Categories and Homological Algebra

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Introduction

The aim of these Notes is to introduce the reader to the language of categories and to present the basic notions of homological algebra, first from an elementary point of view, with the notion of derived functors, next with a more sophisticated approach, with the introduction of triangulated and derived categories.

After having introduced the basic concepts of category theory and in particular those of limits and colimits, also called projective and inductive limits, we treat with some details additive and abelian categories and we construct the derived functors. We also introduce the reader to the concepts of triangulated and derived categories. Our exposition of these topics is rather sketchy, and the reader is encouraged to consult the literature.

These Notes are essentially extracted from [KS06]. Other references are [Mac98, Bor94] for the general theory of categories, [GM96, Wei94] and [KS90, Ch. 1], for homological algebra, including derived categories as well as [Nee01, Yek20] for an exhaustive study of triangulated categories and derived categories, the last reference developing the DG (differential graded) setting. For further developments, see also [SGA4, KS06].

Let us briefly describe the contents of this text.

In **Chapter 1** we expose the basic language of categories and functors. A key point is the Yoneda lemma, which asserts that a category $\mathscr{C}$ may be embedded in the category $\mathscr{C}^\wedge$ of contravariant functors on $\mathscr{C}$ with values in the category **Set** of sets. This naturally leads to the concept of representable functor. Many examples are treated, in particular in relation with the categories **Set** of sets and $\mathrm{Mod}(\mathbf{k})$ of $\mathbf{k}$-modules, for a (non necessarily commutative) ring $\mathbf{k}$.

In **Chapter 2** we construct the limits and colimits, starting with the particular cases of the kernels and cokernels, products and coproducts. We introduce the notions of filtrant categories and cofinal functors, and study with some care filtrant inductive limits in the category **Set**. Finally, we define right or left exact functors and give some examples.

In **Chapter 3** we introduce additive categories and study the category of complexes in such categories. In particular, we introduce the shifted complex, the mapping cone of a morphism, the homotopy of complexes and the simple complex associated with a double complex, with application to bifunctors. We also briefly study the simplicial category and explain how to associate complexes to simplicial objects.

In **Chapter 4** we treat abelian categories. The toy model of such categories is the category $\mathrm{Mod}(\mathbf{k})$ of modules over a ring $\mathbf{k}$ and for sake of simplicity, we shall always argue as if we were working in a full abelian subcategory of a category $\mathrm{Mod}(\mathbf{k})$. We explain the notions of exact sequences, give some basic lemmas such as “the five lemma” and “the snake lemma”, and study injective resolutions. We apply these results in constructing the derived functors of a left exact functor (or bifunctor), assuming that the category admits enough injectives. As an application we get the functors Ext and Tor. Finally, we study Koszul complexes and show how they naturally appear in Algebra and Analysis.

Chapters 1 to 4 may be considered as an elementary introduction to the subject. Chapters 6 to 7 are more difficult, but our study will be more superficial and some proofs will be skipped.

In **Chapter 6**, we construct the localization of a category with respect to a family of morphisms $\mathscr{S}$ satisfying suitable conditions and we construct the localization of functors. Localization of categories appears in particular in the construction of derived categories.

In **Chapter 5**, we introduce triangulated categories, triangulated functors and cohomological functors, and prove some basic results of this theory. We also localize triangulated categories and triangulated functors.

Chapter 7 is devoted to derived categories. The homotopy category $\mathrm{K}(\mathscr{C})$ of an additive category $\mathscr{C}$ is triangulated. When $\mathscr{C}$ is abelian, the cohomology functor $H^0\colon \mathrm{K}(\mathscr{C}) \to \mathscr{C}$ is cohomological and the derived category $\mathrm{D}(\mathscr{C})$ of $\mathscr{C}$ is obtained by localizing $\mathrm{K}(\mathscr{C})$ with respect to the family of quasi-isomorphisms. We explain here this construction, with some examples, and also construct the right derived functor of a left exact functor. We state, without proof, the Brown representability theorem, a fundamental result for applications.

Conventions. In these Notes, all rings are unital and associative but not necessarily commutative. The operations, the zero element, and the unit are denoted by $+, \cdot, 0, 1$, respectively. However, we shall often write for short ab instead of $a \cdot b$. All along these Notes, k will denote a *commutative* ring. (Sometimes, k will be a field.) We denote by $\varnothing$ the empty set and by $\{\mathrm{pt}\}$ a set with one element. We denote by $\mathbb{N}$ the set of non-negative integers, $\mathbb{N} = \{0, 1, \dots\}$.

A comment. These Notes are written in the "classical" language of category theory, that is, 1- or 2-categories. However, some parts of mathematics, especially Algebraic Geometry, are now developed in the language of ∞-categories. The order of difficulty of this last theory is, for the moment, much greater than that of the classical one and there does not exist to our knowledge any accessible text of a reasonable size to this new theory. References are made to [Cis19, Lur, Toë14]. Moreover the classical theory is perfectly suited for the applications we have in mind.

Caution. We will be extremely sketchy with the questions of universes.

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Chapter 1

The language of categories

Caution. All along this book, we shall be rather sketchy with the notion of universes, mentioning when necessary (perhaps not always!) that a category is "small" or "big" with respect to a universe $\mathscr{U}$. Indeed, it not possible to develop category theory without some caution about the size of the objects we consider. An easy and classical illustration of this fact is given in Remark 2.6.12.

In this chapter we start with some reminders on the categories **Set** of sets and $\mathrm{Mod}(A)$ of modules over a (not necessarily commutative) ring A. Then we expose the basic language of categories and functors. A key point is the Yoneda lemma, which asserts that a category $\mathscr{C}$ may be embedded in the category $\mathscr{C}^\wedge$ of contravariant functors on $\mathscr{C}$ with values in the category **Set**. This naturally leads to the concept of representable functor and adjoint functors.. Many examples are treated, in particular in the categories **Set** and $\mathrm{Mod}(A)$.

1.1 Sets and maps

The aim of this section is to fix some notations and to recall some elementary constructions on sets.

If $f\colon X \to Y$ is a map from a set X to a set Y, we shall often say that f is a morphism (of sets) from X to Y. We shall denote by $\mathrm{Hom}_{\mathbf{Set}}(X,Y)$, or simply $\mathrm{Hom}\,(X,Y)$ or also Y^X, the set of all maps from X to Y. If $g\colon Y \to Z$ is another map, we can define the composition $g \circ f\colon X \to Z$. Hence, we get two maps:

$$\begin{aligned} g\circ\colon \mathrm{Hom}\,(X,Y) &\to \mathrm{Hom}\,(X,Z),\\ \circ f\colon \mathrm{Hom}\,(Y,Z) &\to \mathrm{Hom}\,(X,Z). \end{aligned}$$

If f is bijective we shall say that f is an isomorphism and write $f\colon X \xrightarrow{\sim} Y$. This is equivalent to saying that there exists $g\colon Y \to X$ such that $g \circ f$ is the identity of X and $f \circ g$ is the identity of Y. If there exists an isomorphism $f\colon X \xrightarrow{\sim} Y$, we say that X and Y are isomorphic and write $X \simeq Y$.

Notice that if $X = \{x\}$ and $Y = \{y\}$ are two sets with one element each, then there exists a unique isomorphism $X \xrightarrow{\sim} Y$. Of course, if X and Y are finite sets with the same cardinal $\pi > 1$, X and Y are still isomorphic, but the isomorphism is no more unique.

In the sequel we shall denote by $\varnothing$ the empty set and by $\{\mathrm{pt}\}$ a set with one element. Note that for any set X, there is a unique map $\varnothing \to X$ and a unique map $X \to \{\mathrm{pt}\}$.

Let $\{X_i\}_{i\in I}$ be a family of sets indexed by a set I. Their union is denoted by $\bigcup_i X_i$. The product of the X_i's, denoted $\prod_{i\in I} X_i$, or simply $\prod_i X_i$, is defined as

$$\prod_{i\in I} X_i = \{f \in \operatorname{Hom}(I, \bigcup_i X_i); f(i) \in X_i \text{ for all } i \in I\}. \tag{1.1.1}$$

Hence, if $X_i = X$ for all $i \in I$, we get

$$\prod_{i\in I} X_i = X^I.$$

If I is the ordered set $\{1, 2\}$, one sets

$$X_1 \times X_2 = \{(x_1, x_2); x_i \in X_i, i = 1, 2\}, \tag{1.1.2}$$

and there are natural isomorphisms

$$X_1 \times X_2 \simeq \prod_{i\in I} X_i \simeq X_2 \times X_1.$$

This notation and these isomorphisms extend to the case of a finite ordered set I.

If $\{X_i\}_{i\in I}$ is a family of sets indexed by a set I as above, one also considers their disjoint union, also called their coproduct. The coproduct of the X_i's is denoted $\bigsqcup_{i\in I} X_i$ or simply $\bigsqcup_i X_i$. If $X_i = X$ for all $i \in I$, one uses the notation $X^{\sqcup I}$. If $I = \{1, 2\}$, one often writes $X_1 \sqcup X_2$ instead of $X^{1\sqcup 2}$.

For three sets I, X, Y, there is a natural isomorphism

$$\operatorname{Hom}(I, \operatorname{Hom}(X, Y)) = \operatorname{Hom}(X, Y)^I \simeq \operatorname{Hom}(I \times X, Y). \tag{1.1.3}$$

For a set Y, there is a natural isomorphism

$$\operatorname{Hom}(Y, \prod_i X_i) \simeq \prod_i \operatorname{Hom}(Y, X_i). \tag{1.1.4}$$

Note that

$$X \times I \simeq X^{\sqcup I}. \tag{1.1.5}$$

For a set Y, there is a natural isomorphism

$$\operatorname{Hom}(\bigsqcup_i X_i, Y) \simeq \prod_i \operatorname{Hom}(X_i, Y). \tag{1.1.6}$$

In particular,

$$\operatorname{Hom}(X^{\sqcup I}, Y) \simeq \operatorname{Hom}(X, Y^I) \simeq \operatorname{Hom}(X, Y)^I.$$

Consider two sets X and Y and two maps f, g from X to Y. We write for short $f, g\colon X \rightrightarrows Y$. The kernel (or equalizer) of (f, g), denoted $\ker(f, g)$, is defined as

$$\ker(f, g) = \{x \in X; f(x) = g(x)\}. \tag{1.1.7}$$

Note that for a set Z, one has

$$\operatorname{Hom}(Z, \ker(f, g)) \simeq \ker(\operatorname{Hom}(Z, X) \rightrightarrows \operatorname{Hom}(Z, Y)). \tag{1.1.8}$$

Let us recall a few elementary definitions.

- A relation $\mathscr{R}$ on a set X is a subset of $X \times X$. One writes $x\mathscr{R}y$ if $(x, y) \in \mathscr{R}$.
- The opposite relation $\mathscr{R}^{\mathrm{op}}$ is defined by $x\mathscr{R}^{\mathrm{op}}y$ if and only if $y\mathscr{R}x$.
- A relation $\mathscr{R}$ is reflexive if it contains the diagonal, that is, $x\mathscr{R}x$ for all $x \in X$.
- A relation $\mathscr{R}$ is symmetric if $x\mathscr{R}y$, then $y\mathscr{R}x$.
- A relation $\mathscr{R}$ is anti-symmetric if $x\mathscr{R}y$ and $y\mathscr{R}x$, then $x = y$.
- A relation $\mathscr{R}$ is transitive if $x\mathscr{R}y$ and $y\mathscr{R}z$, then $x\mathscr{R}z$.
- A relation $\mathscr{R}$ is an equivalence relation if it is reflexive, symmetric and transitive.
- A relation $\mathscr{R}$ is a pre-order if it is reflexive and transitive. A pre-order is often denoted $\leq$. If the pre-order is anti-symmetric, then one says that $\mathscr{R}$ is an order on X. A set endowed with a pre-order is called a partially ordered set, or, for short, a poset.
- Let $(I, \leq)$ be a poset. One says that $(I, \leq)$ is filtered (one also says "directed" or "filtrant") if I is non empty and for any $i, j \in I$ there exists k with $i \leq k$ and $j \leq k$.
- Assume $(I, \leq)$ is a filtered poset and let $J \subset I$ be a subset. One says that J is cofinal to I if for any $i \in I$ there exists $j \in J$ with $i \leq j$.

If $\mathscr{R}$ is a relation on a set X, there is a smallest equivalence relation which contains $\mathscr{R}$. (Take the intersection of all subsets of $X \times X$ which contain $\mathscr{R}$ and which are equivalence relations.)

Let $\mathscr{R}$ be an equivalence relation on a set X. A subset S of X is saturated if $x \in S$ and $x\mathscr{R}y$ implies $y \in S$. For $x \in X$, the smallest saturated subset $\widehat{x}$ of X containing x is called the equivalence class of x. One then defines a new set $X/\mathscr{R}$ and a canonical map $f\colon X \to X/\mathscr{R}$ as follows: the elements of $X/\mathscr{R}$ are the sets $\widehat{x}$ and the map f associates the set $\widehat{x}$ to $x \in X$.

1.2 Modules and linear maps

All along this book, a ring A means a unital associative ring, but A is not necessarily commutative, and $\mathbf{k}$ denotes a commutative ring. Recall that a $\mathbf{k}$-algebra A is a ring endowed with a morphism of rings $\varphi\colon \mathbf{k} \to A$ such that the image of $\mathbf{k}$ is contained in the center of A (*i.e.*, $\varphi(x)a = a\varphi(x)$ for any $x \in \mathbf{k}$ and $a \in A$). Notice that a ring A is always a $\mathbb{Z}$-algebra. If A is commutative, then A is an A-algebra.

Since we do not assume A commutative, we have to distinguish between left and right structures. Unless otherwise specified, a module M over A means a left A-module.

Recall that an A-module M is an additive group (whose operations and zero element are denoted $+, 0$) endowed with an external law $A \times M \to M$ (denoted

$(a, m) \mapsto a \cdot m$ or simply $(a, m) \mapsto am$) satisfying:

$$\begin{cases} (ab)m = a(bm) \\ (a+b)m = am + bm \\ a(m+m') = am + am' \\ 1 \cdot m = m \end{cases}$$

where $a, b \in A$ and $m, m' \in M$.

Note that, when A is a $\mathbf{k}$-algebra, M inherits a structure of a $\mathbf{k}$-module via φ. In the sequel, if there is no risk of confusion, we shall not write φ.

We denote by A^{op} the ring A with the opposite structure. Hence the product ab in A^{op} is the product ba in A and an A^{op}-module is a right A-module.

Note that if the ring A is a field (here, a field is always commutative), then an A-module is nothing but a vector space.

Examples 1.2.1. (i) The first example of a ring is $\mathbb{Z}$, the ring of integers. Since a field is a ring, $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ are rings. If A is a commutative ring, then $A[x_1, \dots, x_n]$, the ring of polynomials in n variables with coefficients in A, is also a commutative ring. It is a sub-ring of $A[[x_1, \dots, x_n]]$, the ring of formal powers series with coefficients in A.
(ii) Let $\mathbf{k}$ be a field. For $n > 1$, the ring $M_n(\mathbf{k})$ of square matrices of rank n with entries in $\mathbf{k}$ is non-commutative.
(iii) Let $\mathbf{k}$ be a field. The *Weyl algebra* in n variables, denoted $W_n(\mathbf{k})$, is the non commutative ring of polynomials in the variables x_i, ∂_j $(1 \leq i, j \leq n)$ with coefficients in $\mathbf{k}$ and relations :

$$[x_i, x_j] = 0, \ [\partial_i, \partial_j] = 0, \ [\partial_j, x_i] = \delta_j^i$$

where $[p, q] = pq - qp$ and δ_j^i is the Kronecker symbol.

The Weyl algebra $W_n(\mathbf{k})$ may be regarded as the ring of differential operators with coefficients in $\mathbf{k}[x_1, \dots, x_n]$, and $\mathbf{k}[x_1, \dots, x_n]$ becomes a left $W_n(\mathbf{k})$-module: x_i acts by multiplication and ∂_i is the derivation with respect to x_i.

A morphism $f \colon M \to N$ of A-modules is an A-linear map, *i.e.,* f satisfies:

$$\begin{cases} f(m+m') = f(m) + f(m') \quad m, m' \in M, \\ f(am) = af(m) \quad m \in M, a \in A. \end{cases}$$

A morphism f is an isomorphism if there exists a morphism $g : N \to M$ with $f \circ g = \mathrm{id}_N, g \circ f = \mathrm{id}_M$.

If f is bijective, it is easily checked that the inverse map $f^{-1} : N \to M$ is itself A-linear. Hence f is an isomorphism if and only if f is A-linear and bijective.

A submodule N of M is a nonempty subset N of M such that if $n, n' \in N$, then $n + n' \in N$ and if $n \in N, a \in A$, then $an \in N$. A submodule of the A-module A is called an ideal of A. Note that if A is a field, it has no non-trivial ideal, *i.e.,* its only ideals are $\{0\}$ and A. If $A = \mathbb{C}[x]$, then $I = \{P \in \mathbb{C}[x]; P(0) = 0\}$ is a non trivial ideal.

If N is a submodule of M, it defines an equivalence relation: $m\mathscr{R}m'$ if and only if $m - m' \in N$. One easily checks that the quotient set $M/\mathscr{R}$ is naturally endowed with a structure of a left A-module. This module is called the quotient module and is denoted M/N.

Let $f\colon M \to N$ be a morphism of A-modules. One sets:

$$\begin{aligned}\ker f &= \{m \in M; \quad f(m)=0\},\\ \operatorname{Im} f &= \{n \in N; \quad \text{there exists } m \in M, \quad f(m)=n\}.\end{aligned}$$

These are submodules of M and N respectively, called the kernel and the image of f, respectively. One also introduces the cokernel and the coimage of f:

$$\operatorname{Coker} f = N/\operatorname{Im} f, \quad \operatorname{Coim} f = M/\ker f.$$

Note that the natural morphism $\operatorname{Coim} f \to \operatorname{Im} f$ is an isomorphism.

Example 1.2.2. Let $W_n(\mathbf{k})$ denote as above the Weyl algebra. Consider the left $W_n(\mathbf{k})$-linear map $W_n(\mathbf{k}) \to \mathbf{k}[x_1, \dots, x_n]$, $W_n(\mathbf{k}) \ni P \mapsto P(1) \in \mathbf{k}[x_1, \dots, x_n]$. This map is clearly surjective and its kernel is the left ideal generated by $(\partial_1, \cdots, \partial_n)$. Hence, one has the isomorphism of left $W_n(\mathbf{k})$-modules:

$$W_n(\mathbf{k})/\sum_j W_n(\mathbf{k})\partial_j \xrightarrow{\sim} \mathbf{k}[x_1, \dots, x_n]. \tag{1.2.1}$$

Products and direct sums

Let I be a set, and let $\{M_i\}_{i\in I}$ be a family of A-modules indexed by I. The set $\prod_i M_i$ is naturally endowed with a structure of a left A-module by setting

$$\begin{aligned}(m_i)_i + (m'_i)_i &= (m_i + m'_i)_i,\\ a \cdot (m_i)_i &= (a \cdot m_i)_i.\end{aligned}$$

The direct sum $\bigoplus_i M_i$ is the submodule of $\prod_i M_i$ whose elements are the $(x_i)_i$'s such that $x_i = 0$ for all but a finite number of $i \in I$. In particular, if the set I is finite, we have $\bigoplus_i M_i = \prod_i M_i$. If $M_i = M$ for all i, one writes $M^{\oplus I}$ or $M^{(I)}$ instead of $\bigoplus_i M$.

Linear maps

Let M and N be two A-modules. Recall that an A-linear map $f\colon M \to N$ is also called a morphism of A-modules. One denotes by $\operatorname{Hom}_A(M,N)$ the set of A-linear maps $f\colon M \to N$. When A is a $\mathbf{k}$-algebra, $\operatorname{Hom}_A(M,N)$ is a $\mathbf{k}$-module. In fact one defines the action of $\mathbf{k}$ on $\operatorname{Hom}_A(M,N)$ by setting: $(\lambda f)(m) = \lambda(f(m))$. Hence $(\lambda f)(am) = \lambda f(am) = \lambda a f(m) = a\lambda f(m) = a(\lambda f)(m)$, and $\lambda f \in \operatorname{Hom}_A(M,N)$.

There is a natural isomorphism $\operatorname{Hom}_A(A,M) \simeq M$: to $u \in \operatorname{Hom}_A(A,M)$ one associates $u(1)$ and to $m \in M$ one associates the linear map $A \to M, a \mapsto am$. More generally, if I is an ideal of A then $\operatorname{Hom}_A(A/I,M) \simeq \{m \in M; Im = 0\}$.

Note that if A is a $\mathbf{k}$-algebra and $L \in \operatorname{Mod}(\mathbf{k})$, $M \in \operatorname{Mod}(A)$, the $\mathbf{k}$-module $\operatorname{Hom}_{\mathbf{k}}(L,M)$ is naturally endowed with a structure of a left A-module. If N is a right A-module, then $\operatorname{Hom}_{\mathbf{k}}(N,L)$ is naturally endowed with a structure of a left A-module.

Tensor product

Consider a right A-module N, a left A-module M and a $\mathbf{k}$-module L. Let us say that a map $f\colon N \times M \to L$ is $(A, \mathbf{k})$-bilinear if f is additive with respect to each of its arguments and satisfies $f(na, m) = f(n, am)$ and $f(n\lambda, m) = \lambda(f(n, m))$ for all $(n, m) \in N \times M$ and $a \in A, \lambda \in \mathbf{k}$.

Let us identify a set I to a subset of $\mathbf{k}^{(I)}$ as follows: to $i \in I$, we associate $\{l_j\}_{j \in I} \in \mathbf{k}^{(I)}$ given by

$$l_j = \begin{cases} 1 \text{ if } & j = i, \\ 0 \text{ if } & j \neq i. \end{cases} \tag{1.2.2}$$

The tensor product $N \otimes_A M$ is the $\mathbf{k}$-module defined as the quotient of $\mathbf{k}^{(N \times M)}$ by the submodule generated by the following elements (where $n, n' \in N, m, m' \in M, a \in A, \lambda \in \mathbf{k}$ and $N \times M$ is identified to a subset of $\mathbf{k}^{(N \times M)}$):

$$\begin{cases} (n + n', m) - (n, m) - (n', m), \\ (n, m + m') - (n, m) - (n, m'), \\ (na, m) - (n, am), \\ \lambda(n, m) - (n\lambda, m). \end{cases}$$

The image of (n, m) in $N \otimes_A M$ is denoted $n \otimes m$. Hence an element of $N \otimes_A M$ may be written (not uniquely!) as a finite sum $\sum_j n_j \otimes m_j$, $n_j \in N$, $m_j \in M$ and:

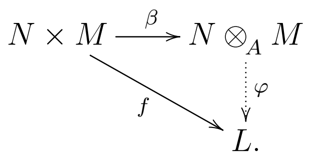
$$\begin{cases} (n + n') \otimes m = n \otimes m + n' \otimes m, \\ n \otimes (m + m') = n \otimes m + n \otimes m', \\ na \otimes m = n \otimes am, \\ \lambda(n \otimes m) = n\lambda \otimes m = n \otimes \lambda m. \end{cases}$$

Denote by $\beta\colon N \times M \to N \otimes_A M$ the natural map which associates $n \otimes m$ to (n, m).

Proposition 1.2.3. *The map β is $(A, \mathbf{k})$-bilinear and for any $\mathbf{k}$-module L and any $(A, \mathbf{k})$-bilinear map $f\colon N \times M \to L$, the map f factorizes uniquely through a $\mathbf{k}$-linear map $\varphi\colon N \otimes_A M \to L$.*

The proof is left to the reader.

Proposition 1.2.3 is visualized by the diagram:



Consider an A-linear map $f\colon M \to L$. It defines a linear map $\mathrm{id}_N \times f\colon N \times M \to N \times L$, hence a $(A, \mathbf{k})$-bilinear map $N \times M \to N \otimes_A L$, and finally a $\mathbf{k}$-linear map

$$\mathrm{id}_N \otimes f\colon N \otimes_A M \to N \otimes_A L.$$

One constructs similarly $g \otimes \mathrm{id}_M$ associated to $g\colon N \to L$.

There are natural isomorphisms $A \otimes_A M \simeq M$ and $N \otimes_A A \simeq N$.

Denote by $\mathrm{Bil(N \times M, L)}$ the $\mathbf{k}$-module of $(A, \mathbf{k})$-bilinear maps from $N \times M$ to L. One has the isomorphisms

$$\begin{aligned}\mathrm{Bil(N \times M, L)} &\simeq \mathrm{Hom}_{\mathbf{k}}(N \otimes_A M, L) \\ &\simeq \mathrm{Hom}_A(M, \mathrm{Hom}_{\mathbf{k}}(N, L)) \\ &\simeq \mathrm{Hom}_A(N, \mathrm{Hom}_{\mathbf{k}}(M, L)).\end{aligned} \tag{1.2.3}$$

For $L \in \mathrm{Mod}(\mathbf{k})$ and $M \in \mathrm{Mod}(A)$, the $\mathbf{k}$-module $L \otimes_{\mathbf{k}} M$ is naturally endowed with a structure of a left A-module. For $M, N \in \mathrm{Mod}(A)$ and $L \in \mathrm{Mod}(\mathbf{k})$, we have the isomorphisms (whose verification is left to the reader):

$$\begin{aligned}\mathrm{Hom}_A(L \otimes_{\mathbf{k}} N, M) &\simeq \mathrm{Hom}_A(N, \mathrm{Hom}_{\mathbf{k}}(L, M)) \\ &\simeq \mathrm{Hom}_{\mathbf{k}}(L, \mathrm{Hom}_A(N, M)).\end{aligned} \tag{1.2.4}$$

If A is commutative, $N \otimes_A M$ is naturally an A-module and there is an isomorphism: $N \otimes_A M \simeq M \otimes_A N$ given by $n \otimes m \mapsto m \otimes n$. Moreover, the tensor product is associative, that is, if L, M, N are A-modules, there are natural isomorphisms $L \otimes_A (M \otimes_A N) \simeq (L \otimes_A M) \otimes_A N$. One simply writes $L \otimes_A M \otimes_A N$.

1.3 Categories and functors

Definition 1.3.1. A category $\mathscr{C}$ consists of:

(i) a set $\mathrm{Ob}(\mathscr{C})$ whose elements are called the objects of $\mathscr{C}$,

(ii) for each $X, Y \in \mathrm{Ob}(\mathscr{C})$, a set $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ whose elements are called the morphisms from X to Y,

(iii) for any $X, Y, Z \in \mathrm{Ob}(\mathscr{C})$, a map, called the composition, $\mathrm{Hom}_{\mathscr{C}}(X, Y) \times \mathrm{Hom}_{\mathscr{C}}(Y, Z) \to \mathrm{Hom}_{\mathscr{C}}(X, Z)$, and denoted $(f, g) \mapsto g \circ f$,

these data satisfying:

(a) $\circ$ is associative,

(b) for each $X \in \mathrm{Ob}(\mathscr{C})$, there exists $\mathrm{id}_X \in \mathrm{Hom}\,(X, X)$ such that for all $f \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$ and $g \in \mathrm{Hom}_{\mathscr{C}}(Y, X)$, $f \circ \mathrm{id}_X = f$, $\mathrm{id}_X \circ g = g$.

Note that $\mathrm{id}_X \in \mathrm{Hom}\,(X, X)$ is characterized by the condition in (b).

Universes

With such a definition of a category, there is no category of sets, since there is no set of "all" sets. The set-theoretical dangers encountered in category theory will be illustrated in Remark 2.6.12.

To overcome this difficulty, one has to be more precise when using the word "set". One way is to use the notion of *universe*. We do not give in this book the exact definition of a universe, only recalling that a universe $\mathscr{U}$ is a set (a very big one) stable by many operations. In particular, $\varnothing \in \mathscr{U}$, $\mathbb{N} \in \mathscr{U}$, $x \in \mathscr{U}$ and $y \in x$ implies $y \in \mathscr{U}$, $x \in \mathscr{U}$ and $y \subset x$ implies $y \in \mathscr{U}$, if $I \in \mathscr{U}$ and $u_i \in \mathscr{U}$ for all $i \in I$, then $\bigcup_{i \in I} u_i \in \mathscr{U}$ and $\prod_{i \in I} u_i \in \mathscr{U}$. See for example [KS06, Def. 1.1.1].

Definition 1.3.2. Let $\mathscr{U}$ be a universe.

(a) A set E is a $\mathscr{U}$-set if it belongs to $\mathscr{U}$.

(b) A set E is $\mathscr{U}$-small if it is isomorphic to a $\mathscr{U}$-set.

(c) A $\mathscr{U}$-category $\mathscr{C}$ is a category such that for any $X, Y \in \mathscr{C}$, the set $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ is $\mathscr{U}$-small.

(d) A $\mathscr{U}$-category $\mathscr{C}$ is $\mathscr{U}$-small if moreover the set $\mathrm{Ob}(\mathscr{C})$ is $\mathscr{U}$-small.

The crucial point is Grothendieck's axiom which says that any set belongs to some universe.

By a "big" category, we mean a category in a bigger universe. Note that, by Grothendieck's axiom, any category is an $\mathscr{V}$-category for a suitable universe $\mathscr{V}$ and one even can choose $\mathscr{V}$ so that $\mathscr{C}$ is $\mathscr{V}$-small.

As far as it has no implication, we shall not always be precise on this matter and the reader may skip the words "small" and "big".

Notation 1.3.3. One often writes $X \in \mathscr{C}$ instead of $X \in \mathrm{Ob}(\mathscr{C})$ and $f\colon X \to Y$ (or else $f\colon Y \leftarrow X$) instead of $f \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$. One calls X the source and Y the target of f.

- A morphism $f\colon X \to Y$ is an *isomorphism* if there exists $g\colon X \leftarrow Y$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$. In such a case, one writes $f\colon X \xrightarrow{\sim} Y$ or simply $X \simeq Y$. Of course g is unique, and one also denotes it by f^{-1}.

- A morphism $f\colon X \to Y$ is a *monomorphism* (resp. an *epimorphism*) if for any morphisms g_1 and g_2, $f \circ g_1 = f \circ g_2$ (resp. $g_1 \circ f = g_2 \circ f$) implies $g_1 = g_2$. One sometimes writes $f\colon X \rightarrowtail Y$ or else $X \hookrightarrow Y$ (resp. $f\colon X \twoheadrightarrow Y$) to denote a monomorphism (resp. an epimorphism).

- Two morphisms f and g are parallel if they have the same sources and targets, visualized by $f, g\colon X \rightrightarrows Y$.

- A category is *discrete* if the only morphisms are the identity morphisms. Note that a set is naturally identified with a discrete category (and conversely).

- A category $\mathscr{C}$ is *finite* if the family of all morphisms in $\mathscr{C}$ (hence, in particular, the family of objects) is a finite set.

- A category $\mathscr{C}$ is a *groupoid* if all morphisms are isomorphisms.

One introduces the *opposite category* $\mathscr{C}^{\mathrm{op}}$:

$$\mathrm{Ob}(\mathscr{C}^{\mathrm{op}}) = \mathrm{Ob}(\mathscr{C}), \quad \mathrm{Hom}_{\mathscr{C}^{\mathrm{op}}}(X, Y) = \mathrm{Hom}_{\mathscr{C}}(Y, X),$$

the identity morphisms and the composition of morphisms being the obvious ones.

A category $\mathscr{C}'$ is a *subcategory* of $\mathscr{C}$, denoted $\mathscr{C}' \subset \mathscr{C}$, if:

(a) $\mathrm{Ob}(\mathscr{C}') \subset \mathrm{Ob}(\mathscr{C})$,

(b) $\mathrm{Hom}_{\mathscr{C}'}(X,Y) \subset \mathrm{Hom}_{\mathscr{C}}(X,Y)$ for any $X, Y \in \mathscr{C}'$, the composition $\circ$ in $\mathscr{C}'$ is induced by the composition in $\mathscr{C}$ and the identity morphisms in $\mathscr{C}'$ are induced by those in $\mathscr{C}$.

- One says that $\mathscr{C}'$ is a *full* subcategory if for all $X, Y \in \mathscr{C}'$, $\mathrm{Hom}_{\mathscr{C}'}(X,Y) = \mathrm{Hom}_{\mathscr{C}}(X,Y)$.
- One says that a full subcategory $\mathscr{C}'$ of $\mathscr{C}$ is *saturated* if $X \in \mathscr{C}$ belongs to $\mathscr{C}'$ as soon as it is isomorphic to an object of $\mathscr{C}'$.

Examples 1.3.4. (i) **Set** is the category of sets and maps (in a given universe $\mathscr{U}$). If necessary, one calls this category $\mathscr{U}$-**Set**. Then $\mathbf{Set}^f$ is the full subcategory consisting of finite sets.
(ii) **Rel** is defined by: $\mathrm{Ob}(\mathbf{Rel}) = \mathrm{Ob}(\mathbf{Set})$ and $\mathrm{Hom}_{\mathbf{Rel}}(X,Y) = \mathscr{P}(X \times Y)$, the set of subsets of $X \times Y$. The composition law is defined as follows. For $f\colon X \to Y$ and $g\colon Y \to Z$, $g \circ f$ is the set

$$\{(x,z) \in X \times Z;\ \text{there exists } y \in Y \text{ with } (x,y) \in f, (y,z) \in g\}.$$

Of course, $\mathrm{id}_X = \Delta \subset X \times X$, the diagonal of $X \times X$.
(iii) Let A be a ring. The category of left A-modules and A-linear maps is denoted $\mathrm{Mod}(A)$. In particular $\mathrm{Mod}(\mathbb{Z})$ is the category of abelian groups.

We shall use the notation $\mathrm{Hom}_A(\,\bullet\,,\,\bullet\,)$ instead of $\mathrm{Hom}_{\mathrm{Mod}(A)}(\,\bullet\,,\,\bullet\,)$.

One denotes by $\mathrm{Mod}^{\mathrm{f}}(A)$ the full subcategory of $\mathrm{Mod}(A)$ consisting of finitely generated A-modules.
(iv) One associates to a pre-ordered set $(I, \le)$ a category, still denoted by I for short, as follows. $\mathrm{Ob}(I) = I$, and the set of morphisms from i to j has a single element if $i \le j$, and is empty otherwise. Note that I^{op} is the category associated with I endowed with the opposite pre-order.
(v) We denote by **Top** the category of topological spaces and continuous maps.
(vi) We shall often represent by the diagram $\bullet \to \bullet$ the category which consists of two objects, say $\{a, b\}$, and one morphism $a \to b$ other than id_a and id_b. We denote this category by **Arr**.
(vii) We represent by $\bullet \rightrightarrows \bullet$ the category with two objects, say $\{a, b\}$, and two parallel morphisms $a \rightrightarrows b$ other than id_a and id_b.
(viii) Let G be a group. We may attach to it the groupoid $\mathscr{G}$ with one object, say $\{a\}$ and morphisms $\mathrm{Hom}_{\mathscr{G}}(a,a) = G$.
(ix) Let X be a topological space locally arcwise connected. We attach to it a category $\widetilde{X}$ as follows: $\mathrm{Ob}(\widetilde{X}) = X$ and for $x, y \in X$, a morphism $f\colon x \to y$ is a path form x to y. (Precise definitions are left to the reader.)

Definition 1.3.5. (i) An object $P \in \mathscr{C}$ is called initial if $\mathrm{Hom}_{\mathscr{C}}(P,X) \simeq \{\mathrm{pt}\}$ for all $X \in \mathscr{C}$. One often denotes by $\varnothing_{\mathscr{C}}$ an initial object in $\mathscr{C}$.

(ii) One says that P is terminal if P is initial in $\mathscr{C}^{\mathrm{op}}$, *i.e.,* for all $X \in \mathscr{C}$, $\mathrm{Hom}_{\mathscr{C}}(X,P) \simeq \{\mathrm{pt}\}$. One often denotes by $\mathrm{pt}_{\mathscr{C}}$ a terminal object in $\mathscr{C}$.

(iii) One says that P is a zero-object if it is both initial and terminal. In such a case, one often denotes it by 0. If $\mathscr{C}$ has a zero object, for any objects $X, Y \in \mathscr{C}$, the morphism obtained as the composition $X \to 0 \to Y$ is still denoted by $0\colon X \to Y$.

Note that initial (resp. terminal) objects are unique up to unique isomorphisms.

Examples 1.3.6. (i) In the category **Set**, $\varnothing$ is initial and $\{\mathrm{pt}\}$ is terminal.
(ii) The zero module 0 is a zero-object in $\mathrm{Mod}(A)$.
(iii) The category associated with the ordered set $(\mathbb{Z}, \leq)$ has neither initial nor terminal object.

Definition 1.3.7. Let $\mathscr{C}$ and $\mathscr{C}'$ be two categories. A functor $F\colon \mathscr{C} \to \mathscr{C}'$ consists of a map $F\colon \mathrm{Ob}(\mathscr{C}) \to \mathrm{Ob}(\mathscr{C}')$ and for all $X, Y \in \mathscr{C}$, of a map still denoted by F: $\mathrm{Hom}_{\mathscr{C}}(X, Y) \to \mathrm{Hom}_{\mathscr{C}'}(F(X), F(Y))$ such that

$$F(\mathrm{id}_X) = \mathrm{id}_{F(X)}, \quad F(f \circ g) = F(f) \circ F(g).$$

A contravariant functor from $\mathscr{C}$ to $\mathscr{C}'$ is a functor from $\mathscr{C}^{\mathrm{op}}$ to $\mathscr{C}'$. In other words, it satisfies $F(g \circ f) = F(f) \circ F(g)$. If one wishes to put the emphasis on the fact that a functor is not contravariant, one says it is covariant.

One denotes by $\mathrm{op} : \mathscr{C} \to \mathscr{C}^{\mathrm{op}}$ the contravariant functor, associated with $\mathrm{id}_{\mathscr{C}^{\mathrm{op}}}$.

Example 1.3.8. Let $\mathscr{C}$ be a category and let $X \in \mathscr{C}$.
(i) $\mathrm{Hom}_{\mathscr{C}}(X, \bullet)$ is a functor from $\mathscr{C}$ to **Set**. To $Y \in \mathscr{C}$, it associates the set $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ and to a morphism $f\colon Y \to Z$ in $\mathscr{C}$, it associates the map

$$\mathrm{Hom}_{\mathscr{C}}(X, f)\colon \mathrm{Hom}_{\mathscr{C}}(X, Y) \xrightarrow{f\circ} \mathrm{Hom}_{\mathscr{C}}(X, Z).$$

(ii) $\mathrm{Hom}_{\mathscr{C}}(\bullet, X)$ is a functor from $\mathscr{C}^{\mathrm{op}}$ to **Set**. To $Y \in \mathscr{C}$, it associates the set $\mathrm{Hom}_{\mathscr{C}}(Y, X)$ and to a morphism $f\colon Y \to Z$ in $\mathscr{C}$, it associates the map

$$\mathrm{Hom}_{\mathscr{C}}(f, X)\colon \mathrm{Hom}_{\mathscr{C}}(Z, X) \xrightarrow{\circ f} \mathrm{Hom}_{\mathscr{C}}(Y, X).$$

Example 1.3.9. Let A be a $\mathbf{k}$-algebra and let $M \in \mathrm{Mod}(A)$. Similarly as in Example 1.3.8, we have the functors

$$\mathrm{Hom}_A(M, \bullet)\colon \mathrm{Mod}(A) \to \mathrm{Mod}(\mathbf{k}),$$
$$\mathrm{Hom}_A(\bullet, M)\colon \mathrm{Mod}(A)^{\mathrm{op}} \to \mathrm{Mod}(\mathbf{k})$$

Clearly, the functor $\mathrm{Hom}_A(M, \bullet)$ commutes with products in $\mathrm{Mod}(A)$, that is,

$$\mathrm{Hom}_A(M, \prod_i N_i) \simeq \prod_i \mathrm{Hom}_A(M, N_i)$$

and the functor $\mathrm{Hom}_A(\bullet, N)$ commutes with direct sums in $\mathrm{Mod}(A)$, that is,

$$\mathrm{Hom}_A(\bigoplus_i M_i, N) \simeq \prod_i \mathrm{Hom}_A(M_i, N).$$

(ii) Let N be a right A-module. Then $N \otimes_A \bullet\colon \mathrm{Mod}(A) \to \mathrm{Mod}(\mathbf{k})$ is a functor. Clearly, the functor $N \otimes_A \bullet$ commutes with direct sums, that is,

$$N \otimes_A (\bigoplus_i M_i) \simeq \bigoplus_i (N \otimes_A M_i),$$

and similarly for the functor $\bullet \otimes_A M$.

Definition 1.3.10. Let $F\colon \mathscr{C} \to \mathscr{C}'$ be a functor.

(i) One says that F is faithful (resp. full, resp. fully faithful) if for $X, Y \in \mathscr{C}$ $\mathrm{Hom}_{\mathscr{C}}(X,Y) \to \mathrm{Hom}_{\mathscr{C}'}(F(X),F(Y))$ is injective (resp. surjective, resp. bijective).

(ii) One says that F is essentially surjective if for each $Y \in \mathscr{C}'$ there exists $X \in \mathscr{C}$ and an isomorphism $F(X) \simeq Y$.

(iii) One says that F is conservative if any morphism $f\colon X \to Y$ in $\mathscr{C}$ is an isomorphism as soon as $F(f)$ is an isomorphism.

Examples 1.3.11. (i) The forgetful functor $\mathit{for}\colon \mathrm{Mod}(A) \to \mathbf{Set}$ associates to an A-module M the set M, and to a linear map f the map f. The functor *for* is faithful and conservative but not fully faithful.
(ii) The forgetful functor $\mathit{for}\colon \mathbf{Top} \to \mathbf{Set}$ (defined similarly as in (i)) is faithful. It is neither fully faithful nor conservative.
(iii) Consider the functor $\mathit{for}\colon \mathbf{Set} \to \mathbf{Rel}$ which is the identity on the objects of these categories and which, to a morphism $f\colon X \to Y$ in $\mathbf{Set}$ associates its graph $\Gamma_f \subset X \times Y$. This forgetful functor is faithful but not fully faithful. It is conservative (this is left as an exercise).

One defines the product of two categories $\mathscr{C}$ and $\mathscr{C}'$ by :

$$\mathrm{Ob}(\mathscr{C} \times \mathscr{C}') = \mathrm{Ob}(\mathscr{C}) \times \mathrm{Ob}(\mathscr{C}')$$
$$\mathrm{Hom}_{\mathscr{C}\times\mathscr{C}'}((X,X'),(Y,Y')) = \mathrm{Hom}_{\mathscr{C}}(X,Y) \times \mathrm{Hom}_{\mathscr{C}'}(X',Y').$$

A bifunctor $F\colon \mathscr{C} \times \mathscr{C}' \to \mathscr{C}''$ is a functor on the product category. This means that for $X \in \mathscr{C}$ and $X' \in \mathscr{C}'$, $F(X, \bullet)\colon \mathscr{C}' \to \mathscr{C}''$ and $F(\bullet, X') : \mathscr{C} \to \mathscr{C}''$ are functors, and moreover for any morphisms $f : X \to Y$ in $\mathscr{C}$, $g : X' \to Y'$ in $\mathscr{C}'$, the diagram below commutes:

$$\begin{array}{ccc} F(X,X') & \xrightarrow{F(X,g)} & F(X,Y') \\ {\scriptstyle F(f,X')}\downarrow & & \downarrow{\scriptstyle F(f,Y')} \\ F(Y,X') & \xrightarrow{F(Y,g)} & F(Y,Y') \end{array}$$

In fact, $(f,g) = (\mathrm{id}_Y, g) \circ (f, \mathrm{id}_{X'}) = (f, \mathrm{id}_{Y'}) \circ (\mathrm{id}_X, g)$.

Examples 1.3.12. (i) $\mathrm{Hom}_{\mathscr{C}}(\bullet\,,\bullet\,) : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathbf{Set}$ is a bifunctor.
(ii) If A is a $\mathbf{k}$-algebra, we have met the bifunctors

$$\mathrm{Hom}_A(\bullet\,,\bullet\,) :\ \mathrm{Mod}(A)^{\mathrm{op}} \times \mathrm{Mod}(A) \to \mathrm{Mod}(\mathbf{k}),$$
$$\bullet \otimes_A \bullet :\ \mathrm{Mod}(A^{\mathrm{op}}) \times \mathrm{Mod}(A) \to \mathrm{Mod}(\mathbf{k}).$$

Definition 1.3.13. Let F_1, F_2 be two functors from $\mathscr{C}$ to $\mathscr{C}'$. A morphism of functors $\theta\colon F_1 \to F_2$ is the data for all $X \in \mathscr{C}$ of a morphism $\theta(X) : F_1(X) \to F_2(X)$ such that for all $f\colon X \to Y$, the diagram below commutes:

(1.3.1)
$$\begin{array}{ccc} F_1(X) & \xrightarrow{\theta(X)} & F_2(X) \\ {\scriptstyle F_1(f)}\downarrow & & \downarrow{\scriptstyle F_2(f)} \\ F_1(Y) & \xrightarrow{\theta(Y)} & F_2(Y). \end{array}$$

A morphism of functors is visualized by a diagram:

$$\mathscr{C} \underset{F_2}{\overset{F_1}{\rightrightarrows}} \mathscr{C}' \quad (\Downarrow\theta)$$

Hence, by considering the family of functors from $\mathscr{C}$ to $\mathscr{C}'$ and the morphisms of such functors, we get a new category.

Notation 1.3.14. (i) We denote by $\mathrm{Fct}(\mathscr{C},\mathscr{C}')$ the category of functors from $\mathscr{C}$ to $\mathscr{C}'$. One may also use the shorter notation $(\mathscr{C}')^{\mathscr{C}}$.

Examples 1.3.15. Let $\mathbf{k}$ be a field and consider the functor

$$\begin{aligned} &{}^*\colon \mathrm{Mod}(\mathbf{k})^{\mathrm{op}} \to \mathrm{Mod}(\mathbf{k}),\\ &V \mapsto V^* = \mathrm{Hom}_{\mathbf{k}}(V,\mathbf{k}), \quad u\colon V \to W \mapsto u^*\colon W^* \to V^*. \end{aligned}$$

Then there is a morphism of functors $\mathrm{id} \to {}^* \circ {}^*$ in $\mathrm{Fct}(\mathrm{Mod}(\mathbf{k}),\mathrm{Mod}(\mathbf{k}))$. Indeed, for any $V \in \mathrm{Mod}(\mathbf{k})$, there is a natural morphism $V \to V^{**}$ and for $u\colon V \to W$ a linear map, the diagram below commutes:

$$\begin{array}{ccc} V & \longrightarrow & V^{**} \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle u^{**}} \\ W & \longrightarrow & W^{**}. \end{array} \tag{1.3.2}$$

(ii) We shall encounter morphisms of functors when considering pairs of adjoint functors (see (1.5.2)).

In particular we have the notion of an isomorphism of categories. A functor $F\colon \mathscr{C} \to \mathscr{C}'$ is an isomorphism of categories if there exists $G : \mathscr{C}' \to \mathscr{C}$ such that: $G \circ F = \mathrm{id}_{\mathscr{C}}$ and $F \circ G = \mathrm{id}_{\mathscr{C}'}$. In particular, for all $X \in \mathscr{C}$, $G \circ F(X) = X$. In practice, such a situation rarely occurs and is not really interesting. There is a weaker notion that we introduce below.

Definition 1.3.16. A functor $F\colon \mathscr{C} \to \mathscr{C}'$ is an equivalence of categories if there exists $G\colon \mathscr{C}' \to \mathscr{C}$ such that: $G \circ F$ is isomorphic to $\mathrm{id}_{\mathscr{C}}$ and $F \circ G$ is isomorphic to $\mathrm{id}_{\mathscr{C}'}$.

We shall not give the proof of the following important result below.

Theorem 1.3.17. *The functor $F\colon \mathscr{C} \to \mathscr{C}'$ is an equivalence of categories if and only if F is fully faithful and essentially surjective.*

If two categories are equivalent, all results and concepts in one of them have their counterparts in the other one. This is why this notion of equivalence of categories plays an important role in Mathematics.

Examples 1.3.18. (i) Let $\mathbf{k}$ be a field and let $\mathscr{C}$ denote the category defined by $\mathrm{Ob}(\mathscr{C}) = \mathbb{N}$ and $\mathrm{Hom}_{\mathscr{C}}(n,m) = M_{m,n}(\mathbf{k})$, the space of matrices of type (m,n) with entries in a field $\mathbf{k}$ (the composition being the usual composition of matrices). Define the functor $F\colon \mathscr{C} \to \mathrm{Mod}^f(\mathbf{k})$ as follows. To $n \in \mathbb{N}$, $F(n)$ associates $\mathbf{k}^n \in \mathrm{Mod}^f(\mathbf{k})$ and to a matrix of type (m,n), F associates the induced linear map from $\mathbf{k}^n$ to

$\mathbf{k}^m$. Clearly F is fully faithful, and since any finite dimensional vector space admits a basis, it is isomorphic to $\mathbf{k}^n$ for some n, hence F is essentially surjective. In conclusion, F is an equivalence of categories.
(ii) let $\mathscr{C}$ and $\mathscr{C}'$ be two categories. There is an equivalence

$$\mathrm{Fct}(\mathscr{C},\mathscr{C}')^{\mathrm{op}} \simeq \mathrm{Fct}(\mathscr{C}^{\mathrm{op}},(\mathscr{C}')^{\mathrm{op}}). \tag{1.3.3}$$

(iii) Let I, J and $\mathscr{C}$ be categories. There are equivalences

$$\mathrm{Fct}(I\times J,\mathscr{C}) \simeq \mathrm{Fct}(J,\mathrm{Fct}(I,\mathscr{C})) \simeq \mathrm{Fct}(I,\mathrm{Fct}(J,\mathscr{C})). \tag{1.3.4}$$

1.4 The Yoneda Lemma

Definition 1.4.1. Let $\mathscr{C}$ be a category. One defines the big categories

$$\mathscr{C}^\wedge = \mathrm{Fct}(\mathscr{C}^{\mathrm{op}},\mathbf{Set}),\ \mathscr{C}^\vee = \mathrm{Fct}(\mathscr{C}^{\mathrm{op}},\mathbf{Set}^{\mathrm{op}}),$$

and the functors

$$\begin{aligned} \mathrm{h}_\mathscr{C} &: \mathscr{C} \to \mathscr{C}^\wedge, \quad X \mapsto \mathrm{Hom}_\mathscr{C}(\,\bullet\,,X) \\ \mathrm{k}_\mathscr{C} &: \mathscr{C} \to \mathscr{C}^\vee, \quad X \mapsto \mathrm{Hom}_\mathscr{C}(X,\,\bullet\,). \end{aligned}$$

Since there is a natural equivalence of categories

$$\mathscr{C}^\vee \simeq \mathscr{C}^{\mathrm{op},\wedge,\mathrm{op}}, \tag{1.4.1}$$

we shall concentrate our study on $\mathscr{C}^\wedge$.

Theorem 1.4.2. (The Yoneda lemma.) *For $A \in \mathscr{C}^\wedge$ and $X \in \mathscr{C}$, there is an isomorphism $\mathrm{Hom}_{\mathscr{C}^\wedge}(\mathrm{h}_\mathscr{C}(X),A) \simeq A(X)$, functorial with respect to X and A.*

Proof. One constructs the morphism $\varphi\colon \mathrm{Hom}_{\mathscr{C}^\wedge}(\mathrm{h}_\mathscr{C}(X),A) \to A(X)$ by the chain of morphisms: $\mathrm{Hom}_{\mathscr{C}^\wedge}(\mathrm{h}_\mathscr{C}(X),A) \to \mathrm{Hom}_{\mathbf{Set}}(\mathrm{Hom}_\mathscr{C}(X,X),A(X)) \to A(X)$, where the last map is associated with id_X.

To construct $\psi\colon A(X) \to \mathrm{Hom}_{\mathscr{C}^\wedge}(\mathrm{h}_\mathscr{C}(X),A)$, it is enough to associate with $s \in A(X)$ and $Y \in \mathscr{C}$ a map from $\mathrm{Hom}_\mathscr{C}(Y,X)$ to $A(Y)$. It is defined by the chain of maps $\mathrm{Hom}_\mathscr{C}(Y,X) \to \mathrm{Hom}_{\mathbf{Set}}(A(X),A(Y)) \to A(Y)$ where the last map is associated with $s \in A(X)$.

One checks that φ and ψ are inverse to each other. □

Corollary 1.4.3. *The functors $\mathrm{h}_\mathscr{C}$ and $\mathrm{k}_\mathscr{C}$ are fully faithful.*

Proof. For $X, Y \in \mathscr{C}$, one has $\mathrm{Hom}_{\mathscr{C}^\wedge}(\mathrm{h}_\mathscr{C}(X),\mathrm{h}_\mathscr{C}(Y)) \simeq \mathrm{h}_\mathscr{C}(Y)(X) = \mathrm{Hom}_\mathscr{C}(X,Y)$. □

One calls $\mathrm{h}_\mathscr{C}$ and $\mathrm{k}_\mathscr{C}$ the Yoneda embeddings.

Hence, one may consider $\mathscr{C}$ as a full subcategory of $\mathscr{C}^\wedge$. In particular, for $X \in \mathscr{C}$, $\mathrm{h}_\mathscr{C}(X)$ determines X up to unique isomorphism, that is, an isomorphism $\mathrm{h}_\mathscr{C}(X) \simeq \mathrm{h}_\mathscr{C}(Y)$ determines a unique isomorphism $X \simeq Y$.

Corollary 1.4.4. *Let $\mathscr{C}$ be a category and let $f\colon X \to Y$ be a morphism in $\mathscr{C}$.*

(i) *Assume that for any* $Z \in \mathscr{C}$*, the map* $\mathrm{Hom}_{\mathscr{C}}(Z, X) \xrightarrow{f\circ} \mathrm{Hom}_{\mathscr{C}}(Z, Y)$ *is bijective. Then* f *is an isomorphism.*

(ii) *Assume that for any* $Z \in \mathscr{C}$*, the map* $\mathrm{Hom}_{\mathscr{C}}(Y, Z) \xrightarrow{\circ f} \mathrm{Hom}_{\mathscr{C}}(X, Z)$ *is bijective. Then* f *is an isomorphism.*

Proof. (i) By the hypothesis, $\mathrm{h}_{\mathscr{C}}(f) : \mathrm{h}_{\mathscr{C}}(X) \to \mathrm{h}_{\mathscr{C}}(Y)$ is an isomorphism in $\mathscr{C}^\wedge$. Since $\mathrm{h}_{\mathscr{C}}$ is fully faithful, this implies that f is an isomorphism (ee Exercise 1.2 (ii)).
(ii) follows by replacing $\mathscr{C}$ with $\mathscr{C}^{\mathrm{op}}$. □

Definition 1.4.5. Let $\mathscr{C}$ and $\mathscr{C}'$ be categories, $F\colon \mathscr{C} \to \mathscr{C}'$ a functor and let $Z \in \mathscr{C}'$.

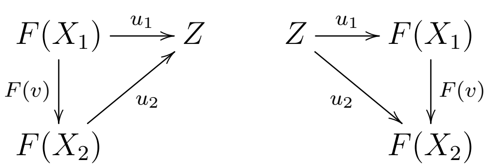
(i) The category $\mathscr{C}_Z$ is defined as follows:

$$\mathrm{Ob}(\mathscr{C}_Z) = \{(X, u); X \in \mathscr{C}, u\colon F(X) \to Z\},$$
$$\mathrm{Hom}_{\mathscr{C}_Z}((X_1, u_1), (X_2, u_2)) = \{v\colon X_1 \to X_2; u_1 = u_2 \circ F(v)\}.$$

(ii) The category $\mathscr{C}^Z$ is defined as follows:

$$\mathrm{Ob}(\mathscr{C}^Z) = \{(X, u); X \in \mathscr{C}, u\colon Z \to F(X))\},$$
$$\mathrm{Hom}_{\mathscr{C}^Z}((X_1, u_1), (X_2, u_2)) = \{v\colon X_1 \to X_2; u_2 = u_1 \circ F(v)\}.$$

Note that the natural functors $(X, u) \mapsto X$ from $\mathscr{C}_Z$ and $\mathscr{C}^Z$ to $\mathscr{C}$ are faithful.
The morphisms in $\mathscr{C}_Z$ (resp. $\mathscr{C}^Z$) are visualized by the commutative diagram on the left (resp. on the right) below:



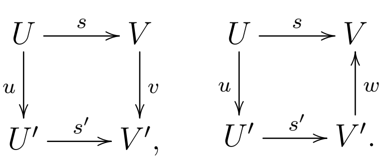
Definition 1.4.6. Let $\mathscr{C}$ be a category. The category $\mathrm{Mor}(\mathscr{C})$ of morphisms in $\mathscr{C}$ is defined as follows.

$$\begin{aligned}&\mathrm{Ob}(\mathrm{Mor}(\mathscr{C})) = \{(U, V, s); U, V \in \mathscr{C}_X, s \in \mathrm{Hom}_{\mathscr{C}}(U, V),\\ &\mathrm{Hom}_{\,\mathrm{Mor}(\mathscr{C})}((s : U \to V), (s' : U' \to V')\\ &\qquad\qquad = \{u : U \to U', v : V \to V'; v \circ s = s' \circ u\}.\end{aligned}$$

The category $\mathrm{Mor}_0(\mathscr{C})$ is defined as follows.

$$\begin{aligned}&\mathrm{Ob}(\mathrm{Mor}_0(\mathscr{C})) = \{(U, V, s); U, V \in \mathscr{C}_X, s \in \mathrm{Hom}_{\mathscr{C}}(U, V),\\ &\mathrm{Hom}_{\,\mathrm{Mor}_0(\mathscr{C})}((s : U \to V), (s' : U' \to V')\\ &\qquad\qquad = \{u : U \to U', w : V' \to V; s = w \circ s' \circ u\}.\end{aligned}$$

A morphism $(s\colon U \to V) \to (s'\colon U' \to V')$ in $\mathrm{Mor}(\mathscr{C})$ (resp. $\mathrm{Mor}_0(\mathscr{C})$) is visualized by the commutative diagram on the left (resp. on the right) below:



1.5 Representable functors, adjoint functors

Representable functors

Definition 1.5.1. (i) One says that a functor F from $\mathscr{C}^{\mathrm{op}}$ to **Set** is representable if there exists $X \in \mathscr{C}$ such that $F(Y) \simeq \mathrm{Hom}_{\mathscr{C}}(Y, X)$ functorially in $Y \in \mathscr{C}$. In other words, $F \simeq \mathrm{h}_{\mathscr{C}}(X)$ in $\mathscr{C}^{\wedge}$. Such an object X is called a representative of F.

(ii) Similarly, a functor $G \colon \mathscr{C} \to \mathbf{Set}$ is representable if there exists $X \in \mathscr{C}$ such that $G(Y) \simeq \mathrm{Hom}_{\mathscr{C}}(X, Y)$ functorially in $Y \in \mathscr{C}$.

It is important to notice that the isomorphisms above determine X up to unique isomorphism. More precisely, given two isomorphisms $F \xrightarrow{\sim} \mathrm{h}_{\mathscr{C}}(X)$ and $F \xrightarrow{\sim} \mathrm{h}_{\mathscr{C}}(X')$ there exists a unique isomorphism $\theta \colon X \xrightarrow{\sim} X'$ making the diagram below commutative:

$$\begin{array}{ccc} & F & \\ {\scriptstyle\sim}\swarrow & & \searrow{\scriptstyle\sim} \\ \mathrm{h}_{\mathscr{C}}(X) & \xrightarrow[\sim]{\mathrm{h}_{\mathscr{C}}(\theta)} & \mathrm{h}_{\mathscr{C}}(X'). \end{array}$$

Representable functors provides a categorical language to deal with universal problems. Let us illustrate this by an example.

Example 1.5.2. Let A be a $\mathbf{k}$-algebra. Let N be a right A-module, M a left A-module and L a $\mathbf{k}$-module. Denote by $B(N \times M, L)$ the set of $(A, \mathbf{k})$-bilinear maps from $N \times M$ to L. Then the functor $F \colon L \mapsto B(N \times M, L)$ is representable by $N \otimes_A M$ by (1.2.3).

Adjoint functors

Definition 1.5.3. Let $F \colon \mathscr{C} \to \mathscr{C}'$ and $G \colon \mathscr{C}' \to \mathscr{C}$ be two functors. One says that (F, G) is a pair of adjoint functors or that F is a left adjoint to G, or that G is a right adjoint to F if there exists an isomorphism of bifunctors:

$$\mathrm{Hom}_{\mathscr{C}'}(F(\,\bullet\,), \,\bullet\,) \simeq \mathrm{Hom}_{\mathscr{C}}(\,\bullet\,, G(\,\bullet\,)) \tag{1.5.1}$$

If G is an adjoint to F, then G is unique up to isomorphism. In fact, $G(Y)$ is a representative of the functor $X \mapsto \mathrm{Hom}_{\mathscr{C}'}(F(X), Y)$.

The isomorphism (1.5.1) gives the isomorphisms

$$\begin{aligned} \mathrm{Hom}_{\mathscr{C}'}(F \circ G(\,\bullet\,), \,\bullet\,) &\simeq \mathrm{Hom}_{\mathscr{C}}(G(\,\bullet\,), G(\,\bullet\,)), \\ \mathrm{Hom}_{\mathscr{C}'}(F(\,\bullet\,), F(\,\bullet\,)) &\simeq \mathrm{Hom}_{\mathscr{C}}(\,\bullet\,, G \circ F(\,\bullet\,)). \end{aligned}$$

In particular, we have morphisms $X \to G \circ F(X)$, functorial in $X \in \mathscr{C}$, and morphisms $F \circ G(Y) \to Y$, functorial in $Y \in \mathscr{C}'$. In other words, we have morphisms of functors

$$F \circ G \to \mathrm{id}_{\mathscr{C}'}, \qquad \mathrm{id}_{\mathscr{C}} \to G \circ F. \tag{1.5.2}$$

Examples 1.5.4. (i) Let $X \in \mathbf{Set}$. Using the bijection (1.1.3), we get that the functor $\mathrm{Hom}_{\mathbf{Set}}(X, \bullet) : \mathbf{Set} \to \mathbf{Set}$ is right adjoint to the functor $\bullet \times X$.

(ii) Let A be a $\mathbf{k}$-algebra and let $L \in \mathrm{Mod}(\mathbf{k})$. Using the first isomorphism in (1.2.4), we get that the functor $\mathrm{Hom}_{\mathbf{k}}(L, \bullet) : \mathrm{Mod}(A)$ to $\mathrm{Mod}(A)$ is right adjoint to the functor $\bullet \otimes_{\mathbf{k}} L$.

(iii) Let A be a $\mathbf{k}$-algebra. Using the isomorphisms in (1.2.4) with $N = A$, we get that the functor $\mathit{for} : \mathrm{Mod}(A) \to \mathrm{Mod}(\mathbf{k})$ which, to an A-module associates the underlying $\mathbf{k}$-module, is right adjoint to the functor $A \otimes_{\mathbf{k}} \bullet : \mathrm{Mod}(\mathbf{k}) \to \mathrm{Mod}(A)$ (extension of scalars).

Exercises to Chapter 1

Exercise 1.1. Prove that the categories $\mathbf{Set}$ and $\mathbf{Set}^{\mathrm{op}}$ are not equivalent and similarly with the categories $\mathbf{Set}^f$ and $(\mathbf{Set}^f)^{\mathrm{op}}$.

(Hint: if $F : \mathbf{Set} \to \mathbf{Set}^{\mathrm{op}}$ were such an equivalence, then $F(\varnothing) \simeq \{\mathrm{pt}\}$ and $F(\{\mathrm{pt}\}) \simeq \varnothing$. Now compare $\mathrm{Hom}_{\mathbf{Set}}(\{\mathrm{pt}\}, X)$ and $\mathrm{Hom}_{\mathbf{Set}^{\mathrm{op}}}(F(\{\mathrm{pt}\}), F(X))$ when X is a set with two elements.)

Exercise 1.2. (i) Let $F : \mathscr{C} \to \mathscr{C}'$ be a faithful functor and let f be a morphism in $\mathscr{C}$. Prove that if $F(f)$ is a monomorphism (resp. an epimorphism), then f is a monomorphism (resp. an epimorphism).

(ii) Assume now that F is fully faithful. Prove that if $F(f)$ is an isomorphism, then f is an isomorphism. In other words, fully faithful functors are conservative.

Exercise 1.3. Is the natural functor $\mathbf{Set} \to \mathbf{Rel}$ full, faithful, fully faithful, conservative?

Exercise 1.4. Prove that the category $\mathscr{C}$ is equivalent to the opposite category $\mathscr{C}^{\mathrm{op}}$ in the following cases:

(i) $\mathscr{C}$ denotes the category of finite abelian groups,

(ii) $\mathscr{C}$ is the category $\mathbf{Rel}$ of relations.

Exercise 1.5. (i) Prove that in the category $\mathbf{Set}$, a morphism f is a monomorphism (resp. an epimorphism) if and only if it is injective (resp. surjective).

(ii) Prove that in the category of rings, the morphism $\mathbb{Z} \to \mathbb{Q}$ is an epimorphism.

(iii) In the category $\mathbf{Top}$, give an example of a morphism which is both a monomorphism and an epimorphism and which is not an isomorphism.

Exercise 1.6. Let $\mathscr{C}$ be a category. We denote by $\mathrm{id}_{\mathscr{C}} : \mathscr{C} \to \mathscr{C}$ the identity functor of $\mathscr{C}$ and by $\mathrm{End}\,(\mathrm{id}_{\mathscr{C}})$ the set of endomorphisms of the identity functor $\mathrm{id}_{\mathscr{C}} : \mathscr{C} \to \mathscr{C}$, that is,

$$\mathrm{End}\,(\mathrm{id}_{\mathscr{C}}) = \mathrm{Hom}_{\,\mathrm{Fct}(\mathscr{C},\mathscr{C})}(\mathrm{id}_{\mathscr{C}}, \mathrm{id}_{\mathscr{C}}).$$

Prove that the composition law on $\mathrm{End}\,(\mathrm{id}_{\mathscr{C}})$ is commutative.

Chapter 2

Limits

After treating the particular cases of kernels and cokernels, products and coproducts, we shall construct limits and colimits, starting with limits in the category **Set**. We also analyze some related notions, in particular those of filtered categories and cofinal functors. Special attention will be paid to filtered colimits in the categories **Set** and Mod(A).

Caution. We may sometimes use the terms "projective limit" or "inductive limits" instead of "limit" or "colimit".

2.1 Products and coproducts

Let $\mathscr{C}$ be a category and consider a family $\{X_i\}_{i\in I}$ of objects of $\mathscr{C}$ indexed by a (small) set I. Consider the two functors

$$\mathscr{C}^{\mathrm{op}} \to \mathbf{Set}, Y \mapsto \prod_i \mathrm{Hom}_{\mathscr{C}}(Y, X_i), \tag{2.1.1}$$

$$\mathscr{C} \to \mathbf{Set}, Y \mapsto \prod_i \mathrm{Hom}_{\mathscr{C}}(X_i, Y). \tag{2.1.2}$$

Definition 2.1.1. (i) Assume that the functor in (2.1.1) is representable. In this case one denotes by $\prod_i X_i$ a representative and calls this object the product of the X_i's. In case I has two elements, say $I = \{1, 2\}$, one simply denotes this object by $X_1 \times X_2$.

(ii) Assume that the functor in (2.1.2) is representable. In this case one denotes by $\coprod_i X_i$ a representative and calls this object the coproduct of the X_i's. In case I has two elements, say $I = \{1, 2\}$, one simply denotes this object by $X_1 \sqcup X_2$.

(iii) If for any family of objects $\{X_i\}_{i\in I}$, the product (resp. coproduct) exists, one says that the category $\mathscr{C}$ admits products (resp. coproducts) indexed by I.

(iv) If $X_i = X$ for all $i \in I$, one writes:

$$X^I := \prod_i X_i,\ X^{\coprod I} := \coprod_i X_i.$$

In case of additive categories (see below), one writes $\oplus_i X_i$ instead of $\coprod_i X_i$ and $X^{(I)}$ or $X^{\oplus I}$ instead of $X^{\coprod I}$. If $\mathscr{C} = \mathbf{Set}$, one often writes $\bigsqcup_i X_i$ instead of $\coprod_i X_i$. and $X^{\sqcup I}$ instead of $X^{\coprod I}$.

Note that the coproduct in $\mathscr{C}$ is the product in $\mathscr{C}^{\mathrm{op}}$.

By this definition, the product or the coproduct exist if and only if one has the isomorphisms, functorial with respect to $Y \in \mathscr{C}$:

$$\mathrm{Hom}_{\mathscr{C}}(Y, \prod_i X_i) \simeq \prod_i \mathrm{Hom}_{\mathscr{C}}(Y, X_i), \tag{2.1.3}$$

$$\mathrm{Hom}_{\mathscr{C}}(\coprod_i X_i, Y) \simeq \prod_i \mathrm{Hom}_{\mathscr{C}}(X_i, Y). \tag{2.1.4}$$

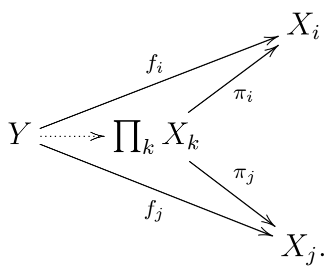
Assume that $\prod_i X_i$ exists. By choosing $Y = \prod_i X_i$ in (2.1.3), we get the morphisms

$$\pi_i \colon \prod_j X_j \to X_i.$$

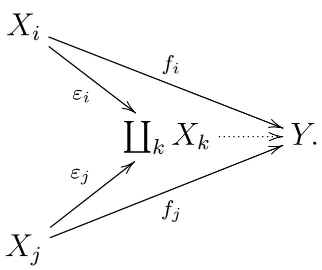
Similarly, assume that $\coprod_i X_i$ exists. By choosing $Y = \coprod_i X_i$ in (2.1.4), we get the morphisms

$$\varepsilon_i \colon X_i \to \coprod_j X_j.$$

The isomorphism (2.1.3) may be translated as follows. Given an object Y and a family of morphisms $f_i \colon Y \to X_i$, this family factorizes uniquely through $\prod_i X_i$. This is visualized by the diagram



The isomorphism (2.1.4) may be translated as follows. Given an object Y and a family of morphisms $f_i \colon X_i \to Y$, this family factorizes uniquely through $\coprod_i X_i$. This is visualized by the diagram



Example 2.1.2. (i) The category **Set** admits products (that is, products indexed by small sets) and the two definitions (that given in (1.1.1) and that given in Definition 2.1.1) coincide.
(ii) The category **Set** admits coproducts indexed by small sets, namely, the disjoint union.
(iii) Let A be a ring. The category $\mathrm{Mod}(A)$ admits products, as defined in § 1.2. The category $\mathrm{Mod}(A)$ also admits coproducts, which are the direct sums defined in § 1.2. and are denoted $\bigoplus$.

(iv) Let X be a set and denote by $\mathfrak{X}$ the category of subsets of X. (The set $\mathfrak{X}$ is ordered by inclusion, hence defines a category.) For $S_1, S_2 \in \mathfrak{X}$, their product in the category $\mathfrak{X}$ is their intersection and their coproduct is their union.

Remark 2.1.3. The forgetful functor $\mathit{for}\colon \mathrm{Mod}(A) \to \mathbf{Set}$ commutes with products but does not commute with coproducts. That is the reason why the coproduct in the category $\mathrm{Mod}(A)$ is called and denoted differently.

2.2 Kernels and cokernels

Let $\mathscr{C}$ be a category and consider two parallel arrows $f, g : X_0 \rightrightarrows X_1$ in $\mathscr{C}$. Consider the two functors

$$\mathscr{C}^{\mathrm{op}} \to \mathbf{Set}, Y \mapsto \ker(\mathrm{Hom}_{\mathscr{C}}(Y, X_0) \rightrightarrows \mathrm{Hom}_{\mathscr{C}}(Y, X_1)), \tag{2.2.1}$$

$$\mathscr{C} \to \mathbf{Set}, Y \mapsto \ker(\mathrm{Hom}_{\mathscr{C}}(X_1, Y) \rightrightarrows \mathrm{Hom}_{\mathscr{C}}(X_0, Y)). \tag{2.2.2}$$

Definition 2.2.1. (i) Assume that the functor in (2.2.1) is representable. In this case one denotes by $\ker(f, g)$ a representative and calls this object a kernel (one also says an equalizer) of (f, g).

(ii) Assume that the functor in (2.2.2) is representable. In this case one denotes by $\mathrm{Coker}(f, g)$ a representative and calls this object a cokernel (one also says a co-equalizer) of (f, g).

(iii) A sequence $Z \to X_0 \rightrightarrows X_1$ (resp. $X_0 \rightrightarrows X_1 \to Z$) is exact if Z is isomorphic to the kernel (resp. cokernel) of $X_0 \rightrightarrows X_1$.

(iv) Assume that the category $\mathscr{C}$ admits a zero-object 0. Let $f\colon X \to Y$ be a morphism in $\mathscr{C}$. A kernel (resp. a cokernel) of f, if it exists, is a kernel (resp. a cokernel) of $f, 0\colon X \rightrightarrows Y$. It is denoted $\ker(f)$ (resp. $\mathrm{Coker}(f)$).

Note that the cokernel in $\mathscr{C}$ is the kernel in $\mathscr{C}^{\mathrm{op}}$.

By this definition, the kernel or the cokernel of $f, g\colon X_0 \rightrightarrows X_1$ exist if and only if one has the isomorphisms, functorial in $Y \in \mathscr{C}$:

$$\mathrm{Hom}_{\mathscr{C}}(Y, \ker(f, g)) \simeq \ker(\mathrm{Hom}_{\mathscr{C}}(Y, X_0) \rightrightarrows \mathrm{Hom}_{\mathscr{C}}(Y, X_1)), \tag{2.2.3}$$

$$\mathrm{Hom}_{\mathscr{C}}(\mathrm{Coker}(f, g), Y) \simeq \ker(\mathrm{Hom}_{\mathscr{C}}(X_1, Y) \rightrightarrows \mathrm{Hom}_{\mathscr{C}}(X_0, Y)). \tag{2.2.4}$$

Assume that $\ker(f, g)$ exists. By choosing $Y = \ker(f, g)$ in (2.2.3), we get the morphism

$$h\colon \ker(X_0 \rightrightarrows X_1) \to X_0.$$

Similarly, assume that $\mathrm{Coker}(f, g)$ exists. By choosing $Y = \mathrm{Coker}(f, g)$ in (2.2.4), we get the morphism

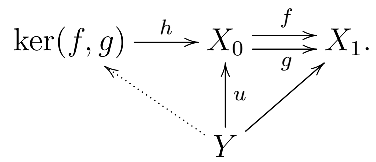
$$k\colon X_1 \to \mathrm{Coker}(X_0 \rightrightarrows X_1).$$

Proposition 2.2.2. *The morphism $h\colon \ker(X_0 \rightrightarrows X_1) \to X_0$ is a monomorphism and the morphism $k\colon X_1 \to \mathrm{Coker}(X_0 \rightrightarrows X_1)$ is an epimorphism.*

Proof. (i) Set $K = \ker(X_0 \rightrightarrows X_1) \to X_0$ and consider a pair of parallel arrows $a, b\colon Y \rightrightarrows K$ such that $h \circ a = h \circ b = w$. Then $f \circ w = f \circ h \circ a = g \circ h \circ a = g \circ h \circ b = g \circ w$. Hence w factors *uniquely* through h, and this implies $a = b$.
(ii) The case of cokernels follows, by reversing the arrows. □

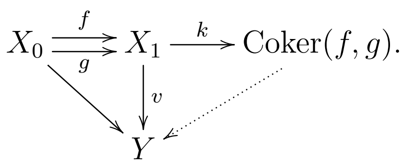
The isomorphism (2.2.3) may be translated as follows. Given an objet Y and a morphism $u\colon Y \to X_0$ such that $f \circ u = g \circ u$, the morphism u factors uniquely through $\ker(f, g)$. This is visualized by the diagram

(2.2.5)

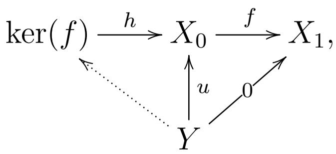


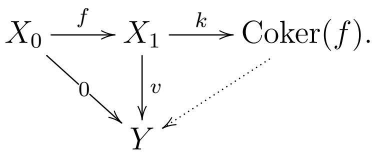
The isomorphism (2.2.4) may be translated as follows. Given an objet Y and a morphism $v\colon X_1 \to Y$ such that $v \circ f = v \circ g$, the morphism v factors uniquely through $\operatorname{Coker}(f, g)$. This is visualized by diagram:

(2.2.6)



Example 2.2.3. (i) The category **Set** admits kernels and the two definitions (that given in (1.1.7) and that given in Definition 2.2.1) coincide.
(ii) The category **Set** admits cokernels. If $f, g\colon Z_0 \rightrightarrows Z_1$ are two maps, the cokernel of (f, g) is the quotient set $Z_1/\mathscr{R}$ where $\mathscr{R}$ is the equivalence relation generated by the relation $x \sim y$ if there exists $z \in Z_0$ with $f(z) = x$ and $g(z) = y$.
(iii) Let A be a ring. The category $\mathrm{Mod}(A)$ admits a zero object. Hence, the kernel or the cokernel of a morphism f means the kernel or the cokernel of $(f, 0)$. As already mentioned, the kernel of a linear map $f\colon M \to N$ is the A-module $f^{-1}(0)$ and the cokernel is the quotient module $M/\operatorname{Im} f$. The kernel and cokernel are visualized by the diagrams





2.3 Limits

Let us generalize and unify the preceding constructions.

Definition 2.3.1. Let I and $\mathscr{C}$ categories with I small. A projective system (resp. an inductive system) in $\mathscr{C}$ indexed by I is nothing but a functor $\alpha\colon I^{\mathrm{op}} \to \mathscr{C}$ (resp. $\beta\colon I \to \mathscr{C}$).

For example, if $(I, \leq)$ is a pre-ordered set, I the associated category, an inductive system indexed by I is the data of a family $(X_i)_{i \in I}$ of objects of $\mathscr{C}$ and for all $i \leq j$, a morphism $X_i \to X_j$ with the natural compatibility conditions.

Projective limits in Set

Assume first that $\mathscr{C}$ is the category **Set** and let us consider projective systems. One sets

$$\lim \beta = \{\{x_i\}_i \in \prod_i \beta(i); \beta(s)(x_j) = x_i \text{ for all } s \in \operatorname{Hom}_I(i,j)\}. \tag{2.3.1}$$

The next result is obvious.

Lemma 2.3.2. *Let $\beta\colon I^{\mathrm{op}} \to$ **Set** be a functor and let $X \in$ **Set**. There is a natural isomorphism*

$$\operatorname{Hom}_{\mathbf{Set}}(X, \lim \beta) \xrightarrow{\sim} \lim \operatorname{Hom}_{\mathbf{Set}}(X, \beta),$$

*where $\operatorname{Hom}_{\mathbf{Set}}(X,\beta)$ denotes the functor $I^{\mathrm{op}} \to$ **Set**, $i \mapsto \operatorname{Hom}_{\mathbf{Set}}(X, \beta(i))$.*

Limits and colimits

Consider now two functors $\beta\colon I^{\mathrm{op}} \to \mathscr{C}$ and $\alpha\colon I \to \mathscr{C}$. For $X \in \mathscr{C}$, we get functors from I^{op} to **Set**:

$$\operatorname{Hom}_{\mathscr{C}}(X,\beta)\colon I^{\mathrm{op}} \ni i \mapsto \operatorname{Hom}_{\mathscr{C}}(X,\beta(i)) \in \mathbf{Set},$$
$$\operatorname{Hom}_{\mathscr{C}}(\alpha,X)\colon I^{\mathrm{op}} \ni i \mapsto \operatorname{Hom}_{\mathscr{C}}(\alpha,X) \in \mathbf{Set}.$$

Definition 2.3.3. (i) Assume that the functor $X \mapsto \lim \operatorname{Hom}_{\mathscr{C}}(X,\beta)$ is representable. We denote by $\lim \beta$ its representative and say that the functor β admits a limit (or "a projective limit") in $\mathscr{C}$. In other words, we have the isomorphism, functorial in $X \in \mathscr{C}$:

$$\operatorname{Hom}_{\mathscr{C}}(X, \lim \beta) \simeq \lim \operatorname{Hom}_{\mathscr{C}}(X,\beta). \tag{2.3.2}$$

(ii) Assume that the functor $X \mapsto \lim \operatorname{Hom}_{\mathscr{C}}(\alpha, X)$ is representable. We denote by $\operatorname{colim} \alpha$ its representative and say that the functor α admits a colimit (or "an inductive limit") in $\mathscr{C}$. In other words, we have the isomorphism, functorial in $X \in \mathscr{C}$:

$$\operatorname{Hom}_{\mathscr{C}}(\operatorname{colim} \alpha, X) \simeq \lim \operatorname{Hom}_{\mathscr{C}}(\alpha, X), \tag{2.3.3}$$

Remark 2.3.4. The limit of the functor β is not only the object $\lim \beta$ but also the isomorphism of functors given in (2.3.2), and similarly with colimits.

When $\mathscr{C} =$ **Set** this definition of $\lim \beta$ coincides with the former one, in view of Lemma 2.3.2.

Notice that both limits and colimits are defined using limits in **Set**.

Assume that $\lim \beta$ exists in $\mathscr{C}$. One gets:

$$\lim \operatorname{Hom}_{\mathscr{C}}(\lim \beta, \beta) \simeq \operatorname{Hom}_{\mathscr{C}}(\lim \beta, \lim \beta)$$

and the identity of $\lim \beta$ defines a family of morphisms

$$\rho_i\colon \lim \beta \to \beta(i).$$

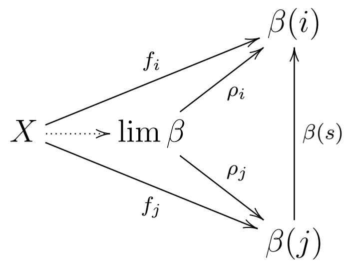
Consider a family of morphisms $\{f_i\colon X \to \beta(i)\}_{i\in I}$ in $\mathscr{C}$ satisfying the compatibility conditions

(2.3.4) $$f_j = f_i \circ f(s) \text{ for all } s \in \mathrm{Hom}_{I}(i,j).$$

This family of morphisms is nothing but an element of $\varprojlim_i \mathrm{Hom}\,(X, \beta(i))$, hence by (2.3.2), an element of $\mathrm{Hom}\,(X, \lim\beta, X)$. Therefore, $\lim\beta$ is characterized by the "universal property":

(2.3.5) $$\begin{cases} \text{for all } X \in \mathscr{C} \text{ and all family of morphisms } \{f_i\colon X \to \beta(i)\}_{i\in I} \\ \text{in } \mathscr{C} \text{ satisfying (2.3.4), all morphisms } f_i\text{'s factorize uniquely} \\ \text{through } \lim\beta. \end{cases}$$

This is visualized by the diagram:



Similarly, assume that $\operatorname{colim}\alpha$ exists in $\mathscr{C}$. One gets:

$$\lim \mathrm{Hom}_{\mathscr{C}}(\alpha, \operatorname{colim}\alpha) \simeq \mathrm{Hom}_{\mathscr{C}}(\operatorname{colim}\alpha, \operatorname{colim}\alpha)$$

and the identity of $\operatorname{colim}\alpha$ defines a family of morphisms

$$\rho_i\colon \alpha(i) \to \operatorname{colim}\alpha.$$

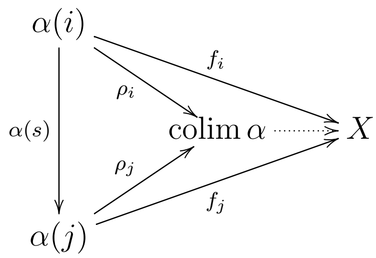
Consider a family of morphisms $\{f_i\colon \alpha(i) \to X\}_{i\in I}$ in $\mathscr{C}$ satisfying the compatibility conditions

(2.3.6) $$f_i = f_j \circ f(s) \text{ for all } s \in \mathrm{Hom}_{I}(i,j).$$

This family of morphisms is nothing but an element of $\varprojlim_i \mathrm{Hom}\,(\alpha(i), X)$, hence by (2.3.3), an element of $\mathrm{Hom}\,(\operatorname{colim}\alpha, X)$. Therefore, $\operatorname{colim}\alpha$ is characterized by the "universal property":

(2.3.7) $$\begin{cases} \text{for all } X \in \mathscr{C} \text{ and all family of morphisms } \{f_i\colon \alpha(i) \to X\}_{i\in I} \\ \text{in } \mathscr{C} \text{ satisfying (2.3.6), all morphisms } f_i\text{'s factorize uniquely} \\ \text{through } \operatorname{colim}\alpha. \end{cases}$$

This is visualized by the diagram:



Example 2.3.5. Let X be a set and let $\mathfrak{X}$ be the category given in Example 2.1.2 (iv). Let $\beta\colon I^{\mathrm{op}} \to \mathfrak{X}$ and $\alpha\colon I \to \mathfrak{X}$ be two functors. Then

$$\lim \beta \simeq \bigcap_i \beta(i),\ \operatorname{colim} \alpha \simeq \bigcup_i \alpha(i).$$

Examples 2.3.6. (i) When the category I is discrete, limits and colimits indexed by I are nothing but products and coproducts indexed by I.
(ii) Consider the category I with two objects and two parallel morphisms other than identities, visualized by $\bullet \rightrightarrows \bullet$. A functor $\alpha\colon I \to \mathscr{C}$ is characterized by two parallel arrows in $\mathscr{C}$:

$$f, g\colon X_0 \rightrightarrows X_1 \tag{2.3.8}$$

In the sequel we shall identify such a functor with the diagram (2.3.8). Then, the kernel (resp. cokernel) of (f, g) is nothing but the limit (resp. colimit) of the functor α.
(iii) If I is the empty category and $\alpha\colon I \to \mathscr{C}$ is a functor, then $\lim \alpha$ exists in $\mathscr{C}$ if and only if $\mathscr{C}$ has a terminal object $\mathrm{pt}_{\mathscr{C}}$, and in this case $\lim \alpha \simeq \mathrm{pt}_{\mathscr{C}}$. Similarly, $\operatorname{colim} \alpha$ exists in $\mathscr{C}$ if and only if $\mathscr{C}$ has an initial object $\varnothing_{\mathscr{C}}$, and in this case $\operatorname{colim} \alpha \simeq \varnothing_{\mathscr{C}}$.
(iv) If I admits a terminal object, say i_o and if $\beta\colon I^{\mathrm{op}} \to \mathscr{C}$ and $\alpha\colon I \to \mathscr{C}$ are functors, then

$$\lim \beta \simeq \beta(i_o),\ \operatorname{colim} \alpha \simeq \alpha(i_o).$$

This follows immediately of (2.3.7) and (2.3.5).

If every functor from I^{op} to $\mathscr{C}$ admits a limit, one says that $\mathscr{C}$ admits limits indexed by I.

Caution We shall often neglect the adjective “small” before the words “limit” and “colimit”.

Remark 2.3.7. Assume that $\mathscr{C}$ admits limits (resp. colimits) indexed by I. Then $\lim : \mathrm{Fct}(I^{\mathrm{op}}, \mathscr{C}) \to \mathscr{C}$ (resp. $\operatorname{colim} : \mathrm{Fct}(I, \mathscr{C}) \to \mathscr{C}$) is a functor.

Definition 2.3.8. One says that a category $\mathscr{C}$ admits small limits (resp. small colimits) if for any small category I and any functor $\beta\colon I^{\mathrm{op}} \to \mathscr{C}$ (resp. $\alpha\colon I \to \mathscr{C}$) $\lim \beta$ (resp. $\operatorname{colim} \alpha$) exists in $\mathscr{C}$.

Similarly one says that $\mathscr{C}$ admits finite limits or colimits if the preceding conditions hold when assuming that I is finite.

Limits as kernels and products

We have seen that products and kernels (resp. coproducts and cokernels) are particular cases of limits (resp. colimits). One can show that conversely, limits can be obtained as kernels of products and colimits can be obtained as cokernels of coproducts.

Recall that for a category I, $\mathrm{Mor}(I)$ denote the set of morphisms in I. There are two natural maps (source and target) from $\mathrm{Mor}(I)$ to $\mathrm{Ob}(I)$:

$$\begin{aligned} \sigma &: \mathrm{Mor}(I) \to \mathrm{Ob}(I), \quad (s\colon i \to j) \mapsto i, \\ \tau &: \mathrm{Mor}(I) \to \mathrm{Ob}(I), \quad (s\colon i \to j) \mapsto j. \end{aligned}$$

Let $\mathscr{C}$ be a category which admits limits and let $\beta\colon I^{\mathrm{op}} \to \mathscr{C}$ be a functor. For $s\colon i \to j$, we get two morphisms in $\mathscr{C}$:

$$\beta(i) \times \beta(j) \underset{\beta(s)}{\overset{\mathrm{id}_{\beta(i)}}{\rightrightarrows}} \beta(i)$$

from which we deduce two morphisms in $\mathscr{C}$: $\prod_{k\in I} \beta(k) \rightrightarrows \beta(\sigma(s))$. These morphisms define the two morphisms in $\mathscr{C}$:

(2.3.9) $$\prod_{k\in I} \beta(k) \underset{b}{\overset{a}{\rightrightarrows}} \prod_{s\in \mathrm{Mor}(I)} \beta(\sigma(s)).$$

Similarly, assume that $\mathscr{C}$ admits colimits and let $\alpha\colon I \to \mathscr{C}$ be a functor. By reversing the arrows, one gets the two morphisms in $\mathscr{C}$:

(2.3.10) $$\coprod_{s\in \mathrm{Mor}(I)} \alpha(\sigma(s)) \underset{b}{\overset{a}{\rightrightarrows}} \coprod_{k\in I} \alpha(k).$$

Proposition 2.3.9. (i) $\lim \beta$ *is the kernel of* (a, b) *in* (2.3.9),

(ii) $\mathrm{colim}\, \alpha$ *is the cokernel of* (a, b) *in* (2.3.10).

Sketch of proof. By the definition of limits and colimits we are reduced to check (i) when $\mathscr{C} = \mathbf{Set}$ and in this case this is obvious. □

In particular, a category $\mathscr{C}$ admits finite limits if and only if it satisfies:

(i) $\mathscr{C}$ admits a terminal object,

(ii) for any $X, Y \in \mathrm{Ob}(\mathscr{C})$, the product $X \times Y$ exists in $\mathscr{C}$,

(iii) for any parallel arrows in $\mathscr{C}$, $f, g\colon X \rightrightarrows Y$, the kernel exists in $\mathscr{C}$.

There is a similar result for finite colimits, replacing a terminal object by an initial object, products by coproducts and kernels by cokernels.

Example 2.3.10. The category **Set** admits small limits and colimits, as well as the category $\mathrm{Mod}(A)$ for a ring A. Indeed, both categories admit small products and coproducts as well as kernels and cokernels.

2.4 Fiber products and coproducts

Consider the category I with three objects $\{a, b, c\}$ and two morphisms other than the identities, visualized by the diagram

$$a \leftarrow c \to b.$$

Let $\mathscr{C}$ be a category. A functor $\beta\colon I^{\mathrm{op}} \to \mathscr{C}$ (resp. $\alpha\colon I \to \mathscr{C}$) is nothing but the data of three objects X_0, X_1, Y and two morphisms (f, g) (resp. (k, l)) visualized by the arrows on the left (resp. on the right)

$$X_0 \xrightarrow{f} Y \xleftarrow{g} X_1,\ X_0 \xleftarrow{k} W \xrightarrow{l} X_1.$$

The fiber product $X_0 \times_Y X_1$ of X_0 and X_1 over Y, if it exists, is the limit of β.

The fiber coproduct $X_0 \sqcup_W X_1$ of X_0 and X_1 over W, if it exists, is the colimit of α.

Consider a commutative diagram in $\mathscr{C}$:

$$\begin{array}{ccc} W & \longrightarrow & X_1 \\ \downarrow & & \downarrow \\ X_0 & \longrightarrow & Y \end{array} \tag{2.4.1}$$

Definition 2.4.1. The square (2.4.1) is Cartesian if $W \simeq X_0 \times_Y X_1$. It is co-Cartesian if $Y \simeq X_0 \sqcup_W X_1$

Proposition 2.4.2. (a) *Assume that $\mathscr{C}$ admits products of two objects and kernels. Then $X_0 \times_Y X_1$ is isomorphic to $\ker(f,g)$, the equalizer of $(f,g)\colon X_0 \times X_1 \rightrightarrows Y$.*

(b) *Assume that $\mathscr{C}$ admits coproducts of two objects and cokernels. Then $X_0 \sqcup_W X_1$ is isomorphic to $\mathrm{Coker}(k,l)$, the co-equalizer of $(k,l)\colon W \rightrightarrows X_0 \sqcup X_1$.*

Proof. It follows from the characterizations of limits and colimits given in (2.3.5) and (2.3.7). □

Proposition 2.4.3. (a) *The category $\mathscr{C}$ admits finite limits if and only if it admits fiber products and a terminal object.*

(b) *The category $\mathscr{C}$ admits finite colimits if and only if it admits fiber coproducts and an initial object.*

Proof. (a) If $\mathscr{C}$ admits finite limits, then it admits fiber products by Proposition 2.4.2 (a). Conversely, if $\mathscr{C}$ admits a terminal object $\mathrm{pt}_{\mathscr{C}}$ and fiber products, then it admits product of two objects (X_0, X_1), namely $X_0 \times_{\mathrm{pt}_{\mathscr{C}}} X_1$. It admits kernels since given $(f,g)\colon X \rightrightarrows Y$, then $\ker(f,g) \simeq X \times_Y X$ again by Proposition 2.4.2 (a).
(b) is deduced from (a) by reversing the arrows. □

To summarize, assuming that $\mathscr{C}$ admits finite limits and colimits, we have for $f\colon X_0 \to Y$, $g\colon X_1 \to Y$ and when $X = X_0 = X_1$

$$X_0 \times_Y X_1 \simeq \ker(X_0 \times X_1 \rightrightarrows Y),\ \ker(f,g) \simeq X \times_Y X, \tag{2.4.2}$$

and for $k\colon W \to X_0$, $l\colon W \to X_1$ and when $X = X_0 = X_1$

$$X_0 \sqcup_W X_1 \simeq \mathrm{Coker}(W \rightrightarrows X_0 \times X_1),\ \mathrm{Coker}(k,l) \simeq X \sqcup_W X. \tag{2.4.3}$$

Moreover

$$X_0 \times X_1 \simeq X_0 \times_{\mathrm{pt}_{\mathscr{C}}} X_1,\ X_0 \sqcup X_1 \simeq X_0 \sqcup_{\varnothing_{\mathscr{C}}} X_1 \tag{2.4.4}$$

Definition 2.4.4. Let $\mathscr{C}$ be a category which admits finite limits and colimits and let $f\colon X \to Y$ be a morphism. One sets

$$\mathrm{Coim}\, f := \mathrm{Coker}(X \times_Y X \rightrightarrows X),\ \mathrm{Im}\, f := \ker(Y \rightrightarrows Y \sqcup_X Y). \tag{2.4.5}$$

Here, the fiber product $X \times_Y X$ as well as the fiber coproduct $Y \sqcup_X Y$ are associated with two copies of the map f.

One calls $\mathrm{Coim}(f)$ and $\mathrm{Im}(f)$ the co-image and the image of f, respectively.

One has a natural epimorphism $s\colon X \to \mathrm{Coim}\, f$ and a natural monomorphism $t\colon \mathrm{Im}\, f \to Y$. Moreover, one can construct a natural morphism $u\colon \mathrm{Coim}(f) \to \mathrm{Im}(f)$ such that the composition $X \to \mathrm{Coim}(f) \to \mathrm{Im}(f) \to Y$ is f (see [KS06, Prop. 5.1.2] and Section 4.1 for a similar construction in the abelian setting).

2.5 Properties of limits

Double limits

For two categories I and $\mathscr{C}$, recall the notation $\mathscr{C}^I := \mathrm{Fct}(I,\mathscr{C})$ and for a third category J, recall the equivalence (1.3.4);

$$\mathrm{Fct}(I\times J,\mathscr{C}) \simeq \mathrm{Fct}(I,\mathrm{Fct}(J,\mathscr{C})).$$

Consider a bifunctor $\beta\colon I^{\mathrm{op}}\times J^{\mathrm{op}}\to\mathscr{C}$ with I and J small. It defines a functor $\beta_J\colon I^{\mathrm{op}}\to\mathscr{C}^{J^{\mathrm{op}}}$ as well as a functor $\beta_I\colon J^{\mathrm{op}}\to\mathscr{C}^{I^{\mathrm{op}}}$. One easily checks that

$$\lim\beta \simeq \lim\lim\beta_J \simeq \lim\lim\beta_I. \tag{2.5.1}$$

Similarly, if $\alpha\colon I\times J\to\mathscr{C}$ is a bifunctor, it defines a functor $\alpha_J\colon I\to\mathscr{C}^J$ as well as a functor $\alpha_I\colon J\to\mathscr{C}^I$ and one has the isomorphisms

$$\mathrm{colim}\,\alpha \simeq \mathrm{colim}\,(\mathrm{colim}\,\alpha_J) \simeq \mathrm{colim}\,(\mathrm{colim}\,\alpha_I). \tag{2.5.2}$$

In other words:

$$\lim_{i,j}\beta(i,j) \simeq \lim_j\lim_i(\beta(i,j)) \simeq \lim_i\lim_j(\beta(i,j)), \tag{2.5.3}$$

$$\mathop{\mathrm{colim}}_{i,j}\alpha(i,j) \simeq \mathop{\mathrm{colim}}_j(\mathop{\mathrm{colim}}_i(\alpha(i,j)) \simeq \mathop{\mathrm{colim}}_i\mathop{\mathrm{colim}}_j(\alpha(i,j)). \tag{2.5.4}$$

Limits with values in a category of functors

Consider another category $\mathscr{A}$ and a functor $\beta\colon I^{\mathrm{op}}\to\mathrm{Fct}(\mathscr{A},\mathscr{C})$. It defines a functor $\widetilde{\beta}\colon I^{\mathrm{op}}\times\mathscr{A}\to\mathscr{C}$, hence for each $A\in\mathscr{A}$, a functor $\widetilde{\beta}(A)\colon I^{\mathrm{op}}\to\mathscr{C}$. Assuming that $\mathscr{C}$ admits limits indexed by I, one checks easily that $A\mapsto\lim\widetilde{\beta}(A)$ is a functor, that is, an object of $\mathrm{Fct}(\mathscr{A},\mathscr{C})$, and is a limit of β. There is a similar result for colimits. Hence:

Proposition 2.5.1. *Let I be a small category and assume that $\mathscr{C}$ admits limits indexed by I. Then for any category $\mathscr{A}$, the category $\mathrm{Fct}(\mathscr{A},\mathscr{C})$ admits limits indexed by I. Moreover, if $\beta\colon I^{\mathrm{op}}\to\mathrm{Fct}(\mathscr{A},\mathscr{C})$ is a functor, then $\lim\beta\in\mathrm{Fct}(\mathscr{A},\mathscr{C})$ is given by*

$$(\lim\beta)(A) = \lim\,(\beta(A)),\quad A\in\mathscr{A}.$$

Similarly, assume that $\mathscr{C}$ admits colimits indexed by I. Then for any category $\mathscr{A}$, the category $\mathrm{Fct}(\mathscr{A},\mathscr{C})$ admits colimits indexed by I. Moreover, if $\alpha\colon I\to\mathrm{Fct}(\mathscr{A},\mathscr{C})$ is a functor, then $\mathrm{colim}\,\alpha\in\mathrm{Fct}(\mathscr{A},\mathscr{C})$ is given by

$$(\mathrm{colim}\,\alpha)(A) = \mathrm{colim}\,(\alpha(A)),\quad A\in\mathscr{A}.$$

Corollary 2.5.2. *Let $\mathscr{C}$ be a category. Then the categories $\mathscr{C}^\wedge$ and $\mathscr{C}^\vee$ admit small limits and colimits.*

Composition of limits

Let $I, \mathscr{C}$ and $\mathscr{C}'$ be categories with I small and let $\alpha\colon I \to \mathscr{C}$, $\beta\colon I^{\rm op} \to \mathscr{C}$ and $F\colon \mathscr{C} \to \mathscr{C}'$ be functors. When $\mathscr{C}$ and $\mathscr{C}'$ admit limits or colimits indexed by I, there are natural morphisms

$$F(\lim \beta) \to \lim (F \circ \beta), \tag{2.5.5}$$

$$\operatorname{colim} (F \circ \alpha) \to F(\operatorname{colim} \alpha). \tag{2.5.6}$$

This follows immediately from (2.3.7) and (2.3.5).

Definition 2.5.3. Let I be a small category and let $F\colon \mathscr{C} \to \mathscr{C}'$ be a functor.

(i) Assume that $\mathscr{C}$ and $\mathscr{C}'$ admit limits indexed by I. One says that F commutes with such limits if (2.5.5) is an isomorphism.

(ii) Similarly, assume that $\mathscr{C}$ and $\mathscr{C}'$ admit colimits indexed by I. One says that F commutes with such colimits if (2.5.6) is an isomorphism.

Examples 2.5.4. (i) Let $\mathscr{C}$ be a category which admits limits indexed by I and let $X \in \mathscr{C}$. By (2.3.2), the functor $\operatorname{Hom}_{\mathscr{C}}(X, \bullet)\colon \mathscr{C} \to \mathbf{Set}$ commutes with limits indexed by I. Similarly, if $\mathscr{C}$ admits colimits indexed by I, then the functor $\operatorname{Hom}_{\mathscr{C}}(\bullet, X)\colon \mathscr{C}^{\rm op} \to \mathbf{Set}$ commutes with limits indexed by I, by (2.3.3).
(ii) Let I and J be two small categories and assume that $\mathscr{C}$ admits limits (resp. colimits) indexed by $I \times J$. Then the functor $\lim\colon \operatorname{Fct}(J^{\rm op}, \mathscr{C}) \to \mathscr{C}$ (resp. $\operatorname{colim}\colon \operatorname{Fct}(J, \mathscr{C}) \to \mathscr{C}$) commutes with limits (resp. colimits) indexed by I. This follows from the isomorphisms (2.5.1) and (2.5.2).
(iii) Let $\mathbf{k}$ be a field, $\mathscr{C} = \mathscr{C}' = \operatorname{Mod}(\mathbf{k})$, and let $X \in \mathscr{C}$. Then the functor $\operatorname{Hom}_{\mathbf{k}}(X, \bullet)$ does not commute with colimit if X is infinite dimensional.

Proposition 2.5.5. *Let $F\colon \mathscr{C} \to \mathscr{C}'$ be a functor and let I be a small category.*

(i) *Assume that $\mathscr{C}$ and $\mathscr{C}'$ admit projective limits indexed by I and F admits a left adjoint $G\colon \mathscr{C}' \to \mathscr{C}$. Then F commutes with limits indexed by I, that is, $F(\lim\limits_{i} \beta(i)) \simeq \lim\limits_{i} F(\beta(i))$.*

(ii) *Similarly, if $\mathscr{C}$ and $\mathscr{C}'$ admit colimits indexed by I and F admits a right adjoint, then F commutes with such colimits.*

Proof. It is enough to prove the first assertion. To check that (2.5.5) is an isomorphism, we apply Corollary 1.4.4. Let $Y \in \mathscr{C}'$. One has the chain of isomorphisms

$$\begin{aligned}
\operatorname{Hom}_{\mathscr{C}'}(Y, F(\lim_i \beta(i))) &\simeq \operatorname{Hom}_{\mathscr{C}}(G(Y), \lim_i \beta(i)) \\
&\simeq \lim_i \operatorname{Hom}_{\mathscr{C}}(G(Y), \beta(i)) \\
&\simeq \lim_i \operatorname{Hom}_{\mathscr{C}'}(Y, F(\beta(i))) \\
&\simeq \operatorname{Hom}_{\mathscr{C}'^\wedge}(Y, \lim_i F(\beta(i))).
\end{aligned}$$

∎

2.6 Filtered colimits

As already seen in Example 2.3.10, the category **Set** admits small colimits. In the category **Set** one uses the notation $\bigsqcup$ rather than $\coprod$.

We shall construct colimits more explicitly.

Let $\alpha\colon I \to \mathbf{Set}$ be a functor (with I small) and consider the relation on $\bigsqcup_{i\in I}\alpha(i)$):

$$(2.6.1)\quad \begin{cases} \alpha(i) \ni x\mathscr{R}y \in \alpha(j) \text{ if there exists } k \in I,\ s\colon i \to k \text{ and } t\colon j \to k \\ \text{with } \alpha(s)(x) = \alpha(t)(y). \end{cases}$$

The relation $\mathscr{R}$ is reflexive and symmetric but is not transitive in general.

Proposition 2.6.1. *With the notations above, denote by $\sim$ the equivalence relation generated by $\mathscr{R}$. Then*

$$\operatorname{colim}\alpha \simeq (\bigsqcup_{i\in I}\alpha(i))/\sim .$$

Proof. Let $S \in \mathbf{Set}$. By the definition of the limit in **Set** we get:

$$\begin{aligned} \lim \operatorname{Hom}(\alpha, S) &\simeq \{\{u_i\}_{i\in I}; u_i\colon \alpha(i) \to S, u_j = u_i \circ \alpha(s) \\ &\qquad \text{if there exists } s\colon i \to j\}, \\ &\simeq \{\{p(i,x)\}_{i\in I, x\in\alpha(i)}; p(i,x) \in S, p(i,x) = p(j,y) \\ &\qquad \text{if there exists } s\colon i \to j \text{ with } \alpha(s)(x) = y\} \\ &\simeq \operatorname{Hom}((\bigsqcup_{i\in I}\alpha(i))/\sim, S). \end{aligned}$$

□

For a ring A, the category $\operatorname{Mod}(A)$ admits coproducts and cokernels. Hence, the category $\operatorname{Mod}(A)$ admits colimits. One shall be aware that the functor *for*: $\operatorname{Mod}(A) \to$ **Set** does not commute with colimits. For example, if I is empty and $\alpha\colon I \to \operatorname{Mod}(A)$ is a functor, then $\alpha(I) = \{0\}$ and *for*$(\{0\})$ is not an initial object in **Set**.

Definition 2.6.2. A small category I is called filtered if it satisfies the conditions (i)–(iii) below.

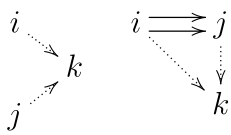
(i) I is non empty,

(ii) for any i and j in I, there exists $k \in I$ and morphisms $i \to k, j \to k$,

(iii) for any parallel morphisms $f, g\colon i \rightrightarrows j$, there exists a morphism $h\colon j \to k$ such that $h \circ f = h \circ g$.

One says that I is cofiltered if I^{op} is filtered.

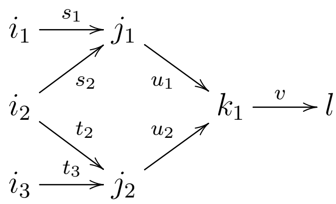
The conditions (ii)–(iii) of being filtered are visualized by the diagrams:



Of course, if $(I, \leq)$ is a non-empty directed ordered set, then the associated category I is filtered.

Proposition 2.6.3. *Let $\alpha\colon I \to \mathbf{Set}$ be a functor, with I filtered. The relation $\mathscr{R}$ given in (2.6.1) on $\coprod_i \alpha(i)$ is an equivalence relation.*

Proof. Let $x_j \in \alpha(i_j)$, $j = 1, 2, 3$ with $x_1 \sim x_2$ and $x_2 \sim x_3$. There exist morphisms visualized by the diagram:



such that $\alpha(s_1)x_1 = \alpha(s_2)x_2$, $\alpha(t_2)x_2 = \alpha(t_3)x_3$, and $v \circ u_1 \circ s_2 = v \circ u_2 \circ t_2$. Set $w_1 = v \circ u_1 \circ s_1$, $w_2 = v \circ u_1 \circ s_2 = v \circ u_2 \circ t_2$ and $w_3 = v \circ u_2 \circ t_3$. Then $\alpha(w_1)x_1 = \alpha(w_2)x_2 = \alpha(w_3)x_3$. Hence $x_1 \sim x_3$. □

Corollary 2.6.4. *Let $\alpha\colon I \to \mathbf{Set}$ be a functor, with I small and filtered.*

(i) *Let S be a finite subset in $\operatorname{colim} \alpha$. Then there exists $i \in I$ such that S is contained in the image of $\alpha(i)$ by the natural map $\alpha(i) \to \operatorname{colim} \alpha$.*

(ii) *Let $i \in I$ and let x and y be elements of $\alpha(i)$ with the same image in $\operatorname{colim} \alpha$. Then there exists $s\colon i \to j$ such that $\alpha(s)(x) = \alpha(s)(y)$ in $\alpha(j)$.*

Proof. (i) Denote by $\lambda\colon \bigsqcup_{i\in I} \alpha(i) \to \operatorname{colim} \alpha$ the quotient map. Let $S = \{x_1, \dots, x_n\}$. For $j = 1, \dots, n$, there exists $y_j \in \alpha(i_j)$ such that $x_j = \lambda(y_j)$. Choose $k \in I$ such that there exist morphisms $s_j\colon \alpha(i_j) \to \alpha(k)$. Then $x_j = \lambda(\alpha(s_j(y_j)))$.
(ii) For $x, y \in \alpha(i)$, $x\mathscr{R}y$ if and only if there exists $s\colon i \to j$ with $\alpha(s)(x) = \alpha(s)(y)$ in $\alpha(j)$. □

Corollary 2.6.5. *Let A be a ring and denote by for the forgetful functor $\operatorname{Mod}(A) \to \mathbf{Set}$. Then the functor for commutes with filtered colimits. In other words, if I is small and filtered and $\alpha\colon I \to \operatorname{Mod}(A)$ is a functor, then*

$$for \circ (\operatorname*{colim}_i \alpha(i)) = \operatorname*{colim}_i (for \circ \alpha(i)).$$

The proof is left as an exercise (see Exercise 2.8).

Colimits with values in **Set** indexed by small filtered categories commute with finite limits. More precisely:

Proposition 2.6.6. *For a small filtered category I, a finite category J and a functor $\alpha\colon I \times J^{\mathrm{op}} \to \mathbf{Set}$, one has $\operatorname*{colim}_i \lim_j \alpha(i,j) \xrightarrow{\sim} \lim_j \operatorname*{colim}_i \alpha(i,j)$. In other words, the functor*

$$\operatorname{colim}\colon \operatorname{Fct}(I, \mathbf{Set}) \to \mathbf{Set}$$

commutes with finite limits.

Proof. It is enough to prove that colim commutes with kernels and with finite products.

(i) colim commutes with kernels. Let $\alpha, \beta \colon I \to \mathbf{Set}$ be two functors and let $f, g \colon \alpha \rightrightarrows \beta$ be two morphisms of functors. Define γ as the kernel of (f, g), that is, we have exact sequences

$$\gamma(i) \to \alpha(i) \rightrightarrows \beta(i).$$

Let Z denote the kernel of $\operatorname{colim}_i \alpha(i) \rightrightarrows \operatorname{colim}_i \beta(i)$. We have to prove that the natural map $\lambda \colon \operatorname{colim}_i \gamma(i) \to Z$ is bijective.

(i) (a) The map λ is surjective. Indeed for $x \in Z$, represent x by some $x_i \in \alpha(i)$. Then $f_i(x_i)$ and $g_i(x_i)$ in $\beta(i)$ having the same image in $\operatorname{colim} \beta$, there exists $s \colon i \to j$ such that $\beta(s) f_i(x_i) = \beta(s) g_i(x_i)$. Set $x_j = \alpha(s) x_i$. Then $f_j(x_j) = g_j(x_j)$, which means that $x_j \in \gamma(j)$. Clearly, $\lambda(x_j) = x$.

(i) (b) The map λ is injective. Indeed, let $x, y \in \operatorname{colim} \gamma$ with $\lambda(x) = \lambda(y)$. We may represent x and y by elements x_i and y_i of $\gamma(i)$ for some $i \in I$. Since x_i and y_i have the same image in $\operatorname{colim} \alpha$, there exists $i \to j$ such that they have the same image in $\alpha(j)$. Therefore their images in $\gamma(j)$ will be the same.

(ii) colim commutes with finite products. The proof is similar to the preceding one and left to the reader. □

Corollary 2.6.7. *Let A be a ring and let I be a small filtered category. Then the functor* $\operatorname{colim} \colon \operatorname{Fct}(I, \operatorname{Mod}(A)) \to \operatorname{Mod}(A)$ *commutes with finite limits.*

Cofinal functors

Let $\varphi \colon J \to I$ be a functor of small categories. If there are no risk of confusion, we still denote by φ the associated functor $\varphi \colon J^{\mathrm{op}} \to I^{\mathrm{op}}$. For two functors $\alpha \colon I \to \mathscr{C}$ and $\beta \colon I^{\mathrm{op}} \to \mathscr{C}$, we have natural morphisms:

$$\lim\,(\beta \circ \varphi) \leftarrow \lim \beta, \tag{2.6.2}$$
$$\operatorname{colim}\,(\alpha \circ \varphi) \to \operatorname{colim} \alpha. \tag{2.6.3}$$

This follows immediately of (2.3.7) and (2.3.5).

Definition 2.6.8. Assume that φ is fully faithful and I is filtered. One says that φ is cofinal if for any $i \in I$ there exists $j \in J$ and a morphism $s \colon i \to \varphi(j)$.

Example 2.6.9. A subset $J \subset \mathbb{N}$ defines a cofinal subcategory of $(\mathbb{N}, \leq)$ if and only if it is infinite.

Proposition 2.6.10. *Let $\varphi \colon J \to I$ be a fully faithful functor. Assume that I is filtered and φ is cofinal. Then*

(i) *for any category $\mathscr{C}$ and any functor $\beta \colon I^{\mathrm{op}} \to \mathscr{C}$, the morphism* (2.6.2) *is an isomorphism,*

(ii) *for any category $\mathscr{C}$ and any functor $\alpha \colon I \to \mathscr{C}$, the morphism* (2.6.3) *is an isomorphism.*

Proof. Let us prove (ii), the other proof being similar. By the hypothesis, for each $i \in I$ we get a morphism $\alpha(i) \to \operatorname*{colim}_{j \in J}(\alpha \circ \varphi(j))$ from which one deduce a morphism

$$\operatorname*{colim}_{i \in I} \alpha(i) \to \operatorname*{colim}_{j \in J}(\alpha \circ \varphi(j)).$$

One checks easily that this morphism is inverse to the morphism in (2.5.6). □

Example 2.6.11. Let X be a topological space, $x \in X$ and denote by I_x the set of open neighborhoods of x in X. We endow I_x with the order: $U \leq V$ if $V \subset U$. Given U and V in I_x, and setting $W = U \cap V$, we have $U \leq W$ and $V \leq W$. Therefore, I_x is filtered.

Denote by $\mathscr{C}^0(U)$ the $\mathbb{C}$-vector space of complex valued continuous functions on U. The restriction maps $\mathscr{C}^0(U) \to \mathscr{C}^0(V), V \subset U$ define an inductive system of $\mathbb{C}$-vector spaces indexed by I_x. One sets

$$\mathscr{C}^0_{X,x} = \operatorname*{colim}_{U \in I_x} \mathscr{C}^0(U). \tag{2.6.4}$$

An element φ of $\mathscr{C}^0_{X,x}$ is called a germ of continuous function at 0. Such a germ is an equivalence class $(U, \varphi_U)/\sim$ with U a neighborhood of x, φ_U a continuous function on U, and $(U, \varphi_U) \sim 0$ if there exists a neighborhood V of x with $V \subset U$ such that the restriction of φ_U to V is the zero function. Hence, a germ of function is zero at x if this function is identically zero in a neighborhood of x.

A set theoretical remark

Remark 2.6.12. As already mentioned, all categories $\mathscr{C}$, $\mathscr{C}'$ etc. belong to a given universe $\mathscr{U}$ and all limits or colimits are indexed by $\mathscr{U}$-small categories I, J, etc. Let us give an example which shows that without some care, we may have troubles.

Let $\mathscr{C}$ be a category which admits products and assume there exist $X, Y \in \mathscr{C}$ such that $\operatorname{Hom}_{\mathscr{C}}(X, Y)$ has more than one element. Set $M = \operatorname{Mor}(\mathscr{C})$, where $\operatorname{Mor}(\mathscr{C})$ denotes the big set of all morphisms in $\mathscr{C}$, and let $\pi = \operatorname{card}(M)$, the cardinal of the set M. We have

$$\operatorname{Hom}_{\mathscr{C}}(X, Y^M) \simeq \operatorname{Hom}_{\mathscr{C}}(X, Y)^M$$

and therefore $\operatorname{card}(\operatorname{Hom}_{\mathscr{C}}(X, Y^M) \geq 2^\pi$. On the other hand, $\operatorname{Hom}_{\mathscr{C}}(X, Y^M) \subset \operatorname{Mor}(\mathscr{C})$ which implies $\operatorname{card}(\operatorname{Hom}_{\mathscr{C}}(X, Y^M) \leq \pi$.

The "contradiction" comes from the fact that $\mathscr{C}$ does not admit products indexed by such a big set as $\operatorname{Mor}(\mathscr{C})$. (This remark is extracted from [Fre64].)

Exercises to Chapter 2

Exercise 2.1. (i) Let I be a small set and $\{X_i\}_{i \in I}$ a family of sets indexed by I. Show that $\coprod_i X_i = \bigsqcup_i X_i$, the disjoint union of the sets X_i.
(ii) Construct the natural map $\bigsqcup_i \operatorname{Hom}_{\mathbf{Set}}(Y, X_i) \to \operatorname{Hom}_{\mathbf{Set}}(Y, \bigsqcup_i X_i)$ and prove it is injective and not surjective in general.

Exercise 2.2. Let $X, Y \in \mathscr{C}$ and consider the category $\mathscr{D}$ whose objects are triplets $Z \in \mathscr{C}, f \colon Z \to X, g \colon Z \to Y$, the morphisms being the natural ones. Prove that this category admits a terminal object if and only if the product $X \times Y$ exists in $\mathscr{C}$, and that in such a case this terminal object is isomorphic to $X \times Y, X \times Y \to X, X \times Y \to Y$. Deduce that if $X \times Y$ exists, it is unique up to unique isomorphism.

Exercise 2.3. Let I and $\mathscr{C}$ be two categories with I small and denote by Δ the functor from $\mathscr{C}$ to $\mathscr{C}^I$ which, to $X \in \mathscr{C}$, associates the constant functor $\Delta(X)\colon I \ni i \mapsto X \in \mathscr{C}$, $(i \to j) \in \mathrm{Mor}(I) \mapsto \mathrm{id}_X$. Assume that any functor from I to $\mathscr{C}$ admits a colimit.
(i) Prove the formula (for $\alpha\colon I \to \mathscr{C}$ and $Y \in \mathscr{C}$):

$$\mathrm{Hom}_{\mathscr{C}}(\operatorname*{colim}_i \alpha(i), Y) \simeq \mathrm{Hom}_{\,\mathrm{Fct}(I,\mathscr{C})}(\alpha, \Delta(Y)).$$

(ii) Replacing I with the opposite category, deduce the formula (assuming limits exist):

$$\mathrm{Hom}_{\mathscr{C}}(X, \lim_i G(i)) \simeq \mathrm{Hom}_{\,\mathrm{Fct}(I^{\mathrm{op}},\mathscr{C})}(\Delta(X), G).$$

Exercise 2.4. Let $\mathscr{C}$ be a category which admits small filtered colimits. One says that an object X of $\mathscr{C}$ is of finite type if for any functor $\alpha\colon I \to \mathscr{C}$ with I filtered, the natural map $\operatorname{colim} \mathrm{Hom}_{\mathscr{C}}(X, \alpha) \to \mathrm{Hom}_{\mathscr{C}}(X, \operatorname{colim} \alpha)$ is injective. Show that this definition coincides with the classical one when $\mathscr{C} = \mathrm{Mod}(A)$, for a ring A.
(Hint: let $X \in \mathrm{Mod}(A)$. To prove that if X is of finite type in the categorical sense then it is of finite type in the usual sense, use the fact that, denoting by $\mathscr{S}$ be the family of submodules of finite type of X ordered by inclusion, we have $\operatorname*{colim}_{V \in \mathscr{S}} X/V \simeq 0$.)

Exercise 2.5. Let $\mathscr{C}$ be a category which admits small filtered colimits. One says that an object X of $\mathscr{C}$ is of finite presentation if for any functor $\alpha\colon I \to \mathscr{C}$ with I small and filtered, the natural map $\operatorname{colim} \mathrm{Hom}_{\mathscr{C}}(X, \alpha) \to \mathrm{Hom}_{\mathscr{C}}(X, \operatorname{colim} \alpha)$ is bijective. Show that this definition coincides with the classical one when $\mathscr{C} = \mathrm{Mod}(A)$, for a ring A.

Exercise 2.6. In the situation of Definition 2.4.4, construct the natural morphism $u\colon \mathrm{Coim}(f) \to \mathrm{Im}(f)$ such that the composition $X \to \mathrm{Coim}(f) \to \mathrm{Im}(f) \to Y$ is f. (See [KS06, Prop. 5.1.2].)

Exercise 2.7. Let I be a filtered ordered set and let $\{A_i\}_{i \in I}$ be an inductive system of rings indexed by I.
(i) Prove that $A := \operatorname*{colim}_i A_i$ is naturally endowed with a ring structure.
(ii) Define the notion of an inductive system M_i of A_i-modules, and define the A-module $\operatorname*{colim}_i M_i$.
(iii) Let N_i (resp. M_i) be an inductive system of right (resp. left) A_i modules. Prove the isomorphism

$$\operatorname*{colim}_i (N_i \otimes_{A_i} M_i) \xrightarrow{\sim} \operatorname*{colim}_i N_i \otimes_A \operatorname*{colim}_i M_i.$$

Exercise 2.8. Let I be a filtered ordered set and let $\{M_i\}_{i \in I}$ be an inductive system of $\mathbf{k}$-modules indexed by I. Let $M = \bigsqcup M_i/\sim$ where $\bigsqcup$ denotes the set-theoretical disjoint union and $\sim$ is the relation $M_i \ni x_i \sim y_j \in M_j$ if there exists $k \geq i, k \geq j$ such that $u_{ki}(x_i) = u_{kj}(y_j)$.

Prove that M is naturally a $\mathbf{k}$-module and is isomorphic to $\operatorname*{colim}_i M_i$.

Exercise 2.9. (i) Let $\mathscr{C}$ be a category which admits colimits indexed by a category I. Let $\alpha\colon I \to \mathscr{C}$ be a functor and let $X \in \mathscr{C}$. Construct the natural morhism

$$\operatorname*{colim}_i \operatorname{Hom}_{\mathscr{C}}(X, \alpha(i)) \to \operatorname{Hom}_{\mathscr{C}}(X, \operatorname*{colim}_i \alpha(i)). \tag{2.6.5}$$

(ii) Let $\mathbf{k}$ be a field and denote by $\mathbf{k}[x]^{\leq n}$ the $\mathbf{k}$-vector space consisting of polynomials of degree $\leq n$. Prove the isomorphism $\mathbf{k}[x] = \operatorname*{colim}_n \mathbf{k}[x]^{\leq n}$ and, noticing that $\mathrm{id}_{\mathbf{k}[x]} \notin \operatorname*{colim}_n \operatorname{Hom}_{\mathbf{k}}(\mathbf{k}[x], \mathbf{k}[x]^{\leq n})$, deduce that the morphism (2.6.5) is not an isomorphism in general.

Exercise 2.10. Let $\mathscr{C}$ be a category and recall (Proposition 2.5.1) that the category $\mathscr{C}^\wedge$ admits colimits. One denotes by "colim" the colimit in $\mathscr{C}^\wedge$. Let $\mathbf{k}$ be a field and let $\mathscr{C} = \operatorname{Mod}(\mathbf{k})$. Prove that the Yoneda functor $\mathrm{h}_{\mathscr{C}}\colon \mathscr{C} \to \mathscr{C}^\wedge$ does not commute with colimits.

Exercise 2.11. Let I be a discrete set and let $\mathscr{J}$ be the set of finite subsets of I, ordered by inclusion. We consider both I and $\mathscr{J}$ as categories. Let $\mathscr{C}$ be a category and $\alpha\colon I \to \mathscr{C}$ a functor. For $J \in \mathscr{J}$ we denote by $\alpha_J\colon J \to \mathscr{C}$ the restriction of α to J.
(i) Prove that the category $\mathscr{J}$ is filtered.
(ii) Prove the isomorphism $\operatorname*{colim}_{J\in\mathscr{J}} \operatorname*{colim}_{j\in J} \alpha_J \xrightarrow{\sim} \operatorname{colim} \alpha$.

Exercise 2.12. Let $\mathscr{C}$ be a category which admits a zero-object and kernels. Prove that a morphism $f\colon X \to Y$ is a monomorphism if and only if $\ker f \simeq 0$.

Chapter 3

Additive categories

Many results or constructions in the category $\mathrm{Mod}(A)$ of modules over a ring A have their counterparts in other contexts, such as finitely generated A-modules, or graded modules over a graded ring, or sheaves of A-modules, etc. Hence, it is natural to look for a common language which avoids to repeat the same arguments. This is the language of additive and abelian categories.

In this chapter we introduce additive categories and study the category of complexes in such categories. In particular, we introduce the shifted complex, the mapping cone of a morphism, the homotopy category and the simple complex associated with a double complex, with application to bifunctors. We also briefly study the simplicial category and explain how to associate complexes to simplicial objects.

3.1 Additive categories

Definition 3.1.1. A category $\mathscr{C}$ is additive if it satisfies conditions (i)-(v) below:

(i) for any $X, Y \in \mathscr{C}, \quad \mathrm{Hom}_{\mathscr{C}}(X, Y) \in \mathrm{Mod}(\mathbb{Z})$,

(ii) the composition law $\circ$ is bilinear,

(iii) there exists a zero object in $\mathscr{C}$,

(iv) the category $\mathscr{C}$ admits finite coproducts,

(v) the category $\mathscr{C}$ admits finite products.

Note that $\mathrm{Hom}_{\mathscr{C}}(X, Y) \neq \varnothing$ since it is a group and for all $X \in \mathscr{C}$, $\mathrm{Hom}_{\mathscr{C}}(X, 0) = \mathrm{Hom}_{\mathscr{C}}(0, X) = 0$. (The morphism 0 should not be confused with the object 0.)

Notation 3.1.2. If X and Y are two objects of $\mathscr{C}$, one denotes by $X \oplus Y$ (instead of $X \coprod Y$) their coproduct, and calls it their direct sum. One denotes as usual by $X \times Y$ their product. This change of notations is motivated by the fact that if A is a ring, the forgetful functor $\mathit{for}\colon \mathrm{Mod}(A) \to \mathbf{Set}$ does not commute with coproducts.

Lemma 3.1.3. *Let $\mathscr{C}$ be a category satisfying conditions* (i)–(iii) *in* Definition 3.1.1. *Consider the condition*

(vi) *for any two objects* X *and* Y *in* $\mathscr{C}$*, there exists* $Z \in \mathscr{C}$ *and morphisms* $i_1\colon X \to Z$*,* $i_2\colon Y \to Z$*,* $p_1\colon Z \to X$ *and* $p_2\colon Z \to Y$ *satisfying*

$$p_1 \circ i_1 = \mathrm{id}_X, \quad p_1 \circ i_2 = 0 \tag{3.1.1}$$

$$p_2 \circ i_2 = \mathrm{id}_Y, \quad p_2 \circ i_1 = 0, \tag{3.1.2}$$

$$i_1 \circ p_1 + i_2 \circ p_2 = \mathrm{id}_Z\,. \tag{3.1.3}$$

Then the conditions (iv)*,* (v) *and* (vi) *are equivalent and the objects* $X \oplus Y$*,* $X \times Y$ *and* Z *are naturally isomorphic.*

Proof. (a) Let us assume condition (iv). The identity of X and the zero morphism $Y \to X$ define the morphism $p_1\colon X \oplus Y \to X$ satisfying (3.1.1). We construct similarly the morphism $p_2\colon X \oplus Y \to Y$ satisfying (3.1.2). To check (3.1.3), we use the fact that if $f\colon X \oplus Y \to X \oplus Y$ satisfies $f \circ i_1 = i_1$ and $f \circ i_2 = i_2$, then $f = \mathrm{id}_{X\oplus Y}$.
(b) Let us assume condition (vi). Let $W \in \mathscr{C}$ and consider morphisms $f\colon X \to W$ and $g\colon Y \to W$. Set $h := f \circ p_1 \oplus g \circ p_2$. Then $h\colon Z \to W$ satisfies $h \circ i_1 = f$ and $h \circ i_2 = g$ and such an h is unique. Hence $Z \simeq X \oplus Y$.
(c) We have proved that conditions (iv) and (vi) are equivalent and moreover that if they are satisfied, then $Z \simeq X \oplus Y$. Replacing $\mathscr{C}$ with $\mathscr{C}^{\mathrm{op}}$, we get that these conditions are equivalent to (v) and $Z \simeq X \times Y$. □

Example 3.1.4. (i) If A is a ring, $\mathrm{Mod}(A)$ and $\mathrm{Mod}^{\mathrm{f}}(A)$ are additive categories.
(ii) **Ban**, the category of $\mathbb{C}$-Banach spaces and linear continuous maps is additive.
(iii) If $\mathscr{C}$ is additive, then $\mathscr{C}^{\mathrm{op}}$ is additive.
(iv) Let I be a small category. If $\mathscr{C}$ is additive, the category $\mathrm{Fct}(I,\mathscr{C})$ of functors from I to $\mathscr{C}$, is additive.
(v) If $\mathscr{C}$ and $\mathscr{C}'$ are additive, then $\mathscr{C} \times \mathscr{C}'$ is additive.

Let $F\colon \mathscr{C} \to \mathscr{C}'$ be a functor of additive categories. One says that F is additive if for $X, Y \in \mathscr{C}$, $\mathrm{Hom}_{\mathscr{C}}(X,Y) \to \mathrm{Hom}_{\mathscr{C}'}(F(X),F(Y))$ is a morphism of groups. We shall not prove here the following result.

Proposition 3.1.5. *Let* $F\colon \mathscr{C} \to \mathscr{C}'$ *be a functor of additive categories. Then* F *is additive if and only if it commutes with direct sum, that is, for* X *and* Y *in* $\mathscr{C}$*:*

$$F(0) \simeq 0$$
$$F(X \oplus Y) \simeq F(X) \oplus F(Y).$$

Unless otherwise specified, functors between additive categories will be assumed to be additive.
Generalization. Let $\mathbf{k}$ be a *commutative* unital ring. One defines the notion of a $\mathbf{k}$-additive category by assuming that for X and Y in $\mathscr{C}$, $\mathrm{Hom}_{\mathscr{C}}(X,Y)$ is a $\mathbf{k}$-module and the composition is $\mathbf{k}$-bilinear.

3.2 Complexes in additive categories

Let $\mathscr{C}$ denote an additive category.

A differential object $(X^\bullet, d_X^\bullet)$ in $\mathscr{C}$ is a sequence of objects X^k and morphisms d^k $(k \in \mathbb{Z})$:

$$\cdots \to X^{k-1} \xrightarrow{d^{k-1}} X^k \xrightarrow{d^k} X^{k+1} \to \cdots . \tag{3.2.1}$$

A morphism of differential objects $f^\bullet \colon X^\bullet \to Y^\bullet$ is visualized by a commutative diagram:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & X^n & \xrightarrow{d_X^n} & X^{n+1} & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle f^n} & & \downarrow{\scriptstyle f^{n+1}} & & \\ \cdots & \longrightarrow & Y^n & \xrightarrow{d_Y^n} & X^{n+1} & \longrightarrow & \cdots \end{array}$$

Hence, the category $\mathrm{Diff}(\mathscr{C})$ of differential objects in $\mathscr{C}$ is nothing but the category $\mathrm{Fct}(\mathbb{Z}, \mathscr{C})$. In particular, it is an additive category.

Definition 3.2.1. (i) A complex is a differential object $(X^\bullet, d_X^\bullet)$ such that $d^n \circ d^{n-1} = 0$ for all $n \in \mathbb{Z}$.

(ii) One denotes by $\mathrm{C}(\mathscr{C})$ the full additive subcategory of $\mathrm{Diff}(\mathscr{C})$ consisting of complexes.

From now on, we shall concentrate our study on the category $\mathrm{C}(\mathscr{C})$.

A complex is bounded (resp. bounded below, bounded above) if $X^n = 0$ for $|n| >> 0$ (resp. $n << 0$, $n >> 0$). One denotes by $\mathrm{C}^*(\mathscr{C})(* = \mathrm{b}, +, -)$ the full additive subcategory of $\mathrm{C}(\mathscr{C})$ consisting of bounded complexes (resp. bounded below, bounded above). We also use the notation $\mathrm{C}^{\mathrm{ub}}(\mathscr{C}) = \mathrm{C}(\mathscr{C})$ (ub for "unbounded"). For $a \in \mathbb{Z}$ we shall denote by $\mathrm{C}^{\geq a}(\mathscr{C})$ the full additive subcategory of $\mathrm{C}(\mathscr{C})$ consisting of objects $X^\bullet$ such that $X^j \simeq 0$ for $j < a$. One defines similarly the categories $\mathrm{C}^{\leq a}(\mathscr{C})$ and, for $a \leq b$, $\mathrm{C}^{[a,b]}(\mathscr{C})$.

One considers $\mathscr{C}$ as a full subcategory of $\mathrm{C}^{\mathrm{b}}(\mathscr{C})$ by identifying an object $X \in \mathscr{C}$ with the complex $X^\bullet$ "concentrated in degree 0":

$$X^\bullet := \cdots \to 0 \to X \to 0 \to \cdots$$

where X stands in degree 0. In other words, one identifies $\mathscr{C}$ and $\mathrm{C}^{[0,0]}(\mathscr{C})$.

Shift functor

Let $\mathscr{C}$ be an additive category, let $X \in \mathrm{C}(\mathscr{C})$ and let $p \in \mathbb{Z}$. One defines the shifted complex $X[p]$ by[1]:

$$\begin{aligned} (X[p])^n &= X^{n+p} \\ d^n_{X[p]} &= (-)^p d_X^{n+p} \end{aligned}$$

If $f \colon X \to Y$ is a morphism in $\mathrm{C}(\mathscr{C})$ one defines $f[p] \colon X[p] \to Y[p]$ by $(f[p])^n = f^{n+p}$.

The shift functor $[1] \colon X \mapsto X[1]$ is an automorphism (*i.e.* an invertible functor) of $\mathrm{C}(\mathscr{C})$.

[1]In these notes, we shall sometimes write $(-)^p$ instead of $(-1)^p$

Mapping cone

Definition 3.2.2. Let $f\colon X \to Y$ be a morphism in $\mathrm{C}(\mathscr{C})$. The mapping cone of f, denoted $\mathrm{Mc}(f)$, is the object of $\mathrm{C}(\mathscr{C})$ defined by:

$$\mathrm{Mc}(f)^n = (X[1])^n \oplus Y^n$$
$$d^n_{\mathrm{Mc}(f)} = \begin{pmatrix} d^n_{X[1]} & 0 \\ f^{n+1} & d^n_Y \end{pmatrix}$$

Of course, before to state this definition, one should check that $d^{n+1}_{\mathrm{Mc}(f)} \circ d^n_{\mathrm{Mc}(f)} = 0$. Indeed:

$$\begin{pmatrix} -d^{n+2}_X & 0 \\ f^{n+2} & d^{n+1}_Y \end{pmatrix} \circ \begin{pmatrix} -d^{n+1}_X & 0 \\ f^{n+1} & d^n_Y \end{pmatrix} = 0$$

Notice that although $\mathrm{Mc}(f)^n = (X[1])^n \oplus Y^n$, $\mathrm{Mc}(f)$ is not isomorphic to $X[1] \oplus Y$ in $\mathrm{C}(\mathscr{C})$ unless f is the zero morphism.

There are natural morphisms of complexes

(3.2.2) $$\alpha(f)\colon Y \to \mathrm{Mc}(f), \quad \beta(f)\colon \mathrm{Mc}(f) \to X[1].$$

and $\beta(f) \circ \alpha(f) = 0$.

If $F\colon \mathscr{C} \to \mathscr{C}'$ is an additive functor, then $F(\mathrm{Mc}(f)) \simeq \mathrm{Mc}(F(f))$.

3.3 Double complexes

Let $\mathscr{C}$ be as above an additive category. A double complex $(X^{\bullet,\bullet}, d_X)$ in $\mathscr{C}$ is the data of

$$\{X^{n,m}, d'^{n,m}_X, d''^{n,m}_X; (n,m) \in \mathbb{Z} \times \mathbb{Z}\}$$

where $X^{n,m} \in \mathscr{C}$ and the "differentials" $d'^{n,m}_X\colon X^{n,m} \to X^{n+1,m}$, $d''^{n,m}_X : X^{n,m} \to X^{n,m+1}$ satisfy:

(3.3.1) $$d'^2_X = d''^2_X = 0, \ \ d' \circ d'' = d'' \circ d'.$$

One can represent a double complex by a commutative diagram:

(3.3.2)
$$\begin{array}{ccccccc}
 & & \downarrow & & \downarrow & & \\
\longrightarrow & X^{n,m} & \xrightarrow{d''^{n,m}} & X^{n,m+1} & \longrightarrow \\
 & \downarrow{\scriptstyle d'^{n,m}} & & \downarrow{\scriptstyle d'^{n,m+1}} & \\
\longrightarrow & X^{n+1,m} & \xrightarrow[d''^{n+1,m}]{} & X^{n+1,m+1} & \longrightarrow \\
 & \downarrow & & \downarrow & &
\end{array}$$

One defines naturally the notion of a morphism of double complexes and one obtains the additive category $\mathrm{C}^2(\mathscr{C})$ of double complexes.

There are two functors $F_I, F_{II} : \mathrm{C}^2(\mathscr{C}) \to \mathrm{C}(\mathrm{C}(\mathscr{C}))$ which associate to a double complex X the complex whose objects are the rows (resp. the columns) of X. These two functors are clearly isomorphisms of categories.

Assume

$$\mathscr{C} \text{ admits countable direct sums.} \tag{3.3.3}$$

One can then associate to the double complex X a simple complex $\mathrm{tot}_\oplus(X)$ by setting:

$$\mathrm{tot}_\oplus(X)^p = \oplus_{m+n=p} X^{n,m}, \quad d^p_{\mathrm{tot}(X)}|_{X^{n,m}} = d'^{n,m} + (-)^n d''^{n,m}. \tag{3.3.4}$$

This is visualized by the diagram:

$$\begin{array}{ccc} X^{n,m} & \xrightarrow{(-)^n d''} & X^{n,m+1} \\ {\scriptstyle d'}\downarrow & & \\ X^{n+1,m} & & \end{array}$$

Similarly, assume

$$\mathscr{C} \text{ admits countable products.} \tag{3.3.5}$$

One can then associate to the double complex X a simple complex $\mathrm{tot}_\pi(X)$ by setting:

$$\mathrm{tot}_\pi(X)^p = \prod_{m+n=p} X^{n,m}, \quad (d_{\mathrm{tot}(X)})^{n+m-1} = d'^{n-1,m} + (-)^n d''^{n,m-1}. \tag{3.3.6}$$

This is visualized by the diagram:

$$\begin{array}{ccc} & & X^{n-1,m} \\ & & \downarrow{\scriptstyle d'} \\ X^{n,m-1} & \xrightarrow{(-)^n d''} & X^{n,m} \end{array}$$

One also encounters the finiteness condition:

$$\text{for all } p \in \mathbb{Z}, \quad \{(m,n) \in \mathbb{Z} \times \mathbb{Z}; X^{n,m} \neq 0, m+n=p\} \text{ is finite.} \tag{3.3.7}$$

To such an X one associates its "total complex" $\mathrm{tot}(X) = \mathrm{tot}_\oplus(X) \simeq \mathrm{tot}_\pi(X)$. In the sequel, we denote by $C^2_f(\mathscr{C})$ the full subcategory of $\mathrm{C}^2(\mathscr{C})$ consisting of objects X satisfying (3.3.7).

Proposition 3.3.1. *Assume* (3.3.3). *Then the differential object* $\{\mathrm{tot}_\oplus(X)^p, d^p_{\mathrm{tot}(X)}\}_{p\in\mathbb{Z}}$ *is a complex* (i.e., $d^{p+1}_{\mathrm{tot}(X)} \circ d^p_{\mathrm{tot}(X)} = 0$) *and* $\mathrm{tot}\colon \mathrm{C}^2(\mathscr{C}) \to \mathrm{C}(\mathscr{C})$ *is a functor of additive categories.*

There is a similar result assuming (3.3.5) *or assuming that* $X \in C^2_f(\mathscr{C})$.

Proof. For $(n,m) \in \mathbb{Z} \times \mathbb{Z}$, one has

$$\begin{aligned} d \circ d|_{X^{n,m}} &= d'' \circ d''|_{X^{n,m}} + d' \circ d'|_{X^{n,m}} \\ &\quad + (-)^{n+1} d'' \circ d'|_{X^{n,m}} + (-)^n d' \circ d''|_{X^{n,m}} \\ &= 0. \end{aligned}$$

It is left to the reader to check that tot is an additive functor. □

Example 3.3.2. Let $f^\bullet : X^\bullet \to Y^\bullet$ be a morphism in $\mathrm{C}(\mathscr{C})$. Consider the double complex $Z^{\bullet,\bullet}$ such that $Z^{-1,\bullet} = X^\bullet$, $Z^{0,\bullet} = Y^\bullet$, $Z^{i,\bullet} = 0$ for $i \neq -1, 0$, with differentials $f^j \colon Z^{-1,j} \to Z^{0,j}$. Then

$$\mathrm{tot}(Z^{\bullet,\bullet}) \simeq \mathrm{Mc}(f^\bullet). \tag{3.3.8}$$

Bifunctor

Let $\mathscr{C}, \mathscr{C}'$ and $\mathscr{C}''$ be additive categories and let $F \colon \mathscr{C} \times \mathscr{C}' \to \mathscr{C}''$ be an additive bifunctor (*i.e.,* $F(\,\bullet\,,\,\bullet\,)$ is additive with respect to each argument). It defines an additive bifunctor $\mathrm{C}^2(F) \colon \mathrm{C}(\mathscr{C}) \times \mathrm{C}(\mathscr{C}') \to \mathrm{C}^2(\mathscr{C}'')$. In other words, if $X \in \mathrm{C}(\mathscr{C})$ and $X' \in \mathrm{C}(\mathscr{C}')$ are complexes, then $\mathrm{C}^2(F)(X, X')$ is a double complex.

Example 3.3.3. Consider the bifunctor $\bullet \otimes \bullet : \mathrm{Mod}(A^{\mathrm{op}}) \times \mathrm{Mod}(A) \to \mathrm{Mod}(\mathbb{Z})$. In the sequel, we shall simply write $\otimes$ instead of $\mathrm{C}^2(\otimes)$. Then, for $X \in \mathrm{C}(\mathrm{Mod}(A^{\mathrm{op}}))$ and $Y \in \mathrm{C}(\mathrm{Mod}(A))$, one has

$$(X \otimes Y)^{n,m} = X^n \otimes Y^m, \quad d'^{n,m} = d_X^n \otimes \mathrm{id}_{Y^m}, d''^{n,m} = \mathrm{id}_{X^n} \otimes d_Y^m,$$
$$(\mathrm{tot}_\oplus(X,Y))^k = \bigoplus_{n+m=k} X^n \otimes Y^m, \quad d_{\mathrm{tot}(X \otimes Y)}|_{X^n \otimes Y^m} = d'^{n,m} + (-)^n d''^{n,m}.$$

The complex $\mathrm{Hom}^\bullet$

Consider the bifunctor $\mathrm{Hom}_{\mathscr{C}} \colon \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathrm{Mod}(\mathbb{Z})$. In the sequel, we shall write $\mathrm{Hom}_{\mathscr{C}}^{\bullet,\bullet}$ instead of $\mathrm{C}^2(\mathrm{Hom}_{\mathscr{C}})$. If X and Y are two objects of $\mathrm{C}(\mathscr{C})$, one has

$$\mathrm{Hom}_{\mathscr{C}}^{\bullet,\bullet}(X,Y)^{n,m} = \mathrm{Hom}_{\mathscr{C}}(X^{-m}, Y^n),$$
$$d'^{n,m} = \mathrm{Hom}_{\mathscr{C}}(X^{-m}, d_Y^n), \quad d''^{m,n} = \mathrm{Hom}_{\mathscr{C}}((-)^m d_X^{-m-1}, Y^n).$$

Note that $\mathrm{Hom}_{\mathscr{C}}^{\bullet,\bullet}(X,Y)$ is a double complex in the category $\mathrm{Mod}(\mathbb{Z})$ and should not be confused with the group $\mathrm{Hom}_{\mathrm{C}(\mathscr{C})}(X,Y)$.

Let $X, Y \in \mathrm{C}(\mathscr{C})$. Using the fact that $\mathrm{Mod}(\mathbb{Z})$ admits countable products, one sets

$$\mathrm{Hom}_{\mathscr{C}}^{\bullet}(X,Y) = \mathrm{tot}_\pi \mathrm{Hom}_{\mathscr{C}}^{\bullet,\bullet}(X,Y), \text{ an object of } \mathrm{C}(\mathrm{Mod}(\mathbb{Z})). \tag{3.3.9}$$

Hence, $\mathrm{Hom}_{\mathscr{C}}(X,Y)^n = \prod_j \mathrm{Hom}_{\mathscr{C}}(X^j, Y^{n+j})$ and $d^n \colon \mathrm{Hom}_{\mathscr{C}}(X,Y)^n \to \mathrm{Hom}_{\mathscr{C}}(X,Y)^{n+1}$ is defined as follows. To $f = \{f^j\}_j \in \prod_{j\in\mathbb{Z}} \mathrm{Hom}_{\mathscr{C}}(X^j, Y^{n+j})$ one associates

$$d^n f = \{g^j\}_j \in \prod_{j\in\mathbb{Z}} \mathrm{Hom}_{\mathscr{C}}(X^j, Y^{n+j+1}), \quad g^j = d'^{n+j,-j} f^j + (-)^{j+n+1} d''^{j+n+1,-j-1} f^{j+1}.$$

In other words, the components of df in $\mathrm{Hom}_{\mathscr{C}}(X,Y)^{n+1}$ will be given by

$$(d^n f)^j = d_Y^{j+n} \circ f^j + (-)^{n+1} f^{j+1} \circ d_X^j. \tag{3.3.10}$$

Note that for $X, Y, Z \in \mathrm{C}(\mathscr{C})$, there is a natural composition map

$$\mathrm{Hom}_{\mathscr{C}}^{\bullet}(X,Y) \times \mathrm{Hom}_{\mathscr{C}}^{\bullet}(Y,Z) \xrightarrow{\circ} \mathrm{Hom}_{\mathscr{C}}^{\bullet}(X,Z) \tag{3.3.11}$$

associated with the map

$$\mathrm{Hom}_{\mathscr{C}}(X,Y)^m \times \mathrm{Hom}_{\mathscr{C}}(Y,Z)^n \to \mathrm{Hom}_{\mathscr{C}}(X,Z)^{m+n},$$
$$\prod_i \mathrm{Hom}_{\mathscr{C}}(X^i, Y^{i+m}) \times \prod_i \mathrm{Hom}_{\mathscr{C}}(Y^{i+m}, Z^{i+m+n}) \to \prod_i \mathrm{Hom}_{\mathscr{C}}(X^i, Z^{i+m+n}).$$

3.4 The homotopy category

Let $\mathscr{C}$ be an additive category.

Definition 3.4.1. (i) A morphism $f\colon X \to Y$ in $\mathrm{C}(\mathscr{C})$ is homotopic to zero if for all p there exists a morphism $s^p\colon X^p \to Y^{p-1}$ such that:

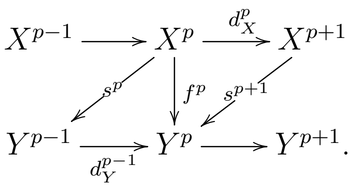
$$f^p = s^{p+1} \circ d_X^p + d_Y^{p-1} \circ s^p.$$

Two morphisms $f, g : X \to Y$ are homotopic if $f - g$ is homotopic to zero.

(ii) An object X in $\mathrm{C}(\mathscr{C})$ is homotopic to 0 if id_X is homotopic to zero.

(iii) A morphism $f\colon X \to Y$ in $\mathrm{C}(\mathscr{C})$ is a homotopy equivalence if there exists $g\colon Y \to X$ such that $g \circ f$ is homotopic to id_X and $f \circ g$ is homotopic to id_Y.

A morphism homotopic to zero is visualized by the diagram (which is not commutative):



Note that an additive functor sends a morphism homotopic to zero to a morphism homotopic to zero.

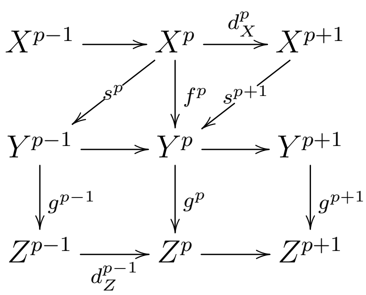
Example 3.4.2. (i) Let $X, Y \in \mathrm{C}(\mathscr{C})$. If both X and Y are homotopic to zero, then so is $X \oplus Y$.

(ii) Let $X \in \mathscr{C}$. Then the complex $0 \to X \xrightarrow{\mathrm{id}_X} X \to 0$ is homotopic to zero.

(iii) In particular, for $X', X'' \in \mathscr{C}$, the complex $0 \to X' \to X' \oplus X'' \to X'' \to 0$ is homotopic to zero.

Lemma 3.4.3. *If $f\colon X \to Y$ and $g\colon Y \to Z$ are two morphisms in $\mathrm{C}(\mathscr{C})$ and if f or g is homotopic to zero, then $g \circ f$ is homotopic to zero.*

Proof. Assume for example the f is homotopic to zero. In this case the proof is visualized by the diagram below.



Indeed, the equality $f^p = s^{p+1} \circ d_X^p + d_Y^{p-1} \circ s^p$ implies

$$g^p \circ f^p = g^p \circ s^{p+1} \circ d_X^p + d_Z^{p-1} \circ g^{p-1} \circ s^p.$$

□

We shall construct a new category by deciding that a morphism in $\mathrm{C}(\mathscr{C})$ homotopic to zero is isomorphic to the zero morphism. Set:

$$Ht(X,Y) = \{f\colon X \to Y; f \text{ is homotopic to } 0\}.$$

Lemma 3.4.3 allows us to state:

Definition 3.4.4. The homotopy category $\mathrm{K}(\mathscr{C})$ is defined by:

$$\mathrm{Ob}(\mathrm{K}(\mathscr{C})) = \mathrm{Ob}(\mathrm{C}(\mathscr{C}))$$
$$\mathrm{Hom}_{\mathrm{K}(\mathscr{C})}(X,Y) = \mathrm{Hom}_{\mathrm{C}(\mathscr{C})}(X,Y)/Ht(X,Y).$$

In other words, a morphism homotopic to zero in $\mathrm{C}(\mathscr{C})$ becomes the zero morphism in $\mathrm{K}(\mathscr{C})$ and a homotopy equivalence becomes an isomorphism.

One defines similarly $\mathrm{K}^*(\mathscr{C})$, $(* = \mathrm{ub}, \mathrm{b}, +, -)$. They are clearly additive categories endowed with an automorphism, the shift functor $[1]\colon X \mapsto X[1]$.

Proposition 3.4.5. *Let $\mathscr{C}$ be an additive category and let $X, Y \in \mathrm{C}(\mathscr{C})$. There are isomorphisms:*

$$Z^0(\mathrm{Hom}^\bullet_{\mathscr{C}}(X,Y)) := \ker d^0 \simeq \mathrm{Hom}_{\mathrm{C}(\mathscr{C})}(X,Y),$$
$$B^0(\mathrm{Hom}^\bullet_{\mathscr{C}}(X,Y)) := \mathrm{Im}\, d^{-1} \simeq Ht(X,Y),$$
$$H^0(\mathrm{Hom}^\bullet_{\mathscr{C}}(X,Y)) := \ker d^0 / \mathrm{Im}\, d^{-1} \simeq \mathrm{Hom}_{\mathrm{K}(\mathscr{C})}(X,Y).$$

Proof. (i) Let us calculate $Z^0(\mathrm{Hom}^\bullet_{\mathscr{C}}(X,Y))$. By (3.3.10), the component of $d^0\{f^j\}_j$ in $\mathrm{Hom}_{\mathscr{C}}(X^j, Y^{j+1})$ will be zero if and only if $d^j_Y \circ f^j = f^{j+1} \circ d^j_X$, that is, if the family $\{f^j\}_j$ defines a morphism of complexes.
(ii) Let us calculate $B^0(\mathrm{Hom}^\bullet_{\mathscr{C}}(X,Y))$. An element $f^j \in \mathrm{Hom}_{\mathscr{C}}(X^j, Y^j)$ will be in the image of d^{-1} if it is in the sum of the image of $\mathrm{Hom}_{\mathscr{C}}(X^j, Y^{j-1})$ by d^{j-1}_Y and the image of $\mathrm{Hom}_{\mathscr{C}}(X^{j+1}, Y^j)$ by d^j_X. Hence, if it can be written as $f^j = d^{j-1}_Y \circ s^j + s^{j+1} \circ d^j_X$.
(iii) The third isomorphism follows. □

Remark 3.4.6. The preceding constructions could be developed in the general setting of DG-categories. Roughly speaking, a DG-category is an additive category in which the morphisms are no more additive groups but are complexes of such groups.

The category $\mathrm{C}(\mathscr{C})$ endowed for each $X, Y \in \mathrm{C}(\mathscr{C})$ with the complex $\mathrm{Hom}^\bullet_{\mathscr{C}}(X,Y)$ and the composition being given by (3.3.11) is an example of such a DG-category. More more details on this subject, see for example [?Ke06, Yek20].

We shall come back to the category $\mathrm{K}(\mathscr{C})$ in § 5.3.

3.5 Simplicial constructions

We shall define the simplicial category and use it to construct complexes and homotopies in additive categories.

[1]§ 3.5 may be skipped.

Definition 3.5.1. (a) The simplicial category, denoted by $\mathbf{\Delta}$, is the category whose objects are the finite totally ordered sets and the morphisms are the order-preserving maps.

(b) We denote by $\mathbf{\Delta}_{inj}$ the subcategory of $\mathbf{\Delta}$ such that $\mathrm{Ob}(\mathbf{\Delta}_{inj}) = \mathrm{Ob}(\mathbf{\Delta})$, the morphisms being the injective order-preserving maps.

For integers n, m denote by $[n, m]$ the totally ordered set $\{k \in \mathbb{Z}; n \le k \le m\}$.

Proposition 3.5.2. (i) *the natural functor $\mathbf{\Delta} \to \mathbf{Set}^f$ is faithful,*

(ii) *the full subcategory of $\mathbf{\Delta}$ consisting of objects $\{[0, n]\}_{n \ge -1}$ is equivalent to $\mathbf{\Delta}$,*

(iii) *$\mathbf{\Delta}$ admits an initial object, namely $\varnothing$, and a terminal object, namely $\{0\}$.*

The proof is obvious.
Let us denote by

$$d_i^n \colon [0, n] \to [0, n+1] \qquad (0 \le i \le n+1)$$

the injective order-preserving map which does not take the value i. In other words

$$d_i^n(k) = \begin{cases} k & \text{for } k < i, \\ k+1 & \text{for } k \ge i. \end{cases}$$

One checks immediately that

$$d_j^{n+1} \circ d_i^n = d_i^{n+1} \circ d_{j-1}^n \text{ for } 0 \le i < j \le n+2. \tag{3.5.1}$$

Indeed, both morphisms are the unique injective order-preserving map which does not take the values i and j.

The category $\mathbf{\Delta}_{inj}$ is visualized by

$$\varnothing \xrightarrow{d_0^{-1}} [0] \underset{d_1^0}{\overset{d_0^0}{\rightrightarrows}} [0,1] \overset{d_0^1,\ d_1^1,\ d_2^1}{\rightrightarrows\!\!\rightarrow} [0,1,2] \cdots \tag{3.5.2}$$

Let $\mathscr{C}$ be an additive category and $F \colon \mathbf{\Delta}_{inj} \to \mathscr{C}$ a functor. We set for $n \in \mathbb{Z}$:

$$F^n = \begin{cases} F([0, n]) & \text{for } n \ge -1, \\ 0 & \text{otherwise,} \end{cases}$$

$$d_F^n \colon F^n \to F^{n+1}, \quad d_F^n = \sum_{i=0}^{n+1} (-)^i F(d_i^n).$$

Consider the differential object

$$F^\bullet := \cdots \to 0 \to F^{-1} \xrightarrow{d_F^{-1}} F^0 \xrightarrow{d_F^0} F^1 \to \cdots \to F^n \xrightarrow{d_F^n} \cdots. \tag{3.5.3}$$

Theorem 3.5.3. (i) *The differential object $F^\bullet$ is a complex.*

(ii) *Assume that there exist morphisms $s_F^n \colon F^n \to F^{n-1}$ $(n \ge 0)$satisfying:*

$$\begin{cases} s_F^{n+1} \circ F(d_0^n) = \mathrm{id}_{F^n} & \textit{for } n \ge -1, \\ s_F^{n+1} \circ F(d_{i+1}^n) = F(d_i^{n-1}) \circ s_F^n & \textit{for } i > 0, n \ge 0. \end{cases}$$

Then $F^\bullet$ is homotopic to zero.

Proof. (i) By (3.5.1), we have

$$\begin{aligned}
d_F^{n+1} \circ d_F^n &= \sum_{j=0}^{n+2} \sum_{i=0}^{n+1} (-)^{i+j} F(d_j^{n+1} \circ d_i^n) \\
&= \sum_{0 \le j \le i \le n+1} (-)^{i+j} F(d_j^{n+1} \circ d_i^n) + \sum_{0 \le i < j \le n+2} (-)^{i+j} F(d_j^{n+1} \circ d_i^n) \\
&= \sum_{0 \le j \le i \le n+1} (-)^{i+j} F(d_j^{n+1} \circ d_i^n) + \sum_{0 \le i < j \le n+2} (-)^{i+j} F(d_i^{n+1} \circ d_{j-1}^n) \\
&= 0\,.
\end{aligned}$$

Here, we have used

$$\begin{aligned}
\sum_{0 \le i < j \le n+2} (-)^{i+j} F(d_i^{n+1} \circ d_{j-1}^n) &= \sum_{0 \le i < j \le n+1} (-)^{i+j+1} F(d_i^{n+1} \circ d_j^n) \\
&= \sum_{0 \le j \le i \le n+1} (-)^{i+j+1} F(d_j^{n+1} \circ d_i^n).
\end{aligned}$$

(ii) We have

$$\begin{aligned}
& s_F^{n+1} \circ d_F^n + d_F^{n-1} \circ s^n \\
&\quad = \sum_{i=0}^{n+1} (-1)^i s_F^{n+1} \circ F(d_i^n) + \sum_{i=0}^{n} (-1)^i F(d_i^{n-1} \circ s_F^n) \\
&\quad = s_F^{n+1} \circ F(d_0^n) + \sum_{i=0}^{n} (-1)^{i+1} s_F^{n+1} \circ F(d_{i+1}^n) + \sum_{i=0}^{n} (-1)^i F(d_i^{n-1} \circ s_F^n) \\
&\quad = \mathrm{id}_{F^n} + \sum_{i=0}^{n} (-1)^{i+1} F(d_i^{n-1} \circ s_F^n) + \sum_{i=0}^{n} (-1)^i F(d_i^{n-1} \circ s_F^n) \\
&\quad = \mathrm{id}_{F^n}.
\end{aligned}$$

□

Exercises to Chapter 3

Exercise 3.1. Let $\mathscr{C}$ be an additive category and let $X \in \mathrm{C}(\mathscr{C})$ with differential d_X. Define the morphism $\delta_X \colon X \to X[1]$ by setting $\delta_X^n = (-1)^n d_X^n$. Prove that δ_X is a morphism in $\mathrm{C}(\mathscr{C})$ and is homotopic to zero.

Exercise 3.2. (see [KS06, Exe. 11.4].) Let $\mathscr{C}$ be an additive category, $f, g \colon X \rightrightarrows Y$ two morphisms in $\mathrm{C}(\mathscr{C})$. Prove that f and g are homotopic if and only if there exists a commutative diagram in $\mathrm{C}(\mathscr{C})$

$$\begin{array}{ccccc}
Y & \xrightarrow[\alpha(f)]{} & \mathrm{Mc}(f) & \xrightarrow[\beta(f)]{} & X[1] \\
\Vert & & \downarrow{\scriptstyle u} & & \Vert \\
Y & \xrightarrow[\alpha(g)]{} & \mathrm{Mc}(g) & \xrightarrow[\beta(g)]{} & X[1].
\end{array}$$

In such a case, prove that u is an isomorphism in $\mathrm{C}(\mathscr{C})$.

Exercise 3.3. (see [KS06, Exe. 11.6].) Let $\mathscr{C}$ be an additive category and let $f\colon X \to Y$ be a morphism in $\mathrm{C}(\mathscr{C})$.
Prove that the following conditions are equivalent:

(a) f is homotopic to zero,

(b) f factors through $\alpha(\mathrm{id}_X)\colon X \to \mathrm{Mc}(\mathrm{id}_X)$,

(c) f factors through $\beta(\mathrm{id}_Y)[-1]\colon \mathrm{Mc}(\mathrm{id}_Y)[-1] \to Y$,

(d) f decomposes as $X \to Z \to Y$ with Z a complex homotopic to zero.

Exercise 3.4. A category with translation $(\mathscr{A}, T)$ is a category $\mathscr{A}$ together with an equivalence $T\colon \mathscr{A} \to \mathscr{A}$. A differential object (X, d_X) in a category with translation $(\mathscr{A}, T)$ is an object $X \in \mathscr{A}$ together with a morphism $d_X\colon X \to T(X)$. A morphism $f\colon (X, d_X) \to (Y, d_Y)$ of differential objects is a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{d_X} & TX \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle T(f)} \\ Y & \xrightarrow{d_Y} & TY. \end{array}$$

One denotes by $\mathscr{A}_d$ the category consisting of differential objects and morphisms of such objects. If $\mathscr{A}$ is additive, one says that a differential object (X, d_X) in $(\mathscr{A}, T)$ is a complex if the composition $X \xrightarrow{d_X} T(X) \xrightarrow{T(d_X)} T^2(X)$ is zero. One denotes by $\mathscr{A}_c$ the full subcategory of $\mathscr{A}_d$ consisting of complexes.
(i) Let $\mathscr{C}$ be a category. Denote by $\mathbb{Z}_d$ the set $\mathbb{Z}$ considered as a discrete category and still denote by $\mathbb{Z}$ the ordered set $(\mathbb{Z}, \leq)$ considered as a category. Prove that $\mathrm{Fct}(\mathbb{Z}_d, \mathscr{C})$ is a category with translation.
(ii) Show that the category $\mathrm{Fct}(\mathbb{Z}, \mathscr{C})$ may be identified to the category of differential objects in $\mathrm{Fct}(\mathbb{Z}_d, \mathscr{C})$.
(iii) Let $\mathscr{C}$ be an additive category. Show that the notions of differential objects and complexes given above coincide with those in Definition 3.2.1 when choosing $\mathscr{A} = \mathrm{C}(\mathscr{C})$ and $T = [1]$.

Exercise 3.5. Consider the catgeory $\mathbf{\Delta}$ and for $n > 0$, denote by

$$s_i^n\colon [0,n] \to [0,n-1] \qquad (0 \leq i \leq n-1)$$

the surjective order-preserving map which takes the same value at i and $i+1$. In other words

$$s_i^n(k) = \begin{cases} k & \text{for } k \leq i, \\ k-1 & \text{for } k > i. \end{cases}$$

Checks the relations:

$$\begin{cases} s_j^n \circ s_i^{n+1} = s_{i-1}^n \circ s_j^{n+1} & \text{for } 0 \leq j < i \leq n, \\ s_j^{n+1} \circ d_i^n = d_i^{n-1} \circ s_{j-1}^n & \text{for } 0 \leq i < j \leq n, \\ s_j^{n+1} \circ d_i^n = \mathrm{id}_{[0,n]} & \text{for } 0 \leq i \leq n+1, i = j, j+1, \\ s_j^{n+1} \circ d_i^n = d_{i-1}^{n-1} \circ s_j^n & \text{for } 1 \leq j+1 < i \leq n+1. \end{cases}$$

Chapter 4

Abelian categories

The toy model of abelian categories is the category $\mathrm{Mod}(A)$ of modules over a ring A and for sake of simplicity, we shall most of the time argue as if we were working in a full abelian subcategory of a category $\mathrm{Mod}(A)$, which is not restrictive in view of a famous theorem of Fred&Mitchell [?Mi60, Fre64]. We explain the notions of exact sequences, give some basic lemmas such as “the five lemma” and “the snake lemma”, and study injective resolutions. We apply these results in constructing the derived functors of a left exact functor (or bifunctor), assuming that the category admits enough injectives. As an application we get the functors Ext and Tor. Finally, we study Koszul complexes and show how they naturally appear in Algebra and Analysis.
Some references: see [CE56, Gro57] for historical references and [Wei94, KS06] for an exposition. Here we shall often follow this last reference.

4.1 Abelian categories

Let $\mathscr{C}$ be an additive category which admits kernels and cokernels (recall Definition 2.2.1). Equivalently, $\mathscr{C}$ admits finite limits and colimits.

Let $f\colon X \to Y$ be a morphism in $\mathscr{C}$. We have already defined the image and co-image of f in Definition 2.4.4. Denote by $h\colon \ker f \to X$ and $k\colon Y \to \mathrm{Coker}\, f$ the natural morphisms.

Lemma 4.1.1. *One has the isomorphisms*

$$\mathrm{Coim}\, f \simeq \mathrm{Coker}\, h,\ \mathrm{Im}\, f \simeq \ker k.$$

Proof. Of course, it is enough to prove the first isomorphism. For $Z \in \mathscr{C}$, one has (see Diagram 2.2.6)

$$\mathrm{Hom}_{\mathscr{C}}(\mathrm{Coim}\, f, Z) = \{u\colon X \to Z; u \circ p_1 = u \circ p_2\},$$

where $p_1, p_2\colon X \times_Y X \to X$ are the two projections. Since $X \times_Y X$ is the kernel of $(f \circ p_1, f \circ p_2)\colon X \times X \rightrightarrows Y$, one also have

$$\mathrm{Hom}_{\mathscr{C}}(\mathrm{Coim}\, f, Z) = \{u\colon X \to Z; u \circ v_1 = u \circ v_2 \text{ for any } W \text{ and } (v_1, v_2)\colon W \rightrightarrows X \text{ such that } f \circ v_1 = f \circ v_2.\}$$

Equivalently,

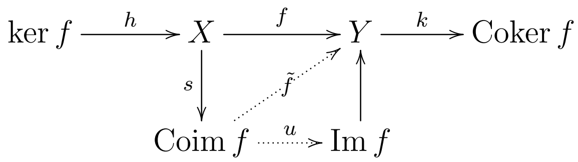
$$\begin{aligned}\mathrm{Hom}_{\mathscr{C}}(\mathrm{Coim}\, f, Z) = \{u \colon X \to Z; u \circ v = 0 \text{ for any } W \text{ and } v \colon W \to X \\ \text{such that} f \circ v = 0.\}\end{aligned}$$

Since such a v factorizes uniquely through h, we get

$$\begin{aligned}\mathrm{Hom}_{\mathscr{C}}(\mathrm{Coim}\, f, Z) &= \{u \colon X \to Z; u \circ h = 0\} \\ &\simeq \mathrm{Hom}_{\mathscr{C}}(\mathrm{Coker}\, h, Z).\end{aligned}$$

Since this isomorphism is functorial in Z (this point being left to the reader), we get the result by the Yoneda lemma. □

Consider the diagram:



Since $f \circ h = 0$, f factors uniquely through $\mathrm{Coim}\, f$, which defines $\tilde{f}$ (see Diagram 2.2.6) and thus $k \circ f$ factors through $k \circ \tilde{f}$. Since $k \circ f = k \circ \tilde{f} \circ s = 0$ and s is an epimorphism, we get that $k \circ \tilde{f} = 0$. Hence $\tilde{f}$ factors through $\ker k = \mathrm{Im}\, f$, which defines u (see Diagram 2.2.5). We have thus constructed a canonical morphism:

(4.1.1) $$\mathrm{Coim}\, f \xrightarrow{u} \mathrm{Im}\, f.$$

Examples 4.1.2. (i) For a ring A and a morphism f in $\mathrm{Mod}(A)$, (4.1.1) is an isomorphism.
(ii) The category **Ban** admits kernels and cokernels. If $f \colon X \to Y$ is a morphism of Banach spaces, define $\ker f = f^{-1}(0)$ and $\mathrm{Coker}\, f = Y/\overline{\mathrm{Im}\, f}$ where $\overline{\mathrm{Im}\, f}$ denotes the closure of the space $\mathrm{Im}\, f$. It is well-known that there exist continuous linear maps $f \colon X \to Y$ which are injective, with dense and non closed image. For such an f, $\ker f =$ $\mathrm{Coker}\, f$ $= 0$ although f is not an isomorphism. Thus $\mathrm{Coim}\, f \simeq X$ and $\mathrm{Im}\, f \simeq Y$. Hence, the morphism (4.1.1) is not an isomorphism.
(iii) Let A be a ring, I an ideal which is not finitely generated and let $M = A/I$. Then the natural morphism $A \to M$ in $\mathrm{Mod}^{\mathrm{f}}(A)$ has no kernel.

Definition 4.1.3. Let $\mathscr{C}$ be an additive category. One says that $\mathscr{C}$ is abelian if:

(i) any $f \colon X \to Y$ admits a kernel and a cokernel,

(ii) for any morphism f in $\mathscr{C}$, the natural morphism $\mathrm{Coim}\, f \to \mathrm{Im}\, f$ is an isomorphism.

Examples 4.1.4. (i) If A is a ring, $\mathrm{Mod}(A)$ is an abelian category. If A is noetherian, then $\mathrm{Mod}^{\mathrm{f}}(A)$ is abelian.
(ii) The category **Ban** admits kernels and cokernels but is not abelian. (See Examples 4.1.2 (ii).)
(iii) If $\mathscr{C}$ is abelian, then $\mathscr{C}^{\mathrm{op}}$ is abelian.

Proposition 4.1.5. *Let I be category and let $\mathscr{C}$ be an abelian category. Then the category $\mathrm{Fct}(I, \mathscr{C})$ of functors from I to $\mathscr{C}$ is abelian.*

Proof. (i) Let $F, G\colon I \to \mathscr{C}$ be two functors and $\varphi\colon F \to G$ a morphism of functors. Let us define a new functor H as follows. For $i \in I$, set $H(i) = \ker(F(i) \to G(i))$. Let $s\colon i \to j$ be a morphism in I. In order to define the morphism $H(s)\colon H(i) \to H(j)$, consider the diagram

$$\begin{array}{ccccc}
H(i) & \xrightarrow{h_i} & F(i) & \xrightarrow{\varphi(i)} & G(i) \\
{\scriptstyle H(s)}\downarrow & & {\scriptstyle F(s)}\downarrow & & \downarrow{\scriptstyle G(s)} \\
H(j) & \xrightarrow{h_j} & F(j) & \xrightarrow{\varphi(i)} & G(j).
\end{array}$$

Since $\varphi(j) \circ F(s) \circ h_i = 0$, the morphism $F(s) \circ h_i$ factorizes uniquely through $H(j)$. This gives $H(s)$. One checks immediately that for a morphism $t\colon j \to k$ in I, one has $H(t) \circ H(s) = H(t \circ s)$. Therefore H is a functor and one also easily cheks that H is a kernel of the morphism of functors φ.
(ii) One defines similarly the functor $\mathrm{Coim}\,\varphi$. Since, for each $i \in I$, the natural morphism $\mathrm{Coim}\,\varphi(i) \to \mathrm{Im}\,\varphi(i)$ is an isomorphism, one deduces that the natural morphism of functors $\mathrm{Coim}\,\varphi \to \mathrm{Im}\,\varphi$ is an isomorphism. □

Corollary 4.1.6. *If $\mathscr{C}$ is abelian, then the categories of complexes $\mathrm{C}^*(\mathscr{C})$ ($* = \mathrm{ub}, \mathrm{b}, +, -$) are abelian.*

Proof. It follows from Proposition 4.1.5 that the category $\mathrm{Diff}(\mathscr{C})$ of differential objects of $\mathscr{C}$ is abelian. One checks immediately that if $f^\bullet : X^\bullet \to Y^\bullet$ is a morphism of complexes, its kernel in the category $\mathrm{Diff}(\mathscr{C})$ is a complex and is a kernel in the category $\mathrm{C}(\mathscr{C})$, and similarly with cokernels. □

For example, if $f\colon X \to Y$ is a morphism in $\mathrm{C}(\mathscr{C})$, the complex Z defined by $Z^n = \ker(f^n\colon X^n \to Y^n)$, with differential induced by those of X, will be a kernel for f, and similarly for $\mathrm{Coker}\, f$.

Note the following results.

- An abelian category admits finite limits and finite colimits. (Indeed, an abelian category admits an initial object, a terminal object, finite products and finite coproducts and kernels and cokernels.)

- In an abelian category, a morphism f is a monomorphism (resp. an epimorphism) if and only if $\ker f \simeq 0$ (resp. $\mathrm{Coker}\, f \simeq 0$) (see Exercise 2.12). Moreover, a morphism $f\colon X \to Y$ is an isomorphism as soon as $\ker f \simeq 0$ and $\mathrm{Coker}\, f \simeq 0$. Indeed, in such a case, $X \xrightarrow{\sim} \mathrm{Coim}\, f$ and $\mathrm{Im}\, f \xrightarrow{\sim} Y$.

Unless otherwise specified, we assume until the end of this chapter that $\mathscr{C}$ is abelian.

Consider a complex $X' \xrightarrow{f} X \xrightarrow{g} X''$ (hence, $g \circ f = 0$). It defines a morphism $\mathrm{Coim}\, f \to \ker g$, hence, $\mathscr{C}$ being abelian, a morphism $\mathrm{Im}\, f \to \ker g$.

Definition 4.1.7. (i) One says that a complex $X' \xrightarrow{f} X \xrightarrow{g} X''$ is exact if $\mathrm{Im}\, f \xrightarrow{\sim} \ker g$.

(ii) More generally, a sequence of morphisms $X^p \xrightarrow{d^p} \cdots \to X^n$ with $d^{i+1} \circ d^i = 0$ for all $i \in [p, n-1]$ is exact if $\operatorname{Im} d^i \xrightarrow{\sim} \ker d^{i+1}$ for all $i \in [p, n-1]$.

(iii) A short exact sequence is an exact sequence $0 \to X' \to X \to X'' \to 0$

Any morphism $f\colon X \to Y$ may be decomposed into short exact sequences:

$$0 \to \ker f \to X \to \operatorname{Coim} f \to 0,$$
$$0 \to \operatorname{Im} f \to Y \to \operatorname{Coker} f \to 0,$$

with $\operatorname{Coim} f \simeq \operatorname{Im} f$.

Proposition 4.1.8. *Let*

$$0 \to X' \xrightarrow{f} X \xrightarrow{g} X'' \to 0 \qquad (4.1.2)$$

be a short exact sequence in $\mathscr{C}$. *Then the conditions* (a) *to* (e) *are equivalent.*

(a) *there exists* $h\colon X'' \to X$ *such that* $g \circ h = \operatorname{id}_{X''}$.

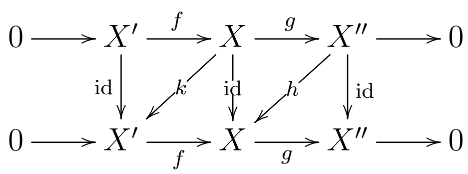
(b) *there exists* $k\colon X \to X'$ *such that* $k \circ f = \operatorname{id}_{X'}$.

(c) *there exists* $\varphi = (k, g)$ *and* $\psi = \begin{pmatrix} f \\ h \end{pmatrix}$ *such that* $X \xrightarrow{\varphi} X' \oplus X''$ *and* $X' \oplus X'' \xrightarrow{\psi} X$ *are isomorphisms inverse to each other.*

(d) *The complex* (4.1.2) *is homotopic to* 0.

(e) *The complex* (4.1.2) *is isomorphic to the complex* $0 \to X' \to X' \oplus X'' \to X'' \to 0$.

Proof. (a) ⇒ (c). Since $g = g \circ h \circ g$, we get $g \circ (\operatorname{id}_X - h \circ g) = 0$, which implies that $\operatorname{id}_X - h \circ g$ factors through $\ker g$, that is, through X'. Hence, there exists $k\colon X \to X'$ such that $\operatorname{id}_X - h \circ g = f \circ k$.
(b) ⇒ (c) follows by reversing the arrows.
(c) ⇒ (a). Since $g \circ f = 0$, we find $g = g \circ h \circ g$, that is $(g \circ h - \operatorname{id}_{X''}) \circ g = 0$. Since g is an epimorphism, this implies $g \circ h - \operatorname{id}_{X''} = 0$.
(c) ⇒ (b) follows by reversing the arrows.
(d) By definition, the complex (4.1.2) is homotopic to zero if and only if there exists a diagram



such that $\operatorname{id}_{X'} = k \circ f$, $\operatorname{id}_{X''} = g \circ h$ and $\operatorname{id}_X = h \circ g + f \circ k$.
(e) is obvious by (c). □

Definition 4.1.9. In the above situation, one says that the exact sequence splits.

Note that an additive functor of abelian categories sends split exact sequences into split exact sequences.

If A is a field, all exact sequences split, but this is not the case in general. For example, the exact sequence of $\mathbb{Z}$-modules

$$0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$$

does not split.

4.2 Exact functors

Definition 4.2.1. Let $F\colon \mathscr{C} \to \mathscr{C}'$ be a functor of abelian categories. One says that

(i) F is left exact if it commutes with finite limits,

(ii) F is right exact if it commutes with finite colimits,

(iii) F is exact if it is both left and right exact.

Lemma 4.2.2. *Consider an additive functor $F\colon \mathscr{C} \to \mathscr{C}'$.*

(a) *The conditions below are equivalent:*

(i) *F is left exact,*

(ii) *F commutes with kernels, that is, for any morphism $f\colon X \to Y$, $F(\ker(f)) \xrightarrow{\sim} \ker(F(f))$,*

(iii) *for any exact sequence $0 \to X' \to X \to X''$ in $\mathscr{C}$, the sequence $0 \to F(X') \to F(X) \to F(X'')$ is exact in $\mathscr{C}'$,*

(iv) *for any exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathscr{C}$, the sequence $0 \to F(X') \to F(X) \to F(X'')$ is exact in $\mathscr{C}'$.*

(b) *The conditions below are equivalent:*

(i) *F is exact,*

(ii) *for any exact sequence $X' \to X \to X''$ in $\mathscr{C}$, the sequence $F(X') \to F(X) \to F(X'')$ is exact in $\mathscr{C}'$,*

(iii) *for any exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathscr{C}$, the sequence $0 \to F(X') \to F(X) \to F(X'') \to 0$ is exact in $\mathscr{C}'$.*

There is a similar result to (a) for right exact functors.

Proof. Since F is additive, it commutes with terminal objects and products of two objects. Hence, by Proposition 2.3.9, F is left exact if and only if it commutes with kernels.
The proof of the other assertions are left as an exercise. □

Proposition 4.2.3. (i) *The functor $\mathrm{Hom}_{\mathscr{C}}\colon \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathrm{Mod}(\mathbb{Z})$ is left exact with respect to each of its arguments.*

(ii) *If a functor $F\colon \mathscr{C} \to \mathscr{C}'$ admits a left (resp. right) adjoint then F is left (resp. right) exact.*

(iii) *Let I be a small category. If $\mathscr{C}$ admits limits indexed by I, then the functor $\lim\colon \mathrm{Fct}(I^{\mathrm{op}}, \mathscr{C}) \to \mathscr{C}$ is left exact. Similarly, if $\mathscr{C}$ admits colimits indexed by I, then the functor $\mathrm{colim}\colon \mathrm{Fct}(I, \mathscr{C}) \to \mathscr{C}$ is right exact.*

(iv) *Let A be a ring and let I be a set. The two functors $\prod_{i\in I}$ and $\bigoplus_{i\in I}$ from $\mathrm{Fct}(I, \mathrm{Mod}(A))$ to $\mathrm{Mod}(A)$ are exact.*

(v) *Let A be a ring and let I be a small filtered category. The functor colim from $\mathrm{Fct}(I, \mathrm{Mod}(A))$ to $\mathrm{Mod}(A)$ is exact.*

Proof. (i) follows from (2.3.2) and (2.3.3).
(ii) Apply Proposition 2.5.5.
(iii) Apply Proposition 2.5.1.
(iv) is left as an exercise (see Exercise 4.1).
(v) follows from Corollary 2.6.7. □

Example 4.2.4. Let A be a ring and let N be a right A-module. Since the functor $N \otimes_A \bullet$ admits a right adjoint, it is right exact. Let us show that the functors $\mathrm{Hom}_A(\bullet, \bullet)$ and $N \otimes_A \bullet$ are not exact in general. In the sequel, we choose $A = \mathbf{k}[x]$, with $\mathbf{k}$ a field, and we consider the exact sequence of A-modules:

$$0 \to A \xrightarrow{\cdot x} A \to A/Ax \to 0, \tag{4.2.1}$$

where $\cdot x$ means multiplication by x.
(i) Apply the functor $\mathrm{Hom}_A(\bullet, A)$ to the exact sequence (4.2.1). We get the sequence:

$$0 \to \mathrm{Hom}_A(A/Ax, A) \to A \xrightarrow{x\cdot} A \to 0$$

which is not exact since $x\cdot$ is not surjective. On the other hand, since $x\cdot$ is injective and $\mathrm{Hom}_A(\bullet, A)$ is left exact, we find that $\mathrm{Hom}_A(A/Ax, A) = 0$.
(ii) Apply $\mathrm{Hom}_A(A/Ax, \bullet)$ to the exact sequence (4.2.1). We get the sequence:

$$0 \to \mathrm{Hom}_A(A/Ax, A) \to \mathrm{Hom}_A(A/Ax, A) \to \mathrm{Hom}_A(A/Ax, A/Ax) \to 0.$$

Since $\mathrm{Hom}_A(A/Ax, A) = 0$ and $\mathrm{Hom}_A(A/Ax, A/Ax) \neq 0$, this sequence is not exact.
(iii) Apply $\bullet \otimes_A A/Ax$ to the exact sequence (4.2.1). We get the sequence:

$$0 \to A/Ax \xrightarrow{x\cdot} A/Ax \to A/xA \otimes_A A/Ax \to 0.$$

Multiplication by x is 0 on A/Ax. Hence this sequence is the same as:

$$0 \to A/Ax \xrightarrow{0} A/Ax \to A/Ax \otimes_A A/Ax \to 0$$

which shows that $A/Ax \otimes_A A/Ax \simeq A/Ax$ and moreover that this sequence is not exact.
(iv) Notice that the functor $\mathrm{Hom}_A(\bullet, A)$ being additive, it sends split exact sequences to split exact sequences. This shows that (4.2.1) does not split.

Example 4.2.5. We shall show that the functor $\varprojlim : \mathrm{Fct}(I^{\mathrm{op}}, \mathrm{Mod}(\mathbf{k})) \to \mathrm{Mod}(\mathbf{k})$ is not right exact in general, even if $\mathbf{k}$ is a field.

Consider as above the $\mathbf{k}$-algebra $A := \mathbf{k}[x]$ over a field $\mathbf{k}$. Denote by $I = A \cdot x$ the ideal generated by x. Notice that $A/I^{n+1} \simeq \mathbf{k}[x]^{\leq n}$, where $\mathbf{k}[x]^{\leq n}$ denotes the $\mathbf{k}$-vector space consisting of polynomials of degree $\leq n$. For $p \leq n$ denote by $v_{pn} \colon A/I^n \twoheadrightarrow A/I^p$ the natural epimorphisms. They define a projective system of A-modules. One checks easily that

$$\varprojlim_n A/I^n \simeq \mathbf{k}[[x]],$$

the ring of formal series with coefficients in $\mathbf{k}$. On the other hand, for $p \leq n$ the monomorphisms $I^n \rightarrowtail I^p$ define a projective system of A-modules and one has

$$\varprojlim_n I^n \simeq 0.$$

Now consider the projective system of exact sequences of A-modules

$$0 \to I^n \to A \to A/I^n \to 0.$$

By taking the (projective) limit of these exact sequences one gets the sequence $0 \to 0 \to \mathbf{k}[x] \to \mathbf{k}[[x]] \to 0$ which is no more exact, neither in the category $\mathrm{Mod}(A)$ nor in the category $\mathrm{Mod}(\mathbf{k})$.

The Mittag-Leffler condition

Let us give a criterion in order that the limit of an exact sequence remains exact in the category $\mathrm{Mod}(A)$. This is a particular case of the so-called "Mittag-Leffler" condition (see [SGA III]).

Proposition 4.2.6. *Let A be a ring and let $0 \to \{M'_n\} \xrightarrow{f_n} \{M_n\} \xrightarrow{g_n} \{M''_n\} \to 0$ be an exact sequence of projective systems of A-modules indexed by $\mathbb{N}$. Assume that for each n, the map $M'_{n+1} \to M'_n$ is surjective. Then the sequence*

$$0 \to \varprojlim_n M'_n \xrightarrow{f} \varprojlim_n M_n \xrightarrow{g} \varprojlim_n M''_n \to 0$$

is exact.

Proof. Let us denote for short by v_p the morphisms $M_p \to M_{p-1}$ which define the projective system $\{M_p\}$, and similarly for v'_p, v''_p. Let $\{x''_p\}_p \in \varprojlim_n M''_n$. Hence $x''_p \in M''_p$, and $v''_p(x''_p) = x''_{p-1}$.

We shall first show that $v_n \colon g_n^{-1}(x''_n) \to g_{n-1}^{-1}(x''_{n-1})$ is surjective. Let $x_{n-1} \in g_{n-1}^{-1}(x''_{n-1})$. Take $x_n \in g_n^{-1}(x''_n)$. Then $g_{n-1}(v_n(x_n) - x_{n-1})) = 0$. Hence $v_n(x_n) - x_{n-1} = f_{n-1}(x'_{n-1})$. By the hypothesis $f_{n-1}(x'_{n-1}) = f_{n-1}(v'_n(x'_n))$ for some x'_n and thus $v_n(x_n - f_n(x'_n)) = x_{n-1}$.

Then we can choose $x_n \in g_n^{-1}(x''_n)$ inductively such that $v_n(x_n) = x_{n-1}$. □

4.3 Injective and projective objects

Definition 4.3.1. Let $\mathscr{C}$ be an abelian category.

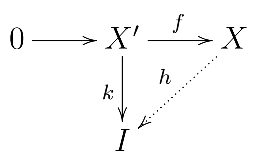
(i) An object I of $\mathscr{C}$ is injective if the functor $\mathrm{Hom}_{\mathscr{C}}(\bullet, I)$ is exact.

(ii) One says that $\mathscr{C}$ has enough injectives if for any $X \in \mathscr{C}$ there exists a monomorphism $X \rightarrowtail I$ with I injective.

(iii) An object P is projective in $\mathscr{C}$ if it is injective in $\mathscr{C}^{\mathrm{op}}$, *i.e.,* if the functor $\mathrm{Hom}_{\mathscr{C}}(P, \bullet)$ is exact.

(iv) One says that $\mathscr{C}$ has enough projectives if for any $X \in \mathscr{C}$ there exists an epimorphism $P \twoheadrightarrow X$ with P projective.

Proposition 4.3.2. *The object $I \in \mathscr{C}$ is injective if and only if, for any $X, Y \in \mathscr{C}$ and any diagram in which the row is exact:*



the dotted arrow may be completed, making the solid diagram commutative.

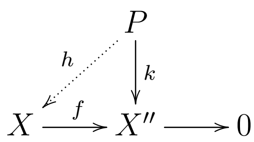
Proof. (i) Assume that I is injective and let X'' denote the cokernel of the morphism $X' \to X$. Applying the functor $\mathrm{Hom}_{\mathscr{C}}(\,\bullet\,, I)$ to the sequence $0 \to X' \to X \to X''$, one gets the exact sequence:

$$\mathrm{Hom}_{\mathscr{C}}(X'', I) \to \mathrm{Hom}_{\mathscr{C}}(X, I) \xrightarrow{\circ f} \mathrm{Hom}_{\mathscr{C}}(X', I) \to 0.$$

Thus there exists $h\colon X \to I$ such that $h \circ f = k$.
(ii) Conversely, consider an exact sequence $0 \to X' \xrightarrow{f} X \xrightarrow{g} X'' \to 0$. Then the sequence $0 \to \mathrm{Hom}_{\mathscr{C}}(X'', I) \xrightarrow{\circ h} \mathrm{Hom}_{\mathscr{C}}(X, I) \xrightarrow{\circ f} \mathrm{Hom}_{\mathscr{C}}(X', I) \to 0$ is exact by the hypothesis.

To conclude, apply Lemma 4.2.2. □

By reversing the arrows, we get that P is projective if and only if for any diagram in which the row is exact:



the dotted arrow may be completed, making the solid diagram commutative.

Lemma 4.3.3. *Let $0 \to X' \xrightarrow{f} X \xrightarrow{g} X'' \to 0$ be an exact sequence in $\mathscr{C}$, and assume that X' is injective. Then the sequence splits.*

Proof. Applying the preceding result with $k = \mathrm{id}_{X'}$, we find $h\colon X \to X'$ such that $k \circ f = \mathrm{id}_{X'}$. Then apply Proposition 4.1.8. □

It follows that if $F\colon \mathscr{C} \to \mathscr{C}'$ is an additive functor of abelian categories, and the hypotheses of the lemma are satisfied, then the sequence $0 \to F(X') \to F(X) \to F(X'') \to 0$ splits and in particular is exact.

Lemma 4.3.4. *Let X', X'' belong to $\mathscr{C}$. Then $X' \oplus X''$ is injective if and only if X' and X'' are injective.*

Proof. It is enough to remark that for two additive functors of abelian categories F and G, the functor $F \oplus G\colon X \mapsto F(X) \oplus G(X)$ is exact if and only if the functors F and G are exact. □

Applying Lemmas 4.3.3 and 4.3.4, we get:

Proposition 4.3.5. *Let $0 \to X' \to X \to X'' \to 0$ be an exact sequence in $\mathscr{C}$ and assume X' and X are injective. Then X'' is injective.*

Example 4.3.6. (i) Let A be a ring. An A-module M is free if it is isomorphic to a direct sum of copies of A, that is, $M \simeq A^{\oplus I}$ for some small set I. It follows from (2.1.4) and Proposition 4.2.3 (iv) that free modules are projective.

Let $M \in \mathrm{Mod}(A)$. For $m \in M$, denote by A_m a copy of A and denote by $1_m \in A_m$ the unit. Define the linear map

$$\psi\colon \bigoplus_{m \in M} A_m \to M$$

by setting $\psi(1_m) = m$ and extending by linearity. This map is clearly surjective. Since the left A-module $\bigoplus_{m\in M} A_m$ is free, it is projective. This shows that the category $\mathrm{Mod}(A)$ has enough projectives.

More generally, if there exists an A-module N such that $M \oplus N$ is free then M is projective (see Exercise 4.3).

One can prove that $\mathrm{Mod}(A)$ has enough injectives (see Exercise 4.4).
(ii) If $\mathbf{k}$ is a field, then any object of $\mathrm{Mod}(\mathbf{k})$ is both injective and projective.
(iii) Let A be a $\mathbf{k}$-algebra and let $M \in \mathrm{Mod}(A^{\mathrm{op}})$. One says that M is flat if the functor $M \otimes_A \bullet \colon \mathrm{Mod}(A) \to \mathrm{Mod}(\mathbf{k})$ is exact. Clearly, projective modules are flat.

4.4 Generators and Grothendieck categories

In this section it is essential to fix a universe $\mathscr{U}$. Hence, a category means a $\mathscr{U}$-category and small means $\mathscr{U}$-small.

Definition 4.4.1. Let $\mathscr{C}$ be a category. A system of generators in $\mathscr{C}$ is a family of objects $\{G_i\}_{i\in I}$ of $\mathscr{C}$ such that I is small and a morphism $f\colon X \to Y$ in $\mathscr{C}$ is an isomorphism as soon as $\mathrm{Hom}_{\mathscr{C}}(G_i, X) \to \mathrm{Hom}_{\mathscr{C}}(G_i, Y)$ is an isomorphism for all $i \in I$.

If the family contains a single element, say G, one says that G is a generator.

If $\{G_i\}_{i\in I}$ is a system of generators, then the functor $\prod_{i\in I} \mathrm{Hom}_{\mathscr{C}}(G_i, \bullet)\colon \mathscr{C} \to$ **Set** is conservative. If $\mathscr{C}$ is additive, these two conditions are equivalent [1]. Moreover, if $\mathscr{C}$ is additive, admits small coproducts and a system of generators as above, then it admits a generator, namely the object $\bigoplus_{i\in I} G_i$.

Lemma 4.4.2. *Let $\mathscr{C}$ be an abelian category which admits small coproducts and a generator G. Let $X \in \mathscr{C}$. Then there exists a small set I and an epimorphism $G^{\oplus I} \twoheadrightarrow X$.*

Proof. In this proof, we write $\mathrm{Hom}\,(Y, Z)$ instead of $\mathrm{Hom}_{\mathscr{C}}(Y, Z)$.

There is a natural isomorphism (see (1.1.3) and (1.1.5))

$$\mathrm{Hom}_{\mathbf{Set}}(\mathrm{Hom}\,(G, X), \mathrm{Hom}\,(G, X)) \simeq \mathrm{Hom}\,(G^{\oplus \mathrm{Hom}\,(G,X)}, X).$$

The identity of $\mathrm{Hom}\,(G, X)$ defines the natural morphism $G^{\oplus \mathrm{Hom}\,(G,X)} \to X$ which, to $(g, s) \in G \times \mathrm{Hom}\,(G, X)$, associates $s(g)$. This morphism defines the morphism

$$\mathrm{Hom}\,(G, G^{\oplus \mathrm{Hom}\,(G,X)}) \to \mathrm{Hom}\,(G, X).$$

This last morphism being obviously surjective, the result follows from Exercise 4.13. □

Definition 4.4.3. A Grothendieck category is an abelian category which admits small limits and small colimits, a generator and such that filtered small colimits are exact.

We shall not give the proof of the important Grothendieck's theorem below, referring to [KS06, Th. 9.6.2]. See [Gro57] for the original proof.

Theorem 4.4.4. *Let $\mathscr{C}$ be an abelian Grothendieck category. Then $\mathscr{C}$ admits enough injectives.*

[1] There was a mistake in [KS06, Def. 5.2.1], see the Errata on the webpage of the author.

4.5 Complexes in abelian categories

One still denotes by $\mathscr{C}$ an abelian category.

Cohomology

Recall that the categories $\mathrm{C}^*(\mathscr{C})$ are abelian for $* = \mathrm{ub}, +, -, \mathrm{b}$.

Let $X \in \mathrm{C}(\mathscr{C})$. One defines the following objects of $\mathscr{C}$:

$$\begin{aligned} Z^n(X) &:= \ker d_X^n \\ B^n(X) &:= \operatorname{Im} d_X^{n-1} \\ H^n(X) &:= Z^n(X)/B^n(X) \quad (:= \operatorname{Coker}(B^n(X) \to Z^n(X))) \end{aligned}$$

One calls $H^n(X)$ the n-th cohomology object of X. If $f\colon X \to Y$ is a morphism in $\mathrm{C}(\mathscr{C})$, then it induces morphisms $Z^n(X) \to Z^n(Y)$ and $B^n(X) \to B^n(Y)$, thus a morphism $H^n(f)\colon H^n(X) \to H^n(Y)$. Clearly, $H^n(X \oplus Y) \simeq H^n(X) \oplus H^n(Y)$. Hence we have obtained an additive functor:

$$H^n(\,\bullet\,) : \mathrm{C}(\mathscr{C}) \to \mathscr{C}.$$

Notice that $H^n(X) = H^0(X[n])$.

There are exact sequences

$$\begin{gathered} X^{n-1} \xrightarrow{d^{n-1}} \ker d_X^n \to H^n(X) \to 0, \\ 0 \to H^n(X) \to \operatorname{Coker} d_X^{n-1} \xrightarrow{d^n} X^{n+1}. \end{gathered}$$

The next result is easily checked.

Lemma 4.5.1. *For $n \in \mathbb{Z}$, the sequences below are exact:*

$$0 \to H^n(X) \to \operatorname{Coker}(d_X^{n-1}) \xrightarrow{d_X^n} \ker d_X^{n+1} \to H^{n+1}(X) \to 0. \tag{4.5.1}$$

One defines the truncation functors:

$$\begin{aligned} \tau^{\leq n}, \widetilde{\tau}^{\leq n} &: C(\mathscr{C}) \to C^-(\mathscr{C}) \\ \tau^{\geq n}, \widetilde{\tau}^{\geq n} &: C(\mathscr{C}) \to C^+(\mathscr{C}) \end{aligned} \tag{4.5.2}$$

as follows. Let $X := \cdots \to X^{n-1} \to X^n \to X^{n+1} \to \cdots$. One sets:

$$\begin{aligned} \tau^{\leq n}(X) &:= \cdots \to X^{n-2} \to X^{n-1} \to \ker d_X^n \to 0 \to \cdots \\ \widetilde{\tau}^{\leq n}(X) &:= \cdots \to X^{n-1} \to X^n \to \operatorname{Im} d_X^n \to 0 \to \cdots \\ \tau^{\geq n}(X) &:= \cdots \to 0 \to \operatorname{Coker} d_X^{n-1} \to X^{n+1} \to X^{n+2} \to \cdots \\ \widetilde{\tau}^{\geq n}(X) &:= \cdots \to 0 \to \operatorname{Im} d_X^{n-1} \to X^n \to X^{n+1} \to \cdots \end{aligned}$$

There is a chain of morphisms in $\mathrm{C}(\mathscr{C})$:

$$\tau^{\leq n} X \to \widetilde{\tau}^{\leq n} X \to X \to \widetilde{\tau}^{\geq n} X \to \tau^{\geq n} X,$$

and there are exact sequences in $\mathrm{C}(\mathscr{C})$:

$$\begin{cases} 0 \to \widetilde{\tau}^{\leq n-1} X \to \tau^{\leq n} X \to H^n(X)[-n] \to 0, \\ 0 \to H^n(X)[-n] \to \tau^{\geq n} X \to \widetilde{\tau}^{\geq n+1} X \to 0, \\ 0 \to \tau^{\leq n} X \to X \to \widetilde{\tau}^{\geq n+1} X \to 0, \\ 0 \to \widetilde{\tau}^{\leq n-1} X \to X \to \tau^{\geq n} X \to 0. \end{cases} \tag{4.5.3}$$

We have the isomorphisms

$$
\begin{aligned}
H^j(\tau^{\leq n}X) &\xrightarrow{\sim} H^j(\widetilde{\tau}^{\leq n}X) \simeq \begin{cases} H^j(X) & j \leq n, \\ 0 & j > n. \end{cases} \\
H^j(\widetilde{\tau}^{\geq n}X) &\xrightarrow{\sim} H^j(\tau^{\geq n}X) \simeq \begin{cases} H^j(X) & j \geq n, \\ 0 & j < n. \end{cases}
\end{aligned} \tag{4.5.4}
$$

The verification is straightforward.

Lemma 4.5.2. *Let $\mathscr{C}$ be an abelian category and let $f\colon X \to Y$ be a morphism in $\mathrm{C}(\mathscr{C})$ homotopic to zero. Then $H^n(f)\colon H^n(X) \to H^n(Y)$ is the 0 morphism.*

Proof. Let $f^n = s^{n+1} \circ d_X^n + d_Y^{n-1} \circ s^n$. Then $d_X^n = 0$ on $\ker d_X^n$ and $d_Y^{n-1} \circ s^n = 0$ on $\ker d_Y^n / \operatorname{Im} d_Y^{n-1}$. Hence $H^n(f)\colon \ker d_X^n / \operatorname{Im} d_X^{n-1} \to \ker d_Y^n / \operatorname{Im} d_Y^{n-1}$ is the zero morphism. □

In view of Lemma 4.5.2, the functor $H^0\colon \mathrm{C}(\mathscr{C}) \to \mathscr{C}$ extends as a functor

$$H^0\colon \mathrm{K}(\mathscr{C}) \to \mathscr{C}.$$

One shall be aware that the additive category $\mathrm{K}(\mathscr{C})$ is not abelian in general.

Definition 4.5.3. One says that a morphism $f\colon X \to Y$ in $\mathrm{C}(\mathscr{C})$ is a quasi-isomorphism (a qis, for short) if $H^k(f)$ is an isomorphism for all $k \in \mathbb{Z}$. In such a case, one says that X and Y are quasi-isomorphic. In particular, $X \in \mathrm{C}(\mathscr{C})$ is qis to 0 if and only if the complex X is exact.

Remark 4.5.4. By Lemma 4.5.2, a complex homotopic to 0 is qis to 0, but the converse is false. In particular, the property for a complex of being homotopic to 0 is preserved when applying an additive functor, contrarily to the property of being qis to 0.

Remark 4.5.5. Consider a bounded complex $X^\bullet$ and denote by $Y^\bullet$ the complex given by $Y^j = H^j(X^\bullet), d_Y^j \equiv 0$. One has:

$$Y^\bullet = \bigoplus_i H^i(X^\bullet)[-i]. \tag{4.5.5}$$

The complexes $X^\bullet$ and $Y^\bullet$ have the same cohomology objects. In other words, $H^j(Y^\bullet) \simeq H^j(X^\bullet)$. However, in general these isomorphisms are neither induced by a morphism from $X^\bullet \to Y^\bullet$, nor by a morphism from $Y^\bullet \to X^\bullet$, and the two complexes $X^\bullet$ and $Y^\bullet$ are not quasi-isomorphic.

Long exact sequence

Lemma 4.5.6. (The "five lemma".) *Consider a commutative diagram:*

$$
\begin{array}{ccccccc}
X^0 & \xrightarrow{\alpha_0} & X^1 & \xrightarrow{\alpha_1} & X^2 & \xrightarrow{\alpha_2} & X^3 \\
\downarrow{\scriptstyle f^0} & & \downarrow{\scriptstyle f^1} & & \downarrow{\scriptstyle f^2} & & \downarrow{\scriptstyle f^3} \\
Y^0 & \xrightarrow[\beta_0]{} & Y^1 & \xrightarrow[\beta_1]{} & Y^2 & \xrightarrow[\beta_2]{} & Y^3
\end{array}
$$

and assume that the rows are exact.

(i) *If f^0 is an epimorphism and f^1, f^3 are monomorphisms, then f^2 is a monomorphism.*

(ii) *If f^3 is a monomorphism and f^0, f^2 are epimorphisms, then f^1 is an epimorphism.*

As already mentioned in the introduction of this Chapter, there is a theorem of Fred and Mitchell [?Mi60, Fre64] which asserts that we may assume that $\mathscr{C}$ is a full abelian subcategory of $\mathrm{Mod}(A)$ for some ring A, what we will do here. Hence we may choose elements in the objects of $\mathscr{C}$.

Proof. (i) Let $x_2 \in X_2$ and assume that $f^2(x_2) = 0$. Then $f^3 \circ \alpha_2(x_2) = 0$ and f^3 being a monomorphism, this implies $\alpha_2(x_2) = 0$. Since the first row is exact, there exists $x_1 \in X_1$ such that $\alpha_1(x_1) = x_2$. Set $y_1 = f^1(x_1)$. Since $\beta_1 \circ f^1(x_1) = 0$ and the second row is exact, there exists $y_0 \in Y^0$ such that $\beta_0(y_0) = f^1(x_1)$. Since f^0 is an epimorphism, there exists $x_0 \in X^0$ such that $y_0 = f^0(x_0)$. Since $f^1 \circ \alpha_0(x_0) = f^1(x_1)$ and f^1 is a monomorphism, $\alpha_0(x_0) = x_1$. Therefore, $x_2 = \alpha_1(x_1) = 0$.
(ii) is nothing but (i) in $\mathscr{C}^{\mathrm{op}}$. □

Lemma 4.5.7. (The snake lemma.) *Consider the commutative diagram in $\mathscr{C}$ below with exact rows:*

$$\begin{array}{ccccccc} & & X' & \xrightarrow{f} & X & \xrightarrow{g} & X'' & \longrightarrow 0 \\ & & {\scriptstyle\alpha}\downarrow & & {\scriptstyle\beta}\downarrow & & {\scriptstyle\gamma}\downarrow & \\ 0 & \longrightarrow & Y' & \xrightarrow{f'} & Y & \xrightarrow{g'} & Y'' & \end{array}$$

Then there exists a morphism $\delta\colon \ker\gamma \to \mathrm{Coker}\,\alpha$ giving rise to an exact sequence:

$$\ker\alpha \to \ker\beta \to \ker\gamma \xrightarrow{\delta} \mathrm{Coker}\,\alpha \to \mathrm{Coker}\,\beta \to \mathrm{Coker}\,\gamma. \tag{4.5.6}$$

Proof. here again, we shall assume that $\mathscr{C}$ is a full abelian subcategory of $\mathrm{Mod}(A)$ for some ring A.
(i) Let us first prove that the sequence $\ker\alpha \to \ker\beta \to \ker\gamma$ is exact. Let $x \in \ker\beta$ with $g(x) = 0$. Using the fact that the first row is exact, there exists $x' \in X'$ with $f(x') = x$. Then $f' \circ \alpha(x') = \beta \circ f(x') = 0$. Since f' is a monomorphism, $\alpha(x') = 0$ and $x' \in \ker\alpha$.
(ii) The sequence $\mathrm{Coker}\,\alpha \to \mathrm{Coker}\,\beta \to \mathrm{Coker}\,\gamma$ is exact. If one works in the abstract setting of abelian categories, this follows from (i) by reversing the arrows. Otherwise, if one wishes to remain in the setting of A-modules, one can adapt the proof of (i)[2].
(iii) Let us construct the map δ making the sequence exact. Let $x'' \in \ker\gamma$ and choose $x \in X$ with $g(x) = x''$. Set $y = \beta(y)$. Since $g'(y) = 0$, there exists $y' \in Y'$ with $f'(y') = y$. One defines $\delta(x'')$ as the image of y' in $\mathrm{Coker}\,\alpha$, that is, in $Y'/\mathrm{Im}\,\alpha$.
The reader will check that the map δ is well-defined (i.e., the construction does not depend on the various choices) and that the sequence (4.5.6) is exact. □

One shall be aware that the morphism δ is not unique. Replacing δ with $-\delta$ does not change the conclusion.

[2]The reader shall be aware that the opposite of an abelian category is still abelian, but the category $\mathrm{Mod}(A)$ is not equivalent to the opposite category $\mathrm{Mod}(A)^{\mathrm{op}}$.

Theorem 4.5.8. *Let $0 \to X' \xrightarrow{f} X \xrightarrow{g} X'' \to 0$ be an exact sequence in $\mathrm{C}(\mathscr{C})$.*

(i) *For each $k \in \mathbb{Z}$, the sequence $H^k(X') \to H^k(X) \to H^k(X'')$ is exact.*

(ii) *For each $k \in \mathbb{Z}$, there exists $\delta^k : H^k(X'') \to H^{k+1}(X')$ making the long sequence*

$$(4.5.7) \qquad \cdots \to H^k(X) \to H^k(X'') \xrightarrow{\delta^k} H^{k+1}(X') \to H^{k+1}(X) \to \cdots$$

exact. Moreover, one can construct δ^k functorial with respect to short exact sequences of $\mathrm{C}(\mathscr{C})$.

Proof. Consider the commutative diagrams:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & H^k(X') & & H^k(X) & & H^k(X'') & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \operatorname{Coker} d^{k-1}_{X'} & \xrightarrow{f} & \operatorname{Coker} d^{k-1}_{X} & \xrightarrow{g} & \operatorname{Coker} d^{k-1}_{X''} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d^k_{X'}} & & \downarrow{\scriptstyle d^k_{X}} & & \downarrow{\scriptstyle d^k_{X''}} & & \\
0 & \longrightarrow & \ker d^{k+1}_{X'} & \xrightarrow{f} & \ker d^{k+1}_{X} & \xrightarrow{g} & \ker d^{k+1}_{X''} & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & H^{k+1}(X') & & H^{k+1}(X) & & H^{k+1}(X'') & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The columns are exact by Lemma 4.5.1 and the rows are exact by the hypothesis. Hence, the result follows from Lemma 4.5.7. □

Corollary 4.5.9. *Consider a morphism $f \colon X \to Y$ in $\mathrm{C}(\mathscr{C})$ and recall that $\mathrm{Mc}(f)$ denotes the mapping cone of f. There is a long exact sequence:*

$$(4.5.8) \qquad \cdots \to H^{k-1}(\mathrm{Mc}(f)) \to H^k(X) \xrightarrow{f} H^k(Y) \to H^k(\mathrm{Mc}(f)) \to \cdots .$$

Proof. Using (3.2.2), we get a complex:

$$(4.5.9) \qquad 0 \to Y \to \mathrm{Mc}(f) \to X[1] \to 0.$$

Clearly, this complex is exact. Indeed, in degree n, it gives the split exact sequence $0 \to Y^n \to Y^n \oplus X^{n+1} \to X^{n+1} \to 0$. Applying Theorem 4.5.8, we find a long exact sequence

$$(4.5.10) \cdots \to H^{k-1}(\mathrm{Mc}(f)) \to H^{k-1}(X[1]) \xrightarrow{\delta^{k-1}} H^k(Y) \to H^k(\mathrm{Mc}(f)) \to \cdots .$$

It remains to check that, up to a sign, the morphism $\delta^{k-1} \colon H^k(X) \to H^k(Y)$ is $H^k(f)$. We shall not give the proof here. □

One shall be aware that the fact that the exact sequences $0 \to Y^n \to Y^n \oplus X^{n+1} \to X^{n+1} \to 0$ split does not imply that the exact sequence of complexes (4.5.9) splits.

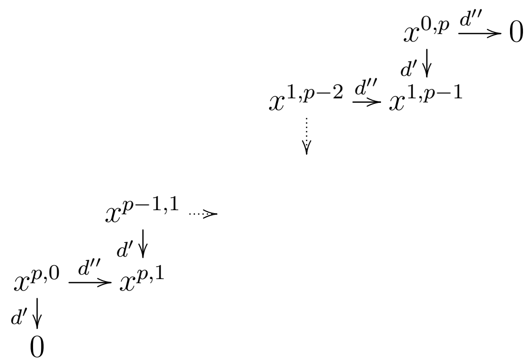
Double complexes

Consider a double complex $X^{\bullet,\bullet}$ as in (3.3.2).

Theorem 4.5.10. *Let $X^{\bullet,\bullet}$ be a double complex. Assume that all rows $X^{j,\bullet}$ and columns $X^{\bullet,j}$ are 0 for $j < 0$ and are exact for $j > 0$. Then $H^p(X^{0,\bullet}) \simeq H^p(X^{\bullet,0})$ for all p.*

Proof. [3] We shall only describe the first isomorphism $H^p(X^{0,\bullet}) \simeq H^p(X^{\bullet,0})$ in the case where $\mathscr{C} = \mathrm{Mod}(A)$, by the so-called "Weil procedure". Let $x^{p,0} \in X^{p,0}$, with $d'x^{p,0} = 0$ which represents $y \in H^p(X^{\bullet,0})$. Define $x^{p,1} = d''x^{p,0}$. Then $d'x^{p,1} = 0$, and the first column being exact, there exists $x^{p-1,1} \in X^{p-1,1}$ with $d'x^{p-1,1} = x^{p,1}$. One can iterate this procedure until getting $x^{0,p} \in X^{0,p}$. Since $d'd''x^{0,p} = 0$, and d' is injective on $X^{0,p}$ for $p > 0$ by the hypothesis, we get $d''x^{0,p} = 0$. The class of $x^{0,p}$ in $H^p(X^{0,\bullet})$ will be the image of y by the Weil procedure. Of course, one has to check that this image does not depend of the various choices we have made, and that it induces an isomorphism.

This can be visualized by the diagram:



$\square$

4.6 Resolutions

Solving linear equations

The aim of this subsection is to illustrate and motivate the constructions which will appear further. In this subsection, we work in the category $\mathrm{Mod}(A)$ for a $\mathbf{k}$-algebra A. Recall that the category $\mathrm{Mod}(A)$ admits enough projectives.

Suppose one is interested in studying a system of linear equations

$$\sum_{j=1}^{N_0} p_{ij}u_j = v_i, \quad (i = 1, \dots, N_1) \tag{4.6.1}$$

where the p_{ij}'s belong to the ring A and u_j, v_i belong to some left A-module S. Using matrix notations, one can write equations (4.6.1) as

$$P_0 u = v \tag{4.6.2}$$

[3]Several proofs of this classical result invoque "spectral sequences", a complicated tool which will never appear in this book.

where P_0 is the matrix (p_{ij}) with N_1 rows and N_0 columns, defining the A-linear map $P_0\cdot : S^{N_0} \to S^{N_1}$. Now consider the right A-linear map

$$\cdot P_0 : A^{N_1} \to A^{N_0}, \tag{4.6.3}$$

where $\cdot P_0$ operates on the right and the elements of A^{N_0} and A^{N_1} are written as rows. Let $(e_1, \dots, e_{N_0})$ and $(f_1, \dots, f_{N_1})$ denote the canonical basis of A^{N_0} and A^{N_1}, respectively. One gets:

$$f_i \cdot P_0 = \sum_{j=1}^{N_0} p_{ij} e_j, \quad (i = 1, \dots, N_1). \tag{4.6.4}$$

Hence $\operatorname{Im} P_0$ is generated by the elements $\sum_{j=1}^{N_0} p_{ij} e_j$ for $i = 1, \dots, N_1$. Denote by M the quotient module $A^{N_0}/A^{N_1} \cdot P_0$ and by $\psi : A^{N_0} \to M$ the natural A-linear map. Let $(u_1, \dots, u_{N_0})$ denote the images by ψ of $(e_1, \dots, e_{N_0})$. Then M is a left A-module with generators $(u_1, \dots, u_{N_0})$ and relations $\sum_{j=1}^{N_0} p_{ij} u_j = 0$ for $i = 1, \dots, N_1$. By construction, we have an exact sequence of left A-modules:

$$A^{N_1} \xrightarrow{\cdot P_0} A^{N_0} \xrightarrow{\psi} M \to 0. \tag{4.6.5}$$

Applying the left exact functor $\operatorname{Hom}_A(\bullet, S)$ to this sequence, we find the exact sequence of $\mathbf{k}$-modules:

$$0 \to \operatorname{Hom}_A(M, S) \to S^{N_0} \xrightarrow{P_0 \cdot} S^{N_1} \tag{4.6.6}$$

(where $P_0\cdot$ operates on the left). Hence, the $\mathbf{k}$-module of solutions of the homogeneous equation associated to (4.6.1) is described by $\operatorname{Hom}_A(M, S)$.

Assume now that A is left Noetherian, that is, any submodule of a free A-module of finite rank is of finite type. In this case, arguing as in the proof of Proposition 4.6.2 below, we construct an exact sequence

$$\cdots \to A^{N_2} \xrightarrow{\cdot P_1} A^{N_1} \xrightarrow{\cdot P_0} A^{N_0} \xrightarrow{\psi} M \to 0.$$

In other words, we have a projective resolution $L^\bullet \to M$ of M by finite free left A-modules:

$$L^\bullet : \cdots \to L^n \to L^{n-1} \to \cdots \to L^0 \to 0.$$

Applying the left exact functor $\operatorname{Hom}_A(\bullet, S)$ to $L^\bullet$, we find the complex of $\mathbf{k}$-modules:

$$0 \to S^{N_0} \xrightarrow{P_0 \cdot} S^{N_1} \xrightarrow{P_1 \cdot} S^{N_2} \to \cdots . \tag{4.6.7}$$

Then

$$\begin{cases} H^0(\operatorname{Hom}_A(L^\bullet, S)) \simeq \ker P_0, \\ H^1(\operatorname{Hom}_A(L^\bullet, S)) \simeq \ker(P_1)/\operatorname{Im}(P_0). \end{cases}$$

Hence, a necessary condition to solve the equation $P_0 u = v$ is that $P_1 v = 0$ and this necessary condition is sufficient if $H^1(\operatorname{Hom}_A(L^\bullet, S)) \simeq 0$. As we shall see in § 4.7, the cohomology groups $H^j(\operatorname{Hom}_A(L^\bullet, S))$ do not depend, up to isomorphisms, of the choice of the projective resolution $L^\bullet$ of M and are denoted $\operatorname{Ext}^j_A(M, S)$.

Resolutions

Definition 4.6.1. Let $\mathscr{J}$ be a full additive subcategory of $\mathscr{C}$. We say that $\mathscr{J}$ is generating[4] if for all X in $\mathscr{C}$, there exist $Y \in \mathscr{J}$ and a monomorphism $X \rightarrowtail Y$.

If $\mathscr{J}$ is generating in $\mathscr{C}^{\mathrm{op}}$, one says that $\mathscr{J}$ is cogenerating in $\mathscr{C}$.

Proposition 4.6.2. *Let $\mathscr{C}$ be an abelian category and let $\mathscr{J}$ be a generating full additive subcategory. Then, for any $X \in \mathscr{C}$, there exists an exact sequence*

$$0 \to X \to J^0 \to \cdots \to J^n \to \cdots \tag{4.6.8}$$

with $J^n \in \mathscr{J}$ for all $n \geq 0$.

Proof. We proceed by induction. Assume to have constructed:

$$0 \to X \to J^0 \to \cdots \to J^n.$$

For $n = 0$ this is the hypothesis. Set $B^n = \mathrm{Coker}(J^{n-1} \to J^n)$ (with $J^{-1} = X$). Then $J^{n-1} \to J^n \to B^n \to 0$ is exact. Embed B^n in an object of $\mathscr{J}$: $0 \to B^n \to J^{n+1}$. Then $J^{n-1} \to J^n \to J^{n+1}$ is exact, and the induction proceeds. □

The sequence

$$J^\bullet := 0 \to J^0 \to \cdots \to J^n \to \cdots \tag{4.6.9}$$

is called a right $\mathscr{J}$-resolution of X. If $\mathscr{J}$ is the category of injective objects in $\mathscr{C}$, one says that $J^\bullet$ is an injective resolution. Note that, identifying X and $J^\bullet$ to objects of $\mathrm{C}^+(\mathscr{C})$,

$$X \to J^\bullet \text{ is a qis.} \tag{4.6.10}$$

Of course, there is a similar result for left resolution. If for any $X \in \mathscr{C}$ there is an exact sequence $Y \to X \to 0$ with $Y \in \mathscr{J}$, then one can construct a left $\mathscr{J}$-resolution of X, that is, a qis $Y^\bullet \to X$, where the Y^n's belong to $\mathscr{J}$. If $\mathscr{J}$ is the category of projective objects of $\mathscr{C}$, one says that $Y^\bullet$ is a projective resolution.

Proposition 4.6.2 is a particular case of the next result whose proof is left as an exercise. (A detailed proof can be found in [KS06, Lem. 13.2.1].)

Proposition 4.6.3. *Assume $\mathscr{J}$ is generating. Then for any $a \in \mathbb{Z}$ and $X^\bullet \in \mathrm{C}^{\geq a}(\mathscr{C})$, there exist $Y^\bullet \in \mathrm{C}^{\geq a}(\mathscr{J})$ and a quasi-isomorphism $X^\bullet \to Y^\bullet$.*

Injective resolutions

In this section, $\mathscr{C}$ denotes an abelian category and $\mathscr{I}_{\mathscr{C}}$ its full additive subcategory consisting of injective objects. We shall asume

(4.6.11) the abelian category $\mathscr{C}$ admits enough injectives.

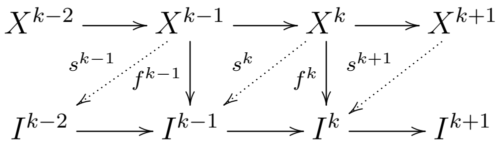
In other words, the category $\mathscr{I}_{\mathscr{C}}$ is generating.

[4] In some texts, such as [KS06, Def. 8.3.21], the words "generating" and "cogenerating" are inverted.

Proposition 4.6.4. (i) *Let $f^\bullet : X^\bullet \to I^\bullet$ be a morphism in $\mathrm{C}^+(\mathscr{C})$. Assume $I^\bullet$ belongs to $\mathscr{C}^+(\mathscr{I}_\mathscr{C})$ and $X^\bullet$ is exact. Then $f^\bullet$ is homotopic to* 0.

(ii) *Let $I^\bullet \in \mathrm{C}^+(\mathscr{I}_\mathscr{C})$ and assume $I^\bullet$ is exact. Then $I^\bullet$ is homotopic to* 0.

Proof. (i) Consider the solid diagram:



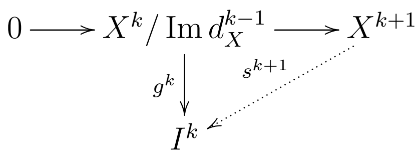
We shall construct by induction morphisms s^k satisfying:

$$f^k = s^{k+1} \circ d_X^k + d_I^{k-1} \circ s^k.$$

For $j << 0$, $s^j = 0$. Assume we have constructed the s^j for $j \le k$. Define $g^k = f^k - d_I^{k-1} \circ s^k$. One has

$$\begin{aligned} g^k \circ d_X^{k-1} &= f^k \circ d_X^{k-1} - d_I^{k-1} \circ s^k \circ d_X^{k-1} \\ &= f^k \circ d_X^{k-1} - d_I^{k-1} \circ f^{k-1} + d_I^{k-1} \circ d_I^{k-2} \circ s^{k-1} \\ &= 0. \end{aligned}$$

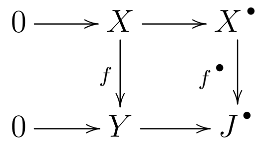
Hence, g^k factorizes through $X^k / \operatorname{Im} d_X^{k-1}$. Since the complex $X^\bullet$ is exact, the sequence $0 \to X^k / \operatorname{Im} d_X^{k-1} \to X^{k+1}$ is exact. Consider



The dotted arrow may be completed by Proposition 4.3.2.
(ii) Apply the result of (i) with $X^\bullet = I^\bullet$ and $f = \mathrm{id}_X$. □

Proposition 4.6.5. (i) *Let $f\colon X \to Y$ be a morphism in $\mathscr{C}$, let $0 \to X \to X^\bullet$ be a resolution of X and let $0 \to Y \to J^\bullet$ be a complex with the J^k's injective. Then there exists a morphism $f^\bullet\colon X^\bullet \to J^\bullet$ making the diagram below commutative:*

(4.6.12)



(ii) *The morphism $f^\bullet$ in $\mathrm{C}(\mathscr{C})$ constructed in* (i) *is unique up to homotopy.*

Proof. (i) Let us denote by d_X (resp. d_Y) the differential of the complex $X^\bullet$ (resp. $J^\bullet$), by d_X^{-1} (resp. d_Y^{-1}) the morphism $X \to X^0$ (resp. $Y \to J^0$) and set $f^{-1} = f$.

We shall construct the f^n's by induction. Morphism f^0 is obtained by Proposition 4.3.2. Assume we have constructed $f^0, \dots, f^n$. Let $g^n = d_Y^n \circ f^n\colon X^n \to J^{n+1}$. The morphism g^n factorizes through $h^n\colon X^n / \operatorname{Im} d_X^{n-1} \to J^{n+1}$. Since $X^\bullet$ is exact, the sequence $0 \to X^n / \operatorname{Im} d_X^{n-1} \to X^{n+1}$ is exact. Since J^{n+1} is injective, h^n extends as $f^{n+1}\colon X^{n+1} \to J^{n+1}$.
(ii) We may assume $f = 0$ and we have to prove that in this case $f^\bullet$ is homotopic to zero. Since the sequence $0 \to X \to X^\bullet$ is exact, this follows from Proposition 4.6.4 (i), replacing the exact sequence $0 \to Y \to J^\bullet$ by the complex $0 \to 0 \to J^\bullet$. □

4.7 Derived functors

Let $\mathscr{C}$ be an abelian category satisfying (4.6.11). Recall that $\mathscr{I}_{\mathscr{C}}$ denotes the full additive subcategory of consisting of injective objects in $\mathscr{C}$. We look at the additive category $\mathrm{K}(\mathscr{I}_{\mathscr{C}})$ as a full additive subcategory of the abelian category $\mathrm{K}(\mathscr{C})$.

Theorem 4.7.1. *Assuming (4.6.11), there exists a functor $\lambda\colon \mathscr{C} \to \mathrm{K}(\mathscr{I}_{\mathscr{C}})$ and for each $X \in \mathscr{C}$, a qis $X \to \lambda(X)$, functorially in $X \in \mathscr{C}$.*

Proof. (i) Let $X \in \mathscr{C}$ and let $I_X^\bullet \in \mathrm{C}^+(\mathscr{I}_{\mathscr{C}})$ be an injective resolution of X. The image of $I_X^\bullet$ in $\mathrm{K}^+(\mathscr{C})$ is unique up to unique isomorphism, by Proposition 4.6.5.

Indeed, consider two injective resolutions $I_X^\bullet$ and $J_X^\bullet$ of X. By Proposition 4.6.5 applied to id_X, there exists a morphism $f^\bullet : I_X^\bullet \to J_X^\bullet$ making the diagram (4.6.12) commutative and this morphism is unique up to homotopy, hence is unique in $\mathrm{K}^+(\mathscr{C})$. Similarly, there exists a unique morphism $g^\bullet : J_X^\bullet \to I_X^\bullet$ in $\mathrm{K}^+(\mathscr{C})$. Hence, $f^\bullet$ and $g^\bullet$ are isomorphisms inverse one to each other.
(ii) Let $f\colon X \to Y$ be a morphism in $\mathscr{C}$, let $I_X^\bullet$ and $I_Y^\bullet$ be injective resolutions of X and Y respectively, and let $f^\bullet : I_X^\bullet \to I_Y^\bullet$ be a morphism of complexes such as in Proposition 4.6.5. Then the image of $f^\bullet$ in $\mathrm{Hom}_{\mathrm{K}^+(\mathscr{I}_{\mathscr{C}})}(I_X^\bullet, I_Y^\bullet)$ does not depend on the choice of $f^\bullet$ by Proposition 4.6.5.

In particular, we get that if $g\colon Y \to Z$ is another morphism in $\mathscr{C}$ and $I_Z^\bullet$ is an injective resolutions of Z, then $g^\bullet \circ f^\bullet = (g \circ f)^\bullet$ as morphisms in $\mathrm{K}^+(\mathscr{I}_{\mathscr{C}})$. □

Let $F\colon \mathscr{C} \to \mathscr{C}'$ be a left exact functor of abelian categories and recall that $\mathscr{C}$ satisfies (4.6.11). Consider the functors

$$\mathscr{C} \xrightarrow{\lambda} \mathrm{K}^+(\mathscr{I}_{\mathscr{C}}) \xrightarrow{F} \mathrm{K}^+(\mathscr{C}') \xrightarrow{H^n} \mathscr{C}'.$$

Definition 4.7.2. One sets

$$R^nF = H^n \circ F \circ \lambda \tag{4.7.1}$$

and calls R^nF the n-th right derived functor of F.

By its definition, the recipe to construct $R^nF(X)$ is as follows:

- choose an injective resolution $I_X^\bullet$ of X, that is, construct an exact sequence $0 \to X \to I_X^\bullet$ with $I_X^\bullet \in \mathrm{C}^+(\mathscr{I}_{\mathscr{C}})$,
- apply F to this resolution,
- take the n-th cohomology.

In other words, $R^nF(X) \simeq H^n(F(I_X^\bullet))$. Note that

- R^nF is an additive functor from $\mathscr{C}$ to $\mathscr{C}'$,
- $R^nF(X) \simeq 0$ for $n < 0$ since $I_X^j = 0$ for $j < 0$,
- $R^0F(X) \simeq F(X)$ since F being left exact, it commutes with kernels,
- $R^nF(X) \simeq 0$ for $n \neq 0$ if F is exact,
- $R^nF(X) \simeq 0$ for $n \neq 0$ if X is injective, by the construction of $R^nF(X)$.

Definition 4.7.3. Let $F\colon \mathscr{C} \to \mathscr{C}'$ be a left exact functor of abelian categories assume that $\mathscr{C}$ admits enough injectives.

(a) One says that F has cohomological dimension $\leq d$ with $d \in \mathbb{N}$ if $R^jF \simeq 0$ for $j > d$. If such an integer d exists, one says that F has finite cohomological

(b) An object X of $\mathscr{C}$ such that $R^kF(X) \simeq 0$ for all $k > 0$ is called F-acyclic.

Hence, injective objects are F-acyclic for all left exact functors F.

Theorem 4.7.4. *Let $0 \to X' \xrightarrow{f} X \xrightarrow{g} X'' \to 0$ be an exact sequence in $\mathscr{C}$. Then there exists a long exact sequence:*

$$0 \to F(X') \to F(X) \to \cdots \to R^kF(X') \to R^kF(X) \to R^kF(X'') \to \cdots .$$

Sketch of the proof. One constructs an exact sequence of complexes $0 \to X'^\bullet \to X^\bullet \to X''^\bullet \to 0$ whose objects are injective and this sequence is quasi-isomorphic to the sequence $0 \to X' \xrightarrow{f} X \xrightarrow{g} X'' \to 0$ in $\mathrm{C}(\mathscr{C})$. Since the objects X'^j are injective, we get a short exact sequence in $\mathrm{C}(\mathscr{C}')$:

$$0 \to F(X'^\bullet) \to F(X^\bullet) \to F(X''^\bullet) \to 0$$

Then one applies Theorem 4.5.8. □

Definition 4.7.5. Let $\mathscr{J}$ be a full additive subcategory of $\mathscr{C}$. One says that $\mathscr{J}$ is F-injective if:

(i) $\mathscr{J}$ is generating,

(ii) for any exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathscr{C}$ with $X' \in \mathscr{J}, X \in \mathscr{J}$, then $X'' \in \mathscr{J}$,

(iii) for any exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathscr{C}$ with $X' \in \mathscr{J}$, the sequence $0 \to F(X') \to F(X) \to F(X'') \to 0$ is exact.

By considering $\mathscr{C}^{\mathrm{op}}$, one obtains the notion of an F-projective subcategory, F being right exact.

Lemma 4.7.6. *Assume $\mathscr{J}$ is F-injective and let $X^\bullet \in \mathrm{C}^+(\mathscr{J})$ be a complex qis to zero* (i.e. *$X^\bullet$ is exact*)*. Then $F(X^\bullet)$ is qis to zero.*

Proof. We decompose $X^\bullet$ into short exact sequences (assuming that this complex starts at step 0 for convenience):

$$\begin{aligned}
&0 \to X^0 \to X^1 \to Z^1 \to 0\\
&0 \to Z^1 \to X^2 \to Z^2 \to 0\\
&\cdots\\
&0 \to Z^{n-1} \to X^n \to Z^n \to 0
\end{aligned}$$

By induction we find that all the Z^j's belong to $\mathscr{J}$, hence all the sequences:

$$0 \to F(Z^{n-1}) \to F(X^n) \to F(Z^n) \to 0$$

are exact. Hence the sequence

$$0 \to F(X^0) \to F(X^1) \to \cdots$$

is exact. □

Theorem 4.7.7. *Assume $\mathscr{J}$ is F-injective and contains the category $\mathscr{I}_{\mathscr{C}}$ of injective objects. Let $X \in \mathscr{C}$ and let $0 \to X \to Y^\bullet$ be a resolution of X with $Y^\bullet \in \mathrm{C}^+(\mathscr{J})$. Then for each n, there is an isomorphism $R^nF(X) \simeq H^n(F(Y^\bullet))$.*

In other words, in order to calculate the derived functors $R^nF(X)$, it is enough to replace X with a right $\mathscr{J}$-resolution.

Proof. Consider a right $\mathscr{J}$-resolution $Y^\bullet$ of X and an injective resolution $I^\bullet$ of X. By the result of Proposition 4.6.5, the identity morphism $X \to X$ will extend to a morphism of complexes $f^\bullet : Y^\bullet \to I^\bullet$ making the diagram below commutative:

$$\begin{array}{ccccc} 0 & \longrightarrow & X & \longrightarrow & Y^\bullet \\ & & \big\downarrow{\scriptstyle \mathrm{id}} & & \big\downarrow{\scriptstyle f^\bullet} \\ 0 & \longrightarrow & X & \longrightarrow & I^\bullet. \end{array}$$

Define the complex $K^\bullet = \mathrm{Mc}(f^\bullet)$, the mapping cone of $f^\bullet$. By the hypothesis, $K^\bullet$ belongs to $\mathrm{C}^+(\mathscr{J})$ and this complex is qis to zero by Corollary 4.5.9. By Lemma 4.7.6, $F(K^\bullet)$ is qis to zero.

On the other-hand, $F(\mathrm{Mc}(f))$ is isomorphic to $\mathrm{Mc}(F(f))$, the mapping cone of $F(f)\colon F(J^\bullet) \to F(I^\bullet)$. Applying Theorem 4.5.8 to this sequence, we find a long exact sequence

$$\cdots \to H^n(F(J^\bullet)) \to H^n(F(I^\bullet)) \to H^n(F(K^\bullet)) \to \cdots.$$

Since $F(K^\bullet)$ is qis to zero, the result follows. □

Example 4.7.8. Let $F\colon \mathscr{C} \to \mathscr{C}'$ be a left exact functor and assume that $\mathscr{C}$ admits enough injectives.
(i) The category $\mathscr{I}_{\mathscr{C}}$ of injective objects of $\mathscr{C}$ is F-injective.
(ii) Denote by $\mathscr{I}_F$ the full subcategory of $\mathscr{C}$ consisting of F-acyclic objects. Then $\mathscr{I}_F$ contains $\mathscr{I}_{\mathscr{C}}$, hence is generating. It easily follows from Theorem 4.7.4 that conditions (ii) and (iii) of Definition 4.7.5 are satisfied. Hence, $\mathscr{I}_F$ is F-injective.

Theorem 4.7.9. *Let $F\colon \mathscr{C} \to \mathscr{C}'$ and $G\colon \mathscr{C}' \to \mathscr{C}''$ be left exact functors of abelian categories and assume that $\mathscr{C}$ and $\mathscr{C}'$ have enough injectives.*

(i) *Assume that G is exact. Then $R^j(G \circ F) \simeq G \circ R^jF$.*

(ii) *Assume that F is exact. There is a natural morphism $R^j(G \circ F) \to (R^jG) \circ F$.*

(iii) *Let $X \in \mathscr{C}$ and assume that $R^jF(X) \simeq 0$ for $j > 0$ and that F sends the injective objects of $\mathscr{C}$ to G-acyclic objects of $\mathscr{C}'$. Then $R^j(G \circ F)(X) \simeq (R^jG)(F(X))$.*

Proof. For $X \in \mathscr{C}$, let $0 \to X \to I_X^\bullet$ be an injective resolution of X. Then $R^j(G \circ F)(X) \simeq H^j(G \circ F(I_X^\bullet))$.
(i) If G is exact, $H^j(G \circ F(I_X^\bullet))$ is isomorphic to $G(H^j(F(I_X^\bullet))$.
(ii) Consider an injective resolution $0 \to F(X) \to J^\bullet_{F(X)}$ of $F(X)$. By the result of Proposition 4.6.5, there exists a morphism $F(I_X^\bullet) \to J^\bullet_{F(X)}$. Applying G we get a morphism of complexes: $(G \circ F)(I_X^\bullet) \to G(J^\bullet_{F(X)})$. Since $H^j((G \circ F)(I_X^\bullet)) \simeq R^j(G \circ F)(X)$ and $H^j(G(J^\bullet_{F(X)})) \simeq R^jG(F(X))$, we get the result.
(iii) Denote by $\mathscr{I}_G$ the full additive subcategory of $\mathscr{C}'$ consisting of G-acyclic objects (see Example 4.7.8). By the hypothesis, $F(I_X^\bullet)$ is qis to $F(X)$ and belongs to $\mathrm{C}^+(\mathscr{I}_G)$. Hence $R^jG(F(X)) \simeq H^j(G(F(I_X^\bullet)))$ by Theorem 4.7.7. Finally, $H^j(G(F(I_X^\bullet))) \simeq R^j(G \circ F)(X)$. □

Derived bifunctor

Let $F\colon \mathscr{C}\times\mathscr{C}' \to \mathscr{C}''$ be a left exact additive bifunctor of abelian categories. Assume that $\mathscr{C}$ and $\mathscr{C}'$ admit enough injectives. For $X \in \mathscr{C}$ and $Y \in \mathscr{C}'$, one can thus construct $(R^jF(X,\bullet))(Y)$ and $(R^jF(\bullet,Y))(X)$.

Theorem 4.7.10. *Assume that*

(a) *for each injective object $I \in \mathscr{C}$, the functor $F(I,\bullet)\colon \mathscr{C}' \to \mathscr{C}''$ is exact,*

(b) *for each injective object $I' \in \mathscr{C}'$, the functor $F(\bullet,I')\colon \mathscr{C} \to \mathscr{C}''$ is exact.*

Then, for $j \in \mathbb{Z}$, $X \in \mathscr{C}$ and $Y \in \mathscr{C}'$, there is an isomorphism, functorial in X and Y: $(R^jF(X,\bullet))(Y) \simeq (R^jF(\bullet,Y))(X)$

Proof. Let $0 \to X \to I_X^\bullet$ and $0 \to Y \to I_Y^\bullet$ be injective resolutions of X and Y, respectively. Consider the double complex:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & 0 & \longrightarrow & F(I_X^0,Y) & \longrightarrow & F(I_X^1,Y) \longrightarrow \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & F(X,I_Y^0) & \longrightarrow & F(I_X^0,I_Y^0) & \longrightarrow & F(I_X^1,I_Y^0) \longrightarrow \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & F(X,I_Y^1) & \longrightarrow & F(I_X^0,I_Y^1) & \longrightarrow & F(I_X^1,I_Y^1) \longrightarrow \\
 & & \downarrow & & \downarrow & & \downarrow
\end{array}$$

The cohomology of the first row (resp. column) calculates $R^kF(\bullet,Y)(X)$ (resp. $R^kF(X,\bullet)(Y)$). Since the other rows and columns are exact by the hypotheses, the result follows from Theorem 4.5.10. □

Assume that $\mathscr{C}$ has enough injectives and enough projectives. Then one can define the j-th derived functor of $\mathrm{Hom}_{\mathscr{C}}(X,\bullet)$ and the j-th derived functor of $\mathrm{Hom}_{\mathscr{C}}(\bullet,Y)$. By Theorem 4.7.10 there exists an isomorphism

$$R^j\mathrm{Hom}_{\mathscr{C}}(X,\bullet)(Y) \simeq R^j\mathrm{Hom}_{\mathscr{C}}(\bullet,Y)(X)$$

functorial with respect to X and Y. Hence, if $\mathscr{C}$ has enough injectives *or* enough projectives, we can denote by the same symbol the derived functor either of the functor $\mathrm{Hom}_{\mathscr{C}}(X,\bullet)$ or of the functor $\mathrm{Hom}_{\mathscr{C}}(\bullet,Y)$.

A similar remark applies to the bifunctor $\otimes_A\colon \mathrm{Mod}(A^{\mathrm{op}})\times\mathrm{Mod}(A) \to \mathrm{Mod}(\mathbf{k})$.

Definition 4.7.11. (i) If $\mathscr{C}$ has enough injectives or enough projectives, one denotes by $\mathrm{Ext}^j_{\mathscr{C}}(\bullet,\bullet)$ the j-th right derived functor of $\mathrm{Hom}_{\mathscr{C}}$.

(ii) For a ring A, one denotes by $\mathrm{Tor}^A_j(\bullet,\bullet)$ the left derived functor of $\bullet\otimes_A\bullet$.

Hence, the derived functors of $\mathrm{Hom}_{\mathscr{C}}$ are calculated as follows. Let $X,Y \in \mathscr{C}$. If $\mathscr{C}$ has enough injectives, one chooses an injective resolution $I_Y^\bullet$ of Y and we get

$$\mathrm{Ext}^j_{\mathscr{C}}(X,Y) \simeq H^j(\mathrm{Hom}_{\mathscr{C}}(X,I_Y^\bullet)). \tag{4.7.2}$$

If $\mathscr{C}$ has enough projectives, one chooses a projective resolution $P_X^\bullet$ of X and we get

$$\mathrm{Ext}^j_{\mathscr{C}}(X,Y) \simeq H^j(\mathrm{Hom}_{\mathscr{C}}(P_X^\bullet, Y)). \tag{4.7.3}$$

If $\mathscr{C}$ admits both enough injectives and projectives, one can choose to use either (4.7.2) or (4.7.3). When dealing with the category $\mathrm{Mod}(A)$, projective resolutions are in general much easier to construct.

Similarly, the derived functors of $\otimes_A$ are calculated as follows. Let $N \in \mathrm{Mod}(A^{\mathrm{op}})$ and $M \in \mathrm{Mod}(A)$. One constructs a projective resolution $P_N^\bullet$ of N or a projective resolution $P_M^\bullet$ of M. Then

$$\mathrm{Tor}_j^A(N,M) \simeq H^{-j}(P_N^\bullet \otimes_A M) \simeq H^{-j}(N \otimes_A P_M^\bullet).$$

In fact, it is enough to take flat resolutions instead of projective ones.

4.8 Koszul complexes

Recall that $\mathbf{k}$ denotes a commutative unital ring. In this section, we do not work in abstract abelian categories but in the category $\mathrm{Mod}(\mathbf{k})$.

If L is a finite free $\mathbf{k}$-module of rank n, one denotes by $\bigwedge^j L$ the $\mathbf{k}$-module consisting of j-multilinear alternate forms on the dual space L^* and calls it the j-th exterior power of L. (Recall that $L^* = \mathrm{Hom}_{\mathbf{k}}(L, \mathbf{k})$.)

Note that $\bigwedge^1 L \simeq L$ and $\bigwedge^n L \simeq \mathbf{k}$. One sets $\bigwedge^0 L = \mathbf{k}$.

If $(e_1, \dots, e_n)$ is a basis of L and $I = \{i_1 < \cdots < i_j\} \subset \{1, \dots, n\}$, one sets

$$e_I = e_{i_1} \wedge \cdots \wedge e_{i_j}.$$

For a subset $I \subset \{1, \dots, n\}$, one denotes by $|I|$ its cardinal. Recall that:

$$\bigwedge^j L \text{ is free with basis } \{e_{i_1} \wedge \cdots \wedge e_{i_j}; 1 \le i_1 < i_2 < \cdots < i_j \le n\}.$$

If $i_1, \dots, i_m$ belong to the set $(1, \dots, n)$, one defines $e_{i_1} \wedge \cdots \wedge e_{i_m}$ by reducing to the case where $i_1 < \cdots < i_j$, using the convention $e_i \wedge e_j = -e_j \wedge e_i$.

Let M be a $\mathbf{k}$-module and let $\varphi = (\varphi_1, \dots, \varphi_n)$ be n $\mathbf{k}$-linear endomorphisms of M which commute with one another:

$$[\varphi_i, \varphi_j] = 0, \ 1 \le i, j \le n.$$

(Recall the notation $[a,b] := ab - ba$.) Set $M^{(j)} = M \otimes \bigwedge^j \mathbf{k}^n$. Hence $M^{(0)} = M$ and $M^{(n)} \simeq M$. Denote by $(e_1, \dots, e_n)$ the canonical basis of $\mathbf{k}^n$. Hence, any element of $M^{(j)}$ may be written uniquely as a sum

$$m = \sum_{|I|=j} m_I \otimes e_I.$$

One defines $d \in \mathrm{Hom}_{\mathbf{k}}(M^{(j)}, M^{(j+1)})$ by:

$$d(m \otimes e_I) = \sum_{i=1}^n \varphi_i(m) \otimes e_i \wedge e_I$$

and extending d by $\mathbf{k}$-linearity. Using the commutativity of the φ_i's one checks easily that $d \circ d = 0$. Hence we get a complex, called a Koszul complex and denoted $K^\bullet(M, \varphi)$:

$$0 \to M^{(0)} \xrightarrow{d} \cdots \to M^{(n)} \to 0.$$

When $n = 1$, the cohomology of this complex gives the kernel and cokernel of φ_1. More generally,

$$\begin{aligned} H^0(K^\bullet(M, \varphi)) &\simeq \ker \varphi_1 \cap \ldots \cap \ker \varphi_n, \\ H^n(K^\bullet(M, \varphi)) &\simeq M/(\varphi_1(M) + \cdots + \varphi_n(M)). \end{aligned}$$

Set $\varphi' = \{\varphi_1, \ldots, \varphi_{n-1}\}$ and denote by d' the differential in $K^\bullet(M, \varphi')$. Then φ_n defines a morphism

$$\widetilde{\varphi}_n : K^\bullet(M, \varphi') \to K^\bullet(M, \varphi') \tag{4.8.1}$$

Lemma 4.8.1. *The complex $K^\bullet(M, \varphi)[1]$ is isomorphic to the mapping cone of $-\widetilde{\varphi}_n$.*

Proof. [5] Consider the diagram

$$\begin{array}{ccc} \mathrm{Mc}(\widetilde{\varphi}_n)^p & \xrightarrow[d_M^p]{} & \mathrm{Mc}(\widetilde{\varphi}_n)^{p+1} \\ {\scriptstyle \lambda^p}\downarrow & & \downarrow{\scriptstyle \lambda^{p+1}} \\ K^{p+1}(M, \varphi) & \xrightarrow[d_K^{p+1}]{} & K^{p+2}(M, \varphi) \end{array}$$

given explicitly by:

$$\begin{array}{ccc} (M \otimes \bigwedge^{p+1} k^{n-1}) \oplus (M \otimes \bigwedge^{p} k^{n-1}) & \xrightarrow[\begin{pmatrix} -d' & 0 \\ -\varphi_n & d' \end{pmatrix}]{} & (M \otimes \bigwedge^{p+2} k^{n-1}) \oplus (M \otimes \bigwedge^{p+1} k^{n-1}) \\ {\scriptstyle \mathrm{id} \oplus (\mathrm{id} \otimes e_n \wedge)}\downarrow & & \downarrow{\scriptstyle \mathrm{id} \oplus (\mathrm{id} \otimes e_n \wedge)} \\ M \otimes \bigwedge^{p+1} k^n & \xrightarrow[-d]{} & M \otimes \bigwedge^{p+2} k^n \end{array}$$

Then

$$\begin{aligned} &d_M^p(a \otimes e_J + b \otimes e_K) = -d'(a \otimes e_J) + (d'(b \otimes e_K) - \varphi_n(a) \otimes e_J), \\ &\lambda^p(a \otimes e_J + b \otimes e_K) = a \otimes e_J + b \otimes e_n \wedge e_K. \end{aligned}$$

(i) The vertical arrows are isomorphisms. Indeed, let us treat the first one. It is described by:

$$\sum_J a_J \otimes e_J + \sum_K b_K \otimes e_K \mapsto \sum_J a_J \otimes e_J + \sum_K b_K \otimes e_n \wedge e_K \tag{4.8.2}$$

with $|J| = p+1$ and $|K| = p$. Any element of $M \otimes \bigwedge^{p+1} k^n$ may uniquely be written as in the right hand side of (4.8.2).

[5] The proof may be skipped

(ii) The diagram commutes. Indeed,

$$\begin{aligned}\lambda^{p+1}\circ d_M^p(a\otimes e_J+b\otimes e_K) &= -d'(a\otimes e_J)+e_n\wedge d'(b\otimes e_K)-\varphi_n(a)\otimes e_n\wedge e_J\\ &= -d'(a\otimes e_J)-d'(b\otimes e_n\wedge e_K)-\varphi_n(a)\otimes e_n\wedge e_J,\\ d_K^{p+1}\circ\lambda^p(a\otimes e_J+b\otimes e_K) &= -d(a\otimes e_J+b\otimes e_n\wedge e_K)\\ &= -d'(a\otimes e_J)-\varphi_n(a)\otimes e_n\wedge e_J-d'(b\otimes e_n\wedge e_K).\end{aligned}$$

□

Theorem 4.8.2. *There exists a* $\mathbf{k}$*-linear long exact sequence*

(4.8.3) $\cdots\to H^j(K^\bullet(M,\varphi'))\xrightarrow{\varphi_n} H^j(K^\bullet(M,\varphi'))\to H^{j+1}(K^\bullet(M,\varphi))\to\cdots$

Proof. Apply Lemma 4.8.1 and the long exact sequence (4.5.8). □

Definition 4.8.3. (i) If for each j, $1\le j\le n$, φ_j is injective as an endomorphism of $M/(\varphi_1(M)+\cdots+\varphi_{j-1}(M))$, one says $(\varphi_1,\dots,\varphi_n)$ is a regular sequence.

(ii) If for each j, $1\le j\le n$, φ_j is surjective as an endomorphism of $\ker\varphi_1\cap\ldots\cap\ker\varphi_{j-1}$, one says $(\varphi_1,\dots,\varphi_n)$ is a coregular sequence.

Corollary 4.8.4. (i) *If* $(\varphi_1,\dots,\varphi_n)$ *is a regular sequence, then* $H^j(K^\bullet(M,\varphi))\simeq 0$ *for* $j\neq n$.

(ii) *If* $(\varphi_1,\dots,\varphi_n)$ *is a coregular sequence, then* $H^j(K^\bullet(M,\varphi))\simeq 0$ *for* $j\neq 0$.

Proof. Assume for example that $(\varphi_1,\dots,\varphi_n)$ is a regular sequence, and let us argue by induction on n. The cohomology of $K^\bullet(M,\varphi')$ is thus concentrated in degree $n-1$ and is isomorphic to $M/(\varphi_1(M)+\cdots+\varphi_{n-1}(M))$. By the hypothesis, φ_n is injective on this group, and Corollary 4.8.4 follows. □

Second proof. Let us give a direct proof of the Corollary in case $n=2$ for coregular sequences. Hence we consider the complex:

$$0\to M\xrightarrow{d} M\times M\xrightarrow{d} M\to 0$$

where $d(x)=(\varphi_1(x),\varphi_2(x))$, $d(y,z)=\varphi_2(y)-\varphi_1(z)$ and we assume φ_1 is surjective on M, φ_2 is surjective on $\ker\varphi_1$.

Let $(y,z)\in M\times M$ with $\varphi_2(y)=\varphi_1(z)$. We look for $x\in M$ solution of $\varphi_1(x)=y,\quad\varphi_2(x)=z$. First choose $x'\in M$ with $\varphi_1(x')=y$. Then $\varphi_2\circ\varphi_1(x')=\varphi_2(y)=\varphi_1(z)=\varphi_1\circ\varphi_2(x')$. Thus $\varphi_1(z-\varphi_2(x'))=0$ and there exists $t\in M$ with $\varphi_1(t)=0,\quad\varphi_2(t)=z-\varphi_2(x')$. Hence $y=\varphi_1(t+x'),\quad z=\varphi_2(t+x')$ and $x=t+x'$ is a solution to our problem. □

Example 4.8.5. Let $\mathbf{k}$ be a field of characteristic 0 and let $A=\mathbf{k}[x_1,\dots,x_n]$.
(i) Denote by $x_i\cdot$ the multiplication by x_i in A. We get the complex:

$$0\to A^{(0)}\xrightarrow{d}\cdots\to A^{(n)}\to 0$$

where:

$$d(\sum_I a_I\otimes e_I)=\sum_{j=1}^n\sum_I x_j\cdot a_I\otimes e_j\wedge e_I.$$

The sequence $(x_1\cdot, \dots, x_n\cdot)$ is a regular sequence. Hence the Koszul complex is exact except in degree n where its cohomology is isomorphic to $\mathbf{k}$.
(ii) Denote by ∂_i the partial derivation with respect to x_i. This is a $\mathbf{k}$-linear map on the $\mathbf{k}$-vector space A. Hence we get a Koszul complex

$$0 \to A^{(0)} \xrightarrow{d} \cdots \xrightarrow{d} A^{(n)} \to 0$$

where:

$$d(\sum_I a_I \otimes e_I) = \sum_{j=1}^n \sum_I \partial_j(a_I) \otimes e_j \wedge e_I.$$

The sequence $(\partial_1\cdot, \dots, \partial_n\cdot)$ is a coregular sequence and the above complex is exact except in degree 0 where its cohomology is isomorphic to $\mathbf{k}$. Writing dx_j instead of e_j, we recognize the "de Rham complex".

Example 4.8.6. Let $\mathbf{k}$ be a field and let $A = \mathbf{k}[x, y]$, $M = \mathbf{k} \simeq A/xA + yA$ and let us calculate the $\mathbf{k}$-modules $\operatorname{Ext}^j_A(M, A)$. Since injective resolutions are not easy to calculate, it is much simpler to calculate a free (hence, projective) resolution of M. Since (x, y) is a regular sequence of endomorphisms of A (viewed as an $\mathbf{k}$-module), M is quasi-isomorphic to the complex:

$$M^\bullet : 0 \to A \xrightarrow{u} A^2 \xrightarrow{v} A \to 0,$$

where $u(a) = (ya, -xa)$, $v(b, c) = xb + yc$ and the module A on the right stands in degree 0. Therefore, $\operatorname{Ext}^j_A(M, N)$ is the j-th cohomology object of the complex $\operatorname{Hom}_A(M^\bullet, N)$, that is:

$$0 \to N \xrightarrow{v'} N^2 \xrightarrow{u'} N \to 0,$$

where $v' = \operatorname{Hom}(v, N)$, $u' = \operatorname{Hom}(u, N)$ and the module N on the left stands in degree 0. Since $v'(n) = (xn, yn)$ and $u'(m, l) = ym - xl$, we find again a Koszul complex. Choosing $N = A$, its cohomology is concentrated in degree 2. Hence, $\operatorname{Ext}^j_A(M, A) \simeq 0$ for $j \neq 2$ and $\simeq \mathbf{k}$ for $j = 2$.

Example 4.8.7. Let $W = W_n(\mathbf{k})$ be the Weyl algebra introduced in Example 1.2.2, and denote by $\cdot\partial_i$ the multiplication on the right by ∂_i. Then $(\cdot\partial_1, \dots, \cdot\partial_n)$ is a regular sequence on W and we get the Koszul complex:

$$0 \to W^{(0)} \xrightarrow{\delta} \cdots \to W^{(n)} \to 0$$

where:

$$\delta(\sum_I a_I \otimes e_I) = \sum_{j=1}^n \sum_I a_I \cdot \partial_j \otimes e_j \wedge e_I.$$

This complex is exact except in degree n where its cohomology is isomorphic to $\mathbf{k}[x]$ (see Exercise 4.9).

Remark 4.8.8. One may also encounter co-Koszul complexes. For $I = (i_1, \dots, i_k)$, introduce

$$e_j \lfloor e_I = \begin{cases} 0 & \text{if } j \notin \{i_1, \dots, i_k\} \\ (-1)^{l+1} e_{I_{\hat{l}}} := (-1)^{l+1} e_{i_1} \wedge \dots \wedge \widehat{e_{i_l}} \wedge \dots \wedge e_{i_k} & \text{if } e_{i_l} = e_j \end{cases}$$

where $e_{i_1} \wedge \dots \wedge \widehat{e_{i_l}} \wedge \dots \wedge e_{i_k}$ means that e_{i_l} should be omitted in $e_{i_1} \wedge \dots \wedge e_{i_k}$. Define δ by:

$$\delta(m \otimes e_I) = \sum_{j=1}^{n} \varphi_j(m) e_j \lfloor e_I.$$

Here again one checks easily that $\delta \circ \delta = 0$, and we get the complex:

$$K_\bullet(M, \varphi) : 0 \to M^{(n)} \xrightarrow{\delta} \cdots \to M^{(0)} \to 0,$$

This complex is in fact isomorphic to a Koszul complex. Consider the isomorphism

$$*\colon \bigwedge^{j} \mathbf{k}^n \xrightarrow{\sim} \bigwedge^{n-j} \mathbf{k}^n$$

which associates $\varepsilon_I m \otimes e_{\hat{I}}$ to $m \otimes e_I$, where $\hat{I} = (1, \dots, n) \setminus I$ and ε_I is the signature of the permutation which sends $(1, \dots, n)$ to $I \sqcup \hat{I}$ (any $i \in I$ is smaller than any $j \in \hat{I}$). Then, up to a sign, $*$ interchanges d and δ.

De Rham complexes

Let E be a real vector space of dimension n and let U be an open subset of E. Denote as usual by $\mathscr{C}^\infty(U)$ the $\mathbb{C}$-algebra of $\mathbb{C}$-valued functions on U of class C^∞. Recall that $\Omega^1(U)$ denotes the $\mathscr{C}^\infty(U)$-module of C^∞-functions on U with values in $E^* \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathrm{Hom}_{\mathbb{R}}(E, \mathbb{C})$. Hence

$$\Omega^1(U) \simeq E^* \otimes_{\mathbb{R}} \mathscr{C}^\infty(U).$$

For $p \in \mathbb{N}$, one sets

$$\begin{aligned} \Omega^p(U) &:= \bigwedge^{p} \Omega^1(U) \\ &\simeq (\bigwedge^{p} E^*) \otimes_{\mathbb{R}} \mathscr{C}^\infty(U). \end{aligned}$$

(The first exterior product is taken over the commutative ring $\mathscr{C}^\infty(U)$ and the second one over $\mathbb{R}$.) Hence, $\Omega^0(U) = \mathscr{C}^\infty(U)$, $\Omega^p(U) = 0$ for $p > n$ and $\Omega^n(U)$ is free of rank 1 over $\mathscr{C}^\infty(U)$. The differential is a $\mathbb{C}$-linear map

$$d\colon \mathscr{C}^\infty(U) \to \Omega^1(U).$$

The differential extends by multilinearity as a $\mathbb{C}$-linear map $d\colon \Omega^p(U) \to \Omega^{p+1}(U)$ satisfying

$$\begin{cases} d^2 = 0, \\ d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-)^p \omega_1 \wedge d\omega_2 \text{ for any } \omega_1 \in \Omega^p(U). \end{cases} \tag{4.8.4}$$

We get a complex, called the De Rham complex, that we denote by $\mathrm{DR}(U)$:

$$\mathrm{DR}(U) := 0 \to \Omega^0(U) \xrightarrow{d} \cdots \to \Omega^n(U) \to 0. \tag{4.8.5}$$

Let us choose a basis $(e_1, \dots, e_n)$ of E and denote by x_i the function which, to $x = \sum_{i=1}^n x_i \cdot e_i \in E$, associates its i-th cordinate x_i. Then $(dx_1, \dots, dx_n)$ is the dual basis on E^* and the differential of a function φ is given by

$$d\varphi = \sum_{i=1}^n \partial_i \varphi \, dx_i.$$

where $\partial_i \varphi := \dfrac{\partial \varphi}{\partial x_i}$. By its construction, the Koszul complex of $(\partial_1, \dots, \partial_n)$ acting on $\mathscr{C}^\infty(U)$ is nothing but the De Rham complex:

$$K^\bullet(\mathscr{C}^\infty(U), (\partial_1, \dots, \partial_n)) = \mathrm{DR}(U).$$

Note that $H^0(\mathrm{DR}(U))$ is the space of locally constant functions on U, and therefore is isomorphic to $\mathbb{C}^{\#cc(U)}$ where $\#cc(U)$ denotes the cardinal of the set of connected components of U. Using sheaf theory, one proves that all cohomology groups $H^j(\mathrm{DR}(U))$ are topological invariants of U.

Holomorphic De Rham complexes

Replacing $\mathbb{R}^n$ with $\mathbb{C}^n$, $\mathscr{C}^\infty(U)$ with $\mathscr{O}(U)$, the space of holomorphic functions on U and the real derivation with the holomorphic derivation, one constructs similarly the holomorphic De Rham complex.

Example 4.8.9. Let $n = 1$ and let $U = \mathbb{C} \setminus \{0\}$. The holomorphic De Rham complex reduces to

$$0 \to \mathscr{O}(U) \xrightarrow{\partial_z} \mathscr{O}(U) \to 0.$$

Its cohomology is isomorphic to $\mathbb{C}$ in degree 0 and 1.

Exercises to Chapter 4

Exercise 4.1. Prove assertion (iv) in Proposition 4.2.3, that is, prove that for a ring A and a set I, the two functors $\prod$ and $\bigoplus$ from $\mathrm{Fct}(I, \mathrm{Mod}(A))$ to $\mathrm{Mod}(A)$ are exact.

Exercise 4.2. Consider two complexes in an abelian category $\mathscr{C}$: $X_1' \to X_1 \to X_1''$ and $X_2' \to X_2 \to X_2''$. Prove that the two sequences are exact if and only if the sequence $X_1' \oplus X_2' \to X_1 \oplus X_2 \to X_1'' \oplus X_2''$ is exact.

Exercise 4.3. (i) Prove that a free module is projective.
(ii) Prove that a module P is projective if and only if it is a direct summand of a free module (*i.e.*, there exists a module K such that $P \oplus K$ is free).
(iii) An A-module M is flat if the functor $\bullet \otimes_A M$ is exact. (One defines similarly flat right A-modules.) Deduce from (ii) that projective modules are flat.

Exercise 4.4. (see [?Go58, Th. 1.2.2]) If M is a $\mathbb{Z}$-module, set $M^\vee = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$.
(i) Prove that $\mathbb{Q}/\mathbb{Z}$ is injective in $\mathrm{Mod}(\mathbb{Z})$.
(ii) Prove that the map $\mathrm{Hom}_{\mathbb{Z}}(M, N) \to \mathrm{Hom}_{\mathbb{Z}}(N^\vee, M^\vee)$ is injective for any $M, N \in \mathrm{Mod}(\mathbb{Z})$.
(iii) Prove that if P is a right projective A-module, then $P^\vee$ is left A-injective.
(iv) Let M be an A-module. Prove that there exists an injective A-module I and a monomorphism $M \to I$.
(Hint: (iii) Use formula (1.2.4). (iv) Prove that $M \mapsto M^{\vee\vee}$ is an injective map using (ii), and replace M with $M^{\vee\vee}$.)

Exercise 4.5. Let $\mathscr{C}$ be an abelian category which admits small colimits and such that small filtered colimits are exact. Let $\{X_i\}_{i\in I}$ be a family of objects of $\mathscr{C}$ indexed by a small set I and let $i_0 \in I$. Prove that the natural morphism $X_{i_0} \to \bigoplus_{i\in I} X_i$ is a monomorphism.

Exercise 4.6. Let $\mathscr{C}$ be an abelian category.
(i) Prove that a complex $0 \to X \to Y \to Z$ is exact iff and only if for any object $W \in \mathscr{C}$ the complex of abelian groups $0 \to \mathrm{Hom}_{\mathscr{C}}(W, X) \to \mathrm{Hom}_{\mathscr{C}}(W, Y) \to \mathrm{Hom}_{\mathscr{C}}(W, Z)$ is exact.
(ii) By reversing the arrows, state and prove a similar statement for a complex $X \to Y \to Z \to 0$.

Exercise 4.7. Recall Definition 2.4.1. Let $\mathscr{C}$ be an abelian category and consider a commutative diagram:

$$\begin{array}{ccc} V & \xrightarrow{f'} & Y \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Z. \end{array}$$

The square is Cartesian if the sequence $0 \to V \to X \times Y \to Z$ is exact, that is, if $V \simeq X \times_Z Y$ (recall that $X \times_Z Y = \ker(f - g)$, where $f - g \colon X \oplus Y \to Z$). The square is co-Cartesian if the sequence $V \to X \oplus Y \to Z \to 0$ is exact, that is, if $Z \simeq X \oplus_V Y$ (recall that $X \oplus_Z Y = \mathrm{Coker}(f' - g')$, where $f' - g' \colon V \to X \times Y$).
(i) Assume the square is Cartesian and f is an epimorphism. Prove that f' is an epimorphism.
(ii) Assume the square is co-Cartesian and f' is a monomorphism. Prove that f is a monomorphism.

Exercise 4.8. Let $\mathscr{C}$ be an abelian category and consider a commutative diagram of complexes

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \to & X'_0 & \to & X_0 & \to & X''_0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \to & X'_1 & \to & X_1 & \to & X''_1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \to & X'_2 & \to & X_2 & \to & X''_2 \end{array}$$

Assume that all rows are exact as well as the second and third column. Prove that all columns are exact.

Exercise 4.9. Let $\mathbf{k}$ be a field of characteristic 0, $W := W_n(\mathbf{k})$ the Weyl algebra in n variables.
(i) Denote by $x_i\cdot : W \to W$ the multiplication on the left by x_i on W (hence, the $x_i\cdot$'s are morphisms of right W-modules). Prove that $\varphi = (x_1\cdot, \dots, x_n\cdot)$ is a regular sequence and calculate $H^j(K^\bullet(W, \varphi))$.
(ii) Denote $\cdot\partial_i$ the multiplication on the right by ∂_i on W. Prove that $\psi = (\cdot\partial_1, \dots, \cdot\partial_n)$ is a regular sequence and calculate $H^j(K^\bullet(W, \psi))$.
(iii) Now consider the left $W_n(\mathbf{k})$-module $\mathscr{O} := \mathbf{k}[x_1, \dots, x_n]$ and the $\mathbf{k}$-linear map $\partial_i : \mathscr{O} \to \mathscr{O}$ (derivation with respect to x_i). Prove that $\lambda = (\partial_1, \dots, \partial_n)$ is a coregular sequence and calculate $H^j(K^\bullet(\mathscr{O}, \lambda))$.

Exercise 4.10. Let $A = W_2(\mathbf{k})$ be the Weyl algebra in two variables. Construct the Koszul complex associated to $\varphi_1 = \cdot x_1$, $\varphi_2 = \cdot\partial_2$ and calculate its cohomology.

Exercise 4.11. Let $\mathbf{k}$ be a field, $A = \mathbf{k}[x, y]$ and consider the A-module $M = \bigoplus_{i \geq 1} \mathbf{k}[x]t^i$, where the action of $x \in A$ is the usual one and the action of $y \in A$ is defined by $y \cdot x^n t^{j+1} = x^n t^j$ for $j \geq 1$, $y \cdot x^n t = 0$. Define the endomorphisms of M, $\varphi_1(m) = x \cdot m$ and $\varphi_2(m) = y \cdot m$. Calculate the cohomology of the Kozsul complex $K^\bullet(M, \varphi)$.

Exercise 4.12. Let $F \colon \mathscr{C} \to \mathscr{C}'$ be a left exact functor of abelian categories. Prove that F is conservative if and only if it is faithful.

Exercise 4.13. Let $\mathscr{C}$ be an abelian category which admits small coproducts and a cogenerator G. Let $f \colon X \to Y$ be a morphism in $\mathscr{C}$ and assume that $\mathrm{Hom}_{\mathscr{C}}(G, X) \to \mathrm{Hom}_{\mathscr{C}}(G, Y)$ is surjective. Prove that f is an epimorphism.

Chapter 5

Triangulated categories

Triangulated categories play an increasing role in mathematics and this subject might deserve a whole book.

However, we have restricted ourselves to describe their main properties as well as some basic results, in particular some related to cohomological functors.

We localize triangulated categories (skipping the proof for which we refer to [KS06]) and triangulated functors with the construction of derived categories in mind.

Finally we show (without proofs) that the homotopy category $\mathrm{K}(\mathscr{C})$ associated to an additive category $\mathscr{C}$, is naturally triangulated.

Note that the fact that the morphism in TR4 (see below) is not unique is the source of many troubles. It is the main obstacle encountered when trying to "glue" derived categories. This difficulty is overcome with the theory of ∞-categories where stable categories play the role of triangulated categories.

Some references: [GM96, KS90, KS06, Nee01, Ver96, Wei94, Yek20].

5.1 Triangulated categories

Definition 5.1.1. Let $\mathscr{D}$ be an additive category endowed with an automorphism T (i.e., an invertible functor $T\colon \mathscr{D} \to \mathscr{D}$). A triangle in $\mathscr{D}$ is a sequence of morphisms:

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T(X). \tag{5.1.1}$$

A morphism of triangles is a commutative diagram:

$$\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & T(X) \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle T(\alpha)} \\
X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & T(X').
\end{array}$$

Example 5.1.2. The triangle $X \xrightarrow{f} Y \xrightarrow{-g} Z \xrightarrow{-h} T(X)$ is isomorphic to the triangle (5.1.1), but the triangle $X \xrightarrow{-f} Y \xrightarrow{-g} Z \xrightarrow{-h} T(X)$ is not isomorphic to the triangle (5.1.1) in general.

Definition 5.1.3. A triangulated category is an additive category $\mathscr{D}$ endowed with an automorphism T and a family of triangles called distinguished triangles (d.t. for short), this family satisfying axioms TR0 - TR5 below.

TR0 A triangle isomorphic to a d.t. is a d.t.

TR1 The triangle $X \xrightarrow{\mathrm{id}_X} X \to 0 \to T(X)$ is a d.t.

TR2 For all $f\colon X \to Y$ there exists a d.t. $X \xrightarrow{f} Y \to Z \to T(X)$.

TR3 A triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T(X)$ is a d.t. if and only if $Y \xrightarrow{g} Z \xrightarrow{h} T(X) \xrightarrow{-T(f)} T(Y)$ is a d.t.

TR4 Given two d.t. $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T(X)$ and $X' \xrightarrow{f'} Y' \xrightarrow{g'} Z' \xrightarrow{h'} T(X')$ and morphisms $\alpha\colon X \to X'$ and $\beta : Y \to Y'$ with $f' \circ \alpha = \beta \circ f$, there exists a morphism $\gamma\colon Z \to Z'$ giving rise to a morphism of d.t.:

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & T(X) \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle T(\alpha)} \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & T(X'), \end{array}$$

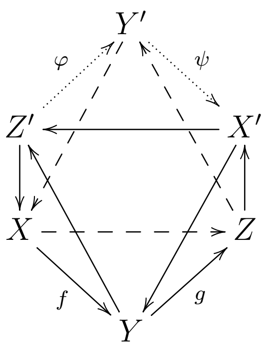
TR5 (Octahedral axiom) Given three d.t.

$$X \xrightarrow{f} Y \xrightarrow{h} Z' \to T(X),$$
$$Y \xrightarrow{g} Z \xrightarrow{k} X' \to T(Y),$$
$$X \xrightarrow{g\circ f} Z \xrightarrow{l} Y' \to T(X),$$

there exists a distinguished triangle $Z' \xrightarrow{\varphi} Y' \xrightarrow{\psi} X' \to T(Z')$ making the diagram below commutative:

(5.1.2)
$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{h} & Z' & \longrightarrow & T(X) \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle \mathrm{id}} \\ X & \xrightarrow{g\circ f} & Z & \xrightarrow{l} & Y' & \longrightarrow & T(X) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle T(f)} \\ Y & \xrightarrow{g} & Z & \xrightarrow{k} & X' & \longrightarrow & T(Y)_{T(h)} \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle l} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow \\ Z' & \xrightarrow{\varphi} & Y' & \xrightarrow{\psi} & X' & \longrightarrow & T(Z') \end{array}$$

Diagram (5.1.2) is often called the octahedron diagram. Indeed, it can be written using the vertexes of an octahedron.



Remark 5.1.4. The morphism γ in TR 4 is not unique and this is the origin of many troubles.

Remark 5.1.5. The category $\mathscr{D}^{\mathrm{op}}$ endowed with the image by the contravariant functor $\mathrm{op}\colon \mathscr{D} \to \mathscr{D}^{\mathrm{op}}$ of the family of the d.t. in $\mathscr{D}$, is a triangulated category.

5.2 Triangulated and cohomological functors

Definition 5.2.1. (i) A triangulated functor of triangulated categories $F\colon (\mathscr{D}, T) \to (\mathscr{D}', T')$ is an additive functor which satisfies $F \circ T \simeq T' \circ F$ and which sends distinguished triangles to distinguished triangles.

(ii) A triangulated subcategory $\mathscr{D}'$ of $\mathscr{D}$ is a subcategory $\mathscr{D}'$ of $\mathscr{D}$ which is triangulated and such that the functor $\mathscr{D}' \to \mathscr{D}$ is triangulated.

(iii) Let $(\mathscr{D}, T)$ be a triangulated category, $\mathscr{C}$ an abelian category, $F\colon \mathscr{D} \to \mathscr{C}$ an additive functor. One says that F is a cohomological functor if for any d.t. $X \to Y \to Z \to T(X)$ in $\mathscr{D}$, the sequence $F(X) \to F(Y) \to F(Z)$ is exact in $\mathscr{C}$.

Remark 5.2.2. By TR3, a cohomological functor gives rise to a long exact sequence:

$$\cdots \to F(X) \to F(Y) \to F(Z) \to F(T(X)) \to \cdots \tag{5.2.1}$$

Proposition 5.2.3. (i) *If* $X \xrightarrow{f} Y \xrightarrow{g} Z \to T(X)$ *is a d.t. then* $g \circ f = 0$.

(ii) *For any* $W \in \mathscr{D}$*, the functors* $\mathrm{Hom}_{\mathscr{D}}(W, \cdot)$ *and* $\mathrm{Hom}_{\mathscr{D}}(\cdot, W)$ *are cohomological.*

Note that (ii) means that if $\varphi : W \to Y$ (resp. $\varphi\colon Y \to W$) satisfies $g \circ \varphi = 0$ (resp. $\varphi \circ f = 0$), then φ factorizes through f (resp. through g).

Proof. (i) Applying TR1 and TR4 we get a commutative diagram:

$$\begin{array}{ccccccc}
X & \xrightarrow{\mathrm{id}} & X & \longrightarrow & 0 & \longrightarrow & T(X) \\
{\scriptstyle \mathrm{id}}\downarrow & & {\scriptstyle f}\downarrow & & \downarrow & & {\scriptstyle \mathrm{id}}\downarrow \\
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & T(X).
\end{array}$$

Then $g \circ f$ factorizes through 0.
(ii) Let $X \to Y \to Z \to T(X)$ be a d.t. and let $W \in \mathscr{D}$. We want to show that

$$\mathrm{Hom}\,(W, X) \xrightarrow{f\circ} \mathrm{Hom}\,(W, Y) \xrightarrow{g\circ} \mathrm{Hom}\,(W, Z)$$

is exact, i.e., for all $\varphi\colon W \to Y$ such that $g \circ \varphi = 0$, there exists $\psi\colon W \to X$ such that $\varphi = f \circ \psi$. This means that the dotted arrow below may be completed, and this follows from the axioms TR4 and TR3.

$$\begin{array}{ccccccc}
W & \xrightarrow{\mathrm{id}} & W & \longrightarrow & 0 & \longrightarrow & T(W) \\
\vdots & & {\scriptstyle \varphi}\downarrow & & \downarrow & & \vdots \\
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & T(X).
\end{array}$$

The proof for $\mathrm{Hom}\,(\cdot, W)$ is similar. $\square$

Proposition 5.2.4. *Consider a morphism of d.t.:*

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & T(X) \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle T(\alpha)} \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & T(X'). \end{array}$$

If α and β are isomorphisms, then so is γ.

Proof. Apply $\mathrm{Hom}\,(W, \cdot)$ to this diagram and write $\tilde{X}$ instead of $\mathrm{Hom}\,(W, X)$, $\tilde{\alpha}$ instead of $\mathrm{Hom}\,(W, \alpha)$, etc. We get the commutative diagram:

$$\begin{array}{ccccccc} \tilde{X} & \xrightarrow{\tilde{f}} & \tilde{Y} & \xrightarrow{\tilde{g}} & \tilde{Z} & \xrightarrow{\tilde{h}} & \widetilde{T(X)} \\ \downarrow{\scriptstyle\tilde{\alpha}} & & \downarrow{\scriptstyle\tilde{\beta}} & & \downarrow{\scriptstyle\tilde{\gamma}} & & \downarrow{\scriptstyle\widetilde{T(\alpha)}} \\ \tilde{X}' & \xrightarrow{\tilde{f}'} & \tilde{Y}' & \xrightarrow{\tilde{g}'} & \tilde{Z}' & \xrightarrow{\tilde{h}'} & \widetilde{T(X')}. \end{array}$$

The rows are exact in view of the preceding proposition, and $\tilde{\alpha}$, $\tilde{\beta}$, $\widetilde{T(\alpha)}$, $\widetilde{T(\beta)}$ are isomorphisms. Therefore $\tilde{\gamma} = \mathrm{Hom}\,(W, \gamma) : \mathrm{Hom}\,(W, Z) \to \mathrm{Hom}\,(W, Z')$ is an isomorphism. This implies that γ is an isomorphism by the Yoneda lemma. □

Corollary 5.2.5. *Let $\mathscr{D}'$ be a full triangulated category of $\mathscr{D}$.*

(i) *Consider a triangle $X \xrightarrow{f} Y \to Z \to T(X)$ in $\mathscr{D}'$ and assume that this triangle is distinguished in $\mathscr{D}$. Then it is distinguished in $\mathscr{D}'$.*

(ii) *Consider a d.t. $X \to Y \to Z \to T(X)$ in $\mathscr{D}$, with X and Y in $\mathscr{D}'$. Then there exists $Z' \in \mathscr{D}'$ and an isomorphism $Z \simeq Z'$.*

Proof. (i) There exists a d.t. $X \xrightarrow{f} Y \to Z' \to T(X)$ in $\mathscr{D}'$. Then Z' is isomorphic to Z by TR4 and Proposition 5.2.4.
(ii) Apply TR2 to the morphism $X \to Y$ in $\mathscr{D}'$. □

Remark 5.2.6. The proof of Proposition 5.2.4 does not make use of axiom TR 5, and this proposition implies that TR 5 is equivalent to the axiom:
TR5': given $f\colon X \to Y$ and $g\colon Y \to Z$, there exists a commutative diagram (5.1.2) such that all rows are d.t.

By Proposition 5.2.4, one gets that the object Z given in TR2 is unique up to isomorphism. However, this isomorphism is not unique, and this is the source of many difficulties (e.g., glueing problems in sheaf theory).

5.3 Applications to the homotopy category

Let $\mathscr{C}$ be an additive category. Recall that the homotopy category $\mathrm{K}(\mathscr{C})$ is defined by identifying to zero the morphisms in $\mathrm{C}(\mathscr{C})$ homotopic to zero.

Recall that if $f\colon X \to Y$ is a morphism in $\mathrm{C}(\mathscr{C})$, one defines its mapping cone $\mathrm{Mc}(f)$, an object of $\mathrm{C}(\mathscr{C})$, and there is a natural triangle

$$Y \xrightarrow{\alpha(f)} \mathrm{Mc}(f) \xrightarrow{\beta(f)} X[1] \xrightarrow{f[1]} Y[1]. \tag{5.3.1}$$

Such a triangle is called a mapping cone triangle. Clearly, a triangle in $\mathrm{C}(\mathscr{C})$ gives rise to a triangle in the homotopy category $\mathrm{K}(\mathscr{C})$.

Definition 5.3.1. A distinguished triangle (d.t. for short) in $\mathrm{K}(\mathscr{C})$ is a triangle isomorphic in $\mathrm{K}(\mathscr{C})$ to a mapping cone triangle.

Theorem 5.3.2. *The category* $\mathrm{K}(\mathscr{C})$ *endowed with the shift functor* $[1]$ *and the family of d.t. is a triangulated category.*

We shall not give the proof of this fundamental result here, referring to [KS06, Th. 11.2.6].

Notation 5.3.3. We shall often write $X \to Y \to Z \xrightarrow{+1}$ instead of $X \to Y \to Z \to X[1]$ to denote a d.t. in $\mathrm{K}(\mathscr{C})$.

Exercises to Chapter 5

Exercise 5.1. Let $\mathscr{D}$ be a triangulated category and consider a commutative diagram in $\mathscr{D}$:

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & T(X) \\ \| & & \| & & \downarrow{\scriptstyle\gamma} & & \| \\ X & \xrightarrow{f} & Y & \xrightarrow{g'} & Z' & \xrightarrow{h'} & T(X), \end{array}$$

Assume that $T(f) \circ h' = 0$ and the first row is a d.t. Prove that the second row is also a d.t. under one of the hypotheses:

(i) for any $P \in \mathscr{D}$, the sequence below is exact:

$$\mathrm{Hom}_{\mathscr{D}}(P, X) \to \mathrm{Hom}_{\mathscr{D}}(P, Y) \to \mathrm{Hom}_{\mathscr{D}}(P, Z') \to \mathrm{Hom}_{\mathscr{D}}(P, T(X)),$$

(ii) for any $P \in \mathscr{D}$, the sequence below is exact:

$$\mathrm{Hom}_{\mathscr{D}}(T(Y), P) \to \mathrm{Hom}_{\mathscr{D}}(T(X), P) \to \mathrm{Hom}_{\mathscr{D}}(Z', P) \to \mathrm{Hom}_{\mathscr{D}}(Y, P).$$

Exercise 5.2. Let $\mathscr{D}$ be a triangulated category and let $X_1 \to Y_1 \to Z_1 \to T(X_1)$ and $X_2 \to Y_2 \to Z_2 \to T(X_2)$ be two d.t. Show that $X_1 \oplus X_2 \to Y_1 \oplus Y_2 \to Z_1 \oplus Z_2 \to T(X_1) \oplus T(X_2)$ is a d.t.

In particular, $X \to X \oplus Y \to Y \xrightarrow{0} T(X)$ is a d.t.

(Hint: Consider a d.t. $X_1 \oplus X_2 \to Y_1 \oplus Y_2 \to H \to T(X_1) \oplus T(X_2)$ and construct the morphisms $H \to Z_1 \oplus Z_2$, then apply the result of Exercise 5.1.)

Exercise 5.3. Let $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T(X)$ be a d.t. in a triangulated category.

(i) Prove that if $h = 0$, this d.t. is isomorphic to $X \to X \oplus Z \to Z \xrightarrow{0} T(X)$.

(ii) Prove the same result by assuming now that there exists $k : Y \to X$ with $k \circ f = \mathrm{id}_X$.

(Hint: to prove (i), construct the morphism $Y \to X \oplus Z$ by TR4, then use Proposition 5.2.4.)

Exercise 5.4. Let $X \xrightarrow{f} Y \to Z \to T(X)$ be a d.t. in a triangulated category. Prove that f is an isomorphism if and only if Z is isomorphic to 0.

Exercise 5.5. Let $f\colon X \to Y$ be a monomorphism in a triangulated category $\mathscr{D}$. Prove that there exist $Z \in \mathscr{D}$ and an isomorphism $h\colon Y \xrightarrow{\sim} X \oplus Z$ such that the composition $X \to Y \to X \oplus Z$ is the canonical morphism.

Chapter 6

Localization

Consider a category $\mathscr{C}$ and a family $\mathscr{S}$ of morphisms in $\mathscr{C}$. The aim of localization is to find a new category $\mathscr{C}_{\mathscr{S}}$ and a functor $Q : \mathscr{C} \to \mathscr{C}_{\mathscr{S}}$ which sends the morphisms belonging to $\mathscr{S}$ to isomorphisms in $\mathscr{C}_{\mathscr{S}}$, $(Q, \mathscr{C}_{\mathscr{S}})$ being "universal" for such a property.

In this chapter, we shall construct the localization of a category when $\mathscr{S}$ satisfies suitable conditions and the localization of functors. A classical reference is [GZ67].

We also show that the localization of a triangulated category by a saturated full triangulated subcategory is naturally triangulated. This result will be essential when constructing derived categories.

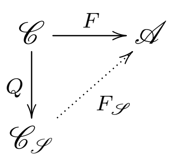
6.1 Localization of categories

Let $\mathscr{C}$ be a category and let $\mathscr{S}$ be a family of morphisms in $\mathscr{C}$.

Definition 6.1.1. A localizaton of $\mathscr{C}$ by $\mathscr{S}$ is the data of a category $\mathscr{C}_{\mathscr{S}}$ and a functor $Q\colon \mathscr{C} \to \mathscr{C}_{\mathscr{S}}$ satisfying:

(a) for all $s \in \mathscr{S}$, $Q(s)$ is an isomorphism,

(b) for any functor $F\colon \mathscr{C} \to \mathscr{A}$ such that $F(s)$ is an isomorphism for all $s \in \mathscr{S}$, there exists a functor $F_{\mathscr{S}}\colon \mathscr{C}_{\mathscr{S}} \to \mathscr{A}$ and an isomorphism $F \simeq F_{\mathscr{S}} \circ Q$,



(c) if G_1 and G_2 are two objects of $\mathrm{Fct}(\mathscr{C}_{\mathscr{S}}, \mathscr{A})$, then the natural map

(6.1.1) $$\mathrm{Hom}_{\,\mathrm{Fct}(\mathscr{C}_{\mathscr{S}},\mathscr{A})}(G_1, G_2) \to \mathrm{Hom}_{\,\mathrm{Fct}(\mathscr{C},\mathscr{A})}(G_1 \circ Q, G_2 \circ Q)$$

is bijective.

Note that (c) means that the functor $\circ Q\colon \mathrm{Fct}(\mathscr{C}_{\mathscr{S}}, \mathscr{A}) \to \mathrm{Fct}(\mathscr{C}, \mathscr{A})$ is fully faithful. This implies that $F_{\mathscr{S}}$ in (b) is unique up to unique isomorphism.

Proposition 6.1.2. (i) *If $\mathscr{C}_{\mathscr{S}}$ exists, it is unique up to equivalence of categories.*

(ii) *If $\mathscr{C}_{\mathscr{S}}$ exists, then, denoting by $\mathscr{S}^{\mathrm{op}}$ the image of $\mathscr{S}$ in $\mathscr{C}^{\mathrm{op}}$ by the functor* op, *$(\mathscr{C}^{\mathrm{op}})_{\mathscr{S}^{\mathrm{op}}}$ exists and there is an equivalence of categories:*

$$(\mathscr{C}_{\mathscr{S}})^{\mathrm{op}} \simeq (\mathscr{C}^{\mathrm{op}})_{\mathscr{S}^{\mathrm{op}}}.$$

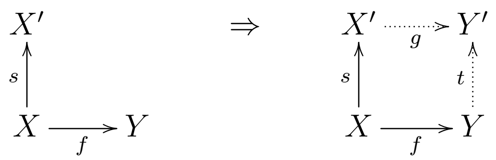
Proof. (i) is obvious.
(ii) Assume $\mathscr{C}_{\mathscr{S}}$ exists. Set $(\mathscr{C}^{\mathrm{op}})_{\mathscr{S}^{\mathrm{op}}} := (\mathscr{C}_{\mathscr{S}})^{\mathrm{op}}$ and define $Q^{\mathrm{op}} \colon \mathscr{C}^{\mathrm{op}} \to (\mathscr{C}^{\mathrm{op}})_{\mathscr{S}^{\mathrm{op}}}$ by $Q^{\mathrm{op}} = \mathrm{op} \circ Q \circ \mathrm{op}$. Then properties (a), (b) and (c) of Definition 6.1.1 are clearly satisfied. □

Definition 6.1.3. One says that $\mathscr{S}$ is a right multiplicative system if it satisfies the axioms S1-S4 below.

S1 For all $X \in \mathscr{C}$, $\mathrm{id}_X \in \mathscr{S}$.

S2 For all $f \in \mathscr{S}, g \in \mathscr{S}$, if $g \circ f$ exists then $g \circ f \in \mathscr{S}$.

S3 Given two morphisms, $f \colon X \to Y$ and $s \colon X \to X'$ with $s \in \mathscr{S}$, there exist $t \colon Y \to Y'$ and $g \colon X' \to Y'$ with $t \in \mathscr{S}$ and $g \circ s = t \circ f$. This can be visualized by the diagram:



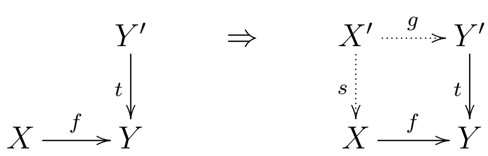
S4 Let $f, g \colon X \to Y$ be two parallel morphisms. If there exists $s \in \mathscr{S} \colon W \to X$ such that $f \circ s = g \circ s$ then there exists $t \in \mathscr{S} \colon Y \to Z$ such that $t \circ f = t \circ g$. This can be visualized by the diagram:

$$W \xrightarrow{s} X \overset{f}{\underset{g}{\rightrightarrows}} Y \overset{t}{\dashrightarrow} Z$$

Notice that these axioms are quite natural if one wants to invert the elements of $\mathscr{S}$. In other words, if the element of $\mathscr{S}$ would be invertible, then these axioms would clearly be satisfied.

Remark 6.1.4. Axioms S1-S2 asserts that $\mathscr{S}$ is the family of morphisms of a subcategory $\widetilde{\mathscr{S}}$ of $\mathscr{C}$ with $\mathrm{Ob}(\widetilde{\mathscr{S}}) = \mathrm{Ob}(\mathscr{C})$.

Remark 6.1.5. One defines the notion of a left multiplicative system $\mathscr{S}$ by reversing the arrows. This means that the condition S3 is replaced by: given two morphisms, $f \colon X \to Y$ and $t \colon Y' \to Y$, with $t \in \mathscr{S}$, there exist $s \colon X' \to X$ and $g \colon X' \to Y'$ with $s \in \mathscr{S}$ and $t \circ g = f \circ s$. This can be visualized by the diagram:



and S4 is replaced by: if there exists $t \in \mathscr{S} \colon Y \to Z$ such that $t \circ f = t \circ g$ then there exists $s \in \mathscr{S} \colon W \to X$ such that $f \circ s = g \circ s$. This is visualized by the diagram

$$W \overset{s}{\dashrightarrow} X \overset{f}{\underset{g}{\rightrightarrows}} Y \xrightarrow{t} Z$$

In the literature, one often calls a multiplicative system a system which is both right and left multiplicative.

Many multiplicative systems that we shall encounter satisfy a useful property that we introduce now.

Definition 6.1.6. Assume that $\mathscr{S}$ satisfies the axioms S1-S2 and let $X \in \mathscr{C}$. One defines the categories $\mathscr{S}_X$ and $\mathscr{S}^X$ as follows.

$$\begin{aligned} \mathrm{Ob}(\mathscr{S}^X) &= \{s\colon X \to X'; s \in \mathscr{S}\} \\ \mathrm{Hom}_{\mathscr{S}^X}((s\colon X \to X'), (s\colon X \to X'')) &= \{h\colon X' \to X''; h \circ s = s'\} \\ \mathrm{Ob}(\mathscr{S}_X) &= \{s\colon X' \to X; s \in \mathscr{S}\} \\ \mathrm{Hom}_{\mathscr{S}_X}((s\colon X' \to X), (s'\colon X'' \to X)) &= \{h\colon X' \to X''; s' \circ h = s\}. \end{aligned}$$

Proposition 6.1.7. *Assume that $\mathscr{S}$ is a right (resp. left) multiplicative system. Then the category $\mathscr{S}^X$ (resp. $\mathscr{S}_X^{\mathrm{op}}$) is filtered.*

Proof. By reversing the arrows, both results are equivalent. We treat the case of $\mathscr{S}^X$.
(a) Let $s\colon X \to X'$ and $s'\colon X \to X''$ belong to $\mathscr{S}$. By S3, there exists $t\colon X' \to X'''$ and $t'\colon X'' \to X'''$ such that $t' \circ s' = t \circ s$, and $t \in \mathscr{S}$. Hence, $t \circ s \in \mathscr{S}$ by S2 and $(X \to X''')$ belongs to $\mathscr{S}^X$.
(b) Let $s\colon X \to X'$ and $s'\colon X \to X''$ belong to $\mathscr{S}$, and consider two morphisms $f, g\colon X' \to X''$, with $f \circ s = g \circ s = s'$. By S4 there exists $t\colon X'' \to W, t \in \mathscr{S}$ such that $t \circ f = t \circ g$. Hence $t \circ s'\colon X \to W$ belongs to $\mathscr{S}^X$. □

One defines the functors:

$$\begin{aligned} &\alpha_X\colon \mathscr{S}^X \to \mathscr{C} \qquad (s\colon X \to X') \mapsto X', \\ &\beta_X\colon \mathscr{S}_X^{\mathrm{op}} \to \mathscr{C} \qquad (s\colon X' \to X) \mapsto X'. \end{aligned}$$

We shall concentrate on right multiplicative system.

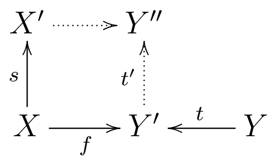
Definition 6.1.8. Let $\mathscr{S}$ be a right multiplicative system, and let $X, Y \in \mathrm{Ob}(\mathscr{C})$. We set

$$\mathrm{Hom}_{\mathscr{C}^r_{\mathscr{S}}}(X, Y) = \operatorname*{colim}_{(Y \to Y') \in \mathscr{S}^Y} \mathrm{Hom}_{\mathscr{C}}(X, Y').$$

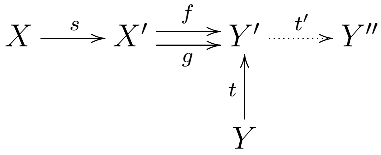
Lemma 6.1.9. *Assume that $\mathscr{S}$ is a right multiplicative system. Let $Y \in \mathscr{C}$ and let $s\colon X \to X' \in \mathscr{S}$. Then s induces an isomorphism*

$$\mathrm{Hom}_{\mathscr{C}^r_{\mathscr{S}}}(X', Y) \xrightarrow[\circ s]{\sim} \mathrm{Hom}_{\mathscr{C}^r_{\mathscr{S}}}(X, Y).$$

Proof. (i) The map $\circ s$ is surjective. This follows from S3, as visualized by the diagram in which $s, t, t' \in \mathscr{S}$:



(ii) The map $\circ s$ is injective. This follows from S4, as visualized by the diagram in which $s, t, t' \in \mathscr{S}$:



□

Using Lemma 6.1.9, we define the composition

$$\text{(6.1.2)} \qquad \mathrm{Hom}_{\mathscr{C}^r_{\mathscr{S}}}(X,Y) \times \mathrm{Hom}_{\mathscr{C}^r_{\mathscr{S}}}(Y,Z) \to \mathrm{Hom}_{\mathscr{C}^r_{\mathscr{S}}}(X,Z)$$

as

$$\begin{aligned}
&\operatorname*{colim}_{Y\to Y'} \mathrm{Hom}_{\mathscr{C}}(X,Y') \times \operatorname*{colim}_{Z\to Z'} \mathrm{Hom}_{\mathscr{C}}(Y,Z') \\
&\qquad\qquad \simeq \operatorname*{colim}_{Y\to Y'} \big(\mathrm{Hom}_{\mathscr{C}}(X,Y') \times \operatorname*{colim}_{Z\to Z'} \mathrm{Hom}_{\mathscr{C}}(Y,Z')\big) \\
&\qquad\qquad \xleftarrow{\sim} \operatorname*{colim}_{Y\to Y'} \big(\mathrm{Hom}_{\mathscr{C}}(X,Y') \times \operatorname*{colim}_{Z\to Z'} \mathrm{Hom}_{\mathscr{C}}(Y',Z')\big) \\
&\qquad\qquad \to \operatorname*{colim}_{Y\to Y'} \operatorname*{colim}_{Z\to Z'} \mathrm{Hom}_{\mathscr{C}}(X,Z') \\
&\qquad\qquad \simeq \operatorname*{colim}_{Z\to Z'} \mathrm{Hom}_{\mathscr{C}}(X,Z')
\end{aligned}$$

Lemma 6.1.10. *The composition* (6.1.2) *is associative.*

The verification is left to the reader.

Hence we get a category $\mathscr{C}^r_{\mathscr{S}}$ whose objects are those of $\mathscr{C}$ and morphisms are given by Definition 6.1.8.

Let us denote by $Q_{\mathscr{S}} \colon \mathscr{C} \to \mathscr{C}^r_{\mathscr{S}}$ the natural functor associated with

$$\mathrm{Hom}_{\mathscr{C}}(X,Y) \to \operatorname*{colim}_{(Y\to Y')\in \mathscr{S}^Y} \mathrm{Hom}_{\mathscr{C}}(X,Y').$$

If there is no risk of confusion, we denote this functor simply by Q.

Lemma 6.1.11. *If $s \colon X \to Y$ belongs to $\mathscr{S}$, then $Q(s)$ is invertible.*

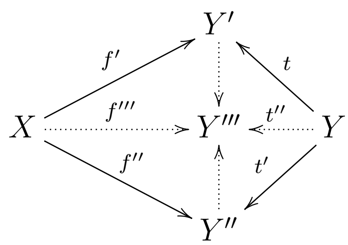
Proof. For any $Z \in \mathscr{C}^r_{\mathscr{S}}$, the map $\mathrm{Hom}_{\mathscr{C}^r_{\mathscr{S}}}(Y,Z) \to \mathrm{Hom}_{\mathscr{C}^r_{\mathscr{S}}}(X,Z)$ is bijective by Lemma 6.1.9. □

A morphism $f \colon X \to Y$ in $\mathscr{C}^r_{\mathscr{S}}$ is thus given by an equivalence class of triplets (Y', t, f') with $t \colon Y \to Y', t \in \mathscr{S}$ and $f' \colon X \to Y'$, that is:

$$X \xrightarrow{f'} Y' \xleftarrow{t} Y$$

the equivalence relation being defined as follows: $(Y', t, f') \sim (Y'', t', f'')$ if there exists (Y''', t'', f''') $(t, t', t'' \in S)$ and a commutative diagram:

(6.1.3)



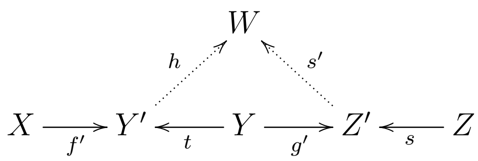
Note that the morphism (Y', t, f') in $\mathscr{C}^r_{\mathscr{S}}$ is $Q(t)^{-1} \circ Q(f')$, that is,

$$f = Q(t)^{-1} \circ Q(f'). \tag{6.1.4}$$

For two parallel arrows $f, g \colon X \rightrightarrows Y$ in $\mathscr{C}$ we have the equivalence

(6.1.5) $Q(f) = Q(g) \in \mathscr{C}^r_{\mathscr{S}} \Leftrightarrow$ there exits $s \colon Y \to Y', s \in \mathscr{S}$ with $s \circ f = s \circ g$.

The composition of two morphisms $(Y', t, f') \colon X \to Y$ and $(Z', s, g') \colon Y \to Z$ is defined by the diagram below in which $t, s, s' \in \mathscr{S}$:



Theorem 6.1.12. *Assume that $\mathscr{S}$ is a right multiplicative system.*

(i) *The category $\mathscr{C}^r_{\mathscr{S}}$ and the functor Q define a localization of $\mathscr{C}$ by $\mathscr{S}$.*

(ii) *For a morphism $f \colon X \to Y$, $Q(f)$ is an isomorphism in $\mathscr{C}^r_{\mathscr{S}}$ if and only if there exist $g \colon Y \to Z$ and $h \colon Z \to W$ such that $g \circ f \in \mathscr{S}$ and $h \circ g \in \mathscr{S}$.*

Notation 6.1.13. From now on, we shall write $\mathscr{C}_{\mathscr{S}}$ instead of $\mathscr{C}^r_{\mathscr{S}}$. This is justified by Theorem 6.1.12.

Remark 6.1.14. (i) In the above construction, we have used the property of $\mathscr{S}$ of being a right multiplicative system. If $\mathscr{S}$ is a left multiplicative system, one sets

$$\mathrm{Hom}_{\mathscr{C}^l_{\mathscr{S}}}(X, Y) = \operatorname*{colim}_{(X' \to X) \in \mathscr{S}_X} \mathrm{Hom}_{\mathscr{C}}(X', Y).$$

By Proposition 6.1.2 (i), the two constructions give equivalent categories.
(ii) If $\mathscr{S}$ is both a right and left multiplicative system,

$$\mathrm{Hom}_{\mathscr{C}_{\mathscr{S}}}(X, Y) \simeq \operatorname*{colim}_{(X' \to X) \in \mathscr{S}_X, (Y \to Y') \in \mathscr{S}^Y} \mathrm{Hom}_{\mathscr{C}}(X', Y').$$

Remark 6.1.15. In general, $\mathscr{C}_{\mathscr{S}}$ is no more a $\mathscr{U}$-category. However, if one assumes that for any $X \in \mathscr{C}$ the category $\mathscr{S}^X$ is small (or more generally, cofinally small, which means that there exists a small category cofinal to it), then $\mathscr{C}_{\mathscr{S}}$ is a $\mathscr{U}$-category, and there is a similar result with the $\mathscr{S}_X$'s.

6.2 Localization of subcategories

Proposition 6.2.1. *Let $\mathscr{C}$ be a category, $\mathscr{I}$ a full subcategory, $\mathscr{S}$ a right multiplicative system in $\mathscr{C}$, $\mathscr{T}$ the family of morphisms in $\mathscr{I}$ which belong to $\mathscr{S}$.*

(i) *Assume that $\mathscr{T}$ is a right multiplicative system in $\mathscr{I}$. Then the functor $\mathscr{I}_{\mathscr{T}} \to \mathscr{C}_{\mathscr{S}}$ is well-defined.*

(ii) *Assume that for every $f \colon Y \to X$, $f \in \mathscr{S}$, $Y \in \mathscr{I}$, there exist $W \in \mathscr{I}$ and $g \colon X \to W$ with $g \circ f \in \mathscr{S}$. Then $\mathscr{T}$ is a right multilplicative system and the functor $\mathscr{I}_{\mathscr{T}} \to \mathscr{C}_{\mathscr{S}}$ is fully faithful.*

Proof. (i) is obvious.
(ii) It is left to the reader to check that $\mathscr{T}$ is a right multpiplicative system. For $X \in \mathscr{I}$, $\mathscr{T}^X$ is the full subcategory of $\mathscr{S}^X$ whose objects are the morphisms $s\colon X \to Y$ with $Y \in \mathscr{I}$. By Proposition 6.1.7 and the hypothesis, the functor $\mathscr{T}^X \to \mathscr{S}^X$ is cofinal, and the result follows from Definition 6.1.8. □

Corollary 6.2.2. *Let $\mathscr{C}$ be a category, $\mathscr{I}$ a full subcategory, $\mathscr{S}$ a right multiplicative system in $\mathscr{C}$, $\mathscr{T}$ the family of morphisms in $\mathscr{I}$ which belong to $\mathscr{S}$. Assume that for any $X \in \mathscr{C}$ there exists $s\colon X \to W$ with $W \in \mathscr{I}$ and $s \in \mathscr{S}$.*

Then $\mathscr{T}$ is a right multpiplicative system and $\mathscr{I}_{\mathscr{T}}$ is equivalent to $\mathscr{C}_{\mathscr{S}}$.

Proof. The natural functor $\mathscr{I}_{\mathscr{T}} \to \mathscr{C}_{\mathscr{S}}$ is fully faithful by Proposition 6.2.1 and is essentially surjective by the assumption. □

6.3 Localization of functors

Let $\mathscr{C}$ be a category, $\mathscr{S}$ a right multiplicative system in $\mathscr{C}$ and $F\colon \mathscr{C} \to \mathscr{A}$ a functor. In general, F does not send morphisms in $\mathscr{S}$ to isomorphisms in $\mathscr{A}$. In other words, F does not factorize through $\mathscr{C}_{\mathscr{S}}$. It is however possible in some cases to define a localization of F as follows.

Definition 6.3.1. A right localization of F (if it exists) is a functor $F_{\mathscr{S}}\colon \mathscr{C}_{\mathscr{S}} \to \mathscr{A}$ and a morphism of functors $\tau\colon F \to F_{\mathscr{S}} \circ Q$ such that for any functor $G\colon \mathscr{C}_{\mathscr{S}} \to \mathscr{A}$ the map

$$\mathrm{Hom}_{\,\mathrm{Fct}(\mathscr{C}_{\mathscr{S}},\mathscr{A})}(F_{\mathscr{S}}, G) \to \mathrm{Hom}_{\,\mathrm{Fct}(\mathscr{C},\mathscr{A})}(F, G \circ Q) \tag{6.3.1}$$

is bijective. (This map is obtained as the composition $\mathrm{Hom}_{\,\mathrm{Fct}(\mathscr{C}_{\mathscr{S}},\mathscr{A})}(F_{\mathscr{S}}, G) \to \mathrm{Hom}_{\,\mathrm{Fct}(\mathscr{C},\mathscr{A})}(F_{\mathscr{S}} \circ Q, G \circ Q) \xrightarrow{\tau} \mathrm{Hom}_{\,\mathrm{Fct}(\mathscr{C},\mathscr{A})}(F, G \circ Q)$.)

We shall say that F is right localizable if it admits a right localization.

One defines similarly the left localization. Since we mainly consider right localization, we shall sometimes omit the word “right” as far as there is no risk of confusion.

If $(\tau, F_{\mathscr{S}})$ exists, it is unique up to unique isomorphisms. Indeed, $F_{\mathscr{S}}$ is a representative of the functor

$$G \mapsto \mathrm{Hom}_{\,\mathrm{Fct}(\mathscr{C},\mathscr{A})}(F, G \circ Q).$$

(This last functor is defined on the category $\mathrm{Fct}(\mathscr{C}_{\mathscr{S}}, \mathscr{A})$ with values in **Set**.)

Proposition 6.3.2. *Let $\mathscr{C}$ be a category, $\mathscr{I}$ a full subcategory, $\mathscr{S}$ a right multiplicative system in $\mathscr{C}$, $\mathscr{T}$ the family of morphisms in $\mathscr{I}$ which belong to $\mathscr{S}$. Let $F\colon \mathscr{C} \to \mathscr{A}$ be a functor. Assume that*

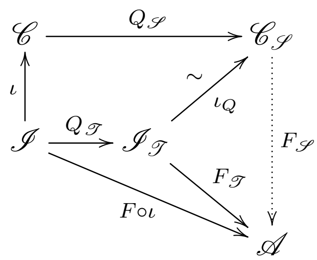
(i) *for any $X \in \mathscr{C}$ there exists $s\colon X \to W$ with $W \in \mathscr{I}$ and $s \in \mathscr{S}$,*

(ii) *for any $t \in \mathscr{T}$, $F(t)$ is an isomorphism.*

Then F is right localizable.

Proof. We shall apply Corollary 6.2.2.

Denote by $\iota\colon \mathscr{I} \to \mathscr{C}$ the natural functor. By the hypothesis, the localization $F_{\mathscr{T}}$ of $F \circ \iota$ exists. Consider the diagram:



Denote by ι_Q^{-1} a quasi-inverse of ι_Q and set $F_{\mathscr{S}} := F_{\mathscr{T}} \circ \iota_Q^{-1}$. Let us show that $F_{\mathscr{S}}$ is the localization of F. Let $G\colon \mathscr{C}_{\mathscr{S}} \to \mathscr{A}$ be a functor. We have the chain of morphisms:

$$\begin{aligned}
\mathrm{Hom}_{\,\mathrm{Fct}(\mathscr{C},\mathscr{A})}(F, G\circ Q_{\mathscr{S}}) &\xrightarrow{\lambda} \mathrm{Hom}_{\,\mathrm{Fct}(\mathscr{I},\mathscr{A})}(F\circ\iota, G\circ Q_{\mathscr{S}}\circ\iota) \\
&\simeq \mathrm{Hom}_{\,\mathrm{Fct}(\mathscr{I},\mathscr{A})}(F_{\mathscr{T}}\circ Q_{\mathscr{T}}, G\circ\iota_Q\circ Q_{\mathscr{T}}) \\
&\simeq \mathrm{Hom}_{\,\mathrm{Fct}(\mathscr{I}_{\mathscr{T}},\mathscr{A})}(F_{\mathscr{T}}, G\circ\iota_Q) \\
&\simeq \mathrm{Hom}_{\,\mathrm{Fct}(\mathscr{C}_{\mathscr{S}},\mathscr{A})}(F_{\mathscr{T}}\circ\iota_Q^{-1}, G) \\
&\simeq \mathrm{Hom}_{\,\mathrm{Fct}(\mathscr{C}_{\mathscr{S}},\mathscr{A})}(F_{\mathscr{S}}, G).
\end{aligned}$$

We shall not prove here that λ is an isomorphism. The first isomomorphism above (after λ) follows from the fact that $Q_{\mathscr{T}}$ is a localization functor (see Definition 6.1.1 (c)). The other isomorphisms are obvious. □

Remark 6.3.3. Let $\mathscr{C}$ (resp. $\mathscr{C}'$) be a category and $\mathscr{S}$ (resp. $\mathscr{S}'$) a right multiplicative system in $\mathscr{C}$ (resp. $\mathscr{C}'$). One checks immediately that $\mathscr{S}\times\mathscr{S}'$ is a right multiplicative system in the category $\mathscr{C}\times\mathscr{C}'$ and $(\mathscr{C}\times\mathscr{C}')_{\mathscr{S}\times\mathscr{S}'}$ is equivalent to $\mathscr{C}_{\mathscr{S}}\times\mathscr{C}'_{\mathscr{S}'}$. Since a bifunctor is a functor on the product $\mathscr{C}\times\mathscr{C}'$, we may apply the preceding results to the case of bifunctors. In the sequel, we shall write $F_{\mathscr{S}\mathscr{S}'}$ instead of $F_{\mathscr{S}\times\mathscr{S}'}$.

6.4 Localization of triangulated categories

Definition 6.4.1. A *null system* $\mathscr{N}$ in $\mathscr{D}$ is a full triangulated saturated subcategory of $\mathscr{D}$.

A null system $\mathscr{N}$ satisfies:

N1 $0 \in \mathscr{N}$,

N2 $X \in \mathscr{N}$ if and only if $T(X) \in \mathscr{N}$,

N3 if $X \to Y \to Z \to T(X)$ is a d.t. in $\mathscr{D}$ and $X, Y \in \mathscr{N}$ then $Z \in \mathscr{N}$.

One easily checks that if $\mathscr{N}$ is a full saturated subcategory of $\mathscr{D}$ satisfying N1-N2-N3, then the restriction of T to $\mathscr{N}$ and the family of d.t. $X \to Y \to Z \to T(X)$ in $\mathscr{D}$ with $X, Y, Z \in \mathscr{N}$ make $\mathscr{N}$ a null system of $\mathscr{D}$. Moreover, it has the property that given a d.t. as above in $\mathscr{D}$, the three objects X, Y, Z belong to $\mathscr{N}$ as soon as two objects among them belong to $\mathscr{N}$.

To a null system one associates a multiplicative system as follows. Define:

$$\mathscr{S} := \{f\colon X \to Y, \text{ there exists a d.t. } X \to Y \to Z \to T(X) \text{ with } Z \in \mathscr{N}\}.$$

Theorem 6.4.2. (i) *$\mathscr{S}$ is a right and left multiplicative system.*

(ii) *Denote as usual by $\mathscr{D}_{\mathscr{S}}$ the localization of $\mathscr{D}$ by $\mathscr{S}$ and by Q the localization functor. Then $\mathscr{D}_{\mathscr{S}}$ is an additive category endowed with an automorphism (the image of T, still denoted by T).*

(iii) *Define a d.t. in $\mathscr{D}_{\mathscr{S}}$ as being isomorphic to the image by Q of a d.t. in $\mathscr{D}$. Then $\mathscr{D}_{\mathscr{S}}$ is a triangulated category.*

(iv) *If $X \in \mathscr{N}$, then $Q(X) \simeq 0$.*

(v) *Let $F\colon \mathscr{D} \to \mathscr{D}'$ be a functor of triangulated categories such that $F(X) \simeq 0$ for any $X \in \mathscr{N}$. Then F factors uniquely through Q.*

The proof being straightforward but tedious, it will not be given here. For a complete proof, see for example [KS06].

Notation 6.4.3. We will write $\mathscr{D}/\mathscr{N}$ instead of $\mathscr{D}_{\mathscr{S}}$.

Now consider a full triangulated subcategory $\mathscr{I}$ of $\mathscr{D}$. denote by $\mathscr{N} \cap \mathscr{I}$ the full subcategory of $\mathscr{D}$ whose objects are $\mathrm{Ob}(\mathscr{N}) \cap \mathrm{Ob}(\mathscr{I})$. This is clearly a null system in $\mathscr{I}$.

Proposition 6.4.4. *Let $\mathscr{D}$ be a triangulated category, $\mathscr{N}$ a null system and $\mathscr{I}$ a full triangulated category of $\mathscr{D}$. Assume condition* (i) *or* (ii) *below*

(i) *any morphism $Y \to Z$ with $Y \in \mathscr{I}$ and $Z \in \mathscr{N}$ factorizes as $Y \to Z' \to Z$ with $Z' \in \mathscr{N} \cap \mathscr{I}$,*

(ii) *any morphism $Z \to Y$ with $Y \in \mathscr{I}$ and $Z \in \mathscr{N}$ factorizes as $Z \to Z' \to Y$ with $Z' \in \mathscr{N} \cap \mathscr{I}$.*

Then the functor $\mathscr{I}/(\mathscr{N} \cap \mathscr{I}) \to \mathscr{D}/\mathscr{N}$ is fully faithful.

Proof. We shall apply Proposition 6.2.1. We may assume (ii), the case (i) being deduced by considering $\mathscr{D}^{\mathrm{op}}$. Let $f\colon Y \to X$ be a morphism in $\mathscr{S}$ with $Y \in \mathscr{I}$. We shall show that there exists $g\colon X \to W$ with $W \in I$ and $g \circ f \in \mathscr{S}$. The morphism f is embedded in a d.t. $Y \to X \to Z \to T(Y)$ with $Z \in \mathscr{N}$. By the hypothesis, the morphism $Z \to T(Y)$ factorizes through an object $Z' \in \mathscr{N} \cap \mathscr{I}$. We may embed $Z' \to T(Y)$ into a d.t. and obtain a commutative diagram of d.t.:

$$\begin{array}{ccccccc}
Y & \xrightarrow{f} & X & \longrightarrow & Z & \longrightarrow & T(Y) \\
\downarrow{\scriptstyle \mathrm{id}} & & \vdots{\scriptstyle g} & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\
Y & \longrightarrow & W & \longrightarrow & Z' & \longrightarrow & T(Y)
\end{array}$$

By TR4, the dotted arrow g may be completed, and Z' belonging to $\mathscr{N}$, this implies that $g \circ f \in \mathscr{S}$. $\square$

Proposition 6.4.5. *Let $\mathscr{D}$ be a triangulated category, $\mathscr{N}$ a null system and $\mathscr{I}$ a full triangulated subcategory of $\mathscr{D}$. Assume conditions* (i) *or* (ii) *below:*

(i) *for any $X \in \mathscr{D}$, there exists a d.t. $X \to Y \to Z \to T(X)$ with $Z \in \mathscr{N}$ and $Y \in \mathscr{I}$,*

(ii) *for any $X \in \mathscr{D}$, there exists a d.t. $Y \to X \to Z \to T(X)$ with $Z \in \mathscr{N}$ and $Y \in \mathscr{I}$.*

Then $\mathscr{I}/\mathscr{N} \cap \mathscr{I} \to \mathscr{D}/\mathscr{N}$ is an equivalence of categories.

Proof. Apply Corollary 6.2.2. □

Localization of triangulated functors

Let $F: \mathscr{D} \to \mathscr{D}'$ be a functor of triangulated categories and let $\mathscr{N}$ be a null system in $\mathscr{D}$. One defines the localization of F similarly as in the usual case, replacing all categories and functors by triangulated ones. Applying Proposition 6.3.2, we get:

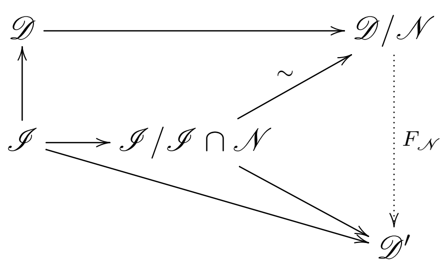
Proposition 6.4.6. *Let $F: \mathscr{D} \to \mathscr{D}'$ be a functor of triangulated categories. Let $\mathscr{N}$ a null system of $\mathscr{D}$ and $\mathscr{I}$ a full triangulated category of $\mathscr{D}$. Assume*

(i) *for any $X \in \mathscr{D}$, there exists a d.t. $X \to Y \to Z \to T(X)$ with $Z \in \mathscr{N}$ and $Y \in \mathscr{I}$,*

(ii) *for any $Y \in \mathscr{N} \cap \mathscr{I}$, $F(Y) \simeq 0$.*

Then F is right localizable.

One can define $F_{\mathscr{N}}$ by the diagram:



If one replace condition (i) in Proposition 6.4.6 by the condition

(i)' for any $X \in \mathscr{D}$, there exists a d.t. $Y \to X \to Z \to T(X)$ with $Z \in \mathscr{N}$ and $Y \in \mathscr{I}$,

one gets that F is left localizable.

Finally, let us consider triangulated bifunctors, i.e., bifunctors which are additive and triangulated with respect to each of their arguments.

Proposition 6.4.7. *Let $\mathscr{D}, \mathscr{N}, \mathscr{I}$ and $\mathscr{D}', \mathscr{N}', \mathscr{I}'$ be as in Proposition 6.4.6. Let $F: \mathscr{D} \times \mathscr{D}' \to \mathscr{D}''$ be a triangulated bifunctor. Assume:*

(i) *for any $X \in \mathscr{D}$, there exists a d.t. $X \to Y \to Z \to T(X)$ with $Z \in \mathscr{N}$ and $Y \in \mathscr{I}$*

(ii) *for any* $X' \in \mathscr{D}'$, *there exists a d.t.* $X' \to Y' \to Z' \to T(X')$ *with* $Z' \in \mathscr{N}'$ *and* $Y' \in \mathscr{I}'$

(iii) *for any* $Y \in \mathscr{I}$ *and* $Y' \in \mathscr{I}' \cap \mathscr{N}'$, $F(Y, Y') \simeq 0$,

(iv) *for any* $Y \in \mathscr{I} \cap \mathscr{N}$ *and* $Y' \in \mathscr{I}'$, $F(Y, Y') \simeq 0$.

Then F *is right localizable.*

One denotes by $F_{\mathscr{N}\mathscr{N}'}$ its localization.

Of course, there exists a similar result for left localizable functors by reversing the arrows in the hypotheses (i) and (ii) above.

Localization and direct sums

Proposition 6.4.8. *Let* $\mathscr{D}$ *be a triangulated category admitting small direct sums (hence, a small direct sum of d. t. is again a d. t.) and let* $\mathscr{N}$ *be a null system in* $\mathscr{D}$ *stable by such direct sums. Then* $\mathscr{D}/\mathscr{N}$ *admits small direct sums and the localization functor* Q *commutes with such direct sums.*

Proof. We shall follow [KS06, Prop. 10.2.8].
Let $\{X_i\}_{i\in I}$ be a small family of objects in $\mathscr{D}$ and let $Y \in \mathscr{D}$. A morphism $u\colon Q(\oplus_i X_i) \to Y$ defines for each i a morphism $u_i\colon Q(X_i) \to Y$. (As far as there is no risk of confusion, we write Y instead of $Q(Y)$.) Hence we have a natural map

$$\theta\colon \mathrm{Hom}_{\mathscr{D}/\mathscr{N}}(Q(\oplus_i X_i), Y) \xrightarrow{\sim} \prod_i \mathrm{Hom}_{\mathscr{D}/\mathscr{N}}(Q(X_i), Y).$$

In order to prove that $Q(\oplus_i X_i)$ is the direct sum of the family $Q(X_i)$, it is enough to check that θ is bijective for any $Y \in \mathscr{D}$.
(i) The map θ is surjective. Consider a family of morphisms $u_i\colon \mathrm{Hom}_{\mathscr{D}/\mathscr{N}}(Q(X_i), Y)$. We represent each u_i by a morphism $v_i\colon X_i' \to Y$ together with a d. t. $X_i' \to X_i \to Z_i \xrightarrow{+1}$ with $Z_i \in \mathscr{N}$. We get a morphism $v\colon \oplus_i X_i' \to Y$ and, for each i, a d. t. $\oplus_i X_i' \to \oplus_i X_i \to \oplus_i Z_i \xrightarrow{+1}$. By the hypothesis, $\oplus_i Z_i \in \mathscr{N}$ and it follows that v defines a morphism $Q(\oplus_i X_i') \to Y$ in $\mathscr{D}/\mathscr{N}$.
(ii) The map θ is injective. Assume that the composition $Q(X_j) \to Q(\oplus_i X_i) \xrightarrow{u} Q(Y)$ is 0 for all $j \in I$. The morphism u may be represented by morphisms $\oplus_i X_i \xrightarrow{v} Y' \xleftarrow{s} Y$ with $s \in \mathscr{S}$ where $\mathscr{S}$ is the multiplicative system associated with $\mathscr{N}$ and the image of the composition $X_j \to \oplus_i X_i \xrightarrow{v} Y'$ is zero in $\mathscr{D}/\mathscr{N}$. By the result of Exercise 6.8 for each i there exists $Z_j \in \mathscr{N}$ such that this composition factorizes as $X_j \to Z_j \to Y'$. Therefore, $\oplus_j X_j \to Y'$ factorizes as $\oplus_j X_j \to \oplus_j Z_j \to Y'$ and thus $Q(u) = 0$. □

Exercises to Chapter 6

Exercise 6.1. Let $\mathscr{S}$ be a right multiplicative system. One says that $\mathscr{S}$ is saturated if it satisfies

S5 for any morphisms $f\colon X \to Y$, $g\colon Y \to Z$ and $h\colon Z \to W$ such that $g \circ f$ and $h \circ g$ belong to $\mathscr{S}$, the morphism f belongs to $\mathscr{S}$.

Prove that if $\mathscr{S}$ is saturated, a morphism f in $\mathscr{C}$ belongs to $\mathscr{S}$ if and only if $Q(f)$ is an isomorphism, where $Q\colon \mathscr{C} \to \mathscr{C}_{\mathscr{S}}$ denotes, as usual, the localization functor.

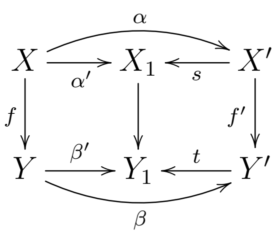
Exercise 6.2. Let $\mathscr{C}$ be a category, $\mathscr{S}$ a right multiplicative system. Let $\mathscr{T}$ be the set of morphisms $f\colon X \to Y$ in $\mathscr{C}$ such that there exist $g\colon Y \to Z$ and $h\colon Z \to W$, with $h \circ g$ and $g \circ f$ in $\mathscr{S}$.

Prove that $\mathscr{T}$ is a right saturated multiplicative system and that the natural functor $\mathscr{C}_{\mathscr{S}} \to \mathscr{C}_{\mathscr{T}}$ is an equivalence.

Exercise 6.3. Let $\mathscr{C}$ be a category, $\mathscr{S}$ a right and left multiplicative system. Prove that $\mathscr{S}$ is saturated if and only if for any $f\colon X \to Y$, $g\colon Y \to Z$, $h\colon Z \to W$, $h \circ g \in \mathscr{S}$ and $g \circ f \in \mathscr{S}$ imply $g \in \mathscr{S}$.

Exercise 6.4. Let $\mathscr{C}$ be a category with a zero object 0, $\mathscr{S}$ a right and left saturated multiplicative system.
(i) Show that $\mathscr{C}_{\mathscr{S}}$ has a zero object (still denoted by 0).
(ii) Prove that $Q(X) \simeq 0$ if and only if the zero morphism $0\colon X \to X$ belongs to $\mathscr{S}$.

Exercise 6.5. Let $\mathscr{C}$ be a category, $\mathscr{S}$ a right multiplicative system. Consider morphisms $f\colon X \to Y$ and $f'\colon X' \to Y'$ in $\mathscr{C}$ and morphisms $\alpha\colon X \to X'$ and $\beta\colon Y \to Y'$ in $\mathscr{C}_{\mathscr{S}}$, and assume that $f' \circ \alpha = \beta \circ f$ in $\mathscr{C}_{\mathscr{S}}$. Prove that there exists a commutative diagram in $\mathscr{C}$



with s and t in $\mathscr{S}$, $\alpha = Q(s)^{-1} \circ Q(\alpha')$ and $\beta = Q(t)^{-1} \circ Q(\beta')$.

Exercise 6.6. Let $F\colon \mathscr{C} \to \mathscr{A}$ be a functor and assume that $\mathscr{C}$ admits finite colimits and F is right exact. Let $\mathscr{S}$ denote the set of morphisms s in $\mathscr{C}$ such that $F(s)$ is an isomorphism.
(i) Prove that $\mathscr{S}$ is a right saturated multiplicative system.
(ii) Prove that the localized functor $F_{\mathscr{S}}\colon \mathscr{C}_{\mathscr{S}} \to \mathscr{A}$ is faithful.

Exercise 6.7. Let $\mathscr{D}$ be a triangulated category, $\mathscr{N}$ a null system and let Y be an object of $\mathscr{D}$ such that $\mathrm{Hom}_{\mathscr{D}}(Z, Y) \simeq 0$ for all $Z \in \mathscr{N}$. Prove that $\mathrm{Hom}_{\mathscr{D}}(X, Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{D}/\mathscr{N}}(X, Y)$.

Exercise 6.8. Let $\mathscr{D}$ be a triangulated category, $\mathscr{N}$ a null system and let $Q\colon \mathscr{D} \to \mathscr{D}/\mathscr{N}$ be the canonical functor.
(i) Let $f\colon X \to Y$ be a morphism in $\mathscr{D}$ and assume that $Q(f) = 0$ in $\mathscr{D}/\mathscr{N}$. Prove that there exists $Z \in \mathscr{N}$ such that f factorizes as $X \to Z \to Y$.
(ii) For $X \in \mathscr{D}$, prove that $Q(X) \simeq 0$ if and only if there exists Y such that $X \oplus Y \in \mathscr{N}$ and this last condition is equivalent to $X \oplus TX \in \mathscr{N}$.

Chapter 7

Derived categories

This chapter is devoted to derived categories. The homotopy category $\mathrm{K}(\mathscr{C})$ of an additive category $\mathscr{C}$ is triangulated (here, we shall admit this fact, referring to [KS06]). When $\mathscr{C}$ is abelian, the cohomology functor $H^0\colon \mathrm{K}(\mathscr{C}) \to \mathscr{C}$ is cohomological and the derived category $\mathrm{D}(\mathscr{C})$ of $\mathscr{C}$ is obtained by localizing $\mathrm{K}(\mathscr{C})$ with respect to the family of quasi-isomorphisms. We explain here this construction, with some examples, and also construct the right derived functor of a left exact functor.

Finally, we state, without proof, the Brown representability theorem, a fundamental result for applications.

The reader shall be aware that in general, the derived category $\mathrm{D}^+(\mathscr{C})$ of a $\mathscr{U}$-category $\mathscr{C}$ is no more a $\mathscr{U}$-category (see Remark 7.2.4).

Some references: [GM96, ?Ha66, KS90, KS06, Ver96, Wei94].

7.1 Derived categories

From now on, $\mathscr{C}$ will denote an abelian category.

Recall that if $f\colon X \to Y$ is a morphism in $\mathrm{C}(\mathscr{C})$, one says that f is a quasi-isomorphism (a qis, for short) if $H^k(f)\colon H^k(X) \to H^k(Y)$ is an isomorphism for all k. One extends this definition to morphisms in $\mathrm{K}(\mathscr{C})$.

If one embeds f into a d.t. $X \xrightarrow{f} Y \to Z \xrightarrow{+1}$, then f is a qis iff $H^k(Z) \simeq 0$ for all $k \in \mathbb{Z}$, that is, if Z is qis to 0.

Proposition 7.1.1. *Let $\mathscr{C}$ be an abelian category. The functor $H^0\colon K(\mathscr{C}) \to \mathscr{C}$ is a cohomological functor.*

Proof. Let $X \xrightarrow{f} Y \to Z \xrightarrow{+1}$ be a d.t. Then it is isomorphic to $X \to Y \xrightarrow{\alpha(f)} \mathrm{Mc}(f) \xrightarrow{\beta(f)} X[1] \xrightarrow{+1}$. Since the sequence in $\mathrm{C}(\mathscr{C})$:

$$0 \to Y \to \mathrm{Mc}(f) \to X[1] \to 0$$

is exact, it follows from Theorem 4.5.8 that the sequence

$$H^k(Y) \to H^k(\mathrm{Mc}(f)) \to H^{k+1}(X)$$

is exact. Therefore, $H^k(Y) \to H^k(Z) \to H^{k+1}(X)$ is exact. □

Corollary 7.1.2. *Let $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ be an exact sequence in $\mathrm{C}(\mathscr{C})$ and define $\varphi\colon \mathrm{Mc}(f) \to Z$ as $\varphi^n = (0, g^n)$. Then φ is a qis.*

Proof. Consider the exact sequence in $C(\mathscr{C})$:

$$0 \to M(\mathrm{id}_X) \xrightarrow{\gamma} \mathrm{Mc}(f) \xrightarrow{\varphi} Z \to 0$$

where $\gamma^n \colon (X^{n+1} \oplus X^n) \to X^{n+1} \oplus Y^n$ is defined by: $\gamma^n = \begin{pmatrix} \mathrm{id}_{X^{n+1}} & 0 \\ 0 & f^n \end{pmatrix}$. Since $H^k(\mathrm{Mc}(\mathrm{id}_X)) \simeq 0$ for all k, we get the result. □

We shall localize $K(\mathscr{C})$ with respect to the family of objects qis to zero (see Section 6.4). Define:

$$N(\mathscr{C}) = \{X \in K(\mathscr{C}); H^k(X) \simeq 0 \text{ for all } k\}.$$

One also defines $N^*(\mathscr{C}) = N(\mathscr{C}) \cap K^*(\mathscr{C})$ for $^* = \mathrm{b}, +, -$.

Clearly, $N^*(\mathscr{C})$ is a null system in $K^*(\mathscr{C})$.

Definition 7.1.3. One defines the derived categories $\mathrm{D}^*(\mathscr{C})$ as $K^*(\mathscr{C})/N^*(\mathscr{C})$, where $* = \mathrm{ub}, \mathrm{b}, +, -$. One denotes by Q the localization functor $K^*(\mathscr{C}) \to \mathrm{D}^*(\mathscr{C})$.

By Theorem 6.4.2, these are triangulated categories. Note that:

- a quasi-isomorphism in $K(\mathscr{C})$ becomes an isomorphism in $\mathrm{D}(\mathscr{C})$,

Recall the truncation functors given in (4.5.2). These functors send a complex homotopic to zero to a complex homotopic to zero, hence are well defined on $\mathrm{K}^+(\mathscr{C})$. Moreover, they send a qis to a qis. Hence the functors below are well defined:

$$\begin{aligned} H^j(\,\bullet\,) \colon \mathrm{D}(\mathscr{C}) &\to \mathscr{C}, \\ \tau^{\le n}, \widetilde{\tau}^{\le n} \colon \mathrm{D}(\mathscr{C}) &\to \mathrm{D}^-(\mathscr{C}), \\ \tau^{\ge n}, \widetilde{\tau}^{\ge n} \colon \mathrm{D}(\mathscr{C}) &\to \mathrm{D}^+(\mathscr{C}). \end{aligned}$$

Note that:

- there are isomorphisms of functors

$$\tau^{\le n} \simeq \widetilde{\tau}^{\le n}, \quad \tau^{\ge n} \simeq \widetilde{\tau}^{\ge n},$$

- $H^j(\,\bullet\,)$ is a cohomological functor on $\mathrm{D}^*(\mathscr{C})$ (apply Proposition 7.1.1).

In particular, if $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{+1}$ is a d.t. in $\mathrm{D}(\mathscr{C})$, we get a long exact sequence:

$$\cdots \to H^k(X) \to H^k(Y) \to H^k(Z) \to H^{k+1}(X) \to \cdots \tag{7.1.1}$$

Let $X \in K(\mathscr{C})$, with $H^j(X) = 0$ for $j > n$. Then the morphism $\tau^{\le n}X \to X$ in $K(\mathscr{C})$ is a qis, hence an isomorphism in $\mathrm{D}(\mathscr{C})$.

It follows from Proposition 6.4.4 that $\mathrm{D}^+(\mathscr{C})$ is equivalent to the full subcategory of $\mathrm{D}(\mathscr{C})$ consisting of objects X satisfying $H^j(X) \simeq 0$ for $j << 0$, and similarly for $\mathrm{D}^-(\mathscr{C}), \mathrm{D}^{\mathrm{b}}(\mathscr{C})$. Moreover, $\mathscr{C}$ is equivalent to the full subcategory of $\mathrm{D}(\mathscr{C})$ consisting of objects X satisfying $H^j(X) \simeq 0$ for $j \neq 0$. For $a, b \in \mathbb{Z} \sqcup \{-\infty\} \sqcup \{+\infty\}$ with $a \le b$, one sets

$$\mathrm{D}^{[a,b]}(\mathscr{C}) := \{X \in \mathrm{D}(\mathscr{C}); H^j(X) \simeq 0 \text{ for } j \notin [a, b]\}. \tag{7.1.2}$$

One defines similarly $\mathrm{D}^{\ge k}(\mathscr{C})$, $\mathrm{D}^{\le k}(\mathscr{C})$, etc.

Definition 7.1.4. Let X, Y be objects of $\mathscr{C}$. One sets

$$\mathrm{Ext}^k_{\mathscr{C}}(X,Y) = \mathrm{Hom}_{\mathrm{D}(\mathscr{C})}(X, Y[k]).$$

We shall see in Theorem 7.4.5 below that if $\mathscr{C}$ has enough injectives, this definition is compatible with Definition 4.7.2.

Notation 7.1.5. Let A be a ring. We shall write for short $\mathrm{D}^*(A)$ instead of $\mathrm{D}^*(\mathrm{Mod}(A))$, for $* = \varnothing, \mathrm{b}, +, -$.

Remark 7.1.6. (i) Let $X \in K(\mathscr{C})$, and let $Q(X)$ denote its image in $\mathrm{D}(\mathscr{C})$. One can prove that:

$$Q(X) \simeq 0 \Leftrightarrow X \text{ is qis to } 0 \text{ in } K(\mathscr{C}).$$

(ii) Let $f\colon X \to Y$ be a morphism in $\mathrm{C}(\mathscr{C})$. Then $f \simeq 0$ in $\mathrm{D}(\mathscr{C})$ iff there exists X' and a qis $g\colon X' \to X$ such that $f \circ g$ is homotopic to 0, or else iff there exists Y' and a qis $h\colon Y \to Y'$ such that $h \circ f$ is homotopic to 0.

Remark 7.1.7. Consider the morphism $\gamma\colon Z \to X[1]$ in $\mathrm{D}(\mathscr{C})$. If X, Y, Z belong to $\mathscr{C}$ (i.e., are concentrated in degree 0), the morphism $H^k(\gamma)\colon H^k(Z) \to H^{k+1}(X)$ is 0 for all $k \in \mathbb{Z}$. However, γ is *not* the zero morphism in $\mathrm{D}(\mathscr{C})$ in general (this happens if the short exact sequence splits). In fact, let us apply the cohomological functor $\mathrm{Hom}_{\mathscr{C}}(W, \bullet)$ to the d.t. above. It gives rise to the long exact sequence:

$$\cdots \to \mathrm{Hom}_{\mathscr{C}}(W,Y) \to \mathrm{Hom}_{\mathscr{C}}(W,Z) \xrightarrow{\tilde{\gamma}} \mathrm{Hom}_{\mathscr{C}}(W,X[1]) \to \cdots$$

where $\tilde{\gamma} = \mathrm{Hom}_{\mathscr{C}}(W,\gamma)$. Since $\mathrm{Hom}_{\mathscr{C}}(W,Y) \to \mathrm{Hom}_{\mathscr{C}}(W,Z)$ is not an epimorphism in general, $\tilde{\gamma}$ is not zero. Therefore γ is not zero in general. The morphism γ may be described as follows.

$$\begin{array}{ccccccccc}
Z := & & 0 & \longrightarrow & 0 & \longrightarrow & Z & \longrightarrow & 0 \\
\uparrow \varphi & & & & \uparrow & & \uparrow & & \\
\mathrm{Mc}(f) := & & 0 & \longrightarrow & X & \xrightarrow{f} & Y & \longrightarrow & 0 \\
\downarrow \beta(f) & & & & \downarrow \mathrm{id} & & \downarrow & & \\
X[1] := & & 0 & \longrightarrow & X & \longrightarrow & 0 & \longrightarrow & 0.
\end{array}$$

Proposition 7.1.8. *Let $X \in \mathrm{D}(\mathscr{C})$.*

(i) *There are d.t. in $\mathrm{D}(\mathscr{C})$:*

$$\begin{aligned}
&\tau^{\leq n}X \to X \to \tau^{\geq n+1}X \xrightarrow{+1}, \\
&\tau^{\leq n-1}X \to \tau^{\leq n}X \to H^n(X)[-n] \xrightarrow{+1}, \\
&H^n(X)[-n] \to \tau^{\geq n}X \to \tau^{\geq n+1}X \xrightarrow{+1}.
\end{aligned} \tag{7.1.3}$$

(ii) *Moreover, $H^n(X)[-n] \simeq \tau^{\leq n}\tau^{\geq n}X \simeq \tau^{\geq n}\tau^{\leq n}X$.*

Corollary 7.1.9. *Let $\mathscr{C}$ be an abelian category and assume that for any $X, Y \in \mathscr{C}$, $\mathrm{Ext}^k_{\mathscr{C}}(X,Y) \simeq 0$ for $k \geq 2$. Let $X \in \mathrm{D}^{\mathrm{b}}(\mathscr{C})$. Then:*

$$X \simeq \bigoplus_j H^j(X)\,[-j].$$

Proof. Call *amplitude* of X the smallest integer k such that $H^j(X) \simeq 0$ for j not belonging to some interval of length k. If $k = 0$, this means that there exists some i with $H^j(X) = 0$ for $j \neq i$, hence $X \simeq H^i(X)[-i]$. Now we argue by induction on the amplitude. Consider the d.t. (7.1.3):

$$\tau^{\leq n-1}X \to \tau^{\leq n}X \to H^n(X)[-n] \xrightarrow{+1}$$

and assume $\tau^{\leq n-1}X \simeq \oplus_{j<n} H^j(X)[-j]$. By the result of Exercise 5.3, it it enough to show that $\mathrm{Hom}_{\mathrm{D^b}(\mathscr{C})}(H^n(X)[-n], H^j(X)[-j+1]) = 0$ for $j < n$. Since $n+1-j \geq 2$, the result follows. □

Example 7.1.10. (i) If a ring A is a principal ideal domain (such as a field, or $\mathbb{Z}$, or $k[x]$ for $\mathbf{k}$ a field), then the category $\mathrm{Mod}(A)$ satisfies the hypotheses of Corollary 7.1.9.
(ii) See Example 7.4.8 for an object which does not split.

7.2 Resolutions

Applying Proposition 4.6.3, we have:

Lemma 7.2.1. *Let $\mathscr{J}$ be an additive subcategory of $\mathscr{C}$ and assume that $\mathscr{J}$ is generating. Then for any $X^\bullet \in \mathrm{K}^+(\mathscr{C})$, there exists $Y^\bullet \in \mathrm{K}^+(\mathscr{J})$ and a qis $X^\bullet \to Y^\bullet$.*

We set $N^+(\mathscr{J}) := N(\mathscr{C}) \cap \mathrm{K}^+(\mathscr{J})$. It is clear that $N^+(\mathscr{J})$ is a null system in $\mathrm{K}^+(\mathscr{J})$.

Proposition 7.2.2. *Assume $\mathscr{J}$ is generating in $\mathscr{C}$. Then the natural functor $\theta\colon \mathrm{K}^+(\mathscr{J})/N^+(\mathscr{J}) \to \mathrm{D}^+(\mathscr{C})$ is an equivalence of categories.*

Proof. Apply Lemma 7.2.1 and Proposition 6.4.4. □

Let us apply the preceding proposition to the category $\mathscr{I}_{\mathscr{C}}$ of injective objects of $\mathscr{C}$.

Corollary 7.2.3. *Assume that $\mathscr{C}$ admits enough injectives. Then $\mathrm{K}^+(\mathscr{I}_{\mathscr{C}}) \to \mathrm{D}^+(\mathscr{C})$ is an equivalence of categories.*

Proof. Recall that if $X^\bullet \in \mathrm{C}^+(\mathscr{I}_{\mathscr{C}})$ is qis to 0, then $X^\bullet$ is homotopic to 0. □

Remark 7.2.4. Assume that $\mathscr{C}$ admits enough injectives. Then $\mathrm{D}^+(\mathscr{C})$ is a $\mathscr{U}$-category.

7.3 Derived functors

In this section, $\mathscr{C}$ and $\mathscr{C}'$ will denote abelian categories. Let $F\colon \mathscr{C} \to \mathscr{C}'$ be a left exact functor. It defines naturally a functor

$$\mathrm{K}^+F\colon \mathrm{K}^+(\mathscr{C}) \to \mathrm{K}^+(\mathscr{C}').$$

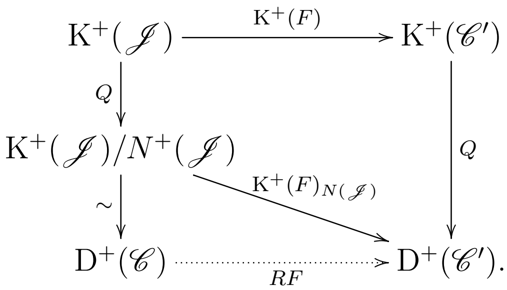
For short, one often writes F instead of K^+F. Applying the results of Chapter 6, we shall construct (under suitable hypotheses) the right localization of F. Recall Definition 4.7.5. By Lemma 4.7.6, $\mathrm{K}^+(F)$ sends $N^+(\mathscr{J})$ to $N^+(\mathscr{C}')$.

Definition 7.3.1. If the functor $\mathrm{K}^+(F)\colon \mathrm{K}^+(\mathscr{C}) \to \mathrm{D}^+(\mathscr{C}')$ admits a right localization (with respect to the qis in $\mathrm{K}^+(\mathscr{C})$), one says that F admits a right derived functor and one denotes by $RF\colon \mathrm{D}^+(\mathscr{C}) \to \mathrm{D}^+(\mathscr{C}')$ the right localization of F.

Theorem 7.3.2. *Let $F\colon \mathscr{C} \to \mathscr{C}'$ be a left exact functor of abelian categories and let $\mathscr{J} \subset \mathscr{C}$ be a full additive subcategory. Assume that $\mathscr{J}$ is F-injective. Then F admits a right derived functor $RF\colon \mathrm{D}^+(\mathscr{C}) \to \mathrm{D}^+(\mathscr{C}')$.*

Proof. This follows immediately from Lemma 7.2.1 and Proposition 6.4.6 applied to $\mathrm{K}^+(F)\colon \mathrm{K}^+(\mathscr{C}) \to \mathrm{D}^+(\mathscr{C}')$. □

It is vizualised by the diagram



Since $\mathrm{Ob}(\mathrm{K}^+(\mathscr{J})/N^+(\mathscr{J})) = \mathrm{Ob}(\mathrm{K}^+(\mathscr{J}))$, we get that for $X \in \mathrm{K}^+(\mathscr{C})$, if there is a qis $X \to Y$ with $Y \in \mathrm{K}^+(\mathscr{J})$, then $RF(X) \simeq F(Y)$ in $\mathrm{D}^+(\mathscr{C}')$.

Note that if $\mathscr{C}$ admits enough injectives, then

$$R^kF = H^k \circ RF. \tag{7.3.1}$$

Recall that the derived functor RF is triangulated, and does not depend on the category $\mathscr{J}$. Hence, if $X' \to X \to X'' \xrightarrow{+1}$ is a d.t. in $\mathrm{D}^+(\mathscr{C})$, then $RF(X') \to RF(X) \to RF(X'') \xrightarrow{+1}$ is a d.t. in $\mathrm{D}^+(\mathscr{C}')$. (Recall that an exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathscr{C}$ gives rise to a d.t. in $\mathrm{D}(\mathscr{C})$.) Applying the cohomological functor H^0, we get the long exact sequence in $\mathscr{C}'$:

$$\cdots \to R^kF(X') \to R^kF(X) \to R^kF(X'') \to R^{k+1}F(X') \to \cdots$$

By considering the category $\mathscr{C}^{\mathrm{op}}$, one defines the notion of left derived functor of a right exact functor F.

Derived functor of a composition

Let $F\colon \mathscr{C} \to \mathscr{C}'$ and $G\colon \mathscr{C}' \to \mathscr{C}''$ be left exact functors of abelian categories. Then $G \circ F\colon \mathscr{C} \to \mathscr{C}''$ is left exact. Using the universal property of the localization, one shows that if F, G and $G \circ F$ are right derivable, then there exists a natural morphism of functors

$$R(G \circ F) \to RG \circ RF. \tag{7.3.2}$$

Proposition 7.3.3. *Assume that there exist full additive subcategories $\mathscr{J} \subset \mathscr{C}$ and $\mathscr{J}' \subset \mathscr{C}'$ such that $\mathscr{J}$ is F-injective, $\mathscr{J}'$ is G-injective and $F(\mathscr{J}) \subset \mathscr{J}'$. Then $\mathscr{J}$ is $(G \circ F)$-injective and the morphism in (7.3.2) is an isomorphism:*

$$R(G \circ F) \simeq RG \circ RF.$$

Proof. (i) The fact that $\mathscr{J}$ is $(G \circ F)$ injective follows immediately from the definition.
(ii) Let $X \in \mathrm{K}^+(\mathscr{C})$ and $Y \in \mathrm{K}^+(\mathscr{J})$ together with a qis $X \to Y$. Then $RF(X)$ is represented by the complex $F(Y)$ which belongs to $\mathrm{K}^+(\mathscr{J}')$. Hence $RG(RF(X))$ is represented by $G(F(Y)) = (G \circ F)(Y)$, and this last complex also represents $R(G \circ F)(Y)$ since $Y \in \mathrm{C}^+(\mathscr{J})$ and $\mathscr{J}$ is $G \circ F$ injective. □

Note that in general F does not send injective objects of $\mathscr{C}$ to injective objects of $\mathscr{C}'$. That is why we had to introduce the notion of "F-injective" category.

7.4 Bifunctors

Now consider three abelian categories $\mathscr{C}, \mathscr{C}', \mathscr{C}''$ and an *additive* bifunctor:

$$F : \mathscr{C} \times \mathscr{C}' \to \mathscr{C}''.$$

We shall assume that F is left exact with respect to each of its arguments.

Let $X \in \mathrm{K}^+(\mathscr{C}), X' \in \mathrm{K}^+(\mathscr{C}')$ and assume that X or X' is homotopic to 0. Then one checks easily that $\mathrm{tot}(F(X, X'))$ is homotopic to zero. Hence one can naturally define:

$$\mathrm{K}^+(F) : \mathrm{K}^+(\mathscr{C}) \times \mathrm{K}^+(\mathscr{C}') \to \mathrm{K}^+(\mathscr{C}'')$$

by setting:

$$\mathrm{K}^+(F)(X, X') = \mathrm{tot}(F(X, X')).$$

If there is no risk of confusion, we shall sometimes write F instead of K^+F.

Definition 7.4.1. One says $(\mathscr{J}, \mathscr{J}')$ is F-injective if:

(i) for all $X \in \mathscr{J}$, $\mathscr{J}'$ is $F(X, \bullet)$-injective.

(ii) for all $X' \in \mathscr{J}'$, $\mathscr{J}$ is $F(\bullet, X')$-injective.

Lemma 7.4.2. *Let $X \in \mathrm{K}^+(\mathscr{J})$ and $X' \in \mathrm{K}^+(\mathscr{J}')$. If X or X' is qis to 0, then $F(X, X')$ is qis to zero.*

Proof. The double complex $F(X, Y)$ will satisfy the hypothesis of Theorem 4.7.10. □

Using Lemma 7.4.2 and Proposition 6.4.7 one gets that F admits a right derived functor,

$$RF : \mathrm{D}^+(\mathscr{C}) \times \mathrm{D}^+(\mathscr{C}') \to \mathrm{D}^+(\mathscr{C}'').$$

Example 7.4.3. Assume $\mathscr{C}$ has enough injectives. Then

$$\mathrm{RHom}_{\mathscr{C}} : \mathrm{D}^-(\mathscr{C})^{\mathrm{op}} \times \mathrm{D}^+(\mathscr{C}) \to \mathrm{D}^+(\mathbb{Z})$$

exists and may be calculated as follows. Let $X \in \mathrm{D}^-(\mathscr{C})$ and $Y \in \mathrm{D}^+(\mathscr{C})$. There exists a qis in $\mathrm{K}^+(\mathscr{C})$, $Y \to I$, the I^j's being injective. Then:

$$\mathrm{RHom}_{\mathscr{C}}(X,Y) \simeq \mathrm{Hom}^{\bullet}_{\mathscr{C}}(X,I).$$

If $\mathscr{C}$ has enough projectives, and $P \to X$ is a qis in $\mathrm{K}^-(\mathscr{C})$, the P^j's being projective, one also has:

$$\mathrm{RHom}_{\mathscr{C}}(X,Y) \simeq \mathrm{Hom}^{\bullet}_{\mathscr{C}}(P,Y).$$

These isomorphisms hold in $\mathrm{D}^+(\mathbb{Z})$.

Example 7.4.4. Let A be a $\mathbf{k}$-algebra. The functor

$$\bullet \overset{\mathrm{L}}{\otimes}_A \bullet : \ \mathrm{D}^-(A^{\mathrm{op}}) \times \mathrm{D}^-(A) \to \mathrm{D}^-(\mathbf{k})$$

is well defined. Moreover,

$$N \overset{\mathrm{L}}{\otimes}_A M \simeq s(N \otimes_A P) \simeq s(Q \otimes_A M)$$

where P (resp. Q) is a complex of projective A-modules qis to M (resp. to N).

In the preceding situation, one has:

$$\mathrm{Tor}_A^{-k}(N,M) \simeq H^k(N \overset{\mathrm{L}}{\otimes}_A M).$$

The functors $\mathrm{RHom}_{\mathscr{C}}$ and $\mathrm{Hom}_{\mathrm{D}(\mathscr{C})}$

Theorem 7.4.5. *Let $\mathscr{C}$ be an abelian category with enough injectives. Then for $X \in \mathrm{D}^-(\mathscr{C})$ and $Y \in \mathrm{D}^+(\mathscr{C})$*

$$H^0\mathrm{RHom}_{\mathscr{C}}(X,Y) \simeq \mathrm{Hom}_{\mathrm{D}(\mathscr{C})}(X,Y).$$

Proof. By Proposition 4.6.3, there exists $I_Y \in \mathrm{C}^+(\mathscr{I})$ and a qis $Y \to I_Y$. Then we have the isomorphisms:

$$\begin{aligned}\mathrm{Hom}_{\mathrm{D}(\mathscr{C})}(X,Y[k]) &\simeq \mathrm{Hom}_{K(\mathscr{C})}(X,I_Y[k]) \\ &\simeq H^0(\mathrm{Hom}^{\bullet}_{\mathscr{C}}(X,I_Y[k])) \\ &\simeq R^k\mathrm{Hom}_{\mathscr{C}}(X,Y),\end{aligned}$$

where the second isomorphism follows from Proposition 3.4.5. □

Theorem 7.4.5 implies the isomorphism

$$\mathrm{Ext}^k_{\mathscr{C}}(X,Y) \simeq H^k\mathrm{RHom}_{\mathscr{C}}(X,Y).$$

Example 7.4.6. Let W be the Weyl algebra in one variable over a field $\mathbf{k}$ of characteristic 0: $W = \mathbf{k}[x,\partial]$ with the relation $[x,\partial] = -1$.

Let $\mathscr{O} = W/W\cdot\partial$, $\Omega = W/\partial\cdot W$ and let us calculate $\Omega \overset{\mathrm{L}}{\otimes}_W \mathscr{O}$. We have an exact sequence: $0 \to W \xrightarrow{\partial\cdot} W \to \Omega \to 0$. Therefore, Ω is qis to the complex

$$0 \to W^{-1} \xrightarrow{\partial\cdot} W^0 \to 0$$

where $W^{-1} = W^0 = W$ and W^0 is in degree 0. Then $\Omega \overset{\mathrm{L}}{\otimes}_W \mathscr{O}$ is qis to the complex

$$0 \to \mathscr{O}^{-1} \xrightarrow{\partial\cdot} \mathscr{O}^0 \to 0,$$

where $\mathscr{O}^{-1} = \mathscr{O}^0 = \mathscr{O}$ and $\mathscr{O}^0$ is in degree 0. Since $\partial\colon \mathscr{O} \to \mathscr{O}$ is surjective and has $\mathbf{k}$ as kernel, we obtain:

$$\Omega \overset{\mathrm{L}}{\otimes}_W \mathscr{O} \simeq \mathbf{k}[1].$$

Example 7.4.7. Let $\mathbf{k}$ be a field and let $A = \mathbf{k}[x_1, \dots, x_n]$. This is a commutative Noetherian ring and it is known (Hilbert) that any finitely generated A-module M admits a finite free presentation of length at most n, *i.e.,* M is qis to a complex:

$$L := \quad 0 \to L^{-n} \to \cdots \xrightarrow{P_0} L^0 \to 0$$

where the L^j's are free of finite rank. Consider the left exact functor

$$\mathrm{Hom}_A(\,\bullet\,, A)\colon \mathrm{Mod}(A)^{\mathrm{op}} \to \mathrm{Mod}(A)$$

and denote for short by * its derived functor:

$$(7.4.1) \qquad {}^* := \mathrm{RHom}_A(\,\bullet\,, A).$$

Since free A-modules are projective, we find that $\mathrm{RHom}_A(M, A)$ is isomorphic in $\mathrm{D^b}(A)$ to the complex

$$L^* := \quad 0 \leftarrow L^{-n*} \leftarrow \cdots \xleftarrow{P_0} L^{0*} \leftarrow 0.$$

Using (7.3.2), we find a natural morphism of functors

$$\mathrm{id} \to {}^{**} := {}^* \circ {}^*.$$

Applying * to the object $\mathrm{RHom}_A(M, A)$ we find:

$$\begin{aligned} \mathrm{RHom}_A(\mathrm{RHom}_A(M, A), A) &\simeq \mathrm{RHom}_A(L^*, A) \\ &\simeq L \simeq M. \end{aligned}$$

In other words, we have proved the isomorphism $M \simeq M^{**}$ in $\mathrm{D^b}(A)$.

Assume now $n = 1$, *i.e.,* $A = \mathbf{k}[x]$ and consider the natural morphism in $\mathrm{Mod}(A)$: $f\colon A \to A/Ax$. Applying the functor *, we get the morphism in $\mathrm{D^b}(A)$:

$$f^*\colon \mathrm{RHom}_A(A/Ax, A) \to A.$$

Remember that $\mathrm{RHom}_A(A/Ax, A) \simeq A/xA\,[-1]$. Hence $H^j(f^*) = 0$ for all $j \in \mathbb{Z}$, although $f^* \neq 0$ since $f^{**} = f$.

Let us give an example of an object of a derived category which is not isomorphic to the direct sum of its cohomology objects (hence, a situation in which Corollary 7.1.9 does not apply).

Example 7.4.8. Let $\mathbf{k}$ be a field and let $A = \mathbf{k}[x_1, x_2]$. Define the A-modules

$$M' = A/(Ax_1 + Ax_2), \quad M = A/(Ax_1^2 + Ax_1x_2), \quad M'' = A/Ax_1.$$

There is an exact sequence

$$0 \to M' \to M \to M'' \to 0 \tag{7.4.2}$$

and this exact sequence does not split since x_1 kills M' and M'' but not M.

Recall the functor * of (7.4.1). We have $M'^* \simeq H^2(M'^*)[-2]$ and $M''^* \simeq H^1(M'^*)[-1]$. The functor * applied to the exact sequence (7.4.2) gives rise to the long exact sequence

$$0 \to H^1(M''^*) \to H^1(M^*) \to 0 \to 0 \to H^2(M^*) \to H^2(M'^*) \to 0.$$

Hence $H^1(M^*)[-1] \simeq H^1(M''^*)[-1] \simeq M''^*$ and $H^2(M^*)[-2] \simeq H^2(M'^*)[-2] \simeq M'^*$. Assume for a while $M^* \simeq \oplus_j H^j(M^*)[-j]$. This implies $M^* \simeq M'^* \oplus M''^*$ hence (by applying again the functor *), $M \simeq M' \oplus M''$, which is a contradiction.

7.5 The Brown representability theorem

We shall follow the exposition of [KS06, § 10.5].

Definition 7.5.1. Let $\mathscr{D}$ be a triangulated category admitting small direct sums. A *system of t-generators* $\mathscr{F}$ in $\mathscr{D}$ is a small family of objects of $\mathscr{D}$ satisfying conditions (i) and (ii) below.

(i) For any $X \in \mathscr{D}$ with $\mathrm{Hom}_{\mathscr{D}}(C, X) \simeq 0$ for all $C \in \mathscr{F}$, we have $X \simeq 0$.

(ii) For any *countable* set I and any family $\{u_i \colon X_i \to Y_i\}_{i \in I}$ of morphisms in $\mathscr{D}$, the map $\mathrm{Hom}_{\mathscr{D}}(C, \oplus_i X_i) \xrightarrow{\oplus_i u_i} \mathrm{Hom}_{\mathscr{D}}(C, \oplus_i Y_i)$ vanishes for every $C \in \mathscr{F}$ as soon as $\mathrm{Hom}_{\mathscr{D}}(C, X_i) \xrightarrow{u_i} \mathrm{Hom}_{\mathscr{D}}(C, Y_i)$ vanishes for every $i \in I$ and every $C \in \mathscr{F}$.

What we call below the Brown representability Theorem is in fact a corollary of such a theorem. See [KS06, Cor. 10.5.3].

Theorem 7.5.2. (The Brown representability Theorem) *Let $\mathscr{D}$ be a triangulated category admitting small direct sums and a system of t-generators. Let $F \colon \mathscr{D} \to \mathscr{D}'$ be a triangulated functor of triangulated categories and assume that F commutes with small direct sums. Then F admits a right adjoint G and G is triangulated.*

Recall Definition 4.4.3 of a Grothendieck category and also recall that such a definition relies on the notion of universe. Hence, all categories in the sequel belong to a given universe $\mathscr{U}$.

We shall apply Theorem 7.5.2 in the particular case of derived categories. Recall Definition 7.5.1 of a system of t-generators in a triangulated category and also recall Definition 4.7.3.

Theorem 7.5.3. (see [KS06, Th. 14.2.1]) *Let $\mathscr{C}$ be a Grothendieck abelian category. Then* $\mathrm{D}(\mathscr{C})$ *admits small direct sums and a system of t-generators.*

Note that the existence of small direct sums follows from Proposition 6.4.8.

From now on, we shall follow [?GS16, § 2.3].

Lemma 7.5.4. *Let $\mathscr{C}$ be a Grothendieck category and let $d \in \mathbb{Z}$. Then the cohomology functor H^d and the truncation functors $\tau^{\leq d}$ and $\tau^{\geq d}$ commute with small direct sums in $\mathrm{D}(\mathscr{C})$. In other words, if $\{X_i\}_{i\in I}$ is a small family of objects of $\mathrm{D}(\mathscr{C})$, then*

$$\bigoplus_i \tau^{\leq d} X_i \xrightarrow{\sim} \tau^{\leq d}(\bigoplus_i X_i) \tag{7.5.1}$$

and similarly with $\tau^{\geq d}$ and H^d.

Proof. (i) Let us treat first the functor H^d. Recall that $Q\colon \mathrm{K}(\mathscr{C}) \to \mathrm{D}(\mathscr{C})$ denotes the localization functor and Q commutes with small direct sums by Proposition 6.4.8. Let us denote for a while by $\widetilde{H}^d\colon \mathrm{K}(\mathscr{C}) \to \mathscr{C}$ the cohomology functor usually denoted by H^d. Then $\widetilde{H}^d \simeq H^d \circ Q$.

Let $\{X_i\}_i$ be a small family of objects in $\mathrm{K}(\mathscr{C})$. Then

$$\begin{aligned} H^d(\oplus_i Q(X_i)) \simeq H^d(Q(\oplus_i X_i)) &\simeq \widetilde{H}^d(\oplus_i X_i) \\ &\simeq \oplus_i \widetilde{H}^d(X_i) \simeq \oplus_i H^d(Q(X_i))). \end{aligned}$$

(ii) The morphism in (7.5.1) is well-defined and it is enough to check that it induces an isomorphism on the cohomology. This follows from (i) since for any object $Y \in \mathrm{D}(\mathscr{C})$, $H^j(\tau^{\leq d} Y)$ is either 0 or $H^j(Y)$. □

Lemma 7.5.5. *Let $\mathscr{C}$ and $\mathscr{C}'$ be two Grothendieck categories and let $\rho\colon \mathscr{C} \to \mathscr{C}'$ be a left exact functor. Let I be a small category. Assume*

(i) *I is either filtrant or discrete,*

(ii) *ρ commutes with inductive limits indexed by I,*

(iii) *inductive limits indexed by I of injective objects in $\mathscr{C}$ are acyclic for the functor ρ.*

Then for all $j \in \mathbb{Z}$, the functor $R^j\rho\colon \mathscr{C} \to \mathscr{C}'$ commutes with inductive limits indexed by I.

Proof. Let $\alpha\colon I \to \mathscr{C}$ be a functor. Denote by $\mathscr{I}$ the full additive subcategory of $\mathscr{C}$ consisting of injective objects. It follows for example from [KS06, Cor. 9.6.6] that there exists a functor $\psi\colon I \to \mathscr{I}$ and a morphism of functors $\alpha \to \psi$ such that for each $i \in I$, $\alpha(i) \to \psi(i)$ is a monomorphism. Therefore one can construct a functor $\Psi\colon I \to \mathrm{C}^+(\mathscr{I})$ and a morphism of functor $\alpha \to \Psi$ such that for each $i \in I$, $\alpha(i) \to \Psi(i)$ is a quasi-isomorphism. Set $X_i = \alpha(i)$ and $G_i^\bullet = \Psi(i)$. We get a qis $X_i \to G_i^\bullet$, hence a qis

$$\operatorname*{colim}_i X_i \to \operatorname*{colim}_i G_i^\bullet.$$

On the other hand, we have

$$\begin{aligned} \operatorname*{colim}_i R^j\rho(X_i) &\simeq \operatorname*{colim}_i H^j(\rho(G_i^\bullet)) \\ &\simeq H^j\rho(\operatorname*{colim}_i G_i^\bullet) \end{aligned}$$

where the second isomorphism follows from the fact that H^j commutes with direct sums and with filtrant inductive limits. Then the result follows from hypothesis (iii).

□

Lemma 7.5.6. *We make the same hypothesis as in Lemma 7.5.5. Let $-\infty < a \le b < \infty$, let I be a small set and let $X_i \in \mathrm{D}^{[a,b]}(\mathscr{C})$. Then*

$$\bigoplus_i R\rho(X_i) \xrightarrow{\sim} R\rho(\bigoplus_i X_i). \tag{7.5.2}$$

Proof. The morphism in (7.5.2) is well-defined and we have to prove it is an isomorphism. If $b = a$, the result follows from Lemma 7.5.5. The general case is deduced by induction on $b - a$ by considering the distinguished triangles

$$H^a(X_i)\,[-a] \to X_i \to \tau^{\ge a+1}X_i \xrightarrow{+1}.$$

□

Theorem 7.5.7. (see [?GS16, Prop. 2.21]) *Let $\mathscr{C}$ and $\mathscr{C}'$ be two Grothendieck categories and let $\rho\colon \mathscr{C} \to \mathscr{C}'$ be a left exact functor. Assume that*

(i) *ρ has finite cohomological dimension,*

(ii) *ρ commutes with small direct sums,*

(iii) *small direct sums of injective objects in $\mathscr{C}$ are acyclic for the functor ρ.*

Then

(a) *the functor $R\rho\colon \mathrm{D}(\mathscr{C}) \to \mathrm{D}(\mathscr{C}')$ commutes with small direct sums,*

(b) *the functor $R\rho\colon \mathrm{D}(\mathscr{C}) \to \mathrm{D}(\mathscr{C}')$ admits a right adjoint $\rho^!\colon \mathrm{D}(\mathscr{C}') \to \mathrm{D}(\mathscr{C})$,*

(c) *the functor $\rho^!$ induces a functor $\rho^!\colon \mathrm{D}^+(\mathscr{C}') \to \mathrm{D}^+(\mathscr{C})$.*

Proof. (a) Let $\{X_i\}_{i\in I}$ be a family of objects of $\mathrm{D}(\mathscr{C})$. It is enough to check that the natural morphism in $\mathrm{D}(\mathscr{C}')$

$$\bigoplus_{i\in I} R\rho(X_i) \to R\rho(\bigoplus_{i\in I} X_i) \tag{7.5.3}$$

induces an isomorphism on the cohomology groups. Assume that ρ has cohomological dimension $\le d$. For $X \in \mathrm{D}(\mathscr{C})$ and for $j \in \mathbb{Z}$, we have

$$\tau^{\ge j}R\rho(X) \simeq \tau^{\ge j}R\rho(\tau^{\ge j-d-1}X).$$

The functor ρ being left exact we get for $k \ge j$:

$$H^kR\rho(X) \simeq H^kR\rho(\tau^{\le k}\tau^{\ge j-d-1}X). \tag{7.5.4}$$

We have the sequence of isomorphisms:

$$\begin{aligned} H^kR\rho(\bigoplus_i X_i) &\simeq H^kR\rho(\tau^{\le k}\tau^{\ge j-d-1}\bigoplus_i X_i) \simeq H^kR\rho(\bigoplus_i \tau^{\le k}\tau^{\ge j-d-1}X_i) \\ &\simeq \bigoplus_i H^kR\rho(\tau^{\le k}\tau^{\ge j-d-1}X_i) \simeq \bigoplus_i H^kR\rho(X_i). \end{aligned}$$

The first and last isomorphisms follow from (7.5.4).
The second isomorphism follows from Lemma 7.5.4.
The third isomorphism follows from Lemma 7.5.6.
(b) follows from (a) and the Brown representability theorem 7.5.2.
(c) This follows from hypothesis (i) and (the well-known) Lemma 7.5.8 below. ☐

Lemma 7.5.8. *Let $\rho\colon \mathscr{C} \to \mathscr{C}'$ be a left exact functor between two Grothendieck categories. Assume that $\rho\colon \mathrm{D}(\mathscr{C}) \to \mathrm{D}(\mathscr{C}')$ admits a right adjoint $\rho^!\colon \mathrm{D}(\mathscr{C}') \to \mathrm{D}(\mathscr{C})$ and assume moreover that ρ has finite cohomological dimension. Then the functor $\rho^!$ sends $\mathrm{D}^+(\mathscr{C}')$ to $\mathrm{D}^+(\mathscr{C})$.*

Proof. By the hypothesis, we have for $X \in \mathrm{D}(\mathscr{C})$ and $Y \in \mathrm{D}(\mathscr{C}')$

$$\mathrm{Hom}_{\mathrm{D}(\mathscr{C}')}(\rho(X), Y) \simeq \mathrm{Hom}_{\mathrm{D}(\mathscr{C})}(X, \rho^!(Y)).$$

Assume that the cohomological dimension of the functor ρ is $\leq r$. Let $Y \in \mathrm{D}^{\geq 0}(\mathscr{C}')$. Then (using Exercise 7.7) $\mathrm{Hom}_{\mathrm{D}(\mathscr{C})}(X, \rho^!(Y)) \simeq 0$ for all $X \in \mathrm{D}^{<-r}(\mathscr{C})$. This implies that $\rho^!(Y) \in \mathrm{D}^{\geq -r}(\mathscr{C}')$. ☐

Exercises to Chapter 7

Exercise 7.1. Let $\mathscr{C}$ be an abelian category with enough injectives. Prove that the two conditions below are equivalent.
(i) For all X and Y in $\mathscr{C}$, $\mathrm{Ext}^j_{\mathscr{C}}(X, Y) \simeq 0$ for all $j > n$.
(ii) For all X in $\mathscr{C}$, there exists an exact sequence $0 \to X \to X^0 \to \cdots \to X^n \to 0$, with the X^j's injective.
In such a situation, one says that $\mathscr{C}$ has homological dimension $\leq n$ and one writes $\mathrm{dh}(\mathscr{C}) \leq n$.
(iii) Assume moreover that $\mathscr{C}$ has enough projectives. Prove that (i) is equivalent to: for all X in $\mathscr{C}$, there exists an exact sequence $0 \to X^n \to \cdots \to X^0 \to X \to 0$, with the X^j's projective.

Exercise 7.2. Let $\mathscr{C}$ be an abelian category with enough injective and such that $\mathrm{dh}(\mathscr{C}) \leq 1$. Let $F\colon \mathscr{C} \to \mathscr{C}'$ be a left exact functor and let $X \in \mathrm{D}^+(\mathscr{C})$.
(i) Construct an isomorphism $H^k(RF(X)) \simeq F(H^k(X)) \oplus R^1F(H^{k-1}(X))$.
(ii) Recall that $\mathrm{dh}(\mathrm{Mod}(\mathbb{Z})) = 1$. Let $X \in \mathrm{D}^-(\mathbb{Z})$, and let $M \in \mathrm{Mod}(\mathbb{Z})$. Deduce the isomorphism

$$H^k(X \overset{\mathrm{L}}{\otimes} M) \simeq (H^k(X) \otimes M) \oplus \mathrm{Tor}_{\mathbb{Z}}^{-1}(H^{k+1}(X), M).$$

Exercise 7.3. Let $\mathscr{C}$ be an abelian category with enough injectives and let $0 \to X' \to X \to X'' \to 0$ be an exact sequence in $\mathscr{C}$. Assuming that $\mathrm{Ext}^1_{\mathscr{C}}(X'', X') \simeq 0$, prove that the sequence splits.

Exercise 7.4. Let $\mathscr{C}$ be an abelian category and let $X \to Y \to Z \xrightarrow{+1}$ be a d.t. in $\mathrm{D}(\mathscr{C})$. Assuming that $\mathrm{Ext}^1_{\mathscr{C}}(Z, X) \simeq 0$, prove that $Y \simeq X \oplus Z$. (Hint: use Exercise 5.3.)

Exercise 7.5. Let $\mathscr{C}$ be an abelian category, let $X \in \mathrm{D}^{\mathrm{b}}(\mathscr{C})$ and let $a < b \in \mathbb{Z}$. Assume that $H^j(X) \simeq 0$ for $j \neq a, b$ and $\mathrm{Ext}_{\mathscr{C}}^{b-a+1}(H^b(X), H^a(X)) \simeq 0$. Prove the isomorphism $X \simeq H^a(X)[-a] \oplus H^b(X)[-b]$. (Hint: use Exercise 7.4 and the d.t. in 7.1.3.

Exercise 7.6. We follow the notations of Exercise 4.9. Hence, $\mathbf{k}$ is a field of characteristic 0 and $W := W_n(\mathbf{k})$ is the Weyl algebra in n variables. Let $1 \leq p \leq n$ and consider the left ideal

$$I_p = W \cdot x_1 + \cdots + W \cdot x_p + W \cdot \partial_{p+1} + \cdots + W \cdot \partial_n.$$

Define similarly the right ideal

$$J_p = x_1 \cdot W + \cdots + x_p \cdot W + \partial_{p+1} \cdot W + \cdots + \partial_n \cdot W.$$

For $1 \leq p \leq q \leq n$, calculate $\mathrm{RHom}_W(W/I_p, W/I_q)$ and $W/J_q \overset{\mathrm{L}}{\otimes}_W W/I_p$.

Exercise 7.7. Let $\mathscr{C}$ be an abelian category.

(a) Let $X \in \mathrm{D}^{<0}(\mathscr{C})$ and $Y \in \mathrm{D}^{\geq 0}(\mathscr{C})$. Prove that $\mathrm{Hom}_{\mathrm{D}(\mathscr{C})}(X, Y) \simeq 0$.

(b) Conversely, let $Y \in \mathrm{D}(\mathscr{C})$ and assume that $\mathrm{Hom}_{\mathrm{D}(\mathscr{C})}(X, Y) \simeq 0$ for all $X \in \mathrm{D}^{<0}(\mathscr{C})$. Prove that $Y \in \mathrm{D}^{\geq 0}(\mathscr{C})$.

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