\bullet(\mathscr{O}, \lambda))$.

(iv) Let $A = W_2(\mathbf{k})$ be the Weyl algebra in two variables. Construct the Koszul complex associated to $\varphi_1 = \cdot x_1$, $\varphi_2 = \cdot\partial_2$ and calculate its cohomology.

Exercise 5.11. Let $\mathbf{k}$ be a field, $A = \mathbf{k}[x, y]$ and consider the A-module $M = \bigoplus_{i \geq 1} \mathbf{k}[x]t^i$, where the action of $x \in A$ is the usual one and the action of $y \in A$ is defined by $y \cdot x^n t^{j+1} = x^n t^j$ for $j \geq 1$, $y \cdot x^n t = 0$. Define the endomorphisms of M, $\varphi_1(m) = x \cdot m$ and $\varphi_2(m) = y \cdot m$. Calculate the cohomology of the Kozsul complex $K^\bullet(M, \varphi)$.

Exercise 5.12. Let $\mathscr{C}$ be an abelian category which admits small direct sums and let I be a small set. For $X, Y \in \mathscr{C}$, prove the isomorphism

$$\mathrm{Hom}_{\mathbf{Set}}(I, \mathrm{Hom}_{\mathscr{C}}(Y, X)) \simeq \mathrm{Hom}_{\mathscr{C}}(Y^{\oplus I}, X).$$

(Hint: see (1.1.3) and (1.1.5).)

Exercise 5.13. Let $F\colon \mathscr{C} \to \mathscr{C}'$ be an additive functor of abelian categories. Prove that if F is faithful then it is conservative. Conversely, assume that F is conservative and exact. Prove that F is faithful. (See [KS06, Exe. 8.25].)

Exercise 5.14. Let $\mathscr{C}$ be an abelian category which admits small coproducts and a generator G. Let $f\colon X \to Y$ be a morphism in $\mathscr{C}$ and assume that $\mathrm{Hom}_{\mathscr{C}}(G, X) \to \mathrm{Hom}_{\mathscr{C}}(G, Y)$ is surjective. Prove that f is an epimorphism.

(Hint: use Lemma 5.4.2.)

Chapter 6

Triangulated categories

Summary

Triangulated categories play an iimportant role in mathematics and this subject would deserve more that the short chapter than we present here. They are a substitute, in some sense, to abelian categories, the distinguished triangles playing the role of the exact sequences, and they are naturally associated to additive (not necessarily abelian) categories. Indeed, as we shall see, the homotopy category $\mathrm{K}(\mathscr{C})$ associated with an additive category $\mathscr{C}$ is naturally triangulated.

We have restricted ourselves to describe the main properties of triangulated categories, presenting only the basic results. In particular, we localize triangulated categories and triangulated functors with the construction of derived categories in mind.

Some tedious proofs are skipped, referring to [KS06].

Remark that the morphism in TR4 (see below) is not unique and this is the source of many troubles. This is the main obstacle encountered when trying to "glue" derived categories. This difficulty is overcome with the theory of ∞-categories where stable categories play the role of triangulated categories.

Some references. For historical comments, see the Introduction. For an non exhaustive list of recent books treating triangulated categories, see [GM96, KS90, KS06, Nee01, Ver96, Wei94, Yek20].

6.1 Triangulated categories

Definition 6.1.1. A category with translation $(\mathscr{D}, T)$ is an additive category $\mathscr{D}$ endowed with an automorphism $T\colon \mathscr{D} \to \mathscr{D}$ (i.e., an invertible functor), called the translation functor.

A triangle in $(\mathscr{D}, T)$ is a sequence of morphisms:

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T(X). \tag{6.1.1}$$

A morphism of triangles is a commutative diagram:

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & T(X) \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle T(\alpha)} \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & T(X'). \end{array}$$

Example 6.1.2. The triangle $X \xrightarrow{f} Y \xrightarrow{-g} Z \xrightarrow{-h} T(X)$ is isomorphic to the triangle (6.1.1), but the triangle $X \xrightarrow{-f} Y \xrightarrow{-g} Z \xrightarrow{-h} T(X)$ is not isomorphic to the triangle (6.1.1) in general.

Definition 6.1.3. A triangulated category is an additive category $\mathscr{D}$ endowed with an automorphism T, called the translation functor, or the shift functor, and a family of triangles called distinguished triangles (d.t.for short), this family satisfying axioms TR0 - TR5 below.

TR0 A triangle isomorphic to a d.t.is a d.t.

TR1 The triangle $X \xrightarrow{\mathrm{id}_X} X \to 0 \to T(X)$ is a d.t.

TR2 For all $f\colon X \to Y$ there exists a d.t.$X \xrightarrow{f} Y \xrightarrow{g} Z \to T(X)$.

TR3 A triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T(X)$ is a d.t.if and only if $Y \xrightarrow{g} Z \xrightarrow{h} T(X) \xrightarrow{-T(f)} T(Y)$ is a d.t.

TR4 Given two d.t.$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T(X)$ and $X' \xrightarrow{f'} Y' \xrightarrow{g'} Z' \xrightarrow{h'} T(X')$ and morphisms $\alpha\colon X \to X'$ and $\beta : Y \to Y'$ with $f' \circ \alpha = \beta \circ f$, there exists a morphism $\gamma\colon Z \to Z'$ giving rise to a morphism of d.t.:

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & T(X) \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle T(\alpha)} \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & T(X'), \end{array}$$

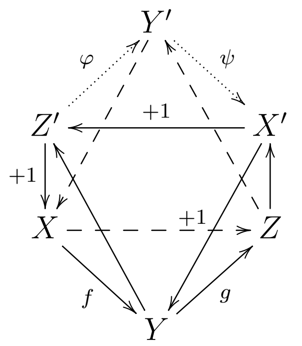
TR5 (Octahedral axiom) Given three d.t.

$$\begin{aligned} X \xrightarrow{f} Y \xrightarrow{h} Z' \to T(X), \\ Y \xrightarrow{g} Z \xrightarrow{k} X' \to T(Y), \\ X \xrightarrow{g\circ f} Z \xrightarrow{l} Y' \to T(X), \end{aligned}$$

there exists a distinguished triangle $Z' \xrightarrow{\varphi} Y' \xrightarrow{\psi} X' \to T(Z')$ making the diagram below commutative:

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{h} & Z' & \longrightarrow & T(X) \\ \downarrow{\scriptstyle\mathrm{id}} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\mathrm{id}} \\ X & \xrightarrow{g\circ f} & Z & \xrightarrow{l} & Y' & \longrightarrow & T(X) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle\mathrm{id}} & & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle T(f)} \\ Y & \xrightarrow{g} & Z & \xrightarrow{k} & X' & \longrightarrow & T(Y) \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle l} & & \downarrow{\scriptstyle\mathrm{id}} & & \downarrow{\scriptstyle T(h)} \\ Z' & \xrightarrow{\varphi} & Y' & \xrightarrow{\psi} & X' & \longrightarrow & T(Z') \end{array} \tag{6.1.2}$$

Diagram (6.1.2) is often called the octahedron diagram. Indeed, it can be written using the vertexes of an octahedron.



In this diagram, the notation $A \xrightarrow{+1} B$ means $A \to T(B)$.

Remark 6.1.4. The category $\mathscr{D}^{\mathrm{op}}$ endowed with the image by the contravariant functor op: $\mathscr{D} \to \mathscr{D}^{\mathrm{op}}$ of the family of the d.t.in $\mathscr{D}$, is a triangulated category.

6.2 Triangulated and cohomological functors

Definition 6.2.1. (i) A triangulated functor of triangulated categories $F\colon (\mathscr{D}, T) \to (\mathscr{D}', T')$ is an additive functor which satisfies $F \circ T \simeq T' \circ F$ and which sends distinguished triangles to distinguished triangles.

(ii) A triangulated subcategory $\mathscr{D}'$ of $\mathscr{D}$ is a subcategory $\mathscr{D}'$ of $\mathscr{D}$ which is triangulated and such that the functor $\mathscr{D}' \to \mathscr{D}$ is triangulated.

(iii) Let $(\mathscr{D}, T)$ be a triangulated category, $\mathscr{C}$ an abelian category, $F\colon \mathscr{D} \to \mathscr{C}$ an additive functor. One says that F is a cohomological functor if for any d.t.$X \to Y \to Z \to T(X)$ in $\mathscr{D}$, the sequence $F(X) \to F(Y) \to F(Z)$ is exact in $\mathscr{C}$.

Remark 6.2.2. By TR3, a cohomological functor gives rise to a long exact sequence:

$$\cdots \to F(X) \to F(Y) \to F(Z) \to F(T(X)) \to \cdots \tag{6.2.1}$$

Proposition 6.2.3. (i) *If* $X \xrightarrow{f} Y \xrightarrow{g} Z \to T(X)$ *is a* d.t.*then* $g \circ f = 0$.

(ii) *For any* $W \in \mathscr{D}$*, the functors* $\mathrm{Hom}_{\mathscr{D}}(W, \cdot)$ *and* $\mathrm{Hom}_{\mathscr{D}}(\cdot, W)$ *are cohomological.*

Note that (ii) means that if $\varphi : W \to Y$ (resp. $\varphi\colon Y \to W$) satisfies $g \circ \varphi = 0$ (resp. $\varphi \circ f = 0$), then φ factorizes through f (resp. through g).

Proof. (i) Applying TR1 and TR4 we get a commutative diagram:

$$\begin{array}{ccccccc}
X & \xrightarrow{\mathrm{id}} & X & \longrightarrow & 0 & \longrightarrow & T(X) \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle f} & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & T(X).
\end{array}$$

Then $g \circ f$ factorizes through 0.

(ii) Let $X \to Y \to Z \to T(X)$ be a d.t.and let $W \in \mathscr{D}$. We want to show that

$$\mathrm{Hom}\,(W, X) \xrightarrow{f\circ} \mathrm{Hom}\,(W, Y) \xrightarrow{g\circ} \mathrm{Hom}\,(W, Z)$$

is exact, i.e., for all $\varphi \colon W \to Y$ such that $g \circ \varphi = 0$, there exists $\psi \colon W \to X$ such that $\varphi = f \circ \psi$. This means that the dotted arrow below may be completed, and this follows from the axioms TR4 and TR3.

$$\begin{array}{ccccccc}
W & \xrightarrow{\mathrm{id}} & W & \longrightarrow & 0 & \longrightarrow & T(W) \\
\vdots & & \downarrow{\scriptstyle\varphi} & & \downarrow & & \vdots \\
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & T(X).
\end{array}$$

The proof for $\mathrm{Hom}\,(\cdot, W)$ is similar. □

Proposition 6.2.4. *Consider a morphism of* d.t.*:*

$$\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & T(X) \\
{\scriptstyle\alpha}\downarrow & & {\scriptstyle\beta}\downarrow & & {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle T(\alpha)} \\
X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & T(X').
\end{array}$$

If α and β are isomorphisms, then so is γ.

Proof. Apply $\mathrm{Hom}\,(W, \cdot)$ to this diagram and write $\tilde{X}$ instead of $\mathrm{Hom}\,(W, X)$, $\tilde{\alpha}$ instead of $\mathrm{Hom}\,(W, \alpha)$, etc. We get the commutative diagram:

$$\begin{array}{ccccccccc}
\tilde{X} & \xrightarrow{\tilde{f}} & \tilde{Y} & \xrightarrow{\tilde{g}} & \tilde{Z} & \xrightarrow{\tilde{h}} & \widetilde{T(X)} & \xrightarrow{\widetilde{Tf}} & \widetilde{T(Y)} \\
{\scriptstyle\tilde{\alpha}}\downarrow & & {\scriptstyle\tilde{\beta}}\downarrow & & {\scriptstyle\tilde{\gamma}}\downarrow & & {\scriptstyle\widetilde{T(\alpha)}}\downarrow & & {\scriptstyle\widetilde{T\beta}}\downarrow \\
\tilde{X}' & \xrightarrow{\tilde{f}'} & \tilde{Y}' & \xrightarrow{\tilde{g}'} & \tilde{Z}' & \xrightarrow{\tilde{h}'} & \widetilde{T(X')} & \xrightarrow{\widetilde{Tf'}} & \widetilde{TY'}.
\end{array}$$

The rows are exact in view of the preceding proposition and $\tilde{\alpha}$, $\tilde{\beta}$, $\widetilde{T(\alpha)}$, $\widetilde{T(\beta)}$ are isomorphisms. Therefore $\tilde{\gamma} = \mathrm{Hom}\,(W, \gamma) : \mathrm{Hom}\,(W, Z) \to \mathrm{Hom}\,(W, Z')$ is an isomorphism. This implies that γ is an isomorphism by the Yoneda lemma. □

Corollary 6.2.5. *Let $\mathscr{D}'$ be a full triangulated category of $\mathscr{D}$.*

(i) *Consider a triangle $X \xrightarrow{f} Y \to Z \to T(X)$ in $\mathscr{D}'$ and assume that this triangle is distinguished in $\mathscr{D}$. Then it is distinguished in $\mathscr{D}'$.*

(ii) *Consider a* d.t.*$X \to Y \to Z \to T(X)$ in $\mathscr{D}$, with X and Y in $\mathscr{D}'$. Then there exists $Z' \in \mathscr{D}'$ and an isomorphism $Z \simeq Z'$.*

Proof. (i) There exists a d.t.$X \xrightarrow{f} Y \to Z' \to T(X)$ in $\mathscr{D}'$. Then it is isomorphic to the original triangle by TR4 and Proposition 6.2.4.

(ii) Apply TR2 to the morphism $X \to Y$ in $\mathscr{D}'$. □

Remark 6.2.6. (a) The morphism γ in TR 4 is not unique and this is the origin of many troubles.

(b) Similarly, it follows from Proposition 6.2.4 that the object Z given in TR2 is unique up to isomorphism. However, this isomorphism is not unique, and again this is the source of many troubles (e.g., glueing problems in sheaf theory).

6.3 Applications to the homotopy category

Let $\mathscr{C}$ be an additive category. Both $\mathrm{C}(\mathscr{C})$ and $\mathrm{K}(\mathscr{C})$ are endowed with a natural translation functor. (Recall that the homotopy category $\mathrm{K}(\mathscr{C})$ is defined by identifying to zero the morphisms in $\mathrm{C}(\mathscr{C})$ homotopic to zero.)

Also recall that if $f\colon X \to Y$ is a morphism in $\mathrm{C}(\mathscr{C})$, one defines its mapping cone $\mathrm{Mc}(f)$, an object of $\mathrm{C}(\mathscr{C})$, and there is a natural triangle

(6.3.1) $$Y \xrightarrow{\alpha(f)} \mathrm{Mc}(f) \xrightarrow{\beta(f)} X[1] \xrightarrow{f[1]} Y[1].$$

Such a triangle is called a mapping cone triangle. Clearly, a triangle in $\mathrm{C}(\mathscr{C})$ gives rise to a triangle in the homotopy category $\mathrm{K}(\mathscr{C})$.

Definition 6.3.1. A distinguished triangle (d.t.for short) in $\mathrm{K}(\mathscr{C})$ is a triangle isomorphic in $\mathrm{K}(\mathscr{C})$ to a mapping cone triangle.

Theorem 6.3.2. *The category* $\mathrm{K}(\mathscr{C})$ *endowed with the shift functor* $[1]$ *and the family of* d.t.*is a triangulated category.*

We shall not give here the proof of this classical and fundamental result, referring to [KS06, Th. 11.2.6].

Notation 6.3.3. We shall often write $X \to Y \to Z \xrightarrow{+1}$ instead of $X \to Y \to Z \to X[1]$ to denote a d.t.in $\mathrm{K}(\mathscr{C})$.

6.4 Localization of triangulated categories

Recall that a full subcategory $\mathscr{C}'$ of a category $\mathscr{C}$ is saturated if $X \in \mathscr{C}'$ and $Y \simeq X$ in $\mathscr{C}$ implies $Y \in \mathscr{C}'$.

Definition 6.4.1. A *null system* $\mathscr{N}$ in $\mathscr{D}$ is a full triangulated saturated subcategory of $\mathscr{D}$.

A null system $\mathscr{N}$ satisfies:

N1 $0 \in \mathscr{N}$,

N2 $X \in \mathscr{N}$ if and only if $T(X) \in \mathscr{N}$,

N3 if $X \to Y \to Z \to T(X)$ is a d.t.in $\mathscr{D}$ and $X, Y \in \mathscr{N}$ then $Z \in \mathscr{N}$.

One easily checks that if $\mathscr{N}$ is a full saturated subcategory of $\mathscr{D}$ satisfying N1-N2-N3, then the restriction of T to $\mathscr{N}$ and the family of d.t.$X \to Y \to Z \to T(X)$ in $\mathscr{D}$ with $X, Y, Z \in \mathscr{N}$ make $\mathscr{N}$ a null system of $\mathscr{D}$. Moreover, it has the property that given a d.t.as above in $\mathscr{D}$, the three objects X, Y, Z belong to $\mathscr{N}$ as soon as two objects among them belong to $\mathscr{N}$.

To a null system one associates a family of morphisms as follows. Define:

(6.4.1) $$\mathscr{S} := \{f\colon X \to Y, \text{ there exists a d.t.} X \to Y \to Z \to T(X) \text{ with } Z \in \mathscr{N}\}.$$

Lemma 6.4.2. $\mathscr{S}$ *is a right and left multiplicative system.*

Proof. By reversing the arrows, it is enough to prove that $\mathscr{S}$ is a right multiplicative system.

S1 is obvious.

S2 follows from the octahedral axiom TR5 (see (6.1.2)).

S3: There exists a d. t. $W \xrightarrow{h} X \to X' \xrightarrow{+1}$ with $W \in \mathscr{N}$. The morphism $h \circ f \colon W \to Y$ gives rise to a d. t. $W \to Y \to Z \xrightarrow{+1}$ and by TR4 there exists a morphism of triangles

$$\begin{array}{ccccccc} W & \xrightarrow{h} & X & \longrightarrow & X' & \xrightarrow{+1} & \\ \| & & \downarrow{\scriptstyle f} & & \downarrow & & \\ W & \longrightarrow & Y & \longrightarrow & Z & \xrightarrow{+1} & \end{array}$$

S4 By replacing f with $f - g$, it is enough to check that if there exists $s \in \mathscr{S} \colon W \to X$ such that $f \circ s = 0$ then there exists $t \in \mathscr{S} \colon Y \to Z$ such that $t \circ f = 0$. Consider the diagram in which the row is a d.t.:

$$\begin{array}{ccccccc} X' & \xrightarrow{s} & X & \xrightarrow{k} & Z & \xrightarrow{+1} & \\ & & & \searrow^{f} & \downarrow{\scriptstyle h} & & \\ & & & & Y & & \\ & & & & \downarrow{\scriptstyle t} & & \\ & & & & Y' & & \end{array}$$

By Proposition 6.2.3 the sequence

$$\mathrm{Hom}\,(Z, Y) \xrightarrow{\circ k} \mathrm{Hom}\,(X, Y) \xrightarrow{\circ s} \mathrm{Hom}\,(X', Y)$$

is exact. Since $f \circ s = 0$, the dotted arrow h may be completed, making the diagram commutative. Then we embed h in a d. t. and obtain the arrow t. Since $t \circ h = 0$, we get $t \circ f = 0$. Since $Z \in \mathscr{N}$, $t \in \mathscr{S}$. □

Theorem 6.4.3. *Let $\mathscr{D}$ be a triangulated category, $\mathscr{N}$ a null system in $\mathscr{D}$ and let $\mathscr{S}$ be as in* (6.4.1). *Then*

(i) *Denote as usual by $\mathscr{D}_{\mathscr{S}}$ the localization of $\mathscr{D}$ by $\mathscr{S}$ and by Q the localization functor. Then $\mathscr{D}_{\mathscr{S}}$ is an additive category endowed with an automorphism* (*the image of T, still denoted by T*).

(ii) *Define a* d.t. *in $\mathscr{D}_{\mathscr{S}}$ as being isomorphic to the image by Q of a* d.t. *in $\mathscr{D}$. Then $\mathscr{D}_{\mathscr{S}}$ is a triangulated category.*

(iii) *If $X \in \mathscr{N}$, then $Q(X) \simeq 0$.*

(iv) *Let $F \colon \mathscr{D} \to \mathscr{D}'$ be a functor of triangulated categories such that $F(X) \simeq 0$ for any $X \in \mathscr{N}$. Then F factors uniquely through Q.*

The proof being straightforward but tedious, it will not be given here. For a complete proof, see for example [KS06].

Notation 6.4.4. We will write $\mathscr{D}/\mathscr{N}$ instead of $\mathscr{D}_{\mathscr{S}}$.

Now consider a full triangulated subcategory $\mathscr{I}$ of $\mathscr{D}$. denote by $\mathscr{N} \cap \mathscr{I}$ the full subcategory of $\mathscr{D}$ whose objects are $\mathrm{Ob}(\mathscr{N}) \cap \mathrm{Ob}(\mathscr{I})$. This is clearly a null system in $\mathscr{I}$.

Proposition 6.4.5. *Let $\mathscr{D}$ be a triangulated category, $\mathscr{N}$ a null system and $\mathscr{I}$ a full triangulated subcategory of $\mathscr{D}$. Assume condition* (i) *or* (ii) *below*

(i) *any morphism $Y \to Z$ with $Y \in \mathscr{I}$ and $Z \in \mathscr{N}$ factorizes as $Y \to Z' \to Z$ with $Z' \in \mathscr{N} \cap \mathscr{I}$,*

(ii) *any morphism $Z \to Y$ with $Y \in \mathscr{I}$ and $Z \in \mathscr{N}$ factorizes as $Z \to Z' \to Y$ with $Z' \in \mathscr{N} \cap \mathscr{I}$.*

Then the functor $\mathscr{I}/(\mathscr{N} \cap \mathscr{I}) \to \mathscr{D}/\mathscr{N}$ is fully faithful.

Proof. We shall apply Proposition 3.2.1. We may assume (ii), the case (i) being deduced by considering $\mathscr{D}^{\mathrm{op}}$. Let $f\colon Y \to X$ be a morphism in $\mathscr{S}$ with $Y \in \mathscr{I}$. We shall show that there exists $g\colon X \to W$ with $W \in \mathscr{I}$ and $g \circ f \in \mathscr{S}$. The morphism f is embedded in a d.t.$Y \to X \to Z \to T(Y)$ with $Z \in \mathscr{N}$. By the hypothesis, the morphism $Z \to T(Y)$ factorizes through an object $Z' \in \mathscr{N} \cap \mathscr{I}$. We may embed $Z' \to T(Y)$ into a d.t.and obtain a commutative diagram of d.t.:

$$\begin{array}{ccccccc} Y & \xrightarrow{f} & X & \longrightarrow & Z & \longrightarrow & T(Y) \\ \downarrow{\scriptstyle \mathrm{id}} & & \vdots{\scriptstyle g} & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ Y & \xrightarrow{g\circ f} & W & \longrightarrow & Z' & \longrightarrow & T(Y) \end{array}$$

By TR4, the dotted arrow g may be completed and Z' belonging to $\mathscr{N}$, this implies that $g \circ f \in \mathscr{S}$. □

Proposition 6.4.6. *Let $\mathscr{D}$ be a triangulated category, $\mathscr{N}$ a null system and $\mathscr{I}$ a full triangulated subcategory of $\mathscr{D}$. Assume conditions* (i) *or* (ii) *below:*

(i) *for any $X \in \mathscr{D}$, there exists a* d.t.$X \to Y \to Z \to T(X)$ *with $Z \in \mathscr{N}$ and $Y \in \mathscr{I}$,*

(ii) *for any $X \in \mathscr{D}$, there exists a* d.t.$Y \to X \to Z \to T(X)$ *with $Z \in \mathscr{N}$ and $Y \in \mathscr{I}$.*

Then $\mathscr{I}/\mathscr{N} \cap \mathscr{I} \to \mathscr{D}/\mathscr{N}$ is an equivalence of categories.

Proof. Apply Corollary 3.2.2. □

Localization of triangulated functors

Let $F\colon \mathscr{D} \to \mathscr{D}'$ be a functor of triangulated categories and let $\mathscr{N}$ be a null system in $\mathscr{D}$. One defines the localization of F similarly as in the usual case, replacing all categories and functors by triangulated ones. Applying Proposition 3.3.2, we get:

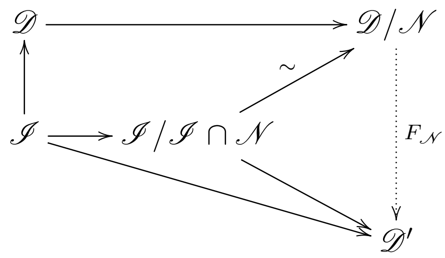
Theorem 6.4.7. *Let $F\colon \mathscr{D} \to \mathscr{D}'$ be a functor of triangulated categories. Let $\mathscr{N}$ a null system of $\mathscr{D}$ and $\mathscr{I}$ a full triangulated subcategory of $\mathscr{D}$. Assume*

(a) *for any* $X \in \mathscr{D}$*, there exists a* d.t.$X \to Y \to Z \to T(X)$ *with* $Z \in \mathscr{N}$ *and* $Y \in \mathscr{I}$*,*

(b) *for any* $Y \in \mathscr{N} \cap \mathscr{I}$*,* $F(Y) \simeq 0$*.*

Then F *is right localizable.*

One defines $F_{\mathscr{N}}$ by the diagram:



If one replaces condition (a) in Theorem 6.4.7 by the condition

(a)' for any $X \in \mathscr{D}$, there exists a d.t.$Y \to X \to Z \to T(X)$ with $Z \in \mathscr{N}$ and $Y \in \mathscr{I}$,

one gets that F is left localizable.

Finally, let us consider triangulated bifunctors, i.e., bifunctors which are additive and triangulated with respect to each of their arguments.

Theorem 6.4.8. *Let* $F \colon \mathscr{D} \times \mathscr{D}' \to \mathscr{D}''$ *be a triangulated bifunctor. Let* $\mathscr{N}$ *and* $\mathscr{N}'$ *be null systems of* $\mathscr{D}$ *and* $\mathscr{D}'$*, respectively, and let* $\mathscr{I}$ *and* $\mathscr{I}'$ *be full triangulated subcategories of* $\mathscr{D}$ *and* $\mathscr{D}'$*, respectively. Assume:*

(a) *for any* $X \in \mathscr{D}$*, there exists a* d.t.$X \to Y \to Z \to T(X)$ *with* $Z \in \mathscr{N}$ *and* $Y \in \mathscr{I}$

(b) *for any* $X' \in \mathscr{D}'$*, there exists a* d.t.$X' \to Y' \to Z' \to T(X')$ *with* $Z' \in \mathscr{N}'$ *and* $Y' \in \mathscr{I}'$

(c) *for any* $Y \in \mathscr{I}$ *and* $Y' \in \mathscr{I}' \cap \mathscr{N}'$*,* $F(Y, Y') \simeq 0$*,*

(d) *for any* $Y \in \mathscr{I} \cap \mathscr{N}$ *and* $Y' \in \mathscr{I}'$*,* $F(Y, Y') \simeq 0$*.*

Then F *is right localizable.*

The proof is similar to that of Theorem 6.4.7 and left to the reader.

One denotes by $F_{\mathscr{N}\mathscr{N}'}$ its localization.

Of course, there exists a similar result for left localizable functors by reversing the arrows in the hypotheses (a) and (b) above.

Localization and direct sums

Proposition 6.4.9. *Let* $\mathscr{D}$ *be a triangulated category admitting small direct sums.*

(a) *The shift functor commutes with small direct sum and a small direct sum of* d.t. *is again a* d.t.

(b) *Let $\mathscr{N}$ be a null system in $\mathscr{D}$ stable by small direct sums. Then $\mathscr{D}/\mathscr{N}$ admits small direct sums and the localization functor Q commutes with such direct sums.*

Proof. We shall follow [KS06, Prop. 10.1.19 , 10.2.8].
(a) The functor T commutes with direct sums since is is an automorphism of $\mathscr{D}$. Let I be a small set. Consider a family of d.t.'s indexed by I

$$D_i := X_i \to Y_i \to Z_i \to TX_i$$

and consider the (not necessarily distinguished) triangle in which the composition of two arrows is 0:

$$D := \oplus_i X_i \to \oplus_i Y_i \to \oplus_i Z_i \to T(\oplus_i X_i).$$

On the other hand, consider a d.t..

$$D' := \oplus_i X_i \to \oplus_i Y_i \to Z \to T(\oplus_i X_i).$$

By TR3, there exist morphisms of triangles $D_i \to D'$ and they induce a morphism $u\colon D \to D'$ in $\mathscr{D}$. In order to show that this morphism is an isomorphism, it is enough to prove that for any $W \in \mathscr{D}$, it induces an isomorphism

$$\mathrm{Hom}_{\mathscr{D}}(u, W)\colon \mathrm{Hom}_{\mathscr{D}}(D', W) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{D}}(D, W).$$

Here, we write for short $\mathrm{Hom}_{\mathscr{D}}(D, W)$ instead of the sequence

$$(6.4.2)\quad \cdots \to \mathrm{Hom}_{\mathscr{D}}(\oplus_i Z_i, W) \to \mathrm{Hom}_{\mathscr{D}}(\oplus_i Y_i, W) \to \mathrm{Hom}_{\mathscr{D}}(\oplus_i X_i, W) \to \cdots$$

and similarly with $\mathrm{Hom}_{\mathscr{D}}(D', W)$.

The sequence $\mathrm{Hom}_{\mathscr{D}}(D', W)$ is exact since the functor Hom is cohomological and D' is a d.t.. The sequence (6.4.2) is also exact since it is isomorphic to the exact sequences

$$\cdots \to \prod_i \mathrm{Hom}_{\mathscr{D}}(Z_i, W) \to \prod_i \mathrm{Hom}_{\mathscr{D}}(Y_i, W) \to \prod_i \mathrm{Hom}_{\mathscr{D}}(X_i, W) \to \cdots$$

Hence, the sequence $\mathrm{Hom}_{\mathscr{D}}(D, W)$ is exact by Lemma 5.5.7.
(b)–(i) Let $\{X_i\}_{i\in I}$ be a small family of objects in $\mathscr{D}$ and let $Y \in \mathscr{D}$. A morphism $u\colon Q(\oplus_i X_i) \to Q(Y)$ defines for each i a morphism $u_i\colon Q(X_i) \to Q(Y)$. Hence we have a natural map

$$\theta\colon \mathrm{Hom}_{\mathscr{D}/\mathscr{N}}(Q(\oplus_i X_i), Q(Y)) \xrightarrow{\sim} \prod_i \mathrm{Hom}_{\mathscr{D}/\mathscr{N}}(Q(X_i), Q(Y)).$$

In order to prove that $Q(\oplus_i X_i)$ is the direct sum of the family $Q(X_i)$, it is enough to check that θ is bijective for any $Y \in \mathscr{D}$.
(b)–(ii) The map θ is surjective. Consider morphisms $u_i\colon \mathrm{Hom}_{\mathscr{D}/\mathscr{N}}(Q(X_i), Q(Y))$. We represent each u_i by a morphism $v_i\colon X'_i \to Y$ together with a d. t. $X'_i \to X_i \to Z_i \xrightarrow{+1}$ with $Z_i \in \mathscr{N}$. We get a morphism $v\colon \oplus_i X'_i \to Y$ and a d. t. $\oplus_i X'_i \to \oplus_i X_i \to \oplus_i Z_i \xrightarrow{+1}$. By the hypothesis, $\oplus_i Z_i \in \mathscr{N}$ and it follows that v defines a morphism $Q(\oplus_i X_i) \to Q(Y)$ in $\mathscr{D}/\mathscr{N}$.

(b)–(iii) The map θ is injective. Assume that the composition $Q(X_j) \to Q(\oplus_i X_i) \xrightarrow{u} Q(Y)$ is 0 for all $j \in I$. The morphism u may be represented by morphisms $\oplus_i X_i \xrightarrow{v} Y' \xleftarrow{s} Y$ with $s \in \mathscr{S}$ where $\mathscr{S}$ is the multiplicative system associated with $\mathscr{N}$ and the image of the composition $X_j \to \oplus_i X_i \xrightarrow{v} Y'$ is zero in $\mathscr{D}/\mathscr{N}$. By the result of Exercise 6.7, for each i there exists $Z_j \in \mathscr{N}$ such that this composition factorizes as $X_j \to Z_j \to Y'$. Therefore, $\oplus_j X_j \to Y'$ factorizes as $\oplus_j X_j \to \oplus_j Z_j \to Y'$ and thus $Q(u) = 0$. $\square$

Exercises to Chapter 6

Exercise 6.1. Let $\mathscr{D}$ be a triangulated category and consider a commutative diagram in $\mathscr{D}$:

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & T(X) \\ \| & & \| & & \downarrow{\scriptstyle\gamma} & & \| \\ X & \xrightarrow{f} & Y & \xrightarrow{g'} & Z' & \xrightarrow{h'} & T(X). \end{array}$$

Assume that $T(f) \circ h' = 0$ and the first row is a d.t.Prove that the second row is also a d.t.under one of the hypotheses:
(i) for any $P \in \mathscr{D}$, the sequence below is exact:

$$\mathrm{Hom}_{\mathscr{D}}(P, X) \to \mathrm{Hom}_{\mathscr{D}}(P, Y) \to \mathrm{Hom}_{\mathscr{D}}(P, Z') \to \mathrm{Hom}_{\mathscr{D}}(P, T(X)),$$

(ii) for any $P \in \mathscr{D}$, the sequence below is exact:

$$\mathrm{Hom}_{\mathscr{D}}(T(Y), P) \to \mathrm{Hom}_{\mathscr{D}}(T(X), P) \to \mathrm{Hom}_{\mathscr{D}}(Z', P) \to \mathrm{Hom}_{\mathscr{D}}(Y, P).$$

Exercise 6.2. Let $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T(X)$ be a d.t.in a triangulated category.
(i) Prove that if $h = 0$, this d.t.is isomorphic to $X \to X \oplus Z \to Z \xrightarrow{0} T(X)$.
(ii) Prove the same result by assuming now that there exists $k : Y \to X$ with $k \circ f = \mathrm{id}_X$.
(Hint: to prove (i), construct the morphism $Y \to X \oplus Z$ by TR4, then use Proposition 6.2.4.)

Exercise 6.3. Let X and Y be objects of a triangulated category. Prove that $X \to X \oplus Y \to Y \xrightarrow{0} T(X)$ is a d.t.

Exercise 6.4. Let $X \xrightarrow{f} Y \to Z \to T(X)$ be a d.t.in a triangulated category. Prove that f is an isomorphism if and only if Z is isomorphic to 0.

Exercise 6.5. Let $f \colon X \to Y$ be a monomorphism in a triangulated category $\mathscr{D}$. Prove that there exist $Z \in \mathscr{D}$ and an isomorphism $h \colon Y \xrightarrow{\sim} X \oplus Z$ such that the composition $X \to Y \to X \oplus Z$ is the canonical morphism.

Exercise 6.6. Let $\mathscr{D}$ be a triangulated category, $\mathscr{N}$ a null system and let Y be an object of $\mathscr{D}$ such that $\mathrm{Hom}_{\mathscr{D}}(Z, Y) \simeq 0$ for all $Z \in \mathscr{N}$. Prove that $\mathrm{Hom}_{\mathscr{D}}(X, Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{D}/\mathscr{N}}(X, Y)$.

Exercise 6.7. Let $\mathscr{D}$ be a triangulated category, $\mathscr{N}$ a null system and let $Q\colon \mathscr{D} \to \mathscr{D}/\mathscr{N}$ be the canonical functor.

(i) Let $f\colon X \to Y$ be a morphism in $\mathscr{D}$ and assume that $Q(f) = 0$ in $\mathscr{D}/\mathscr{N}$. Prove that there exists $Z \in \mathscr{N}$ such that f factorizes as $X \to Z \to Y$.

(ii) For $X \in \mathscr{D}$, prove that $Q(X) \simeq 0$ if and only if there exists Y such that $X \oplus Y \in \mathscr{N}$ and this last condition is equivalent to $X \oplus TX \in \mathscr{N}$.

Chapter 7

Derived categories

Summary

This chapter is devoted to derived categories. Recall that the homotopy category $\mathrm{K}(\mathscr{C})$ of an additive category $\mathscr{C}$ is triangulated. When $\mathscr{C}$ is abelian, the cohomology functor $H^0\colon \mathrm{K}(\mathscr{C}) \to \mathscr{C}$ is cohomological and the derived category $\mathrm{D}(\mathscr{C})$ of $\mathscr{C}$ is obtained by localizing $\mathrm{K}(\mathscr{C})$ with respect to the family of quasi-isomorphisms. We explain here this construction, with some examples. We also construct the right derived functor of a left exact functor as well as a bifunctor. Some classical examples are discussed.

Finally, we state, without proof, the Brown representability theorem, a fundamental result for applications.

Some references. We refer to the Introduction for a brief history of the genesis of theory. Derived categories are constructed in many places, among which [GM96, Har66, KS90, KS06, Ver96, Wei94, Yek20].

7.1 Derived categories

Construction of the derived category

From now on, $\mathscr{C}$ will denote an abelian category.

Recall that if $f\colon X \to Y$ is a morphism in $\mathrm{C}(\mathscr{C})$, one says that f is a quasi-isomorphism (a qis, for short) if $H^k(f)\colon H^k(X) \to H^k(Y)$ is an isomorphism for all $k \in \mathbb{Z}$. One extends this definition to morphisms in $\mathrm{K}(\mathscr{C})$.

If one embeds f into a d.t.$X \xrightarrow{f} Y \to Z \xrightarrow{+1}$, then f is a qis iff $H^k(Z) \simeq 0$ for all $k \in \mathbb{Z}$, that is, if Z is qis to 0.

Proposition 7.1.1. *Let $\mathscr{C}$ be an abelian category. The functor $H^0\colon K(\mathscr{C}) \to \mathscr{C}$ is a cohomological functor.*

Proof. Let $X \xrightarrow{f} Y \to Z \xrightarrow{+1}$ be a d.t.Then it is isomorphic to $X \to Y \xrightarrow{\alpha(f)} \mathrm{Mc}(f) \xrightarrow{\beta(f)} X[1] \xrightarrow{+1}$. Since the sequence in $\mathrm{C}(\mathscr{C})$:

$$0 \to Y \to \mathrm{Mc}(f) \to X[1] \to 0$$

is exact, it follows from Theorem 5.5.9 that the sequence

$$H^k(Y) \to H^k(\mathrm{Mc}(f)) \to H^{k+1}(X)$$

is exact. Therefore, $H^k(Y) \to H^k(Z) \to H^{k+1}(X)$ is exact. □

Corollary 7.1.2. *Let* $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ *be an exact sequence in* $\mathrm{C}(\mathscr{C})$ *and define* $\varphi\colon \mathrm{Mc}(f) \to Z$ *as* $\varphi^n = (0, g^n)$. *Then* φ *is a qis.*

Proof. Consider the exact sequence in $\mathrm{C}(\mathscr{C})$:

$$0 \to M(\mathrm{id}_X) \xrightarrow{\gamma} \mathrm{Mc}(f) \xrightarrow{\varphi} Z \to 0$$

where $\gamma^n\colon (X^{n+1} \oplus X^n) \to X^{n+1} \oplus Y^n$ is defined by: $\gamma^n = \begin{pmatrix} \mathrm{id}_{X^{n+1}} & 0 \\ 0 & f^n \end{pmatrix}$. Since $H^k(\mathrm{Mc}(\mathrm{id}_X)) \simeq 0$ for all k, we get the result by Proposition 7.1.1. □

We shall localize $\mathrm{K}(\mathscr{C})$ with respect to the family of objects qis to zero (see Section 6.4). Define:

$$N(\mathscr{C}) = \{X \in \mathrm{K}(\mathscr{C}); H^k(X) \simeq 0 \ \text{ for all } k\}.$$

One also defines $N^*(\mathscr{C}) = N(\mathscr{C}) \cap \mathrm{K}^*(\mathscr{C})$ for $^* = \mathrm{b}, +, -$.

Clearly, $N^*(\mathscr{C})$ is a null system in $\mathrm{K}^*(\mathscr{C})$. Denote by $\mathscr{S}^*(\mathscr{C})$ the multiplicative system associated with $N^*(\mathscr{C})$ as in (6.4.1) and recall Definition 3.1.18 of a multiplicative system.

Lemma 7.1.3. *For* $^* = \mathrm{ub}, \mathrm{b}, +, -$, *the multiplicative system* $\mathscr{S}^*(\mathscr{C})$ *is saturated.*

Proof. It is enough to teat the case $* = \mathrm{ub}$. Hence, let $f\colon X \to Y$, $g\colon Y \to Z$ and $h\colon Z \to W$ be morphisms in $\mathrm{K}(\mathscr{C})$. Assume that $g \circ f$ and $h \circ g$ are qis. This means that for all $k \in \mathbb{Z}$, the morphisms $H^k(g \circ f)\colon H^k(X) \to H^k(Z)$ and $H^k(h \circ g)\colon H^k(Y) \to H^k(W)$ are isomorphisms. Since $H^k(g \circ f) = H^k(g) \circ H^k(f)$ and $H^k(h \circ g) = H^k(h) \circ H^k(G)$, the result follows from Exercise 1.1. □

Definition 7.1.4. One defines the derived categories $\mathrm{D}^*(\mathscr{C})$ as $\mathrm{K}^*(\mathscr{C})/N^*(\mathscr{C})$, where $* = \mathrm{ub}, \mathrm{b}, +, -$. One denotes by Q the localization functor $\mathrm{K}^*(\mathscr{C}) \to \mathrm{D}^*(\mathscr{C})$.

Remark 7.1.5. One shall be aware that in general, the derived category $\mathrm{D}^+(\mathscr{C})$ of a $\mathscr{U}$-category $\mathscr{C}$ is no more a $\mathscr{U}$-category (see Remark 7.2.7).

By Theorem 6.4.3, the categories $\mathrm{D}^*(\mathscr{C})$ are triangulated.

Applying Lemma 7.1.3 and Corollary 3.1.19, we get:

Proposition 7.1.6. *Let* $X \in \mathrm{K}(\mathscr{C})$ *and let* $Q(X)$ *denote its image in* $\mathrm{D}(\mathscr{C})$. *Then* $Q(X) \simeq 0$ *in* $\mathrm{D}(\mathscr{C})$ *if and only if* X *is qis to* 0 *in* $\mathrm{K}(\mathscr{C})$.

Recall the truncation functors given in (5.5.10). These functors send a complex homotopic to zero to a complex homotopic to zero, hence are well defined on $\mathrm{K}^+(\mathscr{C})$. Moreover, they send a qis to a qis. Hence the functors below are well defined:

$$\begin{aligned} &H^j(\,\bullet\,)\colon \mathrm{D}(\mathscr{C}) \to \mathscr{C}, \\ &\tau^{\leq n}, \widetilde{\tau}^{\leq n}\colon \mathrm{D}(\mathscr{C}) \to \mathrm{D}^-(\mathscr{C}), \\ &\tau^{\geq n}, \widetilde{\tau}^{\geq n}\colon \mathrm{D}(\mathscr{C}) \to \mathrm{D}^+(\mathscr{C}). \end{aligned} \tag{7.1.1}$$

Note that:

- there are isomorphisms of functors
$$\tau^{\leq n} \simeq \widetilde{\tau}^{\leq n}, \quad \tau^{\geq n} \simeq \widetilde{\tau}^{\geq n},$$
- $H^j(\,\cdot\,)$ is a cohomological functor on $\mathrm{D}^*(\mathscr{C})$ (apply Proposition 7.1.1).

In particular, if $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{+1}$ is a d.t.in $\mathrm{D}(\mathscr{C})$, we get a long exact sequence:

$$\cdots \to H^k(X) \to H^k(Y) \to H^k(Z) \to H^{k+1}(X) \to \cdots \tag{7.1.2}$$

Let $X \in \mathrm{K}(\mathscr{C})$, with $H^j(X) = 0$ for $j > n$. Then the morphism $\tau^{\leq n}X \to X$ in $\mathrm{K}(\mathscr{C})$ is a qis, hence an isomorphism in $\mathrm{D}(\mathscr{C})$.

It follows from Proposition 6.4.5 that $\mathrm{D}^+(\mathscr{C})$ is equivalent to the full subcategory of $\mathrm{D}(\mathscr{C})$ consisting of objects X satisfying $H^j(X) \simeq 0$ for $j << 0$, and similarly for $\mathrm{D}^-(\mathscr{C}), \mathrm{D}^{\mathrm{b}}(\mathscr{C})$. Moreover, $\mathscr{C}$ is equivalent to the full subcategory of $\mathrm{D}(\mathscr{C})$ consisting of objects X satisfying $H^j(X) \simeq 0$ for $j \neq 0$. For $a, b \in \mathbb{Z} \sqcup \{-\infty\} \sqcup \{+\infty\}$ with $a \leq b$, one sets

$$\mathrm{D}^{[a,b]}(\mathscr{C}) := \{X \in \mathrm{D}(\mathscr{C}); H^j(X) \simeq 0 \text{ for } j \notin [a,b]\}. \tag{7.1.3}$$

One defines similarly $\mathrm{D}^{\geq k}(\mathscr{C})$, $\mathrm{D}^{\leq k}(\mathscr{C})$, etc.

Definition 7.1.7. Let X, Y be objects of $\mathscr{C}$ and let $k \in \mathbb{Z}$. One sets

$$\mathrm{Ext}^k_{\mathscr{C}}(X, Y) = \mathrm{Hom}_{\mathrm{D}(\mathscr{C})}(X, Y[k]).$$

Of course, $\mathrm{Ext}^k_{\mathscr{C}}(X, Y)$ vanishes for $k < 0$.

Notation 7.1.8. Let A be a ring. We shall write for short $\mathrm{D}^*(A)$ instead of $\mathrm{D}^*(\mathrm{Mod}(A))$, for $^* = \emptyset, \mathrm{b}, +, -$.

Remark 7.1.9. Let $f\colon X \to Y$ be a morphism in $\mathrm{C}(\mathscr{C})$. Then $f = 0$ in $\mathrm{D}(\mathscr{C})$ iff there exists X' and a qis $g\colon X' \to X$ such that $f \circ g$ is homotopic to 0, or else iff there exists Y' and a qis $h\colon Y \to Y'$ such that $h \circ f$ is homotopic to 0.

Remark 7.1.10. Consider an exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathscr{C}$. It gives rise to a d.t.$X \to Y \to Z \to X[1]$ in $\mathrm{D}(\mathscr{C})$. Consider the morphism $\gamma\colon Z \to X[1]$ in $\mathrm{D}(\mathscr{C})$. It defines morphisms $H^k(\gamma)\colon H^k(Z) \to H^{k+1}(X)$ is 0 for all $k \in \mathbb{Z}$ and X and Z being concentrated in degree 0, we get that $H^k(\gamma) = 0$ for all $k \in \mathbb{Z}$. However, γ is *not* the zero morphism in $\mathrm{D}(\mathscr{C})$ in general (this happens only if the short exact sequence splits). In fact, let us apply the cohomological functor $\mathrm{Hom}_{\mathrm{D}(\mathscr{C})}(W, \,\cdot\,)$ to the d.t.above. It gives rise to the long exact sequence:

$$\cdots \to \mathrm{Hom}_{\mathrm{D}(\mathscr{C})}(W, Y) \to \mathrm{Hom}_{\mathrm{D}(\mathscr{C})}(W, Z) \xrightarrow{\tilde{\gamma}} \mathrm{Hom}_{\mathrm{D}(\mathscr{C})}(W, X[1]) \to \cdots$$

where $\tilde{\gamma} = \mathrm{Hom}_{\mathrm{D}(\mathscr{C})}(W, \gamma)$. Since $\mathrm{Hom}_{\mathrm{D}(\mathscr{C})}(W, Y) \to \mathrm{Hom}_{\mathrm{D}(\mathscr{C})}(W, Z)$ is not an epimorphism in general, $\tilde{\gamma}$ is not zero. Therefore γ is not zero in general (see Theorem 7.4.10 below). The morphism γ may be described as follows (see Example 4.2.4), where φ is a qis in $\mathrm{C}(\mathscr{C})$:

$$\begin{array}{ccccccccc}
Z := & & 0 & \to & 0 & \to & Z & \to & 0 \\
\uparrow{\scriptstyle\varphi} & & & & \uparrow & & \uparrow & & \\
\mathrm{Mc}(f) := & & 0 & \to & X & \xrightarrow{f} & Y & \to & 0 \\
\downarrow{\scriptstyle\beta(f)} & & & & \downarrow{\scriptstyle\mathrm{id}} & & \downarrow & & \\
X[1] := & & 0 & \to & X & \to & 0 & \to & 0.
\end{array}$$

By using the exact sequences (5.5.11), we get:

Proposition 7.1.11. *Let $X \in \mathrm{D}(\mathscr{C})$.*

(i) *There are* d.t. *in* $\mathrm{D}(\mathscr{C})$*:*

$$\begin{aligned} &\tau^{\leq n}X \to X \to \tau^{\geq n+1}X \xrightarrow{+1}, \\ &\tau^{\leq n-1}X \to \tau^{\leq n}X \to H^n(X)[-n] \xrightarrow{+1}, \\ &H^n(X)[-n] \to \tau^{\geq n}X \to \tau^{\geq n+1}X \xrightarrow{+1}. \end{aligned} \tag{7.1.4}$$

(ii) *Moreover,* $H^n(X)[-n] \simeq \tau^{\leq n}\tau^{\geq n}X \simeq \tau^{\geq n}\tau^{\leq n}X$.

Corollary 7.1.12. *Let $\mathscr{C}$ be an abelian category and assume that for any $Y, Z \in \mathscr{C}$, $\mathrm{Ext}^k_{\mathscr{C}}(Y,Z) \simeq 0$ for $k \geq 2$. Let $X \in \mathrm{D}^{\mathrm{b}}(\mathscr{C})$. Then:*

$$X \simeq \bigoplus_j H^j(X)\,[-j].$$

Proof. We may assume that $H^j(X) \simeq 0$ if $j \notin [0,n]$ for some integer n. We argue by induction on $n \geq 0$, the case $n = 0$ being obvious. The second d.t. in (7.1.4) gives the d.t.:

$$X \to H^n(X)\,[-n] \to \tau^{\leq n-1}X\,[1] \xrightarrow{+1}.$$

By the induction hypothesis, $\tau^{\leq n-1}X \simeq \oplus_{j<n} H^j(X)\,[-j]$. Now we have

$$\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(\mathscr{C})}(H^n(X)[-n], H^j(X)\,[-j+1]) \simeq \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(\mathscr{C})}(H^n(X), H^j(X)\,[n-j+1])$$

and these groups are 0 for $j < n$ by the hypothesis. Therefore,

$$\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(\mathscr{C})}(H^n(X)[-n], \tau^{\leq n-1}X\,[1]) \simeq 0$$

and the result follows from the result of Exercise 6.2, □

Example 7.1.13. If a ring A is a principal ideal domain (such as a field, or $\mathbb{Z}$, or $\mathbf{k}[x]$ for $\mathbf{k}$ a field), then the category $\mathrm{Mod}(A)$ satisfies the hypotheses of Corollary 7.1.12.

7.2 Resolutions

Definition 7.2.1. Let $\mathscr{J}$ be a full additive subcategory of $\mathscr{C}$. We say that $\mathscr{J}$ is cogenerating if for all X in $\mathscr{C}$, there exist $Y \in \mathscr{J}$ and a monomorphism $X \rightarrowtail Y$.

If $\mathscr{J}$ is cogenerating in $\mathscr{C}^{\mathrm{op}}$, one says that $\mathscr{J}$ is generating in $\mathscr{C}$.

Theorem 7.2.2. *Assume $\mathscr{J}$ is cogenerating. Then for any $a \in \mathbb{Z}$ and $X^\bullet \in \mathrm{C}^{\geq a}(\mathscr{C})$, there exist $Y^\bullet \in \mathrm{C}^{\geq a}(\mathscr{J})$ and a quasi-isomorphism $X^\bullet \to Y^\bullet$.*

Proof. We shall follows the proof of [KS06, Lem. 13.2.1].

Let $X^\bullet \in \mathrm{C}^{\geq a}(\mathscr{C})$. We shall construct by induction on p a complex $Y^\bullet_{\leq p}$ in $\mathscr{J}$ and a morphism $f\colon X^\bullet \to Y^\bullet_{\leq p}$ such that $H^k(X^\bullet) \to H^k(Y^\bullet_{\leq p})$ is an isomorphism for $k < p$ and is a monomorphism for $k = p$, visualized by the diagram

$$\begin{array}{lcccccccc}
X^\bullet := & \cdots \longrightarrow & X^{p-1} & \xrightarrow{d_X^{p-1}} & X^p & \xrightarrow{d_X^p} & X^{p+1} & \xrightarrow{d_X^{p+1}} & \cdots \\
& & \downarrow{\scriptstyle f^{p-1}} & & \downarrow{\scriptstyle f^p} & & & & \\
Y^\bullet_{\leq p} := & \cdots \longrightarrow & Y^{p-1} & \xrightarrow{d_Y^{p-1}} & Y^p & & & &
\end{array}$$

For $p < a$, choose $Y^\bullet_{\leq p} = 0$. Now assume that $Y^\bullet_{\leq p}$ has been constructed. Set

$$Z^p = \operatorname{Coker} d_Y^{p-1} \oplus_{\operatorname{Coker} d_X^{p-1}} \operatorname{Ker} d_X^{p+1},$$
$$W^p = \operatorname{Coker} d_Y^{p-1} \oplus_{\operatorname{Coker} d_X^{p-1}} X^{p+1}.$$

(Recall that in an abelian category, given two morphisms $Z \to X$ and $Z \to Y$, $X \oplus_Z Y$ is the cokernel of $Z \to X \oplus Y$.) Hence, there is a monomorphism $Z^p \hookrightarrow W^p$. Consider the commutative diagram

$$\text{(7.2.1)}\quad \begin{array}{ccccccccc}
0 \longrightarrow & H^p(X^\bullet) & \longrightarrow & \operatorname{Coker} d_X^{p-1} & \longrightarrow & \operatorname{Ker} d_X^{p+1} & \longrightarrow & H^{p+1}(X^\bullet) & \longrightarrow 0 \\
& \downarrow{\scriptstyle =} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle =} & \\
0 \longrightarrow & H^p(X^\bullet) & \longrightarrow & \operatorname{Coker} d_Y^{p-1} & \longrightarrow & Z^p & \longrightarrow & H^{p+1}(X^\bullet) & \longrightarrow 0
\end{array}$$

The top row is clearly exact. The sequence $\operatorname{Coker} d_X^{p-1} \to \operatorname{Coker} d_Y^{p-1} \oplus \operatorname{Ker} d_X^{p+1} \to H^{p+1}(X^\bullet) \to 0$ defines the morphism $Z^p \to H^{p+1}(X^\bullet)$ and one checks that the sequence $\operatorname{Coker} d_Y^{p-1} \to Z^p \to H^{p+1}(X^\bullet) \to 0$ is exact. Denote by K^p the kernel of $\operatorname{Coker} d_Y^{p-1} \to Z^p$. We get a morphism $u\colon H^p(X^\bullet) \to K^p$ which is a monomorphism by the induction hypothesis and which is an epimorphism thanks to the fact that the middle square in (7.2.1) is co-Cartesian (see [KS06, Exe. 8.21]). Therefore, Diagram 7.2.1 is exact. Since $\mathscr{J}$ is cogenerating, we may find a monomorphism $W^p \hookrightarrow Y^{p+1}$ with $Y^{p+1} \in \mathscr{J}$. The natural morphisms $X^{p+1} \to W^p$ and $Y^p \to W^p$ define the morphisms $f^{p+1}\colon X^{p+1} \to Y^{p+1}$ and $d_Y^p\colon Y^p \to Y^{p+1}$. Let $Y^\bullet_{\leq p+1}$ be the complex so constructed. Then

$$H^p(Y^\bullet_{\leq p+1}) \simeq \mathrm{Ker}\big(\mathrm{Coker}\, \mathrm{d}_Y^{p-1} \to \mathrm{Y}^{p+1}\big) \simeq \mathrm{Ker}\big(\mathrm{Coker}\, \mathrm{d}_Y^{p-1} \to \mathrm{Z}^p\big) \simeq \mathrm{H}^p(\mathrm{X}^\bullet).$$

Finally, the monomorphism $Z^p \to Y^{p+1}$ induces the monomorphism

$$H^{p+1}(X^\bullet) \simeq \mathrm{Coker}\big(\mathrm{Coker}\, \mathrm{d}_Y^{p-1} \to \mathrm{Z}^p\big)$$
$$\to \mathrm{Coker}\big(\mathrm{Coker}\, \mathrm{d}_Y^{p-1} \to \mathrm{Y}^{p+1}\big) \simeq \mathrm{H}^{p+1}(\mathrm{Y}^\bullet_{\leq p+1}).$$

□

We shall also have to consider the following situation. Consider the hypothesis

$$\text{(7.2.2)}\quad \begin{cases} \text{there exists } d \in \mathbb{N}_{>0} \text{ such that for any exact sequence} \\ Y_1 \to \cdots Y_d \to Y \to 0, \text{ with } Y_j \in \mathscr{J},\ 1 \leq j \leq d, \text{ we have } Y \in \mathscr{J}. \end{cases}$$

Corollary 7.2.3. *Assume $\mathscr{J}$ is cogenerating and satisfies (7.2.2). Then for any $X^\bullet \in \mathrm{C}^{[a,b]}(\mathscr{C})$, there exist $Y^\bullet \in \mathrm{C}^{[a,b+d+1]}(\mathscr{J})$ and a quasi-isomorphism $X^\bullet \to Y^\bullet$.*

Proof. Let $X^\bullet \to Y^\bullet$ be a quasi-isomorphism given by Theorem 7.2.2, with $Y^\bullet \in \mathrm{C}^{\geq a}(\mathscr{J})$. Consider the truncation functor $\tau^{\leq j}$ of (5.5.10). It induces an isomorphism $\tau^{\leq j}(X^\bullet) \xrightarrow{\sim} X^\bullet$ for $j > b$ and a quasi-isomorphism for all j:

$$\tau^{\leq j}(X^\bullet) \xrightarrow{qis} \tau^{\leq j}(Y^\bullet).$$

Moreover, the sequence $Y^{b+1} \to Y^{b+2} \to \cdots$ is exact. Thanks to the hypothesis, we get that $\tau^{\leq b+d}(Y^\bullet)$ belongs to $\mathrm{C}^{[a,b+d+1]}(\mathscr{J})$ and this complex is qis to $X^\bullet$. □

In the sequel, for $\mathscr{J}$ an additive subcategory of $\mathscr{C}$, we set

$$N^+(\mathscr{J}) := N(\mathscr{C}) \cap \mathrm{K}^+(\mathscr{J}). \tag{7.2.3}$$

It is clear that $N^+(\mathscr{J})$ is a null system in $\mathrm{K}^+(\mathscr{J})$.

Applying Proposition 6.4.6, we get:

Corollary 7.2.4. *Let $\mathscr{J}$ be an additive subcategory of $\mathscr{C}$ and assume that $\mathscr{J}$ is cogenerating. Then*

(a) *For any $X^\bullet \in \mathrm{K}^+(\mathscr{C})$, there exists $Y^\bullet \in \mathrm{K}^+(\mathscr{J})$ and a qis $X^\bullet \to Y^\bullet$. Moreover, the natural functor $\theta\colon \mathrm{K}^+(\mathscr{J})/N^+(\mathscr{J}) \to \mathrm{D}^+(\mathscr{C})$ is an equivalence of categories.*

(b) *Assume moreover that $\mathscr{J}$ satisfies (7.2.2) and $X^\bullet \in \mathrm{K}^{\mathrm{b}}(\mathscr{C})$. Then we may choose $Y^\bullet \in \mathrm{K}^{\mathrm{b}}(\mathscr{J})$ and the natural functor $\theta\colon \mathrm{K}^{\mathrm{b}}(\mathscr{J})/N^{\mathrm{b}}(\mathscr{J}) \to \mathrm{D}^{\mathrm{b}}(\mathscr{C})$ is an equivalence of categories.*

Injective resolutions

In this subsection, $\mathscr{C}$ denotes an abelian category and $\mathscr{I}_{\mathscr{C}}$ its full additive subcategory consisting of injective objects. We shall asume

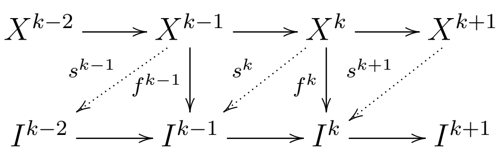
(7.2.4) the abelian category $\mathscr{C}$ admits enough injectives.

In other words, the category $\mathscr{I}_{\mathscr{C}}$ is cogenerating.

Proposition 7.2.5. (i) *Let $f^\bullet\colon X^\bullet \to I^\bullet$ be a morphism in $\mathrm{C}^+(\mathscr{C})$. Assume $I^\bullet$ belongs to $\mathrm{C}^+(\mathscr{I}_{\mathscr{C}})$ and $X^\bullet$ is exact. Then $f^\bullet$ is homotopic to 0.*

(ii) *Let $I^\bullet \in \mathrm{C}^+(\mathscr{I}_{\mathscr{C}})$ and assume $I^\bullet$ is exact. Then $I^\bullet$ is homotopic to 0.*

Proof. (i) Consider the diagram:



We shall construct by induction morphisms s^k satisfying:

$$f^k = s^{k+1} \circ d_X^k + d_I^{k-1} \circ s^k.$$

For $j << 0$, $s^j = 0$. Assume we have constructed the s^j for $j \leq k$. Define $g^k = f^k - d_I^{k-1} \circ s^k$. One has

$$\begin{aligned} g^k \circ d_X^{k-1} &= f^k \circ d_X^{k-1} - d_I^{k-1} \circ s^k \circ d_X^{k-1} \\ &= f^k \circ d_X^{k-1} - d_I^{k-1} \circ f^{k-1} + d_I^{k-1} \circ d_I^{k-2} \circ s^{k-1} \\ &= 0. \end{aligned}$$

Hence, g^k factorizes through $X^k / \operatorname{Im} d_X^{k-1}$. Since the complex $X^\bullet$ is exact, the sequence $0 \to X^k / \operatorname{Im} d_X^{k-1} \to X^{k+1}$ is exact. Consider

$$\begin{array}{ccccc} 0 & \longrightarrow & X^k / \operatorname{Im} d_X^{k-1} & \longrightarrow & X^{k+1} \\ & & {\scriptstyle g^k}\downarrow & \swarrow{\scriptstyle s^{k+1}} & \\ & & I^k & & \end{array}$$

The dotted arrow may be completed by Proposition 5.3.2.
(ii) Apply the result of (i) with $X^\bullet = I^\bullet$ and $f = \mathrm{id}_X$. □

Corollary 7.2.6. *Assume that $\mathscr{C}$ admits enough injectives. Then $\mathrm{K}^+(\mathscr{I}_\mathscr{C}) \to \mathrm{D}^+(\mathscr{C})$ is an equivalence of categories.*

Proof. According to Notation 7.2.3, $N^+(\mathscr{I}_\mathscr{C})$ is the subcategory of $\mathrm{K}^+(\mathscr{I}_\mathscr{C})$ consisting of complexes qis to 0. By Corollary 7.2.4, the natural functor $\mathrm{K}^+(\mathscr{I}_\mathscr{C})/N^+(\mathscr{I}_\mathscr{C}) \to \mathrm{D}^+(\mathscr{C})$ is an equivalence. To conclude, remark that the objects of $N^+(\mathscr{I}_\mathscr{C})$ are isomorphic to 0 in $\mathrm{K}^+(\mathscr{I}_\mathscr{C})$. Hence, $\mathrm{K}^+(\mathscr{I}_\mathscr{C})/N^+(\mathscr{I}_\mathscr{C})$ is equivalent to $\mathrm{K}^+(\mathscr{I}_\mathscr{C})$. □

Remark 7.2.7. Assume that $\mathscr{C}$ admits enough injectives. Then $\mathrm{D}^+(\mathscr{C})$ is a $\mathscr{U}$-category.

7.3 Derived functors

In this section, $\mathscr{C}$ and $\mathscr{C}'$ will denote abelian categories and $F\colon \mathscr{C} \to \mathscr{C}'$ a left exact functor. We shall construct the right derived functor $RF\colon \mathrm{D}^+(\mathscr{C}) \to \mathrm{D}^+(\mathscr{C}')$ under suitable conditions, and in particular the j-th derived functor $R^jF\colon \mathscr{C} \to \mathscr{C}'$. Note that we do not assume that $\mathscr{C}$ admits enough injectives.

The functor F defines naturally a functor

$$\mathrm{K}^+F\colon \mathrm{K}^+(\mathscr{C}) \to \mathrm{K}^+(\mathscr{C}').$$

For short, one often writes F instead of K^+F. Applying the results of § 6.4, we shall construct (under suitable hypotheses) the right localization of F.

Definition 7.3.1. (a) If the functor $\mathrm{K}^+F\colon \mathrm{K}^+(\mathscr{C}) \to \mathrm{D}^+(\mathscr{C}')$ admits a right localization (with respect to the qis in $\mathrm{K}^+(\mathscr{C})$), one says that F admits a right derived functor, or is right derivable, and one denotes by $RF\colon \mathrm{D}^+(\mathscr{C}) \to \mathrm{D}^+(\mathscr{C}')$ the right localization of F.

(b) If F admits a right derived functor, one sets for $X \in \mathscr{C}$, $R^jF(X) = H^j(RF(X))$. (Note that $R^0F \simeq F$.)

(c) An object $X \in \mathscr{C}$ satisfying $R^jF(X) \simeq 0$ for all $j \neq 0$ is called F-acyclic.

(d) Assume that F admits a right derived functor. One says that F has cohomological dimension $\leq d$ with $d \in \mathbb{N}$ if for any $X \in \mathscr{C}$, $R^j F(X) \simeq 0$ for $j > d$. If such an integer d exists, one says that F has finite cohomological dimension.

There is a similar definition for right exact functor. The exact formulation is left to the reader.

Note that if $F : \mathscr{C} \to \mathscr{C}'$ is exact, it admits a right derived functor as well as a left derived functor and both coincide. In this case, one still denotes by F its localization.

Recall that if RF exists, then it sends distinguished triangles in $\mathrm{D}^+(\mathscr{C})$ to distinguished triangles in $\mathrm{D}^+(\mathscr{C}')$. In particular, we get:

Proposition 7.3.2. *Let $F : \mathscr{C} \to \mathscr{C}'$ a left exact functor as above and let $0 \to X' \xrightarrow{f} X \xrightarrow{g} X'' \to 0$ be an exact sequence in $\mathscr{C}$. Then there exists a long exact sequence:*

$$0 \to F(X') \to F(X) \to \cdots \to R^k F(X') \to R^k F(X) \to R^k F(X'') \to \cdots .$$

Definition 7.3.3. Let $\mathscr{J}$ be a full additive subcategory of $\mathscr{C}$. One says that $\mathscr{J}$ is F-injective, or is injective with respect to F, if:

(i) $\mathscr{J}$ is cogenerating (Definition 7.2.1),

(ii) for any exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathscr{C}$, if $X', X \in \mathscr{J}$, then $X'' \in \mathscr{J}$,

(iii) for any exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathscr{C}$ with $X' \in \mathscr{J}$, the sequence $0 \to F(X') \to F(X) \to F(X'') \to 0$ is exact.

By considering $\mathscr{C}^{\mathrm{op}}$, one obtains the notion of an F-projective subcategory, F being right exact.

Example 7.3.4. I the category $\mathscr{I}_{\mathscr{C}}$ of injective objects of $\mathscr{C}$ is cogenerating, then it is F-injective.

Lemma 7.3.5. *Assume $\mathscr{J}$ is F-injective and let $X^\bullet \in \mathrm{C}^+(\mathscr{J})$ be a complex qis to zero* (i.e. *$X^\bullet$ is exact*)*. Then $F(X^\bullet)$ is qis to zero.*

Proof. We decompose $X^\bullet$ into short exact sequences (assuming that this complex starts at step 0 for convenience):

$$\begin{aligned}
&0 \to X^0 \to X^1 \to Z^1 \to 0,\\
&0 \to Z^1 \to X^2 \to Z^2 \to 0,\\
&\cdots\\
&0 \to Z^{n-1} \to X^n \to Z^n \to 0.
\end{aligned}$$

By induction we find that all the Z^j's belong to $\mathscr{J}$, hence all the sequences:

$$0 \to F(Z^{n-1}) \to F(X^n) \to F(Z^n) \to 0$$

are exact. Hence the sequence $0 \to F(X^0) \to F(X^1) \to \cdots$ is exact. □

Theorem 7.3.6. *Let $F : \mathscr{C} \to \mathscr{C}'$ be a left exact functor of abelian categories and let $\mathscr{J} \subset \mathscr{C}$ be a full additive subcategory. Assume that $\mathscr{J}$ is F-injective. Then*

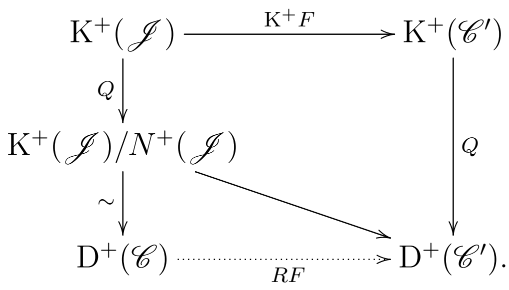
(a) *F admits a right derived functor $RF\colon \mathrm{D}^+(\mathscr{C}) \to \mathrm{D}^+(\mathscr{C}')$.*

(b) *If moreover $\mathscr{J}$ satisfies (7.2.2), then RF induces a functor $\mathrm{D}^{\mathrm{b}}(\mathscr{C}) \to \mathrm{D}^{\mathrm{b}}(\mathscr{C}')$.*

By reversing the arrows, one obtains a similar result for right exact functors.

Proof. (i) We shall apply Theorem 6.4.7. Condition (a) is satisfied thanks to Corollary 7.2.4. Condition (b) is satisfied thanks to Lemma 7.3.5.
(ii) The proof of (b) is similar. □

Recall that the construction of RF is vizualised by the diagram



Recall that the derived functor RF is triangulated, and does not depend on the category $\mathscr{J}$. Hence, if $X' \to X \to X'' \xrightarrow{+1}$ is a d.t.in $\mathrm{D}^+(\mathscr{C})$, then $RF(X') \to RF(X) \to RF(X'') \xrightarrow{+1}$ is a d.t.in $\mathrm{D}^+(\mathscr{C}')$.

Also recall that an exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathscr{C}$ gives rise to a d.t.in $\mathrm{D}(\mathscr{C})$. Applying the cohomological functor H^0, we get the long exact sequence in $\mathscr{C}'$:

$$\cdots \to R^kF(X') \to R^kF(X) \to R^kF(X'') \to R^{k+1}F(X') \to \cdots$$

By considering the category $\mathscr{C}^{\mathrm{op}}$, one defines the notion of left derived functor of a right exact functor F.

Remark 7.3.7. Consider a functor $F\colon \mathrm{K}^+(\mathscr{C}) \to \mathrm{K}^+(\mathscr{C}')$ and assume that there exists an additive subcategory $\mathscr{I}$ of $\mathscr{C}$ satisfying the following properties:

(7.3.1)
(i) any $X \in \mathrm{K}^+(\mathscr{C})$ is qis to an object of $\mathrm{K}^+(\mathscr{I})$,
(ii) if $X \in \mathrm{K}^+(\mathscr{I})$ is qis to 0, then $F(X)$ is qis to 0.

Then the conclusion of Theorem 7.3.6 (a) holds. If moreover, any $X \in \mathrm{K}^{\mathrm{b}}(\mathscr{C})$ is qis to an object of $\mathrm{K}^{\mathrm{b}}(\mathscr{I})$, then (b) holds.

Remark 7.3.8. Let $F\colon \mathscr{C} \to \mathscr{C}'$ be a left exact functor and assume that F admits a right derived functor. Denote by $\mathscr{I}_F$ the full additive subcategory of $\mathscr{C}$ consisting of F-acyclic objects and assume that this category if cogenerating. Then $\mathscr{I}_F$ is F-injective. Indeed, conditions (ii) and (iii) of Definition 7.3.3 are satisfied thanks to Proposition 7.3.2.

The next result follows immediately from the construction of RF and gives an explicit construction of the derived functor.

Proposition 7.3.9. *Assume $\mathscr{J}$ is F-injective. Let $X \in \mathscr{C}$ and let $0 \to X \to Y^\bullet$ be a resolution of X with $Y^\bullet \in \mathrm{C}^+(\mathscr{J})$. Then for each n, there is an isomorphism $R^nF(X) \simeq H^n(F(Y^\bullet))$.*

In other words, in order to calculate the derived functors $R^jF(X)$ for $X \in \mathscr{C}$, it is enough to replace X with a right $\mathscr{J}$-resolution, apply F to this complex and take the j-th cohomology. This construction applies in particular if $\mathscr{C}$ has enough injectives.

Derived functor of a composition

Let $F\colon \mathscr{C} \to \mathscr{C}'$ and $G\colon \mathscr{C}' \to \mathscr{C}''$ be left exact functors of abelian categories. Then $G \circ F\colon \mathscr{C} \to \mathscr{C}''$ is left exact. Using the universal property of the localization, one shows that if F, G and $G\circ F$ are right derivable, then there exists a natural morphism of functors

$$R(G \circ F) \to RG \circ RF. \tag{7.3.2}$$

Theorem 7.3.10. *Assume that there exist full additive subcategories $\mathscr{J} \subset \mathscr{C}$ and $\mathscr{J}' \subset \mathscr{C}'$ such that $\mathscr{J}$ is F-injective, $\mathscr{J}'$ is G-injective and $F(\mathscr{J}) \subset \mathscr{J}'$. Then $\mathscr{J}$ is $(G \circ F)$-injective and the morphism in (7.3.2) is an isomorphism: $R(G \circ F) \xrightarrow{\sim} RG \circ RF$.*

Proof. (i) The fact that $\mathscr{J}$ is $(G \circ F)$ injective follows immediately from the definition.
(ii) Let $X \in \mathrm{K}^+(\mathscr{C})$ and $Y \in \mathrm{K}^+(\mathscr{J})$ together with a qis $X \to Y$. Then $RF(X)$ is represented by the complex $F(Y)$ which belongs to $\mathrm{K}^+(\mathscr{J}')$. Hence $RG(RF(X))$ is represented by $G(F(Y)) = (G \circ F)(Y)$, and this last complex also represents $R(G \circ F)(Y)$ since $Y \in \mathrm{C}^+(\mathscr{J})$ and $\mathscr{J}$ is $G \circ F$ injective. □

Note that in general F does not send injective objects of $\mathscr{C}$ to injective objects of $\mathscr{C}'$. That is why the notion of an "F-injective" category is important.

Corollary 7.3.11. *Assume that there exists a full additive subcategory $\mathscr{J} \subset \mathscr{C}$ such that $\mathscr{J}$ is F-injective and assume that G is exact. Then $\mathscr{J}$ is $(G\circ F)$-injective and the morphism in (7.3.2) is an isomorphism.*

Proof. If G is exact, then $\mathscr{C}'$ is G-injective. Then apply Theorem 7.3.10 with $\mathscr{J}' = \mathscr{C}'$. □

Corollary 7.3.12. *In the situation of Theorem 7.3.10, let $X \in \mathscr{C}$ and assume that $R^jF(X) \simeq 0$ for $j > 0$. Then $R^j(G \circ F)(X) \simeq (R^jG)(F(X))$.*

Proof. We have $RF(X) \simeq F(X)$ in $\mathrm{D}^+(\mathscr{C}')$. Then Theorem 7.3.10 gives $R(G \circ F)(X) \simeq (RG \circ RF)(X) \simeq RG(F(X))$. □

7.4 Bifunctors

Now consider three abelian categories $\mathscr{C}, \mathscr{C}', \mathscr{C}''$ and an *additive* bifunctor:

$$F\colon \mathscr{C} \times \mathscr{C}' \to \mathscr{C}''.$$

We shall assume that F is left exact with respect to each of its arguments.

Let $X \in \mathrm{C}^+(\mathscr{C}), X' \in \mathrm{C}^+(\mathscr{C}')$. Then the double complex $F(X, X')$ satisfies the finiteness condition (4.3.7) and $\mathrm{tot}(F(X, X') \in \mathrm{C}^+(\mathscr{C}'')$ is well-defined. Now assume that X or X' is homotopic to 0. Then one checks easily that $\mathrm{tot}(F(X,X'))$ is homotopic to zero. Hence one can naturally define:

$$\mathrm{K}^+F\colon \mathrm{K}^+(\mathscr{C})\times\mathrm{K}^+(\mathscr{C}')\to\mathrm{K}^+(\mathscr{C}''),\quad \mathrm{K}^+F(X,X')=\mathrm{tot}(F(X,X')).$$

If there is no risk of confusion, we shall sometimes write F instead of K^+F.

Definition 7.4.1. If the functor $\mathrm{K}^+F\colon \mathrm{K}^+(\mathscr{C})\times\mathrm{K}^+(\mathscr{C}')\to\mathrm{D}^+(\mathscr{C}'')$ admits a right localization (with respect to the qis in $\mathrm{K}^+(\mathscr{C})$ and $\mathrm{K}^+(\mathscr{C}')$), one says that F admits a right derived functor and one denotes by $RF\colon \mathrm{D}^+(\mathscr{C})\times\mathrm{D}^+(\mathscr{C}')\to\mathrm{D}^+(\mathscr{C}'')$ the right localization of F.

One defines similarly the notion of left derived functor for a right exact bifunctor.

Definition 7.4.2. Let $\mathscr{J}$ and $\mathscr{J}'$ be additive subcategories of $\mathscr{C}$ and $\mathscr{C}'$, respectively. One says $(\mathscr{J},\mathscr{J}')$ is F-injective if:

(i) for all $X'\in\mathscr{J}'$, $\mathscr{J}$ is $F(\,\bullet\,,X')$-injective,

(ii) for all $X\in\mathscr{J}$, $\mathscr{J}'$ is $F(X,\,\bullet\,)$-injective.

Note that if $(\mathscr{J},\mathscr{J}')$ is F-injective, then $\mathscr{J}$ and $\mathscr{J}'$ are cogenerating in $\mathscr{C}$ and $\mathscr{C}'$, respectively.

One defines similarly the notion of being G-projective for a right exact bifunctor G.

Theorem 7.4.3. *Let $F\colon\mathscr{C}\times\mathscr{C}'\to\mathscr{C}''$ be a left exact bifunctor of abelian categories and let $\mathscr{J}$ and $\mathscr{J}'$ be additive subcategories of $\mathscr{C}$ and $\mathscr{C}'$, respectively. Assume that $(\mathscr{J},\mathscr{J}')$ is F-injective. Then F admits a right derived functor $RF\colon \mathrm{D}^+(\mathscr{C})\times\mathrm{D}^+(\mathscr{C}')\to\mathrm{D}^+(\mathscr{C}'')$.*

Proof. Let $X\in\mathrm{K}^+(\mathscr{J})$ and $X'\in\mathrm{K}^+(\mathscr{J}')$. If X or X' is qis to 0, then all rows or all columns of $F(X,X')$ are exact and it follows that $\mathrm{tot}(F(X,X'))$ is qis to zero by Corollary 5.6.2. To conclude, apply Theorem 6.4.8 with $\mathscr{I}=\mathrm{K}^+(\mathscr{J})$ and $\mathscr{I}'=\mathrm{K}^+(\mathscr{J}')$. □

There is a similar statement for a right exact bifunctor G, replacing F-injective with G-projective.

The next result is obvious by the construction of RF.

Corollary 7.4.4. *In the situation of Theorem 7.4.3, assume moreover that $\mathscr{J}$ and $\mathscr{J}'$ satisfy (7.2.2). Then RF induces a functor $RF\colon \mathrm{D}^{\mathrm{b}}(\mathscr{C})\times\mathrm{D}^{\mathrm{b}}(\mathscr{C}')\to\mathrm{D}^{\mathrm{b}}(\mathscr{C}'')$.*

Theorem 7.4.5. *Let $F\colon\mathscr{C}\times\mathscr{C}'\to\mathscr{C}''$ be a left exact bifunctor of abelian categories and let $\mathscr{J}$ and $\mathscr{J}'$ be additive subcategories of $\mathscr{C}$ and $\mathscr{C}'$, respectively. Assume that for all $X'\in\mathscr{C}'$, $\mathscr{J}$ is $F(\,\bullet\,,X')$-injective, and for all $X\in\mathscr{C}$, $\mathscr{J}'$ is $F(X,\,\bullet\,)$-injective. Then for all $X\in\mathscr{C}$, $Y\in\mathscr{C}'$, one has:*

$$RF(X,Y)\simeq R_{II}F(X,\,\bullet\,)(Y)\simeq R_IF(\,\bullet\,,Y)(X). \tag{7.4.1}$$

Here, $R_{II}F(X,\,\bullet\,)$ is the derived functor of the functor $F(X,\,\bullet\,)$ and similarly with $R_IF(\,\bullet\,,Y)$.

Proof. Let $X \in \mathscr{C}$, $Y \in \mathscr{C}'$ and let $0 \to X \to I_X^\bullet$ and $0 \to Y \to I_Y^\bullet$ be resolutions of X and Y in $\mathscr{J}$ and $\mathscr{J}'$, respectively. The object $RF(X, X')$ is the image by the localization functor Q of the object $\mathrm{tot}(F(I_X^\bullet, I_Y^\bullet))$ of $\mathrm{K}^+(\mathscr{C}'')$. Consider the double complex:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & F(I_X^0, Y) & \longrightarrow & F(I_X^1, Y) & \longrightarrow & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F(X, I_Y^0) & \longrightarrow & F(I_X^0, I_Y^0) & \longrightarrow & F(I_X^1, I_Y^0) & \longrightarrow & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F(X, I_Y^1) & \longrightarrow & F(I_X^0, I_Y^1) & \longrightarrow & F(I_X^1, I_Y^1) & \longrightarrow & \\
 & & \downarrow & & \downarrow & & \downarrow & &
\end{array}
$$

By the hypotheses, all rows and columns are exact with the exception of the 0-row and the 0-column (each starting with $0 \to 0$). By Corollary 5.6.2, $\mathrm{tot}(F(I_X^\bullet, I_Y^\bullet))$ is qis to the cohomology of the 0-column, which calculates $R_{II}F(X, \bullet)(Y)$, as well as the cohomology of the 0-row, which calculates $R_I F(X, Y)$. □

Corollary 7.4.6. *Let $F \colon \mathscr{C} \times \mathscr{C}' \to \mathscr{C}''$ be a left exact bifunctor of abelian categories and let $\mathscr{J}$ be an additive subcategory of $\mathscr{C}$. Assume that for any $X' \in \mathscr{C}'$, the category $\mathscr{J}$ is $F(\bullet, X')$-injective and for any $X \in \mathscr{J}$, the functor $F(X, \bullet)$ is exact. Then F admits a right derived functor $RF \colon \mathrm{D}^+(\mathscr{C}) \times \mathrm{D}^+(\mathscr{C}') \to \mathrm{D}^+(\mathscr{C}'')$. If moreover, $\mathscr{J}$ satisfies (7.2.2), then RF induces a functor $RF \colon \mathrm{D}^{\mathrm{b}}(\mathscr{C}) \times \mathrm{D}^{\mathrm{b}}(\mathscr{C}') \to \mathrm{D}^{\mathrm{b}}(\mathscr{C}'')$.*

Proof. Apply Theorem 7.4.5 and Corollary 7.4.4 with $\mathscr{J}' = \mathscr{C}'$. □

Proposition 7.4.7. *Let $F \colon \mathscr{C} \times \mathscr{C}' \to \mathscr{C}''$ be a left exact bifunctor of abelian categories and let $\mathscr{J}$, $\mathscr{J}'$ and $\mathscr{J}''$ be additive subcategories of $\mathscr{C}$, $\mathscr{C}'$ and $\mathscr{C}''$, respectively. Let $G \colon \mathscr{C}'' \to \mathscr{C}'''$ be a left exact functor of abelian categories. Assume that $(\mathscr{J}, \mathscr{J}')$ is F-injective, $\mathscr{J}''$ is G-injective and $F(\mathscr{J}, \mathscr{J}') \subset \mathscr{J}''$. Then the derived functor $R(G \circ F)$ exists and moreover, $R(G \circ F) \simeq RG \circ RF$.*

The proof is straightforward.

One naturally extends Definition 7.3.1 to bifunctors. The exact statement is left to the reader.

Example 7.4.8. Assume $\mathscr{C}$ has enough injectives. Then

$$\mathrm{RHom}_{\mathscr{C}} \colon \mathrm{D}^-(\mathscr{C})^{\mathrm{op}} \times \mathrm{D}^+(\mathscr{C}) \to \mathrm{D}^+(\mathbb{Z})$$

exists and may be calculated as follows. Let $X \in \mathrm{C}^-(\mathscr{C})$ and $Y \in \mathrm{C}^+(\mathscr{C})$. There exists a qis in $\mathrm{K}^+(\mathscr{C})$, $Y \to I$, the I^j's being injective. Then:

$$\mathrm{RHom}_{\mathscr{C}}(X, Y) \simeq \mathrm{Hom}^\bullet_{\mathscr{C}}(X, I).$$

If $\mathscr{C}$ has enough projectives, and $P \to X$ is a qis in $\mathrm{K}^-(\mathscr{C})$, the P^j's being projective, one also has:

$$\mathrm{RHom}_{\mathscr{C}}(X, Y) \simeq \mathrm{Hom}^\bullet_{\mathscr{C}}(P, Y).$$

These isomorphisms hold in $\mathrm{D}^+(\mathbb{Z})$.

Example 7.4.9. Let A be a $\mathbf{k}$-algebra. By choosing the category of projective modules for $\mathscr{J}$ and $\mathscr{J}'$ in Theorem 7.4.3, we get that the bifunctor

$$\bullet \overset{\mathrm{L}}{\otimes}_A \bullet \colon \ \mathrm{D}^-(A^{\mathrm{op}}) \times \mathrm{D}^-(A) \to \mathrm{D}^-(\mathbf{k})$$

is well defined. Moreover,

$$N \overset{\mathrm{L}}{\otimes}_A M \simeq \mathrm{tot}(N \otimes_A P) \simeq \mathrm{tot}(Q \otimes_A M)$$

where P (resp. Q) is a complex of projective A-modules qis to M (resp. to N).

Note that instead of choosing the category of projective modules, we could have chosen that of flat modules. When working with sheaves, there are not enough projective modules in general, although there are enough flat modules.

In the preceding situation, one defines:

$$\mathrm{Tor}_k^A(N, M) = H^{-k}(N \overset{\mathrm{L}}{\otimes}_A M).$$

The functors $\mathrm{RHom}_{\mathscr{C}}$ and $\mathrm{Hom}_{\mathrm{D}(\mathscr{C})}$

Theorem 7.4.10. *Let $\mathscr{C}$ be an abelian category with enough injectives. Then for $X \in \mathrm{D}^-(\mathscr{C})$, $Y \in \mathrm{D}^+(\mathscr{C})$ and $j \in \mathbb{Z}$:*

$$H^j \mathrm{RHom}_{\mathscr{C}}(X, Y) \simeq \mathrm{Hom}_{\mathrm{D}(\mathscr{C})}(X, Y[j]).$$

Proof. By Proposition 7.2.2, there exists $I_Y \in \mathrm{C}^+(\mathscr{I})$ and a qis $Y \to I_Y$. Then we have the isomorphisms:

$$\begin{aligned}\mathrm{Hom}_{\mathrm{D}(\mathscr{C})}(X, Y[j]) &\simeq \mathrm{Hom}_{K(\mathscr{C})}(X, I_Y[j]) \\ &\simeq H^0(\mathrm{Hom}^{\bullet}_{\mathscr{C}}(X, I_Y[j])) \\ &\simeq R^j \mathrm{Hom}_{\mathscr{C}}(X, Y),\end{aligned}$$

where the second isomorphism follows from Proposition 4.4.5. □

Recall that one has set

$$\mathrm{Ext}^j_{\mathscr{C}}(X, Y) := H^j \mathrm{RHom}_{\mathscr{C}}(X, Y).$$

Example 7.4.11. Let W be the Weyl algebra in one variable over a field $\mathbf{k}$ of characteristic 0: $W = \mathbf{k}[x, \partial]$ with the relation $[x, \partial] = -1$.

Let $\mathscr{O} = W/W \cdot \partial$, $\Omega = W/\partial \cdot W$ and let us calculate $\Omega \overset{\mathrm{L}}{\otimes}_W \mathscr{O}$. We have an exact sequence: $0 \to W \xrightarrow{\partial \cdot} W \to \Omega \to 0$. Therefore, Ω is qis to the complex

$$0 \to W^{-1} \xrightarrow{\partial \cdot} W^0 \to 0$$

where $W^{-1} = W^0 = W$ and W^0 is in degree 0. Then $\Omega \overset{\mathrm{L}}{\otimes}_W \mathscr{O}$ is qis to the complex

$$0 \to \mathscr{O}^{-1} \xrightarrow{\partial \cdot} \mathscr{O}^0 \to 0,$$

where $\mathscr{O}^{-1} = \mathscr{O}^0 = \mathscr{O}$ and $\mathscr{O}^0$ is in degree 0. Since $\partial \colon \mathscr{O} \to \mathscr{O}$ is surjective and has $\mathbf{k}$ as kernel, we obtain:

$$\Omega \overset{\mathrm{L}}{\otimes}_W \mathscr{O} \simeq \mathbf{k}[1].$$

Example 7.4.12. Let $\mathbf{k}$ be a field and let $A = \mathbf{k}[x_1, \dots, x_n]$. This is a commutative Noetherian ring and it is known (Hilbert) that any finitely generated A-module M admits a finite free presentation of length at most n, *i.e.,* M is qis to a complex:

$$L := \quad 0 \to L^{-n} \to \cdots \xrightarrow{P_0} L^0 \to 0$$

where the L^j's are free of finite rank. Consider the left exact functor

$$\mathrm{Hom}_A(\,\bullet\,, A)\colon \mathrm{Mod}(A)^{\mathrm{op}} \to \mathrm{Mod}(A)$$

and denote for short by * its derived functor:

$$ ^* := \mathrm{RHom}_A(\,\bullet\,, A). \tag{7.4.2}$$

Since free A-modules are projective, we find that $\mathrm{RHom}_A(M, A)$ is isomorphic in $\mathrm{D^b}(A)$ to the complex

$$L^* := \quad 0 \leftarrow L^{-n*} \leftarrow \cdots \xleftarrow{P_0} L^{0*} \leftarrow 0.$$

Using (7.3.2), we find a natural morphism of functors

$$\mathrm{id} \to {}^{**} := {}^* \circ {}^*.$$

Applying * to the object $\mathrm{RHom}_A(M, A)$ we find:

$$\begin{aligned}\mathrm{RHom}_A(\mathrm{RHom}_A(M, A), A) &\simeq \mathrm{RHom}_A(L^*, A)\\ &\simeq L \simeq M.\end{aligned}$$

In other words, we have proved the isomorphism $M \simeq M^{**}$ in $\mathrm{D^b}(A)$.

Assume now $n = 1$, *i.e.,* $A = \mathbf{k}[x]$ and consider the natural morphism in $\mathrm{Mod}(A)$: $f\colon A \to A/Ax$. Applying the functor *, we get the morphism in $\mathrm{D^b}(A)$:

$$f^*\colon \mathrm{RHom}_A(A/Ax, A) \to A.$$

Remember that $\mathrm{RHom}_A(A/Ax, A) \simeq A/xA\,[-1]$. Hence $H^j(f^*) = 0$ for all $j \in \mathbb{Z}$, although $f^* \neq 0$ since $f^{**} = f$.

Let us give an example of an object of a derived category which is not isomorphic to the direct sum of its cohomology objects (hence, a situation in which Corollary 7.1.12 does not apply).

Example 7.4.13. Let $\mathbf{k}$ be a field and let $A = \mathbf{k}[x_1, x_2]$. Define the A-modules

$$M' = A/(Ax_1 + Ax_2), \quad M = A/(Ax_1^2 + Ax_1x_2), \quad M'' = A/Ax_1.$$

There is an exact sequence

$$0 \to M' \to M \to M'' \to 0 \tag{7.4.3}$$

and this exact sequence does not split since x_1 kills M' and M'' but not M.

Recall the functor * of (7.4.2). We have $M'^* \simeq H^2(M'^*)[-2]$ and $M''^* \simeq H^1(M''^*)[-1]$. The functor * applied to the exact sequence (7.4.3) gives rise to the long exact sequence

$$0 \to H^1(M''^*) \to H^1(M^*) \to 0 \to 0 \to H^2(M^*) \to H^2(M'^*) \to 0.$$

Hence $H^1(M^*)[-1] \simeq H^1(M''^*)[-1] \simeq M''^*$ and $H^2(M^*)[-2] \simeq H^2(M'^*)[-2] \simeq M'^*$. Assume for a while $M^* \simeq \oplus_j H^j(M^*)[-j]$. This implies $M^* \simeq M'^* \oplus M''^*$ hence (by applying again the functor *), $M \simeq M' \oplus M''$, which is a contradiction.

7.5 The Brown representability theorem

We shall follow the exposition of [KS06, § 10.5].

Definition 7.5.1. Let $\mathscr{D}$ be a triangulated category admitting small direct sums. A *system of t-generators* $\mathscr{F}$ in $\mathscr{D}$ is a small family of objects of $\mathscr{D}$ satisfying conditions (i) and (ii) below.

(i) For any $X \in \mathscr{D}$ with $\mathrm{Hom}_{\mathscr{D}}(C, X) \simeq 0$ for all $C \in \mathscr{F}$, we have $X \simeq 0$.

(ii) For any *countable* set I and any family $\{u_i : X_i \to Y_i\}_{i \in I}$ of morphisms in $\mathscr{D}$, the map $\mathrm{Hom}_{\mathscr{D}}(C, \oplus_i X_i) \xrightarrow{\oplus_i u_i} \mathrm{Hom}_{\mathscr{D}}(C, \oplus_i Y_i)$ vanishes for every $C \in \mathscr{F}$ as soon as $\mathrm{Hom}_{\mathscr{D}}(C, X_i) \xrightarrow{u_i} \mathrm{Hom}_{\mathscr{D}}(C, Y_i)$ vanishes for every $i \in I$ and every $C \in \mathscr{F}$.

What we call below the Brown representability Theorem is in fact a corollary of such a theorem. See [KS06, Cor. 10.5.3].

Theorem 7.5.2 (The Brown representability Theorem)**.** *Let $\mathscr{D}$ be a triangulated category admitting small direct sums and a system of t-generators. Let $F : \mathscr{D} \to \mathscr{D}'$ be a triangulated functor of triangulated categories and assume that F commutes with small direct sums. Then F admits a right adjoint G and G is triangulated.*

Recall Definition 5.4.3 of a Grothendieck category and also recall that such a definition relies on the notion of universe. Hence, all categories in the sequel belong to a given universe $\mathscr{U}$.

We shall apply Theorem 7.5.2 in the particular case of derived categories.

Theorem 7.5.3 (see [KS06, Th. 14.3.1])**.** *Let $\mathscr{C}$ be a Grothendieck abelian category.*

(a) *The category $\mathrm{D}(\mathscr{C})$ admits small direct sums and a system of t-generators.*

(b) *Let $\mathscr{D}$ be a triangulated category and $F : \mathrm{K}(\mathscr{C}) \to \mathscr{D}$ a triangulated functor. Then F admits a right localization $RF : \mathrm{D}(\mathscr{C}) \to \mathscr{D}$.*

Note that the existence of small direct sums follows from Proposition 6.4.9.

From now on, we shall follow [GS16, § 2.3].

Lemma 7.5.4. *Let $\mathscr{C}$ be a Grothendieck category and let $d \in \mathbb{Z}$. Then the cohomology functor H^d and the truncation functors $\tau^{\leq d}$ and $\tau^{\geq d}$ commute with small direct sums in $\mathrm{D}(\mathscr{C})$. In other words, if $\{X_i\}_{i \in I}$ is a small family of objects of $\mathrm{D}(\mathscr{C})$, then*

$$\bigoplus_i \tau^{\leq d} X_i \xrightarrow{\sim} \tau^{\leq d}(\bigoplus_i X_i) \tag{7.5.1}$$

and similarly with $\tau^{\geq d}$ and H^d.

Proof. (i) Let us treat first the functor H^d. Recall that $Q : \mathrm{K}(\mathscr{C}) \to \mathrm{D}(\mathscr{C})$ denotes the localization functor and Q commutes with small direct sums by Proposition 6.4.9. Let us denote for a while by $\widetilde{H}^d : \mathrm{K}(\mathscr{C}) \to \mathscr{C}$ the cohomology functor usually denoted by H^d. Then $\widetilde{H}^d \simeq H^d \circ Q$.

Let $\{X_i\}_i$ be a small family of objects in $\mathrm{K}(\mathscr{C})$. Then

$$\begin{aligned} H^d(\oplus_i Q(X_i)) &\simeq H^d(Q(\oplus_i X_i)) \simeq \widetilde{H}^d(\oplus_i X_i) \\ &\simeq \oplus_i \widetilde{H}^d(X_i) \simeq \oplus_i H^d(Q(X_i))). \end{aligned}$$

(ii) The morphism in (7.5.1) is well-defined and it is enough to check that it induces an isomorphism on the cohomology. This follows from (i) since for any object $Y \in \mathrm{D}(\mathscr{C})$, $H^j(\tau^{\leq d} Y)$ is either 0 or $H^j(Y)$. □

Lemma 7.5.5. *Let $\mathscr{C}$ and $\mathscr{C}'$ be two Grothendieck categories and let $\rho\colon \mathscr{C} \to \mathscr{C}'$ be a left exact functor. Let I be a small category. Assume*

(i) *I is either filtrant or discrete,*

(ii) *ρ commutes with inductive limits indexed by I,*

(iii) *inductive limits indexed by I of injective objects in $\mathscr{C}$ are acyclic for the functor ρ.*

Then for all $j \in \mathbb{Z}$, the functor $R^j\rho\colon \mathscr{C} \to \mathscr{C}'$ commutes with inductive limits indexed by I.

Proof. Let $\alpha\colon I \to \mathscr{C}$ be a functor. Denote by $\mathscr{I}$ the full additive subcategory of $\mathscr{C}$ consisting of injective objects. It follows for example from [KS06, Cor. 9.6.6] that there exists a functor $\psi\colon I \to \mathscr{I}$ and a morphism of functors $\alpha \to \psi$ such that for each $i \in I$, $\alpha(i) \to \psi(i)$ is a monomorphism. Therefore one can construct a functor $\Psi\colon I \to \mathrm{C}^+(\mathscr{I})$ and a morphism of functor $\alpha \to \Psi$ such that for each $i \in I$, $\alpha(i) \to \Psi(i)$ is a quasi-isomorphism. Set $X_i = \alpha(i)$ and $G_i^\bullet = \Psi(i)$. We get a qis $X_i \to G_i^\bullet$, hence a qis

$$\operatorname*{colim}_i X_i \to \operatorname*{colim}_i G_i^\bullet.$$

On the other hand, we have

$$\begin{aligned} \operatorname*{colim}_i R^j\rho(X_i) &\simeq \operatorname*{colim}_i H^j(\rho(G_i^\bullet)) \\ &\simeq H^j\rho(\operatorname*{colim}_i G_i^\bullet) \end{aligned}$$

where the second isomorphism follows from the fact that H^j commutes with direct sums and with filtrant inductive limits. Then the result follows from hypothesis (iii). □

Lemma 7.5.6. *We make the same hypothesis as in Lemma 7.5.5. Let $-\infty < a \leq b < \infty$, let I be a small set and let $X_i \in \mathrm{D}^{[a,b]}(\mathscr{C})$. Then*

$$\bigoplus_i R\rho(X_i) \xrightarrow{\sim} R\rho(\bigoplus_i X_i). \tag{7.5.2}$$

Proof. The morphism in (7.5.2) is well-defined and we have to prove it is an isomorphism. If $b = a$, the result follows from Lemma 7.5.5. The general case is deduced by induction on $b - a$ by considering the distinguished triangles

$$H^a(X_i)\,[-a] \to X_i \to \tau^{\geq a+1} X_i \xrightarrow{+1}.$$

□

Theorem 7.5.7 (see [GS16, Prop. 2.21])**.** *Let $\mathscr{C}$ and $\mathscr{C}'$ be two Grothendieck categories and let $\rho\colon \mathscr{C} \to \mathscr{C}'$ be a left exact functor. Assume that*

(i) *ρ has finite cohomological dimension,*

(ii) *ρ commutes with small direct sums,*

(iii) *small direct sums of injective objects in $\mathscr{C}$ are acyclic for the functor ρ.*

Then

(a) *the functor $R\rho\colon \mathrm{D}(\mathscr{C}) \to \mathrm{D}(\mathscr{C}')$ commutes with small direct sums,*

(b) *the functor $R\rho\colon \mathrm{D}(\mathscr{C}) \to \mathrm{D}(\mathscr{C}')$ admits a right adjoint $\rho^!\colon \mathrm{D}(\mathscr{C}') \to \mathrm{D}(\mathscr{C})$,*

(c) *the functor $\rho^!$ induces a functor $\rho^!\colon \mathrm{D}^+(\mathscr{C}') \to \mathrm{D}^+(\mathscr{C})$.*

Proof. (a) Let $\{X_i\}_{i\in I}$ be a family of objects of $\mathrm{D}(\mathscr{C})$. It is enough to check that the natural morphism in $\mathrm{D}(\mathscr{C}')$

$$\bigoplus_{i\in I} R\rho(X_i) \to R\rho(\bigoplus_{i\in I} X_i) \tag{7.5.3}$$

induces an isomorphism on the cohomology groups. Assume that ρ has cohomological dimension $\leq d$. For $X \in \mathrm{D}(\mathscr{C})$ and for $j \in \mathbb{Z}$, we have

$$\tau^{\geq j} R\rho(X) \simeq \tau^{\geq j} R\rho(\tau^{\geq j-d-1} X).$$

The functor ρ being left exact we get for $k \geq j$:

$$H^k R\rho(X) \simeq H^k R\rho(\tau^{\leq k}\tau^{\geq j-d-1} X). \tag{7.5.4}$$

We have the sequence of isomorphisms:

$$\begin{aligned} H^k R\rho(\bigoplus_i X_i) &\simeq H^k R\rho(\tau^{\leq k}\tau^{\geq j-d-1} \bigoplus_i X_i) \simeq H^k R\rho(\bigoplus_i \tau^{\leq k}\tau^{\geq j-d-1} X_i) \\ &\simeq \bigoplus_i H^k R\rho(\tau^{\leq k}\tau^{\geq j-d-1} X_i) \simeq \bigoplus_i H^k R\rho(X_i). \end{aligned}$$

The first and last isomorphisms follow from (7.5.4).
The second isomorphism follows from Lemma 7.5.4.
The third isomorphism follows from Lemma 7.5.6.

(b) follows from (a) and the Brown representability theorem 7.5.2.

(c) This follows from hypothesis (i) and (the well-known) Lemma 7.5.8 below. □

Lemma 7.5.8. *Let $\rho\colon \mathscr{C} \to \mathscr{C}'$ be a left exact functor between two Grothendieck categories. Assume that $\rho\colon \mathrm{D}(\mathscr{C}) \to \mathrm{D}(\mathscr{C}')$ admits a right adjoint $\rho^!\colon \mathrm{D}(\mathscr{C}') \to \mathrm{D}(\mathscr{C})$ and assume moreover that ρ has finite cohomological dimension. Then the functor $\rho^!$ sends $\mathrm{D}^+(\mathscr{C}')$ to $\mathrm{D}^+(\mathscr{C})$.*

Proof. By the hypothesis, we have for $X \in \mathrm{D}(\mathscr{C})$ and $Y \in \mathrm{D}(\mathscr{C}')$

$$\mathrm{Hom}_{\mathrm{D}(\mathscr{C}')}(\rho(X), Y) \simeq \mathrm{Hom}_{\mathrm{D}(\mathscr{C})}(X, \rho^!(Y)).$$

Assume that the cohomological dimension of the functor ρ is $\leq r$. Let $Y \in \mathrm{D}^{\geq 0}(\mathscr{C}')$. Then (using Exercise 7.7) $\mathrm{Hom}_{\mathrm{D}(\mathscr{C})}(X, \rho^!(Y)) \simeq 0$ for all $X \in \mathrm{D}^{<-r}(\mathscr{C})$. This implies that $\rho^!(Y) \in \mathrm{D}^{\geq -r}(\mathscr{C}')$. □

Exercises to Chapter 7

Exercise 7.1. Let $\mathscr{C}$ be an abelian category with enough injectives. Prove that the two conditions below are equivalent.
(i) For all X and Y in $\mathscr{C}$, $\mathrm{Ext}^j_{\mathscr{C}}(X,Y)\simeq 0$ for all $j>n$.
(ii) For all X in $\mathscr{C}$, there exists an exact sequence $0\to X\to X^0\to\cdots\to X^n\to 0$, with the X^j's injective.
In such a situation, one says that $\mathscr{C}$ has homological dimension $\leq n$ and one writes $\mathrm{dh}(\mathscr{C})\leq n$.
(iii) Assume moreover that $\mathscr{C}$ has enough projectives. Prove that (i) is equivalent to: for all X in $\mathscr{C}$, there exists an exact sequence $0\to X^n\to\cdots\to X^0\to X\to 0$, with the X^j's projective.

Exercise 7.2. Let $\mathscr{C}$ be an abelian category with enough injectives and such that $\mathrm{dh}(\mathscr{C})\leq 1$. Let $F\colon\mathscr{C}\to\mathscr{C}'$ be a left exact functor and let $X\in\mathrm{D}^+(\mathscr{C})$.
(i) Construct an isomorphism $H^k(RF(X))\simeq F(H^k(X))\oplus R^1F(H^{k-1}(X))$.
(ii) Recall that $\mathrm{dh}(\mathrm{Mod}(\mathbb{Z}))=1$. Let $X\in\mathrm{D}^-(\mathbb{Z})$, and let $M\in\mathrm{Mod}(\mathbb{Z})$. Deduce the isomorphism

$$H^k(X\overset{\mathrm{L}}{\otimes}_{\mathbb{Z}}M)\simeq(H^k(X)\otimes_{\mathbb{Z}}M)\oplus\mathrm{Tor}_{\mathbb{Z}}^{-1}(H^{k+1}(X),M).$$

Exercise 7.3. Let $\mathscr{C}$ be an abelian category with enough injectives and let $0\to X'\to X\to X''\to 0$ be an exact sequence in $\mathscr{C}$. Assuming that $\mathrm{Ext}^1_{\mathscr{C}}(X'',X')\simeq 0$, prove that the sequence splits.

Exercise 7.4. Let $\mathscr{C}$ be an abelian category and let $X\to Y\to Z\xrightarrow{+1}$ be a d.t. in $\mathrm{D}(\mathscr{C})$. Assuming that $\mathrm{Ext}^1_{\mathscr{C}}(Z,X)\simeq 0$, prove that $Y\simeq X\oplus Z$. (Hint: use Exercise 6.2.)

Exercise 7.5. Let $\mathscr{C}$ be an abelian category, let $X\in\mathrm{D}^{\mathrm{b}}(\mathscr{C})$ and let $a<b\in\mathbb{Z}$. Assume that $H^j(X)\simeq 0$ for $j\neq a,b$ and $\mathrm{Ext}^{b-a+1}_{\mathscr{C}}(H^b(X),H^a(X))\simeq 0$. Prove the isomorphism $X\simeq H^a(X)[-a]\oplus H^b(X)[-b]$. (Hint: use Exercise 7.4 and the d.t. in 7.1.4.

Exercise 7.6. We follow the notations of Exercise 5.10. Hence, $\mathbf{k}$ is a field of characteristic 0 and $W:=W_n(\mathbf{k})$ is the Weyl algebra in n variables. Let $1\leq p\leq n$ and consider the left ideal

$$I_p=W\cdot x_1+\cdots+W\cdot x_p+W\cdot\partial_{p+1}+\cdots+W\cdot\partial_n.$$

Define similarly the right ideal

$$J_p=x_1\cdot W+\cdots+x_p\cdot W+\partial_{p+1}\cdot W+\cdots+\partial_n\cdot W.$$

For $1\leq p\leq q\leq n$, calculate $\mathrm{RHom}_W(W/I_p,W/I_q)$ and $(W/J_q)\overset{\mathrm{L}}{\otimes}_W(W/I_p)$.

Exercise 7.7. Let $\mathscr{C}$ be an abelian category.
(a) Let $X\in\mathrm{D}^{<0}(\mathscr{C})$ and $Y\in\mathrm{D}^{\geq 0}(\mathscr{C})$. Prove that $\mathrm{Hom}_{\mathrm{D}(\mathscr{C})}(X,Y)\simeq 0$.
(b) Conversely, let $Y\in\mathrm{D}(\mathscr{C})$ and assume that $\mathrm{Hom}_{\mathrm{D}(\mathscr{C})}(X,Y)\simeq 0$ for all $X\in\mathrm{D}^{<0}(\mathscr{C})$. Prove that $Y\in\mathrm{D}^{\geq 0}(\mathscr{C})$.

Exercise 7.8. Let $F\colon \mathscr{C} \to \mathscr{C}'$ be a left exact functor of abelian categories and assume that F has a right derived functor and has cohomological dimension $\leq d$. Denote by $\mathscr{J}$ the additive subcategory of $\mathscr{C}$ consisting of F-injective objects and consider an exact sequence $0 \to X \to X^0 \to X^1 \to \cdots \to X^d \to 0$ with $X^j \in \mathscr{J}$ for $0 \leq j < d$. Prove that $X^d \in \mathscr{J}$.
(Hint: decompose the exact sequence into short exact sequences $0 \to Z^{j-1} \to X^j \to Z^j \to 0$ with $Z^{-1} = X$ and show that $R^jF(Z^k) \simeq 0$ for $j > d - k$.)

Exercise 7.9. Recall Definition 5.1.11 and Exercise 5.7.

Let $\mathscr{C}$ be an abelian category and $\mathscr{J}$ a full abelian subcategory, the embedding $\mathscr{J} \to \mathscr{C}$ being exact. Denote by $\mathrm{D}^{\mathrm{b}}_{\mathscr{J}}(\mathscr{C})$ the full subcategory of $\mathrm{D}^{\mathrm{b}}(\mathscr{C})$ consisting of objects X such that for all $j \in \mathbb{Z}$, $H^j(X)$ is isomorphic to an object of $\mathscr{J}$.

(a) Assume that $\mathscr{J}$ is thick in $\mathscr{C}$. Prove that $\mathrm{D}^{\mathrm{b}}_{\mathscr{J}}(\mathscr{C})$ is triangulated.

(b) Assume moreover that for any monomorphism $Y \to X$ with $Y \in \mathscr{J}$, there exists a morphism $X \to Z$ with $Z \in \mathscr{J}$ such that the composition $Y \to X \to Z$ is a monomorphism. Then prove that the natural functor $\mathrm{D}^{\mathrm{b}}(\mathscr{J}) \to \mathrm{D}^{\mathrm{b}}_{\mathscr{J}}(\mathscr{C})$ is an equivalence of categories. (Hint: use Proposition 6.4.5 or see [KS90, Prop. 1.7.11].)

Exercise 7.10. Assume that $\mathbf{k}$ (which has finite global dimension) is Noetherian and denote by $\mathrm{D}^{\mathrm{b}}_f(\mathbf{k})$ the full triangulated subcategory of $\mathrm{D}^{\mathrm{b}}(\mathbf{k})$ consisting of objects whose cohomologies are finitely generated. Let $L, M \in \mathrm{D}^{\mathrm{b}}(\mathbf{k})$ and let $N \in \mathrm{D}^{\mathrm{b}}_f(\mathbf{k})$. Prove the isomorphism in $\mathrm{D}^{-}(\mathbf{k})$:

$$\mathrm{RHom}\,(L, M)\overset{\mathrm{L}}{\otimes}N \xrightarrow{\sim} \mathrm{RHom}\,(L, M\overset{\mathrm{L}}{\otimes}N).$$

(Hint: Represent N by a bounded for above complex of projective modules of finite rank.)

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