Ind-Sheaves, distributions, and microlocalization

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1 Introduction

If $\mathcal{C}$ is an abelian category, the category $\mathrm{Ind}(\mathcal{C})$ of ind-objects of $\mathcal{C}$ has many remarkable properties: it is much bigger than $\mathcal{C}$, it contains $\mathcal{C}$, and furthermore it is dual (in a certain sense) to $\mathcal{C}$.

We introduce here the category of ind-sheaves on a locally compact space X as the category of ind-objects of the category of sheaves with compact supports. This construction has some analogy with that of distributions: the space of distributions is bigger than that of functions, and is dual to that of functions with compact support. This last condition implies the local nature of distributions, and similarly, one proves that the category of ind-sheaves defines a stack.

In our opinion, what makes the theory of ind-sheaves on manifolds really interesting is twofold.

(i) Ind-sheaves allows us to treat in the formalism of sheaves (the "six operations") functions with growth conditions. For example, on a complex manifold X, one can define the ind-sheaf of "tempered holomorphic functions" $\mathcal{O}_X^t$ or the ind-sheaf of "Whitney holomorphic functions" $\mathcal{O}_X^{\mathrm{w}}$, and obtain for example the sheaf of Schwartz's distributions using Sato's construction of hyperfunctions, simply replacing $\mathcal{O}_X$ with $\mathcal{O}_X^t$.

(ii) On a real manifold, one can construct a microlocalization functor μ_X which sends sheaves (i.e. objects of the derived category of sheaves) on X to ind-sheaves on T^*X, and the Sato functor of microlocalization along a submanifold $M \subset X$ (see [9]) becomes the usual functor $\mathcal{H}om\,(\pi^{-1}(k_M), \cdot)$ (where $\pi : T^*X \to X$ is the projection) composed with $\mu_X(\cdot)$.

When combining (i) and (ii), one can treat in a unified way various objects of classical analysis.

The results presented here are extracted from [7]. We refer to [5] for an exposition on derived categories and sheaves, and to [3] for the theory of ind-objects.

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2 Ind-objects

Let $\mathcal{U}$ be a universe (see [8]). If $\mathcal{C}$ is a $\mathcal{U}$-category, one denotes by $\mathcal{C}^\vee$ the category of contravariant functors from $\mathcal{C}$ to **Set**, the category of ($\mathcal{U}$-small) sets. One sends $\mathcal{C}$ into $\mathcal{C}^\vee$ by the fully faithful functor $h^\vee : X \mapsto \mathrm{Hom}_{\mathcal{C}}(\cdot, X)$.

Let I be a small filtrant category and let $i \mapsto X_i$ be an inductive system in $\mathcal{C}$ indexed by I. One denotes by "$\varinjlim\limits_i$"X_i the object of $\mathcal{C}^\vee$ defined by

$$\mathcal{C} \ni Y \mapsto \varinjlim_{i \in I} \mathrm{Hom}_{\mathcal{C}}(Y, X_i).$$

An ind-object in $\mathcal{C}$ is an object of $\mathcal{C}^\vee$ which is isomorphic to such a "$\varinjlim\limits_i$"X_i. One denotes by $\mathrm{Ind}(\mathcal{C})$ the full subcategory of $\mathcal{C}^\vee$ consisting of ind-objects and identifies $\mathcal{C}$ with a full subcategory of $\mathrm{Ind}(\mathcal{C})$ by the functor $h^\vee$.

From now on we shall assume that $\mathcal{C}$ is abelian. The subcategory $\mathcal{C}^{\vee,add}$ of $\mathcal{C}^\vee$ consisting of additive functors is clearly abelian and the functor $h^\vee : \mathcal{C} \longrightarrow \mathcal{C}^{\vee,add}$ is left exact.

Since the functor Hom is left exact and filtrant inductive limits are exact, any ind-object defines a left exact functor on $\mathcal{C}$. The converse holds if $\mathcal{C}$ is small.

Example 2.1. Let k be a field. Define the inductive system in $\mathrm{Mod}(k)$ indexed by $\mathbb{N}$ by setting $X_n = k^n$, with the natural injections $X_n \hookrightarrow X_{n+1}$. Then "$\varinjlim\limits_n$"$X_n$ is the functor $Y \mapsto \varinjlim\limits_n \mathrm{Hom}_k(Y, X_n)$. This object does not belong to $\mathrm{Mod}(k)$ since there are no vector space Z such that $\varinjlim\limits_n \mathrm{Hom}_k(Y, X_n) \simeq \mathrm{Hom}_k(Y, Z)$ for all vector spaces Y.

If $f : X \longrightarrow Y$ is a morphism in $\mathrm{Ind}(\mathcal{C})$, one can construct a small filtrant category I such that $X =$ "$\varinjlim\limits_i$"X_i, $Y =$ "$\varinjlim\limits_i$"Y_i and $f =$ "$\varinjlim\limits_i$"f_i, with $f_i : X_i \longrightarrow Y_i$. Then "$\varinjlim\limits_i$" $\ker f_i$ and "$\varinjlim\limits_i$" $\mathrm{coker}\, f_i$ are the kernel and the cokernel of f in $\mathrm{Ind}(\mathcal{C})$.

Theorem 2.2. (i) *The category* $\mathrm{Ind}(\mathcal{C})$ *is abelian.*

(ii) *The natural functor* $\mathcal{C} \longrightarrow \mathrm{Ind}(\mathcal{C})$ *is fully faithful and exact, and* $\mathcal{C}$ *is thick in* $\mathrm{Ind}(\mathcal{C})$.

(iii) *The natural functor* $\mathrm{Ind}(\mathcal{C}) \longrightarrow \mathcal{C}^{\vee,add}$ *is fully faithful and left exact.*

(iv) *The category* $\mathrm{Ind}(\mathcal{C})$ *admits right exact small inductive limits. Moreover, inductive limits over small filtrant categories are exact.*

(v) *If small products of objects of* $\mathcal{C}$ *exist in* $\mathrm{Ind}(\mathcal{C})$, *then* $\mathrm{Ind}(\mathcal{C})$ *admits small projective limits and such limits are left exact.*

We shall denote by "$\varinjlim$" the usual inductive limit in $\mathrm{Ind}(\mathcal{C})$. With this convention, one can identify $\mathcal{C}$ with its image in $\mathrm{Ind}(\mathcal{C})$ by $h^\vee$ without risk of confusion.

Consider an additive functor $F : \mathcal{C} \longrightarrow \mathcal{C}'$. It defines an additive functor $IF : \mathrm{Ind}(\mathcal{C}) \longrightarrow \mathrm{Ind}(\mathcal{C}')$. If there is no risk of confusion, we shall write F instead of IF. If F is right exact, or left exact, or fully faithhful, so is IF.

We shall have to consider the derived category of $\mathrm{Ind}(\mathcal{C})$. One proves that the natural functor $D^b(\mathcal{C}) \longrightarrow D^b_{\mathcal{C}}(\mathrm{Ind}(\mathcal{C}))$ is an equivalence.

Even if $\mathcal{C}$ has enough injectives, we cannot prove that $\mathrm{Ind}(\mathcal{C})$ has the same property. However, quasi-injective objects, i.e. ind-objects which are exact functors on $\mathcal{C}$, are sufficient for many applications.

3 Ind-sheaves

Let X be a locally compact topological space which is countable at infinity. Let k be a field and let $\mathcal{A}$ be a sheaf of commutative k-unitary algebras on X.

One denotes by $\mathrm{Mod}(\mathcal{A})$ the abelian category of sheaves of $\mathcal{A}$-modules, and by $\mathrm{Mod}^c(\mathcal{A})$ the full subcategory consisting of sheaves with compact support.

We call an object of $\mathrm{Ind}(\mathrm{Mod}^c(\mathcal{A}))$ an ind-sheaf on X and we set for short:

$$\mathrm{I}(\mathcal{A}) := \mathrm{Ind}(\mathrm{Mod}^c(\mathcal{A})).$$

If U is an open subset of X, one defines the restriction of F to U as follows. If $F = \text{``}\varinjlim_i\text{''} F_i$, one sets

$$F|_U = \text{``}\varinjlim_{i, V\subset\subset U}\text{''}(F_{iV}|_U).$$

Then one proves that for F and G in $\mathrm{I}(\mathcal{A})$, the presheaf $U \mapsto \mathrm{Hom}_{\mathrm{I}(\mathcal{A}|_U)}(F|_U, G|_U)$ is a sheaf. We shall denote it by $\mathcal{H}om\,(F, G)$.

In fact, there is a better result: $U \mapsto \mathrm{I}(\mathcal{A}|_U)$ is a stack. Roughly speaking, this means a sheaf of categories.

Note that $\mathrm{Ind}(\mathrm{Mod}(\mathcal{A}))$ is not a stack.

Example 3.1. Let $X = \mathbb{R}$, and let $\mathcal{A} = k_X$. Let $F = k_X, G_n = k_{[n,+\infty[}, G = \text{``}\varinjlim_n\text{''} G_n$. Then $G|_U = 0$ for any relatively compact open subset U of X. On the other hand, $\mathrm{Hom}_{\mathrm{Ind}(\mathrm{Mod}(k_X))}(k_X, G) \simeq \varinjlim_n \mathrm{Hom}_{k_X}(k_X, G_n) \simeq k$.

We construct the functors

$$\iota_X : \mathrm{Mod}(\mathcal{A}) \longrightarrow \mathrm{I}(\mathcal{A}), \qquad \iota_X F = \text{``}\varinjlim_{U\subset\subset X}\text{''} F_U,$$
$$\alpha_X : \mathrm{I}(\mathcal{A}) \longrightarrow \mathrm{Mod}(\mathcal{A}), \qquad \alpha_X(\text{``}\varinjlim_i\text{''} F_i) = \varinjlim_i F_i.$$

The functor α_X admits a left adjoint $\beta_X : \mathrm{Mod}(\mathcal{A}) \to \mathrm{I}(\mathcal{A})$.

These functors satisfy:

(i) ι_X and α_X are exact, fully faithful, and commute with $\varinjlim$ and $\varprojlim$,

(ii) β_X is right exact, fully faithful and commutes with $\varinjlim$, and if $\mathcal{A}$ is left coherent, β_X is exact.

(iii) α_X is left adjoint to ι_X and is right adjoint to β_X,

(iv) $\alpha_X \circ \iota_X \simeq \mathrm{id}$ and $\alpha_X \circ \beta_X \simeq \mathrm{id}$.

If Z is locally closed in X and F is a sheaf on X, recall that the sheaf F_Z is 0 on $X \setminus Z$ and is isomorphic to $F|_Z$ on Z. We set

$$\widetilde{\mathcal{A}_Z} = \beta_X(\mathcal{A}_Z).$$

Since β_X is right exact and commutes with $\varinjlim$, it is characterized by its values on the sheaves $\mathcal{A}_U$, U open. If U is open and S is closed, one has

$$\widetilde{\mathcal{A}_U} \simeq \text{“}\varinjlim_{V \subset\subset U}\text{”}\mathcal{A}_V, \quad V \text{ open, and} \qquad \widetilde{\mathcal{A}_S} \simeq \text{“}\varinjlim_{V \supset S}\text{”}\mathcal{A}_{\bar{V}}, \quad V \text{ open.}$$

Note that $\widetilde{\mathcal{A}_U} \to \mathcal{A}_U$ is a monomorphism and $\widetilde{\mathcal{A}_S} \to \mathcal{A}_S$ is an epimorphism.

Example 3.2. Let X be a real manifold of dimension $n \geq 1$ and let $\mathcal{A} = k_X$. For short, we shall write k_Z instead of $(k_X)_Z$. Let $a \in X$. Define $N_a \in \mathrm{I}(k_X)$ by the exact sequence

$$0 \to N_a \to \widetilde{k_{\{a\}}} \to k_{\{a\}} \to 0. \tag{3.1}$$

The derived functor $\mathrm{RHom}_{\mathrm{I}(k_X)}(\cdot, \cdot)$ is well-defined and moreover if G and F_i are sheaves on X,

$$H^k\mathrm{RHom}_{\mathrm{I}(k_X)}(G, \text{“}\varinjlim_i\text{”} F_i) \simeq \varinjlim_i H^k\mathrm{RHom}_{k_X}(G, F_i). \tag{3.2}$$

Since $\varinjlim_{a \in V} H^n_{\{a\}}(X; k_{\overline{V}}) \neq 0$, we find that the morphism $\widetilde{k_{\{a\}}} \to k_{\{a\}}$ is not an isomorphism, hence $N_a \neq 0$.

On the other hand, for any open neighborhood U of a, we have

$$\mathrm{RHom}_{\mathrm{I}(k_X)}(k_U, N_a) \simeq 0.$$

4 Operations on ind-sheaves

We define the functors internal tensor product, denoted $\otimes$, and internal $\mathcal{H}om$, denoted $\mathcal{I}hom$

$$\otimes : \mathrm{I}(\mathcal{A}) \times \mathrm{I}(\mathcal{A}) \longrightarrow \mathrm{I}(\mathcal{A}),$$
$$\mathcal{I}hom : \mathrm{I}(\mathcal{A})^{\mathrm{op}} \times \mathrm{I}(\mathcal{A}) \longrightarrow \mathrm{I}(\mathcal{A}),$$

by the formulas:

$$\begin{aligned} \text{“}\varinjlim_i\text{”} F_i \otimes \text{“}\varinjlim_j\text{”} G_j &= \text{“}\varinjlim_{i,j}\text{”} (F_i \otimes G_j), \\ \mathcal{I}hom\,(\text{“}\varinjlim_i\text{”} F_i, \text{“}\varinjlim_j\text{”} G_j) &= \varprojlim_i \text{“}\varinjlim_j\text{”} \mathcal{H}om\,(F_i, G_j). \end{aligned}$$

One has:

$$\alpha_X \mathcal{I}hom\,(F, G) \simeq \mathcal{H}om\,(F, G).$$

The functor $\otimes$ is right exact and commutes with “$\varinjlim$” and the functor $\mathcal{I}hom$ is left exact. The functors $\otimes$ and $\mathcal{I}hom$ are adjoint:

$$\mathrm{Hom}_{\mathrm{I}(\mathcal{A})}(F \otimes_{\mathcal{A}} K, G) \simeq \mathrm{Hom}_{\mathrm{I}(\mathcal{A})}(G, \mathcal{I}hom_{\mathcal{A}}(K, G)),$$

Now consider a continuous map $f : X \longrightarrow Y$ of locally compact spaces, and assume $\mathcal{A} = f^{-1}\mathcal{B}$, for a sheaf of rings $\mathcal{B}$ on Y.

If $G = \text{“}\varinjlim_i\text{”} G_i$ is an ind-sheaf on Y, we define

$$f^{-1}G = \text{“}\varinjlim_i\text{”} (f^{-1}G_i)_U,\ U \subset\subset X,\ \ U \text{ open}.$$

The functor $f^{-1} : \mathrm{I}(\mathcal{B}) \longrightarrow \mathrm{I}(\mathcal{A})$ is exact and commutes with “$\varinjlim$” and $\otimes$. Moreover, it commutes with the functors ι, α and β.

If $F = \text{“}\varinjlim_i\text{”} F_i$ is an ind-sheaf on X, we define:

$$f_*F = \varprojlim_K \text{“}\varinjlim_i\text{”} f_* F_{iK},\ \ K \text{ compact}.$$

The functor $f_* : \mathrm{I}(\mathcal{A}) \longrightarrow \mathrm{I}(\mathcal{B})$ is left exact and commutes with $\varprojlim$. Moreover, it commutes with the functors ι and α.

The two functors f^{-1} and f_* are adjoint. More precisely, for $F \in \mathrm{I}(\mathcal{A})$ and $G \in \mathrm{I}(\mathcal{B})$, we have

$$\mathrm{Hom}_{\mathrm{I}(\mathcal{A})}(f^{-1}G, F) \simeq \mathrm{Hom}_{\mathrm{I}(\mathcal{B})}(G, f_*F).$$

If $F = \text{``}\varinjlim_i\text{''} F_i$ is an ind-sheaf on X, we define $f_{!!}F$ by the formula

$$f_{!!}\text{``}\varinjlim_i\text{''} F_i = \text{``}\varinjlim_i\text{''} f_! F_i.$$

Note that the natural morphism $f_{!!}\iota_X F \longrightarrow \iota_Y f_! F$ is not an isomorphism in general. The functor $f_{!!}$ is left exact and commutes with "$\varinjlim$". Moreover, it commutes with the functor α.

There is a base change formula as well as a projection formula for ind-sheaves.

Example 4.1. For $F \in \mathrm{I}(k_X)$ and $x \in X$ set $F_x = j_x^{-1} F$, where $j_x : \{x\} \hookrightarrow X$. Let N_a be as in Example 3.2. Then $(N_a)_x \simeq 0$ for all $x \in X$. On the other hand, one shows that the functor $F \mapsto \prod_{x\in X}(F \otimes \widetilde{k_{\{x\}}})$ is faithful.

5 Construction of ind-sheaves

We assume that X is a real analytic manifold and denote by $\mathcal{T}$ the family of open relatively compact subanalytic subsets of X. We denote by $\mathbb{R}\text{-}\mathrm{C}(k_X)$ the (small) category of $\mathbb{R}$-constructible sheaves of k-vector spaces on X and by $\mathbb{R}\text{-}\mathrm{C}^c(k_X)$ the full subcategory consisting of $\mathbb{R}$-constructible sheaves with compact support. Note that for any $F \in \mathbb{R}\text{-}\mathrm{C}^c(k_X)$ there is an exact sequence $F^1 \longrightarrow F^0 \longrightarrow F \longrightarrow 0$, with F^0 and F^1 finite direct sums of sheaves k_U with $U \in \mathcal{T}$.

We set

$$\mathrm{I}_{\mathbb{R}\text{-}c}(k_X) = \mathrm{Ind}(\mathbb{R}\text{-}\mathrm{C}^c(k_X)).$$

The natural functor $\iota : \mathbb{R}\text{-}\mathrm{C}^c(k_X) \longrightarrow \mathrm{Mod}^c(k_X)$ defines the fully faithful exact functor

$$\iota : \mathrm{I}_{\mathbb{R}\text{-}c}(k_X) \longrightarrow \mathrm{I}(k_X). \tag{5.1}$$

On the other hand, the forgetful functor $\mathrm{Mod}^c(k_X)^{\vee,add} \longrightarrow \mathbb{R}\text{-}\mathrm{C}^c(k_X)^{\vee,add}$ induces a functor

$$\rho : \mathrm{I}(k_X) \longrightarrow \mathrm{I}_{\mathbb{R}\text{-}c}(k_X). \tag{5.2}$$

The next theorem formulates a previous result of [6] in the language of ind-sheaves.

Theorem 5.1. *Let F be a presheaf of k-vector spaces on $\mathcal{T}$. Assume:*

(i) $F(\varnothing) = 0$,

(ii) *for any U and V in $\mathcal{T}$, the sequence $0 \longrightarrow F(U \cup V) \longrightarrow F(U) \oplus F(V) \longrightarrow F(U \cap V)$ is exact.*

Then there exists $F^+_{\mathcal{T}} \in \mathrm{I}_{\mathbb{R}\text{-}c}(k_X)$ *such that for any* $U \in \mathcal{T}$,

$$\mathrm{Hom}_{\mathrm{I}_{\mathbb{R}\text{-}c}(k_X)}(k_U, F^+_{\mathcal{T}}) \simeq F(U). \tag{5.3}$$

We denote by F^+ the image of $F^+_{\mathcal{T}}$ in $\mathrm{I}(k_X)$ by the functor ι of (5.1). If $\mathcal{D}$ is a sheaf of (not necessarily commutative) k-algebras and F a presheaf of $\mathcal{D}$-modules, then F^+ belongs to $\mathrm{Mod}(\mathcal{D}, \mathrm{I}(k_X))$, the subcategory of $\mathrm{I}(k_X)$ of ind-sheaves endowed with a $\mathcal{D}$-action. Recall that $F \in \mathrm{Mod}(\mathcal{D}, \mathrm{I}(k_X))$ means that F is endowed with a morphism of sheaves of unitary rings $\mathcal{D} \longrightarrow \mathcal{E}nd_{\mathrm{I}(k_X)}(F)$.

6 Some classical ind-sheaves

In this section, the base field k is $\mathbb{C}$. We denote by $\mathcal{C}^\infty_X$ the sheaf of complex valued functions of class $\mathcal{C}^\infty$, and by $\mathcal{D}_X$ the sheaf of real analytic finite-order differential operators.

(a) Let $f \in \mathcal{C}^\infty(U)$. One says that f has *polynomial growth* at $p \in X$ if for a local coordinate system $(x_1, \cdots, x_n)$ around p, there exist a sufficiently small compact neighborhood K of p and a positive integer N such that

$$\sup_{x \in K \cap U} \big(dist(x, X \setminus U)\big)^N |f(x)| < \infty\,. \tag{6.1}$$

We say that f is *tempered at* p if all of its derivatives are of polynomial growth at p. We say that f is *tempered* if it is tempered at any point of X. One denotes by $\mathcal{C}^{\infty,t}(U)$ the $\mathbb{C}$-vector subspace of $\mathcal{C}^\infty(U)$ consisting of tempered functions. By a theorem of Lojaciewicz, the contravariant functor $\mathcal{C}^{\infty,t}(\cdot)$ defined on the category $\mathcal{T}$ of open relatively compact subanalytic sets is exact. Hence, we may apply Theorem 5.1, and we get an ind-sheaf:

$$\mathcal{C}^{\infty,t}_X \in \mathrm{Mod}(\mathcal{D}_X, \mathrm{I}(\mathbb{C}_X)).$$

(b) If S is closed in X, one denotes by $\mathcal{I}^\infty_S(X)$ the ideal of $\mathcal{C}^\infty(X)$ consisting of functions which vanish on S with infinite order, and for an open subset U in X, we set $\Gamma(X; \mathbb{C}_U \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_X) = \mathcal{I}^\infty_{X\setminus U}(X)$. Again by a theorem of Lojaciewicz, the (covariant) functor $\Gamma(X; \cdot \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_X)$ defined on the category $\mathcal{T}$ is exact. It extends to an exact functor $\Gamma(X; \cdot \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_X)$ on the category $\mathbb{R}\text{-}\mathrm{C}^c(\mathbb{C}_X)$ and we may define

$$\mathcal{C}^{\infty,\mathrm{w}}_X \in \mathrm{Mod}(\mathcal{D}_X, \mathrm{I}(\mathbb{R}\text{-}\mathrm{C}(\mathbb{C}_X)))$$

by the formula

$$\mathcal{C}^{\infty,\mathrm{w}}_X(F) = \Gamma(X; H^0(D'F) \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_X)$$

where $F \in \mathbb{R}\text{-}\mathrm{C}^c(\mathbb{C}_X)$ and $D'F = R\mathcal{H}om_{\mathbb{C}_X}(F, \mathbb{C}_X)$.

(c) Similarly, replacing $\overset{\mathrm{w}}{\otimes}$ by $\otimes$ in the above formula, we find an ind-sheaf $\mathcal{C}_X^{\infty,\omega}$ which, in fact, is isomorphic to $\beta_X\mathcal{C}_X^{\infty}$.

(d) Now assume that X is a complex manifold with structure sheaf $\mathcal{O}_X$, denote by $\overline{X}$ the complex conjugate manifold and by $X^{\mathbb{R}}$ the underlying real manifold, identified with the diagonal of $X \times \overline{X}$. We define the objects $\mathcal{O}_X^t$, $\mathcal{O}_X^{\mathrm{w}}$ and by $\mathcal{O}_X^{\omega}$ considering the Dolbeault complexes with coefficients in $\mathcal{C}_X^{\infty,t}$, $\mathcal{C}_X^{\infty,\mathrm{w}}$ and $\mathcal{C}_X^{\infty,\omega}$:

$$\mathcal{O}_X^{\lambda} = R\mathcal{H}om_{\mathcal{D}_{\overline{X}}}(\mathcal{O}_{\overline{X}}, \mathcal{C}_X^{\infty,\lambda}),\ \lambda = t, \mathrm{w}, \omega.$$

Then, $\mathcal{O}_X^t, \mathcal{O}_X^{\mathrm{w}}$ and $\mathcal{O}_X^{\omega}$ belong to $D^b(\mathrm{Mod}(\mathcal{D}_X, \mathrm{I}(\mathbb{C}_X)))$. Note that if F is $\mathbb{R}$-constructible,

$$\begin{aligned} R\mathcal{H}om\,(F, \mathcal{O}_X^t) &\simeq \mathcal{T}\mathcal{H}om\,(F, \mathcal{O}_X) \\ R\mathcal{H}om\,(F, \mathcal{O}_X^{\mathrm{w}}) &\simeq D'F \overset{\mathrm{w}}{\otimes} \mathcal{O}_X \\ R\mathcal{H}om\,(F, \mathcal{O}_X^{\omega}) &\simeq D'F \otimes \mathcal{O}_X, \end{aligned}$$

where $\mathcal{T}\mathcal{H}om\,(\cdot, \mathcal{O}_X)$ and $\cdot \overset{\mathrm{w}}{\otimes} \mathcal{O}_X$ are the functors of tempered and formal cohomology of [4] and [6], respectively.

In particular, let M be a real analytic manifold and assume that X is a complexification of M. We find:

$$\begin{aligned} R\mathcal{H}om\,(D'\mathbb{C}_M, \mathcal{O}_X) &\simeq \mathcal{B}_M \text{ (Sato's hyperfunctions)}, \\ R\mathcal{H}om\,(D'\mathbb{C}_M, \mathcal{O}_X^t) &\simeq \mathcal{D}b_M \text{ (Schwartz's distributions)}, \\ R\mathcal{H}om\,(D'\mathbb{C}_M, \mathcal{O}_X^{\mathrm{w}}) &\simeq \mathcal{C}_M^{\infty} \text{ (C^{∞}-functions)}, \\ R\mathcal{H}om\,(D'\mathbb{C}_M, \mathcal{O}_X^{\omega}) &\simeq \mathcal{A}_M \text{ (real analytic functions)}. \end{aligned}$$

Replacing $\mathcal{H}om$ by $\mathcal{I}hom$ we find new ind-sheaves. For example, we can define the ind-sheaf $\mathcal{D}b_M^t$ of tempered distributions by setting

$$\mathcal{D}b_M^t = \mathrm{R}\mathcal{I}hom\,(D'\mathbb{C}_M, \mathcal{O}_X^t).$$

Note that $\alpha_X(\mathcal{D}b_M^t) \simeq \mathcal{D}b_M$.

Remark 6.1. The object $\mathcal{O}_X^t$ is not concentrated in degree 0 if $\dim X > 1$. In fact if $\mathcal{C}$ is an abelian category, a complex $A' \underset{f}{\longrightarrow} A \underset{g}{\longrightarrow} A''$ in $\mathrm{Ind}(\mathcal{C})$ is exact if and only if the dotted arrows in the diagram below with $X \in \mathcal{C}$ may be completed with $Y \in \mathcal{C}$ in such a way that the morphism $\alpha : Y \longrightarrow X$ is an epimorphism:

$$\begin{tikzcd} A' \arrow[r, "f"] & A \arrow[r, "g"] & A'' \\ Y \arrow[u, dotted, "t"] \arrow[r, dotted, two heads, "\alpha"'] & X \arrow[u, "s"] \arrow[ur, "0"] & \end{tikzcd}$$

Let us apply this result to the complex

$$\mathcal{C}_X^{\infty,t,(p-1)} \xrightarrow[\overline{\partial}]{} \mathcal{C}_X^{\infty,t,(p)} \xrightarrow[\overline{\partial}]{} \mathcal{C}_X^{\infty,t,(p+1)}$$

and choose $F = \mathbb{C}_U$, U open in X. Assume $\mathcal{O}_X^t$ is concentrated in degree 0. Then, for any $s \in \mathcal{C}^{\infty,t,(p)}(U)$ solution of $\overline{\partial} s = 0$, there exists an epimorphism $\alpha : G \twoheadrightarrow F$ and $t \in \mathrm{Hom}\,(G, \mathcal{C}^{\infty,t,(p-1)})$ such that $s \circ \alpha = \overline{\partial} t$. We may assume G is a finite direct sum of sheaves $\mathbb{C}_{U_j}$, with $U = \cup_j U_j$. We thus find that $s|_{U_j} = \overline{\partial} t_j$, which is in general not possible.

Note that the same argument hold with the sheaf $\mathcal{O}_X$, which shows that the functor ρ in (5.2) is not exact.

7 Microlocalization

In this section, X denotes a real analytic manifold. We denote by p_1 and p_2 the first and second projection defined on $X \times X$ and by Δ the diagonal. We denote by $\tau : TX \longrightarrow X$ and $\pi : T^*X \longrightarrow X$ its tangent and cotangent bundles, respectively. We denote by $T_Y X$ and $T_Y^* X$ the normal and conormal bundle to a closed submanifold Y of X. In particular, $T_X X$ and $T_X^* X$ are the zero-sections of these bundles. We shall identify $T_\Delta(X \times X)$ with TX by the first projection on $TX \times TX$. If S is a subset of X, one denotes by $C_Y(S)$ the Whitney normal cone of S along Y, a closed cone in $T_Y X$.

We denote by ω_X the dualizing complex on X. (Hence, $\omega_X \simeq or_X[\dim X]$.) The micro-support $SS(F)$ of a sheaf F and the functor μhom are defined in [5].

For a section $s : X \longrightarrow T^*X$ of π, one can construct an object $L_s \in D^b(\mathrm{I}(k_{X \times X}))$ with the properties that $\mathrm{supp}\, L_s = \Delta$ and such that

$$\begin{aligned} &L_s \otimes \widetilde{\mathbb{C}}_{s^{-1}(T_X^* X)} \simeq \widetilde{\mathbb{C}}_{s^{-1}(T_X^* X)} \\ &L_s \simeq \text{“}\varinjlim_{U,V}\text{”} k_{U \cap \overline{V}} \otimes p_2^{-1}\omega_X \quad \text{over } X \setminus s^{-1}(T_X^* X), \end{aligned}$$

where V ranges through the family of open neighborhoods of Δ and U through the family of open subsets of $X \times X$ such that:

$$C_\Delta(U) \subset \{v \in T_x X; \langle v, s(x) \rangle < 0\} \cup T_X X.$$

One defines

$$L_s \circ G = Rp_{1!!}(L_s \otimes p_2^{-1} G),$$

and one proves that for $F, G \in \mathrm{I}(k_X)$

$$s^{-1}\mu hom(F, G) \simeq R\mathcal{H}om\,(F, L_s \circ G).$$

Now assume that $s : T^*X \to T^*(T^*X)$ is a section. Then $s^{-1}\mu hom(\pi^{-1}F, \pi^{-1}G) \simeq \mu hom(F,G)$. Taking as s the canonical 1-form α_X on T^*X, one can then construct an object $K_X \in D^b(\mathrm{I}(k_{T^*X\times T^*X}))$ and define the microlocalization functor

$$\begin{aligned}\mu_X : D^b(\mathrm{I}(k_X)) &\to D^b(\mathrm{I}(k_{T^*X})),\\ \mu_X(F) &= K_X \circ \pi^{-1}F.\end{aligned}$$

Theorem 7.1. *Let $F, G \in D^b(k_X)$. Then*

$$\begin{aligned}SS(F) &= \operatorname{supp}\mu_X(F),\\ \mu hom(F,G) &\simeq R\mathcal{H}om\,(\mu_X(F), \mu_X(G)),\\ &\simeq R\mathcal{H}om\,(\pi^{-1}F, \mu_X(G)).\end{aligned}$$

8 Applications

Let X denote a complex manifold of complex dimension n. On T^*X there are some classical sheaves associated with a sheaf F on X: the sheaf $\mu hom(F, \mathcal{O}_X)$, or ($F$ being $\mathbb{R}$-constructible) the sheaves $t\mu hom(F, \mathcal{O}_X)$ of [1] and $w\mu hom(F, \mathcal{O}_X)$ of [2]. We can obtain all these sheaves in a unified way:

$$\begin{aligned}\mu hom(F, \mathcal{O}_X) &\simeq R\mathcal{H}om\,(\pi^{-1}F, \mu_X(\mathcal{O}_X)),\\ t\mu hom(F, \mathcal{O}_X) &\simeq R\mathcal{H}om\,(\pi^{-1}F, \mu_X(\mathcal{O}^t_X)),\\ w\mu hom(F, \mathcal{O}_X) &\simeq R\mathcal{H}om\,(\pi^{-1}F, \mu_X(\mathcal{O}^{\mathrm{w}}_X)).\end{aligned}$$

In particular, the sheaf $\mathcal{E}^{\mathbb{R}}_X$ of microlocal operators of [9] is isomorphic to the sheaf $R\mathcal{H}om\,(\pi^{-1}\mathbb{C}_\Delta, \mu_{X\times X}(\mathcal{O}^{(0,n)}_{X\times X}))[n]$.

Theorem 8.1. *The complex $\mu_X(\mathcal{O}_X)[n]$ is concentrated in degree 0 on $\dot{T}^*X := T^*X \setminus T^*_XX$. Moreover, $\mu_X(\mathcal{O}_X)[n] \in \mathrm{Mod}(\mathcal{E}^{\mathbb{R}}_X, \mathrm{I}(\mathbb{C}_{\dot{T}^*X}))$.*

Corollary 8.2. *Let $F \in D^b(\mathbb{C}_X)$. The object $\mu hom(F, \mathcal{O}_X)$ is well-defined in the derived category $D^b(\mathcal{E}^{\mathbb{R}}_X)$.*

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