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Regular holonomic $\mathscr{D}[[\hbar]]$-modules

Dedicated to Professor Mikio Sato on the occasion of his 80th birthday with our deep admiration and warmest regards

By

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Abstract

We describe the category of regular holonomic modules over the ring $\mathscr{D}[[\hbar]]$ of linear differential operators with a formal parameter $\hbar$. In particular, we establish the Riemann-Hilbert correspondence and discuss the additional t-structure related to $\hbar$-torsion.

Introduction

On a complex manifold X, we will be interested in the study of holonomic modules over the ring $\mathscr{D}_X[[\hbar]]$ of differential operators with a formal parameter $\hbar$. Such modules naturally appear when studying deformation quantization modules (DQ-modules) along a smooth Lagrangian submanifold of a complex symplectic manifold (see [13, Chapter 7]).

In this paper, after recalling the tools from loc. cit. that we shall use, we explain some basic notions of $\mathscr{D}_X[[\hbar]]$-modules theory. For example, it follows easily from general results on modules over $\mathbb{C}[[\hbar]]$-algebras that given two

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holonomic $\mathscr{D}_X[[\hbar]]$-modules $\mathscr{M}$ and $\mathscr{N}$, the complex $\mathrm{R}\mathscr{H}om_{\mathscr{D}_X[[\hbar]]}(\mathscr{M},\mathscr{N})$ is constructible over $\mathbb{C}[[\hbar]]$ and that the microsupport of the solution complex $\mathrm{R}\mathscr{H}om_{\mathscr{D}_X[[\hbar]]}(\mathscr{M},\mathscr{O}_X[[\hbar]])$ coincides with the characteristic variety of $\mathscr{M}$.

Then we establish our main result, the Riemann-Hilbert correspondence for regular holonomic $\mathscr{D}_X[[\hbar]]$-modules, an $\hbar$-variant of Kashiwara's classical theorem. In other words, we show that the solution functor with values in $\mathscr{O}_X[[\hbar]]$ induces an equivalence between the derived category of regular holonomic $\mathscr{D}_X[[\hbar]]$-modules and that of constructible sheaves over $\mathbb{C}[[\hbar]]$. A quasi-inverse is obtained by constructing the "sheaf" of holomorphic functions with temperate growth and a formal parameter $\hbar$ in the subanalytic site. This needs some care since the literature on this subject is written in the framework of sheaves over a field and does not immediately apply to the ring $\mathbb{C}[[\hbar]]$.

We also discuss the t-structure related to $\hbar$-torsion. Indeed, as we work over the ring $\mathbb{C}[[\hbar]]$ and not over a field, the derived category of holonomic $\mathscr{D}_X[[\hbar]]$-modules (or, equivalently, that of constructible sheaves over $\mathbb{C}[[\hbar]]$) has an additional t-structure related to $\hbar$-torsion. We will show how the duality functor interchanges it with the natural t-structure.

We end this paper by describing some natural links between the ring $\mathscr{D}_X[[\hbar]]$ and deformation quantization algebras, as mentioned above.

Historical remark. As it is well-known, holonomic modules play an essential role in Mathematics. They appeared independently in the work of M. Kashiwara [4] and J. Bernstein [1], but they were first invented by Mikio Sato in a series of (unfortunately unpublished) lectures at Tokyo University in the 60's. (See [17] for a more detailed history.)

Notations and conventions

We shall mainly follow the notations of [12]. In particular, if $\mathscr{C}$ is an abelian category, we denote by $\mathsf{D}(\mathscr{C})$ the derived category of $\mathscr{C}$ and by $\mathsf{D}^*(\mathscr{C})$ ($*=+,-,\mathrm{b}$) the full triangulated subcategory consisting of objects with bounded from below (resp. bounded from above, resp. bounded) cohomology.

For a sheaf of rings $\mathscr{R}$ on a topological space X, or more generally a site, we denote by $\mathrm{Mod}(\mathscr{R})$ the category of left $\mathscr{R}$-modules and we write $\mathsf{D}^*(\mathscr{R})$ instead of $\mathsf{D}^*(\mathrm{Mod}(\mathscr{R}))$ ($*=\emptyset,+,-,\mathrm{b}$). We denote by $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{R})$ the full abelian subcategory of $\mathrm{Mod}(\mathscr{R})$ of coherent objects, and by $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{R})$ the full triangulated subcategory of $\mathsf{D}^{\mathrm{b}}(\mathscr{R})$ of objects with coherent cohomology groups.

If R is a ring (a sheaf of rings over a point), we write for short $\mathsf{D}^{\mathrm{b}}_f(R)$ instead of $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(R)$.

§1. Formal deformations (after [13])

We review here some definitions and results from [13] that we shall use in this paper.

Modules over $\mathbb{Z}[\hbar]$-algebras. Let X be a topological space. One says that a sheaf of $\mathbb{Z}_X[\hbar]$-modules $\mathscr{M}$ has no $\hbar$-torsion if $\hbar\colon \mathscr{M} \to \mathscr{M}$ is injective and one says that $\mathscr{M}$ is $\hbar$-complete if $\mathscr{M} \to \varprojlim_n \mathscr{M}/\hbar^n\mathscr{M}$ is an isomorphism.

Let $\mathscr{R}$ be a sheaf of $\mathbb{Z}_X[\hbar]$-algebras, and assume that $\mathscr{R}$ has no $\hbar$-torsion. One sets

$$\mathscr{R}^{\mathrm{loc}} := \mathbb{Z}[\hbar, \hbar^{-1}] \otimes_{\mathbb{Z}[\hbar]} \mathscr{R}, \qquad \mathscr{R}_0 := \mathscr{R}/\hbar\mathscr{R},$$

and considers the functors

$$(\,\bullet\,)^{\mathrm{loc}}\colon \mathrm{Mod}(\mathscr{R}) \to \mathrm{Mod}(\mathscr{R}^{\mathrm{loc}}), \quad \mathscr{M} \mapsto \mathscr{M}^{\mathrm{loc}} := \mathscr{R}^{\mathrm{loc}} \otimes_{\mathscr{R}} \mathscr{M},$$

$$\mathrm{gr}_\hbar\colon \mathsf{D}(\mathscr{R}) \to \mathsf{D}(\mathscr{R}_0), \quad \mathscr{M} \mapsto \mathrm{gr}_\hbar(\mathscr{M}) := \mathscr{R}_0 \overset{\mathrm{L}}{\otimes}_{\mathscr{R}} \mathscr{M}.$$

Note that $(\,\bullet\,)^{\mathrm{loc}}$ is exact and that for $\mathscr{M}, \mathscr{N} \in \mathsf{D}^{\mathrm{b}}(\mathscr{R})$ and $\mathscr{P} \in \mathsf{D}^{\mathrm{b}}(\mathscr{R}^{\mathrm{op}})$ one has isomorphisms:

$$\mathrm{gr}_\hbar(\mathscr{P} \overset{\mathrm{L}}{\otimes}_{\mathscr{R}} \mathscr{M}) \simeq \mathrm{gr}_\hbar \mathscr{P} \overset{\mathrm{L}}{\otimes}_{\mathscr{R}_0} \mathrm{gr}_\hbar \mathscr{M}, \tag{1.1}$$

$$\mathrm{gr}_\hbar(\mathrm{R}\mathscr{H}om_{\mathscr{R}}(\mathscr{M}, \mathscr{N})) \simeq \mathrm{R}\mathscr{H}om_{\mathscr{R}_0}(\mathrm{gr}_\hbar(\mathscr{M}), \mathrm{gr}_\hbar(\mathscr{N})). \tag{1.2}$$

Here, the functor $\mathrm{gr}_\hbar$ on the left hand side acts on $\mathbb{Z}_X[\hbar]$-modules.

Cohomologically $\hbar$-complete sheaves.

Definition 1.1. *One says that an object $\mathscr{M}$ of $\mathsf{D}(\mathscr{R})$ is cohomologically $\hbar$-complete if $\mathrm{R}\mathscr{H}om_{\mathscr{R}}(\mathscr{R}^{\mathrm{loc}}, \mathscr{M}) = 0$.*

Hence, the full subcategory of cohomologically $\hbar$-complete objects is triangulated. In fact, it is the right orthogonal to the full subcategory $\mathsf{D}(\mathscr{R}^{\mathrm{loc}})$ of $\mathsf{D}(\mathscr{R})$.

Remark that $\mathscr{M} \in \mathsf{D}(\mathscr{R})$ is cohomologically $\hbar$-complete if and only if its image in $\mathsf{D}(\mathbb{Z}_X[\hbar])$ is cohomologically $\hbar$-complete.

Proposition 1.2. *Let $\mathscr{M} \in \mathsf{D}(\mathscr{R})$. Then $\mathscr{M}$ is cohomologically $\hbar$-complete if and only if*

$$\varinjlim_{U \ni x} \mathrm{Ext}^j_{\mathbb{Z}[\hbar]}\big(\mathbb{Z}[\hbar, \hbar^{-1}], H^i(U; \mathscr{M})\big) = 0,$$

for any $x \in X$, any integer $i \in \mathbb{Z}$ and any $j = 0, 1$. Here, U ranges over an open neighborhood system of x.

Corollary 1.3. *Let $\mathscr{M} \in \mathrm{Mod}(\mathscr{R})$. Assume that $\mathscr{M}$ has no $\hbar$-torsion, is $\hbar$-complete and there exists a base $\mathfrak{B}$ of open subsets such that $H^i(U;\mathscr{M}) = 0$ for any $i > 0$ and any $U \in \mathfrak{B}$. Then $\mathscr{M}$ is cohomologically $\hbar$-complete.*

The functor $\mathrm{gr}_\hbar$ is conservative on the category of cohomologically $\hbar$-complete objects:

Proposition 1.4. *Let $\mathscr{M} \in \mathsf{D}(\mathscr{R})$ be a cohomologically $\hbar$-complete object. If $\mathrm{gr}_\hbar(\mathscr{M}) = 0$, then $\mathscr{M} = 0$.*

Proposition 1.5. *If $\mathscr{M} \in \mathsf{D}(\mathscr{R})$ is cohomologically $\hbar$-complete, then $\mathrm{R}\mathscr{H}om_{\mathscr{R}}(\mathscr{N},\mathscr{M}) \in \mathsf{D}(\mathbb{Z}_X[\hbar])$ is cohomologically $\hbar$-complete for any $\mathscr{N} \in \mathsf{D}(\mathscr{R})$.*

Proposition 1.6. *Let $f\colon X \to Y$ be a continuous map, and $\mathscr{M} \in \mathsf{D}(\mathbb{Z}_X[\hbar])$. If $\mathscr{M}$ is cohomologically $\hbar$-complete, then so is $\mathrm{R}f_*\mathscr{M}$.*

Reductions to $\hbar = 0$. Now we assume that X is a Hausdorff locally compact topological space.

By a basis $\mathfrak{B}$ of compact subsets of X, we mean a family of compact subsets such that for any $x \in X$ and any open neighborhood U of x, there exists $K \in \mathfrak{B}$ such that $x \in \mathrm{Int}(K) \subset U$.

Let $\mathscr{A}$ be a $\mathbb{Z}[\hbar]$-algebra, and recall that we set $\mathscr{A}_0 = \mathscr{A}/\hbar\mathscr{A}$. Consider the following conditions:

(i) $\mathscr{A}$ has no $\hbar$-torsion and is $\hbar$-complete,

(ii) $\mathscr{A}_0$ is a left Noetherian ring,

(iii) there exists a basis $\mathfrak{B}$ of compact subsets of X and a prestack $U \mapsto \mathrm{Mod}_{\mathrm{good}}(\mathscr{A}_0|_U)$ (U open in X) such that

  (a) for any $K \in \mathfrak{B}$ and any open subset U such that $K \subset U$, there exists $K' \in \mathfrak{B}$ such that $K \subset \mathrm{Int}(K') \subset K' \subset U$,

  (b) $U \mapsto \mathrm{Mod}_{\mathrm{good}}(\mathscr{A}_0|_U)$ is a full subprestack of $U \mapsto \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_0|_U)$,

  (c) for any $K \in \mathfrak{B}$, any open set U containing K, any $j > 0$ and any $\mathscr{M} \in \mathrm{Mod}_{\mathrm{good}}(\mathscr{A}_0|_U)$, one has $H^j(K;\mathscr{M}) = 0$,

  (d) for any open subset U and any $\mathscr{M} \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_0|_U)$, if $\mathscr{M}|_V$ belongs to $\mathrm{Mod}_{\mathrm{good}}(\mathscr{A}_0|_V)$ for any relatively compact open subset V of U, then $\mathscr{M}$ belongs to $\mathrm{Mod}_{\mathrm{good}}(\mathscr{A}_0|_U)$,

(e) for any U open in X, $\mathrm{Mod}_{\mathrm{good}}(\mathscr{A}_0|_U)$ is stable by subobjects, quotients and extensions in $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_0|_U)$,

(f) for any U open in X and any $\mathscr{M} \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_0|_U)$, there exists an open covering $U = \bigcup_i U_i$ such that $\mathscr{M}|_{U_i} \in \mathrm{Mod}_{\mathrm{good}}(\mathscr{A}_0|_{U_i})$,

(g) $\mathscr{A}_0 \in \mathrm{Mod}_{\mathrm{good}}(\mathscr{A}_0)$,

(iii)' there exists a basis $\mathfrak{B}$ of open subsets of X such that for any $U \in \mathfrak{B}$, any $\mathscr{M} \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_0|_U)$ and any $j > 0$, one has $H^j(U;\mathscr{M}) = 0$.

We will suppose that $\mathscr{A}$ and $\mathscr{A}_0$ satisfy either Assumption 1.7 or Assumption 1.8.

Assumption 1.7. *$\mathscr{A}$ and $\mathscr{A}_0$ satisfy conditions (i), (ii) and (iii) above.*

Assumption 1.8. *$\mathscr{A}$ and $\mathscr{A}_0$ satisfy conditions (i), (ii) and (iii)' above.*

Theorem 1.9.

(i) *$\mathscr{A}$ is a left Noetherian ring.*

(ii) *Any coherent $\mathscr{A}$-module $\mathscr{M}$ is $\hbar$-complete.*

(iii) *Let $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A})$. Then $\mathscr{M}$ is cohomologically $\hbar$-complete.*

Corollary 1.10. *The functor $\mathrm{gr}_\hbar \colon \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}) \to \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_0)$ is conservative.*

Theorem 1.11. *Let $\mathscr{M} \in \mathsf{D}^+(\mathscr{A})$ and assume:*

(a) *$\mathscr{M}$ is cohomologically $\hbar$-complete,*

(b) $\mathrm{gr}_\hbar(\mathscr{M}) \in \mathsf{D}^+_{\mathrm{coh}}(\mathscr{A}_0)$.

Then, $\mathscr{M} \in \mathsf{D}^+_{\mathrm{coh}}(\mathscr{A})$ and for all $i \in \mathbb{Z}$ we have the isomorphism

$$H^i(\mathscr{M}) \xrightarrow{\sim} \varprojlim_n H^i(\mathscr{A}/\hbar^n\mathscr{A} \overset{\mathrm{L}}{\otimes}_{\mathscr{A}} \mathscr{M}).$$

Theorem 1.12. *Assume that $\mathscr{A}_0^{\mathrm{op}} = \mathscr{A}^{\mathrm{op}}/\hbar\mathscr{A}^{\mathrm{op}}$ is a Noetherian ring and the flabby dimension of X is finite. Let $\mathscr{M}$ be an $\mathscr{A}$-module. Assume the following conditions:*

(a) *$\mathscr{M}$ has no $\hbar$-torsion,*

(b) *$\mathscr{M}$ is cohomologically $\hbar$-complete,*

(c) *$\mathscr{M}/\hbar\mathscr{M}$ is a flat $\mathscr{A}_0$-module.*

Then $\mathscr{M}$ is a flat $\mathscr{A}$-module.

If moreover $\mathscr{M}/\hbar\mathscr{M}$ is a faithfully flat $\mathscr{A}_0$-module, then $\mathscr{M}$ is a faithfully flat $\mathscr{A}$-module.

Theorem 1.13. *Let $d \in \mathbb{N}$. Assume that $\mathscr{A}_0$ is d-syzygic,* i.e., *that any coherent $\mathscr{A}_0$-module locally admits a projective resolution of length $\leq d$ by free $\mathscr{A}_0$-modules of finite rank. Then*

(a) *$\mathscr{A}$ is $(d+1)$-syzygic.*

(b) *Let $\mathscr{M}^\bullet$ be a complex of $\mathscr{A}$-modules concentrated in degrees $[a,b]$ and with coherent cohomology groups. Then, locally there exists a quasi-isomorphism $\mathscr{L}^\bullet \to \mathscr{M}^\bullet$ where $\mathscr{L}^\bullet$ is a complex of free $\mathscr{A}$-modules of finite rank concentrated in degrees $[a-d-1,b]$.*

Proposition 1.14. *Let $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A})$ and let $a \in \mathbb{Z}$. The conditions below are equivalent:*

(i) $H^a(\mathrm{gr}_\hbar(\mathscr{M})) \simeq 0$,

(ii) *$H^a(\mathscr{M}) \simeq 0$ and $H^{a+1}(\mathscr{M})$ has no $\hbar$-torsion.*

Cohomologically $\hbar$-complete sheaves on real manifolds. Let now X be a real analytic manifold. Recall from [9] that the microsupport of $F \in \mathsf{D}^{\mathrm{b}}(\mathbb{Z}_X)$ is a closed involutive subset of the cotangent bundle T^*X denoted by $\mathrm{SS}(F)$. The microsupport is additive on $\mathsf{D}^{\mathrm{b}}(\mathbb{Z}_X)$ (cf Definition 3.3 (ii) below). Considering the distinguished triangle $F \xrightarrow{\hbar} F \to \mathrm{gr}_\hbar F \xrightarrow{+1}$, one gets the estimate

$$\mathrm{SS}(\mathrm{gr}_\hbar(F)) \subset \mathrm{SS}(F). \tag{1.3}$$

Proposition 1.15. *Let $F \in \mathsf{D}^{\mathrm{b}}(\mathbb{Z}_X[\hbar])$ and assume that F is cohomologically $\hbar$-complete. Then*

$$\mathrm{SS}(F) = \mathrm{SS}(\mathrm{gr}_\hbar(F)). \tag{1.4}$$

Proof. It is enough to show that $\mathrm{SS}(F) \subset \mathrm{SS}(\mathrm{gr}_\hbar(F))$. For $V \subset U$ open subsets, consider the distinguished triangle

$$\mathrm{R}\Gamma(U;F) \to \mathrm{R}\Gamma(V;F) \to G \xrightarrow{+1}.$$

By Proposition 1.6, $\mathrm{R}\Gamma(U;F)$ and $\mathrm{R}\Gamma(U;F)$ are cohomologically $\hbar$-complete, and thus so is G. One has the distinguished triangle

$$\mathrm{R}\Gamma(U;\mathrm{gr}_\hbar F) \to \mathrm{R}\Gamma(V;\mathrm{gr}_\hbar F) \to \mathrm{gr}_\hbar G \xrightarrow{+1} .$$

By the definition of microsupport, it is enough to prove that $\mathrm{gr}_\hbar G = 0$ implies $G = 0$. This follows from Proposition 1.4. □

For $\mathbb{K}$ a commutative unital Noetherian ring, one denotes by $\mathrm{Mod}_{\mathbb{R}\text{-c}}(\mathbb{K}_X)$ the full subcategory of $\mathrm{Mod}(\mathbb{K}_X)$ consisting of $\mathbb{R}$-constructible sheaves and by $\mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{K}_X)$ the full triangulated subcategory of $\mathsf{D}^{\mathrm{b}}(\mathbb{K}_X)$ consisting of objects with $\mathbb{R}$-constructible cohomology (see [9, §8.4]). In this paper, we shall mainly be interested with the case where $\mathbb{K}$ is either $\mathbb{C}$ or the ring of formal power series in an indeterminate $\hbar$, that we denote by

$$\mathbb{C}^\hbar := \mathbb{C}[[\hbar]].$$

Proposition 1.16. *Let $F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathbb{C}^\hbar_X)$. Then F is cohomologically $\hbar$-complete.*

Proof. This follows from Proposition 1.2 since for any $x \in X$ one has $\mathrm{R}\Gamma(U;F) \xrightarrow{\sim} F_x$ for U in a fundamental system of neighborhoods of x. □

Corollary 1.17. *The functor $\mathrm{gr}_\hbar\colon \mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathbb{C}^\hbar_X) \to \mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathbb{C}_X)$ is conservative.*

Corollary 1.18. *For $F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathbb{C}^\hbar_X)$, one has the equality*

$$\mathrm{SS}(\mathrm{gr}_\hbar(F)) = \mathrm{SS}(F).$$

Proposition 1.19. *For $F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathbb{C}^\hbar_X)$ and $i \in \mathbb{Z}$ one has $\mathrm{supp}\, H^i(F) \subset \mathrm{supp}\, H^i(\mathrm{gr}_\hbar F)$. In particular if $H^i(\mathrm{gr}_\hbar F) = 0$ then $H^i(F) = 0$.*

Proof. We apply Proposition 1.14 to F_x for any $x \in X$. □

§2. Formal extension

Let X be a topological space, or more generally a site, and let $\mathscr{R}_0$ be a sheaf of rings on X. In this section, we let

$$\mathscr{R} := \mathscr{R}_0[[\hbar]] = \prod_{n\geq 0} \mathscr{R}_0 \hbar^n$$

be the formal extension of $\mathscr{R}_0$, whose sections on an open subset U are formal series $r = \sum_{n=0}^{\infty} r_n \hbar^n$, with $r_n \in \Gamma(U;\mathscr{R}_0)$. Consider the associated functor

$$
\begin{aligned}
(2.1) \qquad & (\,\bullet\,)^{\hbar}\colon \mathrm{Mod}(\mathscr{R}_0) \to \mathrm{Mod}(\mathscr{R}), \\
& \mathscr{N} \mapsto \mathscr{N}[[\hbar]] = \varprojlim_n (\mathscr{R}_n \otimes_{\mathscr{R}_0} \mathscr{N}),
\end{aligned}
$$

where $\mathscr{R}_n := \mathscr{R}/\hbar^{n+1}\mathscr{R}$ is regarded as an $(\mathscr{R},\mathscr{R}_0)$-bimodule. Since $\mathscr{R}_n$ is free of finite rank over $\mathscr{R}_0$, the functor $(\,\bullet\,)^{\hbar}$ is left exact. We denote by $(\,\bullet\,)^{\mathrm{R}\hbar}$ its right derived functor.

Proposition 2.1. *For $\mathscr{N} \in \mathsf{D}^{\mathrm{b}}(\mathscr{R}_0)$ one has*

$$\mathscr{N}^{\mathrm{R}\hbar} \simeq \mathrm{R}\mathscr{H}om_{\mathscr{R}_0}(\mathscr{R}^{\mathrm{loc}}/\hbar\mathscr{R}, \mathscr{N}),$$

where $\mathscr{R}^{\mathrm{loc}}/\hbar\mathscr{R}$ is regarded as an $(\mathscr{R}_0,\mathscr{R})$-bimodule.

Proof. It is enough to prove that for $\mathscr{N} \in \mathrm{Mod}(\mathscr{R}_0)$ one has

$$\mathscr{N}^{\hbar} \simeq \mathscr{H}om_{\mathscr{R}_0}(\mathscr{R}^{\mathrm{loc}}/\hbar\mathscr{R}, \mathscr{N}).$$

Using the right $\mathscr{R}_0$-module structure of $\mathscr{R}_n$, set $\mathscr{R}_n^* = \mathscr{H}om_{\mathscr{R}_0}(\mathscr{R}_n,\mathscr{R}_0)$. Then $\mathscr{R}_n^*$ is an $(\mathscr{R}_0,\mathscr{R})$-bimodule, and

$$\mathscr{N}^{\hbar} = \varprojlim_n (\mathscr{R}_n \otimes_{\mathscr{R}_0} \mathscr{N}) \simeq \mathscr{H}om_{\mathscr{R}_0}(\varinjlim_n \mathscr{R}_n^*, \mathscr{N}).$$

Since

$$\mathscr{R}^{\mathrm{loc}}/\hbar\mathscr{R} \simeq \varinjlim_n (\hbar^{-n}\mathscr{R}/\hbar\mathscr{R}),$$

it is enough to prove that there is an isomorphism of $(\mathscr{R}_0,\mathscr{R})$-bimodules

$$\mathscr{H}om_{\mathscr{R}_0}(\mathscr{R}_n,\mathscr{R}_0) \simeq \hbar^{-n}\mathscr{R}/\hbar\mathscr{R}.$$

Recalling that $\mathscr{R}_n = \mathscr{R}/\hbar^{n+1}\mathscr{R}$, this follows from the pairing

$$(\mathscr{R}/\hbar^{n+1}\mathscr{R}) \otimes_{\mathscr{R}_0} (\hbar^{-n}\mathscr{R}/\hbar\mathscr{R}) \to \mathscr{R}_0, \quad f \otimes g \mapsto \mathrm{Res}_{\hbar=0}(fg\,d\hbar/\hbar).$$

□

Note that the isomorphism of $(\mathscr{R},\mathscr{R}_0)$-bimodules

$$\mathscr{R} \simeq (\mathscr{R}_0)^{\hbar} = \mathscr{H}om_{\mathscr{R}_0}(\mathscr{R}^{\mathrm{loc}}/\hbar\mathscr{R}, \mathscr{R}_0)$$

induces a natural morphism

$$(2.2) \qquad \mathscr{R} \overset{\mathrm{L}}{\otimes}_{\mathscr{R}_0} \mathscr{N} \to \mathscr{N}^{\mathrm{R}\hbar}, \quad \text{for } \mathscr{N} \in \mathsf{D}^{\mathrm{b}}(\mathscr{R}_0).$$

Proposition 2.2. *For $\mathscr{N} \in \mathsf{D}^{\mathrm{b}}(\mathscr{R}_0)$, its formal extension $\mathscr{N}^{\mathrm{R}\hbar}$ is cohomologically $\hbar$-complete.*

Proof. The statement follows from $(\mathscr{R}^{\mathrm{loc}}/\hbar\mathscr{R}) \overset{\mathrm{L}}{\otimes}_{\mathscr{R}} \mathscr{R}^{\mathrm{loc}} \simeq 0$ and from the isomorphism

$$\mathrm{R}\mathscr{H}om_{\mathscr{R}}(\mathscr{R}^{\mathrm{loc}}, \mathscr{N}^{\mathrm{R}\hbar}) \simeq \mathrm{R}\mathscr{H}om_{\mathscr{R}_0}((\mathscr{R}^{\mathrm{loc}}/\hbar\mathscr{R}) \overset{\mathrm{L}}{\otimes}_{\mathscr{R}} \mathscr{R}^{\mathrm{loc}}, \mathscr{N}).$$

□

Lemma 2.3. *Assume that $\mathscr{R}_0$ is an $\mathscr{S}_0$-algebra, for $\mathscr{S}_0$ a commutative sheaf of rings, and let $\mathscr{S} = \mathscr{S}_0[[\hbar]]$. For $\mathscr{M}, \mathscr{N} \in \mathsf{D}^{\mathrm{b}}(\mathscr{R}_0)$ we have an isomorphism in $\mathsf{D}^{\mathrm{b}}(\mathscr{S})$*

$$\mathrm{R}\mathscr{H}om_{\mathscr{R}_0}(\mathscr{M}, \mathscr{N})^{\mathrm{R}\hbar} \simeq \mathrm{R}\mathscr{H}om_{\mathscr{R}_0}(\mathscr{M}, \mathscr{N}^{\mathrm{R}\hbar}).$$

Proof. Note the isomorphisms

$$\mathscr{R}^{\mathrm{loc}}/\hbar\mathscr{R} \simeq \mathscr{R}_0 \otimes_{\mathscr{S}_0} (\mathscr{S}^{\mathrm{loc}}/\hbar\mathscr{S}) \simeq \mathscr{R}_0 \overset{\mathrm{L}}{\otimes}_{\mathscr{S}_0} (\mathscr{S}^{\mathrm{loc}}/\hbar\mathscr{S})$$

as $(\mathscr{R}_0, \mathscr{S})$-bimodules. Then one has

$$\begin{aligned}
\mathrm{R}\mathscr{H}om_{\mathscr{R}_0}(\mathscr{M}, \mathscr{N})^{\mathrm{R}\hbar} &= \mathrm{R}\mathscr{H}om_{\mathscr{S}_0}(\mathscr{S}^{\mathrm{loc}}/\hbar\mathscr{S}, \mathrm{R}\mathscr{H}om_{\mathscr{R}_0}(\mathscr{M}, \mathscr{N})) \\
&\simeq \mathrm{R}\mathscr{H}om_{\mathscr{R}_0}(\mathscr{M}, \mathrm{R}\mathscr{H}om_{\mathscr{S}_0}(\mathscr{S}^{\mathrm{loc}}/\hbar\mathscr{S}, \mathscr{N})) \\
&\simeq \mathrm{R}\mathscr{H}om_{\mathscr{R}_0}(\mathscr{M}, \mathrm{R}\mathscr{H}om_{\mathscr{R}_0}(\mathscr{R}^{\mathrm{loc}}/\hbar\mathscr{R}, \mathscr{N})) \\
&= \mathrm{R}\mathscr{H}om_{\mathscr{R}_0}(\mathscr{M}, \mathscr{N}^{\mathrm{R}\hbar}).
\end{aligned}$$

□

Lemma 2.4. *Let $f\colon X \to Y$ be a morphism of sites, and assume that $(f^{-1}\mathscr{R}_0)^{\hbar} \simeq f^{-1}\mathscr{R}$. Then the functors $\mathrm{R}f_*$ and $(\bullet)^{\mathrm{R}\hbar}$ commute, that is, for $\mathscr{P} \in \mathsf{D}^{\mathrm{b}}(f^{-1}\mathscr{R}_0)$ we have $(\mathrm{R}f_*\mathscr{P})^{\mathrm{R}\hbar} \simeq \mathrm{R}f_*(\mathscr{P}^{\mathrm{R}\hbar})$ in $\mathsf{D}^{\mathrm{b}}(\mathscr{R})$.*

Proof. One has the isomorphism

$$\begin{aligned}
\mathrm{R}f_*(\mathscr{P}^{\mathrm{R}\hbar}) &= \mathrm{R}f_*\mathrm{R}\mathscr{H}om_{f^{-1}\mathscr{R}_0}(f^{-1}(\mathscr{R}^{\mathrm{loc}}/\hbar\mathscr{R}), \mathscr{P}) \\
&\simeq \mathrm{R}\mathscr{H}om_{\mathscr{R}_0}(\mathscr{R}^{\mathrm{loc}}/\hbar\mathscr{R}, \mathrm{R}f_*\mathscr{P}) \\
&= \mathrm{R}f_*(\mathscr{P})^{\mathrm{R}\hbar}.
\end{aligned}$$

□

Proposition 2.5. *Let $\mathscr{T}$ be either a basis of open subsets of the site X or, assuming that X is a locally compact topological space, a basis of compact subsets. Denote by $J_{\mathscr{T}}$ the full subcategory of* $\mathrm{Mod}(\mathscr{R}_0)$ *consisting of $\mathscr{T}$-acyclic objects,* i.e., *sheaves $\mathscr{N}$ for which $H^k(S;\mathscr{N}) = 0$ for all $k > 0$ and all $S \in \mathscr{T}$. Then $J_{\mathscr{T}}$ is injective with respect to the functor $(\,\bullet\,)^\hbar$. In particular, for $\mathscr{N} \in J_{\mathscr{T}}$, we have $\mathscr{N}^\hbar \simeq \mathscr{N}^{\mathrm{R}\hbar}$.*

Proof. (i) Since injective sheaves are $\mathscr{T}$-acyclic, $J_{\mathscr{T}}$ is cogenerating.
(ii) Consider an exact sequence $0 \to \mathscr{N}' \to \mathscr{N} \to \mathscr{N}'' \to 0$ in $\mathrm{Mod}(\mathscr{R}_0)$. Clearly, if both $\mathscr{N}'$ and $\mathscr{N}$ belong to $J_{\mathscr{T}}$, then so does $\mathscr{N}''$.
(iii) Consider an exact sequence as in (ii) and assume that $\mathscr{N}' \in J_{\mathscr{T}}$. We have to prove that $0 \to \mathscr{N}'^{,\hbar} \to \mathscr{N}^\hbar \to \mathscr{N}''^{,\hbar} \to 0$ is exact. Since $(\,\bullet\,)^\hbar$ is left exact, it is enough to prove that $\mathscr{N}^\hbar \to \mathscr{N}''^{,\hbar}$ is surjective. Noticing that $\mathscr{N}^\hbar \simeq \prod_{\mathbb{N}} \mathscr{N}$ as $\mathscr{R}_0$-modules, it is enough to prove that $\prod_{\mathbb{N}} \mathscr{N} \to \prod_{\mathbb{N}} \mathscr{N}''$ is surjective.
(iii)-(a) Assume that $\mathscr{T}$ is a basis of open subsets. Any open subset $U \subset X$ has a cover $\{U_i\}_{i\in I}$ by elements $U_i \in \mathscr{T}$. For any $i \in I$, the morphism $\mathscr{N}(U_i) \to \mathscr{N}''(U_i)$ is surjective. The result follows taking the product over $\mathbb{N}$.
(iii)-(b) Assume that $\mathscr{T}$ is a basis of compact subsets. For any $K \in \mathscr{T}$, the morphism $\mathscr{N}(K) \to \mathscr{N}''(K)$ is surjective. Hence, there exists a basis $\mathscr{V}$ of open subsets such that for any $x \in X$ and any $V \ni x$ in $\mathscr{V}$, there exists $V' \in \mathscr{V}$ with $x \in V' \subset V$ and the image of $\mathscr{N}(V') \to \mathscr{N}''(V')$ contains the image of $\mathscr{N}''(V)$ in $\mathscr{N}''(V')$. The result follows as in (iii)-(a) taking the product over $\mathbb{N}$. ☐

Corollary 2.6. *The following sheaves are acyclic for the functor $(\,\bullet\,)^\hbar$:*

(i) $\mathbb{R}$-*constructible sheaves of* $\mathbb{C}$-*vector spaces on a real analytic manifold X,*

(ii) *coherent modules over the ring $\mathscr{O}_X$ of holomorphic functions on a complex analytic manifold X,*

(iii) *coherent modules over the ring $\mathscr{D}_X$ of linear differential operators on a complex analytic manifold X.*

Proof. The statements follow by applying Proposition 2.5 for the following choices of $\mathscr{T}$.
(i) Let F be an $\mathbb{R}$-constructible sheaf. Then for any $x \in X$ one has $F_x \xleftarrow{\sim} \mathrm{R}\Gamma(U_x;F)$ for U_x in a fundamental system of open neighborhoods of x. Take for $\mathscr{T}$ the union of these fundamental systems.

(ii) Take for $\mathscr{T}$ the family of open Stein subsets.
(iii) Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module. The problem being local, we may assume that $\mathscr{M}$ is endowed with a good filtration. Then take for $\mathscr{T}$ the family of compact Stein subsets. □

Example 2.7. *Let $X = \mathbb{R}$, $\mathscr{R}_0 = \mathbb{C}_X$, $Z = \{1/n\colon n = 1, 2, \dots\} \cup \{0\}$ and $U = X \setminus Z$. One has the isomorphisms $(\mathbb{C}^\hbar)_X \simeq (\mathbb{C}_X)^\hbar \simeq (\mathbb{C}_X)^{\mathrm{R}\hbar}$ and $(\mathbb{C}^\hbar)_U \simeq (\mathbb{C}_U)^\hbar$. Considering the exact sequences*

$$\begin{aligned}&0 \to (\mathbb{C}^\hbar)_U \to (\mathbb{C}^\hbar)_X \to (\mathbb{C}^\hbar)_Z \to 0,\\ &0 \to (\mathbb{C}_U)^\hbar \to (\mathbb{C}_X)^\hbar \to (\mathbb{C}_Z)^\hbar \to H^1(\mathbb{C}_U)^{\mathrm{R}\hbar} \to 0,\end{aligned}$$

we get $H^1(\mathbb{C}_U)^{\mathrm{R}\hbar} \simeq (\mathbb{C}_Z)^\hbar/(\mathbb{C}^\hbar)_Z$, whose stalk at the origin does not vanish. Hence $\mathbb{C}_U$ is not acyclic for the functor $(\,\bullet\,)^\hbar$.

Assume now that

$$\mathscr{A}_0 = \mathscr{R}_0 \quad \text{and} \quad \mathscr{A} = \mathscr{R}_0[[\hbar]]$$

satisfy either Assumption 1.7 or Assumption 1.8 (where condition (i) is clear) and that $\mathscr{A}_0$ is syzygic. Note that by Proposition 2.5 one has $\mathscr{A} \simeq (\mathscr{A}_0)^{\mathrm{R}\hbar}$.

Proposition 2.8. *For $\mathscr{N} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_0)$:*

(i) there is an isomorphism $\mathscr{N}^{\mathrm{R}\hbar} \xrightarrow{\sim} \mathscr{A} \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_0} \mathscr{N}$ induced by (2.2)

(ii) there is an isomorphism $\mathrm{gr}_\hbar(\mathscr{N}^{\mathrm{R}\hbar}) \simeq \mathscr{N}$.

Proof. Since $\mathscr{A}_0$ is syzygic, we may locally represent $\mathscr{N}$ by a bounded complex $\mathscr{L}^\bullet$ of free $\mathscr{A}_0$-modules of finite rank. Then (i) is obvious. As for (ii), both complexes are isomorphic to the mapping cone of $\hbar\colon (\mathscr{L}^\bullet)^\hbar \to (\mathscr{L}^\bullet)^\hbar$. □

In particular, the functor $(\,\bullet\,)^\hbar$ is exact on $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_0)$ and preserves coherence. One thus gets a functor

$$(\,\bullet\,)^{\mathrm{R}\hbar}\colon \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_0) \to \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}).$$

The subanalytic site. The subanalytic site associated to an analytic manifold X has been introduced and studied in [11, Chapter 7] (see also [15] for a detailed and systematic study as well as for complementary results). Denote by Op_X the category of open subsets of X, the morphisms being the inclusion morphisms, and by $\mathrm{Op}_{X_{\mathrm{sa}}}$ the full subcategory consisting of relatively compact subanalytic open subsets of X. The site X_{sa} is the presite $\mathrm{Op}_{X_{\mathrm{sa}}}$ endowed with the Grothendieck topology for which the coverings are those admitting a finite subcover. One calls X_{sa} the subanalytic site associated to X. Denote by $\rho\colon X \to X_{\mathrm{sa}}$ the natural morphism of sites. Recall that the inverse image functor ρ^{-1}, besides the usual right adjoint given by the direct image functor ρ_*, admits a left adjoint denoted $\rho_!$. Consider the diagram

$$\begin{array}{ccc} \mathsf{D}^{\mathrm{b}}(\mathbb{C}_X) & \underset{\rho^{-1}}{\overset{\mathrm{R}\rho_*}{\rightleftarrows}} & \mathsf{D}^{\mathrm{b}}(\mathbb{C}_{X_{\mathrm{sa}}}) \\ \downarrow{\scriptstyle (\,\bullet\,)^{\mathrm{R}\hbar}} & & {\scriptstyle (\,\bullet\,)^{\mathrm{R}\hbar}}\downarrow \\ \mathsf{D}^{\mathrm{b}}(\mathbb{C}^{\hbar}_X) & \underset{\rho^{-1}}{\overset{\mathrm{R}\rho_*}{\rightleftarrows}} & \mathsf{D}^{\mathrm{b}}(\mathbb{C}^{\hbar}_{X_{\mathrm{sa}}}). \end{array}$$

Lemma 2.9.

(i) *The functors* ρ^{-1} *and* $(\,\bullet\,)^{\mathrm{R}\hbar}$ *commute, that is, for* $G \in \mathsf{D}^{\mathrm{b}}(\mathbb{C}_{X_{\mathrm{sa}}})$ *we have* $(\rho^{-1}G)^{\mathrm{R}\hbar} \simeq \rho^{-1}(G^{\mathrm{R}\hbar})$ *in* $\mathsf{D}^{\mathrm{b}}(\mathbb{C}^{\hbar}_X)$.

(ii) *The functors* $\mathrm{R}\rho_*$ *and* $(\,\bullet\,)^{\mathrm{R}\hbar}$ *commute, that is, for* $F \in \mathsf{D}^{\mathrm{b}}(\mathbb{C}_X)$ *we have* $(\mathrm{R}\rho_* F)^{\mathrm{R}\hbar} \simeq \mathrm{R}\rho_*(F^{\mathrm{R}\hbar})$ *in* $\mathsf{D}^{\mathrm{b}}(\mathbb{C}^{\hbar}_{X_{\mathrm{sa}}})$.

Proof. (i) Since it admits a left adjoint, the functor ρ^{-1} commutes with projective limits. It follows that for $G \in \mathrm{Mod}(\mathbb{C}_{X_{\mathrm{sa}}})$ one has an isomorphism

$$\rho^{-1}(G^{\hbar}) \to (\rho^{-1}G)^{\hbar}.$$

To conclude, it remains to show that $(\rho^{-1}(\,\bullet\,))^{\mathrm{R}\hbar}$ is the derived functor of $(\rho^{-1}(\,\bullet\,))^{\hbar}$. Recall that an object G of $\mathrm{Mod}(\mathbb{C}_{X_{\mathrm{sa}}})$ is quasi-injective if the functor $\mathrm{Hom}_{\mathbb{C}_{X_{\mathrm{sa}}}}(\,\bullet\,, G)$ is exact on the category $\mathrm{Mod}_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$. By a result of [15], if $G \in \mathrm{Mod}(\mathbb{C}_{X_{\mathrm{sa}}})$ is quasi-injective, then $\rho^{-1}G$ is soft. Hence, $\rho^{-1}G$ is injective for the functor $(\,\bullet\,)^{\hbar}$ by Proposition 2.5.

(ii) By (i) we can apply Lemma 2.4. □

§3. $\mathscr{D}[[\hbar]]$-modules and propagation

Let now X be a complex analytic manifold of complex dimension d_X. As usual, denote by $\mathbb{C}_X$ the constant sheaf with stalk $\mathbb{C}$, by $\mathscr{O}_X$ the structure sheaf and by $\mathscr{D}_X$ the ring of linear differential operators on X. We will use the notations

$$
\begin{aligned}
\mathrm{D}'\colon \mathsf{D}^{\mathrm{b}}(\mathbb{C}_X)^{\mathrm{op}} \to \mathsf{D}^{\mathrm{b}}(\mathbb{C}_X), \quad & F \mapsto \mathrm{R}\mathscr{H}om_{\mathbb{C}_X}(F, \mathbb{C}_X),\\
\mathbb{D}\colon \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)^{\mathrm{op}} \to \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X), \quad & \mathscr{M} \mapsto \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}_X \otimes_{\mathscr{O}_X} \Omega_X^{\otimes -1})\,[d_X],\\
\mathrm{Sol}\colon \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)^{\mathrm{op}} \to \mathsf{D}^{\mathrm{b}}(\mathbb{C}_X), \quad & \mathscr{M} \mapsto \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X),\\
\mathrm{DR}\colon \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X) \to \mathsf{D}^{\mathrm{b}}(\mathbb{C}_X), \quad & \mathscr{M} \mapsto \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{O}_X, \mathscr{M}),
\end{aligned}
$$

where Ω_X denotes the line bundle of holomorphic forms of maximal degree and $\Omega_X^{\otimes -1}$ the dual bundle.

As shown in Corollary 2.6, the sheaves $\mathbb{C}_X$, $\mathscr{O}_X$ and $\mathscr{D}_X$ are all acyclic for the functor $(\,\bullet\,)^{\hbar}$. We will be interested in the formal extensions

$$
\mathbb{C}_X^{\hbar} = \mathbb{C}_X[[\hbar]], \quad \mathscr{O}_X^{\hbar} = \mathscr{O}_X[[\hbar]], \quad \mathscr{D}_X^{\hbar} = \mathscr{D}_X[[\hbar]].
$$

In the sequel, we shall treat left $\mathscr{D}_X^{\hbar}$-modules, but all results apply to right modules since the categories $\mathrm{Mod}(\mathscr{D}_X^{\hbar})$ and $\mathrm{Mod}(\mathscr{D}_X^{\hbar,\mathrm{op}})$ are equivalent.

Proposition 3.1. *Assumption 1.7 is satisfied by the $\mathbb{C}^{\hbar}$-algebras $\mathscr{D}_X^{\hbar}$ and $\mathscr{D}_X^{\hbar,\mathrm{op}}$.*

Proof. Assumption 1.7 holds for $\mathscr{A} = \mathscr{D}_X^{\hbar}$, $\mathscr{A}_0 = \mathscr{D}_X$, $\mathrm{Mod}_{\mathrm{good}}(\mathscr{A}_0|_U)$ the category of good $\mathscr{D}_U$-modules (see [7]) and for $\mathfrak{B}$ the family of Stein compact subsets of X. □

In particular, by Theorem 1.9 one has that $\mathscr{D}_X^{\hbar}$ is right and left Noetherian (and thus coherent). Moreover, by Theorem 1.13 any object of $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X^{\hbar})$ can be locally represented by a bounded complex of free $\mathscr{D}_X^{\hbar}$-modules of finite rank.

We will use the notations

$$
\begin{aligned}
\mathrm{D}'_{\hbar}\colon \mathsf{D}^{\mathrm{b}}(\mathbb{C}_X^{\hbar})^{\mathrm{op}} \to \mathsf{D}^{\mathrm{b}}(\mathbb{C}_X^{\hbar}), \quad & F \mapsto \mathrm{R}\mathscr{H}om_{\mathbb{C}_X^{\hbar}}(F, \mathbb{C}_X^{\hbar}),\\
\mathbb{D}_{\hbar}\colon \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X^{\hbar})^{\mathrm{op}} \to \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X^{\hbar}), \quad & \mathscr{M} \mapsto \mathrm{R}\mathscr{H}om_{\mathscr{D}_X^{\hbar}}(\mathscr{M}, \mathscr{D}_X^{\hbar} \otimes_{\mathscr{O}_X} \Omega_X^{\otimes -1})\,[d_X],\\
\mathrm{Sol}_{\hbar}\colon \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X^{\hbar})^{\mathrm{op}} \to \mathsf{D}^{\mathrm{b}}(\mathbb{C}^{\hbar}), \quad & \mathscr{M} \mapsto \mathrm{R}\mathscr{H}om_{\mathscr{D}_X^{\hbar}}(\mathscr{M}, \mathscr{O}_X^{\hbar}),\\
\mathrm{DR}_{\hbar}\colon \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X^{\hbar}) \to \mathsf{D}^{\mathrm{b}}(\mathbb{C}^{\hbar}), \quad & \mathscr{M} \mapsto \mathrm{R}\mathscr{H}om_{\mathscr{D}_X^{\hbar}}(\mathscr{O}_X^{\hbar}, \mathscr{M}).
\end{aligned}
$$

By Proposition 2.8 and Lemma 2.3, for $\mathscr{N} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$ one has

$$
\mathscr{N}^{\mathrm{R}\hbar} \simeq \mathscr{D}_X^{\hbar} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{N}, \tag{3.1}
$$

$$
\mathrm{gr}_{\hbar}(\mathscr{N}^{\mathrm{R}\hbar}) \simeq \mathscr{N}, \tag{3.2}
$$

$$
\mathrm{Sol}_{\hbar}(\mathscr{N}^{\mathrm{R}\hbar}) \simeq \mathrm{Sol}(\mathscr{N})^{\mathrm{R}\hbar}. \tag{3.3}
$$

Definition 3.2. *For $\mathscr{M} \in \mathrm{Mod}(\mathscr{D}_X^\hbar)$, denote by $\mathscr{M}_{\hbar\text{-}tor}$ its submodule consisting of sections locally annihilated by some power of $\hbar$ and set $\mathscr{M}_{\hbar\text{-}tf} = \mathscr{M}/\mathscr{M}_{\hbar\text{-}tor}$. We say that $\mathscr{M} \in \mathrm{Mod}(\mathscr{D}_X^\hbar)$ is an $\hbar$-torsion module if $\mathscr{M}_{\hbar\text{-}tor} \xrightarrow{\sim} \mathscr{M}$ and that $\mathscr{M}$ has no $\hbar$-torsion (or is $\hbar$-torsion free) if $\mathscr{M} \xrightarrow{\sim} \mathscr{M}_{\hbar\text{-}tf}$.*

Denote by ${}_n\mathscr{M}$ the kernel of $\hbar^{n+1}\colon \mathscr{M} \to \mathscr{M}$. Then $\mathscr{M}_{\hbar\text{-tor}}$ is the sheaf associated with the increasing union of the ${}_n\mathscr{M}$'s. Hence, if $\mathscr{M}$ is coherent, the increasing family $\{{}_n\mathscr{M}\}_n$ is locally stationary and $\mathscr{M}_{\hbar\text{-tor}}$ as well as $\mathscr{M}_{\hbar\text{-tf}}$ are coherent.

Characteristic variety. Recall the following definition

Definition 3.3. *(i) For $\mathscr{C}$ an abelian category, a function $c\colon \mathrm{Ob}(\mathscr{C}) \to \mathrm{Set}$ is called additive if $c(M) = c(M') \cup c(M'')$ for any short exact sequence $0 \to M' \to M \to M'' \to 0$.*
(ii) For $\mathscr{T}$ a triangulated category, a function $c\colon \mathrm{Ob}(\mathscr{T}) \to \mathrm{Set}$ is called additive if $c(M) = c(M[1])$ and $c(M) \subset c(M') \cup c(M'')$ for any distinguished triangle $M' \to M \to M'' \xrightarrow{+1}$.

Note that an additive function c on $\mathscr{C}$ naturally extends to the derived category $\mathsf{D}(\mathscr{C})$ by setting $c(M) = \bigcup_i c(H^i(M))$.

For $\mathscr{N}$ a coherent $\mathscr{D}_X$-module, denote by $\mathrm{char}(\mathscr{N})$ its characteristic variety, a closed involutive subvariety of the cotangent bundle T^*X. The characteristic variety is additive on $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X)$. For $\mathscr{N} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$ one sets $\mathrm{char}(\mathscr{N}) = \bigcup_i \mathrm{char}(H^i(\mathscr{N}))$.

Definition 3.4. *The characteristic variety of $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X^\hbar)$ is defined by*

$$\mathrm{char}_\hbar(\mathscr{M}) = \mathrm{char}(\mathrm{gr}_\hbar(\mathscr{M})).$$

To $\mathscr{M} \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X^\hbar)$ one associates the coherent $\mathscr{D}_X$-modules

$$ {}_0\mathscr{M} = \mathrm{Ker}(\hbar\colon \mathscr{M} \to \mathscr{M}) = H^{-1}(\mathrm{gr}_\hbar \mathscr{M}), \tag{3.4}$$
$$ \mathscr{M}_0 = \mathrm{Coker}(\hbar\colon \mathscr{M} \to \mathscr{M}) = H^{0}(\mathrm{gr}_\hbar \mathscr{M}). \tag{3.5}$$

Lemma 3.5. *For $\mathscr{M} \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X^\hbar)$ an $\hbar$-torsion module, one has*

$$\mathrm{char}_\hbar(\mathscr{M}) = \mathrm{char}(\mathscr{M}_0) = \mathrm{char}({}_0\mathscr{M}).$$

Proof. By definition, $\mathrm{char}_\hbar(\mathscr{M}) = \mathrm{char}(\mathscr{M}_0) \cup \mathrm{char}({}_0\mathscr{M})$. It is thus enough to prove the equality $\mathrm{char}(\mathscr{M}_0) = \mathrm{char}({}_0\mathscr{M})$.

Since the statement is local we may assume that $\hbar^N \mathscr{M} = 0$ for some $N \in \mathbb{N}$. We proceed by induction on N.

For $N = 1$ we have $\mathscr{M} \simeq \mathscr{M}_0 \simeq {}_0\mathscr{M}$, and the statement is obvious.

Assume that the statement has been proved for $N - 1$. The short exact sequence

$$0 \to \hbar\mathscr{M} \to \mathscr{M} \to \mathscr{M}_0 \to 0 \tag{3.6}$$

induces the distinguished triangle

$$\mathrm{gr}_\hbar \hbar\mathscr{M} \to \mathrm{gr}_\hbar \mathscr{M} \to \mathrm{gr}_\hbar \mathscr{M}_0 \xrightarrow{+1} .$$

Noticing that $\mathscr{M}_0 \simeq (\mathscr{M}_0)_0 \simeq {}_0(\mathscr{M}_0)$, the associated long exact cohomology sequence gives

$$0 \to {}_0(\hbar\mathscr{M}) \to {}_0\mathscr{M} \to \mathscr{M}_0 \to (\hbar\mathscr{M})_0 \to 0.$$

By inductive hypothesis we have $\mathrm{char}({}_0(\hbar\mathscr{M})) = \mathrm{char}((\hbar\mathscr{M})_0)$, and we deduce $\mathrm{char}(\mathscr{M}_0) = \mathrm{char}(\mathscr{M}_0)$ by additivity of char. □

Proposition 3.6. *(i) For $\mathscr{M} \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X^\hbar)$ one has*

$$\mathrm{char}_\hbar(\mathscr{M}) = \mathrm{char}(\mathscr{M}_0).$$

(ii) The characteristic variety $\mathrm{char}_\hbar$ *is additive both on* $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X^\hbar)$ *and on* $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X^\hbar)$.

Proof. (i) As $\mathrm{char}(\mathrm{gr}_\hbar \mathscr{M}) = \mathrm{char}(\mathscr{M}_0) \cup \mathrm{char}({}_0\mathscr{M})$, it is enough to prove the inclusion

$$\mathrm{char}({}_0\mathscr{M}) \subset \mathrm{char}(\mathscr{M}_0). \tag{3.7}$$

Consider the short exact sequence $0 \to \mathscr{M}_{\hbar\text{-tor}} \to \mathscr{M} \to \mathscr{M}_{\hbar\text{-tf}} \to 0$. Since $\mathscr{M}_{\hbar\text{-tf}}$ has no $\hbar$-torsion, ${}_0(\mathscr{M}_{\hbar\text{-tf}}) = 0$. The associated long exact cohomology sequence thus gives

$${}_0(\mathscr{M}_{\hbar\text{-tor}}) \simeq {}_0\mathscr{M}, \quad 0 \to (\mathscr{M}_{\hbar\text{-tor}})_0 \to \mathscr{M}_0 \to (\mathscr{M}_{\hbar\text{-tf}})_0 \to 0.$$

We deduce

$$\mathrm{char}({}_0\mathscr{M}) = \mathrm{char}({}_0(\mathscr{M}_{\hbar\text{-tor}})) = \mathrm{char}((\mathscr{M}_{\hbar\text{-tor}})_0) \subset \mathrm{char}(\mathscr{M}_0),$$

where the second equality follows from Lemma 3.5.

(ii) It is enough to prove the additivity on $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X^\hbar)$, i.e. the equality

$$\mathrm{char}_\hbar(\mathscr{M}) = \mathrm{char}_\hbar(\mathscr{M}') \cup \mathrm{char}_\hbar(\mathscr{M}'')$$

for $0 \to \mathscr{M}' \to \mathscr{M} \to \mathscr{M}'' \to 0$ a short exact sequence of coherent $\mathscr{D}_X^\hbar$-modules.

The associated distinguished triangle $\mathrm{gr}_\hbar \mathscr{M}' \to \mathrm{gr}_\hbar \mathscr{M} \to \mathrm{gr}_\hbar \mathscr{M}'' \xrightarrow{+1}$ induces the long exact cohomology sequence

$${}_0(\mathscr{M}'') \to (\mathscr{M}')_0 \to \mathscr{M}_0 \to (\mathscr{M}'')_0 \to 0.$$

By additivity of $\mathrm{char}(\bullet)$, the exactness of this sequence at the first, second and third term from the right, respectively, gives:

$$\begin{aligned}
\mathrm{char}_\hbar(\mathscr{M}'') &\subset \mathrm{char}_\hbar(\mathscr{M}),\\
\mathrm{char}_\hbar(\mathscr{M}) &\subset \mathrm{char}_\hbar(\mathscr{M}') \cup \mathrm{char}_\hbar(\mathscr{M}''),\\
\mathrm{char}_\hbar(\mathscr{M}') &\subset \mathrm{char}({}_0(\mathscr{M}'')) \cup \mathrm{char}_\hbar(\mathscr{M}).
\end{aligned}$$

Finally, note that $\mathrm{char}({}_0(\mathscr{M}'')) \subset \mathrm{char}_\hbar(\mathscr{M}'') \subset \mathrm{char}_\hbar(\mathscr{M})$. □

In view of Proposition 3.6 (i), in order to define the characteristic variety of a coherent $\mathscr{D}_X^\hbar$-module $\mathscr{M}$ one could avoid derived categories considering $\mathrm{char}(\mathscr{M}_0)$ instead of $\mathrm{char}(\mathrm{gr}_\hbar \mathscr{M})$. The next lemma shows that these definitions are still compatible for $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X^\hbar)$.

Lemma 3.7. *For $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X^\hbar)$ one has*

$$\bigcup\nolimits_i \mathrm{char}(H^i(\mathrm{gr}_\hbar \mathscr{M})) = \bigcup\nolimits_i \mathrm{char}((H^i\mathscr{M})_0).$$

Proof. By additivity of char, the short exact sequence

$$0 \to (H^i\mathscr{M})_0 \to H^i(\mathrm{gr}_\hbar \mathscr{M}) \to {}_0(H^{i+1}\mathscr{M}) \to 0 \tag{3.8}$$

from [13, Lemma 1.4.2] induces the estimates

$$\begin{aligned}
\mathrm{char}((H^i\mathscr{M})_0) &\subset \mathrm{char}(H^i(\mathrm{gr}_\hbar \mathscr{M})),\\
\mathrm{char}(H^i(\mathrm{gr}_\hbar \mathscr{M})) &= \mathrm{char}((H^i\mathscr{M})_0) \cup \mathrm{char}({}_0(H^{i+1}\mathscr{M})).
\end{aligned}$$

One concludes by noticing that (3.7) gives

$$\mathrm{char}({}_0(H^{i+1}\mathscr{M})) \subset \mathrm{char}((H^{i+1}\mathscr{M})_0).$$

□

Proposition 3.8. *Let $\mathscr{M} \in \mathrm{Mod}(\mathscr{D}_X^\hbar)$ be an $\hbar$-torsion module. Then $\mathscr{M}$ is coherent as a $\mathscr{D}_X^\hbar$-module if and only if it is coherent as a $\mathscr{D}_X$-module, and in this case one has $\mathrm{char}_\hbar(\mathscr{M}) = \mathrm{char}(\mathscr{M})$.*

Proof. As in the proof of Lemma 3.5 we assume that $\hbar^N \mathscr{M} = 0$ for some $N \in \mathbb{N}$. Since coherence is preserved by extension and since the characteristic varieties of $\mathscr{D}_X^\hbar$-modules and $\mathscr{D}_X$-modules are additive, we can argue by induction on N using the exact sequence (3.6). We are thus reduced to the case $N = 1$, where $\mathscr{M} = \mathscr{M}_0$ and the statement becomes obvious. □

It follows from (3.2) that

Proposition 3.9. *For $\mathscr{N} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$ one has $\mathrm{char}_\hbar(\mathscr{N}^\hbar) = \mathrm{char}(\mathscr{N})$.*

Holonomic modules. Recall that a coherent $\mathscr{D}_X$-module (or an object of the derived category) is called holonomic if its characteristic variety is isotropic. We refer e.g. to [7, Chapter 5] for the notion of regularity.

Definition 3.10. *We say that $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X^\hbar)$ is holonomic, or regular holonomic, if so is $\mathrm{gr}_\hbar(\mathscr{M})$. We denote by $\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X^\hbar)$ the full triangulated subcategory of $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X^\hbar)$ of holonomic objects and by $\mathsf{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathscr{D}_X^\hbar)$ the full triangulated subcategory of regular holonomic objects.*

Note that a coherent $\mathscr{D}_X^\hbar$-module is holonomic if and only if its characteristic variety is isotropic.

Example 3.11. *Let $\mathscr{N}$ be a regular holonomic $\mathscr{D}_X$-module. Then*
(i) $\mathscr{N}$ itself, considered as a $\mathscr{D}_X^\hbar$-module, is regular holonomic, as follows from the isomorphism $\mathrm{gr}_\hbar \mathscr{N} \simeq \mathscr{N} \oplus \mathscr{N}[1]$;
(ii) $\mathscr{N}^\hbar$ is a regular holonomic $\mathscr{D}_X^\hbar$-module, as follows from the isomorphism $\mathrm{gr}_\hbar \mathscr{N}^\hbar \simeq \mathscr{N}$. In particular, $\mathscr{O}_X^\hbar$ is regular holonomic.

Remark 3.12. *We denote by $\mathrm{Mod}_{\mathrm{rh}}(\mathscr{D}_X)$ the category of regular holonomic $\mathscr{D}_X$-modules and by $\mathrm{Mod}_{\mathrm{rh}}(\mathscr{D}_X^\hbar)$ the subcategory of $\mathrm{Mod}(\mathscr{D}_X^\hbar)$ of regular holonomic objects in the above sense. The proofs of Lemma 3.5 and Proposition 3.6 adapt to the notion of regular holonomy and give the following results:*

(i) For $\mathscr{M} \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X^\hbar)$ an $\hbar$-torsion module,

$$\mathscr{M} \in \mathrm{Mod}_{\mathrm{rh}}(\mathscr{D}_X^\hbar) \Longleftrightarrow \mathscr{M}_0 \in \mathrm{Mod}_{\mathrm{rh}}(\mathscr{D}_X) \Longleftrightarrow {}_0\mathscr{M} \in \mathrm{Mod}_{\mathrm{rh}}(\mathscr{D}_X).$$

(ii) For $\mathscr{M} \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X^\hbar)$,

$$\mathscr{M} \in \mathrm{Mod}_{\mathrm{rh}}(\mathscr{D}_X^\hbar) \Longleftrightarrow \mathscr{M}_0 \in \mathrm{Mod}_{\mathrm{rh}}(\mathscr{D}_X).$$

In this case, ${}_0\mathscr{M} \in \mathrm{Mod}_{\mathrm{rh}}(\mathscr{D}_X)$.

Now for $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X^\hbar)$ *the exact sequence* (3.8) *shows that, for any* i,

$$H^i(\mathrm{gr}_\hbar \mathscr{M}) \in \mathrm{Mod}_{\mathrm{rh}}(\mathscr{D}_X) \iff (H^i\mathscr{M})_0 \text{ and } {}_0(H^{i+1}\mathscr{M}) \in \mathrm{Mod}_{\mathrm{rh}}(\mathscr{D}_X).$$

By the above we deduce that $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathscr{D}_X^\hbar)$ *if and only if* $(H^i\mathscr{M})_0 \in \mathrm{Mod}_{\mathrm{rh}}(\mathscr{D}_X)$ *for all* i. *This is again equivalent to* $H^i\mathscr{M} \in \mathrm{Mod}_{\mathrm{rh}}(\mathscr{D}_X^\hbar)$ *for all* i.

Propagation. Denote by $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathbb{C}_X^\hbar)$ the full triangulated subcategory of $\mathsf{D}^{\mathrm{b}}(\mathbb{C}_X^\hbar)$ consisting of objects with $\mathbb{C}$-constructible cohomology over the ring $\mathbb{C}^\hbar$.

Theorem 3.13. *Let* $\mathscr{M}, \mathscr{N} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X^\hbar)$. *Then*

$$\mathrm{SS}\big(\mathrm{R}\mathscr{H}om_{\mathscr{D}_X^\hbar}(\mathscr{M},\mathscr{N})\big) = \mathrm{SS}\big(\mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathrm{gr}_\hbar(\mathscr{M}), \mathrm{gr}_\hbar(\mathscr{N}))\big).$$

If moreover $\mathscr{M}$ *and* $\mathscr{N}$ *are holonomic, then* $\mathrm{R}\mathscr{H}om_{\mathscr{D}_X^\hbar}(\mathscr{M},\mathscr{N})$ *is an object of* $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(\mathbb{C}_X^\hbar)$.

Proof. Set $F = \mathrm{R}\mathscr{H}om_{\mathscr{D}_X^\hbar}(\mathscr{M},\mathscr{N})$. By Theorem 1.9 and Proposition 1.5, F is cohomologically $\hbar$-complete. Hence $\mathrm{SS}(F) = \mathrm{SS}(\mathrm{gr}_\hbar(F))$ by Proposition 1.15. If moreover $\mathscr{M}$ and $\mathscr{N}$ are holonomic, then $\mathrm{gr}_\hbar F$ is $\mathbb{C}$-constructible. The equality $\mathrm{SS}(F) = \mathrm{SS}(\mathrm{gr}_\hbar(F))$ implies that F is weakly $\mathbb{C}$-constructible. Moreover, the finiteness of the stalks $\mathrm{gr}_\hbar(F)_x \simeq \mathrm{gr}_\hbar(F_x)$ over $\mathbb{C}$ implies the finiteness of F_x over $\mathbb{C}^\hbar$ by Theorem 1.11 applied with $X = \{\mathrm{pt}\}$ and $\mathscr{A} = \mathbb{C}^\hbar$. □

Applying Theorem 3.13, and [9, Theorem 11.3.3], we get:

Corollary 3.14. *Let* $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X^\hbar)$. *Then*

$$\mathrm{SS}(\mathrm{Sol}_\hbar(\mathscr{M})) = \mathrm{SS}(\mathrm{DR}_\hbar(\mathscr{M})) = \mathrm{char}_\hbar(\mathscr{M}).$$

If moreover $\mathscr{M}$ *is holonomic, then* $\mathrm{Sol}_\hbar(\mathscr{M})$ *and* $\mathrm{DR}_\hbar(\mathscr{M})$ *belong to* $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(\mathbb{C}_X^\hbar)$.

Theorem 3.15. *Let* $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X^\hbar)$. *Then there is a natural isomorphism in* $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(\mathbb{C}_X^\hbar)$

$$\mathrm{Sol}_\hbar(\mathscr{M}) \simeq \mathrm{D}'_\hbar(\mathrm{DR}_\hbar(\mathscr{M})). \tag{3.9}$$

Proof. The natural $\mathbb{C}^\hbar$-linear morphism

$$\mathrm{R}\mathscr{H}om_{\mathscr{D}_X^\hbar}(\mathscr{O}_X^\hbar, \mathscr{M}) \overset{\mathrm{L}}{\otimes}_{\mathbb{C}_X^\hbar} \mathrm{R}\mathscr{H}om_{\mathscr{D}_X^\hbar}(\mathscr{M}, \mathscr{O}_X^\hbar) \\ \to \mathrm{R}\mathscr{H}om_{\mathscr{D}_X^\hbar}(\mathscr{O}_X^\hbar, \mathscr{O}_X^\hbar) \simeq \mathbb{C}_X^\hbar$$

induces the morphism in $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathbb{C}_X^\hbar)$

$$\alpha\colon \mathrm{R}\mathscr{H}om_{\mathscr{D}_X^\hbar}(\mathscr{M}, \mathscr{O}_X^\hbar) \to \mathrm{D}'_\hbar(\mathrm{R}\mathscr{H}om_{\mathscr{D}_X^\hbar}(\mathscr{O}_X^\hbar, \mathscr{M})). \tag{3.10}$$

(Note that, choosing $\mathscr{M} = \mathscr{D}_X^\hbar$, this morphism defines the morphism $\mathscr{O}_X^\hbar \to \mathrm{D}'_\hbar(\Omega_X^\hbar[-d_X])$.) The morphism (3.10) induces an isomorphism

$$\mathrm{gr}_\hbar(\alpha)\colon \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathrm{gr}_\hbar(\mathscr{M}), \mathscr{O}_X) \to \mathrm{D}'(\mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{O}_X, \mathrm{gr}_\hbar(\mathscr{M}))).$$

It is thus an isomorphism by Corollary 1.17. □

§4. Formal extension of tempered functions

Let us start by reviewing after [11, Chapter 7] the construction of the sheaves of tempered distributions and of C^∞-functions with temperate growth on the subanalytic site.

Let X be a real analytic manifold, and U an open subset. One says that a function $f \in \mathscr{C}_X^\infty(U)$ has *polynomial growth* at $p \in X$ if, for a local coordinate system $(x_1, \dots, x_n)$ around p, there exist a sufficiently small compact neighborhood K of p and a positive integer N such that

$$\sup_{x \in K \cap U} \big(\mathrm{dist}(x, K \setminus U)\big)^N |f(x)| < \infty\,.$$

One says that f is *tempered* at p if all its derivatives are of polynomial growth at p. One says that f is tempered if it is tempered at any point of X. One denotes by $\mathscr{C}_X^{\infty,t}(U)$ the $\mathbb{C}$-vector subspace of $\mathscr{C}^\infty(U)$ consisting of tempered functions. It then follows from a theorem of Lojaciewicz that $U \mapsto \mathscr{C}_X^{\infty,t}(U)$ ($U \in \mathrm{Op}_{X_{\mathrm{sa}}}$) is a sheaf on X_{sa}. We denote it by $\mathscr{C}_{X_{\mathrm{sa}}}^{\infty,t}$ or simply $\mathscr{C}_X^{\infty,t}$ if there is no risk of confusion.

Lemma 4.1. *One has $H^j(U; \mathscr{C}_X^{\infty,t}) = 0$ for any $U \in \mathrm{Op}_{X_{\mathrm{sa}}}$ and any $j > 0$.*

This result is well-known (see [10, Chapter 1]), but we recall its proof for the reader's convenience.

Proof. Consider the full subcategory $\mathscr{J}$ of $\mathrm{Mod}(\mathbb{C}_{X_{\mathrm{sa}}})$ whose objects are sheaves F such that for any pair $U, V \in \mathrm{Op}_{X_{\mathrm{sa}}}$, the Mayer-Vietoris sequence

$$0 \to F(U \cup V) \to F(U) \oplus F(V) \to F(U \cap V) \to 0$$

is exact. Let us check that this category is injective with respect to the functor $\Gamma(U; \bullet)$. The only non obvious fact is that if $0 \to F' \to F \to F'' \to 0$ is an exact sequence and that F' belongs to $\mathscr{J}$, then $F(U) \to F''(U)$ is surjective. Let $t \in F''(U)$. There exist a finite covering $U = \bigcup_{i \in I} U_i$ and $s_i \in F(U_i)$ whose image in $F''(U_i)$ is $t|_{U_i}$. Then the proof goes by induction on the cardinal of I using the property of F' and standard arguments. To conclude, note that $\mathscr{C}_X^{\infty,t}$ belongs to $\mathscr{J}$ thanks to Lojaciewicz's result (see [14]). □

Let $\mathscr{D}b_X$ be the sheaf of distributions on X. For $U \in \mathrm{Op}_{X_{\mathrm{sa}}}$, denote by $\mathscr{D}b_X^t(U)$ the space of tempered distributions on U, defined by the exact sequence

$$0 \to \Gamma_{X \setminus U}(X; \mathscr{D}b_X) \to \Gamma(X; \mathscr{D}b_X) \to \mathscr{D}b_X^t(U) \to 0.$$

Again, it follows from a theorem of Lojaciewicz that $U \mapsto \mathscr{D}b^t(U)$ is a sheaf on X_{sa}. We denote it by $\mathscr{D}b^t_{X_{\mathrm{sa}}}$ or simply $\mathscr{D}b_X^t$ if there is no risk of confusion. The sheaf $\mathscr{D}b_X^t$ is quasi-injective, that is, the functor $\mathscr{H}om_{\mathbb{C}_{X_{\mathrm{sa}}}}(\bullet, \mathscr{D}b_X^t)$ is exact in the category $\mathrm{Mod}_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$. Moreover, for $U \in \mathrm{Op}_{X_{\mathrm{sa}}}$, $\mathscr{H}om_{\mathbb{C}_{X_{\mathrm{sa}}}}(\mathbb{C}_U, \mathscr{D}b_X^t)$ is also quasi-injective and $\mathrm{R}\mathscr{H}om_{\mathbb{C}_{X_{\mathrm{sa}}}}(\mathbb{C}_U, \mathscr{D}b_X^t)$ is concentrated in degree 0. Note that the sheaf

$$\Gamma_{[U]}\mathscr{D}b_X := \rho^{-1}\mathscr{H}om_{\mathbb{C}_{X_{\mathrm{sa}}}}(\mathbb{C}_U, \mathscr{D}b_X^t)$$

is a $\mathscr{C}_X^\infty$-module, so that in particular $\mathrm{R}\Gamma(V; \Gamma_{[U]}\mathscr{D}b_X)$ is concentrated in degree 0 for $V \subset X$ an open subset.

Formal extensions. By Proposition 2.5 the sheaves $\mathscr{C}_X^{\infty,t}$, $\mathscr{D}b_X^t$ and $\Gamma_{[U]}\mathscr{D}b_X$ are acyclic for the functor $(\bullet)^\hbar$. We set

$$\mathscr{C}_X^{\infty,t,\hbar} := (\mathscr{C}_X^{\infty,t})^\hbar, \qquad \mathscr{D}b_X^{t,\hbar} := (\mathscr{D}b_X^t)^\hbar, \qquad \Gamma_{[U]}\mathscr{D}b_X^\hbar := (\Gamma_{[U]}\mathscr{D}b_X)^\hbar.$$

Note that, by Lemmas 2.3 and 2.9,

$$\Gamma_{[U]}\mathscr{D}b_X^\hbar \simeq \rho^{-1}\mathscr{H}om_{\mathbb{C}_{X_{\mathrm{sa}}}}(\mathbb{C}_U, \mathscr{D}b_X^{t,\hbar}).$$

By Proposition 2.2 we get:

Proposition 4.2. *The sheaves $\mathscr{C}_X^{\infty,t,\hbar}$, $\mathscr{D}b_X^{t,\hbar}$ and $\Gamma_{[U]}\mathscr{D}b_X^\hbar$ are cohomologically $\hbar$-complete.*

Now assume X is a complex manifold. Denote by $\overline{X}$ the complex conjugate manifold and by $X^{\mathbb{R}}$ the underlying real analytic manifold, identified with the diagonal of $X \times \overline{X}$. One defines the sheaf (in fact, an object of the derived category) of tempered holomorphic functions by

$$\mathscr{O}_X^t := \mathrm{R}\mathscr{H}om_{\rho_!\mathscr{D}_{\overline{X}}}(\rho_!\mathscr{O}_{\overline{X}}, \mathscr{C}_X^{\infty,t}) \xrightarrow{\sim} \mathrm{R}\mathscr{H}om_{\rho_!\mathscr{D}_{\overline{X}}}(\rho_!\mathscr{O}_{\overline{X}}, \mathscr{D}b_X^t).$$

Here and in the sequel, we write $\mathscr{C}_X^{\infty,t}$ and $\mathscr{D}b_X^t$ instead of $\mathscr{C}_{X^{\mathbb{R}}}^{\infty,t}$ and $\mathscr{D}b_{X^{\mathbb{R}}}^t$, respectively. We set

$$\mathscr{O}_X^{t,\hbar} := (\mathscr{O}_X^t)^{\mathrm{R}\hbar},$$

a cohomologically $\hbar$-complete object of $\mathsf{D}^{\mathrm{b}}(\mathbb{C}^{\hbar}_{X_{\mathrm{sa}}})$. By Lemma 2.3,

$$\mathscr{O}_X^{t,\hbar} \simeq \mathrm{R}\mathscr{H}om_{\rho_!\mathscr{D}_{\overline{X}}}(\rho_!\mathscr{O}_{\overline{X}}, \mathscr{C}_X^{\infty,t,\hbar}) \xrightarrow{\sim} \mathrm{R}\mathscr{H}om_{\rho_!\mathscr{D}_{\overline{X}}}(\rho_!\mathscr{O}_{\overline{X}}, \mathscr{D}b_X^{t,\hbar}).$$

Note that $\mathrm{gr}_\hbar(\mathscr{O}_X^{t,\hbar}) \simeq \mathscr{O}_X^t$ in $\mathsf{D}^{\mathrm{b}}(\mathbb{C}_{X_{\mathrm{sa}}})$.

§5. Riemann-Hilbert correspondence

Let X be a complex analytic manifold. Consider the functors

$$\mathrm{TH}\colon \mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathbb{C}_X) \to \mathsf{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathscr{D}_X)^{\mathrm{op}}, \qquad F \mapsto \rho^{-1}\mathrm{R}\mathscr{H}om_{\mathbb{C}_{X_{\mathrm{sa}}}}(\rho_*F, \mathscr{O}_X^t),$$
$$\mathrm{TH}_\hbar\colon \mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathbb{C}^\hbar_X) \to \mathsf{D}^{\mathrm{b}}(\mathscr{D}^\hbar_X)^{\mathrm{op}}, \qquad F \mapsto \rho^{-1}\mathrm{R}\mathscr{H}om_{\mathbb{C}^\hbar_{X_{\mathrm{sa}}}}(\rho_*F, \mathscr{O}_X^{t,\hbar}).$$

The classical Riemann-Hilbert correspondence of Kashiwara [6] states that the functors Sol and TH are equivalences of categories between $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathbb{C}_X)$ and $\mathsf{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathscr{D}_X)^{\mathrm{op}}$ quasi-inverse to each other. In order to obtain a similar statement for $\mathbb{C}_X$ and $\mathscr{D}_X$ replaced with $\mathbb{C}^\hbar_X$ and $\mathscr{D}^\hbar_X$, respectively, we start by establishing some lemmas.

Lemma 5.1. *For $\mathscr{M}, \mathscr{N} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}^\hbar_X)$ one has a natural isomorphism in $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(\mathbb{C}^\hbar_X)$*

$$\mathrm{R}\mathscr{H}om_{\mathscr{D}^\hbar_X}(\mathscr{M}, \mathscr{N}) \xrightarrow{\sim} \mathrm{R}\mathscr{H}om_{\mathbb{C}^\hbar_X}(\mathrm{Sol}_\hbar(\mathscr{N}), \mathrm{Sol}_\hbar(\mathscr{M})).$$

Proof. Applying the functor $\mathrm{gr}_\hbar$ to this morphism, we get an isomorphism by the classical Riemann-Hilbert correspondence. Then the result follows from Corollary 1.17 and Theorem 3.13. □

Note that there is an isomorphism in $\mathsf{D}^{\mathrm{b}}(\mathscr{D}_X)$

$$\mathrm{gr}_\hbar(\mathrm{TH}_\hbar(F)) \simeq \mathrm{TH}(\mathrm{gr}_\hbar(F)). \tag{5.1}$$

Lemma 5.2. *The functor* $\mathrm{TH}_\hbar$ *induces a functor*

$$\mathrm{TH}_\hbar\colon \mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(\mathbb{C}^\hbar_X) \to \mathsf{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathscr{D}^\hbar_X)^{\mathrm{op}}. \tag{5.2}$$

Proof. Let $F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(\mathbb{C}^\hbar_X)$. By (5.1) and the classical Riemann-Hilbert correspondence we know that $\mathrm{gr}_\hbar(\mathrm{TH}_\hbar(F))$ is regular holonomic, and in particular coherent. It is thus left to prove that $\mathrm{TH}_\hbar(F)$ is coherent. Note that our problem is of local nature.

We use the Dolbeault resolution of $\mathscr{O}^{t,\hbar}_X$ with coefficients in $\mathscr{D}b^{t,\hbar}_X$ and we choose a resolution of F as given in Proposition A.2 (i). We find that $\mathrm{TH}_\hbar(F)$ is isomorphic to a bounded complex $\mathscr{M}^\bullet$, where the $\mathscr{M}^i$ are locally finite sums of sheaves of the type $\Gamma_{[U]}\mathscr{D}b^{t,\hbar}$ with $U \in \mathrm{Op}_{X_{\mathrm{sa}}}$. It follows from Proposition 4.2 that $\mathrm{TH}_\hbar(F)$ is cohomologically $\hbar$-complete, and we conclude by Theorem 1.11 with $\mathscr{A} = \mathscr{D}^\hbar_X$. □

Lemma 5.3. *We have* $\mathrm{R}\mathscr{H}om_{\rho_!\mathscr{D}^\hbar_X}(\rho_!\mathscr{O}^\hbar_X, \mathscr{O}^{t,\hbar}_X) \simeq \mathbb{C}^\hbar_{X_{\mathrm{sa}}}$.

Proof. This isomorphism is given by the sequence

$$\begin{aligned}\mathrm{R}\mathscr{H}om_{\rho_!\mathscr{D}^\hbar_X}(\rho_!\mathscr{O}^\hbar_X, \mathscr{O}^{t,\hbar}_X) &\simeq \mathrm{R}\mathscr{H}om_{\rho_!\mathscr{D}_X}(\rho_!\mathscr{O}_X, \mathscr{O}^{t,\hbar}_X)\\ &\simeq \mathrm{R}\mathscr{H}om_{\rho_!\mathscr{D}_X}(\rho_!\mathscr{O}_X, \mathscr{O}^{t}_X)^{\mathrm{R}\hbar}\\ &\simeq (\rho_*\mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{O}_X, \mathscr{O}_X))^{\mathrm{R}\hbar} \simeq (\mathbb{C}_{X_{\mathrm{sa}}})^{\mathrm{R}\hbar} \simeq \mathbb{C}^\hbar_{X_{\mathrm{sa}}},\end{aligned}$$

where the first isomorphism is an extension of scalars, the second one follows from Lemma 2.3 and the third one is given by the adjunction between $\rho_!$ and ρ^{-1}. □

Theorem 5.4. *The functors* $\mathrm{Sol}_\hbar$ *and* $\mathrm{TH}_\hbar$ *are equivalences of categories between* $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(\mathbb{C}^\hbar_X)$ *and* $\mathsf{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathscr{D}^\hbar_X)^{\mathrm{op}}$ *quasi-inverse to each other.*

Proof. In view of Lemma 5.1, we know that the functor $\mathrm{Sol}_\hbar$ is fully faithful. It is then enough to show that $\mathrm{Sol}_\hbar(\mathrm{TH}_\hbar(F)) \simeq F$ for $F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathbb{C}^\hbar_X)$. Since we already know by Lemma 5.2 that $\mathrm{TH}_\hbar(F)$ is holonomic, we may use (3.9). We have the sequence of isomorphisms:

$$\begin{aligned}\rho_*\mathrm{R}\mathscr{H}om_{\mathscr{D}^\hbar_X}(\mathscr{O}^\hbar_X, \mathrm{TH}_\hbar(F)) &= \rho_*\mathrm{R}\mathscr{H}om_{\mathscr{D}^\hbar_X}(\mathscr{O}^\hbar_X, \rho^{-1}\mathrm{R}\mathscr{H}om_{\mathbb{C}^\hbar_{X_{\mathrm{sa}}}}(\rho_*F, \mathscr{O}^{t,\hbar}_X))\\ &\simeq \mathrm{R}\mathscr{H}om_{\rho_!\mathscr{D}^\hbar_X}(\rho_!\mathscr{O}^\hbar_X, \mathrm{R}\mathscr{H}om_{\mathbb{C}^\hbar_{X_{\mathrm{sa}}}}(\rho_*F, \mathscr{O}^{t,\hbar}_X))\\ &\simeq \mathrm{R}\mathscr{H}om_{\mathbb{C}^\hbar_{X_{\mathrm{sa}}}}(\rho_*F, \mathrm{R}\mathscr{H}om_{\rho_!\mathscr{D}^\hbar_X}(\rho_!\mathscr{O}^\hbar_X, \mathscr{O}^{t,\hbar}_X))\\ &\simeq \mathrm{R}\mathscr{H}om_{\mathbb{C}^\hbar_{X_{\mathrm{sa}}}}(\rho_*F, \mathbb{C}^\hbar_{X_{\mathrm{sa}}}) \simeq \mathrm{R}\mathscr{H}om_{\mathbb{C}^\hbar_{X_{\mathrm{sa}}}}(\rho_*F, \rho_*\mathbb{C}^\hbar_X)\\ &\simeq \rho_*\mathrm{D}'_\hbar F,\end{aligned}$$

where we have used the adjunction between $\rho_!$ and ρ^{-1}, the isomorphism of Lemma 5.3 and the commutation of ρ_* with $\mathrm{R}\mathscr{H}om$. One concludes by recalling the isomorphism of functors $\rho^{-1}\rho_* \simeq \mathrm{id}$. □

t-structure. Recall the definition of the middle perversity t-structure for complex constructible sheaves. Let $\mathbb{K}$ denote either the field $\mathbb{C}$ or the ring $\mathbb{C}^\hbar$. For $F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathbb{K}_X)$, we have $F \in {}^p\mathsf{D}^{\leq 0}_{\mathbb{C}\text{-c}}(\mathbb{K}_X)$ if and only if

$$\forall i \in \mathbb{Z} \qquad \dim \operatorname{supp} H^i(F) \leq d_X - i, \tag{5.3}$$

and $F \in {}^p\mathsf{D}^{\geq 0}_{\mathbb{C}\text{-c}}(\mathbb{K}_X)$ if and only if, for any locally closed complex analytic subset $S \subset X$,

$$H^i_S(F) = 0 \text{ for all } i < d_X - \dim(S). \tag{5.4}$$

One denotes by $\mathrm{Perv}(\mathbb{K}_X)$ the heart of this t-structure.

With the above convention, the de Rham functor

$$\mathrm{DR}\colon \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X) \to {}^p\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathbb{C}_X)$$

is t-exact, when $\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X)$ is equipped with the natural t-structure.

Theorem 5.5. *The de Rham functor $\mathrm{DR}_\hbar\colon \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}^\hbar_X) \to {}^p\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(\mathbb{C}^\hbar_X)$ is t-exact, and induces an equivalence of categories when restricted to $\mathsf{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathscr{D}^\hbar_X)$.*

Proof. (i) Let $\mathscr{M} \in \mathsf{D}^{\leq 0}_{\mathrm{hol}}(\mathscr{D}^\hbar_X)$. Let us prove that $\mathrm{DR}_\hbar\mathscr{M} \in {}^p\mathsf{D}^{\leq 0}_{\mathbb{C}\text{-c}}(\mathbb{C}^\hbar_X)$. Since $\mathrm{DR}_\hbar\mathscr{M}$ is constructible, by Proposition 1.19 it is enough to check (5.3) for $\mathrm{gr}_\hbar(\mathrm{DR}_\hbar\mathscr{M}) \simeq \mathrm{DR}(\mathrm{gr}_\hbar \mathscr{M})$. In other words, it is enough to check that $\mathrm{DR}(\mathrm{gr}_\hbar \mathscr{M}) \in {}^p\mathsf{D}^{\leq 0}_{\mathbb{C}\text{-c}}(\mathbb{C}_X)$. Since $\mathrm{gr}_\hbar \mathscr{M} \in \mathsf{D}^{\leq 0}_{\mathrm{hol}}(\mathscr{D}_X)$, this result follows from the t-exactness of the functor DR.

(ii) Let $\mathscr{M} \in \mathsf{D}^{\geq 0}_{\mathrm{hol}}(\mathscr{D}^\hbar_X)$. Let us prove that $\mathrm{DR}_\hbar\mathscr{M} \in {}^p\mathsf{D}^{\geq 0}_{\mathbb{C}\text{-c}}(\mathbb{C}^\hbar_X)$. We set $\mathscr{N} = (H^0\mathscr{M})_{\hbar\text{-tor}}$. We have a morphism $u\colon \mathscr{N} \to \mathscr{M}$ induced by $H^0\mathscr{M} \to \mathscr{M}$ and we let $\mathscr{M}'$ be the mapping cone of u. We have a distinguished triangle

$$\mathrm{DR}_\hbar\mathscr{N} \to \mathrm{DR}_\hbar\mathscr{M} \to \mathrm{DR}_\hbar\mathscr{M}' \xrightarrow{+1}$$

so that it is enough to show that $\mathrm{DR}_\hbar\mathscr{N}$ and $\mathrm{DR}_\hbar\mathscr{M}'$ belong to ${}^p\mathsf{D}^{\geq 0}_{\mathbb{C}\text{-c}}(\mathbb{C}^\hbar_X)$.

(ii-a) By Proposition 3.6 (ii) and Proposition 3.8, $\mathscr{N}$ is holonomic as a $\mathscr{D}_X$-module. Hence $\mathrm{DR}_\hbar\mathscr{N} \simeq \mathrm{DR}\mathscr{N}$ is a perverse sheaf (over $\mathbb{C}$) and satisfies (5.4). Since (5.4) does not depend on the coefficient ring, $\mathrm{DR}_\hbar\mathscr{N} \in {}^p\mathsf{D}^{\geq 0}_{\mathbb{C}\text{-c}}(\mathbb{C}^\hbar_X)$.

(ii-b) We note that $H^0\mathscr{M}' \simeq (H^0\mathscr{M})_{\hbar\text{-tf}}$. Hence by Proposition 1.14, $\mathrm{gr}_\hbar\,\mathscr{M}' \in \mathsf{D}^{\geq 0}_{\mathrm{hol}}(\mathscr{D}_X)$ and $\mathrm{DR}(\mathrm{gr}_\hbar\,\mathscr{M}') \in {}^p\mathsf{D}^{\geq 0}_{\mathbb{C}\text{-c}}(\mathbb{C}_X)$, that is, $\mathrm{DR}(\mathrm{gr}_\hbar\,\mathscr{M}')$ satisfies (5.4). Let $S \subset X$ be a locally closed complex subanalytic subset. We have

$$\mathrm{R}\Gamma_S(\mathrm{DR}(\mathrm{gr}_\hbar\,\mathscr{M}')) \simeq \mathrm{gr}_\hbar(\mathrm{R}\Gamma_S(\mathrm{DR}_\hbar\mathscr{M}'))$$

and it follows from Proposition 1.19 that $\mathrm{DR}_\hbar\mathscr{M}'$ also satisfies (5.4) and thus belongs to ${}^p\mathsf{D}^{\geq 0}_{\mathbb{C}\text{-c}}(\mathbb{C}^\hbar_X)$.

(iii) The fact that $\mathrm{DR}_\hbar$ is an equivalence follows from Theorems 5.4 and 3.15, in view of Lemma A.1. □

§6. Duality and $\hbar$-torsion

The duality functors $\mathbb{D}$ on $\mathsf{D}_{\mathrm{rh}}(\mathscr{D}_X)$ and D' on ${}^p\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathbb{C}_X)$ are t-exact. We will discuss here the finer t-structures needed in order to obtain a similar result when replacing $\mathbb{C}_X$ and $\mathscr{D}_X$ by their formal extensions $\mathbb{C}^\hbar_X$ and $\mathscr{D}^\hbar_X$.

Following [2, Chapter I.2], let us start by recalling some facts related to torsion pairs and t-structures. We need in particular Proposition 6.2 below, which can also be found in [3].

Definition 6.1. *Let $\mathscr{C}$ be an abelian category. A torsion pair on $\mathscr{C}$ is a pair $(\mathscr{C}_{tor}, \mathscr{C}_{tf})$ of full subcategories such that*

(i) for all objects T in $\mathscr{C}_{tor}$ and F in $\mathscr{C}_{tf}$, we have $\mathrm{Hom}_{\mathscr{C}}(T, F) = 0$,

(ii) for any object M in $\mathscr{C}$, there are objects M_{tor} in $\mathscr{C}_{tor}$ and M_{tf} in $\mathscr{C}_{tf}$ and a short exact sequence $0 \to M_{tor} \to M \to M_{tf} \to 0$.

Proposition 6.2. *Let D be a triangulated category endowed with a t-structure $({}^p\mathsf{D}^{\leq 0}, {}^p\mathsf{D}^{\geq 0})$. Let us denote its heart by $\mathscr{C}$ and its cohomology functors by ${}^pH^i \colon \mathsf{D} \to \mathscr{C}$. Suppose that $\mathscr{C}$ is endowed with a torsion pair $(\mathscr{C}_{tor}, \mathscr{C}_{tf})$. Then we can define a new t-structure $({}^\pi\mathsf{D}^{\leq 0}, {}^\pi\mathsf{D}^{\geq 0})$ on D by setting:*

$$\begin{aligned} {}^\pi\mathsf{D}^{\leq 0} &= \{M \in {}^p\mathsf{D}^{\leq 1} \colon {}^pH^1(M) \in \mathscr{C}_{tor}\}, \\ {}^\pi\mathsf{D}^{\geq 0} &= \{M \in {}^p\mathsf{D}^{\geq 0} \colon {}^pH^0(M) \in \mathscr{C}_{tf}\}. \end{aligned}$$

With the notations of Definition 3.2, there is a natural torsion pair attached to $\mathrm{Mod}(\mathscr{D}^\hbar_X)$ given by the full subcategories

$$\begin{aligned} \mathrm{Mod}(\mathscr{D}^\hbar_X)_{\hbar\text{-tor}} &= \{\mathscr{M} \colon \mathscr{M}_{\hbar\text{-tor}} \xrightarrow{\sim} \mathscr{M}\}, \\ \mathrm{Mod}(\mathscr{D}^\hbar_X)_{\hbar\text{-tf}} &= \{\mathscr{M} \colon \mathscr{M} \xrightarrow{\sim} \mathscr{M}_{\hbar\text{-tf}}\}. \end{aligned}$$

Definition 6.3.

(a) *We call the torsion pair on* $\mathrm{Mod}(\mathscr{D}_X^\hbar)$ *defined above, the* $\hbar$*-torsion pair.*

(b) *We denote by* $\big(\mathsf{D}^{\leq 0}(\mathscr{D}_X^\hbar), \mathsf{D}^{\geq 0}(\mathscr{D}_X^\hbar)\big)$ *the natural* t*-structure on* $\mathsf{D}(\mathscr{D}_X^\hbar)$*.*

(c) *We denote by* $\big({}^t\mathsf{D}^{\leq 0}(\mathscr{D}_X^\hbar), {}^t\mathsf{D}^{\geq 0}(\mathscr{D}_X^\hbar)\big)$ *the* t*-structure on* $\mathsf{D}^{\mathrm{b}}(\mathscr{D}_X^\hbar)$ *associated via Proposition 6.2 with the* $\hbar$*-torsion pair on* $\mathrm{Mod}(\mathscr{D}_X^\hbar)$*.*

Proposition 1.14 implies the following equivalences for $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X^\hbar)$:

$$\mathscr{M} \in {}^t\mathsf{D}^{\geq 0}(\mathscr{D}_X^\hbar) \iff \mathrm{gr}_\hbar \mathscr{M} \in \mathsf{D}^{\geq 0}(\mathscr{D}_X), \tag{6.1}$$

$$\mathscr{M} \in \mathsf{D}^{\leq 0}(\mathscr{D}_X^\hbar) \iff \mathrm{gr}_\hbar \mathscr{M} \in \mathsf{D}^{\leq 0}(\mathscr{D}_X). \tag{6.2}$$

Proposition 6.4. *Let* $\mathscr{M}$ *be a holonomic* $\mathscr{D}_X^\hbar$*-module.*

(i) *If* $\mathscr{M}$ *has no* $\hbar$*-torsion, then* $\mathbb{D}_\hbar \mathscr{M}$ *is concentrated in degree* 0 *and has no* $\hbar$*-torsion.*

(ii) *If* $\mathscr{M}$ *is an* $\hbar$*-torsion module, then* $\mathbb{D}_\hbar \mathscr{M}$ *is concentrated in degree* 1 *and is an* $\hbar$*-torsion module.*

Proof. By (1.2) we have $\mathrm{gr}_\hbar(\mathbb{D}_\hbar \mathscr{M}) \simeq \mathbb{D}(\mathrm{gr}_\hbar \mathscr{M})$. Since $\mathrm{gr}_\hbar \mathscr{M}$ is concentrated in degrees 0 and -1, with holonomic cohomology, $\mathbb{D}(\mathrm{gr}_\hbar \mathscr{M})$ is concentrated in degrees 0 and 1. By Proposition 1.14, $\mathbb{D}_\hbar \mathscr{M}$ itself is concentrated in degrees 0 and 1 and $H^0(\mathbb{D}_\hbar \mathscr{M})$ has no $\hbar$-torsion.

(i) The short exact sequence

$$0 \to \mathscr{M} \xrightarrow{\hbar} \mathscr{M} \to \mathscr{M}/\hbar\mathscr{M} \to 0$$

induces the long exact sequence

$$\cdots \to H^1(\mathbb{D}_\hbar(\mathscr{M}/\hbar\mathscr{M})) \to H^1(\mathbb{D}_\hbar \mathscr{M}) \xrightarrow{\hbar} H^1(\mathbb{D}_\hbar \mathscr{M}) \to 0.$$

By Nakayama's lemma $H^1(\mathbb{D}_\hbar \mathscr{M}) = 0$ as required.

(ii) Since $\mathscr{M}$ is locally annihilated by some power of $\hbar$, the cohomology groups $H^i(\mathbb{D}_\hbar \mathscr{M})$ also are $\hbar$-torsion modules. As $H^0(\mathbb{D}_\hbar \mathscr{M})$ has no $\hbar$-torsion, we get $H^0(\mathbb{D}_\hbar \mathscr{M}) = 0$. □

Theorem 6.5. *The duality functor* $\mathbb{D}_\hbar \colon \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X^\hbar)^{\mathrm{op}} \to {}^t\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X^\hbar)$ *is* t*-exact. In other words,* $\mathbb{D}_\hbar$ *interchanges* $\mathsf{D}^{\leq 0}_{\mathrm{hol}}(\mathscr{D}_X^\hbar)$ *with* ${}^t\mathsf{D}^{\geq 0}_{\mathrm{hol}}(\mathscr{D}_X^\hbar)$ *and* $\mathsf{D}^{\geq 0}_{\mathrm{hol}}(\mathscr{D}_X^\hbar)$ *with* ${}^t\mathsf{D}^{\leq 0}_{\mathrm{hol}}(\mathscr{D}_X^\hbar)$*.*

Proof. (i) Let us first prove for $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}^\hbar_X)$:

$$\mathscr{M} \in \mathsf{D}^{\leq 0}_{\mathrm{hol}}(\mathscr{D}^\hbar_X) \Longleftrightarrow \mathbb{D}_\hbar(\mathscr{M}) \in {}^t\mathsf{D}^{\geq 0}_{\mathrm{hol}}(\mathscr{D}^\hbar_X). \tag{6.3}$$

By (1.2) we have $\mathrm{gr}_\hbar(\mathbb{D}_\hbar \mathscr{M}) \simeq \mathbb{D}(\mathrm{gr}_\hbar \mathscr{M})$ and we know that the analog of (6.3) holds true for $\mathscr{D}_X$-modules:

$$\mathscr{N} \in \mathsf{D}^{\leq 0}_{\mathrm{hol}}(\mathscr{D}_X) \Longleftrightarrow \mathbb{D}(\mathscr{N}) \in \mathsf{D}^{\geq 0}_{\mathrm{hol}}(\mathscr{D}_X).$$

Hence (6.3) follows easily from (6.1) and (6.2).

(ii) We recall the general fact for a t-structure $(\mathsf{D}, \mathsf{D}^{\leq 0}, \mathsf{D}^{\geq 0})$ and $A \in \mathsf{D}$:

$$\begin{aligned} A \in \mathsf{D}^{\leq 0} &\Longleftrightarrow \mathrm{Hom}\,(A,B) = 0 \text{ for any } B \in \mathsf{D}^{\geq 1},\\ A \in \mathsf{D}^{\geq 0} &\Longleftrightarrow \mathrm{Hom}\,(B,A) = 0 \text{ for any } B \in \mathsf{D}^{\leq -1}.\end{aligned}$$

Since $\mathbb{D}_\hbar$ is an involutive equivalence of categories we deduce from (6.3) the dual statement:

$$\mathscr{M} \in \mathsf{D}^{\geq 0}_{\mathrm{hol}}(\mathscr{D}^\hbar_X) \Longleftrightarrow \mathbb{D}_\hbar(\mathscr{M}) \in {}^t\mathsf{D}^{\leq 0}_{\mathrm{hol}}(\mathscr{D}^\hbar_X).$$

□

Remark 6.6. *The above result can be stated as follows in the language of quasi-abelian categories of [19]. We will follow the same notations as in [8, Chapter 2]. The category $\mathscr{C} = \mathrm{Mod}(\mathscr{D}^\hbar_X)_{\hbar\text{-}tf}$ is quasi-abelian. Hence its derived category has a natural generalized t-structure $(\mathsf{D}^{\leq s}(\mathscr{C}), \mathsf{D}^{>s-1}(\mathscr{C}))_{s\in\frac{1}{2}\mathbb{Z}}$. Note that $\mathsf{D}^{[-1/2,0]}(\mathscr{C})$ is equivalent to $\mathrm{Mod}(\mathscr{D}^\hbar_X)$, and that $\mathsf{D}^{[0,1/2]}(\mathscr{C})$ is equivalent to the heart of ${}^t\mathsf{D}^{\mathrm{b}}(\mathscr{D}^\hbar_X)$. Then Theorem 6.5 states that the duality functor $\mathbb{D}_\hbar$ is t-exact on $\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{C})$.*

Recall that $\mathrm{Perv}(\mathbb{C}^\hbar_X)$ denotes the heart of the middle perversity t-structure on $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathbb{C}^\hbar_X)$. Consider the full subcategories of $\mathrm{Perv}(\mathbb{C}^\hbar_X)$

$$\begin{aligned}\mathrm{Perv}(\mathbb{C}^\hbar_X)_{\hbar\text{-tor}} &= \{F\colon \text{locally } \hbar^N F = 0 \text{ for some } N \in \mathbb{N}\},\\ \mathrm{Perv}(\mathbb{C}^\hbar_X)_{\hbar\text{-tf}} &= \{F\colon F \text{ has no non zero subobjects in } \mathrm{Perv}(\mathbb{C}^\hbar_X)_{\hbar\text{-tor}}\}.\end{aligned}$$

Lemma 6.7.

(i) *Let $F \in \mathrm{Perv}(\mathbb{C}^\hbar_X)$. Then the inductive system of sub-perverse sheaves $\mathrm{Ker}(\hbar^n\colon F \to F)$ is locally stationary.*

(ii) *The pair $\big(\mathrm{Perv}(\mathbb{C}^\hbar_X)_{\hbar\text{-}tor}, \mathrm{Perv}(\mathbb{C}^\hbar_X)_{\hbar\text{-}tf}\big)$ is a torsion pair.*

Proof. (i) Set $\mathscr{M} = \mathbb{D}_\hbar \mathrm{TH}_\hbar(F)$. By the Riemann-Hilbert correspondence, one has $\mathrm{Ker}(\hbar^n\colon F \to F) \simeq \mathrm{DR}_\hbar(\mathrm{Ker}(\hbar^n\colon \mathscr{M} \to \mathscr{M}))$. Since $\mathscr{M}$ is coherent, the inductive system $\mathrm{Ker}(\hbar^n\colon \mathscr{M} \to \mathscr{M})$ is locally stationary. Hence so is the system $\mathrm{Ker}(\hbar^n\colon F \to F)$.
(ii) By (i) it makes sense to define for $F \in \mathrm{Perv}(\mathbb{C}_X^\hbar)$:

$$F_{\hbar\text{-tor}} = \bigcup_n \mathrm{Ker}(\hbar^n\colon F \to F), \quad F_{\hbar\text{-tf}} = F/F_{\hbar\text{-tor}}.$$

It is easy to check that $F_{\hbar\text{-tor}} \in \mathrm{Perv}(\mathbb{C}_X^\hbar)_{\hbar\text{-tor}}$ and $F_{\hbar\text{-tf}} \in \mathrm{Perv}(\mathbb{C}_X^\hbar)_{\hbar\text{-tf}}$. Then property (ii) in Definition 6.1 is clear. For property (i) let $u\colon F \to G$ be a morphism in $\mathrm{Perv}(\mathbb{C}_X^\hbar)$ with $F \in \mathrm{Perv}(\mathbb{C}_X^\hbar)_{\hbar\text{-tor}}$ and $G \in \mathrm{Perv}(\mathbb{C}_X^\hbar)_{\hbar\text{-tf}}$. Then $\mathrm{Im}\, u$ also is in $\mathrm{Perv}(\mathbb{C}_X^\hbar)_{\hbar\text{-tor}}$ and so it is zero by definition of $\mathrm{Perv}(\mathbb{C}_X^\hbar)_{\hbar\text{-tf}}$. ☐

Denote by $\left({}^\pi\mathsf{D}^{\leq 0}_{\mathbb{C}\text{-c}}(\mathbb{C}_X^\hbar), {}^\pi\mathsf{D}^{\geq 0}_{\mathbb{C}\text{-c}}(\mathbb{C}_X^\hbar)\right)$ the t-structure on $\mathsf{D}_{\mathbb{C}\text{-c}}(\mathbb{C}_X^\hbar)$ induced by the perversity t-structure and this torsion pair as in Proposition 6.2. We also set ${}^\pi\mathrm{Perv}(\mathbb{C}_X^\hbar) = {}^\pi\mathsf{D}^{\leq 0}_{\mathbb{C}\text{-c}}(\mathbb{C}_X^\hbar) \cap {}^\pi\mathsf{D}^{\geq 0}_{\mathbb{C}\text{-c}}(\mathbb{C}_X^\hbar)$.

Theorem 6.8. *There is a quasi-commutative diagram of t-exact functors*

$$\begin{array}{ccc} \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X^\hbar)^{\mathrm{op}} & \xrightarrow{\mathrm{DR}_\hbar} & {}^p\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(\mathbb{C}_X^\hbar)^{\mathrm{op}} \\ \downarrow{\scriptstyle \mathbb{D}_\hbar} & & \downarrow{\scriptstyle \mathrm{D}'_\hbar} \\ {}^t\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X^\hbar) & \xrightarrow{\mathrm{DR}_\hbar} & {}^\pi\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(\mathbb{C}_X^\hbar) \end{array}$$

where the duality functors are equivalences of categories and the de Rham functors become equivalences when restricted to the subcategories of regular objects.

Example 6.9. *Let $X = \mathbb{C}$, $U = X \setminus \{0\}$ and denote by $j\colon U \hookrightarrow X$ the embedding. Let L be the local system on U with stalk $\mathbb{C}^\hbar$ and monodromy $1+\hbar$. The sheaf $\mathrm{R}j_*L \simeq \mathrm{D}'_h(j_!(\mathrm{D}'_h L))$ is perverse for both t-structures, as is the sheaf $H^0(\mathrm{R}j_*L) = j_*L \simeq j_!L$. The sheaf $H^1(\mathrm{R}j_*L) \simeq \mathbb{C}_{\{0\}}$ has $\hbar$-torsion. From the distinguished triangle $j_*L \to \mathrm{R}j_*L \to \mathbb{C}_{\{0\}}[-1] \xrightarrow{+1}$, one gets the short exact sequences*

$$\begin{aligned} &0 \to j_*L \to \mathrm{R}j_*L \to \mathbb{C}_{\{0\}}[-1] \to 0 \quad \textit{in } \mathrm{Perv}(\mathbb{C}_X^\hbar), \\ &0 \to \mathbb{C}_{\{0\}}[-2] \to j_*L \to \mathrm{R}j_*L \to 0 \quad \textit{in } {}^\pi\mathrm{Perv}(\mathbb{C}_X^\hbar). \end{aligned}$$

§7. $\mathscr{D}((\hbar))$-modules

Denote by

$$\mathbb{C}^{\hbar,\mathrm{loc}} := \mathbb{C}((\hbar)) = \mathbb{C}[\hbar^{-1},\hbar]]$$

the field of Laurent series in $\hbar$, that is the fraction field of $\mathbb{C}^{\hbar}$. Recall the exact functor

$$(\bullet)^{\mathrm{loc}}\colon \mathrm{Mod}(\mathbb{C}^{\hbar}_X) \to \mathrm{Mod}(\mathbb{C}^{\hbar,\mathrm{loc}}_X), \quad F \mapsto \mathbb{C}^{\hbar,\mathrm{loc}} \otimes_{\mathbb{C}^{\hbar}} F, \tag{7.1}$$

and note that by [9, Proposition 5.4.14] one has the estimate

$$\mathrm{SS}(F^{\mathrm{loc}}) \subset \mathrm{SS}(F). \tag{7.2}$$

For $G \in \mathsf{D}^{\mathrm{b}}(\mathbb{C}_X)$, we write $G^{\hbar,\mathrm{loc}}$ instead of $(G^{\hbar})^{\mathrm{loc}}$. We will consider in particular

$$\mathscr{O}^{\hbar,\mathrm{loc}}_X = \mathscr{O}_X((\hbar)), \qquad \mathscr{D}^{\hbar,\mathrm{loc}}_X = \mathscr{D}_X((\hbar)).$$

Lemma 7.1. *Let $\mathscr{M}$ be a coherent $\mathscr{D}^{\hbar,\mathrm{loc}}_X$-module. Then $\mathscr{M}$ is pseudo-coherent over $\mathscr{D}^{\hbar}_X$. In other word, if $\mathscr{L} \subset \mathscr{M}$ is a finitely generated $\mathscr{D}^{\hbar}_X$-module, then $\mathscr{L}$ is $\mathscr{D}^{\hbar}_X$-coherent.*

Proof. The proof follows from [7, Appendix. A1]. □

Definition 7.2. *A lattice $\mathscr{L}$ of a coherent $\mathscr{D}^{\hbar,\mathrm{loc}}_X$-module $\mathscr{M}$ is a coherent $\mathscr{D}^{\hbar}_X$-submodule of $\mathscr{M}$ which generates it.*

Since $\mathscr{M}$ has no $\hbar$-torsion, any of its lattices has no $\hbar$-torsion. In particular, one has $\mathscr{M} \simeq \mathscr{L}^{\mathrm{loc}}$ and $\mathrm{gr}_{\hbar}\,\mathscr{L} \simeq \mathscr{L}_0 = \mathscr{L}/\hbar\mathscr{L}$.

It follows from Lemma 7.1 that lattices locally exist: for a local system of generators $(m_1,\dots,m_N)$ of $\mathscr{M}$, define $\mathscr{L}$ as the $\mathscr{D}^{\hbar}_X$-submodule with the same generators.

Lemma 7.3. *Let $0 \to \mathscr{M}' \to \mathscr{M} \to \mathscr{M}'' \to 0$ be an exact sequence of coherent $\mathscr{D}^{\hbar,\mathrm{loc}}_X$-modules. Locally there exist lattices $\mathscr{L}'$, $\mathscr{L}$, $\mathscr{L}''$ of $\mathscr{M}'$, $\mathscr{M}$, $\mathscr{M}''$, respectively, inducing an exact sequence of $\mathscr{D}^{\hbar}_X$-modules*

$$0 \to \mathscr{L}' \to \mathscr{L} \to \mathscr{L}'' \to 0.$$

Proof. Let $\mathscr{L}$ be a lattice of $\mathscr{M}$ and let $\mathscr{L}''$ be its image in $\mathscr{M}''$. We set $\mathscr{L}' := \mathscr{L} \cap \mathscr{M}'$. These sub-$\mathscr{D}^{\hbar}_X$-modules give rise to an exact sequence.

Since $\mathscr{L}''$ is of finite type over $\mathscr{D}^{\hbar}_X$, it is a lattice of $\mathscr{M}''$. Let us show that $\mathscr{L}'$ is a lattice of $\mathscr{M}'$. Being the kernel of a morphism $\mathscr{L} \to \mathscr{L}''$ between coherent $\mathscr{D}^{\hbar}_X$-modules, $\mathscr{L}'$ is coherent. To show that $\mathscr{L}'$ generates $\mathscr{M}'$, note that any $m' \in \mathscr{M}' \subset \mathscr{M}$ may be written as $m' = \hbar^{-N} m$ for some $N \geq 0$ and $m \in \mathscr{L}$. Hence $m = \hbar^N m' \in \mathscr{M}' \cap \mathscr{L} = \mathscr{L}'$. □

For an abelian category $\mathscr{C}$, we denote by $\mathrm{K}(\mathscr{C})$ its Grothendieck group. For an object M of $\mathscr{C}$, we denote by $[M]$ its class in $\mathrm{K}(\mathscr{C})$. We let $\mathscr{K}(\mathscr{D}_X)$ be the sheaf on X associated to the presheaf

$$U \mapsto \mathrm{K}(\mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X|_U)).$$

We define $\mathscr{K}(\mathscr{D}_X^{\hbar,\mathrm{loc}})$ in the same way.

Lemma 7.4. *Let $\mathscr{L}$ be a coherent $\mathscr{D}_X^{\hbar}$-module without $\hbar$-torsion. Then, for any $i > 0$, the $\mathscr{D}_X$-module $\mathscr{L}/\hbar^i\mathscr{L}$ is coherent, and we have the equality $[\mathscr{L}/\hbar^i\mathscr{L}] = i \cdot [\mathrm{gr}_\hbar(\mathscr{L})]$ in $\mathrm{K}(\mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X))$.*

Proof. Since the functor $(\bullet) \otimes_{\mathbb{C}^\hbar} \mathbb{C}^\hbar/\hbar^i\mathbb{C}^\hbar$ is right exact, $\mathscr{L}/\hbar^i\mathscr{L}$ is a coherent $\mathscr{D}_X$-module. Since $\mathscr{L}$ has no $\hbar$-torsion, multiplication by $\hbar^i$ induces an isomorphism $\mathscr{L}/\hbar\mathscr{L} \xrightarrow{\sim} \hbar^i\mathscr{L}/\hbar^{i+1}\mathscr{L}$. We conclude by induction on i with the exact sequence

$$0 \to \hbar^i\mathscr{L}/\hbar^{i+1}\mathscr{L} \to \mathscr{L}/\hbar^{i+1}\mathscr{L} \to \mathscr{L}/\hbar^i\mathscr{L} \to 0.$$

□

Lemma 7.5. *For $\mathscr{M} \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X^{\hbar,\mathrm{loc}})$, $U \subset X$ an open set and $\mathscr{L} \subset \mathscr{M}|_U$ a lattice of $\mathscr{M}|_U$, the class $[\mathrm{gr}_\hbar(\mathscr{L})] \in \mathrm{K}(\mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X|_U))$ only depends on $\mathscr{M}$. This defines a morphism of abelian sheaves $\mathscr{K}(\mathscr{D}_X^{\hbar,\mathrm{loc}}) \to \mathscr{K}(\mathscr{D}_X)$.*

Proof. (i) We first prove that $[\mathrm{gr}_\hbar(\mathscr{L})]$ only depends on $\mathscr{M}$. We consider another lattice $\mathscr{L}'$ of $\mathscr{M}|_U$. Since $\mathscr{L}$ is a $\mathscr{D}_X^{\hbar}$-module of finite type, and $\mathscr{L}'$ generates $\mathscr{M}$, there exists $n > 1$ such that $\mathscr{L} \subset \hbar^{-n}\mathscr{L}'$. Similarly, there exists $m > 1$ with $\mathscr{L}' \subset \hbar^{-m}\mathscr{L}$, so that we have the inclusions

$$\hbar^{m+n+2}\mathscr{L} \subset \hbar^{m+n+1}\mathscr{L} \subset \hbar^{m+1}\mathscr{L}' \subset \hbar^m\mathscr{L}' \subset \mathscr{L}.$$

Any inclusion $A \subset B \subset C$ yields an identity $[C/A] = [C/B] + [B/A]$ in the Grothendieck group, and we obtain in particular:

$$\begin{aligned}
[\hbar^m\mathscr{L}'/\hbar^{m+n+1}\mathscr{L}] &= [\hbar^m\mathscr{L}'/\hbar^{m+1}\mathscr{L}'] + [\hbar^{m+1}\mathscr{L}'/\hbar^{m+n+1}\ shl] \\
[\mathscr{L}/\hbar^{m+n+1}\mathscr{L}] &= [\mathscr{L}/\hbar^{m+1}\mathscr{L}'] + [\hbar^{m+1}\mathscr{L}'/\hbar^{m+n+1}\ shl] \\
[\mathscr{L}/\hbar^{m+n+2}\mathscr{L}] &= [\mathscr{L}/\hbar^{m+1}\mathscr{L}'] + [\hbar^{m+1}\mathscr{L}'/\hbar^{m+n+2}\ shl].
\end{aligned}$$

Note that we have isomorphisms of the type $\hbar^k \mathscr{M}_1/\hbar^k \mathscr{M}_2 \simeq \mathscr{M}_1/\mathscr{M}_2$ for modules without $\hbar$-torsion. Then Lemma 7.4 and the above equalities give:

$$\begin{aligned}
[\mathscr{L}'/\hbar^{n+1}\mathscr{L}] &= [\mathrm{gr}_\hbar(\mathscr{L}')] + [\mathscr{L}'/\hbar^n\mathscr{L}] \\
(m+n+1)[\mathrm{gr}_\hbar(\mathscr{L})] &= [\mathscr{L}/\hbar^{m+1}\mathscr{L}'] + [\mathscr{L}'/\hbar^n\mathscr{L}] \\
(m+n+2)[\mathrm{gr}_\hbar(\mathscr{L})] &= [\mathscr{L}/\hbar^{m+1}\mathscr{L}'] + [\mathscr{L}'/\hbar^{n+1}\mathscr{L}].
\end{aligned}$$

A suitable combination of these lines gives $[\mathrm{gr}_\hbar(\mathscr{L})] = [\mathrm{gr}_\hbar(\mathscr{L}')]$, as desired.

(ii) Now we consider an open subset $V \subset X$ and $\mathscr{M} \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X^{\hbar,\mathrm{loc}}|_V)$. We choose an open covering $\{U_i\}_{i\in I}$ of V such that for each $i \in I$ $\mathscr{M}|_{U_i}$ admits a lattice, say $\mathscr{L}^i$. We have seen that $[\mathrm{gr}_\hbar(\mathscr{L}^i)] \in \mathrm{K}(\mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X|_{U_i}))$ only depends on $\mathscr{M}$. This implies that

$$[\mathrm{gr}_\hbar(\mathscr{L}^i)]|_{U_{i,j}} = [\mathrm{gr}_\hbar(\mathscr{L}^j)]|_{U_{i,j}} \text{ in } \mathrm{K}(\mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X|_{U_{i,j}})).$$

Hence the $[\mathrm{gr}_\hbar(\mathscr{L}^i)]$'s define a section, say $c(\mathscr{M})$, of $\mathscr{K}(\mathscr{D}_X)$ over V. By Lemma 7.3, $c(\mathscr{M})$ only depends on the class $[\mathscr{M}]$ in $\mathrm{K}(\mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X^{\hbar,\mathrm{loc}}|_V))$, and $\mathscr{M} \mapsto c(\mathscr{M})$ induces the morphism $\mathscr{K}(\mathscr{D}_X^{\hbar,\mathrm{loc}}) \to \mathscr{K}(\mathscr{D}_X)$. □

By Lemma 7.5, the following definition is well posed.

Definition 7.6. *The characteristic variety of a coherent $\mathscr{D}_X^{\hbar,\mathrm{loc}}$-module $\mathscr{M}$ is defined by*

$$\mathrm{char}_{\hbar,\mathrm{loc}}(\mathscr{M}) = \mathrm{char}_\hbar(\mathscr{L}),$$

for $\mathscr{L} \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X^\hbar)$ a (local) lattice. For $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X^{\hbar,\mathrm{loc}})$, one sets $\mathrm{char}_{\hbar,\mathrm{loc}}(\mathscr{M}) = \bigcup_j \mathrm{char}_{\hbar,\mathrm{loc}}(H^j(\mathscr{M}))$.

Proposition 7.7. *The characteristic variety $\mathrm{char}_{\hbar,\mathrm{loc}}$ is additive both on $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X^{\hbar,\mathrm{loc}})$ and on $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X^{\hbar,\mathrm{loc}})$.*

Proof. This follows from Proposition 3.6 (ii) and Lemma 7.3. □

Consider the functor

$$\mathrm{Sol}_{\hbar,\mathrm{loc}} \colon \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X^{\hbar,\mathrm{loc}})^{\mathrm{op}} \to \mathsf{D}^{\mathrm{b}}(\mathbb{C}_X^{\hbar,\mathrm{loc}}), \quad \mathscr{M} \mapsto \mathrm{R}\mathscr{H}om_{\mathscr{D}_X^{\hbar,\mathrm{loc}}}(\mathscr{M}, \mathscr{O}_X^{\hbar,\mathrm{loc}}).$$

Proposition 7.8. *Let $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X^{\hbar,\mathrm{loc}})$. Then*

$$\mathrm{SS}\big(\mathrm{Sol}_{\hbar,\mathrm{loc}}(\mathscr{M})\big) \subset \mathrm{char}_{\hbar,\mathrm{loc}}(\mathscr{M}).$$

Proof. By dévissage, we can assume that $\mathscr{M} \in \mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X^{\hbar,\mathrm{loc}})$. Moreover, since the problem is local, we may assume that $\mathscr{M}$ admits a lattice $\mathscr{L}$.

One has the isomorphism $\mathrm{Sol}_{\hbar,\mathrm{loc}}(\mathscr{M}) \simeq \mathrm{R}\mathscr{H}om_{\mathscr{D}_X^{\hbar}}(\mathscr{L}, \mathscr{O}_X^{\hbar,\mathrm{loc}})$ by extension of scalars. Taking a local resolution of $\mathscr{L}$ by free $\mathscr{D}_X^{\hbar}$-modules of finite type, we deduce that $\mathrm{Sol}_{\hbar,\mathrm{loc}}(\mathscr{M}) \simeq F^{\mathrm{loc}}$ for $F = \mathrm{Sol}_{\hbar}(\mathscr{L})$. The statement follows by (7.2) and Corollary 3.14. ☐

One says that $\mathscr{M}$ is holonomic if its characteristic variety is isotropic.

Proposition 7.9. *The functor* $\mathrm{Sol}_{\hbar,\mathrm{loc}}$ *induces a functor*

$$\mathrm{Sol}_{\hbar,\mathrm{loc}} \colon \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X^{\hbar,\mathrm{loc}})^{\mathrm{op}} \to \mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(\mathbb{C}_X^{\hbar,\mathrm{loc}}).$$

Proof. By the same arguments and with the same notations as in the proof of Proposition 7.8, we reduce to the case $\mathrm{Sol}_{\hbar,\mathrm{loc}}(\mathscr{M}) \simeq F^{\mathrm{loc}}$, for $F = \mathrm{Sol}_{\hbar}(\mathscr{L})$ and $\mathscr{L}$ a lattice of $\mathscr{M} \in \mathrm{Mod}_{\mathrm{hol}}(\mathscr{D}_X^{\hbar,\mathrm{loc}})$. Hence $\mathscr{L}$ is a holonomic $\mathscr{D}_X^{\hbar}$-module, and $F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathbb{C}_X^{\hbar})$. ☐

Remark 7.10. *In general the functor*

$$\mathrm{Sol}_{\hbar,\mathrm{loc}} \colon \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X^{\hbar,\mathrm{loc}})^{\mathrm{op}} \to \mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(\mathbb{C}_X^{\hbar,\mathrm{loc}})$$

is not locally essentially surjective. In fact, consider the quasi-commutative diagram of categories

$$\begin{array}{ccc} \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X^{\hbar})^{\mathrm{op}} & \xrightarrow{\mathrm{Sol}_{\hbar}} & \mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(\mathbb{C}_X^{\hbar}) \\ {\scriptstyle (\bullet)^{\mathrm{loc}}}\downarrow & & \downarrow{\scriptstyle (\bullet)^{\mathrm{loc}}} \\ \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X^{\hbar,\mathrm{loc}})^{\mathrm{op}} & \xrightarrow{\mathrm{Sol}_{\hbar,\mathrm{loc}}} & \mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(\mathbb{C}_X^{\hbar,\mathrm{loc}}). \end{array}$$

By the local existence of lattices the left vertical arrow is locally essentially surjective. If $\mathrm{Sol}_{\hbar,\mathrm{loc}}$ *were also locally essentially surjective, so should be the right vertical arrow. The following example shows that it is not the case.*

One can interpret this phenomenon by remarking that $\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X^{\hbar,\mathrm{loc}})$ *is equivalent to the localization of the category* $\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X^{\hbar})$ *with respect to the morphism* $\hbar$*, contrarily to the category* $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(\mathbb{C}_X^{\hbar,\mathrm{loc}})$.

Example 7.11. *Let* $X = \mathbb{C}$, $U = X \setminus \{0\}$ *and denote by* $j \colon U \hookrightarrow X$ *the embedding. Set* $F = \mathrm{R}j_! L$*, where* L *is the local system on* U *with stalk* $\mathbb{C}^{\hbar,\mathrm{loc}}$ *and monodromy* $\hbar$ *around the origin. Since* $\hbar$ *is not invertible in* $\mathbb{C}^{\hbar}$*, there is no* $F_0 \in \mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(\mathbb{C}_X^{\hbar})$ *such that* $F \simeq (F_0)^{\mathrm{loc}}$.

§8. Links with deformation quantization

In this last section, we shall briefly explain how the study of deformation quantization algebras on complex symplectic manifolds is related to $\mathscr{D}_X^\hbar$. We follow the terminology of [13].

The cotangent bundle $\mathfrak{X} = T^*X$ to the complex manifold X has a structure of a complex symplectic manifold and is endowed with the $\mathbb{C}^\hbar$-algebra $\widehat{\mathscr{W}}_{\mathfrak{X}}$, a non homogeneous version of the algebra of microdifferential operators. Its subalgebra $\widehat{\mathscr{W}}_{\mathfrak{X}}(0)$ of operators of order at most zero is a deformation quantization algebra. In a system (x, u) of local symplectic coordinates, $\widehat{\mathscr{W}}_{\mathfrak{X}}(0)$ is identified with the star algebra $(\mathscr{O}_{\mathfrak{X}}^\hbar, \star)$ in which the star product is given by the Leibniz product:

$$f \star g = \sum_{\alpha \in \mathbb{N}^n} \frac{\hbar^{|\alpha|}}{\alpha!} (\partial_u^\alpha f)(\partial_x^\alpha g), \quad \text{for } f, g \in \mathscr{O}_{\mathfrak{X}}. \tag{8.1}$$

In this section we will set for short $\mathscr{A} := \widehat{\mathscr{W}}_{\mathfrak{X}}(0)$, so that $\mathscr{A}^{\mathrm{loc}} \simeq \widehat{\mathscr{W}}_{\mathfrak{X}}$. Note that $\mathscr{A}$ satisfies Assumption 1.8.

Let us identify X with the zero section of the cotangent bundle $\mathfrak{X}$. Recall that X is a local model for any smooth Lagrangian submanifold of $\mathfrak{X}$, and that $\mathscr{O}_X^\hbar$ is a local model of any simple $\mathscr{A}$-module along X. As $\mathscr{O}_X^\hbar$ has both a $\mathscr{D}_X^\hbar$-module and an $\mathscr{A}|_X$-module structure, there are morphisms of $\mathbb{C}^\hbar$-algebras

$$\mathscr{D}_X^\hbar \to \mathscr{E}nd_{\mathbb{C}^\hbar}(\mathscr{O}_X^\hbar) \leftarrow \mathscr{A}|_X. \tag{8.2}$$

Lemma 8.1. *The morphisms in* (8.2) *are injective and induce an embedding* $\mathscr{A}|_X \hookrightarrow \mathscr{D}_X^\hbar$.

Proof. Since the problem is local, we may choose a local symplectic coordinate system (x, u) on $\mathfrak{X}$ such that $X = \{u = 0\}$. Then $\mathscr{A}|_X$ is identified with $\mathscr{O}_{\mathfrak{X}}^\hbar|_X$. As the action of u_i on $\mathscr{O}_X^\hbar$ is given by $\hbar\partial_{x_i}$, the morphism $\mathscr{A}|_X \to \mathscr{E}nd_{\mathbb{C}^\hbar}(\mathscr{O}_X^\hbar)$ factors through $\mathscr{D}_X^\hbar$, and the induced morphism $\mathscr{A}|_X \to \mathscr{D}_X^\hbar$ is described by

$$\sum_{i \in \mathbb{N}} f_i(x, u)\hbar^i \mapsto \sum_{j \in \mathbb{N}} \left(\sum_{\alpha \in \mathbb{N}^n,\ |\alpha| \leq j} \partial_u^\alpha f_{j-|\alpha|}(x, 0)\partial_x^\alpha \right) \hbar^j, \tag{8.3}$$

which is clearly injective. □

Consider the following subsheaves of $\mathscr{D}_X^\hbar$

$$\mathscr{D}_X^{\hbar,m} = \prod_{i \geq 0} (F_{i+m}\mathscr{D}_X)\,\hbar^i, \quad \mathscr{D}_X^{\hbar,\mathrm{f}} = \bigcup_{m \geq 0} \mathscr{D}_X^{\hbar,m}.$$

Note that $\mathscr{D}_X^{\hbar,0}$ and $\mathscr{D}_X^{\hbar,\mathrm{f}}$ are subalgebras of $\mathscr{D}_X^{\hbar}$, that $\mathscr{D}_X^{\hbar,0}$ is $\hbar$-complete while $\mathscr{D}_X^{\hbar,\mathrm{f}}$ is not and that $\mathscr{D}_X^{\hbar,0,\mathrm{loc}} \simeq \mathscr{D}_X^{\hbar,\mathrm{f},\mathrm{loc}}$. By (8.3), the image of $\mathscr{A}|_X$ in $\mathscr{D}_X^{\hbar}$ is contained in $\mathscr{D}_X^{\hbar,0}$. (The ring $\mathscr{D}_X^{\hbar,0}$ should be compared with the ring $\mathscr{R}_{X\times\mathbb{C}}$ of [16].)

Remark 8.2. *More precisely, denote by* $\mathscr{O}_{\mathfrak{X}}^{\hbar}\hat{|}_X \simeq (\mathscr{O}_{\mathfrak{X}}\hat{|}_X)^{\hbar}$ *the formal completion of* $\mathscr{O}_{\mathfrak{X}}^{\hbar}$ *along the submanifold* X*. Then the star product in* (8.1) *extends to this sheaf, and* (8.3) *induces an isomorphism* $(\mathscr{O}_{\mathfrak{X}}^{\hbar}\hat{|}_X, \star) \simeq \mathscr{D}_X^{\hbar,0}$.

Summarizing, one has the compatible embeddings of algebras

$$\begin{array}{ccccccc}
\mathscr{A}^{\mathrm{loc}}|_X & \hookrightarrow & \mathscr{D}_X^{\hbar,0,\mathrm{loc}} & \xrightarrow{\sim} & \mathscr{D}_X^{\hbar,\mathrm{f},\mathrm{loc}} & \hookrightarrow & \mathscr{D}_X^{\hbar,\mathrm{loc}} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\mathscr{A}|_X & \hookrightarrow & \mathscr{D}_X^{\hbar,0} & \hookrightarrow & \mathscr{D}_X^{\hbar,\mathrm{f}} & \hookrightarrow & \mathscr{D}_X^{\hbar}
\end{array}$$

One has

$$\mathrm{gr}_\hbar\, \mathscr{A}|_X \simeq \mathscr{O}_{\mathfrak{X}}|_X, \quad \mathrm{gr}_\hbar\, \mathscr{D}_X^{\hbar,0} \simeq \mathscr{O}_{\mathfrak{X}}\hat{|}_X, \quad \mathrm{gr}_\hbar\, \mathscr{D}_X^{\hbar,\mathrm{f}} \simeq \mathrm{gr}_\hbar\, \mathscr{D}_X^{\hbar} \simeq \mathscr{D}_X.$$

Proposition 8.3.

(i) *The algebra* $\mathscr{D}_X^{\hbar,0}$ *is faithfully flat over* $\mathscr{A}|_X$.

(ii) *The algebra* $\mathscr{D}_X^{\hbar,\mathrm{loc}}$ *is flat over* $\mathscr{A}^{\mathrm{loc}}|_X$.

Proof. (i) follows from Theorem 1.12.
(ii) follows from (i) and the isomorphism $(\mathscr{D}_X^{\hbar,0})^{\mathrm{loc}} \simeq \mathscr{D}_X^{\hbar,\mathrm{loc}}$. □

The next examples show that the scalar extension functor

$$\mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X^{\hbar,0}) \to \mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X^{\hbar})$$

is neither exact nor full.

Example 8.4. *Let* $X = \mathbb{C}^2$ *with coordinates* (x, y)*. Then* $\hbar\partial_y$ *is injective as an endomorphism of* $\mathscr{D}_X^{\hbar,0}/\langle\hbar\partial_x\rangle$ *but it is not injective as an endomorphism of* $\mathscr{D}_X^{\hbar}/\langle\hbar\partial_x\rangle$*, since* ∂_x *belongs to its kernel. This shows that* $\mathscr{D}_X^{\hbar}$ *is not flat over* $\mathscr{D}_X^{\hbar,0}$.

Example 8.5. *This example was communicated to us by Masaki Kashiwara. Let* $X = \mathbb{C}$ *with coordinate* x*, and denote by* (x, u) *the symplectic coordinates on* $\mathfrak{X} = T^*\mathbb{C}$*. Consider the cyclic* $\mathscr{A}$*-modules*

$$\mathscr{M} = \mathscr{A}/\langle x - u\rangle, \quad \mathscr{N} = \mathscr{A}/\langle x\rangle,$$

and their images in $\mathrm{Mod}(\mathscr{D}_X^\hbar)$

$$\mathscr{M}' = \mathscr{D}_X^\hbar/\langle x - \hbar\partial_x\rangle, \quad \mathscr{N}' = \mathscr{D}_X^\hbar/\langle x\rangle.$$

As their supports in $\mathfrak{X}$ *differ,* $\mathscr{M}$ *and* $\mathscr{N}$ *are not isomorphic as* $\mathscr{A}$*-modules. On the other hand, in* $\mathscr{D}_X^\hbar$ *one has the relation*

$$x \cdot e^{\hbar\partial_x^2/2} = e^{\hbar\partial_x^2/2} \cdot (x - \hbar\partial_x), \tag{8.4}$$

and hence an isomorphism $\mathscr{M}' \xrightarrow{\sim} \mathscr{N}'$ *given by* $[P] \mapsto [P \cdot e^{-\hbar\partial_x^2/2}]$*. In fact, one checks that*

$$\mathscr{H}om_{\mathscr{A}}(\mathscr{M}, \mathscr{N})|_X = 0, \quad \mathscr{H}om_{\mathscr{D}_X^\hbar}(\mathscr{M}', \mathscr{N}') \simeq \mathbb{C}_X^\hbar.$$

§A. Complements on constructible sheaves

Let us review some results, well-known from the specialists (see *e.g.*,[18, Proposition 3.10]), but which are usually stated over a field, and we need to work here over the ring $\mathbb{C}^\hbar$.

Let $\mathbb{K}$ be a commutative unital Noetherian ring of finite global dimension. Assume that $\mathbb{K}$ is syzygic, i.e. that any finitely generated $\mathbb{K}$-module admits a finite projective resolution by finite free modules. (For our purposes we will either have $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{C}^\hbar$).

Let X be a real analytic manifold. Denote by $\mathrm{Mod}_{\mathbb{R}\text{-c}}(\mathbb{K}_X)$ the abelian category of $\mathbb{R}$-constructible sheaves on X and by $\mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{K}_X)$ the bounded derived category of sheaves of $\mathbb{K}$-modules with $\mathbb{R}$-constructible cohomology. Under the above assumptions on the base ring, by [9, Propositions 3.4.3, 8.4.9] one has

Lemma A.1. *The duality functor* $\mathrm{D}'_{\mathbb{K}}(\bullet) = \mathrm{R}\mathscr{H}om_{\mathbb{K}_X}(\bullet, \mathbb{K}_X)$ *induces an involution of* $\mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{K}_X)$*.*

For the next proposition we recall some notations and results of [6, 9]. We consider a simplicial complex $\mathbf{S} = (S, \Delta)$, with set of vertices S and set of simplices Δ. We let $|\mathbf{S}|$ be the realization of $\mathbf{S}$. Thus $|\mathbf{S}|$ is the disjoint union of the realizations $|\sigma|$ of the simplices. For a simplex $\sigma \in \Delta$, the open set $U(\sigma)$ is defined in [9, (8.1.3)]. A sheaf F of $\mathbb{K}$-modules on $|\mathbf{S}|$ is said weakly $\mathbf{S}$-constructible if $F|_{|\sigma|}$ is constant for any $\sigma \in \Delta$. An object $F \in \mathsf{D}^{\mathrm{b}}(\mathbb{K}_{|\mathbf{S}|})$ is said weakly $\mathbf{S}$-constructible if its cohomology sheaves are so. If moreover, all stalks F_x are perfect complexes, F is said $\mathbf{S}$-constructible. By [9, Proposition 8.1.4] we have isomorphisms, for a weakly $\mathbf{S}$-constructible sheaf F and for any $\sigma \in \Delta$

and $x \in |\sigma|$:

$$\Gamma(U(\sigma); F) \xrightarrow{\sim} \Gamma(|\sigma|; F) \xrightarrow{\sim} F_x, \tag{A.1}$$

$$H^j(U(\sigma); F) = H^j(|\sigma|; F) = 0, \quad \text{for } j \neq 0. \tag{A.2}$$

It follows that, for a weakly $\mathbf{S}$-constructible $F \in \mathsf{D}^{\mathrm{b}}(\mathbb{K}_{|\mathbf{S}|})$, the natural morphisms of complexes of $\mathbb{K}$-modules

$$\Gamma(U(\sigma); F) \to \Gamma(|\sigma|; F) \to F_x \tag{A.3}$$

are quasi-isomorphisms.

For $U \subset X$ an open subset, we denote by $\mathbb{K}_U := (\mathbb{K}_X)_U$ the extension by 0 of the constant sheaf on U.

Proposition A.2. *Let $F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathbb{K}_X)$. Then*

(i) *F is isomorphic to a complex*

$$0 \to \bigoplus_{i_a \in I_a} \mathbb{K}_{U_{a,i_a}} \to \cdots \to \bigoplus_{i_b \in I_b} \mathbb{K}_{U_{b,i_b}} \to 0,$$

where the $\{U_{k,i_k}\}_{k,i_k}$'s are locally finite families of relatively compact subanalytic open subsets of X.

(ii) *F is isomorphic to a complex*

$$0 \to \bigoplus_{i_a \in I_a} \Gamma_{V_{a,i_a}} \mathbb{K}_X \to \cdots \to \bigoplus_{i_b \in I_b} \Gamma_{V_{b,i_b}} \mathbb{K}_X \to 0,$$

where the $\{V_{k,i_k}\}_{k,i_k}$'s are locally finite families of relatively compact subanalytic open subsets of X.

Proof. (i) By the triangulation theorem for subanalytic sets (see for example [9, Proposition 8.2.5]) we may assume that F is an $\mathbf{S}$-constructible object in $\mathsf{D}^{\mathrm{b}}(\mathbb{K}_{|\mathbf{S}|})$ for some simplicial complex $\mathbf{S} = (S, \Delta)$. For i an integer, let $\Delta_i \subset \Delta$ be the subset of simplices of dimension $\leq i$ and set $\mathbf{S}_i = (S, \Delta_i)$. We denote by $\mathsf{K}^{\mathrm{b}}(\mathbb{K})$ (resp. $\mathsf{K}^{\mathrm{b}}(\mathbb{K}_{|\mathbf{S}|})$) the category of bounded complexes of $\mathbb{K}$-modules (resp. sheaves of $\mathbb{K}$-modules on $|\mathbf{S}|$) with morphisms up to homotopy. We shall prove by induction on i that there exists a morphism $u_i \colon G_i \to F$ in $\mathsf{K}^{\mathrm{b}}(\mathbb{K}_{|\mathbf{S}|})$ such that:

(a) the G_i^k are finite direct sums of $\mathbb{K}_{U(\sigma_\alpha)}$'s for some $\sigma_\alpha \in \Delta_i$,

(b) $u_i|_{|\mathbf{S}_i|} \colon G_i|_{|\mathbf{S}_i|} \to F|_{|\mathbf{S}_i|}$ is a quasi-isomorphism.

The desired result is obtained for i equal to the dimension of X.

(i)-(1) For $i = 0$ we consider $F|_{|\mathbf{S}_0|} \simeq \bigoplus_{\sigma\in\Delta_0} F_\sigma$. The complexes $\Gamma(U(\sigma);F)$, $\sigma \in \Delta_0$, have finite bounded cohomology by the quasi-isomorphisms (A.3). Hence we may choose bounded complexes of finite free $\mathbb{K}$-modules, $R_{0,\sigma}$, and morphisms $u_{0,\sigma}\colon R_{0,\sigma} \to \Gamma(U(\sigma);F)$ which are quasi-isomorphisms.

We have the natural isomorphism $\Gamma(U(\sigma);F) \simeq a_*\mathscr{H}om_{\mathsf{K}^{\mathrm{b}}(\mathbb{K}_{|\mathbf{S}|})}(\mathbb{K}_{U(\sigma)}, F)$ in $\mathsf{K}^{\mathrm{b}}(\mathbb{K})$, where $a\colon |\mathbf{S}| \to \mathrm{pt}$ is the projection and $\mathscr{H}om$ is the internal Hom functor. We deduce the adjunction formula, for $R \in \mathsf{K}^{\mathrm{b}}(\mathbb{K})$, $F \in \mathsf{K}^{\mathrm{b}}(\mathbb{K}_{|\mathbf{S}|})$:

$$\mathrm{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathbb{K})}(R, \Gamma(U(\sigma);F)) \simeq \mathrm{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathbb{K}_{|\mathbf{S}|})}(R_{U(\sigma)}, F). \tag{A.4}$$

Hence the $u_{0,\sigma}$ induce $u_0\colon G_0 := \bigoplus_{\sigma\in\Delta_0}(R_{0,\sigma})_{U(\sigma)} \to F$. By (A.3) $(u_0)_x$ is a quasi-isomorphism for all $x \in |\mathbf{S}_0|$, so that $u_0|_{|\mathbf{S}_0|}$ also is a quasi-isomorphism, as required.

(i)-(2) We assume that u_i is built and let $H_i = M(u_i)[-1]$ be the mapping cone of u_i, shifted by -1. By the distinguished triangle in $\mathsf{K}^{\mathrm{b}}(\mathbb{K}_{|\mathbf{S}|})$

$$H_i \xrightarrow{v_i} G_i \xrightarrow{u_i} F \xrightarrow{+1} \tag{A.5}$$

$H_i|_{|\mathbf{S}_i|}$ is quasi-isomorphic to 0. Hence $\bigoplus_{\sigma\in\Delta_{i+1}\setminus\Delta_i}(H_i)_{|\sigma|} \to H_i|_{|\mathbf{S}_{i+1}|}$ is a quasi-isomorphism. As above we choose quasi-isomorphisms $u_{i+1,\sigma}\colon R_{i+1,\sigma} \to \Gamma(U(\sigma);H_i)$, $\sigma \in \Delta_{i+1}\setminus\Delta_i$, where the $R_{i+1,\sigma}$ are bounded complexes of finite free $\mathbb{K}$-modules. By (A.4) again the $u_{i+1,\sigma}$ induce a morphism in $\mathsf{K}^{\mathrm{b}}(\mathbb{K}_{|\mathbf{S}|})$

$$u'_{i+1}\colon G'_{i+1} := \bigoplus_{\sigma\in\Delta_{i+1}\setminus\Delta_i}(R_{i+1,\sigma})_{U(\sigma)} \to H_i.$$

For $x \in |\mathbf{S}_{i+1}|\setminus|\mathbf{S}_i|$, $(u'_{i+1})_x$ is a quasi-isomorphism by (A.3), and, for $x \in |\mathbf{S}_i|$, this is trivially true. Hence $u'_{i+1}|_{|\mathbf{S}_{i+1}|}$ is a quasi-isomorphism.

Now we let H_{i+1} and G_{i+1} be the mapping cones of u'_{i+1} and $v_i \circ u'_{i+1}$, respectively. We have distinguished triangles in $\mathsf{K}^{\mathrm{b}}(\mathbb{K}_{|\mathbf{S}|})$

$$G'_{i+1} \xrightarrow{u'_{i+1}} H_i \to H_{i+1} \xrightarrow{+1}, \qquad G'_{i+1} \xrightarrow{v_i\circ u'_{i+1}} G_i \to G_{i+1} \xrightarrow{+1}. \tag{A.6}$$

By the construction of the mapping cone, the definition of G'_{i+1} and the induction hypothesis, G_{i+1} satisfies property (a) above. The octahedral axiom applied to triangles (A.5) and (A.6) gives a morphism $u_{i+1}\colon G_{i+1} \to F$ and a distinguished triangle $H_{i+1} \to G_{i+1} \xrightarrow{u_{i+1}} F \xrightarrow{+1}$. By construction $H_{i+1}|_{|\mathbf{S}_{i+1}|}$ is quasi-isomorphic to 0 so that u_{i+1} satisfies property (b) above.

(ii) Set $G = \mathrm{D}'_{\mathbb{K}}(F)$, and represent it by a bounded complex as in (i). Since U_{k,i_k} corresponds to an open subset of the form $U(\sigma)$ in $|\mathbf{S}|$, the sheaves $\mathbb{K}_{U_{k,i_k}}$ are acyclic for the functor $\mathrm{D}'_{\mathbb{K}}$. Hence $F \simeq \mathrm{D}'_{\mathbb{K}}(G)$ can be represented as claimed. $\square$

Lemma A.3. *Let $F \to G \to 0$ be an exact sequence in $\mathrm{Mod}_{\mathbb{R}\text{-}c}(\mathbb{K}_X)$. Then for any relatively compact subanalytic open subset $U \subset X$, there exists a finite covering $U = \bigcup_{i \in I} U_i$ by subanalytic open subsets such that, for each $i \in I$, the morphism $F(U_i) \to G(U_i)$ is surjective.*

Proof. As in the proof of Proposition A.2 we may assume that F, G and $\mathbb{K}_U$ are constructible sheaves on the realization of some finite simplicial complex (S, Δ). For $\sigma \in \Delta$ the morphism $\Gamma(U(\sigma); F) \to \Gamma(U(\sigma); G)$ is surjective, by (A.1). Since the image of U in $|\mathbf{S}|$ is a finite union of $U(\sigma)$'s, this proves the lemma. $\square$

§B. Complements on subanalytic sheaves

We review here some well-known results (see [11, Chapter 7] and [15]) but which are usually stated over a field, and we need to work here over the ring $\mathbb{C}^{\hbar}$.

Let $\mathbb{K}$ be a commutative unital Noetherian ring of finite global dimension (for our purposes we will either have $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{C}^{\hbar}$). Let X be a real analytic manifold, and consider the natural morphism $\rho \colon X \to X_{\mathrm{sa}}$.

Lemma B.1. *The functor $\rho_* \colon \mathrm{Mod}_{\mathbb{R}\text{-}c}(\mathbb{K}_X) \to \mathrm{Mod}(\mathbb{K}_{X_{\mathrm{sa}}})$ is exact and $\rho^{-1}\rho_*$ is isomorphic to the canonical functor $\mathrm{Mod}_{\mathbb{R}\text{-}c}(\mathbb{K}_X) \to \mathrm{Mod}(\mathbb{K}_X)$.*

Proof. Being a direct image functor, ρ_* is left exact. It is right exact thanks to Lemma A.3. The composition $\rho^{-1}\rho_*$ is isomorphic to the identity on $\mathrm{Mod}(\mathbb{K}_X)$ since the open sets of the site X_{sa} give a basis of the topology of X. $\square$

In the sequel, we denote by $\mathrm{Mod}_{\mathbb{R}\text{-c}}(\mathbb{K}_{X_{\mathrm{sa}}})$ the image by the functor ρ_* of $\mathrm{Mod}_{\mathbb{R}\text{-c}}(\mathbb{K}_X)$ in $\mathrm{Mod}(\mathbb{K}_{X_{\mathrm{sa}}})$. Hence ρ_* induces an equivalence of categories $\mathrm{Mod}_{\mathbb{R}\text{-c}}(\mathbb{K}_X) \simeq \mathrm{Mod}_{\mathbb{R}\text{-c}}(\mathbb{K}_{X_{\mathrm{sa}}})$. We also denote by $\mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{K}_{X_{\mathrm{sa}}})$ the full triangulated subcategory of $\mathsf{D}^{\mathrm{b}}(\mathbb{K}_{X_{\mathrm{sa}}})$ consisting of objects with cohomology in $\mathrm{Mod}_{\mathbb{R}\text{-c}}(\mathbb{K}_{X_{\mathrm{sa}}})$.

Corollary B.2. *The subcategory $\mathrm{Mod}_{\mathbb{R}\text{-}c}(\mathbb{K}_{X_{\mathrm{sa}}})$ of $\mathrm{Mod}(\mathbb{K}_{X_{\mathrm{sa}}})$ is thick.*

Proof. Since ρ_* is fully faithful and exact, $\mathrm{Mod}_{\mathbb{R}\text{-c}}(\mathbb{K}_{X_{\mathrm{sa}}})$ is stable by kernel and cokernel. It remains to see that, for $F, G \in \mathrm{Mod}_{\mathbb{R}\text{-c}}(\mathbb{K}_X)$

$$\mathrm{Ext}^1_{\mathrm{Mod}_{\mathbb{R}\text{-c}}(\mathbb{K}_X)}(F,G) \simeq \mathrm{Ext}^1_{\mathrm{Mod}(\mathbb{K}_{X_{\mathrm{sa}}})}(\rho_*F, \rho_*G).$$

By [6] we know that the first Ext^1 may as well be computed in $\mathrm{Mod}(\mathbb{K}_X)$. Note that the functors ρ^{-1} and $\mathrm{R}\rho_*$ between $\mathsf{D}^{\mathrm{b}}(\mathbb{K}_X)$ and $\mathsf{D}^{\mathrm{b}}(\mathbb{K}_{X_{\mathrm{sa}}})$ are adjoint, and moreover $\rho^{-1}\mathrm{R}\rho_* \simeq \mathrm{id}$. Thus, for $F', G' \in \mathsf{D}^{\mathrm{b}}(\mathbb{K}_X)$ we have

$$\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathbb{K}_{X_{\mathrm{sa}}})}(\mathrm{R}\rho_*F', \mathrm{R}\rho_*G') \simeq \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathbb{K}_X)}(F', G'),$$

and this gives the result. □

This corollary gives the equivalence $\mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{K}_X) \simeq \mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{K}_{X_{\mathrm{sa}}})$, both categories being equivalent to $\mathsf{D}^{\mathrm{b}}(\mathrm{Mod}_{\mathbb{R}\text{-c}}(\mathbb{K}_X))$.

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