Boundary values, Fourier-Sato tranform and Laplace transform

Pierre Schapira

Abstract

In the first part of this paper we describe the well-known Fourier-Sato's transform of conic sheaves and we show how this theory permits to define the boundary value of holomorphic cohomology classes associated to conic sheaves and in particular cohomology classes supported by non necessarily convex cones. As an example, we treat the case of quadratic cones.

Next, we recall the recent result of [7] on the Laplace transform of holomorphic cohomology classes associated to constructible sheaves, and give a link between all these constructions.

1 Introduction

Let M be a real analytic manifold of dimension n, X a complexification of M, γ a cone in T_MX, the normal bundle to M in X. When γ is open and convex (i.e. the fibers of γ over M are convex), Sato's theory, as explained in [9], allows one to define the boundary value of holomorphic functions defined in tuboids with profile γ. But in fact, this theory also applies to the non open and non convex case, even if it has never been written down explicitely, and one of the aims of this paper is to fill up this gap of the literature. Hence, it should be clear that the results obtained here are more or less well-known from some specialists. Note that boundary values of holomorphic cohomology classes defined in non convex cones are constructed "à la main" in some situations in [2].

In order to describe our results, consider a (non necessarily convex) locally closed cone γ as above, and let $\mathbb{C}_\gamma^\wedge$ denote the Fourier-Sato transform of the sheaf $\mathbb{C}_\gamma$. Assume that for some local system L on M, some integer d, and some locally closed cone λ in T^*_MX, one has:

$$H^{n-d}(\mathbb{C}_\gamma^\wedge) \simeq \mathbb{C}_\lambda \otimes L.$$

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(Here, we write L instead of $\pi^{-1}L$ for short.) Then:

$$H^d_\gamma(T_MX;\nu_M(\mathcal{O}_X))\simeq\Gamma_\lambda(T^*_MX;\mathcal{C}_M\otimes L^*\otimes or_M)$$

where $\nu_M(\mathcal{O}_X)$ is the specialization of the sheaf $\mathcal{O}_X$ along M, $\mathcal{C}_M$ is the sheaf of Sato's microfunctions, L^* is the dual of L and or_M is the orientation sheaf. As we shall see, this formula appears as an immediate corollary of Sato's theory.

Then comes the problem of calculating the Fourier-Sato transform $\mathbb{C}^\wedge_\gamma$ of $\mathbb{C}_\gamma$. We consider various examples and in particular the case of a quadratic cone. This last example has already been considered in a different language and with a different method by Faraut-Gindikin [3].

We end this paper by briefly recalling the main theorem of [7] on the Laplace transform of cohomology classes associated to conic $\mathbb{R}$-constructible sheaves (and in particular to tempered distributions supported by non convex cones) and show a link between all the above constructions.

2 Review on the Fourier-Sato transform

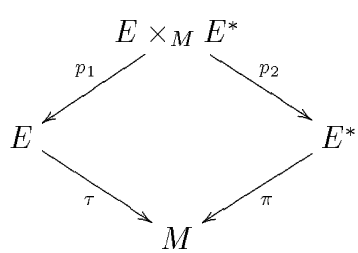
The Fourier transform for sheaves on sphere bundles has been introduced by Sato (see [9]), then extended to conic sheaves on vector bundles by [1]. We refer to [5] for a detailed construction and for historical comments.

On a topological space X, one denotes by $\mathbf{D}^{\mathrm{b}}(\mathbb{C}_X)$ the derived category of the category of bounded complexes of sheaves of $\mathbb{C}$-vector spaces on X. If X is a real manifold, one denotes by or_X the orientation sheaf on X and by ω_X the dualizing complex, $\omega_X\simeq or_X[dimX]$. If $f:Y\longrightarrow X$ is a morphism of manifolds, one sets $or_{Y/X}=or_Y\otimes f^{-1}or_X$.

Let $\tau:E\longrightarrow M$ be a finite dimensional real vector bundle over a real manifold M with fiber dimension n, $\pi:E^*\longrightarrow M$ the dual vector bundle. Denote by p_1 and p_2 the first and second projection defined on $E\times_M E^*$, and define:

$$P=\{(x,y)\in E\times_M E^*;\langle x,y\rangle\geq 0\}$$
$$P'=\{(x,y)\in E\times_M E^*;\langle x,y\rangle\leq 0\}$$

Consider the diagram:



Denote by $\mathbf{D}^{b}_{\mathbb{R}^+}(\mathbb{C}_E)$ the full triangulated subcategory consisting of conic objects, that is, of objects F such that for all j, $H^j(F)$ is locally constant on the orbits of the action of $\mathbb{R}^+$ on E.

Definition 2.1. Let $F \in \mathbf{D}^{b}_{\mathbb{R}^+}(\mathbb{C}_E)$, $G \in \mathbf{D}^{b}_{\mathbb{R}^+}(\mathbb{C}_{E^*})$. One sets:

$$\begin{aligned} F^\wedge &= Rp_{2!}(p_1^{-1}F)_{P'} \\ G^\vee &= Rp_{1!}(p_2^{-1}G)_P \otimes or_{E/M}\,[n] \end{aligned}$$

The main result of the theory is the following.

Theorem 2.2. *(i) There is a natural isomorphism*

$$F^\wedge \simeq Rp_{2*}R\Gamma_P(p_1^{-1}F),$$

(ii) the two functors $(\cdot)^\wedge$ and $(\cdot)^\vee$ are inverse to each other, hence define equivalences of categories

$$\mathbf{D}^{b}_{\mathbb{R}^+}(\mathbb{C}_E) \simeq \mathbf{D}^{b}_{\mathbb{R}^+}(\mathbb{C}_{E^*}).$$

(iii) In particular, let F_1 and F_2 belong to $\mathbf{D}^{b}_{\mathbb{R}^+}(\mathbb{C}_E)$. Then there is a natural isomorphism:

$$\mathrm{RHom}\,(F_1, F_2) \simeq \mathrm{RHom}\,(F_1^\wedge, F_2^\wedge).$$

Moreover, the following formulas are of constant use.

$$R\tau_! F \simeq R\Gamma_M(F)|_M \simeq R\pi_*(F^\wedge) \simeq (F^\wedge)|_M \tag{2.1}$$

$$R\tau_* F \simeq F|_M \simeq R\pi_!(F^\wedge) \otimes or_{E^*/M}[n] \simeq R\Gamma_M(F^\wedge) \otimes or_{E^*/M} \tag{2.2}$$

Example 2.3. (i) Let γ be a closed proper convex cone in E with $M \subset \gamma$. Then:

$$(\mathbb{C}_\gamma)^\wedge \simeq \mathbb{C}_{Int\gamma^\circ}.$$

Here γ° is the polar cone to γ, a closed convex cone in E^* and $Int\gamma^\circ$ denotes its interior.

(ii) Let γ be an open convex cone in E. Then:

$$(\mathbb{C}_\gamma)^\wedge \simeq \mathbb{C}_{\gamma^{\circ a}} \otimes or_{E^*/M}\,[-n].$$

Here $\lambda^a = -\lambda$, the image of λ by the antipodal map.

(iii) In particular, applying Theorem 2.2 (iii), we get for F a conic sheaf on E, and γ an open convex cone:

$$R\Gamma(\gamma; F) \simeq R\Gamma_{\gamma^{\circ a}}(E^*; F^\wedge \otimes or_{E^*/M})\,[n].$$

A useful property of the Fourier-Sato transform is that it commutes to base change. More precely, consider a morphism of manifolds: $f : N \longrightarrow M$. It defines morphisms of vector bundles:

$$\begin{array}{ccc} f^*E & \xrightarrow{f_\tau} & E \\ \downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle\tau} \\ N & \xrightarrow{f} & M \end{array} \qquad \begin{array}{ccc} f^*E^* & \xrightarrow{f_\pi} & E^* \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ N & \xrightarrow{f} & M \end{array}$$

Theorem 2.4. *Let $F \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathbb{C}_E)$ and let $G \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathbb{C}_{f^*E})$. Then there are natural isomorphisms:*

$$(f_\tau^{-1}F)^\wedge \simeq f_\pi^{-1}(F^\wedge)$$
$$(Rf_{\tau !}G)^\wedge \simeq Rf_{\pi !}(G^\wedge)$$

3 Review on specialization and microlocalization

We shall describe here the functor of specialization and its Fourier transform, Sato's functor of microlocalization. For a detailed exposition, see [5].

Let X be a real manifold (let's say of class $\mathcal{C}^\infty$), M a closed submanifold. Denote by $\tau : T_MX \longrightarrow M$ and $\pi : T^*_MX \longrightarrow M$ the normal bundle and the conormal bundle to M in X, respectively. Let $F \in \mathbf{D}^{\mathrm{b}}(\mathbb{C}_X)$. The specialization of F along M, denoted $\nu_M(F)$, is an object of $\mathbf{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathbb{C}_{T_MX})$. Its cohomology objects are described as follows. If V is an open cone in T_MX, then

$$H^j(V; \nu_M(F)) \simeq \varinjlim_U H^j(U; F)$$

where U ranges over the family of open subsets of X which are "tangent" to V, that is, open tuboids in X with wedge M whose "profiles" is V. (For a precise definition, refer to [5].)

The microlocalization of F along M, denoted $\mu_M(F)$, is the Fourier-Sato transform of $\nu_M(F)$, hence is an object of $\mathbf{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathbb{C}_{T^*_MX})$. It satisfies:

$$R\pi_*\mu_M(F) \simeq R\Gamma_M(F),$$
$$H^j(\mu_M(F))_{(x_0;\xi_0)} \simeq \varinjlim_{U,Z} H^j_{U\cap Z}(U; F).$$

In the last formula, $(x_0; \xi_0) \in T^*_MX$, U ranges over the family of open neighborghoods of x_0 in X and Z ranges over the family of closed tuboids in X with wedge M whose profiles λ in T_MX satisfy $(x_0; \xi_0) \in Int\lambda^{\circ a}$. (For a precise definition, refer to [5].)

Now assume M is a real analytic manifold of dimension n and X is a complexification of M. (This situation may be immediately generalized by considering "totally real submanifolds" of X.) In such a case, one of the main results of Sato's theory asserts that the object $\mu_M(\mathcal{O}_X)$ is concentrated in degree n (see [9] and also [8]). This leads to the following definition.

Definition 3.1. (i) The sheaf of Sato's hyperfunctions on M is defined by:

$$\mathcal{B}_M = H^n_M(\mathcal{O}_X) \otimes or_M,$$

(ii) The sheaf of Sato's microfunctions on $T^*_M X$ is defined by:

$$\mathcal{C}_M = H^n(\mu_M(\mathcal{O}_X)) \otimes or_M.$$

Hence, a hyperfunction is nothing but a microfunction globally defined in the fibers of π. Denote by *spec* the natural isomorphism:

$$spec : \mathcal{B}_M \simeq \pi_* \mathcal{C}_M.$$

If u is an hyperfunction, Sato defines its analytic wave front set as:

$$WF(u) = \mathrm{supp}(spec(u)),$$

a closed conic subset of $T^*_M X$.

Since $\Gamma_M(T^*_M X; \mathcal{C}_M) \simeq \mathcal{A}_M$ (the sheaf of real analytic functions on M), one sees that a hyperfunction is a real analytic function if and only if its wave front set is contained in the zero-section.

An important property (proved by Kashiwara) is that the sheaf $\mathcal{C}_M$ it is conically flabby (see [9]).

Applying Theorem 2.2, one gets:

Corollary 3.2. *Let* $F \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathbb{C}_{T_M X})$. *Then:*

$$R\tau_* R\mathcal{H}om\,(F, \nu_M(\mathcal{O}_X)) \simeq R\pi_* R\mathcal{H}om\,(F^\wedge \otimes or_M\,[n], \mathcal{C}_M).$$

This corollary applies in particular when $F = \mathbb{C}_\gamma$, γ being a cone in $T_M X$.

4 Boundary values

In this section, we shall apply Corollary 3.2 to various geometrical situations.

As above, let M be a real analytic manifold of dimension n and let X be a complexification of M. First, assume to be given a morphism (a "boundary values morphism"):

$$b : or_M\,[-n] \longrightarrow F. \tag{4.1}$$

It defines $\mathbb{C}_M \longrightarrow F \otimes or_M\,[n]$ and also, applying the Fourier-Sato transform on the vector bundle $T_M X$:

$$b : \mathbb{C}_M \longrightarrow F^\wedge \otimes or_M\,[n].$$

Using Corollary 3.2, we get the commutative diagram:

$$\begin{array}{ccc} R\tau_* R\mathcal{H}om\,(F, \nu_M(\mathcal{O}_X)) & \xrightarrow{\ \sim\ } & R\pi_* R\mathcal{H}om\,(F^\wedge \otimes or_M\,[n], \mathcal{C}_M) \\ & {\scriptstyle b}\searrow \quad \swarrow & \\ & \mathcal{B}_M & \end{array}$$

Next, we specialize our study to the case where F is associated to a cone. Let γ be a locally closed cone of $T_M X$, d an integer, and assume that there exists a locally closed cone λ in $T^*_M X$ and a local system L on M such that

$$H^{n-d}(\mathbb{C}^\wedge_\gamma) \simeq \mathbb{C}_\lambda \otimes L. \tag{4.2}$$

Denote by L^* the dual local system to L.

Proposition 4.1. *Assume (4.2). Then there is a natural isomorphism:*

$$H^d_\gamma(T_M X; \nu_M(\mathcal{O}_X)) \simeq \Gamma_\lambda(T^*_M X; \mathcal{C}_M \otimes or_M \otimes L^*). \tag{4.3}$$

Proof. Applying Corollary 3.2, we get the isomorphisms:

$$\begin{aligned} \mathrm{R}\Gamma_\gamma(T_M X; \nu_M(\mathcal{O}_X)) &\simeq \mathrm{RHom}\,(\mathbb{C}_\gamma, \nu_M(\mathcal{O}_X)) \\ &\simeq \mathrm{RHom}\,(\mathbb{C}^\wedge_\gamma \otimes or_M\,[n], \mathcal{C}_M). \end{aligned}$$

Now apply H^d to both sides. Since the sheaf $\mathcal{C}_M$ is conically flabby, it is injective in the category of conic sheaves of $\mathbb{C}$-vector spaces on $T^*_M X$. Hence we get the isomorphism:

$$H^d_\gamma(T_M X; \nu_M(\mathcal{O}_X)) \simeq \mathrm{Hom}\,(H^{n-d}(\mathbb{C}^\wedge_\gamma) \otimes or_M, \mathcal{C}_M), \tag{4.4}$$

and the result follows. q.e.d.

Remark 4.2. (i) In practice, L will be of rank one, and locally, it can be forgotten, as well as or_M.

(ii) The result is of particular interest when λ is a closed cone and L is of rank one. In such a case, the right hand-side of (4.3) is nothing but the space of hyperfunctions on M with wave front set contained in λ (locally on M, see the remark above).

(iii) If for some j, $H^{n-j}(\mathbb{C}^\wedge_\gamma) = 0$, then $H^j(\gamma; \nu_M(\mathcal{O}_X)) = 0$.

(iv) Using formula (2.1), we get that if γ is an open cone in $T_M X$, then $H^{n-d}(M; R\tau_! \mathbb{C}_\gamma \otimes or_M) \simeq H^{n-d}(T^*_M X; \mathbb{C}^\wedge_\gamma \otimes or_M)$. Using equation (4.4), we get the pairing:

$$H^{n-d}(M; R\tau_! \mathbb{C}_\gamma \otimes or_M) \otimes H^d(\gamma; \nu_M(\mathcal{O}_X)) \longrightarrow \Gamma(M; \mathcal{B}_M).$$

Now assume M is a vector space, identify X to $M \times \sqrt{-1}M$ and let $\gamma = M \times \tilde{\gamma}$, where $\tilde{\gamma}$ is an open cone of $\sqrt{-1}M$. Then $H^{n-d}(M; R\tau_!\mathbb{C}_\gamma \otimes or_M) \simeq H_c^{n-d}(\sqrt{-1}M; \mathbb{C}_{\tilde{\gamma}}) \simeq H_d(\tilde{\gamma})$ (homology of $\tilde{\gamma}$ in degree d). Hence we get a pairing:

$$H_d(\tilde{\gamma}) \otimes H^d(\gamma; \nu_M(\mathcal{O}_X)) \longrightarrow \Gamma(M; \mathcal{B}_M).$$

Given a cycle $c \in H_d(\tilde{\gamma})$, we get a map $H^d(\gamma; \nu_M(\mathcal{O}_X)) \longrightarrow \Gamma(M; \mathcal{B}_M)$. This map is already considered (in some special situations) in [2] where its injectivity (which, of course, does not hold in general) is discussed.

(v) Let γ be an open tube in X, $\bar{\gamma}$ its closure, and assume that γ is topologically convex and $\bar{\gamma}$ contains M. The natural morphism $\mathbb{C}_{\bar{\gamma}} \longrightarrow \mathbb{C}_M$ defines by duality the boundary values morphism (of 4.1) $or_M[-n] \longrightarrow \mathbb{C}_\gamma$, from which one deduces the morphism $\Gamma(\gamma; \mathcal{O}_X) \longrightarrow \mathcal{B}(M)$. This purely topological construction of the boundary value morphism first appeared in [10] (see also [5] pp. 467).

Example 4.3. (i) Let γ be an open convex cone in $T_M X$. Then:

$$\Gamma(\gamma; \nu_M(\mathcal{O}_X)) \simeq \Gamma_{\gamma^{\circ a}}(T_M^* X; \mathcal{C}_M),$$

and $H^j((\gamma; \nu_M(\mathcal{O}_X)) = 0$ for $j \neq 0$.

(ii) Assume $n > 1$ and choose $\gamma = T_M X \setminus M$. The Fourier-Sato's functor applied to the exact sequence of sheaves $\mathbb{C}_\gamma \longrightarrow \mathbb{C}_{T_M X} \longrightarrow \mathbb{C}_M \underset{+1}{\longrightarrow}$ gives rise to the distinguished triangle: $\mathbb{C}_\gamma^\wedge \longrightarrow or_M[-n] \longrightarrow \mathbb{C}_{T_M^* X} \underset{+1}{\longrightarrow}$. Hence, $H^1(\mathbb{C}_\gamma^\wedge) \simeq \mathbb{C}_{T_M^* X}$, $H^{n-1}(\mathbb{C}_\gamma^\wedge) \simeq or_M$, and the other groups are zero. We get:

$$\begin{aligned} H^0(\gamma; \nu_M(\mathcal{O}_X)) &\simeq \Gamma_M(T_M^* X; \mathcal{C}_M) \simeq \Gamma(M; \mathcal{A}_M) \\ H^{n-1}(\gamma; \nu_M(\mathcal{O}_X)) &\simeq \Gamma(T_M^* X; \mathcal{C}_M \otimes or_M) \simeq \Gamma(M; \mathcal{B}_M \otimes or_M) \\ H^j(\gamma; \nu_M(\mathcal{O}_X)) &\simeq 0 \text{ for } j \neq 1, n-1. \end{aligned}$$

(Of course, this result is classical, and can easily be obtained without the explicit use of the Fourier-Sato transform.)

5 Fourier-Sato transform of quadratic cones

Let E denote the the Euclidian space $\mathbb{R}^n$, with coordinates $x = (x_1, \ldots, x_n)$, and set $x = (x', x'')$ with $x' = (x_1, \ldots, x_p)$ and $x'' = (x_{p+1}, \ldots, x_n)$. Let $q = n - p$ and assume $q > 1$. We denote by $u = (u', u'')$ the dual coordinates on the dual space E^*.

Let γ denote the closed solid cone in E,

$$\gamma = \{x; x'^2 - x''^2 \geq 0\}$$

and let λ denote the closed solid cone in E^*:

$$\lambda = \{u; u'^2 - u''^2 \leq 0\}.$$

Lemma 5.1. *We have:*

$$\mathbb{C}_\gamma^\wedge \simeq \mathbb{C}_\lambda\,[-p].$$

Proof. For $u \in E^*$, set:

$$\gamma_u = \{x \in \gamma; \langle x, u\rangle \le 0\}.$$

Then $(\mathbb{C}_\gamma^\wedge)_u \simeq \mathrm{R}\Gamma_c(E; \mathbb{C}_{\gamma_u})$. We have (see [5], Ex III 5):

$$\mathrm{R}\Gamma_c(E; \mathbb{C}_\gamma) \simeq \mathbb{C}[-p].$$

Hence it is enough to check that:
(i) if $u \notin \lambda$, $\mathrm{R}\Gamma_c(E; \mathbb{C}_{\gamma_u}) = 0$,
(ii) if $u \in \lambda \setminus \{0\}$, the morphism

$$\mathrm{R}\Gamma_c(E; \mathbb{C}_\gamma) \longrightarrow \mathrm{R}\Gamma_c(E; \mathbb{C}_{\gamma_u})$$

is an isomorphism.

Let us prove (i). Let $u = (u', u'')$. We may assume $u'' = 0$. Let f be the projection $\mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^p$, $(x', x'') \mapsto x'$, and set $\tilde{\gamma}_u = f(\gamma_u)$. The fibers of f above $\tilde{\gamma}_u$ are closed balls and $\tilde{\gamma}_u$ is a closed half plane. Hence $\mathrm{R}\Gamma_c(\tilde{\gamma}_u; \mathbb{C}_{\tilde{\gamma}_u}) = 0$, and (i) follows.

Let us prove (ii). We may assume $u = (0, \dots, 0, 1)$. Let f be the projection $\mathbb{R}^n \to \mathbb{R}^{n-1}$, $x \mapsto (x_1, \cdots, x_{n-1})$. Set $\tilde{\gamma}_u^- = f(\gamma \setminus \gamma_u)$. Then the fibers of f above $\tilde{\gamma}_u^-$ are intervals $(0, a]$ for some $a \in \mathbb{R}$. Hence $Rf_!\mathbb{C}_{\gamma\setminus\gamma_u} = 0$, and we obtain $\mathrm{R}\Gamma_c(M; \mathbb{C}_{\gamma\setminus\gamma_u}) = 0$, which implies (ii). q.e.d.

Now assume M is an open subset of E, X a complexification of M (of course this situation could be generalized to manifolds). We change our notations and denote by γ the cone $M + \sqrt{-1}\{y; y'^2 - y''^2 \ge 0\}$ in $T_M X$ and similarly for λ in $T_M^* X$.

Applying Lemma 5.1 and Proposition 4.1, we get:

Proposition 5.2. *One has the isomorphism:*

$$H^q_\gamma(T_M X; \nu_M \mathcal{O}_X) \;\simeq\; \Gamma_\lambda(T_M^* X; \mathcal{C}_M).$$

Remark 5.3. The case of quadratic cones has already been considered by Faraut-Gindikin [3], with a different formulation and a different method. Their method consists in representing cohomology classes in the non convex tube γ as an integral of holomorphic functions in convex tubes.

6 Formal and temperate cohomology

In order to state the next result on the Laplace transform, we need to briefly recall the functors of formal (see [6]) and temperate cohomology (see [4]).

Let M be a real manifold, and let $\mathbb{R}$-cons$(\mathbb{C}_M)$ denote the category of $\mathbb{R}$-constructible sheaves on M, $\mathrm{D}^b_{\mathbb{R}-c}(\mathbb{C}_M)$ its derived category. The functors $\mathcal{T}hom(\cdot, \mathcal{D}b_M)$ of [4] and the dual functor $\cdot \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_M$ of [6], are defined on the category $\mathbb{R}$-cons$(\mathbb{C}_M)$, with values in the category $\mathrm{Mod}(\mathcal{D}_M)$ of $\mathcal{D}_M$-modules on M. (The first functor is contravariant).

They are characterized as follows. Denote by $\mathcal{D}b_M$ the sheaf of Schwartz's distributions on M and by $\mathcal{C}^\infty_M$ the sheaf of $\mathcal{C}^\infty$ functions on M. Let Z (resp. U) be a closed (resp. open) subanalytic subset of M. Then these two functors are exact and moreover:

$$\begin{aligned} \mathcal{T}hom(\mathbb{C}_Z, \mathcal{D}b_M) &= \Gamma_Z(\mathcal{D}b_M), \\ \mathbb{C}_U \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_M &= \mathcal{I}^\infty_{M\setminus U}, \end{aligned}$$

where $\Gamma_Z(\mathcal{D}b_M)$ denotes as usual the subsheaf of $\mathcal{D}b_M$ of sections supported by Z and $\mathcal{I}^\infty_{M\setminus U}$ denotes the ideal of $\mathcal{C}^\infty_M$ of sections vanishing up to order infinity on $M \setminus U$.

These functors being exact, they extend naturally to the derived category $\mathrm{D}^b_{\mathbb{R}-c}(\mathbb{C}_X)$. We keep the same notations to denote the derived functors.

Now let X be a complex manifold and denote by $\overline{X}$ the complex conjugate manifold and by $X_\mathbb{R}$ the real underlying manifold. Let $\mathcal{O}_X$ be the sheaf of holomorphic functions on X, let $\mathcal{D}_X$ be the sheaf of finite order holomorphic differential operators on X. The functors of moderate and formal cohomology are defined for $F \in \mathrm{D}^b_{\mathbb{R}-c}(\mathbb{C}_{X_\mathbb{R}})$ by:

$$\begin{aligned} \mathcal{T}hom(F, \mathcal{O}_X) &= R\mathcal{H}om_{\mathcal{D}_{\overline{X}}}(\mathcal{O}_{\overline{X}}, \mathcal{T}hom(F, \mathcal{D}b_{X_\mathbb{R}})) \\ F \overset{\mathrm{w}}{\otimes} \mathcal{O}_X &= R\mathcal{H}om_{\mathcal{D}_{\overline{X}}}(\mathcal{O}_{\overline{X}}, F \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_{X_\mathbb{R}}). \end{aligned}$$

7 Laplace transform

Consider now a complex vector space $\mathbb{E}$ of complex dimension n, and denote by $j : \mathbb{E} \hookrightarrow \mathbb{P}$ its projective compactification. Let $\mathrm{D}^b_{\mathbb{R}-c,\mathbb{R}^+}(\mathbb{C}_\mathbb{E})$ denote the full triangulated subcategory of $\mathrm{D}^b_{\mathbb{R}-c}(\mathbb{C}_\mathbb{E})$ consisting of $\mathbb{R}^+$-conic objects (i.e. objects whose cohomology is $\mathbb{R}$-constructible and locally constant on the orbits of the action of $\mathbb{R}^+$ on $\mathbb{E}$).

Let $F \in \mathrm{D}^b_{\mathbb{R}-c,\mathbb{R}^+}(\mathbb{C}_\mathbb{E})$ and set for short

$$\begin{aligned} \mathrm{THom}(F, \mathcal{O}_\mathbb{E}) &= \mathrm{R}\Gamma(\mathbb{P}; \mathcal{T}hom(j_!F, \mathcal{O}_\mathbb{P})) \\ F \overset{\mathrm{W}}{\otimes} \mathcal{O}_\mathbb{E} &= \mathrm{R}\Gamma(\mathbb{P}; j_!F \overset{\mathrm{w}}{\otimes} \mathcal{O}_\mathbb{P}) \end{aligned}$$

Remark that if E is a real n-dimensional vector space and $\mathbb{E}$ its complexification, then $\mathrm{THom}(or_E[-n], \mathcal{O}_\mathbb{E}) \simeq \mathcal{S}'(E)$, the Schwartz's space of tempered distributions on E and $\mathbb{C}_E \overset{\mathrm{W}}{\otimes} \mathcal{O}_\mathbb{E} \simeq \mathcal{S}(E)$, the space of rapidly decreasing C^∞-functions on E.

Theorem 7.1. *([7]) The Laplace transform extends naturally as isomorphisms:*

$$TL : \mathrm{THom}\,(F, \mathcal{O}_{\mathbb{E}}) \simeq \mathrm{THom}\,(F^{\wedge}[n], \mathcal{O}_{\mathbb{E}^*}) \tag{7.1}$$

$$WL : F \overset{\mathrm{W}}{\otimes} \mathcal{O}_{\mathbb{E}} \simeq F^{\wedge}[n] \overset{\mathrm{W}}{\otimes} \mathcal{O}_{\mathbb{E}^*}. \tag{7.2}$$

We may summarize the previous isomorphisms by the following diagrams.

$$\begin{array}{ccc}
E + \sqrt{-1}E^* \simeq T^*_E\mathbb{E} & \longleftrightarrow & \sqrt{-1}E^* + E \simeq T^*_{\sqrt{-1}E^*}\mathbb{E}^* \\
\updownarrow & & \updownarrow \\
E + \sqrt{-1}E \simeq T_E\mathbb{E} & \longleftrightarrow & \sqrt{-1}E^* + E^* \simeq T_{\sqrt{-1}E^*}\mathbb{E}^* \\
\downarrow & & \downarrow \\
E & \longleftrightarrow & \sqrt{-1}E^*
\end{array}$$

Let $F \in \mathrm{D}^b_{\mathbb{R}-c,\mathbb{R}^+}(\mathbb{C}_{\mathbb{E}})$ and assume to be given a "boundary value morphism" $b : \mathbb{C}_E\,[-n] \to F$. We get the diagram, in which we neglect or_E for short:

$$\begin{array}{ccc}
\mathrm{RHom}\,(F^{\wedge}\,[n], \mathcal{C}_E) & & \mathrm{RHom}\,(F, \mathcal{C}_{\sqrt{-1}E^*}) \\
\sim \updownarrow \scriptstyle{Sato} & & \sim \updownarrow \scriptstyle{Sato} \\
\mathrm{RHom}\,(F, \nu_E(\mathcal{O}_{\mathbb{E}})) & & \mathrm{RHom}\,(F^{\wedge}\,[n], \nu_{\sqrt{-1}E^*}(\mathcal{O}^*_{\mathbb{E}})) \\
\uparrow & & \uparrow \\
\mathrm{THom}\,(F, \mathcal{O}_{\mathbb{E}}) & \xrightarrow[Laplace]{\sim} & \mathrm{THom}\,(F^{\wedge}[n], \mathcal{O}_{\mathbb{E}^*}) \\
{\scriptstyle b}\downarrow & & {\scriptstyle b}\downarrow \\
\mathcal{S}'(E) & \xrightarrow[Fourier]{\sim} & \mathcal{S}'(\sqrt{-1}E^*)
\end{array}$$

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Institut de Mathématiques; Analyse Algébrique; Université Pierre et Marie Curie; Case 247; 4, place Jussieu; F-75252 Paris Cedex 05
email: `schapira@math.jussieu.fr`