Modules over deformation quantization algebroids: an overview

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Abstract

This paper is essentially an overview of a forthcoming paper in which we study coherent modules over deformation quantization algebroids on complex Poisson manifolds.

First, we construct the convolution of coherent kernels over such algebroids, and prove that this convolution preserves coherency and commutes with duality.

Next, we define the Hochschild class of coherent modules and prove that the Hochschild class of the convolution of two coherent kernels is the convolution of their Hochschild classes.

Finally, we study with some details the case of symplectic deformations and apply these results to the Euler class of coherent $\mathscr{D}$-modules.

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Introduction

The notion of a star product is now a classical subject studied by many authors and naturally appearing in various contexts. Two cornerstones of its history are the paper [3] (see also [1, 2]) who defines $\star$-products and the fundamental result of [21] which, roughly speaking, asserts that any real

Poisson manifold may be "quantized", that is, endowed with a star algebra to which the Poisson structure is associated. It is now a well-known fact (see [15, 22]) that, in order to quantize complex Poisson manifolds, sheaves of algebras are not well-suited and have to be replaced by algebroid stacks. We refer to [9, 32] for further developments.

In this paper, we consider complex manifolds endowed with DQ-algebroids, that is, algebroid stacks locally associated to sheaves of star-algebras, and study modules over such algebroids. Our main results are a finiteness theorem, which asserts that the convolution of two coherent kernels is coherent under suitable properness assumptions (a kind of Grauert's theorem), the construction of the dualizing complex and a duality theorem, which asserts that duality commutes with convolution, the construction of the Hochschild class of coherent DQ-modules and the theorem which asserts that Hochschild class commutes with convolution. We also make a link (in the symplectic case) with $\mathscr{D}$-module theory and the Euler classes of $\mathscr{D}$-modules of [29].

Let us describe this paper with some details.

Set $\mathbf{k}_0 := \mathbb{C}[[\hbar]]$, $\mathbf{k} := \mathbb{C}((\hbar)) = \mathbf{k}_0[\hbar^{-1}]$. In [19], we define a DQ-algebra $\mathscr{A}_X$ on a complex manifold X as a sheaf of $\mathbf{k}_0$-algebras locally isomorphic to $(\mathscr{O}_X[[\hbar]], \star)$, where $\star$ is a star-product, and we define a DQ-algebroid as a $\mathbf{k}_0$-algebroid stack locally equivalent to the algebroid associated with a DQ-algebra. (Here, DQ stands for "deformation quantization".)

For a DQ-algebroid $\mathscr{A}_X$, we denote by $\mathscr{A}_{X^a}$ the opposite algebroid $(\mathscr{A}_X)^{\mathrm{op}}$ and we denote by $\mathscr{A}_{X_1\times X_2}$ the external product of the algebroids $\mathscr{A}_{X_i}$ ($i = 1, 2$). An object of $\mathrm{D^b}(\mathscr{A}_{X_1\times X_2^a})$, the bounded derived category of the abelian category of $\mathscr{A}_{X_1\times X_2^a}$-modules, is sometimes called a kernel.

There exist a canonical $\mathscr{A}_{X\times X^a}$-module $\mathscr{C}_X$ on $X \times X^a$ supported by the diagonal, and a dualizing complex $\omega_X^{\mathscr{A}}$ associated to $\mathscr{A}_X$. Consider now three complex manifolds X_i endowed with DQ-algebroids $\mathscr{A}_{X_i}$ ($i = 1, 2, 3$). Let $\mathscr{K}_i \in \mathrm{D^b}(\mathscr{A}_{X_i\times X_{i+1}^a})$ ($i = 1, 2$) be two kernels. Their convolution is defined as

$$\mathscr{K}_1 \circ \mathscr{K}_2 := \mathrm{R}p_{14!}\big((\mathscr{K}_1 \boxtimes \mathscr{K}_2) \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_{X_2\times X_2^a}} \mathscr{C}_{X_2}\big).$$

Here, p_{14} denotes the projection of the product $X_1 \times X_2^a \times X_2 \times X_3^a$ to $X_1 \times X_3^a$. The main results of [19] assert that if $\mathscr{K}_1$ and $\mathscr{K}_2$ are coherent and $\mathrm{Supp}(\mathscr{K}_1) \times_{X_2} \mathrm{Supp}(\mathscr{K}_2)$ is proper over $X_1 \times X_3^a$, then $\mathscr{K}_1 \circ \mathscr{K}_2$ is coherent and the convolution commutes with duality.

In [20], we introduce the Hochschild homology $\mathcal{HH}(\mathscr{A}_X)$ of the algebroid $\mathscr{A}_X$:

$$\mathcal{HH}(\mathscr{A}_X) := \mathscr{C}_{X^a} \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_{X\times X^a}} \mathscr{C}_X, \text{ an object of } \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{0X}),$$

and, using the dualizing complex, we construct a natural convolution morphism

$$\underset{X_2}{\circ} : \mathrm{R}p_{13!}(p_{12}^{-1}\mathcal{HH}(\mathscr{A}_{X_1\times X_2^a}) \overset{\mathrm{L}}{\otimes} p_{23}^{-1}\mathcal{HH}(\mathscr{A}_{X_2\times X_3^a})) \to \mathcal{HH}(\mathscr{A}_{X_1\times X_3^a}).$$

To an object $\mathscr{M}$ of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X)$, we naturally associate its Hochschild class $\mathrm{hh}_X(\mathscr{M})$, an element of $H^0_{\mathrm{Supp}(\mathscr{M})}(X; \mathcal{HH}(\mathscr{A}_X))$. The main result of [20] is Theorem 2.8 which asserts that taking the Hochschild class commutes with the convolution:

$$\mathrm{hh}_{X_1\times X_3^a}(\mathscr{K}_1 \circ \mathscr{K}_2) = \mathrm{hh}_{X_1\times X_2^a}(\mathscr{K}_1) \underset{X_2}{\circ} \mathrm{hh}_{X_2\times X_3^a}(\mathscr{K}_2).$$

When the Poisson structure associated to the deformation is symplectic, we prove that the dualizing complex $\omega_X^{\mathscr{A}}$ is isomorphic to $\mathscr{C}_X$ shifted by d_X, the complex dimension of X, and we construct canonical morphisms

$$\hbar^{d_X/2}\mathbf{k}_{0X}\,[d_X] \to \mathcal{HH}(\mathscr{A}_X) \to \hbar^{-d_X/2}\mathbf{k}_{0X}\,[d_X] \tag{0.1}$$

whose composition coincides with the canonical inclusion.

The first morphism in (0.1) gives an intrinsic construction of a canonical class in $H^{-d_X}(X; \mathcal{HH}(\mathscr{A}_X))$ studied and used by several authors (see [5, 4, 12]).

Setting $\mathscr{A}_X^{\mathrm{loc}} := \mathbf{k} \otimes_{\mathbf{k}_0} \mathscr{A}_X$, there is an isomorphism $\mathcal{HH}(\mathscr{A}_X^{\mathrm{loc}}) \simeq \mathbf{k}_X\,[d_X]$ which allows us to define the Euler class $\mathrm{eu}_X(\mathscr{M}) \in H^{d_X}_{\mathrm{Supp}(\mathscr{M})}(X; \mathbf{k}_X)$ of $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X^{\mathrm{loc}})$. Again, the Euler class of the convolution is the convolution of the Euler classes.

When $X = T^*M \xrightarrow{\pi} M$ is the cotangent bundle to a complex manifold M, there is a canonical DQ-algebra denoted by $\widehat{\mathscr{W}}_X$ and a well-defined morphism of $\mathbb{C}$-algebras $\pi^{-1}\mathscr{D}_M \hookrightarrow \widehat{\mathscr{W}}_X$. Then the Euler class of $\widehat{\mathscr{W}}_X$-modules allows us to recover the Euler class of $\mathscr{D}$-modules and to recover the results of [29] on the functoriality of these Euler classes.

The results of § 1 and § 2 are presented, with complete proofs, in [19] and [20], respectively.

1 Modules over DQ-algebroids

In this section, we shall review the main constructions and results of [19].

Recall that we set $\mathbf{k}_0 := \mathbb{C}[[\hbar]]$ and $\mathbf{k} = \mathbb{C}((\hbar))$, the fraction field of $\mathbf{k}_0$.

Algebroid

Let X be a topological space and $\mathbb{K}$ a commutative unital ring.

Recall that a $\mathbb{K}$-algebroid (introduced in [22]) $\mathscr{A}$ on X is a $\mathbb{K}$-linear stack (see [18] for an exposition on stacks) locally non empty and such that for any open subset U of X, two objects of $\mathscr{A}(U)$ are locally isomorphic.

For a $\mathbb{K}$-algebra A, we denote by A^+ the category with one object and having A as the set of endomorphisms of this object. If $\mathscr{A}$ is a sheaf of $\mathbb{K}$-algebras, we denote by $\mathscr{A}^+$ the stack associated to the prestack $U \mapsto \mathscr{A}(U)^+$ (U open in X). Then $\mathscr{A}^+$ is an algebroid and is called the $\mathbb{K}$-algebroid associated with $\mathscr{A}$. The category $\mathscr{A}^+(X)$ is equivalent to the full subcategory of $\mathrm{Mod}(\mathscr{A}^{\mathrm{op}})$ consisting of objects locally isomorphic to $\mathscr{A}^{\mathrm{op}}$.

Convention 1.1. *If $\mathscr{A}$ is a sheaf of algebras and if there is no risk of confusion, we shall keep the same notation $\mathscr{A}$ to denote the associated algebroid.*

For an algebroid $\mathscr{A}$, one defines the $\mathbb{K}$-abelian category $\mathrm{Mod}(\mathscr{A})$, whose objects are called $\mathscr{A}$-modules, by setting

$$\mathrm{Mod}(\mathscr{A}) := \mathrm{Fct}_{\mathbb{K}}(\mathscr{A}, \mathfrak{Mod}(\mathbb{K}_X)). \tag{1.1}$$

Here $\mathfrak{Mod}(\mathbb{K}_X)$ is the stack of sheaves of $\mathbb{K}$-modules on X, and $\mathrm{Fct}_{\mathbb{K}}$ is the category of $\mathbb{K}$-linear functors of stacks. For a $\mathbb{K}$-algebroid $\mathscr{A}$, $\mathrm{Mod}(\mathscr{A} \otimes_{\mathbb{K}} \mathscr{A}^{\mathrm{op}})$ has a canonical object given by

$$\mathscr{A} \otimes_{\mathbb{K}} \mathscr{A}^{\mathrm{op}} \ni (\sigma, \sigma'^{\mathrm{op}}) \mapsto \mathscr{H}om_{\mathscr{A}}(\sigma', \sigma) \in \mathfrak{Mod}(\mathbb{K}_X).$$

We denote this object by the same letter $\mathscr{A}$.

We denote by $\mathrm{D}^{\mathrm{b}}(\mathscr{A})$ the bounded derived category of the category $\mathrm{Mod}(\mathscr{A})$.

DQ-algebras and DQ-algebroids

From now on, X will be a complex manifold. We denote by d_X its complex dimension.

Set $\mathscr{O}_X[[\hbar]] := \varprojlim_n \mathscr{O}_X \otimes_{\mathbb{C}} (\mathbf{k}_0/\hbar^n\mathbf{k}_0)$. An associative multiplication law $\star$ on $\mathscr{O}_X[[\hbar]]$ is a star-product if it is $\mathbf{k}_0$-bilinear and satisfies

$$f \star g = \sum_{i\geq 0} P_i(f,g)\hbar^i \quad \text{for } f,g \in \mathscr{O}_X, \tag{1.2}$$

where the P_i's are bi-differential operators, $P_0(f,g) = fg$ and $P_i(f,1) = P_i(1,f) = 0$ for $i > 0$.

We call $(\mathscr{O}_X[[\hbar]], \star)$ a *star-algebra*.

Let $\star'$ be another star-product and let $\varphi\colon (\mathscr{O}_X[[\hbar]], \star) \to (\mathscr{O}_X[[\hbar]], \star')$ be a morphism of $\mathbb{C}[[\hbar]]$-algebras. Then there exists a unique sequence of differential operators $\{R_i\}_{i\geq 0}$ such that $R_0 = 1$ and $\varphi(f) = \sum_{i\geq 0} R_i(f)\hbar^i$ for any $f \in \mathscr{O}_X$. In particular, φ is an isomorphism.

Definition 1.2. (a) A DQ-algebra $\mathscr{A}$ on X is a $\mathbf{k}_0$-algebra locally isomorphic, as a $\mathbf{k}_0$-algebra, to a star-algebra.

(b) A DQ-algebroid $\mathscr{A}$ on X is a $\mathbf{k}_0$-algebroid such that for each open set $U \subset X$ and each $\sigma \in \mathscr{A}(U)$, the $\mathbf{k}_0$-algebra $\mathscr{H}om_{\mathscr{A}}(\sigma, \sigma)$ is a DQ-algebra on U.

For a DQ-algebra, there is a $\mathbb{C}$-algebra isomorphism $\mathscr{A}/\hbar\mathscr{A} \xrightarrow{\sim} \mathscr{O}_X$. We denote by $\sigma_0\colon \mathscr{A} \to \mathscr{O}_X$ the $\mathbf{k}_0$-algebra morphism so defined.

Theorem 1.3. *Any* DQ-*algebra $\mathscr{A}$ is right and left Noetherian* (*in particular, coherent*). *Moreover, an $\mathscr{A}$-module $\mathscr{M}$ is coherent if and only if $\hbar^n\mathscr{M}/\hbar^{n+1}\mathscr{M}$ is a coherent $\mathscr{O}_X$-module for any $n \geq 0$ and $\mathscr{M} \to \varprojlim_n(\mathscr{M}/\hbar^n\mathscr{M})$ is an isomorphism.*

If X is endowed with a DQ-algebroid $\mathscr{A}_X$, then we denote by X^a the manifold X endowed with the DQ-algebroid $\mathscr{A}_{X^a} := (\mathscr{A}_X)^{\mathrm{op}}$.

Let X and Y be complex manifolds endowed with DQ-algebroids $\mathscr{A}_X$ and $\mathscr{A}_Y$ respectively. There is a canonical DQ-algebroid $\mathscr{A}_X \underline{\boxtimes} \mathscr{A}_Y$ on $X \times Y$ which contains $\mathscr{A}_X \boxtimes \mathscr{A}_Y$ as a subalgebroid, and we set $\mathscr{A}_{X\times Y} := \mathscr{A}_X \underline{\boxtimes} \mathscr{A}_Y$.

Let $\mathscr{A}_X$ be a DQ-algebroid on X and let $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}(\mathscr{A}_X)$. Its *dual* $\mathrm{D}'_{\mathscr{A}}\mathscr{M} \in \mathrm{D}^{\mathrm{b}}(\mathscr{A}_{X^a})$ is defined by

$$\mathrm{D}'_{\mathscr{A}}\mathscr{M} := \mathrm{R}\mathscr{H}om_{\mathscr{A}_X}(\mathscr{M}, \mathscr{A}_X). \tag{1.3}$$

Here, $\mathscr{A}_X$ is regarded as an $\mathscr{A}_X \otimes \mathscr{A}_{X^a}$-module.

Let $\mathscr{A}_X$ be a DQ-algebroid on X. We denote by $\mathrm{gr}_\hbar(\mathscr{A}_X)$ the $\mathbb{C}$-algebroid locally associated with the sheaf of algebras $\mathscr{A}_X/\hbar\mathscr{A}_X$. Then $\mathrm{Mod}(\mathrm{gr}_\hbar(\mathscr{A}_X))$ is equivalent to the full subcategory of $\mathrm{Mod}(\mathscr{A}_X)$ consisting of objects $\mathscr{M}$ such that $\hbar\colon \mathscr{M} \to \mathscr{M}$ vanishes. The functor $\mathit{for}\colon \mathrm{Mod}(\mathrm{gr}_\hbar(\mathscr{A}_X)) \to \mathrm{Mod}(\mathscr{A}_X)$ admits a left adjoint functor $\mathscr{M} \mapsto \mathscr{M}/\hbar\mathscr{M} \simeq \mathbb{C} \otimes_{\mathbf{k}_0} \mathscr{M}$.

The left derived functor of the functor $\mathscr{M} \mapsto \mathscr{M}/\hbar\mathscr{M}$ is denoted by $\mathrm{gr}_\hbar\colon \mathrm{D}^{\mathrm{b}}(\mathscr{A}_X) \to \mathrm{D}^{\mathrm{b}}(\mathrm{gr}_\hbar(\mathscr{A}_X))$. For $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}(\mathscr{A}_X)$, we call $\mathrm{gr}_\hbar(\mathscr{M})$ the graded module associated to $\mathscr{M}$.

The functor $\mathrm{gr}_\hbar$ induces a functor:

$$\mathrm{gr}_\hbar\colon \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathrm{gr}_\hbar(\mathscr{A}_X)). \tag{1.4}$$

Proposition 1.4. *The functor* $\mathrm{gr}_\hbar$ *in* (1.4) *is conservative.*

Let $\mathscr{K}_i \in \mathrm{D}^{\mathrm{b}}(\mathscr{A}_{X_i \times X_{i+1}^a})$ $(i = 1, 2)$. Then

$$\mathrm{gr}_\hbar(\mathscr{K}_1 \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_{X_2}} \mathscr{K}_2) \simeq \mathrm{gr}_\hbar(\mathscr{K}_1) \overset{\mathrm{L}}{\otimes}_{\mathrm{gr}_\hbar(\mathscr{A}_{X_2})} \mathrm{gr}_\hbar(\mathscr{K}_2). \tag{1.5}$$

Let $\mathscr{K}_i \in \mathrm{D}^{\mathrm{b}}(\mathscr{A}_{X_i \times X_{i+1}})$ $(i = 1, 2)$. Then

$$\mathrm{gr}_\hbar(\mathrm{R}\mathscr{H}om_{\mathscr{A}_{X_2}}(\mathscr{K}_1, \mathscr{K}_2)) \simeq \mathrm{R}\mathscr{H}om_{\mathrm{gr}_\hbar(\mathscr{A}_{X_2})}(\mathrm{gr}_\hbar(\mathscr{K}_1), \mathrm{gr}_\hbar(\mathscr{K}_2)). \tag{1.6}$$

Let Λ be a smooth submanifold of X and let $\mathscr{L}$ be a coherent $\mathscr{A}_X$-module supported by Λ. One says that $\mathscr{L}$ is *simple* along Λ if $\mathrm{gr}_\hbar(\mathscr{L})$ is concentrated in degree 0 and $H^0(\mathrm{gr}_\hbar(\mathscr{L}))$ is an invertible $\mathscr{O}_\Lambda \otimes_{\mathscr{O}_X} \mathrm{gr}_\hbar(\mathscr{A}_X)$-module. (In particular, $\mathscr{L}$ is without $\hbar$-torsion.)

Proposition 1.5. *Let* Λ *be a closed smooth submanifold of* X *of codimension* l *and let* $\mathscr{L}$ *be a coherent* $\mathscr{A}_X$*-module simple along* Λ*. Then* $\mathscr{E}xt^j_{\mathscr{A}_X}(\mathscr{L}, \mathscr{A}_X)$ *vanishes for* $j \neq l$*, and it is a simple* $\mathscr{A}_{X^a}$*-module along* Λ *for* $j = l$*.*

Recall that d_X denotes the complex dimension of X.

Proposition 1.6. *Let* $\mathscr{A}_X$ *be a* DQ*-algebra. Then, any coherent* $\mathscr{A}_X$*-module locally admits a resolution by free modules of finite rank of length* $\leq d_X + 1$*.*

DQ-modules supported by the diagonal

We denote by Δ_X the diagonal of $X \times X^a$, by $\delta_X \colon X \hookrightarrow X \times X^a$ the diagonal embedding, and by $\mathrm{Mod}_{\Delta_X}(\mathscr{A}_X \boxtimes \mathscr{A}_{X^a})$ the category of $(\mathscr{A}_X \boxtimes \mathscr{A}_{X^a})$-modules supported by the diagonal. Then

$$\delta_{X*} \colon \mathrm{Mod}(\mathscr{A}_X \otimes \mathscr{A}_{X^a}) \to \mathrm{Mod}_{\Delta_X}(\mathscr{A}_X \boxtimes \mathscr{A}_{X^a})$$

gives an equivalence of categories and we shall often identify these two categories.

The algebroid $\mathscr{A}_X$ may be regarded as an object of $\mathrm{Mod}(\mathscr{A}_X \otimes \mathscr{A}_{X^a})$, and the $\mathscr{A}_X \boxtimes \mathscr{A}_{X^a}$-module $\delta_{X*}\mathscr{A}_X$ has a natural structure of an $\mathscr{A}_{X\times X^a}$-module, simple along the diagonal. We set

$$\mathscr{C}_X := \delta_{X*}\mathscr{A}_X, \text{ an object of } \mathrm{Mod}(\mathscr{A}_{X\times X^a}). \tag{1.7}$$

A coherent $\mathscr{A}_{X\times X^a}$-module simple along the diagonal is called a *bi-invertible* $\mathscr{A}_{X\times X^a}$-module. Then, the category of bi-invertible $\mathscr{A}_{X\times X^a}$-modules is a tensor category and $\mathscr{C}_X$ is a unit object. More generally, we say that $\mathscr{P} \in \mathrm{D}^{\mathrm{b}}(\mathscr{A}_{X\times X^a})$ is bi-invertible if it is concentrated in a single degree, say n, and $H^n(\mathscr{P})$ is bi-invertible. If $\mathscr{P}$ is bi-invertible, we set

$$\mathscr{P}^{\otimes -1} := \mathrm{R}\mathscr{H}om_{\mathscr{A}_X}(\mathscr{P}, \mathscr{A}_X). \tag{1.8}$$

Then we have

$$\mathscr{P}^{\otimes -1} \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_X} \mathscr{P} \simeq \mathscr{P} \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_X} \mathscr{P}^{\otimes -1} \simeq \mathscr{C}_X.$$

$\hbar$-localization

To a DQ-algebroid $\mathscr{A}_X$ we associate its *$\hbar$-localization*, the $\mathbf{k}$-algebroid

$$\mathscr{A}_X^{\mathrm{loc}} = \mathbf{k} \otimes_{\mathbf{k}_0} \mathscr{A}_X. \tag{1.9}$$

There exists a pair of adjoint exact functors $(\mathbf{k} \otimes_{\mathbf{k}_0} \bullet,\ \mathit{for})$:

$$\mathrm{Mod}(\mathscr{A}_X^{\mathrm{loc}}) \underset{\mathbf{k}\otimes_{\mathbf{k}_0}\bullet}{\overset{\mathit{for}}{\rightleftarrows}} \mathrm{Mod}(\mathscr{A}_X). \tag{1.10}$$

The algebroid $\mathscr{A}_X^{\mathrm{loc}}$ is Noetherian.

If $\mathscr{M}_0$ is an $\mathscr{A}_X$-submodule of an $\mathscr{A}_X^{\rm loc}$-module $\mathscr{M}$ and $\mathscr{M}_0 \otimes_{\mathbf{k}_0} \mathbf{k} \xrightarrow{\sim} \mathscr{M}$, then we shall say that $\mathscr{M}_0$ generates $\mathscr{M}$.

A coherent $\mathscr{A}_X^{\rm loc}$-module $\mathscr{M}$ is *good* if, for any relatively compact open subset U of X, there exists a coherent $(\mathscr{A}_X|_U)$-module which generates $\mathscr{M}|_U$.

One denotes by $\mathrm{Mod}_{\rm gd}(\mathscr{A}_X^{\rm loc})$ the full subcategory of $\mathrm{Mod}_{\rm coh}(\mathscr{A}_X^{\rm loc})$ consisting of good modules. As in [16, Prop. 4.23], one proves that $\mathrm{Mod}_{\rm gd}(\mathscr{A}_X^{\rm loc})$ is a thick subcategory of $\mathrm{Mod}_{\rm coh}(\mathscr{A}_X^{\rm loc})$.

We denote by $\mathrm{D}^{\rm b}_{\rm coh}(\mathscr{A}_X^{\rm loc})$ (resp. $\mathrm{D}^{\rm b}_{\rm gd}(\mathscr{A}_X^{\rm loc})$) the full triangulated subcategory of $\mathrm{D}^{\rm b}(\mathscr{A}_X^{\rm loc})$ consisting of objects $\mathscr{M}$ such that $H^j(\mathscr{M})$ is coherent (resp. good) for all $j \in \mathbb{Z}$. The notion of good $\mathscr{A}_X^{\rm loc}$-module is similar to that of good $\mathscr{D}$-module of loc. cit.

Deformation quantization of $\mathscr{D}_X$ and dualizing complex

The ring $\mathscr{D}_X[[\hbar]]$ of differential operators on $\mathscr{O}_X[[\hbar]]$ is naturally regarded as a subsheaf of $\mathscr{E}nd_{\mathbf{k}_0}(\mathscr{O}_X[[\hbar]])$. There is a canonical equivalence of $\mathbf{k}_0$-algebroids $\mathscr{E}nd_{\mathbf{k}_0}(\mathscr{O}_X[[\hbar]]) \simeq \mathscr{E}nd_{\mathbf{k}_0}(\mathscr{A}_X)$. (In particular, the algebroid $\mathscr{E}nd_{\mathbf{k}_0}(\mathscr{A}_X)$ is associated to a sheaf of algebras.) We denote by $\mathscr{D}_X^{\mathscr{A}}$ the substack of $\mathscr{E}nd_{\mathbf{k}_0}(\mathscr{A}_X)$, the image of $\mathscr{D}_X[[\hbar]]$ by this equivalence. This is a $\mathbf{k}_0$-algebroid that we call the algebroid of differential operators (associated with $\mathscr{A}_X$). Then $\mathscr{A}_X$ may be regarded as an object of $\mathrm{Mod}(\mathscr{D}_X^{\mathscr{A}})$. We have morphisms of algebroids: $\mathscr{A}_X \otimes \mathscr{A}_{X^a} \to \delta_X^{-1}\mathscr{A}_{X\times X^a} \to \mathscr{D}_X^{\mathscr{A}}$. The object $\mathrm{R}\mathscr{H}om_{\mathscr{D}_X^{\mathscr{A}}}(\mathscr{A}_X, \mathscr{D}_X^{\mathscr{A}})$ of $\mathrm{D}^{\rm b}((\mathscr{D}_X^{\mathscr{A}})^{\rm op})$ is concentrated in degree d_X. Through $\mathscr{A}_{X\times X^a} \to (\mathscr{D}_X^{\mathscr{A}})^{\rm op}$, we set:

$$
(1.11)\qquad \begin{aligned} \Omega_X^{\mathscr{A}} &= \mathscr{E}xt^{d_X}_{\mathscr{D}_X^{\mathscr{A}}}(\mathscr{A}_X, \mathscr{D}_X^{\mathscr{A}}) \in \mathrm{Mod}(\mathscr{A}_{X\times X^a}), \\ \omega_X^{\mathscr{A}} &:= \Omega_X^{\mathscr{A}}[d_X] \in \mathrm{D}^{\rm b}(\mathscr{A}_{X\times X^a}). \end{aligned}
$$

We call $\omega_X^{\mathscr{A}}$ the $\mathscr{A}_X$-*dualizing sheaf*. It is a bi-invertible $\mathscr{A}_{X\times X^a}$-module. Note that one has the morphisms:

$$
(1.12)\qquad \begin{aligned} \Omega_X^{\mathscr{A}} \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_{X\times X^a}} \mathscr{C}_X[-d_X] &\to \Omega_X^{\mathscr{A}} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X^{\mathscr{A}}} \mathscr{A}_X[-d_X] \\ &\simeq \mathrm{R}\mathscr{H}om_{\mathscr{D}_X^{\mathscr{A}}}(\mathscr{A}_X, \mathscr{A}_X) \simeq \mathbf{k}_{0X}. \end{aligned}
$$

In [19, Theorem 8.5], one proves the isomorphism

$$
(1.13)\qquad \omega_X^{\mathscr{A}} \simeq (\mathrm{D}'_{\mathscr{A}}(\mathscr{C}_{X^a}))^{\otimes -1} \quad \text{in } \mathrm{D}^{\rm b}(\mathscr{A}_{X\times X^a}).
$$

Note that in this formula, $\mathrm{D}'_{\mathscr{A}}$ is the dual over $\mathscr{A}_{X\times X^a}$ and $(\bullet)^{\otimes -1}$ is given in (1.8).

Remark 1.7. The fact that $\mathrm{D}'_{\mathscr{A}}\mathscr{C}_X$ is concentrated in a single degree and plays the role of a dualizing complex in the sense of [31] was already proved by [10].

Let Y be another manifold endowed with a DQ-algebroid $\mathscr{A}_Y$. We introduce the notation:

$$\omega^{\mathscr{A}}_{X\times Y/Y} = \omega^{\mathscr{A}}_X \overset{\mathrm{L}}{\underline{\boxtimes}} \mathscr{C}_Y \in \mathrm{D}^{\mathrm{b}}(\mathscr{A}_{X\times X^a\times Y\times Y^a}).$$

Then $\omega^{\mathscr{A}}_{X\times Y/Y}$ also belongs to $\mathrm{D}^{\mathrm{b}}({\mathscr{D}^{\mathscr{A}}_X}^{\mathrm{op}}\boxtimes\mathscr{A}_{Y\times Y^a})$, and we have $\omega^{\mathscr{A}}_{X\times Y/Y}\overset{\mathrm{L}}{\otimes}_{\mathscr{D}^{\mathscr{A}}_X}\mathscr{A}_X \simeq \mathbf{k}_{0X}\boxtimes\mathscr{A}_Y$. Hence we have a canonical morphism

$$\omega^{\mathscr{A}}_{X\times Y/Y}\overset{\mathrm{L}}{\otimes}_{\mathscr{A}_{X\times X^a}}\mathscr{C}_X \to (\mathbf{k}_{0X}\boxtimes\mathscr{C}_Y)[2d_X] \tag{1.14}$$

in $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{0X}\boxtimes\mathscr{A}_{Y\times Y^a})$.

Convolution of kernels

For two complex manifolds X_i ($i=1,2$) endowed with DQ-algebroids $\mathscr{A}_{X_i}$ and for $\mathscr{M}_i\in\mathrm{D}^{\mathrm{b}}(\mathscr{A}_{X_i})$, we defined their external product

$$\mathscr{M}_1\overset{\mathrm{L}}{\underline{\boxtimes}}\mathscr{M}_2 := \mathscr{A}_{X_1\times X_2}\otimes_{(\mathscr{A}_{X_1}\boxtimes\mathscr{A}_{X_2})}(\mathscr{M}_1\overset{\mathrm{L}}{\boxtimes}\mathscr{M}_2).$$

Consider now three complex manifolds X_i ($i=1,2,3$) endowed with DQ-algebroids $\mathscr{A}_{X_i}$. We denote by p_i the i-th projection and by p_{ij} the (i,j)-th projection. For $\Lambda_i\subset X_i\times X_{i+1}$ ($i=1,2$), we set

$$\Lambda_1\circ\Lambda_2 = p_{13}(p_{12}^{-1}\Lambda_1\cap p_{23}^{-1}\Lambda_2). \tag{1.15}$$

We shall write for short $\mathscr{A}_i$ instead of $\mathscr{A}_{X_i}$, $\mathscr{A}_{ij^a}$ instead of $\mathscr{A}_{X_{ij^a}}$, etc. (See also Notations 2.5 below.)

Definition 1.8. Let $\mathscr{K}_i \in \mathrm{D^b}(\mathscr{A}_{X_i\times X_{i+1}^a})$ $(i=1,2)$. We set

$$
\begin{aligned}
(1.16)\quad \mathscr{K}_1 \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_2} \mathscr{K}_2 &:= p_{12}^{-1}\mathscr{K}_1 \overset{\mathrm{L}}{\otimes}_{p_2^{-1}\mathscr{A}_2} p_{23}^{-1}\mathscr{K}_2 \\
&\simeq (\mathscr{K}_1 \overset{\mathrm{L}}{\boxtimes} \mathscr{K}_2)\overset{\mathrm{L}}{\otimes}_{\mathscr{A}_2\boxtimes\mathscr{A}_{2^a}} \mathscr{C}_2 \in \mathrm{D^b}(p_{13}^{-1}(\mathscr{A}_1 \boxtimes \mathscr{A}_{3^a})),\\
(1.17)\quad \mathscr{K}_1 \underline{\overset{\mathrm{L}}{\otimes}}_{\mathscr{A}_2} \mathscr{K}_2 &:= (\mathscr{K}_1 \underline{\overset{\mathrm{L}}{\boxtimes}} \mathscr{K}_2)\overset{\mathrm{L}}{\otimes}_{\mathscr{A}_{22^a}} \mathscr{C}_2 \\
&\simeq p_{12}^{-1}\mathscr{K}_1 \overset{\mathrm{L}}{\otimes}_{p_{12}^{-1}\mathscr{A}_{1^a2}} \mathscr{A}_{123} \overset{\mathrm{L}}{\otimes}_{p_{23}\mathscr{A}_{23^a}} p_{23}^{-1}\mathscr{K}_2 \in \mathrm{D^b}(p_{13}^{-1}\mathscr{A}_{13^a}),\\
(1.18)\quad \mathscr{K}_1 \underset{X_2}{\circ} \mathscr{K}_2 &:= \mathrm{R}p_{13!}(\mathscr{K}_1 \underline{\overset{\mathrm{L}}{\otimes}}_{\mathscr{A}_2} \mathscr{K}_2) \in \mathrm{D^b}(\mathscr{A}_{X_1\times X_3^a}),\\
(1.19)\quad \mathscr{K}_1 \underset{X_2}{*} \mathscr{K}_2 &:= \mathrm{R}p_{13*}(\mathscr{K}_1 \underline{\overset{\mathrm{L}}{\otimes}}_{\mathscr{A}_2} \mathscr{K}_2) \in \mathrm{D^b}(\mathscr{A}_{X_1\times X_3^a}).
\end{aligned}
$$

If there is no risk of confusion we write $\mathscr{K}_1 \circ \mathscr{K}_2$ for $\mathscr{K}_1 \underset{X_2}{\circ} \mathscr{K}_2$ and similarly with $*$. We call $\mathscr{K}_1 \circ \mathscr{K}_2$ and $\mathscr{K}_1 * \mathscr{K}_2$ the convolution product of the kernels $\mathscr{K}_1$ and $\mathscr{K}_2$.

When $X_1 = \mathrm{pt}$ or $X_3 = \mathrm{pt}$ we get $\mathscr{K}_1 \otimes_{\mathscr{A}_2} \mathscr{K}_2 \xrightarrow{\sim} \mathscr{K}_1 \underline{\overset{\mathrm{L}}{\otimes}}_{\mathscr{A}_2} \mathscr{K}_2$.
There are canonical isomorphisms

$$
(1.20)\qquad \mathscr{K}_1 \underset{X_2}{\circ} \mathscr{C}_{X_2} \simeq \mathscr{K}_1 \quad\text{and}\quad \mathscr{C}_{X_1} \underset{X_1}{\circ} \mathscr{K}_1 \simeq \mathscr{K}_1.
$$

One shall be aware that $\circ$ and $*$ are not associative in general. However, if $\mathscr{L}$ is an invertible $\mathscr{A}_{X_2}$-module, $\mathscr{E}$ is a bi-invertible $\mathscr{A}_{X_2\times X_2^a}$-module and the $\mathscr{K}_i$ are as above $(i=1,2)$, then there are natural isomorphisms

$$
\begin{aligned}
&\mathscr{K}_1 \underset{X_2}{\circ} \mathscr{L} \simeq \mathscr{K}_1 \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_{X_2}} \mathscr{L}, \quad \mathscr{L} \underset{X_2}{\circ} \mathscr{K}_2 \simeq \mathscr{L} \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_{X_2}} \mathscr{K}_2,\\
&(\mathscr{K}_1 \underset{X_2}{\circ} \mathscr{E}) \underset{X_2}{\circ} \mathscr{K}_2 \simeq \mathscr{K}_1 \underset{X_2}{\circ} (\mathscr{E} \underset{X_2}{\circ} \mathscr{K}_2).
\end{aligned}
$$

Note that the functor $\mathrm{gr}_\hbar$ in (1.4) commutes with the convolution of kernels.

Theorem 1.9. *Let $\mathscr{K}_i \in \mathrm{D^b_{coh}}(\mathscr{A}_{X_i\times X_{i+1}^a})$ $(i=1,2)$. Assume that the projection p_{13} defined on $X_1 \times X_2 \times X_3$ is proper on $p_{12}^{-1}\operatorname{Supp}(\mathscr{K}_1) \cap p_{23}^{-1}\operatorname{Supp}(\mathscr{K}_2)$. Then*

(a) *the object* $\mathscr{K}_1 \circ \mathscr{K}_2$ *belongs to* $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_{X_1\times X_3^a})$ *and* $\mathrm{Supp}(\mathscr{K}_1 \circ \mathscr{K}_2) \subset \mathrm{Supp}(\mathscr{K}_1) \circ \mathrm{Supp}(\mathscr{K}_2)$*,*

(b) *we have a natural isomorphism*

$$\mathrm{D}'_{\mathscr{A}}(\mathscr{K}_1) \underset{X_2^a}{\circ} \omega^{\mathscr{A}}_{X_2^a} \underset{X_2^a}{\circ} \mathrm{D}'_{\mathscr{A}}(\mathscr{K}_2) \xrightarrow{\sim} \mathrm{D}'_{\mathscr{A}}(\mathscr{K}_1 \underset{X_2}{\circ} \mathscr{K}_2) \tag{1.21}$$

in $\mathrm{D}^{\mathrm{b}}(\mathscr{A}_{X_1^a\times X_3})$*.*

Corollary 1.10. *Let* $\mathscr{M}$ *and* $\mathscr{N}$ *be two objects of* $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X)$ *and assume that* $\mathrm{Supp}(\mathscr{M}) \cap \mathrm{Supp}(\mathscr{N})$ *is compact. Then* $\mathrm{RHom}_{\mathscr{A}_X}(\mathscr{M},\mathscr{N}) \in \mathrm{D}^{\mathrm{b}}_f(\mathbf{k}_0)$*, and there is a natural isomorphism in* $\mathrm{D}^{\mathrm{b}}_f(\mathbf{k}_0)$*:*

$$\mathrm{RHom}_{\mathscr{A}_X}(\mathscr{N}, \omega^{\mathscr{A}}_X \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_X} \mathscr{M}) \simeq (\mathrm{RHom}_{\mathscr{A}_X}(\mathscr{M},\mathscr{N}))^{\star},$$

where * *is the duality functor in* $\mathrm{D}^{\mathrm{b}}_f(\mathbf{k}_0)$*.*

In particular, if X is compact, then $\mathscr{M} \mapsto \omega^{\mathscr{A}}_X \otimes_{\mathscr{A}_X} \mathscr{M}$ is a Serre functor of the triangulated category $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X)$.

2 Hochschild class

In this section, we shall review with some details the main constructions and results of [20].

Construction of the Hochschild class

Let X be a complex manifold and let $\mathscr{A}_X$ be a DQ-algebroid. Recall that $\delta\colon X \to X \times X^a$ is the diagonal embedding. We define the Hochschild homology $\mathcal{HH}(\mathscr{A}_X)$ of $\mathscr{A}_X$ by:

$$\mathcal{HH}(\mathscr{A}_X) := \delta^{-1}(\mathscr{C}_{X^a} \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_{X\times X^a}} \mathscr{C}_X),\ \text{an object of } \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{0X}). \tag{2.1}$$

Note that, using (1.13), we get the isomorphisms:

$$\begin{aligned}\mathcal{HH}(\mathscr{A}_X) &\simeq \delta^{-1}\mathrm{R}\mathscr{H}om_{\mathscr{A}_{X\times X^a}}(\mathrm{D}'_{\mathscr{A}}(\mathscr{C}_{X^a}), \mathscr{C}_X)\\ &\simeq \delta^{-1}\mathrm{R}\mathscr{H}om_{\mathscr{A}_{X\times X^a}}(\omega_X^{\mathscr{A}\otimes -1}, \mathscr{C}_X).\end{aligned}$$

We have also the isomorphisms

$$\begin{aligned}
\mathrm{R}\mathscr{H}om_{\mathscr{A}_{X\times X^a}}(\omega_X^{\mathscr{A}\otimes -1},\mathscr{C}_X) &\simeq \mathrm{R}\mathscr{H}om_{\mathscr{A}_{X\times X^a}}(\omega_X^{\mathscr{A}} \underset{X}{\circ} \omega_X^{\mathscr{A}\otimes -1}, \omega_X^{\mathscr{A}} \underset{X}{\circ} \mathscr{C}_X)\\
&\simeq \mathrm{R}\mathscr{H}om_{\mathscr{A}_{X\times X^a}}(\mathscr{C}_X,\omega_X^{\mathscr{A}}).
\end{aligned}$$

One shall be aware that there are two different isomorphisms

$$\mathrm{R}\mathscr{H}om_{\mathscr{A}_{X\times X^a}}(\omega_X^{\mathscr{A}\otimes -1},\mathscr{C}_X) \simeq \mathrm{R}\mathscr{H}om_{\mathscr{A}_{X\times X^a}}(\mathscr{C}_X,\omega_X^{\mathscr{A}})$$

according as one applies the functor $\bullet \underset{X}{\circ} \omega_X^{\mathscr{A}}$ or the functor $\omega_X^{\mathscr{A}} \underset{X}{\circ} \bullet$.

Let $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X)$. We have the chain of morphisms

$$\begin{aligned}
\mathrm{R}\mathscr{H}om_{\mathscr{A}_X}(\mathscr{M},\mathscr{M}) &\xleftarrow{\sim} \mathrm{D}'_{\mathscr{A}}\mathscr{M} \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_X} \mathscr{M}\\
&\simeq \mathscr{C}_{X^a} \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_{X\times X^a}} (\mathscr{M} \underline{\boxtimes} \mathrm{D}'_{\mathscr{A}}\mathscr{M})\\
&\to \mathscr{C}_{X^a} \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_{X\times X^a}} \mathscr{C}_X = \mathcal{HH}(\mathscr{A}_X).
\end{aligned} \tag{2.2}$$

We get a map

$$\mathrm{Hom}_{\mathscr{A}_X}(\mathscr{M},\mathscr{M}) \to H^0_{\mathrm{Supp}(\mathscr{M})}(X;\mathcal{HH}(\mathscr{A}_X)).$$

For $u \in \mathrm{End}(\mathscr{M})$, the image of u gives an element

$$\mathrm{hh}_X((\mathscr{M},u)) \in H^0_{\mathrm{Supp}(\mathscr{M})}(X;\mathcal{HH}(\mathscr{A}_X)). \tag{2.3}$$

Definition 2.1. Let $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X)$. We set $\mathrm{hh}_X(\mathscr{M}) = \mathrm{hh}_X((\mathscr{M},\mathrm{id}_{\mathscr{M}}))$ and call it the Hochschild class of $\mathscr{M}$.

Let $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X)$. There are natural morphisms in $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_{X\times X^a})$:

$$\omega_X^{\mathscr{A}\otimes -1} \longrightarrow \mathscr{M} \overset{\mathrm{L}}{\underline{\boxtimes}} \mathrm{D}'_{\mathscr{A}}\mathscr{M} \longrightarrow \mathscr{C}_X. \tag{2.4}$$

Lemma 2.2. *The composition of the two morphisms in* (2.4) *coincides with the Hochschild class* $\mathrm{hh}_X(\mathscr{M})$ *when identifying* $\mathrm{R}\mathscr{H}om_{\mathscr{A}_{X\times X^a}}(\omega_X^{\mathscr{A}\otimes -1},\mathscr{C}_X)$ *with* $\mathcal{HH}(\mathscr{A}_X)$.

Remark 2.3. For the additivity of the Hochschild class with respect to distinguished triangles, that is, for the equality $\mathrm{hh}_X(\mathscr{M}) = \mathrm{hh}_X(\mathscr{M}') + \mathrm{hh}_X(\mathscr{M}'')$ when $\mathscr{M}' \to \mathscr{M} \to \mathscr{M}'' \xrightarrow{+1}$ is a distinguished triangle in $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X)$, we refer to May [25].

Composition of Hochschild classes

Let X_i be complex manifolds endowed with DQ-algebroids $\mathscr{A}_{X_i}$ ($i = 1, 2, 3$) and we denote as usual by p_{ij} the projection from $X_1 \times X_2 \times X_3$ to $X_i \times X_j$ ($1 \le i < j \le 3$).

Proposition 2.4. *There is a natural morphism*

$$\circ\colon \mathrm{R}p_{13!}(p_{12}^{-1}\mathcal{HH}(\mathscr{A}_{X_1\times X_2^a})\overset{\mathrm{L}}{\otimes} p_{23}^{-1}\mathcal{HH}(\mathscr{A}_{X_2\times X_3^a})) \to \mathcal{HH}(\mathscr{A}_{X_1\times X_3^a}).$$

Sketch of proof. (i) Set $Z_i = X_i \times X_i^a$. We shall denote by the same letter p_{ij} the projection from $Z_1 \times Z_2 \times Z_3$ to $Z_i \times Z_j$.

We have

$$\mathcal{HH}(\mathscr{A}_{X_i\times X_j^a}) \simeq \mathrm{R}\mathscr{H}om_{\mathscr{A}_{Z_i\times Z_j^a}}(\omega_{X_i}^{\mathscr{A}\otimes -1}\underline{\overset{\mathrm{L}}{\boxtimes}}\mathscr{C}_{X_j^a}, \mathscr{C}_{X_i}\underline{\overset{\mathrm{L}}{\boxtimes}}\omega_{X_j{}^a}^{\mathscr{A}}).$$

Set $S_{ij} := \omega_{X_i}^{\mathscr{A}\otimes -1}\underline{\overset{\mathrm{L}}{\boxtimes}}\mathscr{C}_{X_j^a} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_{Z_i\times Z_j^a})$ and $K_{ij} := \mathscr{C}_{X_i}\underline{\overset{\mathrm{L}}{\boxtimes}}\omega_{X_j^a}^{\mathscr{A}} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_{Z_i\times Z_j^a})$. Then we get

$$\mathcal{HH}(\mathscr{A}_{X_i\times X_j^a}) \simeq \mathrm{R}\mathscr{H}om_{\mathscr{A}_{Z_i\times Z_j^a}}(S_{ij}, K_{ij}).$$

Thus we obtain

$$\begin{aligned}
&p_{12}^{-1}\mathcal{HH}(\mathscr{A}_{X_1\times X_2^a})\overset{\mathrm{L}}{\otimes} p_{23}^{-1}\mathcal{HH}(\mathscr{A}_{X_2\times X_3^a})\\
&\qquad\simeq p_{12}^{-1}\mathrm{R}\mathscr{H}om_{\mathscr{A}_{Z_1\times Z_2^a}}(S_{12}, K_{12})\overset{\mathrm{L}}{\otimes} p_{23}^{-1}\mathrm{R}\mathscr{H}om_{\mathscr{A}_{Z_2\times Z_3^a}}(S_{23}, K_{23})\\
&\qquad\to \mathrm{R}\mathscr{H}om_{p_{13}^{-1}\mathscr{A}_{Z_1\times Z_3^a}}\big(S_{12}\underline{\overset{\mathrm{L}}{\otimes}}_{\mathscr{A}_{Z_2}}S_{23}, K_{12}\underline{\overset{\mathrm{L}}{\otimes}}_{\mathscr{A}_{Z_2}}K_{23}\big).
\end{aligned}$$

Hence, we get a chain of morphisms

$$\begin{aligned}
&\mathrm{R}p_{13!}(p_{12}^{-1}\mathcal{HH}(\mathscr{A}_{X_1\times X_2^a})\overset{\mathrm{L}}{\otimes} p_{23}^{-1}\mathcal{HH}(\mathscr{A}_{X_2\times X_3^a}))\\
&\quad\to \mathrm{R}p_{13!}\mathrm{R}\mathscr{H}om_{p_{13}^{-1}\mathscr{A}_{Z_1\times Z_3^a}}\big(S_{12}\underline{\overset{\mathrm{L}}{\otimes}}_{\mathscr{A}_{Z_2}}S_{23}, K_{12}\underline{\overset{\mathrm{L}}{\otimes}}_{\mathscr{A}_{Z_2}}K_{23}\big)\\
&\quad\to \mathrm{R}\mathscr{H}om_{\mathscr{A}_{Z_1\times Z_3^a}}\big(\mathrm{R}p_{13*}(S_{12}\underline{\overset{\mathrm{L}}{\otimes}}_{\mathscr{A}_{Z_2}}S_{23}), \mathrm{R}p_{13!}(K_{12}\underline{\overset{\mathrm{L}}{\otimes}}_{\mathscr{A}_{Z_2}}K_{23})\big).
\end{aligned} \tag{2.5}$$

(ii) To conclude, we construct the morphisms

$$S_{13} \to \mathrm{R}p_{13*}(S_{12}\underline{\overset{\mathrm{L}}{\otimes}}_{\mathscr{A}_{Z_2}}S_{23}), \tag{2.6}$$

$$\mathrm{R}p_{13!}(K_{12}\underline{\overset{\mathrm{L}}{\otimes}}_{\mathscr{A}_{Z_2}}K_{23}) \to K_{13}. \tag{2.7}$$

Q.E.D.

Main theorem

Consider four manifolds X_i endowed with DQ-algebroids $\mathscr{A}_{X_i}$ $(i=1,2,3,4)$.

Notation 2.5. In the sequel and until the end of this section, when there is no risk of confusion, we use the following conventions.

(i) For $i,j\in\{1,2,3,4\}$, we set $X_{ij}:=X_i\times X_j$, $X_{ij^a}:=X_i\times X_j^a$ and similarly with X_{ijk}, etc.

(ii) We sometimes omit the symbols $p_{ij}, p_{ij_*}, p_{ij}^{-1}$, etc.

(iii) We write $\mathscr{A}_i$ instead of $\mathscr{A}_{X_i}$, $\mathscr{A}_{ij^a}$ instead of $\mathscr{A}_{X_{ij^a}}$ and similarly with $\mathscr{C}_i, \omega_i^{\mathscr{A}}$, etc. Moreover, we even sometimes write $\mathscr{H}om_i$ instead of $\mathscr{H}om_{\mathscr{A}_i}$ and $\otimes_i$ instead of $\otimes_{\mathscr{A}_i}$ and similarly with ij^a, ijk, etc.

(iv) We write D' instead of $\mathrm{D}'_{\mathscr{A}}$ and ω_X instead of $\omega_X^{\mathscr{A}}$.

(v) We often identify an invertible object of $\mathrm{D^b}(\mathscr{A}_X\otimes\mathscr{A}_{X^a})$ with an object of $\mathrm{D^b}(\mathscr{A}_{X\times X^a})$ supported by the diagonal.

(vi) We identify $(X_i\times X_j^a)^a$ with $X_j\times X_i^a$.

For a closed subset Λ of X, we set

$$\mathrm{HH}_\Lambda(\mathscr{A}_X):=H^0\mathrm{R}\Gamma_\Lambda(X;\mathcal{HH}(\mathscr{A}_X)). \tag{2.8}$$

Let $\Lambda_{ij}\subset X_{ij}$ $(i=1,2,\ j=i+1)$ be a closed subset and assume that $p_{12}^{-1}\Lambda_{12}\cap p_{23}^{-1}\Lambda_{23}$ is proper over $X_1\times X_3$. Using Proposition 2.4, we get a map

$$\underset{2}{\circ}:\mathrm{HH}_{\Lambda_{12}}(\mathscr{A}_{X_{12^a}})\times\mathrm{HH}_{\Lambda_{23}}(\mathscr{A}_{X_{23^a}})\longrightarrow\mathrm{HH}_{\Lambda_{12}\circ\Lambda_{23}}(\mathscr{A}_{X_{13^a}}). \tag{2.9}$$

For $C_{ij}\in\mathrm{HH}_{\Lambda_{ij}}(\mathscr{A}_{X_{ij^a}})$ $(i=1,2,\ j=i+1)$, we obtain a class

$$C_{12}\underset{2}{\circ}C_{23}\in\mathrm{HH}_{\Lambda_{12}\circ\Lambda_{23}}(\mathscr{A}_{X_{13^a}}). \tag{2.10}$$

The morphism $(\mathscr{C}_{1^a}\overset{\mathrm{L}}{\underline{\otimes}}_{11^a}\mathscr{C}_1)\overset{\mathrm{L}}{\boxtimes}(\mathscr{C}_{2^a}\overset{\mathrm{L}}{\underline{\otimes}}_{22^a}\mathscr{C}_2)\to(\mathscr{C}_{2^a1^a}\overset{\mathrm{L}}{\underline{\otimes}}_{121^a2^a}\mathscr{C}_{12})$ induces the exterior product

$$\boxtimes:\mathrm{HH}_{\Lambda_1}(\mathscr{A}_{X_1})\times\mathrm{HH}_{\Lambda_2}(\mathscr{A}_{X_2})\to\mathrm{HH}_{\Lambda_1\times\Lambda_2}(\mathscr{A}_{X_1\times X_2}) \tag{2.11}$$

for $\Lambda_i\subset X_i$ $(i=1,2)$.

Now let $\Lambda_{ij}\subset X_{ij}$ $(i=1,2,3,\ j=i+1)$ and assume that $p_{ij}^{-1}\Lambda_{ij}\cap p_{jk}^{-1}\Lambda_{jk}$ is proper over X_{ik} $(i=1,2,\ j=i+1,\ k=j+1)$.

Lemma 2.6. *Let* $C_{ij} \in \mathrm{HH}_{\Lambda_{ij}}(\mathscr{A}_{X_{ij^a}})$ $(i = 1,2,3,\ j = i+1)$.

(a) *One has* $(C_{12} \underset{2}{\circ} C_{23}) \underset{3}{\circ} C_{34} = C_{12} \underset{2}{\circ} (C_{23} \underset{3}{\circ} C_{34})$.

(b) *for* $C_{245} \in \mathrm{HH}(\mathscr{A}_{X_{245^a}})$ *we have*

$$(C_{12} \boxtimes C_{34}) \underset{24}{\circ} C_{245} = C_{12} \underset{2}{\circ} (C_{34} \underset{4}{\circ} C_{245}).$$

(c) *Set* $C_{\Delta_i} = \mathrm{hh}_{X_{ii^a}}(\mathscr{C}_{X_i})$. *Then* $C_{12} \underset{2}{\circ} C_{\Delta_2} = C_{\Delta_1} \underset{1}{\circ} C_{12} = C_{12}$.

(d) $(C_{12} \boxtimes C_{\Delta_3}) \underset{23^a}{\circ} C_{23} = C_{12} \underset{2}{\circ} C_{23}$. *Here* $C_{12} \boxtimes C_{\Delta_3} \in \mathrm{HH}_{\Lambda_{12} \times \Delta_3}(\mathscr{A}_{X_{12^a 33^a}})$ *is regarded as an element of* $\mathrm{HH}_{\Lambda_{12} \times \Delta_3}(\mathscr{A}_{X_{(13^a)(23^a)^a}})$.

Let $\mathscr{K} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_{X_{12^a}})$. Using (2.4) one constructs natural morphisms in $\mathrm{D}^{\mathrm{b}}(\mathscr{A}_{X_{11^a}})$:

$$\omega_1^{\otimes -1} \to \mathscr{K} \underset{2}{*} \mathrm{D}'\mathscr{K}, \tag{2.12}$$

$$\mathscr{K} \underset{2}{\circ} \omega_2 \underset{2}{\circ} \mathrm{D}'\mathscr{K} \to \mathscr{C}_1. \tag{2.13}$$

For the sake of brevity, we shall write $\Gamma_\Lambda \mathrm{Hom}$ instead of $H^0(\mathrm{R}\Gamma_\Lambda \mathrm{R}\mathscr{H}om)$.

Let Λ_{12} be a closed subset of $X_1 \times X_2^a$ and Λ_2 a closed subset of X_2. Let $\mathscr{K} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_{X_{12^a}})$ with support Λ_{12}. We assume

(2.14) $\quad \Lambda_{12} \times_{X_2} \Lambda_2$ is proper over X_1.

We define the map

$$\Phi_{\mathscr{K}} \colon \mathrm{HH}_{\Lambda_2}(\mathscr{A}_{X_2}) \longrightarrow \mathrm{HH}_{\Lambda_{12} \circ \Lambda_2}(\mathscr{A}_{X_1}) \tag{2.15}$$

as the composition of the sequence of maps

$$\begin{aligned}
&\mathrm{HH}_{\Lambda_2}(\mathscr{A}_2) \simeq \Gamma_{\Lambda_2}\mathrm{Hom}_{22^a}(\omega_2^{\otimes -1}, \mathscr{C}_2) \\
&\to \Gamma_{\Lambda_{12} \times_{X_2} \Lambda_2}\mathrm{Hom}_{11^a}(\mathscr{K} \overset{\mathrm{L}}{\underline{\otimes}}_2 \omega_2^{\otimes -1} \underset{2}{\circ} \omega_2 \underset{2}{\circ} \mathrm{D}'\mathscr{K}, \mathscr{K} \overset{\mathrm{L}}{\underline{\otimes}}_2 \mathscr{C}_2 \underset{2}{\circ} \omega_2 \underset{2}{\circ} \mathrm{D}'\mathscr{K}) \\
&\to \Gamma_{\Lambda_{12} \circ \Lambda_2}\mathrm{Hom}_{11^a}\big(\mathrm{R}p_{1*}(\mathscr{K} \overset{\mathrm{L}}{\underline{\otimes}}_2 (\omega_2^{\otimes -1} \underset{2}{\circ} \omega_2 \underset{2}{\circ} \mathrm{D}'\mathscr{K})), \mathrm{R}p_{1!}(\mathscr{K} \overset{\mathrm{L}}{\underline{\otimes}}_2 (\mathscr{C}_2 \underset{2}{\circ} \omega_2 \underset{2}{\circ} \mathrm{D}'\mathscr{K}))\big) \\
&\simeq \Gamma_{\Lambda_{12} \circ \Lambda_2}\mathrm{Hom}_{11^a}(\mathscr{K} \underset{2}{*} \mathrm{D}'\mathscr{K}, \mathscr{K} \underset{2}{\circ} \omega_2 \underset{2}{\circ} \mathrm{D}'\mathscr{K}) \\
&\to \Gamma_{\Lambda_{12} \circ \Lambda_2}\mathrm{Hom}_{11^a}(\omega_1^{\otimes -1}, \mathscr{C}_1) \simeq \mathrm{HH}_{\Lambda_{12} \circ \Lambda_2}(\mathscr{A}_1).
\end{aligned}$$

The first arrow is obtained by applying the functor $\mathscr{L} \mapsto \mathscr{K} \overset{\mathrm{L}}{\underline{\otimes}}_2 (\mathscr{L} \underset{2}{\circ} \omega_2 \underset{2}{\circ} \mathrm{D}'\mathscr{K})$, The last arrow is associated with the morphisms in (2.12) and (2.13).

Lemma 2.7. *The map* $\Phi_{\mathscr{K}}\colon \mathrm{HH}_{\Lambda_2}(\mathscr{A}_{X_2}) \longrightarrow \mathrm{HH}_{\Lambda_{12}\circ\Lambda_2}(\mathscr{A}_{X_1})$ *in* (2.15) *is the map* $\mathrm{hh}_{X_{12^a}}(\mathscr{K})\circ$ *given in* (2.10).

Theorem 2.8. *Let* $\mathscr{K}_i \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_{X_i\times X^a_{i+1}})$ $(i = 1, 2)$ *and set* $\Lambda_i = \mathrm{Supp}(\mathscr{K}_i)$. *Assume that* $\Lambda_1 \times_{X_2} \Lambda_2$ *is proper over* $X_1 \times X_3$. *Then*

$$\mathrm{hh}_{X_{13^a}}(\mathscr{K}_1 \circ \mathscr{K}_2) = \mathrm{hh}_{X_{12^a}}(\mathscr{K}_1) \circ \mathrm{hh}_{X_{23^a}}(\mathscr{K}_2) \tag{2.16}$$

as elements of $\mathrm{HH}_{\Lambda_1\circ\Lambda_2}(\mathscr{A}_{X_1\times X^a_3})$.

Proof. For the sake of simplicity, we assume that $X_3 = \mathrm{pt}$. Consider the diagram in which we set $\lambda_2 = \mathrm{hh}_2(\mathscr{K}_2)$:

$$\begin{tikzcd}
\omega_1^{\otimes -1} \arrow[r] \arrow[dddr] & \mathscr{K}_1 \underset{2}{\circ} \omega_2^{\otimes -1} \underset{2}{\circ} \omega_2 \underset{2}{\circ} \mathrm{D}'\mathscr{K}_1 \arrow[r, "\lambda_2"] \arrow[d] & \mathscr{K}_1 \underset{2}{\circ} \mathscr{C}_2 \underset{2}{\circ} \omega_2 \underset{2}{\circ} \mathrm{D}'\mathscr{K}_1 \arrow[r] & \mathscr{C}_1 \\
 & \mathscr{K}_1 \underset{2}{\circ} (\mathscr{K}_2 \underline{\boxtimes} \mathrm{D}'\mathscr{K}_2) \underset{2}{\circ} \omega_2 \underset{2}{\circ} \mathrm{D}'\mathscr{K}_1 \arrow[ur] \arrow[d, "\sim"] & & \\
 & (\mathscr{K}_1 \underset{2}{\circ} \mathscr{K}_2) \underline{\boxtimes} \mathrm{D}'\mathscr{K}_2 \underset{2}{\circ} \omega_2 \underset{2}{\circ} \mathrm{D}'\mathscr{K}_1 \arrow[d, "\sim"] & & \\
 & (\mathscr{K}_1 \underset{2}{\circ} \mathscr{K}_2) \underline{\boxtimes} \mathrm{D}'(\mathscr{K}_1 \underset{2}{\circ} \mathscr{K}_2) \arrow[uuurr] & &
\end{tikzcd}$$

Here, the left horizontal arrow on the top is the composition of morphisms $\omega_1^{\otimes -1} \to \mathscr{K}_1 \underset{2}{\circ} \mathrm{D}'\mathscr{K}_1 \to \mathscr{K}_1 \underset{2}{\circ} \omega_2^{\otimes -1} \underset{2}{\circ} \omega_2 \underset{2}{\circ} \mathrm{D}'\mathscr{K}_1$. The composition of the arrows on the bottom is $\mathrm{hh}_1(\mathscr{K}_1 \circ \mathscr{K}_2)$ by Lemma 2.2 and the composition of the arrows on the top is $\Phi_{\mathscr{K}_1}(\mathrm{hh}_2(\mathscr{K}_2))$ by Lemma 2.7. Hence, the assertion follows from the commutativity of the diagram. Q.E.D.

Remark 2.9. (i) The fact that Hochschild homology of $\mathscr{O}$-modules is functorial is well-known, see *e.g.,* [6, 14].
(ii) In [8], its authors interpret Hochschild homology as a morphism of functors and the action of kernels as a 2-morphism in a suitable 2-category. Its authors claim that the the relation $\Phi_{\mathscr{K}_1} \circ \Phi_{\mathscr{K}_1} = \Phi_{\mathscr{K}_1 \circ \mathscr{K}_2}$ follows by general arguments on 2-categories. Their result applies in a general framework including in particular $\mathscr{O}$-modules in the algebraic case and presumably DQ-modules but the precise axioms are not specified in loc. cit. See also [30] for

related results. Note that, as far as we understand, these authors do not introduce the convolution of Hochschild homologies and they did not consider Lemma 2.7 nor Theorem 2.8.

As a particular case of Theorem 2.8, consider two objects $\mathscr{M}$ and $\mathscr{N}$ in $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X)$ and assume that $\mathrm{Supp}(\mathscr{M})\cap\mathrm{Supp}(\mathscr{N})$ is compact. Then the cohomology groups of the complex $\mathrm{RHom}_{\mathscr{A}_X}(\mathscr{M},\mathscr{N})$ are finitely generated $\mathbf{k}_0$-modules and

$$\begin{aligned}\chi(\mathrm{RHom}_{\mathscr{A}_X}(\mathscr{M},\mathscr{N})) &= \mathrm{hh}_{X^a}(\mathrm{D}'_{\mathscr{A}}\mathscr{M})\underset{X}{\circ}\mathrm{hh}_X(\mathscr{N})\\ &= \mathrm{hh}_X(\mathscr{M})\underset{X}{\circ}\mathrm{hh}_X(\mathscr{N}).\end{aligned}$$

Graded and localized Hochschild classes

Similarly to the case of $\mathscr{A}_X$, one defines

$$\begin{aligned}\mathcal{HH}(\mathrm{gr}_\hbar(\mathscr{A}_X)) &:= \mathrm{gr}_\hbar(\mathscr{C}_{X^a})\overset{\mathrm{L}}{\otimes}_{\mathrm{gr}_\hbar(\mathscr{A}_{X\times X^a})}\mathrm{gr}_\hbar(\mathscr{C}_X),\\ \mathcal{HH}(\mathscr{A}_X^{\mathrm{loc}}) &:= \mathscr{A}_{X^a}^{\mathrm{loc}}\overset{\mathrm{L}}{\otimes}_{\mathscr{A}^{\mathrm{loc}}_{X\times X^a}}\mathscr{A}_X^{\mathrm{loc}}.\end{aligned}$$

Note that

$$\mathcal{HH}(\mathrm{gr}_\hbar(\mathscr{A}_X))\simeq\mathbb{C}\overset{\mathrm{L}}{\otimes}_{\mathbf{k}_0}\mathcal{HH}(\mathscr{A}_X),\quad \mathcal{HH}(\mathscr{A}_X^{\mathrm{loc}})\simeq\mathbf{k}\otimes_{\mathbf{k}_0}\mathcal{HH}(\mathscr{A}_X)$$

and there are natural morphisms

$$\mathrm{gr}_\hbar\colon\mathcal{HH}(\mathscr{A}_X)\to\mathcal{HH}(\mathrm{gr}_\hbar(\mathscr{A}_X)),\quad \mathrm{loc}\colon\mathcal{HH}(\mathscr{A}_X)\to\mathcal{HH}(\mathscr{A}_X^{\mathrm{loc}}).$$

For $\mathscr{F}\in\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathrm{gr}_\hbar(\mathscr{A}_X))$ (resp. $\mathscr{F}\in\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X^{\mathrm{loc}})$), one defines its Hochschild class $\mathrm{hh}^{\mathrm{gr}}_{\mathrm{X}}(\mathscr{F})$ (resp. $\mathrm{hh}^{\mathrm{loc}}_{\mathrm{X}}(\mathscr{F})$) by the same construction as for $\mathscr{A}_X$-modules. For $\mathscr{M}\in\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X)$, setting $\mathscr{M}^{\mathrm{loc}}=\mathbf{k}\otimes_{\mathbf{k}_0}\mathscr{M}$, we have:

$$\mathrm{gr}_\hbar(\mathrm{hh}_X(\mathscr{M}))=\mathrm{hh}^{\mathrm{gr}}_{\mathrm{X}}(\mathrm{gr}_\hbar(\mathscr{M})),\quad (\mathrm{hh}_{\mathrm{X}}(\mathscr{M}))^{\mathrm{loc}}=\mathrm{hh}^{\mathrm{loc}}_{\mathrm{X}}(\mathscr{M}^{\mathrm{loc}}).\tag{2.17}$$

Hochschild class on symplectic manifolds

Consider first the case where X is an open subset of T^*M, M being affine, that is, M is open in some finite-dimensional $\mathbb{C}$-vector space. Denote by $(x)=(x_1,\dots,x_n)$ a coordinate system on M and by $(x;u)$ the associated symplectic coordinate system on T^*M. Let $f, g \in \mathscr{O}_X[[\hbar]]$. One defines a star-product on $\mathscr{O}_X[[\hbar]]$, hence a DQ-algebra $\widehat{\mathscr{W}}_{T^*M}(0)$, by setting:

$$f \star g \;=\; \sum_{\alpha\in\mathbb{N}^n} \frac{\hbar^{|\alpha|}}{\alpha!}(\partial_u^\alpha f)(\partial_x^\alpha g). \tag{2.18}$$

This product is similar to the product of the total symbols of differential operators (see § 3).

It is a well-known fact that if $\mathscr{A}_X$ is a DQ-algebra and the associated Poisson structure is symplectic, then X is locally isomorphic to an open subset of a cotangent bundle T^*M (Darboux's theorem) and $\mathscr{A}_X$ is locally isomorphic to $\widehat{\mathscr{W}}_{T^*M}(0)$.

Throughout this section, X denotes a complex manifold endowed with a DQ-algebroid $\mathscr{A}_X$ such that the associated Poisson structure is symplectic. Hence, X is symplectic and we denote by α_X the symplectic 2-form on X.

We set $2n = d_X$, $Z = X \times X^a$ and we denote by dv the volume form on X given by $dv = \alpha_X^n/n!$. Recall that the objects $\Omega_X^{\mathscr{A}}$ and $\omega_X^{\mathscr{A}}$ are defined in (1.11).

It follows from a classical result of [28] that two simple $\mathscr{A}_X$-modules $\mathscr{L}_i$ $(i = 0, 1)$ along a smooth Lagrangian submanifold Λ of X are locally isomorphic and the natural morphism $\mathbf{k}_0 \to \mathscr{H}om_{\mathscr{A}_X}(\mathscr{L}_0, \mathscr{L}_0)$ is an isomorphism. It follows that there exists a local system L of rank one over $\mathbf{k}_{0X}$ such that $\Omega_X^{\mathscr{A}} \simeq L \otimes_{\mathbf{k}_{0X}} \mathscr{C}_X$ in $\mathrm{Mod}(\mathscr{A}_{X\times X^a})$. Hence we obtain

$$\begin{aligned}\mathcal{HH}(\mathscr{A}_X) &= \mathscr{C}_{X^a} \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_Z} \mathscr{C}_X \simeq \mathrm{R}\mathscr{H}om_{\mathscr{A}_Z}(\mathrm{D}'_{\mathscr{A}}\mathscr{C}_X, \mathscr{C}_X)\\ &\simeq L \otimes \mathrm{R}\mathscr{H}om_{\mathscr{A}_Z}(\mathscr{C}_X, \mathscr{C}_X)[d_X]\end{aligned}$$

and

$$\begin{aligned}\mathcal{HH}(\mathscr{A}_X) &= \mathscr{C}_{X^a} \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_Z} \mathscr{C}_X \simeq L^{\otimes -1} \otimes \Omega_X^{\mathscr{A}} \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_Z} \mathscr{C}_X\\ &\to L^{\otimes -1} \otimes \Omega_X^{\mathscr{A}} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_Z^{\mathscr{A}}} \mathscr{C}_X \simeq L^{\otimes -1}[d_X].\end{aligned}$$

Therefore, we get the morphisms:

$$L[d_X] \to \mathcal{HH}(\mathscr{A}_X) \to L^{\otimes -1}[d_X]. \tag{2.19}$$

Moreover, one proves the isomorphism

$$L \simeq \hbar^{d_X/2}\mathbf{k}_{0X}, \tag{2.20}$$

from which one deduces:

Theorem 2.10. *Assume that X is symplectic.*

(i) *There is a canonical $\mathscr{A}_Z$-linear isomorphism $\Omega_X^{\mathscr{A}} \xrightarrow{\sim} \hbar^{d_X/2}\mathbf{k}_0 \otimes_{\mathbf{k}_0} \mathscr{C}_X$.*

(ii) *This isomorphism induces canonical morphisms*

$$\hbar^{d_X/2}\mathbf{k}_{0X}\,[d_X] \xrightarrow{\iota_X} \mathcal{HH}(\mathscr{A}_X) \xrightarrow{\tau_X} \hbar^{-d_X/2}\mathbf{k}_{0X}\,[d_X] \tag{2.21}$$

and the composition $\tau_X \circ \iota_X$ is the canonical morphism $\hbar^{d_X/2}\mathbf{k}_{0X}\,[d_X] \to \hbar^{-d_X/2}\mathbf{k}_{0X}\,[d_X]$.

(iii) *$H^j(\mathcal{HH}(\mathscr{A}_X)) \simeq 0$ unless $-d_X \le j \le 0$ and the morphism ι_X induces an isomorphism*

$$\iota_X : \hbar^{d_X/2}\mathbf{k}_{0X} \xrightarrow{\sim} H^{-d_X}(\mathcal{HH}(\mathscr{A}_X)). \tag{2.22}$$

In particular, there is a canonical non-zero section in $H^{-d_X}(X;\mathcal{HH}(\mathscr{A}_X))$.

Remark 2.11. The existence of a canonical section in $H^{-d_X}(X;\mathcal{HH}(\mathscr{A}_X))$ is well known (see in particular [5, 12]). It is intensively used by many authors (see in particular [4, 11]), some of them calling it the "trace density map".

Remark 2.12. The Hochschild class of coherent $\mathscr{O}$-modules has been studied by many authors. Let us quote in particular [6, 7, 8, 14, 24, 27, 30].

3 Euler class and applications to $\mathscr{D}$-modules

Euler class on symplectic manifolds

Denote as above by X a complex manifold endowed with a DQ-algebroid $\mathscr{A}_X$ such that the associated Poisson structure is symplectic.

Recall (see (1.9)) that $\mathbf{k} := \mathbb{C}((\hbar))$ and $\mathscr{A}_X^{\mathrm{loc}} = \mathbf{k} \otimes_{\mathbf{k}_0} \mathscr{A}_X$.

The following result is easily deduced from Theorem 2.10.

Theorem 3.1. *The complex $\mathcal{HH}(\mathscr{A}_X^{\mathrm{loc}})$ is concentrated in degree $-d_X$, the morphisms ι_X and τ_X in Theorem 2.10 induce isomorphisms*

$$\mathbf{k}_X\,[d_X] \underset{\iota_X}{\xrightarrow{\sim}} \mathcal{HH}(\mathscr{A}_X^{\mathrm{loc}}) \underset{\tau_X}{\xrightarrow{\sim}} \mathbf{k}_X[d_X] \tag{3.1}$$

and the composition $\tau_X \circ \iota_X$ is the identity.

Definition 3.2. Let $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X^{\mathrm{loc}})$. We set

$$\mathrm{eu}_X(\mathscr{M}) = \tau_X(\mathrm{hh}_X(\mathscr{M})) \in H^{d_X}_{\mathrm{Supp}(\mathscr{M})}(X;\mathbf{k}_X) \tag{3.2}$$

and call $\mathrm{eu}_X(\mathscr{M})$ the Euler class of $\mathscr{M}$.

Consider the diagram

$$\begin{tikzcd}
p_{13!}(p_{12}^{-1}\mathcal{HH}(\mathscr{A}^{\mathrm{loc}}_{X_1\times X_2^a}) \otimes p_{23}^{-1}\mathcal{HH}(\mathscr{A}^{\mathrm{loc}}_{X_2\times X_3^a})) \arrow[r, "\star"] \arrow[d, "\tau_{12^a}\otimes\tau_{23^a}"] & \mathcal{HH}(\mathscr{A}^{\mathrm{loc}}_{X_1\times X_3^a}) \arrow[d, "\tau_{13^a}"] \\
p_{13!}(p_{12}^{-1}\mathbf{k}_{X_{12}}[d_{12}] \otimes p_{23}^{-1}\mathbf{k}_{X_{23}}[d_{23}]) \arrow[r, "\int_2(\cdot\cup\cdot)"] & \mathbf{k}_{X_{13}}[d_{13}].
\end{tikzcd} \tag{3.3}$$

Here, the horizontal arrow in the bottom denoted by $\int_2(\bullet \cup \bullet)$ is obtained by taking the cup product and integrating on X_2 (Poincaré duality), using the fact that the manifold X_2 has real dimension $2\,d_2$ and is oriented. The arrow in the top denoted by $\star$ is obtained by Proposition 2.4 (ii). The two vertical arrows are given by the Euler classes.

Proposition 3.3. Diagram 3.3 *commutes.*

Proof. Since X_1 and X_3 play the role of parameter spaces, we may assume that $X_1 = X_3 = \{\mathrm{pt}\}$. We set $X_2 = X$, $d_X = d$ and denote by a_X the projection $X \to \{\mathrm{pt}\}$. We are reduce to prove the commutativity of the diagram below:

$$\begin{tikzcd}
a_{X!}(\mathcal{HH}(\mathscr{A}_X^{\mathrm{loc}}) \otimes \mathcal{HH}(\mathscr{A}_X^{\mathrm{loc}})) \arrow[d, "\tau\otimes\tau"'] \arrow[dr, "\star"] & \\
a_{X!}(\mathbf{k}_X\,[d_X] \otimes \mathbf{k}_X\,[d_X]) \arrow[r, "\int_X(\cdot\cup\cdot)"'] & \mathbf{k}.
\end{tikzcd} \tag{3.4}$$

This will follow by applying the functor $a_{X!}$ to Diagram 3.5 bellow. Q.E.D.

Lemma 3.4. *The diagram below commutes.*

$$\begin{tikzcd}
\mathcal{HH}(\mathscr{A}_X^{\mathrm{loc}}) \otimes \mathcal{HH}(\mathscr{A}_X^{\mathrm{loc}}) \arrow[d, "\tau\otimes\tau"'] \arrow[dr, "\star"] & \\
\mathbf{k}_X\,[d] \otimes \mathbf{k}_X\,[d] \arrow[r] & \mathbf{k}_X\,[2d].
\end{tikzcd} \tag{3.5}$$

Corollary 3.5. *Let $\mathscr{K}_i \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}^{\mathrm{loc}}_{X_i \times X^a_{i+1}})$ $(i = 1, 2)$. Assume that the projection p_{13} defined on $X_1 \times X_2 \times X_3$ is proper on $p_{12}^{-1} \operatorname{Supp}(\mathscr{K}_1) \cap p_{23}^{-1} \operatorname{Supp}(\mathscr{K}_2)$. Then*

$$\mathrm{eu}_{X_{13^a}}(\mathscr{K}_1 \underset{2}{\circ} \mathscr{K}_2) = \int_{X_2} \mathrm{eu}_{X_{12^a}}(\mathscr{K}_1) \cup \mathrm{eu}_{X_{23^a}}(\mathscr{K}_2). \tag{3.6}$$

Applications to $\mathscr{D}$-modules

From now on, $(M, \mathscr{O}_M)$ denotes a complex manifold. As usual, we denote by $\mathscr{D}_M$ the $\mathbb{C}$-algebra of differential operators on M. This is a right and left Noetherian sheaf of rings.

One says that a coherent $\mathscr{D}_M$-module $\mathscr{M}$ is *good* (see [16]) if, for any open relatively compact set $U \subset M$, there exists a coherent sub-$\mathscr{O}_U$-module $\mathscr{F}$ of $\mathscr{M}|_U$ which generates it on U as a $\mathscr{D}_M$-module. One denotes by $\mathrm{D}^b_{\mathrm{gd}}(\mathscr{D}_M)$ the full sub-triangulated category of $\mathrm{D}^b_{\mathrm{coh}}(\mathscr{D}_M)$ consisting of objects with good cohomology.

Let $\pi\colon T^*M \to M$ denote the cotangent bundle to M. The manifold T^*M is naturally endowed with the conic sheaf $\widehat{\mathscr{E}}_{T^*M}$ of formal microdifferential operators of [28] and with its subring $\widehat{\mathscr{E}}_{T^*M}(0)$ consisting of operators of order ≤ 0. It is also endowed with a canonical DQ-algebra, denoted by $\widehat{\mathscr{W}}_{T^*M}(0)$, which may be constructed using the sheaf $\widehat{\mathscr{E}}_{T^*(M\times\mathbb{C})}(0)$ (see [26]). Its localization is denoted by $\widehat{\mathscr{W}}_{T^*M}$ and there are natural morphisms of algebras

$$\pi_M^{-1}\mathscr{D}_M \hookrightarrow \widehat{\mathscr{E}}_{T^*M} \hookrightarrow \widehat{\mathscr{W}}_{T^*M}. \tag{3.7}$$

If M is endowed with a local coordinate systems $(x) = (x_1, \dots, x_n)$, the composition of the morphisms in (3.7) is given by

$$\varphi(x) \mapsto \varphi(x), \qquad \partial_{x_i} \mapsto \hbar^{-1} u_i.$$

Recall that for a coherent $\mathscr{D}_M$-module $\mathscr{M}$, the support of $\widehat{\mathscr{E}}_{T^*M} \otimes_{\pi_M^{-1}\mathscr{D}_M} \pi_M^{-1}\mathscr{M}$ is called its characteristic variety and denoted by $\mathrm{char}(\mathscr{M})$.

From now on, we set $X = T^*M$. One defines the Hochschild homology $\mathcal{HH}(\widehat{\mathscr{E}}_X)$ of $\widehat{\mathscr{E}}_X$ and the Hochschild class $\mathrm{hh}_X(\mathscr{M})$ of a coherent $\widehat{\mathscr{E}}_X$-module $\mathscr{M}$ similarly as for $\mathcal{HH}(\mathscr{A}_X)$.

In the sequel, we identify a coherent $\mathscr{D}_M$-module with its image in $\pi_*(\widehat{\mathscr{E}}_X)$. In particular, we define by this way the Hochschild class $\mathrm{hh}_X(\mathscr{M})$ of a coherent $\mathscr{D}$-module $\mathscr{M}$. Hence

$$\mathrm{hh}_X(\mathscr{M}) \in H^{d_X}_{\mathrm{char}(\mathscr{M})}(X; \mathcal{HH}(\widehat{\mathscr{E}}_X)). \tag{3.8}$$

Lemma 3.6. *There is a natural isomorphism*

$$\mathcal{HH}(\widehat{\mathscr{E}}_X) \xrightarrow{\sim} \mathbb{C}_X\,[d_X] \tag{3.9}$$

which makes the diagram below commutative:

$$\begin{tikzcd}
\mathcal{HH}(\widehat{\mathscr{E}}_X) \arrow[r, "\sim"] \arrow[d] & \mathbb{C}_X[d_X] \arrow[d] \\
\mathcal{HH}(\widehat{\mathscr{W}}_X) \arrow[r, "\sim", "\tau"'] & \mathbf{k}_X\,[d_X].
\end{tikzcd}$$

Definition 3.7. Let $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\widehat{\mathscr{E}}_X)$. We denote by $\mathrm{eu}_X(\mathscr{M})$ the image of $\mathrm{hh}_X(\mathscr{M})$ in $H^{d_X}_{\mathrm{char}(\mathscr{M})}(X;\mathbb{C}_X)$ by the morphism in (3.9) and call it the Euler class of $\mathscr{M}$.

We now introduce the functor

$$\begin{aligned}
(\,\bullet\,)^{\mathrm{W}}\colon \mathrm{Mod}(\mathscr{D}_M) &\to \mathrm{Mod}(\widehat{\mathscr{W}}_X) && (3.10)\\
\mathscr{M} &\mapsto \widehat{\mathscr{W}}_X \otimes_{\pi_M^{-1}\mathscr{D}_M} \pi_M^{-1}\mathscr{M}. && (3.11)
\end{aligned}$$

The next result shows that one can, in some sense, reduce the study of $\mathscr{D}$-modules to that of $\widehat{\mathscr{W}}$-modules.

Proposition 3.8. *The functor $\mathscr{M} \mapsto \mathscr{M}^{\mathrm{W}}|_M$ is exact and faithful.*

It follows that $(\,\bullet\,)^W$ sends $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_M)$ to $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\widehat{\mathscr{W}}_X)$ and $\mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{D}_M)$ to $\mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\widehat{\mathscr{W}}_X)$. The next result immediately follows from Lemma 3.6.

Proposition 3.9. *For $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_M)$, $\mathrm{eu}_X(\mathscr{M}^{\mathrm{W}})$ is the image of $\mathrm{eu}_X(\mathscr{M})$ by the natural map $H^{d_X}_{\mathrm{char}(\mathscr{M})}(X;\mathbb{C}_X) \to H^{d_X}_{\mathrm{char}(\mathscr{M})}(X;\mathbf{k}_X)$.*

Let M and N be two complex manifolds, set $X = T^*M$, $Y = T^*N$. Denote by q_i the i-th projection defined on $M \times N$ and by p_i the i-th projection defined on $X \times Y$ $(i = 1, 2)$. Let $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}(\mathscr{D}_M)$ and $\mathscr{L} \in \mathrm{D}^{\mathrm{b}}(\mathscr{D}_{M^a\times N})$. Set

$$\mathscr{M} \underset{M}{\circ} \mathscr{L} := \mathrm{R}q_{2!}(\mathscr{L} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}} q_1^{-1}\mathscr{M}).$$

Theorem 3.10. *Assume that $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{D}_M)$, $\mathscr{L} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{D}_{M^a\times N})$ and assume that p_2 is proper on $p_1^{-1}\,\mathrm{char}(\mathscr{M}) \cap \mathrm{char}(\mathscr{L})$. Then $\mathscr{M} \underset{M}{\circ} \mathscr{L} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{D}_N)$ and*

$$(\mathscr{M} \underset{M}{\circ} \mathscr{L})^{\mathrm{W}} \xrightarrow{\sim} \mathscr{M}^{\mathrm{W}} \underset{X}{\circ} \mathscr{L}^{\mathrm{W}}. \tag{3.12}$$

The proof is straightforward and is left to the reader.

Corollary 3.11. *Let $\mathscr{M}$ and $\mathscr{L}$ be as above. Then*

$$\mathrm{eu}_Y(\mathscr{M} \underset{M}{\circ} \mathscr{L}) = \mathrm{eu}_X(\mathscr{M}) \circ \mathrm{eu}_{X\times Y}(\mathscr{L}). \tag{3.13}$$

This formula is equivalent to the results of [29] on the functoriality of the Euler class of $\mathscr{D}$-modules. Note that the results of loc. cit. also deal with constructible sheaves.

Remark 3.12. Recall that the functoriality of the Chern class of the graded modules associated to coherent $\mathscr{D}$-modules (in the algebraic settings) is proved in [23] as a corollary of the Riemann-Roch-Grothendieck theorem. It can be shown that the Hochschild class of DQ-modules allows us to recover this result.

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