Topologies and sheaves on causal manifolds

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Abstract

A causal manifold (M,γ) is a manifold M endowed with a closed proper cone γ in the tangent bundle TM such that the projection $TM \to M$ is surjective when restricted to $\mathrm{Int}(\gamma)$. Let $\lambda = \gamma^{\circ a} \subset T^*M$ be the antipodal of the polar cone of γ. An open set U of M is called γ-open if its Whitney normal cone contains $\mathrm{Int}\gamma$. Similarly, U is called λ-open if the micro-support of the constant sheaf on U is contained in λ. We begin by proving that the two notions coincide.

Next, we prove that if (M,γ) admits a "future time function" (in particular if it is globally hyperbolic) the functor of direct images establishes an equivalence of triangulated categories between the derived category of sheaves on M micro-supported by λ and the derived category of sheaves on M_γ, the manifold M endowed with the γ-topology. This generalizes a result of [KS90] which dealt with the case of a constant cone in a vector space.

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Introduction

We choose a commutative unital ring $\mathbf{k}$ of finite global dimension and we shall consider sheaves of $\mathbf{k}$-modules (in the derived setting). Our main reference on this subject is the book [KS90].

Consider first a real finite dimensional vector space $\mathbb{V}$ and a closed convex proper cone $\theta \subset \mathbb{V}$ with vertex at 0. Set $\gamma = \mathbb{V} \times \theta$. The γ-topology was first defined in loc. cit. § 3.5: an open subset V of $\mathbb{V}$ is γ-open if $V = V + \theta$. Denote by $\mathbb{V}_\gamma$ the set $\mathbb{V}$

endowed with the γ-topology and denote by $\varphi_\gamma\colon \mathbb{V} \to \mathbb{V}_\gamma$ the continuous map associated with the identity. Denote by $\mathrm{D}^{\mathrm{b}}_{\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}})$ the bounded derived category of sheaves on $\mathbb{V}$ micro-supported by $\gamma^{\circ a}$, the antipodal of the polar cone to γ. It is proved in loc. cit. that the functor of direct image $\mathrm{R}\varphi_{\gamma_*}$ induces an equivalence of this triangulated category with the derived category $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{V}_\gamma})$.

The aim of this paper is to generalize this result to causal manifolds.

There is a vast body of literature on spacetimes and Lorentzian manifolds, particularly concerning the wave equation and its solutions. Usually, such spacetimes are defined through a quadratic form of signature $(+,-,\cdots,-)$ but, following [JS16], we prefer a weaker formulation which only makes use of a cone in the tangent space.

A causal manifold (M,γ) is a manifold M endowed with a closed convex proper cone γ in the tangent bundle TM with the property that the projection $\tau_M\colon TM \to M$ is surjective when restricted to $\mathrm{Int}(\gamma)$. In other words, for any $x_0 \in M$, $\mathrm{Int}(\gamma_{x_0})$ is non-empty. One also naturally defines the notion of a morphism of causal manifolds.

The γ-topology M_γ on M and the map $\varphi_\gamma\colon M \to M_\gamma$ were extended to causal manifolds in [JS16]. A subset Z of M is a γ-set if its Whitney normal cone contains $\mathrm{Int}\gamma$. A γ-open subset is an open subset of M which is a γ-set. Our first result asserts that U is γ-open if and only if $\mathrm{SS}(\mathbf{k}_U)$, the micro-support of the constant sheaf on U, is contained in $\lambda = \gamma^{\circ a}$, the antipodal of the polar cone to γ.

One can then define the λ-future $L^+_\lambda(x_0)$ of $x_0 \in M$ as the smallest closed subset of M containing x_0 and whose micro-support is contained in λ^a. This set is also the closure of $I^+_\gamma(x_0)$, the smallest γ-set of M containing x_0. We get two preorders $\preceq_\lambda$ and $\preceq_\gamma$ on M and $x \preceq_\gamma y$ implies $x \preceq_\lambda y$.

Next we introduce the notion of a future time function $q\colon M \to \mathbb{R}$, a variant of the classical notion of a time function. This is a causal submersive morphism with the property that, for any $x_0 \in M$, it is proper on the sets $L^+_\lambda(x_0)$ and $q^{-1}q_*(x_0) \cap L^+_\lambda(x_0) = \{x_0\}$. Recall that globally hyperbolic manifolds admit time functions (see [Ger70, MS08, FS11] among others). Also recall that we have used time functions in [JS16] to obtain global propagation results for sheaves whose micro-support satisfy suitable conditions. As a by-product, we globally solve in loc. cit. the Cauchy problem for hyperbolic $\mathscr{D}$-modules with hyperfunctions solution.

Here, we use future time functions to prove our main result: for a causal manifold (M,γ) endowed with a future time function q, the functor of direct images $\mathrm{R}\varphi_{\gamma_*}\colon \mathrm{D}^{\mathrm{b}}_{\gamma^{\circ a}}(\mathbf{k}_M) \to \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{M_\gamma})$ is an equivalence of triangulated categories with quasi-inverse φ_γ^{-1}.

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1 Notations and background

Unless otherwise specified, a manifold means a real C^∞-manifold and a morphism of manifolds $f\colon M \to N$ is a map of class C^∞.

Vector bundles

Let $p\colon E \to M$ be a real (finite-dimensional) vector bundle over M.

- As usual, one denotes by $a\colon E \to E$ the antipodal map, $(x;\xi) \mapsto (x;-\xi)$. For a subset $A \subset E$, one sets $A^a = a(A)$.
- A subset γ of E is *conic* (or is a cone) if it is invariant by the action of $\mathbb{R}_{>0}$, that is, $\gamma_x \subset E_x$ is a cone for each $x \in M$. Here, γ_x is the restriction of γ to the fibre E_x. If γ is closed and conic, then its projection by p on M coincides with its intersection with the zero-section of E and we identify the zero-section of E with M.
- A cone γ is proper if γ_x contains no line for any $x \in M$.
- For a cone $\gamma \subset E$, one defines the polar cone $\gamma^\circ \subset E^*$ by

  (1.1) $$\gamma_x^\circ = \{\eta \in E_x^*; \langle \eta, \xi\rangle \geq 0 \text{ for all } \xi \in \gamma_x\}.$$

One shall be aware that for a cone $\gamma \subset E$, its closure $\overline{\gamma}$ may be different from the union $\bigcup_{x\in M} \overline{\gamma_x}$ and similarly with its interior $\mathrm{Int}(\gamma)$.

Cotangent bundle

Let M be a real manifold of class C^∞. We shall use the classical notations below.

- $\tau_M\colon TM \to M$ is the tangent bundle, $\pi_M\colon T^*M \to M$ the cotangent bundle,
- for N a submanifold of M, T_NM is the normal bundle, T^*_NM the conormal bundle,
- M is identified with T^*_MM, (resp. T_MM) the zero-section of T^*M (resp. of TM),
- Δ_M, or simply Δ, is the diagonal of $M \times M$,
- for two manifolds M and N one denotes by q_1 and q_2 the first and second projections defined on $M \times N$,
- one denotes by p_1 and p_2 the first and second projections defined on $T^*(M\times N) \simeq T^*M \times T^*N$. One also uses the notation p_2^a for the composition of p_2 and the antipodal map on T^*N.

To a morphism $f\colon M \to N$ of manifolds, one associates the maps

(1.2) $$\begin{tikzcd} TM \arrow[r, "f'"] \arrow[dr, "\tau"'] & M\times_N TN \arrow[r, "f_\tau"] \arrow[d, "\tau"] & TN \arrow[d, "\tau"] \\ & M \arrow[r, "f"] & N, \end{tikzcd} \qquad \begin{tikzcd} T^*M \arrow[dr, "\pi"'] & M\times_N T^*N \arrow[l, "f_d"'] \arrow[r, "f_\pi"] \arrow[d, "\pi"] & T^*N \arrow[d, "\pi"] \\ & M \arrow[r, "f"] & N. \end{tikzcd}$$

We set

(1.3) $$Tf := f_\tau \circ f'\colon TM \to TN,$$

and call Tf the tangent map to f.

One denotes by Γ_f the graph of f and sets

$$\Lambda_f := T^*_{\Gamma_f}(M \times N). \tag{1.4}$$

Then there is a commutative diagram:

$$\begin{tikzcd}
 & \Lambda_f \arrow[dl, "p_1"'] \arrow[d, "\sim"] \arrow[dr, "p_2^a"] & \\
T^*M & M \times_N T^*N \arrow[l, "f_d"] \arrow[r, "f_\pi"'] & T^*N.
\end{tikzcd} \tag{1.5}$$

Sheaves

We shall mainly follow the notations of [KS90] and use some of the main results of this book. Here, $\mathbf{k}$ denotes a commutative unitary ring with finite global dimension. One denotes by $Derb(\mathbf{k}_M)$ the bouneded derived category of sheaves on $\mathbf{k}$-modules on M and calls an object of this category, a sheaf. For a sheaf F, we shall in particular use its micro-support $\mathrm{SS}(F)$, a closed $\mathbb{R}^+$-conic subset of T^*M.

2 The linear case

Let $\mathbb{V}$ be a finite dimensional real vector space and let θ be a closed convex proper cone with non-empty interior. We set $\gamma = \mathbb{V} \times \theta$.

Let $V \subset \mathbb{V}$ be an open subset of $\mathbb{V}$. Recall, after [KS90, §3.5] with slightly different notations, that V is γ-open if $V = V + \theta$. The family of γ-open subsets of $\mathbb{V}$ defines the γ-topology $\mathbb{V}_\gamma$ on the set $\mathbb{V}$. One denotes by $\varphi_\gamma\colon \mathbb{V} \to \mathbb{V}_\gamma$ the continuous map associated with the identity of the set $\mathbb{V}$.

Definition 2.1. Let $\mathbb{V}$ and γ be as above, let U be open in $\mathbb{V}$ and let $x_0 \in \partial U$. We say that U is γ-open in a neighborhood of x_0 if there exists a γ-open set V and an open neighborhood U_0 of x_0 such that $U \cap U_0 = V \cap U_0$.

Lemma 2.2. *Let $\mathbb{V}$ and γ be as above. Let U be open in $\mathbb{V}$ and let $x_0 \in \partial U$. Assume that for an open neighborhood U_0 of x_0, $U_0 \times_{\mathbb{V}} \mathrm{SS}(\mathbf{k}_U) \subset U_0 \times \theta^{\circ a}$. Then there exists an open subset W of $\mathbb{V}$ which coincides with U in a neighborhood of x_0 and such that $\mathrm{SS}(\mathbf{k}_W) \subset \mathbb{V} \times \theta^{\circ a}$.*

Proof. We may assume that $\mathbb{V}$ is the Euclidian space $\mathbb{R}^n$ and $x_0 = 0$. We denote by $B_\varepsilon(x)$ the open ball of center x and radius $\varepsilon > 0$. We may assume that, setting $\xi_0 = (1, 0, \dots, 0)$, $\xi_0 \in \mathrm{Int}(\theta)$. Also set $y_\varepsilon = (\varepsilon, 0, \dots, 0)$ so that x_0 belongs to the boundary of $B_\varepsilon(y_\varepsilon)$. For $\eta \in \mathbb{R}$, set

$$H_\eta = \{x \in \mathbb{V}; x_1 \leq \eta\}, \quad L_\eta = \{x \in \mathbb{V}; x_1 = \eta\}.$$

Let $Z_{\varepsilon,\eta}$ be the lens

$$Z_{\varepsilon,\eta} = B_{\varepsilon+\eta}(y_\varepsilon) \cap H_\eta,$$

and set

$$U_{\varepsilon,\eta} = U \cap Z_{\varepsilon,\eta}.$$

We fix $\varepsilon > 0$. For $0 < \eta < \varepsilon$, $x_0 \in \mathrm{Int} Z_{\varepsilon,\eta}$. Moreover, for η small enough, $\mathrm{SS}(\mathbf{k}_{Z_{\varepsilon,\eta}}) \subset \mathbb{V} \times \theta^{\circ a}$. Since $U_0 \times_{\mathbb{V}} \mathrm{SS}(\mathbf{k}_U) \subset U_0 \times \theta^{\circ a}$, it follows (see [KS90, Prop. 5.4.14]) that for η small enough,

$$\mathrm{SS}(\mathbf{k}_{U_{\varepsilon,\eta}}) \subset \mathbb{V} \times \theta^{\circ a},$$

and $U_{\varepsilon,\eta} = U$ in a neighborhood of x_0.

Now set

$$W_{\varepsilon,\eta} = \bigcup_{x \in L_\eta \cap U} x + \mathrm{Int}\theta.$$

Then $W_{\varepsilon,\eta}$ is open, $\mathrm{SS}(\mathbf{k}_{W_{\varepsilon,\eta}}) \subset \mathbb{V} \times \theta^{\circ a}$ and $U_{\varepsilon,\eta} \cup W_{\varepsilon,\eta}$ satisfies the requested properties. Q.E.D.

Lemma 2.3. *Let $\mathbb{V}$ and γ be as above and let U be open in $\mathbb{V}$. Then U is γ-open in $\mathbb{V}$ if and only if* $\mathrm{SS}(\mathbf{k}_U) \subset \mathbb{V} \times \theta^{\circ a}$.

Proof. (i) Assume that U is θ-open. Then $N(U)^\circ \subset \mathbb{V} \times \theta^\circ$ by [KS90, Prop. 5.3.7 (ii)] and thus $\mathrm{SS}(\mathbf{k}_U) \subset \mathbb{V} \times \theta^{\circ a}$ by [KS90, Prop. 5.3.8].
(ii) Assume that $\mathrm{SS}(\mathbf{k}_U) \subset \mathbb{V} \times \theta^{\circ a}$. It follows from [KS90, Prop. 5.2.3] that $\mathbf{k}_U \simeq \varphi_\theta^{-1} \mathrm{R}\varphi_{\theta *} \mathbf{k}_U$. By Proposition 3.5.3 (i) of loc. cit., for V open and convex, we have the isomorphism $\Gamma((V+\theta); \mathrm{R}\varphi_{\theta *}\mathbf{k}_U) \simeq \Gamma(V; \varphi_\theta^{-1}\mathrm{R}\varphi_{\theta *}\mathbf{k}_U)$. Therefore, $\Gamma((V+\theta); \mathbf{k}_U) \simeq \Gamma(V; \mathbf{k}_U)$, hence $\Gamma((V+\theta); \mathbf{k}_U) \neq 0$ for any open set $V \subset U$. This last condition implies $(V+\theta) \subset U$ hence $U = U + \theta$. Q.E.D.

Let $\mathbb{V}$ and γ be as above.

Lemma 2.4. *Let $U \subset \mathbb{V}$ be an open subset and let $x_0 \in \partial U$. Then U is γ-open in a neighborhood of x_0 if and only if there exists an open neighborhood W of x_0 such that $W \times_{\mathbb{V}} \mathrm{SS}(\mathbf{k}_U) \subset W \times \theta^{\circ a}$.*

Proof. Since both properties are local on the boundary of U in W, we may assume by Lemma 2.2 that $W = \mathbb{V}$. Then apply Lemma 2.4. Q.E.D.

Definition 2.5. Let U be a convex open subset of $\mathbb{V}$. One denotes by U_γ the set U endowed with the topology induced by $\mathbb{V}_\gamma$ and by $\varphi_{U_\gamma} \colon U \to U_\gamma$ the continuous map associated with the identity map of U.

Note that an open subset V of U is open in U_γ if and only if $V = (V+\theta) \cap U$. Indeed, if W is open and $V = (W+\theta) \cap U$, then $V \subset (V+\theta) \cap U \subset (W+\theta) \cap U$.

One has a commutative diagram of topological spaces

$$\begin{tikzcd} \mathbb{V} \arrow[r, "\varphi_\gamma"] & \mathbb{V}_\gamma \\ U \arrow[u, "i_U"] \arrow[r, "\varphi_{U_\gamma}"'] & U_\gamma \arrow[u, "i_\gamma"'] \end{tikzcd} \tag{2.1}$$

One shall be aware that U and $\mathbb{V}$ are locally compact spaces Hausdorff contrarily to U_γ and $\mathbb{V}_\gamma$ which are not Hausdorff and the classical results on proper direct images do not apply to these spaces.

Lemma 2.6. *Let U be a convex open subset of $\mathbb{V}$. The functor $\varphi_{U_\gamma}^{-1}\colon \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{U_\gamma}) \to \mathrm{D}^{\mathrm{b}}(\mathbf{k}_U)$ takes its values in $\mathrm{D}^{\mathrm{b}}_{U\times\theta^{\circ a}}(\mathbf{k}_U)$.*

Proof. Let $G \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{U_\gamma})$. Consider Diagram (2.1). Then

$$\varphi_{U_\gamma}^{-1}G \;\simeq\; \varphi_{U_\gamma}^{-1} i_\gamma^{-1}\mathrm{R}i_{\gamma *}G \simeq i_U^{-1}\varphi_\gamma^{-1}(\mathrm{R}i_{\gamma *}G).$$

Since $\mathrm{SS}(\varphi_\gamma^{-1}(\mathrm{R}i_{\gamma *}G)) \subset \mathbb{V}\times\theta^{\circ a}$ by [KS90, Prop. 5.2.3], we get the result. Q.E.D.

Proposition 2.7. *Let U be a convex open subset of $\mathbb{V}$. The functor $\mathrm{R}\varphi_{U_\gamma *}\colon \mathrm{D}^{\mathrm{b}}_{U\times\theta^{\circ a}}(\mathbf{k}_U) \to \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{U_\theta})$ is an equivalence of triangulated categories with quasi-inverse $\varphi_{U_\gamma}^{-1}$.*

When $U = \mathbb{V}$, this result is an immediate consequence of [KS90, Prop. 3.5.3, Prop. 5.2.3]. A careful reading of these propositions shows that the proofs extend verbatim to the case where $\mathbb{V}$ is replaced with a convex open subset.

Note that we shall not use Proposition 2.7 and that Theorem 5.5 will generalize it.

3 Topologies on causal manifolds

Causal manifolds

Consider first a real manifold M (of class C^∞) endowed with a closed convex proper cone $\gamma \subset TM$ containing the zero-section.

Definition 3.1. (a) A causal manifold (M,γ) is a manifold M equipped with a closed convex proper cone $\gamma \subset TM$ with the property that the projection $\tau_M\colon \mathrm{Int}(\gamma) \to M$ is surjective.

(b) To a causal manifold (M,γ) one associates the cone $\lambda = \gamma^{\circ a} \subset T^*M$.

(c) A morphism of causal manifolds $f\colon (M,\gamma_M) \to (N,\gamma_N)$ is a morphism of manifolds $f\colon M \to N$ such that the map $Tf\colon TM \to TN$ satisfies $Tf(\gamma_M) \subset \gamma_N$.

(d) Equivalently, a morphism of causal manifolds $f\colon (M,\lambda_M) \to (N,\lambda_N)$ is a morphism of manifolds $f\colon M \to N$ such that $f_d f_\pi^{-1}\lambda_N \subset \lambda_M$.

(e) For I an open interval of $\mathbb{R}$, we denote by $(I,+)$ the causal manifold (I,γ) where $\gamma = \{(t;v) \in TI; v \geq 0\}$.

The equivalence of (c) and (d) is proved in [JS16, Prop. 1.12].

Also note that the cone λ satisfies

$$\lambda \cap \lambda^a = M, \quad \lambda + \lambda = \lambda. \tag{3.1}$$

Definition 3.2. (i) A constant cone contained in γ is a triple (φ, U, θ) where U is open in M, $\varphi\colon U \to \mathbb{R}^d$ is a chart, $\theta \subset \mathbb{R}^d$ is a closed convex proper cone with non-empty interior such that in this chart $U \times \theta \subset \gamma$ (that is, $\varphi(U) \times \theta \subset T\varphi(U \times_M \gamma)$). A constant cone (φ, U, θ) will often simply be denoted by $U \times \theta$.

(ii) A *basis of constant cones* for γ is a family of constant cones $\{U_i \times \theta_i\}_i$ contained in γ such that $\bigcup_i U_i = M$ and for any $x_0 \in M$, $\mathrm{Int}(\gamma_{x_0}) = \bigcup_{x_0 \in U_i} \theta_i$.

Remark 3.3. For a constant cone (φ, U, θ), we shall often identify U with its image by φ in $\mathbb{R}^d$.

One shall be aware that in [JS16, Def 1.13] the cone γ was open.

Lemma 3.4. *Let (M, γ) be a manifold M equipped with a closed convex proper cone $\gamma \subset TM$. Then (M, γ) is a causal manifold if and only if there exists a basis of constant cones.*

Proof. (i) Assume that there exists a basis of constant cones $\{U_i \times \theta_i\}_i$. Then $\mathrm{Int}(\gamma_{x_0}) \neq \varnothing$ since $\bigcup_i U_i = M$, $\mathrm{Int}(\gamma_{x_0}) = \bigcup_{x_0 \in U_i} \theta_i$ and θ_i has non-empty interior.

(ii) Conversely, assume that $\mathrm{Int}(\gamma_{x_0}) \neq \varnothing$ for all $x_0 \in M$. We may assume that M is open in $\mathbb{R}^d$. Then for any open convex cone θ with $\overline{\theta} \subset \mathrm{Int}(\gamma_{x_0})$, there exists an open neighborhood U of x_0 such that $U \times \theta \subset \mathrm{Int}(\gamma)$. Q.E.D.

γ-topology

Let (M, γ) be a causal manifold and recall that we have set $\lambda = \gamma^{\circ a}$.

For two subsets $A, B \subset M$ and $x \in M$, the normal cone $C_x(A, B) \subset T_xX$ is defined in [KS90, Def 4.1.1]. In a local coordinate system

$$(x_0; v_0) \in C_{x_0}(A, B) \quad \Leftrightarrow \quad \begin{array}{l} \text{there exists a sequence } \{(x_n, y_n, c_n)\}_n \subset A \times B \times \mathbb{R}^+ \\ \quad \text{with } c_n(x_n - y_n) \xrightarrow{n} v_0, x_n \xrightarrow{n} x_0, y_n \xrightarrow{n} x_0. \end{array}$$

The strict normal cone $N(A) \subset TM$ in defined in [KS90, Def 5.3.6] by

$$\begin{aligned} N_x(A) &= T_xX \setminus C_x(M \setminus A, A), \\ N(A) &= \bigcup_{x \in M} N_x(A). \end{aligned}$$

The cone $N(A)$ is open and convex.

Definition 3.5 ([JS16, Def 1.18])**.** Let (M, γ) be a causal manifold.

(a) A subset $A \subset M$ is a γ-set if $\gamma \subset N(A)$.

(b) A subset $U \subset M$ is γ-open if it is both an open subset and a γ-set.

It follows from [KS90, (5.3.3)] that

$$A \text{ is a } \gamma\text{-set} \Leftrightarrow \begin{cases} \text{there exists a basis of constant cones } U \times \theta \text{ for } \gamma \\ \text{such that } U \cap (U \cap A + \mathrm{Int}\theta) \subset A. \end{cases} \tag{3.2}$$

Remark that the property of being γ-open is a local property on the boundary ∂U.

One needs to check that this definition is in accordance with the previous definition of § 2.

Lemma 3.6. *Assume that $M = \mathbb{V}$ is a finite dimensional vector space and $\gamma = \mathbb{V} \times \theta$ for an open convex proper cone θ with non-empty interior. Then an open subset $U \subset \mathbb{V}$ is γ-open if and only if $U = U + \theta$.*

Proof. (i) Assume that $U = U + \theta$. Let us choose $U \times \theta$ as a basis of constant cones. Then $U = U + \mathrm{Int}\theta$. Therefore, U is γ-open by (3.2).
(ii) Conversely, asssume that U is γ-open. This implies that for any $x_0 \in \partial U$ and any closed convex cone $\lambda \subset \mathrm{Int}\theta$, there exists an open neighborhood V of x_0 such that $V \cap (V \cap U + \lambda) \subset U$. Hence, $U = U + \lambda$ for any any closed convex cone $\lambda \subset \mathrm{Int}\theta$, which implies $U = U + \theta$. Q.E.D.

Proposition 3.7 ([JS16, Prop 1.19, 1.21])**.** *Let (M, γ) be a causal manifold.*

(i) *A set A is a γ-set if and only if $\gamma_x \subset N_x(A)$ for every $x \in \partial A$.*

(ii) *A set A is a γ-set if and only if $M \setminus A$ is a γ^a-set.*

(iii) *The family of γ-sets is closed under arbitrary unions and intersections.*

(iv) *The family of γ-sets is closed under taking closure and interior.*

(v) *If A is a γ-set, then $\overline{\mathrm{Int}A} = \overline{A}$ and $\mathrm{Int}\overline{A} = \mathrm{Int}A$.*

(vi) *If A is a γ-set and $\mathrm{Int}A \subset B \subset \overline{A}$, then B is a γ-set.*

The γ-topology on M was first defined in [KS90, § 3.5] when $M = \mathbb{V}$ is a vector space and $\gamma = M \times \theta$ for a closed convex cone $\theta \subset \mathbb{V}$.

Definition 3.8 ([KS90, § 3.5] and [JS16, Def 1.23])**.** Let (M, γ) be a causal manifold.

(a) The γ-topology on M is the topology for which the open sets are the γ-open sets.

(b) One denotes by M_γ the space M endowed with the γ-topology and by $\varphi_\gamma \colon M \to M_\gamma$ the continuous map associated with the identity of the set M.

One shall be aware that the closed sets for the γ-topology are not γ-sets in general. They are γ^a-sets. That is why it is better to avoid the terminology "a closed γ-set" whose meaning could be a set which is closed for the γ-topology as well as a closed set which is a γ-set.

Remark 3.9. When $\mathbb{V}$ is a vector space, $U \subset \mathbb{V}$ is a convex open subset and θ is a closed convex proper cone, Definition 3.8 gives $U_\theta = U_{U\times\theta}$ and $\varphi_{U_\theta} = \varphi_{U\times\theta}$. As far as there is no risk of confusion, we keep the notations U_θ and φ_θ.

γ-topology and λ-topology

Definition 3.10. Let (M, γ) be a causal manifold and let $\lambda = \gamma^{\circ a}$.

(a) A locally closed set $A \subset M$ is a λ-set if $\mathrm{SS}(\mathbf{k}_A) \subset \lambda$.

(b) A λ-open set (resp. a λ-closed set) is an open set (resp. a closed set) which is also a λ-set.

Note that U is λ-open if and only if $M \setminus U$ is λ-closed.

Remark 3.11. In [JS16, Not. 1.10] we have set $\lambda = \gamma^{\circ}$, contrarily to the above notation.

Proposition 3.12. *Let (M, γ) be a causal manifold.*

(a) *The family of λ-open sets is closed under arbitrary unions and finite intersections. In particular, the family of λ-open subsets of M defines a topology on M.*

(b) *Similarly, the family of λ-closed sets is closed under arbitrary intersections and finite unions.*

Proof. Since an open set U is λ-open if and only if $M \setminus U$ is λ-closed, it is enough to prove (a).
(i) If U_1 and U_2 are λ-open, so is $U_1 \cap U_2$ since $\mathbf{k}_{U_1 \cap U_2} \simeq \mathbf{k}_{U_1} \otimes \mathbf{k}_{U_2}$ and $\mathrm{SS}(\mathbf{k}_{U_1} \otimes \mathbf{k}_{U_2}) \subset \lambda$ by (3.1).
(ii) If U_1 and U_2 are λ-open, so is $U_1 \cup U_2$. This follows from (i) and the distinguished triangle $\mathbf{k}_{U_1 \cap U_2} \to \mathbf{k}_{U_1} \oplus \mathbf{k}_{U_2} \to \mathbf{k}_{U_1 \cup U_2} \xrightarrow{+1}$.
(iii) Let $\{U_i\}_{i \in I}$ be a family of λ-open subsets. Let us order I by the relation $i \leq j$ if $U_i \subset U_j$. We may assume that I is non empty and by (ii) we may assume that $(I, \leq)$ is directed. It then follows from [KS90, Exe. 5.7] that, setting $U = \bigcup_{i \in I} U_i$, $\mathrm{SS}(\mathbf{k}_U) \subset \lambda$.
(iv) Since $\mathrm{SS}(\mathbf{k}_M) = T^*_M M \subset \lambda$ and $\mathrm{SS}(\mathbf{k}_\varnothing) = \varnothing$, both M and $\varnothing$ are λ-open and the proof is complete. Q.E.D.

Theorem 3.13. *Let (M, γ) be a causal manifold and recall that we have set $\lambda = \gamma^{\circ a}$.*

(a) *An open set U of M is a γ-set if and only if it is a λ-set.*

(b) *A closed subset Z of M is a γ^a-set if and only if it is a λ-set. In particular, a closed λ-set is closed for the γ-topology.*

Proof. (a)–(i) We know by [KS90, Prop. 5.3.8] that $\mathrm{SS}(\mathbf{k}_U) \subset N(U)^{\circ a}$. If U is a γ-set, then $\mathrm{Int}\gamma \subset N(U)$ and thus $\mathrm{SS}(\mathbf{k}_U) \subset N(U)^{\circ a} \subset \gamma^{\circ a} = \lambda$.
(a)–(ii) Now assume that $\mathrm{SS}(\mathbf{k}_U) \subset \gamma^{\circ a}$. The problem is local on M and we may assume that M is open in a vector space $\mathbb{V}$. Let $U \subset M$ be an open subset and assume that $\mathrm{SS}(\mathbf{k}_U) \subset \lambda$. Let W be an open and convex set and θ a open convex cone with $W \times_M \lambda \subset W \times \theta^{\circ a}$. Then $U \cap W$ is γ-open by Lemma 2.4. It follows that U is γ-open by (3.2).
(b) Set $U = M \setminus Z$. Then Z is a λ-set if and only if so is U. On the other-hand, U is a γ-set if and only if Z is a γ^a-set. Q.E.D.

Definition 3.14. Let (M, γ) be a causal manifold. The λ-topology on M is the topology for which the open sets are the λ-open sets. Hence, the closed sets are the closed λ^a-sets.

It follows from Theorem 3.13 that the γ-topology and the λ-topology coincide.

4 Preorders on causal manifolds

Preorders

Consider a preorder $\preceq$ on a manifold M and its graph $\Delta_{\preceq} \subset M \times M$. Then

$$\Delta \subset \Delta_{\preceq} \quad \Delta_{\preceq} \circ \Delta_{\preceq} = \Delta_{\preceq}\,.$$

For a subset $A \subset M$, one sets

$$(4.1) \qquad \begin{cases} J^+_{\preceq}(A) = q_2(q_1^{-1}(A) \cap \Delta_{\preceq}) = \{x \in M;\ \text{there exists } y \in A \text{ with } y \preceq x\}, \\ J^-_{\preceq}(A) = q_1(q_2^{-1}(A) \cap \Delta_{\preceq}) = \{x \in M;\ \text{there exists } y \in A \text{ with } x \preceq y\}. \end{cases}$$

For $x \in M$, we write $J^+_{\preceq}(x)$ and $J^-_{\preceq}(x)$ instead of $J^+_{\preceq}(\{x\})$ and $J^-_{\preceq}(\{x\})$ respectively. One calls $J^+_{\preceq}(A)$ the future of A and $J^-_{\preceq}(A)$ the past of A (for the preorder $\preceq$).

Note that $J^+_{\preceq}(A) = \bigcup_{x\in A} J^+_{\preceq}(x)$ and $J^-_{\preceq}(A) = \bigcup_{x\in A} J^-_{\preceq}(x)$.

By its definition, one has

$$(4.2) \qquad \Delta_{\preceq} = \bigcup_{x\in M} J^-_{\preceq}(x) \times J^+_{\preceq}(x) = \bigcup_{x\in M} \{x\} \times J^+_{\preceq}(x).$$

Definition 4.1. Let $\preceq$ be a preorder on M.

(a) The preorder is closed if $\Delta_{\preceq}$ is closed in $M \times M$.

(b) The preorder is proper if q_{13} is proper on $\Delta_{\preceq} \times_M \Delta_{\preceq}$. Equivalently, the preorder is proper if for any two compact subsets A and B of M, the so-called causal diamond $J^+_{\preceq}(A) \cap J^-_{\preceq}(B)$ is compact.

If the preorder $\preceq$ is closed, then for any compact subset A of M the sets $J^-_{\preceq}(A)$ and $J^+_{\preceq}(A)$ are closed. If the preorder is proper, then it is closed. A preorder is proper as soon as $J^+_{\preceq}(x) \cap J^-_{\preceq}(y)$ is compact for any two $x, y \in M$.

The γ-preorder

This subsection is, with some modifications, extracted from [JS16, § 1.4].

Let (M, γ) be a causal manifold.

Definition 4.2. For $A \subset M$, we denote by $I^+_{\gamma}(A)$ the intersection of all the γ-sets which contain A and call it the γ-future of A. We set $I^+_{\gamma}(x) = I^+_{\gamma}(\{x\})$.

Note that a set A is a γ-set if and only if $I^+_{\gamma}(A) = A$.

Let us set

$$x \preceq_{\gamma} y \text{ if and only if } y \in I^+_{\gamma}(x).$$

Lemma 4.3. (a) *The relation $x \preceq_\gamma y$ is a preorder.*

(b) *For $A \subset M$, $I^+_\gamma(A) = \bigcup_{x\in A} I^+_\gamma(x)$. In other words, with the preceding notations, $I^+_\gamma(A) = J^+_{\preceq_\gamma}(A)$.*

Proof. (a) Let $y \in I^+_\gamma(x)$ and $z \in I^+_\gamma(y)$. Then $I^+_\gamma(x)$ is a γ-set which contains y and $I^+_\gamma(y)$ is the smallest γ-set which contains y. Therefore, $I^+_\gamma(y) \subset I^+_\gamma(x)$ and $z \in I^+_\gamma(x)$.
(b) The term on the right-hand side is a γ-set, contains A and is contained in $I^+_\gamma(A)$ (since $I^+_\gamma(x) \subset I^+_\gamma(A)$ for $x \in A$), whence the equality. It follows that $y \in I^+_\gamma(A)$ if and only if there exists $x \in A$ such that $x \preceq_\gamma y$. Q.E.D.

According to (4.1), one has

$$
\begin{aligned}
&I^+_\gamma(x) = J^+_{\preceq_\gamma}(x),\\
&J^-_{\preceq_\gamma}(y) = J^-_{\preceq_\gamma}(\{y\}) := \{x; x \preceq_\gamma y\}, \quad J^-_{\preceq_\gamma}(A) = \textstyle\bigcup_{y\in A} J^-_{\preceq_\gamma}(y).
\end{aligned} \tag{4.3}
$$

Remark 4.4. Recall the causal manifold $(I, +)$ of Definition 3.1 (e). On $(I, +)$ the preorder $\preceq_\gamma$ is the usual order $\leq$.

Proposition 4.5. *Let $A \subset M$ be a closed subset of M. Then $I^+_\gamma(A) \setminus A$ is open.*

Proof. One has $\mathrm{Int}(I^+_\gamma(A)) \subset \mathrm{Int}(I^+_\gamma(A)) \cup A \subset I^+_\gamma(A)$. Applying Proposition 3.7 (iv), we get that $\mathrm{Int}(I^+_\gamma(A)) \cup A$ is a γ-set. Since it contains A, it contains $I^+_\gamma(A)$. Therefore $\mathrm{Int}(I^+_\gamma(A)) \cup A = I^+_\gamma(A)$ and $L^+_\lambda(A) \setminus A = \mathrm{Int}(I^+_\gamma(A)) \setminus A$ is open. Q.E.D.

Corollary 4.6. *Let U be open in M. Then $I^+_\gamma(U)$ is γ-open.*

Proof. One has $I^+_\gamma(U) = \bigcup_{x\in U}(I^+_\gamma(x) \setminus \{x\}) \cup U$. Q.E.D.

Definition 4.7. (a) Let $I = [0, 1]$. If a function $c\colon I \to M$ is left (resp. right) differentiable, we denote its left (resp. right) differential by c'_l (resp. c'_r).

(b) A path $c\colon I \to M$ is a piecewise smooth map.

(c) A path c is causal if $c'_l(t), c'_r(t) \in \gamma_{c(t)}$ for any $t \in I$ and is strictly causal if $c'_l(t), c'_r(t) \in (\mathrm{Int}\gamma)_{c(t)}$ for any $t \in I$.

Note that if c_1 and c_2 are two causal (resp. strictly causal) paths on I with $c_1(1) = c_2(0)$, the concatenation $c = c_1 \cup c_2$ (defined by glueing the two paths as usual) is causal (resp. strictly causal).

The next result is obvious (see also [JS16, Lem. 1.30]).

Lemma 4.8. *Let (M, γ) be a causal manifold and consider a constant cone $U \times \theta$ contained in γ. Then, for $y, z \in U$ with $z - y \in \theta$, we have $z \in I^+_\gamma(y)$.*

Lemma 4.9 (see [JS16, Lem. 1.34]). *Let $c\colon I \to M$ be a strictly causal path. Then for $t_1 \leq t_2$ with $t_1, t_2 \in I$ we have $c(t_1) \preceq_\gamma c(t_2)$.*

Proof. It is enough to prove that for any $t_0 \in I$, there exists $\alpha > 0$ such that $c(t_0) \preceq_\gamma c(t)$ for $t \in (t_0, t_0 + \alpha)$ and similarly $c(t) \preceq_\gamma c(t_0)$ for $t \in (t_0 - \alpha, t_0)$. We may assume $t_0 = 0$. There exists a constant cone $U \times \theta$ contained in γ and containing $(c(0), c'_r(0))$. There exists $\alpha > 0$ such that $c(t) - c(0) \in \theta$ for $t \in (0, \alpha)$. By Lemma 4.8, this implies $c(0) \preceq_\gamma c(t)$ for $t \in (0, \alpha)$. The other case is similar, using $c'_l(0)$. Q.E.D.

Proposition 4.10 (see [JS16, Lem. 1.35])**.** *Let $A \subset M$. One has $y \in I^+_\gamma(A)$ if and only if $y \in A$ or there exists a strictly causal path $c\colon I \to M$ such that $c(0) \in A$, $c(1) = y$.*

Proof. Let B be the union of A with the set of points that can be reached from A by a strictly causal path. We shall prove that $I^+_\gamma(A) = B$.

(i) To prove that $B \supset I^+_\gamma(A)$, it is enough to check that B is a γ-set. Choose a constant cone $U \times \theta$ contained in γ with U convex. By (3.2), il is enough to prove that $U \cap (B \cap U + \theta) \subset B$. Let $y' \in B \cap U$ and $v' \in \theta$ with $y' + v' \in U$. Since U is convex, $c\colon I \to U, t \mapsto y' + tv'$ is a strictly causal path for I a small enough neighborhood of $[0, 1]$. Since $y' \in B$, there exists a strictly causal path $\tilde{c}$ with $\tilde{c}(0) \in A$ and $\tilde{c}(1) = y'$. Therefore, concatenating $\tilde{c}$ and c proves that $y' + v' \in B$.

(ii) Let us prove that $B \subset I^+_\gamma(A)$. Let $y \in B, y \notin A$. There exist $x \in A$ and a strictly causal curve c going from x to y. Then $y \in I^+_\gamma(x)$ by Lemma 4.9. Hence, $y \in I^+_\gamma(A)$. Q.E.D.

Corollary 4.11 (see [JS16, Lem. 1.41])**.** *One has $J^-_{\preceq_\gamma}(y) = I^+_{\gamma^a}(y)$.*

Proof. This follows immediately from Proposition 4.10. Q.E.D.

The λ-preorder

Let (M, γ) be a causal manifold. Recall that $\lambda = \gamma^{\circ a}$.

Definition 4.12. (a) For $A \subset M$, we denote by $L^+_\lambda(A)$ the intersection of all closed λ^a-sets which contain A:

$$L^+_\lambda(A) = \bigcap Z \text{ with } A \subset Z \text{ and } Z \text{ is a closed } \lambda^a\text{-set.}$$

(b) For $x \in M$, we set $L^+_\lambda(x) = L^+_\lambda(\{x\})$.

In other words, $L^+_\lambda(A)$ is the closure of A for the λ-topology, hence, for the γ-topology. In particular, $L^+_\lambda(A) = L^+_\lambda(\overline{A})$. Let us set

$$x \preceq_\lambda y \text{ if and only if } y \in L^+_\lambda(x).$$

Lemma 4.13. *The relation $x \preceq_\lambda y$ is a preorder.*

Proof. If $y \in L^+_\lambda(x)$, then $L^+_\lambda(y) \subset L^+_\lambda(x)$. Indeed, $L^+_\lambda(x)$ is a closed λ^a-set which contains y. Hence, it contains $L^+_\lambda(y)$. Q.E.D.

Notation 4.14. We call $\preceq_\gamma$ the γ-preorder and $\preceq_\lambda$ the λ-preorder.

According to (4.1), one has

$$
\begin{aligned}
&J^+_{\preceq_\gamma}(x) = I^+_\gamma(x)\\
&J^-_{\preceq_\lambda}(y) = J^-_{\preceq_\lambda}(\{y\}) := \{x; x \preceq_\lambda y\}, \quad J^-_{\preceq_\lambda}(A) = \bigcup_{y\in A} J^-_{\preceq_\lambda}(y).
\end{aligned}
\tag{4.4}
$$

Proposition 4.15. *For $A \subset M$, one has $L^+_\lambda(A) = \overline{I^+_\gamma(A)}$. In particular, for $x, y \in M$, $L^+_\lambda(x) = \overline{I^+_\gamma(x)}$ and $x \preceq_\gamma y$ implies $x \preceq_\lambda y$.*

Proof. (i) Let us prove the inclusion $\overline{I^+_\gamma(A)} \subset L^+_\lambda(A)$. The closed set $L^+_\lambda(A)$ contains A and being a λ^a-set, it is a γ-set by Theorem 3.13. Therefore, $I^+_\gamma(A) \subset L^+_\lambda(A)$ and the result follows.

(ii) Let us prove the inclusion $L^+_\lambda(A) \subset \overline{I^+_\gamma(A)}$. The closed set $\overline{I^+_\gamma(A)}$ is a γ-set by Proposition 3.7, hence a λ^a-set by Theorem 3.13. Q.E.D.

Corollary 4.16. *One has $\overline{\bigcup_{x\in A} L^+_\lambda(x)} = L^+_\lambda(A)$.*

Proof. (i) The inclusion $L^+_\lambda(x) \subset L^+_\lambda(A)$ implies the inclusion $\subset$ in the statement.

(ii) The left-hand side is a γ-set thanks to Propositions 3.7 and 4.15. It is thus a λ^a-set by Theorem 3.13. Therefore, it contains $L^+_\lambda(A)$ by Definition 4.12. Q.E.D.

In the sequel, we set $\Delta_\gamma = \Delta_{\preceq_\gamma}$ and $\Delta_\lambda = \Delta_{\preceq_\lambda}$.

Corollary 4.17. *One has*

$$\Delta_\gamma = I^+_{\gamma^a\times\gamma}(\Delta) \subset \Delta_\lambda \subset L^+_{\lambda^a\times\lambda}(\Delta) = \overline{\Delta_\gamma}.$$

Proof. (i) $I^+_{\gamma^a\times\gamma}(\Delta) \subset \Delta_\gamma$. Since Δ_γ contains the diagonal, il is enough to prove that it is a $(\gamma^a \times \gamma)$-set. Each $I^+_{\gamma^a}(x) \times I^+_\gamma(y)$ is a $(\gamma^a \times \gamma)$-set by [JS16, Prop. 1.19 (iii)]. By using (4.2) we get the result by Proposition 3.7.

(ii) $\Delta_\gamma \subset I^+_{\gamma^a\times\gamma}(\Delta)$. Let $(x, y) \in \Delta_\gamma$, that is $x \preceq_\gamma y$. By Proposition 4.10, there exists a strictly causal path $c\colon I \to M$ with $c(0) = x$ and $c(1) = y$. Consider the path $\widetilde{c} = (c_1, c_2)\colon I \to M \times M$ given by $c_1(t) = c(1-t)$ and $c_2(t) = c(t)$. Then $\widetilde{c}$ is a strictly causal path in $(M \times M, \gamma^a \times \gamma)$. Since $\widetilde{c}(1/2, 1/2) \in \Delta$, we get by Proposition 4.10 that $(x, y) \in I^+_{\gamma^a\times\gamma}(\Delta)$.

(iii) $\Delta_\gamma \subset \Delta_\lambda$ by Proposition 4.15.

(iv) $L^+_{\lambda^a\times\lambda}(\Delta) = \overline{I^+_{\gamma^a\times\gamma}(\Delta)}$ by Proposition 4.15. Hence, $L^+_{\lambda^a\times\lambda}(\Delta) = \overline{\Delta_\gamma}$ by (i)–(ii).

(v) $\Delta_\lambda \subset L^+_{\lambda^a\times\lambda}$. Indeed, $\Delta_\lambda = \bigcup_x(\{x\} \times L^+_\lambda(x))$ by (4.2). Since $\{x\} \times L^+_\lambda(x) = \overline{\{x\} \times I^+_\gamma(x)}$, we get $\{x\} \times L^+_\lambda(x) \subset \overline{\Delta_\gamma}$. Q.E.D.

Remark 4.18. One has $y \in I^+_\gamma(x) \Leftrightarrow x \in I^+_{\gamma^a}(y)$ and $y \in L^+_\lambda(x) \Leftrightarrow y \in \overline{I^+_\gamma(x)}$. However, the equivalence $y \in \overline{I^+_\gamma(x)} \Leftrightarrow x \in \overline{I^+_{\gamma^a}(y)}$ is false, as shown by the following example well-known to specialists of the field. .

Example 4.19. Let (x_1, x_2) be the coordinates on $\mathbb{R}^2$, $Z = \{(x_1, x_2); x_1 \leq 0, x_2 = 0\}$, $M = \mathbb{R}^2 \setminus Z$. Let $\theta \subset \mathbb{R}^2$ be the closed cone $\theta = \{(x_1, x_2); x_2 \geq |x_1|\}$ and let $\gamma = M \times \theta \subset TM$. Now let $x = (-1, -1) \in M$, $y = (1, 1) \in M$. Then

$$x \in L^+_{\lambda^a}(y) \text{ but } y \notin L^+_\lambda(x).$$

To check this point, use Propositions 4.15 and 4.10

Future time functions

Definition 4.20 is a variation on [JS16, Def. 1.50]. The terminology G-causal below is inspired by the name of Geroch.

Definition 4.20. (a) A future time function on a causal manifold (M, γ) is a surjective and submersive causal morphism $q\colon (M, \gamma) \to (\mathbb{R}, +)$ which is proper on the sets $L_\lambda^+(K)$ for all compact subsets K of M.

(b) A future time function q is strict if for any $x_0 \in M$, setting $t_0 = q(x_0)$, the family $\{I_\gamma^+(U) \cap q^{-1}(t_0 - a, t_0 + a)\}_{U,a}$ is a neighborhood system of x_0 when U ranges over a neighborhood system of x_0 and $a \in \mathbb{R}_{>0}$.

(c) A G^+-causal manifold (M, γ, q) is a causal manifold endowed with a strict future time function q.

A time function is a function q which is a future time function both for γ and γ^a.

We do not recall here the definition of a globally hyperbolic spacetime and its links with that of G-causal manifold. It is known, after the works (among others) of [Ger70, MS08, FS11] that a globally hyperbolic manifold admits time functions.

Lemma 4.21. *Let (M, γ) be a causal manifold and q a future time function. Then*

(a) *the function q is strictly causal,*

(b) *$x \preceq_\gamma y$ implies $q(x) \leq q(y)$.*

Proof. (a) follows from [JS16, Prop. 1.9] and (b) from loc. cit. Lemma 1.46. Q.E.D.

5 An equivalence

Micro-support of inverse images

Let (M, γ) be a causal manifold and consider a constant cone (U, θ) contained in γ. We denote by U_γ the set U endowed with the topology induced by M_γ. We write U_θ instead of $U_{U\times\theta}$ and φ_θ instead of $\varphi_{U\times\theta}$.

We get the commutative diagram of topological spaces:

(5.1)
$$\begin{tikzcd} U \arrow[r, "\varphi_\theta"] \arrow[d, "i_U"'] & U_\theta \arrow[r, "\rho"] & U_\gamma \arrow[d, "i_{U_\gamma}"] \\ M \arrow[rr, "\varphi_\gamma"] & & M_\gamma. \end{tikzcd}$$

Thanks to Lemma 2.6, it gives rise to the commutative diagram

(5.2)
$$\begin{tikzcd} \mathrm{D^b}(\mathbf{k}_M) \arrow[d, "i_U^{-1}"'] & & & \mathrm{D^b}(\mathbf{k}_{M_\gamma}) \arrow[lll, "\varphi_\gamma^{-1}"'] \arrow[d, "i_{U_\gamma}^{-1}"] \\ \mathrm{D^b}(\mathbf{k}_U) & \mathrm{D^b}_{U\times\theta^{\circ a}}(\mathbf{k}_U) \arrow[l, "\iota"] & \mathrm{D^b}(\mathbf{k}_{U_\theta}) \arrow[l, "\varphi_\theta^{-1}"] & \mathrm{D^b}(\mathbf{k}_{U_\gamma}) \arrow[l, "\rho^{-1}"] \end{tikzcd}$$

Proposition 5.1. *Let* (M,γ) *be a causal manifold. The functor* $\varphi_\gamma^{-1}\colon \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{M_\gamma}) \to \mathrm{D}^{\mathrm{b}}(\mathbf{k}_M)$ *takes its values in* $\mathrm{D}^{\mathrm{b}}_{\gamma^{\circ a}}(\mathbf{k}_M)$.

Proof. The problem is local on M and we may assume that M is open in a vector space. Consider a constant cone (U,θ) contained in γ with U convex so that we may apply Lemma 2.6. Then

$$i_U^{-1}\circ\varphi_\gamma^{-1} \simeq \iota\circ\varphi_\theta^{-1}\circ\rho^{-1}\circ i_{U_\gamma}^{-1} \tag{5.3}$$

It follows that for $G\in\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{M_\gamma})$, $\mathrm{SS}(\varphi_\gamma^{-1}G|_U)\subset U\times\theta^{\circ a}$.

Let $(x_0;\xi_0)\in\mathrm{SS}(G)$. There exists a family of constant cones $\{(U_i\times\theta_i)\}_i$ such that $\mathrm{Int}\gamma_{x_0}=\bigcup_i\theta_i$. Hence, $\gamma_{x_0}^{\circ a}=\bigcap_i\theta_i^{\circ a}$ in $T^*_{x_0}M$. Since $(x_0;\xi_0)\in\{x_0\}\times\theta_i^{\circ a}$ for all i, we get that $(x_0;\xi_0)\in\gamma_{x_0}^{\circ a}$. Q.E.D.

Full faithfulness of direct image

Recall that for $x_0\in M$, the set $L_\lambda^+(x_0)$ is λ^a-closed hence is closed for the γ-topology,

Lemma 5.2. *Let* (M,γ,q) *be a* G^+*-causal manifold and let* $F\in\mathrm{D}^{\mathrm{b}}_{\gamma^{\circ a}}(\mathbf{k}_M)$. *Let* $x_0\in M$, $t_0=q(x_0)$. *Let* $a>0$ *and let* U *be an open relatively compact neighborhood of* x_0 *such that* $q(U)\subset(t_0-a,t_0+a)$. *Then* $\mathrm{R}\Gamma(I_\gamma^+(U);F)\xrightarrow{\sim}\mathrm{R}\Gamma(I_\gamma^+(U)\cap q^{-1}(t_0-a,t_0+a));F)$.

Proof. (i) Set $U^+=I_\gamma^+(U)$, $G=\mathrm{R}q_*\mathrm{R}\Gamma_{U^+}F$. Note that, by the hypotheses, the function q is proper on $\overline{I_\gamma^+(U)}$. Moreover, it is increasing for the order $\preceq_\gamma$ by Lemma 4.21, hence $q(U^+)\subset[t_0-a,+\infty)$.
(ii) The set U^+ being a γ-open subset, we get $\mathrm{SS}(\mathrm{R}\Gamma_{U^+}F)\subset\lambda$. Since q is proper on $\mathrm{supp}(\mathrm{R}\Gamma_{U^+}F)$, $\mathrm{SS}(G)\subset\{(t;\tau);t\geq t_0-a,\tau\leq 0\}$ and it follows that $\mathrm{R}\Gamma(\mathbb{R};G)\simeq\mathrm{R}\Gamma((t_0-a,+\infty);G)\xrightarrow{\sim}\mathrm{R}\Gamma((t_0-a,t_0+a);G)$.
(iii) One has $\mathrm{R}\Gamma(I_\gamma^+(U);F)\simeq\mathrm{R}\Gamma(M;\mathrm{R}\Gamma_{U^+}F)\simeq\mathrm{R}\Gamma(\mathbb{R};G)$ and $\mathrm{R}\Gamma((t_0-a,t_0+a);G)\simeq\mathrm{R}\Gamma(I_\gamma^+(U)\cap q^{-1}(t_0-a,t_0+a));F)$. Q.E.D.

Proposition 5.3. *Let* (M,γ,q) *be a* G^+*-causal manifold. Then the functor* $\mathrm{R}\varphi_{\gamma_*}\colon\mathrm{D}^{\mathrm{b}}_{\gamma^{\circ a}}(\mathbf{k}_M)\to\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{M_\gamma})$ *is fully faithful,*

Proof. It is enough to prove the isomorphisms for all $j\in\mathbb{Z}$ and $x_0\in M$

$$H^j(\varphi_\gamma^{-1}\mathrm{R}\varphi_{\gamma_*}F)_{x_0}\xrightarrow{\sim}H^j(F_{x_0}).$$

The left-hand side is obtained by applying H^j and taking the colimit with respect to the open neighborhoods U of x_0 of $\mathrm{R}\Gamma(I_\gamma^+(U);F)$. The right-hand side is similarly obtained when replacing $\mathrm{R}\Gamma(I_\gamma^+(U);F)$ with $\mathrm{R}\Gamma(U;F)$.

Since the family $\{I_\gamma^+(U)\cap q^{-1}(t_0-a,t_0+a))\}_{U,a}$ is a basis of open neighborhoods of x_0 by Definition 4.20, the result follows from Lemma 5.2. Q.E.D.

Main theorem

Proposition 5.4. *Let $f\colon X\to Y$ tbe a continuous map. and let $\mathscr{T}$ be a full triangulated subcategory of $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$. Assume:*

(a) *f is surjective,*

(b) *the functor $f^{-1}\colon \mathrm{D}^{\mathrm{b}}(\mathbf{k}_Y)\to\mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$ takes its values in $\mathscr{T}$,*

(c) *the functor $\mathrm{R}f_*\colon\mathscr{T}\to\mathrm{D}^{\mathrm{b}}(\mathbf{k}_Y)$ is fully faithful.*

Then $\mathrm{R}f_*\colon\mathscr{T}\to\mathrm{D}^{\mathrm{b}}(\mathbf{k}_Y)$ is an equivalence.

Proof. It is enough to prove that f^{-1} is fully faithful, that is $\mathrm{R}f_*\circ f^{-1}\simeq\mathrm{id}$.

Let $G\in\mathrm{D}^{\mathrm{b}}(\mathbf{k}_Y)$ and set $F=f^{-1}G$. Let $y\in Y$ and choose $x\in X$ with $f(x)=y$. By the hypothesis, $(f^{-1}\mathrm{R}f_*F)_x\simeq F_x$, hence $(\mathrm{R}f_*F)_y\simeq F_x$. Therefore, $(\mathrm{R}f_*f^{-1}G)_y\simeq(f^{-1}G)_x$. The result follows since $(f^{-1}G)_x\simeq G_y$. Q.E.D.

Applying Propositions 5.3 and 5.4, we get:

Theorem 5.5. *Let (M,γ,q) be a G^+-causal manifold. Then the functor $\mathrm{R}\varphi_{\gamma *}\colon\mathrm{D}^{\mathrm{b}}_{\gamma^{\circ a}}(\mathbf{k}_M)\to\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{M_\gamma})$ is an equivalence of categories with quasi-inverse φ_γ^{-1}.*

Example 5.6. Let $M=\mathbb{S}^1$ be the circle and denote by x a coordinate (hence, $x=x+2\pi$) on M, $(x;\xi)$ a coordinates system on T^*M. Consider the cone $\gamma=\{(x;w);w\geq 0\}\subset TN$. Then (M,γ) is a causal manifold, $\gamma^{\circ a}=\{(x;\xi);\xi\leq 0\}$. The only γ-open sets are M and $\varnothing$. Therefore, the only sheaves on M_γ are the constant sheaves. The functor φ_γ^{-1} is not essentially surjective. Indeed, the sheaf $\mathbf{k}_I$ belongs to $\mathrm{D}^{\mathrm{b}}_{\gamma^{\circ a}}(\mathbf{k}_M)$ for I the interval $(a,b]$, $|b-a|<2\pi$. This example does not contradict Theorem 5.5 since (M,γ) is not G^+-causal.

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