A property of the interleaving distance for sheaves

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Abstract

Let X be a real analytic manifold endowed with a distance satisfying suitable properties and let $\mathbf{k}$ be a field. In [PS20], the authors construct a pseudo-distance on the derived category of sheaves of $\mathbf{k}$-modules on X, generalizing a previous construction of [KS18]. We prove here that if the distance between two constructible sheaves with compact support (or more generally, constructible sheaves up to infinity) on X is zero, then these two sheaves are isomorphic. This answers in particular a question of [KS18].

1 Introduction

The interleaving distance was introduced in [CCSG$^+$] and provides a pseudo-metric on the category of persistent modules. It was generalized to multi-persistence modules by M. Lesnick in [Les12, Les15]. In his thesis [Cur14], J. Curry showed how to interpret the notion of persistent modules in the classical language of sheaves. This allows one to interpret the interleaving pseudo-distance as a pseudo-metric on the category of γ-sheaves on a finite-dimensional real vector space where γ is a convex proper cone. In [KS18], M. Kashiwara and P. Schapira systematically treat persistent homology in the framework of *derived* sheaf theory. In particular, they introduce the so-called convolution pseudo-distance for derived sheaves on real normed vector spaces. There, they asked if this pseudo-distance is a distance [KS18, Rem. 2.3], that is if two sheaves with convolution distance zero are necessarily isomorphic. A similar question had been studied by Lesnick in [Les15] where he proved that the interleaving pseudo-distance restricted to finitely presented persistent modules is a distance. This result is related to the aforementioned question as it follows from [BP21] that the restriction of the convolution pseudo-distance to the subcategory of γ-sheaves is equal to the interleaving pseudo-distance. In [BG18], N. Berkouk and G. Ginot proved that the convolution pseudo-distance is in general not a distance and established, by constructing an appropriate matching distance, that it is a distance on constructible sheaves on $\mathbb{R}$. Here,

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following [PS20], we consider a generalization of the convolution pseudo-distance for sheaves on "good" metric spaces (as for instance complete Riemannian manifolds with strictly positive convexity radius). We prove that on a real analytic manifold endowed with a "good" distance, the pseudo-distance on sheaves is a distance when restricted to constructible sheaves with compact support (or more generally, constructible sheaves up to infinity).

2 Review

Throughout this paper, $\mathbf{k}$ is a field. We shall mainly follow the notations of [KS90] for sheaf theory. For a topological space X we denote by Δ the diagonal of $X \times X$ and by a_X the map $X \to \mathrm{pt}$. We denote by $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$ the bounded derived category of sheaves of $\mathbf{k}$-modules. If X is a real analytic manifold, we denote by $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_X)$ the full triangulated subcategory of $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$ consisting of $\mathbb{R}$-constructible sheaves. We denote by ω_X the dualizing complex.

2.1 Kernels

Recall that a topological space X is good if it is Hausdorff, locally compact, countable at infinity and of finite flabby dimension.

Given topological spaces X_i $(i = 1, 2, 3)$ we set $X_{ij} = X_i \times X_j$, $X_{123} = X_1 \times X_2 \times X_3$. We denote by $q_i\colon X_{ij} \to X_i$ and $q_{ij}\colon X_{123} \to X_{ij}$ the projections.

For $A_{ij} \subset X_{ij}$ with $i = 1, 2$, $j = i + 1$, one defines $A_{12} \circ A_{23} \subset X_{13}$ as

$$A_{12} \underset{2}{\circ} A_{23} := q_{13}(q_{12}^{-1} A_{12} \cap q_{23}^{-1} A_{23}). \tag{2.1}$$

For good topological spaces X_i's, one often calls an object $K_{ij} \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{X_{ij}})$ *a kernel*. One defines as usual the composition of kernels by

$$K_{12} \underset{2}{\circ} K_{23} := \mathrm{R}q_{13!}(q_{12}^{-1} K_{12} \overset{\mathrm{L}}{\otimes} q_{23}^{-1} K_{23}). \tag{2.2}$$

If there is no risk of confusion, we write $\circ$ instead of $\underset{2}{\circ}$.

2.2 Distances

For a metric space (X, d), $x_0 \in X$ and $a \in \mathbb{R}_{\geq 0}$, we set

$$\begin{aligned} &B_a(x_0) = \{x \in X; d(x_0, x) \leq a\}, && B_a^\circ(x_0) = \{x \in X; d(x_0, x) < a\} \\ &\Delta_a = \{(x, y) \in X \times X; d(x, y) \leq a\}, && \Delta_a^\circ = \{(x, y) \in X \times X; d(x, y) < a\}, \\ &\Delta^+ = \{(x, y, t) \in X \times X \times \mathbb{R}; d(x, y) \leq t\}, && \Delta^{+,\circ} = \{(x, y, t) \in X \times X \times \mathbb{R}; d(x, y) < t\}. \end{aligned}$$

Following [PS20], we say that a metric space (X, d) is good, or simply that the distance is good, if the underlying topological space is good and moreover

$$\text{(2.3)}\quad \begin{cases} \text{there exists some } \alpha_X > 0 \text{ such that for all } 0 \le a, b \text{ with } a+b \le \alpha_X\text{:} \\ \text{(i) for any } x_1, x_2 \in X,\ B_a(x_1) \cap B_b(x_2) \text{ is contractible or empty (in} \\ \text{particular, for any } x \in X,\ B_a(x) \text{ is contractible),} \\ \text{(ii) the two projections } q_1 \text{ and } q_2 \text{ are proper on } \Delta_a, \\ \text{(iii) } \Delta_a \circ \Delta_b = \Delta_{a+b}. \end{cases}$$

In order to construct a distance on sheaves in this situation, the main idea of [PS20] is to use the kernel $\mathbf{k}_{\Delta_a}$ when $0 \le a < \alpha_X$ and to replace it by a composition of kernels $\mathbf{k}_{\Delta_b}$ with $0 \le b < \alpha_X$ otherwise.

We shall also consider the hypotheses (2.4) below which insure that the kernels $\mathbf{k}_{\Delta_a}$ are invertible (see [PS20, Prop. 2.2.3]).

Let U be an open subset of a real C^0-manifold M. We say that U is locally topologically convex (l.t.c. for short) in M if each $x \in M$ admits an open neighborhood W such that there exists a topological isomorphism $\varphi\colon W \xrightarrow{\sim} V$, with V open in a real vector space, such that $\varphi(W \cap U)$ is convex. Clearly, if U is l.t.c. then it is l.c.t.

$$\text{(2.4)}\quad \begin{cases} \text{The good metric space } X \text{ is a } C^0\text{-manifold and} \\ \text{(a) for } 0 < a \le \alpha_X, \text{ the set } \Delta_a^\circ \text{ is l.t.c. in } X \times X, \\ \text{(b) the set } \Delta^{+,\circ} \text{ is l.t.c. in } X \times X \times]-\infty, \alpha_X[. \\ \text{(c) For } x, y \in X, \text{ setting } Z_a(x,y) = B_a(x) \cap B_a^\circ(y), \text{ one has} \\ \quad \mathrm{R}\Gamma(X; \mathbf{k}_{Z_a(x,y)}) \simeq 0 \text{ for } x \ne y \text{ and } 0 < a \le \alpha_X. \end{cases}$$

In loc. cit. it is shown that complete Riemannian manifolds with strictly positive convexity radius as well as normed vector spaces satisfy conditions (2.3) and (2.4). Moreover, given a good metric space (X, d), one can naturally associate a pseudo-distance on the objects of the derived category $\mathrm{D^b}(\mathbf{k}_X)$, generalizing the convolution distance first introduced in [KS18].

2.3 Constructible sheaves

Here, we work in the framework of real analytic manifolds and use the notions of being subanalytic or constructible "up to infinity", following [Sch20].

Recall that a b-analytic manifold $X_\infty = (X, \widehat{X})$ is the data of real analytic manifold $\widehat{X}$ and a subanalytic open relatively compact subset $X \subset \widehat{X}$. One denotes by $j_X\colon X \hookrightarrow \widehat{X}$ the embedding. A morphism of b-analytic manifolds is a morphism of real analytic manifolds $f\colon X \to Y$ whose graph is subanalytic in $\widehat{X} \times \widehat{Y}$.

One defines naturally the full triangulated subcategory $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{X_\infty})$ of $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_X)$ consisting of sheaves constructible up to infinity. These are the objects F of $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_X)$ such that $j_{X!}F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{\widehat{X}})$. The advantage of the notion of being constructible up to infinity is that it is stable under the six operations, in particular, by direct images of morphisms of b-analytic manifolds [Sch20].

In the sequel, instead of writing "subanalytic up to infinity " or "constructible up to infinity", we shall write "b-subanalytic" or "b-constructible".

Let $F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_X)$. Recall that F is b-constructible if $j_{X!}F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{\widehat{X}})$. One denotes by $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{X_\infty})$ the full triangulated subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_X)$ consisting of sheaves b-constructible.

3 Main theorem

In this section, X_∞ is a b-analytic manifold endowed with a good distance.

Recall the definition of being a-isomorphic and the associated pseudo-distance dist, following [KS18] generalized in [PS20].

Definition 3.1. Let $F, G \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$ and let $a > 0$. One says that F and G are a-isomorphic if there exist morphisms $u_a \colon F \circ \mathbf{k}_{\Delta_a} \to G$, $v_a \colon G \circ \mathbf{k}_{\Delta_a} \to F$ such that $F \circ \mathbf{k}_{\Delta_{2a}} \to G \circ \mathbf{k}_{\Delta_a} \to F$ and $G \circ \mathbf{k}_{\Delta_{2a}} \to F \circ \mathbf{k}_{\Delta_a} \to G$ are the natural morphisms associated with $\mathbf{k}_{\Delta_{2a}} \to \mathbf{k}_{\Delta}$.

One sets

$$\mathrm{dist}(F, G) = \inf\{a \in [0, +\infty]; F \text{ and } G \text{ are } a\text{-isomorphic}\}.$$

In the sequel, when considering thickenings of the diagonal Δ_a, we shall assume $0 \le a < \alpha_X$, where α_X is given in (2.3). For $0 \le a \le b < \alpha_X$, the morphism

$$\mathbf{k}_{\Delta_b} \to \mathbf{k}_{\Delta_a}$$

define the morphisms

$$\mathrm{RHom}\,(F \circ \mathbf{k}_{\Delta_a}, G) \to \mathrm{RHom}\,(F \circ \mathbf{k}_{\Delta_b}, G), \tag{3.1}$$

$$\mathrm{RHom}\,(F, G \circ \mathbf{k}_{\Delta_b}) \to \mathrm{RHom}\,(F, G \circ \mathbf{k}_{\Delta_a}). \tag{3.2}$$

One of the central step of the proof is to establish that the morphisms (3.1) and (3.2) are isomorphisms. For that purpose, we will prove that these morphisms spaces can be written as the cohomology of a constructible sheaf on $\mathbb{R}$ supported by balls of radius a and b. The constructibility will imply that, for $0 \le a \le b$ sufficiently small, these cohomology groups are isomorphic which implies the result.

Denote by $q_1 \colon X \times \mathbb{R} \to X$ and $q_2 \colon X \times \mathbb{R} \to \mathbb{R}$ the projections. Also denote by I_a the closed interval $[-a, a]$ of $\mathbb{R}$ and by I_a° the open interval.

Lemma 3.2. *Let $F, G \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{X_\infty})$.*

(i) *For $a \ge 0$ one has,*

$$\mathrm{R}\Gamma_{I_a}(\mathbb{R}; \mathrm{R}q_{2*}\mathrm{R}\mathscr{H}om\,(F \circ \mathbf{k}_{\Delta^+}, q_1^! G)) \simeq \mathrm{RHom}\,(F \circ \mathbf{k}_{\Delta_a}, G).$$

(ii) *For $a > 0$ one has,*

$$\mathrm{R}\Gamma(I_a^\circ; \mathrm{R}q_{2*}\mathrm{R}\mathscr{H}om\,(F \circ \mathbf{k}_{\Delta^{+,\circ}}, q_1^! G)) \simeq \mathrm{RHom}\,(F \circ \mathbf{k}_{\Delta_a^\circ}, G)[1].$$

Proof. (i) Set $K := \mathrm{R}\mathscr{H}om\,(F \circ \mathbf{k}_{\Delta^+}, q_1^! G)$. Then

$$\begin{aligned}\mathrm{R}\Gamma_{I_a}\mathrm{R}q_{2*}K &\simeq \mathrm{R}q_{2*}\mathrm{R}\Gamma_{q_2^{-1}I_a}K \\ &\simeq \mathrm{R}q_{2*}\mathrm{R}\mathscr{H}om\,((F \circ \mathbf{k}_{\Delta^+}) \otimes \mathbf{k}_{q_2^{-1}I_a}, q_1^! G)).\end{aligned}$$

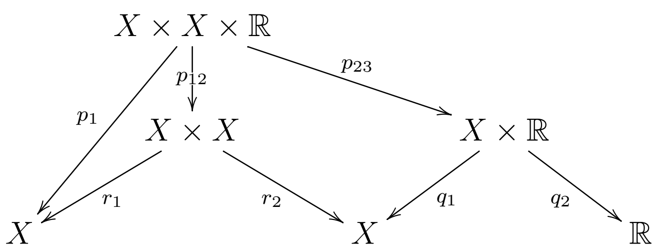
Therefore,

$$\begin{aligned}\mathrm{R}\Gamma_{I_a}(\mathbb{R}; \mathrm{R}q_{2*}K) &\simeq \mathrm{R}a_{X*}\mathrm{R}q_{1*}\mathrm{R}\mathscr{H}om\,((F \circ \mathbf{k}_{\Delta^+}) \otimes \mathbf{k}_{q_2^{-1}I_a}, q_1^! G)) \\ &\simeq \mathrm{R}a_{X*}\mathrm{R}\mathscr{H}om\,(\mathrm{R}q_{1!}\big((F \circ \mathbf{k}_{\Delta^+}) \otimes \mathbf{k}_{q_2^{-1}I_a}\big), G).\end{aligned}$$

To conclude, let us check the isomorphism

$$\mathrm{R}q_{1!}\big((F \circ \mathbf{k}_{\Delta^+}) \otimes \mathbf{k}_{q_2^{-1}I_a}\big) \simeq F \circ \mathbf{k}_{\Delta_a}. \tag{3.3}$$

Consider the diagram



One has

$$\begin{aligned}\mathrm{R}q_{1!}\big((F \circ \mathbf{k}_{\Delta^+}) \otimes \mathbf{k}_{q_2^{-1}I_a}\big) &\simeq \mathrm{R}q_{1!}\mathrm{R}p_{23!}\big(p_1^{-1}F \otimes \mathbf{k}_{\Delta^+} \otimes p_{23}^{-1}\mathbf{k}_{q_2^{-1}I_a}\big) \\ &\simeq \mathrm{R}r_{2!}\mathrm{R}p_{12!}\big(p_{12}^{-1}r_1^{-1}F \otimes \mathbf{k}_{\Delta^+ \cap p_{23}^{-1}q_2^{-1}I_a})\big) \\ &\simeq \mathrm{R}r_{2!}\big(r_1^{-1}F \otimes \mathrm{R}p_{12!}\mathbf{k}_{\Delta^+ \cap p_{23}^{-1}q_2^{-1}I_a}\big)\end{aligned}$$

We now remark that $\Delta^+ \cap p_{23}^{-1}q_2^{-1}I_a \subset p_{12}^{-1}(\Delta_a)$ and p_{12} restricted to $\Delta^+ \cap p_{23}^{-1}q_2^{-1}I_a$ is proper. Hence, there are natural morphisms (the first morphism is obtained by adjunction)

$$\mathbf{k}_{\Delta_a} \to \mathrm{R}p_{12*}p_{12}^{-1}\mathbf{k}_{\Delta_a} \to \mathrm{R}p_{12*}\mathbf{k}_{\Delta^+ \cap p_{23}^{-1}q_2^{-1}I_a} \xleftarrow{\sim} \mathrm{R}p_{12!}\mathbf{k}_{\Delta^+ \cap p_{23}^{-1}q_2^{-1}I_a} \tag{3.4}$$

which define

$$\mathbf{k}_{\Delta_a} \to \mathrm{R}p_{12!}\mathbf{k}_{\Delta^+ \cap p_{23}^{-1}q_2^{-1}I_a} \tag{3.5}$$

It remains to prove that this last morphism is an isomorphism. One has $p_{12}(\Delta^+ \cap p_{23}^{-1}q_2^{-1}I_a) = \Delta_a$ and the fibers of p_{12} above $(x, y) \in X \times X$ is the interval $[d(x, y), a]$ which is contractible or empty. This, together with [KS90, Prop. 2.5.2], proves that (3.5) is an isomorphism, hence proves (3.3).

(ii) Replacing I_a with I_a° and Δ^+ with $\Delta^{+,\circ}$ in the proof of (i) it remains to show

$$\mathrm{R}q_{1!}\big((F \circ \mathbf{k}_{\Delta^{+,\circ}}) \otimes \mathbf{k}_{q_2^{-1}I_a^\circ}\big) \simeq F \circ \mathbf{k}_{\Delta_a^\circ}[-1]. \tag{3.6}$$

By the same proof as for (3.3) we have reduced to showing:

$$\mathrm{R}p_{12!}\mathbf{k}_{\Delta^{+,\circ}\cap p_{23}^{-1}q_2^{-1}I_a^\circ} \simeq \mathbf{k}_{\Delta_a^\circ}[-1]. \tag{3.7}$$

This isomorphism is deduced from (3.5) by duality. □

Lemma 3.3. *Let $F, G \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{X_\infty})$ and let $0 \le a < \alpha_X$. Then there exists $c > a$ such that for $a \le b \le c$, (3.1) and (3.2) are isomorphisms.*

Proof. (i) Let us treat (3.1). Set $H := \mathrm{R}q_{2*}\mathrm{R}\mathscr{H}om\,(F \circ \mathbf{k}_{\Delta^+}, q_1^! G)$. Since $F, G \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{X_\infty})$, then $H \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{\mathbb{R}})$ by [Sch20, §2.] and therefore by [KS90, Lem. 8.4.7],

$$\mathrm{R}\Gamma_{I_a}(\mathbb{R}; H) \xrightarrow{\sim} \mathrm{R}\Gamma_{I_b}(\mathbb{R}; H)$$

for $a \le b \le c$ for some $c > a$. Applying Lemma 3.2 (i), we get the result.
(ii) Let us treat (3.2). First, using [PS20, Prop. 2.2.3], we obtain

$$\begin{aligned}\mathrm{RHom}\,(F, G \circ \mathbf{k}_{\Delta_a}) &\simeq \mathrm{RHom}\,(F \circ (\mathbf{k}_{\Delta_a^\circ} \otimes r_2^{-1}\omega_X), G)\\ &\simeq \mathrm{RHom}\,((F \otimes \omega_X) \circ \mathbf{k}_{\Delta_a^\circ}, G).\end{aligned}$$

Second, set $H := \mathrm{R}q_{2*}\mathrm{R}\mathscr{H}om\,((F \otimes \omega_X) \circ \mathbf{k}_{\Delta^{+,\circ}}, q_1^! G))$. Since $F, G \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{X_\infty})$, then $H \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{\mathbb{R}})$ and therefore by [KS90, Lem. 8.4.7]

$$\mathrm{R}\Gamma(I_a^\circ; H) \xrightarrow{\sim} \mathrm{R}\Gamma(I_b^\circ; H)$$

for $a \le b \le c$ for some $c > a$. Applying Lemma 3.2 (ii), we get the result.

□

Theorem 3.4. *Let X_∞ be a b-analytic manifold endowed with a good distance and satisfying (2.4). Let $F, G \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{X_\infty})$. If $\mathrm{dist}(F, G) \le a$ with $0 \le a < \alpha_X$, then F and G are a-isomorphic. In particular, if $\mathrm{dist}(F, G) = 0$, then F and G are isomorphic.*

Proof. The proof proceeds in four steps. Let $0 \le a \le b$ with b small enough so that Lemma 3.3 holds.
(i) Consider Diagram (3.8) below

$$\begin{array}{ccc}
\mathrm{Hom}\,(F\circ\mathbf{k}_{\Delta_{2a}}, G\circ\mathbf{k}_{\Delta_a}) \times \mathrm{Hom}\,(G\circ\mathbf{k}_{\Delta_a}, F) & \xrightarrow{\circ} & \mathrm{Hom}\,(F\circ\mathbf{k}_{\Delta_{2a}}, F)\\
\downarrow & ① & \downarrow\\
\mathrm{Hom}\,(F\circ\mathbf{k}_{\Delta_{a+b}}, G\circ\mathbf{k}_{\Delta_b}) \times \mathrm{Hom}\,(G\circ\mathbf{k}_{\Delta_b}, F\circ\mathbf{k}_{\Delta_{b-a}}) & \xrightarrow{\circ} & \mathrm{Hom}\,(F\circ\mathbf{k}_{\Delta_{a+b}}, F\circ\mathbf{k}_{\Delta_{b-a}})\\
\downarrow & ② & \downarrow\\
\mathrm{Hom}\,(F\circ\mathbf{k}_{\Delta_{a+b}}, G\circ\mathbf{k}_{\Delta_b}) \times \mathrm{Hom}\,(G\circ\mathbf{k}_{\Delta_b}, F) & \xrightarrow{\circ} & \mathrm{Hom}\,(F\circ\mathbf{k}_{\Delta_{a+b}}, F)\\
\downarrow & ③ & \downarrow\\
\mathrm{Hom}\,(F\circ\mathbf{k}_{\Delta_{2b}}, G\circ\mathbf{k}_{\Delta_b}) \times \mathrm{Hom}\,(G\circ\mathbf{k}_{\Delta_b}, F) & \xrightarrow{\circ} & \mathrm{Hom}\,(F\circ\mathbf{k}_{\Delta_{2b}}, F)
\end{array} \tag{3.8}$$

Let us show that this diagram commutes.

Diagram ① is obtained by applying the functor $\cdot \circ \mathbf{k}_{\Delta_{b-a}}$ to the first line.

Diagram ② is obtained by composing the second line with the canonical morphism $F \circ \mathbf{k}_{\Delta_{b-a}} \to F$.

Diagram ③ is obtained by composing the third line with the canonical morphism $F \circ \mathbf{k}_{\Delta_{2b}} \to F \circ \mathbf{k}_{\Delta_{a+b}}$.

Hence, Diagram (3.8) commutes.

(ii) Assume $\mathrm{dist}(F, G) \leq a$, let $b > a$ and let $u_b \in \mathrm{Hom}\,(F \circ \mathbf{k}_{\Delta_b}, G)$ and $v_b \in \mathrm{Hom}\,(G \circ \mathbf{k}_{\Delta_b}, F)$ be as in Definition 3.1. Denote by ε_b the natural morphism associated with $\mathbf{k}_{\Delta_b} \to \mathbf{k}_\Delta$, by $\Psi_b \colon \mathrm{D^b}(\mathbf{k}_X) \to \mathrm{D^b}(\mathbf{k}_X)$ the functor $L \mapsto L \circ \mathbf{k}_{\Delta_b}$ and by $\Psi_F \colon \mathrm{D^b}(\mathbf{k}_{X\times X}) \to \mathrm{D^b}(\mathbf{k}_X)$, $K \mapsto F \circ K$. Then, the b-isomorphism equations are explicitely given by

$$v_b \circ \Psi_b(u_b) = \Psi_F(\varepsilon_{2b}). \tag{3.9}$$

(iii) The vertical arrows in Diagram ① are isomorphisms thanks to (2.4) (see [PS20, Prop. 2.2.3]). The vertical arrows in Diagram ② and ③ are isomorphisms thanks to Lemma 3.3.

(iv) Hence, there exist $u_a \in \mathrm{Hom}\,(F \circ \mathbf{k}_{\Delta_a}, G)$ and $v_a \in \mathrm{Hom}\,(G \circ \mathbf{k}_{\Delta_a}, F)$ whose images are the morphisms u_b and v_b. Using the vertical isomorphism, equation (3.9) translates to the same equation with b replaced by a. The same result holds with F and G interchanged. This complete the proof. □

Corollary 3.5. *Let X be a real analytic manifold endowed with a good distance and satisfying (2.4). Let $F, G \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_X)$, both with compact support. Assume that* $\mathrm{dist}(F, G) = 0$. *Then $F \simeq G$.*

Proof. Let Y be an open subanalytic subset of X containing the supports of F and G. Then regard $Y_\infty = (Y, X)$ as a b-analytic manifold and apply Theorem 3.4. □

Remark 3.6. When the space X is a finite dimensional real vector space endowed with a closed proper convex subanalytic cone γ with nonempty interior and a vector v in the interior of γ, then, thanks to [BP21, Cor. 5.9], Theorem 3.4 implies the same results for γ-sheaves endowed with the interleaving distance associated with the pair (γ, v) (see [BP21, Def. 4.8]) .

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