Advanced Studies in Pure Mathematics ,

pp. 1–9

Masaki Kashiwara and Algebraic Analysis

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It is a great honor[1] to present here some aspects of the work of Masaki Kashiwara.

Recall that Masaki's work covers many fields of mathematics, algebraic and microlocal analysis of course, but also representation theory, Hodge theory, integrable systems, quantum groups and so on. Also recall that Masaki had many collaborators, among whom Daniel Barlet, Jean-Luc Brylinski, Etsurio Date, Ryoji Hotta, Michio Jimbo, Seok-Jin Kang, Takahiro Kawai, Tetsuji Miwa, Kiyosato Okamoto, Toshio Oshima, Mikio Sato, myself, Toshiyuki Tanisaki and Michèle Vergne.

In each of the domain he approached, Masaki has given essential contributions and made important discoveries, such as, for example, the existence of crystal bases in quantum groups. But in this talk, I will restrict myself to describe some part of his work related to microlocal and algebraic analysis.

The story begins long ago, in the early sixties, when Mikio Sato created a new branch of mathematics now called "Algebraic Analysis". In 1959/60, M. Sato published two papers on hyperfunction theory [24] and then developed his vision of analysis and linear partial differential equations in a series of lectures at Tokyo University (see [1]). If M is a real analytic manifold and X is a complexification of M, hyperfunctions on M are cohomology classes supported by M of the sheaf $\mathcal{O}_X$ of holomorphic functions on X. It is difficult to realize now how Sato's point of view was revolutionary at that time. Sato's hyperfunctions are constructed using tools from sheaf theory and complex analysis, when people were totally addicted to functional analysis, and when the separation between real and complex analysis was very strong.

Then came Kashiwara's thesis, dated December 1970 (of course written in Japanese, but translated in English and published by the French

Key words and phrases. D-modules, microlocal analysis, algebraic analysis.

[1]talk given in honor of Masaki Kashiwara's 60th birthday, Kyoto, June 27, 2007.

Fig. 1. Mikio Sato around 1972 and with Masaki Kashiwara more recently

Mathematical Society [7]) in which he settles the foundations of analytic $\mathcal{D}$-module theory and obtains almost all basic results of the theory (compare with [13]). With $\mathcal{D}$-module theory (also constructed independently in the algebraic settings by J. Bernstein [3]), one finally has the tools to treat general systems of linear partial differential equations, as opposed to one equation with one unknown, or to some very particular overdetermined systems. In particular, Kashiwara succeeds in formulating (and solving, but the difficult problem is the functorial formulation) the Cauchy problem for $\mathcal{D}$-modules, obtaining what is now called the Cauchy-Kowalesky-Kashiwara theorem.

After the first revolution of hyperfunction theory, Sato made a second one, ten years later, by creating microlocal analysis, a way to analyse objects of a manifold X in the cotangent bundle T^*X. With Kashiwara and Kawai, they wrote a long paper [25], quoted everywhere as SKK, whose influence has been considerable during the whole seventies among the analysts (and not only the analysts), although very few of them even tried to read the paper. The SKK paper contains Sato's construction of the sheaf $\mathcal{C}_M$ of microfunctions, and as a byproduct, the definition of the wave front set. This is essentially what the analysts, led by Hörmander, remember of this theory (see [6]). But, to my opinion, this is certainly not the only key point of the SKK paper. Another essential fact is that all constructions are made functorially. For example, microfunctions are obtained by first constructing the microlocalization functor μ_M, and then applying it to the sheaf $\mathcal{O}_X$ of holomorphic functions on a complex manifold X. When you take for M a real analytic manifold of whom X is a complexification, you get the sheaf $\mathcal{C}_M$ (living on $T^*_M X$, the cornormal bundle to M in X), but if you replace the embedding $M \hookrightarrow X$ by the diagonal embedding $\Delta \hookrightarrow X \times X$, then you get the sheaf of microdifferential operators (on $T^*_\Delta(X \times X) \simeq T^*X$) whose theory was developed by Kashiwara and Kawai. With this approach, you can adapt the six Grothendieck operations to Analysis and obtain a completely new point of view to classical problems (*e.g.* the Fourier-Sato transformation).

Moreover, the SKK paper contains at least two fundamental and extremely deep results, first the involutivity of characteristics, second the structure of systems of microdifferential equations at the generic points of the characteristic variety. More precisely, let X be a complex manifold and let $\mathcal{E}_X$ be the sheaf of rings of microdifferential operators (a kind of localization of the sheaf $\mathcal{D}_X$ of differential operators). A microdifferential system $\mathcal{M}$ on an open subset U of T^*X is a coherent $\mathcal{E}_X|_U$-module. Then

- the support char($\mathcal{M}$) of $\mathcal{M}$, also called its characteristic variety, is a closed complex analytic *involutive* (that is, co-isotropic) subset of U. Of course, the involutivity theorem has a longer history, including the previous work of Guillemin-Quillen-Sternberg [5], and culminating with the purely algebraic proof of Gabber [4].
- At *generic points* of char($\mathcal{M}$), (after using complex quantized contact transformations and infinite order microdifferential operators) $\mathcal{M}$ is isomorphic to a partial de Rham system:

  $$\partial_{x_i} u = 0,\ (i = 1, \dots, p).$$

  In the real case, $\mathcal{M}$ is isomorphic to a mixture of de Rham, Dolbeault and Hans Lewy systems:

  $$\begin{cases} \partial_{x_i} u = 0,\ (i = 1, \dots, p) \\ (\partial_{y_j} + \sqrt{-1}\partial_{y_{j+1}})u = 0, (j = 1, \dots, q) \\ (\partial_{t_k} + \sqrt{-1}t_k\partial_{t_{k+1}})u = 0, (k = 1, \dots, r). \end{cases}$$

From 1970 to 1980, Kashiwara solved almost all fundamental questions of $\mathcal{D}$-module theory, proving in particular the rationality of the zeroes of b-functions [10] and also stating and solving almost all questions

Fig. 2. Masaki Kashiwara, Teresa Monteiro Fernandes and myself around 1975

related to regular holonomic modules, in particular the Riemann-Hilbert problem.

Let us give some details on this part of Kashiwara's work. In 1975, he proved that the complex $F = R\mathcal{H}om_{\mathcal{D}}(\mathcal{M}, \mathcal{O}_X)$ of holomorphic solutions of a holonomic $\mathcal{D}$-module $\mathcal{M}$ has constructible cohomology and satisfies properties which are now translated by saying that F is perverse [9]. Moreover, two years *before* [8], in 1973, he calculated the local Euler-Poincaré index of F using the characteristic cycle associated to $\mathcal{M}$ and in fact, defining first what is now called the local Euler obstruction, or equivalently, the intersection of Lagrangian cycles. In 1977 he gave a precise statement of what should be the Riemann-Hilbert correspondence (see [23, p. 287]), the difficulty being to define a suitable class of holonomic $\mathcal{D}$-modules, the so-called regular holonomic modules, what he does in the microlocal setting with Kawai [15] (after related work with Oshima [16]). Then, in 1979, he announces at the 1979/1980 Seminar of Ecole Polytechnique [11] the theorem, giving with some details the main steps of the proof.

$$\begin{tikzcd}
\mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)^{\mathrm{op}} \arrow[r, "(1975)"] & \mathrm{D}^{\mathrm{b}}_{\mathbb{C}-\mathrm{c}}(\mathbb{C}_X) \arrow[dl, "\sim", "(1979-80)"'] \\
\mathrm{D}^{\mathrm{b}}_{\mathrm{holreg}}(\mathcal{D}_X)^{\mathrm{op}} \arrow[u, hook, "(1977)"] &
\end{tikzcd}$$

Unfortunately, Masaki did not publish the whole proof before 1984 [12] and some people tried to make his result their own. As everyone knows, if the platonic world of Mathematics is pure and rigorous, these qualities definitely do not apply to the world of mathematicians.

Of course, Kashiwara did a lot of other things during this period 1970/80, in particular in the theory of microdifferential equations, but he did not always take the time to publish his results. I remember that I had once in 1978 at Oberwolfach the opportunity to explain to Hörmander the so-called "watermelon cut theorem" and you can now find it in [6, Th. 9.6.6]. This beautiful theorem asserts in particular that if a hyperfunction u is supported by a half space $f \geq 0$, then the analytic wave front set of u above the boundary $f = 0$ is invariant by the Hamiltonian vector field H_f.

After that, essentially from 1980 to 1990, came another period in which I am more involved.

Indeed, we developed together the microlocal theory of sheaves (see [18]). To a sheaf F (not necessarily constructible) on a *real* manifold M, we associate a closed conic subset $\mathrm{SS}(F)$[2] of the cotangent bundle, the microsupport of F, which describes the directions of non propagation of F. The idea of microsupport emerged when, on one side, Masaki noticed that it was possible to recover the characteristic variety of a holonomic $\mathcal{D}$-module from the knowledge of the complex of its holomorphic solutions by using the vanishing cycle functor, and when, on my side, I was lead to this notion by remarking that our previous results on propagation for hyperbolic systems was of purely geometrical nature and had almost nothing to do with partial differential equations.

One of the main result of the theory asserts that the microsupport of a sheaf F is an involutive subset of the cotangent bundle, but now we are working on real manifolds. In case one works on a complex manifold and F is the sheaf of solutions of a coherent $\mathcal{D}$-module $\mathcal{M}$, the microsupport of F coincides with the characteristic variety of $\mathcal{M}$. This gives an alternative proof of the involutivity of characteristics of $\mathcal{D}$-modules.

[2] $\mathrm{SS}(F)$ stands for singular support.

Fig. 3. Masaki Kashiwara and myself around 1980

Moreover, constructible sheaves on a real manifold are sheaves whose microsupport is subanalytic and Lagrangian. This allowed Kashiwara to adapt to the real case the notion of characteristic cycle of a $\mathcal{D}$-module and to define the Lagrangian cycle of an $\mathbb{R}$-constructible sheaf. The group of Lagrangian cycles is isomorphic to the Grothendieck group of the abelian category of $\mathbb{R}$-constructible sheaves and is also isomorphic to the group of constructible functions. Lagrangian cycles play a basic role in many questions and have been recently extended to higher K-theory by Beilinson [2].

After 90, Kashiwara concentrated mainly on other subjects such as crystal bases, but nevertheless we wrote several papers together. In order to overcome some difficulties related to the microlocalization functor, we were led to generalize the notion of sheaves and to define ind-sheaves [19]. This theory required a lot of technology from category theory, and, as a byproduct, we wrote a whole book on this subject [20].

Algebraic Analysis and Microlocal Analysis are still actively developing in various directions. Let us mention three of them.

(i) Recall that Masaki was the first, in 1996, to introduce algebroid stacks in microlocal analysis [14]. Indeed, on a complex contact manifold the sheaf of microdifferential operators does not exist in general and one has to replace sheaves with stacks. Such algebroid stacks are now commonly used on complex symplectic manifolds where microdifferential operators are replaced by a variant involving a central parameter $\hbar$. Note that Masaki and Raphael Rouquier recently used such rings of operators to make a surprising link with Cherednik algebras [17].

(ii) As a particular case of the theory of ind-sheaves, one gets the theory of usual sheaves on the subanalytic site. Personally, (I am not sure that Masaki shares this point of view) I am convinced that the subanalytic topology is particularly well suited to treat many problems in Analysis and that there are lot of interesting results to be obtained in this direction.

(iii) Another very promising direction is the link between the microlocal theory of sheaves and Fukaya's category. On one side, Nadler and Zaslow [22, 21], adapting the construction of Lagrangian cycles, constructed a category equivalent to Fukaya's category (on cotangent bundles, not on general real symplectic manifolds). On the other side, Tamarkin [26] also constructed a category which should play this role. Tamarkin's idea is to add a variable $t \in \mathbb{R}$ whose dual variable τ plays the role of the inverse of $\hbar$ and to work "microlocally" with the category of constructible sheaves on $X \times \mathbb{R}$ in the open set $\tau > 0$ of $T^*(X \times \mathbb{R})$.

I hope that this very sketchy panorama of almost fifty years of Algebraic Analysis (perhaps one should now better call it "Functorial Analysis") will have convinced you of the importance of the theory and of the fact that Masaki plays the main role in it since the early seventies.

References

[1] E. Andronikof, *Interview with Mikio Sato,* Notices of the AMS, **54** Vol 2, (2007).

[2] S. Beilinson, *Topological ε-factors,* math.AG/0610055.

[3] J. Bernstein, *Modules over a ring of differential operators. Study of fundamental solutions of equations with constant coefficients,* Funct. Analysis Appl. **5**(1971) 89–101.

[4] O. Gabber, *The integrability of the characteristic variety,* Amer. Journ. Math. **103** (1981) 445–468.

[5] V. Guillemin, D. Quillen and S. Sternberg, *The integrability of characteristics,* Comm. Pure and Appl. Math. **23** (1970) 39–77.

[6] L. Hörmander, *The analysis of partial differential operators I,* Grundlehren der Mathematischen Wissenschaften **256** Springer-Verlag, (1983).

[7] M. Kashiwara, *Algebraic study of systems of partial differential equations,* Tokyo 1970 (in Japanese), Mém. Soc. Math. France **63** (1995) (translated by A. D'Agnolo and J-P.Schneiders).

[8] ——— *Index theorem for a maximally overdetermined system of linear differential equations,* Proc. Japan Acad. **49** (1973) 803–804.

[9] ——— *On the maximally overdetermined system of linear differential equations I,* Publ. Res. Inst. Math. Sci. **10** (1974/75) 563–579.

[10] ——— *B-functions and holonomic systems. Rationality of roots of B-functions,* Invent. Math. **38** (1976/77) 33–53.

[11] ——— *Faisceaux constructibles et systèmes holonômes d'équations aux dérivées partielles linéaires à points singuliers réguliers,* Séminaire Goulaouic-Schwartz, 1979–1980 (French), Exp. No. **19** École Polytech., Palaiseau, 1980.

[12] ——— *The Riemann Hilbert problem for holonomic systems,* Publ. Res. Inst. Math. Sci. **26** (1984) 319–365.

[13] ——— *$\mathcal{D}$-modules and microlocal calculus,* Translations of Mathematical Monographs, **217** American Math. Soc. (2003).

[14] ——— *Quantization of contact manifolds,* Publ. RIMS, Kyoto Univ. **32** (1996) 1–5.

[15] M. Kashiwara and T. Kawai, *On holonomic systems of microdifferential equations III. Systems with regular singularities,* Publ. Res. Inst. Math. Sci. **17** (1981) 813–979.

[16] M. Kashiwara and T. Oshima, *Systems of differential equations with regular singularities and their boundary value problems,* Ann. of Math. **106** (1977) 145–200.

[17] M. Kashiwara and R. Rouquier *Microlocalization of rational Cherednik algebras,* `arXiv:math.RT/0705.1245`, to appear.

[18] M. Kashiwara and P. Schapira, *Sheaves on manifolds,* Grundlehren der Mathematischen Wissenschaften **292** Springer-Verlag, (1990).

[19] ——— *Ind-sheaves,* Astérisque Soc. Math. France **271** (2001).

[20] ——— *Categories and sheaves,* Grundlehren der Mathematischen Wissenschaften **332** Springer-Verlag, (2006).

[21] D. Nadler, *Microlocal branes are constructible sheaves,* `arXiv:math/0612399`.

[22] D. Nadler and E. Zaslow, *Constructible sheaves and the Fukaya Category,* `arXiv:math/0604379`. To appear in J. Amer. Math. Soc.

[23] J-P. Ramis, *Additif II à "variations sur le thème GAGA",* Lecture Notes Math. **694** Springer (1978).

[24] M. Sato, *Theory of hyperfunctions I & II,* Journ. Fac. Sci. Univ. Tokyo, **8** (1959–1960)139–193 487–436.

[25] M. Sato, T. Kawai, and M. Kashiwara, *Hyperfunctions and pseudo-differential equations.* In Komatsu (ed.), *Hyperfunctions and pseudo-differential equations,* Proceedings Katata 1971, Lecture Notes in Mathematics Springer **287** (1973) 265–529.

[26] D. Tamarkin, *Microlocal condition for non-displaceablility,* `arXiv:0809.1584`.

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