Micro-support and Cauchy problem for temperate solutions of regular $\mathcal{D}$-Modules

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Abstract

Let X be a complex manifold, V a smooth involutive submanifold of T^*X, $\mathcal{M}$ a microdifferential system regular along V, and F an $\mathbb{R}$-constructible sheaf on X. We study the complex of temperate microfunction solutions of $\mathcal{M}$ associated with F, that is, the complex $R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{T}\mu hom(F, \mathcal{O}_X))$. We give a bound to its micro-support and solve the Cauchy problem under a suitable hyperbolicity assumption.

1 Introduction

The Cauchy problem for solutions of linear differential operators as well as the problem of propagation of singularities, are closely related subjects which have been intensively studied in the 80'th. In the analytic case, it is shown in [8] that these problems may be reduced to purely geometric ones, using sheaf theory, the only analytic tool being the Cauchy-Kowalevski theorem.

To be more precise, recall that a system of linear partial differential operators on a complex manifold X is the data of a coherent module $\mathcal{M}$ over the sheaf of rings $\mathcal{D}_X$ of holomorphic differential operators. Let F be a complex of sheaves on X with $\mathbb{R}$-constructible cohomologies (one says an $\mathbb{R}$-constructible sheaf, for short). The complex of "generalized functions" associated with F is described by the complex $R\mathcal{H}om(F, \mathcal{O}_X)$, and the complex

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of solutions of $\mathcal{M}$ with values in this complex is described by the complex

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, R\mathcal{H}om(F, \mathcal{O}_X)).$$

One may also microlocalize the problem by replacing $R\mathcal{H}om(F, \mathcal{O}_X)$ with $\mu hom(F, \mathcal{O}_X)$. In (loc cit) one shows that most of the properties of this complex, especially those related to propagation or Cauchy problem, are encoded in two geometric objects, both living in the cotangent bundle T^*X, the characteristic variety of the system $\mathcal{M}$, denoted by $\operatorname{char}(\mathcal{M})$, and the micro-support of F, denoted by $SS(F)$.

The complex $R\mathcal{H}om(F, \mathcal{O}_X)$ allows us to treat various situations. For example if M is a real manifold and X is a complexification of M, by taking as F the dual $D'(\mathbb{C}_M)$ of the constant sheaf on M, one obtains the sheaf $\mathcal{B}_M$ of Sato's hyperfunctions. If Z is a complex analytic hypersurface of X and $F = \mathbb{C}_Z[-1]$ is the (shifted) constant sheaf on Z, one obtains the sheaf of holomorphic functions with singularities on Z.

However, the complex $R\mathcal{H}om(F, \mathcal{O}_X)$ does not allow us to treat sheaves associated with holomorphic functions with temperate growth such as Schwartz's distributions or meromorphic functions with poles on Z. To consider such cases, one has to replace it by the complex $\mathcal{T}hom(F, \mathcal{O}_X)$ of temperate cohomology, introduced in [4] or its microlocalization $\mathcal{T}\mu hom(F, \mathcal{O}_X)$ constructed by Andronikof [1]. At this stage, a serious difficulty appears: the geometric methods of [8] do not apply any more, and indeed, it is a well known fact that to solve for example the Cauchy problem for distributions requires more informations than the data of the characteristic variety of the system.

In fact, very little is known concerning the problems of propagation of singularities and the Cauchy problem in the space of distributions, apart the case where $\mathcal{M}$ has real simple characteristics (see [2, 10] for a formulation in the language of sheaves and $\mathcal{D}$-modules) and some very specific situation (e.g., operators with constant coefficients on $\mathbb{R}^n$). We refer to [3] for historical and bibliographical comments.

In this paper we give an estimate for the microsupport of the sheaf of temperate microfunction solutions associated with an $\mathbb{R}$-constructible object F, when $\mathcal{M}$ has regular singularities along an involutive manifold V in the sense of [7]. More precisely, we prove the estimate

$$SS(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{T}\mu hom(F, \mathcal{O}_X))) \subset \rho_V^{-1}(C_V(SS(F))^a), \tag{1}$$

where $\rho_V\colon V \times_{T^*X} T^*(T^*X) \xrightarrow{\sim} V \times_{T^*X} T(T^*X) \to T_V(T^*X)$ is the projection and a is the antipodal map of $T_V(T^*X)$ as a vector bundle over V.

One can translate this result as follows. For a bicharacteristic leaf Σ of V, one has $\Sigma \times_{T^*X} T_V(T^*X) \simeq T^*\Sigma$, and $\Sigma \times_V C_V(SS(F))^a$ may be regarded as a subset of $T^*\Sigma$. Then for any Σ, (1) implies

$$(2) \qquad SS(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{T}\mu hom(F, \mathcal{O}_X))|_\Sigma) \subset \Sigma \times_V C_V(SS(F))^a.$$

What makes this paper original, in our opinion, is that we treat general $\mathbb{R}$-constructible sheaves F. Let us illustrate our results by an example.

We consider a smooth morphism $f: X \to Y$, we set $V = X \times_Y T^*Y$, and we assume that $\mathcal{M}$ is a coherent module regular along V. Let M be a real analytic manifold with complexification X, S a closed subanalytic subset of M. We obtain the estimate

$$SS(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \Gamma_S(\mathcal{D}b_M))) \subset V \widehat{+} (SS(\mathbb{C}_S))^a,$$

where $\mathcal{D}b_M$ denote the sheaf of distributions on M, and the operation $\widehat{+}$ is defined in [8] and recalled in Section 2.

2 Notations and main results

We will mainly follow the notations in [8].

Let X be a real analytic manifold. We shall denote by $\tau: TX \to X$ the tangent bundle to X and by $\pi: T^*X \to X$ the cotangent bundle. Set $\dot{T}^*X = T^*X \setminus X$ and $\dot{\pi}: \dot{T}^*X \to X$ the projection $\dot{T}^*X \to X$. For a smooth submanifold Y of X, T_YX denotes the normal bundle to Y and T^*_YX the conormal bundle. In particular, T^*_XX is identified with X, the zero section.

For a submanifold Y of X and a subset S of X, we denote by $C_Y(S)$ the normal cone to S along Y, a conic subset of T_YX.

If A and B are two conic subsets of T^*X, the operation $A\widehat{+}B$ is defined in (loc cit) and will be recalled below. The set A^a denotes the image of A by the antipodal map, $(x;\xi) \mapsto (x;-\xi)$.

For a cone $\gamma \subset TX$, the polar cone γ° to γ is the closed convex cone of T^*X defined by

$$\gamma^\circ = \{(x;\xi); x \in \pi(\gamma), \langle v, \xi\rangle \geq 0 \text{ for any } (x;v) \in \gamma\}.$$

Let $f: X \to Y$ be a morphism of complex manifolds. One has two natural morphisms

$$T^*X \underset{f_d}{\longleftarrow} X \times_Y T^*Y \underset{f_\pi}{\longrightarrow} T^*Y$$

(In [7], f_d is denoted by ${}^t f'$.)

We denote by $D(\mathbb{C}_X)$ (respectively $D^b(\mathbb{C}_X)$, $D^b_{\mathbb{R}-c}(\mathbb{C}_X)$) the derived category of complexes of sheaves of $\mathbb{C}$-vector spaces (respectively with bounded cohomologies, with bounded and $\mathbb{R}$-constructible cohomologies).

We denote by D'_X the duality functor on $D^b(\mathbb{C}_X)$, defined by

$$D'_X(F) = R\mathcal{H}om\,(F, \mathbb{C}_X).$$

If F is an object of $D^b(\mathbb{C}_X)$, $SS(F)$ denotes its micro-support, a closed $\mathbb{R}^+$-conic involutive subset of T^*X.

On a complex manifold X we consider the sheaf $\mathcal{O}_X$ of holomorphic functions, the sheaf Θ_X of holomorphic vector fields, the sheaf $\mathcal{D}_X$ of linear holomorphic differential operators of finite order, and its subsheaves $\mathcal{D}_X(m)$ of operators of order at most m. We shall also consider the sheaf $\mathcal{E}_X$ on T^*X of microdifferential operators of finite order ([14] and [13] for an exposition) and its subsheaves $\mathcal{E}_X(m)$ of operators of order at most m. We denote by $\mathrm{Mod}_{\mathrm{coh}}(\mathcal{D}_X)$ (respectively by $\mathrm{Mod}_{\mathrm{coh}}(\mathcal{E}_X)$) the abelian category of coherent $\mathcal{D}_X$-modules (respectively coherent $\mathcal{E}_X$-modules). We denote by $D^b(\mathcal{D}_X)$ the bounded derived category of left $\mathcal{D}_X$-modules and by $D^b_{\mathrm{coh}}(\mathcal{D}_X)$ its full triangulated category consisting of objects with coherent cohomologies. We define similarly $D^b(\mathcal{E}_X)$ and $D^b_{\mathrm{coh}}(\mathcal{E}_X)$.

The notion of regularity of an $\mathcal{E}_X$-module along an involutive submanifold V of T^*X will be recalled in Section 3.

The main purpose of this paper is to prove the three following results.

Let V be an involutive vector subbundle of T^*X. Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module regular along V and let $F \in D^b_{\mathbb{R}-c}(\mathbb{C}_X)$,

Theorem 2.1. *We have the estimate:*

$$SS\big(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{T}hom(F, \mathcal{O}_X))\big) \subset V \widehat{+} SS(F)^a. \tag{3}$$

Note that $V \widehat{+} SS(F)^a$ coincides with the closure of $V + SS(F)^a$ by Lemma 4.1.

As a particular case of Theorem 2.1, assume that X is the complexification of a real analytic manifold M and let S be a closed subanalytic subset of M. We obtain the estimate:

$$SSR\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \Gamma_S(\mathcal{D}b_M)) \subset (SS(\mathbb{C}_S))^a \widehat{+} V.$$

For a complex submanifold Z of X, we denote by $\mathcal{M}_Z$ the induced system of $\mathcal{M}$ on Z.

Theorem 2.2. *Assume* $T^*_Z X \cap ((SSF)^a \widehat{+} V) \subset T^*_X X$. *Then*

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{T}hom(F, \mathcal{O}_X))|_Z \simeq R\mathcal{H}om_{\mathcal{D}_Z}(\mathcal{M}_Z, \mathcal{T}hom(F_Z, \mathcal{O}_Z)).$$

Let U be a conic open subset of $\dot{T}^*X$, and let V be a closed smooth conic regular involutive submanifold in U. We denote by

$$\rho_V : V \times_{T^*X} T^*(T^*X) \xrightarrow{\sim} V \times_{T^*X} T(T^*X) \to T_V T^*X$$

the canonical projection.

For an $\mathbb{R}$-constructible sheaf F, the cohomology of $\mathcal{T}\mu hom(F, \mathcal{O}_X)$ is provided with an action of $\mathcal{E}_X$ as proved in [1], therefore, when $\mathcal{T}\mu hom(F, \mathcal{O}_X)$ is concentrated in a single degree, we regard it as an $\mathcal{E}_X$-module.

Theorem 2.3. *Let U and V be as above and let $\mathcal{M}$ be a coherent $\mathcal{E}_X$-module defined on U and regular along V. Assume that $F \in D^b_{\mathbb{R}-c}(\mathbb{C}_X)$ and $\mathcal{T}\mu hom(F, \mathcal{O}_X)|_U$ is concentrated in a single degree. Then*

$$SS(R\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{T}\mu hom(F, \mathcal{O}_X))) \subset {\rho_V}^{-1}(C_V(SS(F))^a). \tag{4}$$

For a bicharacteristic leaf Σ of V, one has $\Sigma \times_{T^*X} T_V(T^*X) \simeq T^*\Sigma$, and $\Sigma \times_V C_V(SS(F))^a$ may be regarded as a subset of $T^*\Sigma$.

Corollary 2.4. *Let Σ be a bicharacteristic leaf of V. Then one has*

$$SS(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{T}\mu hom(F, \mathcal{O}_X))|_\Sigma) \subset \Sigma \times_V C_V(SS(F))^a.$$

Proof. The map ρ_V decomposes as $V \times_{T^*X} T^*(T^*X) \underset{j_d}{\longrightarrow} T^*V \underset{h}{\to} T_V T^*X$. Here $j : V \hookrightarrow T^*X$ is the embedding. Set $\mathcal{S} = R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{T}\mu hom(F, \mathcal{O}_X))$. Then the support of $\mathcal{S}$ is contained in V, and

$$SS(\mathcal{S}|_V) = j_d(SS(\mathcal{S})) \subset h^{-1}(C_V(SS(F))^a).$$

Hence the corollary follows from the following lemma. q.e.d.

Lemma 2.5. *Let $f\colon X \to Y$ be a smooth morphism of real analytic manifolds. Let $T^*(X/Y)$ be the relative cotangent bundle and $h\colon T^*X \to T^*(X/Y)$ be the canonical projection. Let S be a closed conic subset of $T^*(X/Y)$, and let $F \in D^b(\mathbb{C}_X)$. Then the following two conditions are equivalent.*

(i) $SS(F) \subset h^{-1}(S)$.

(ii) *for any $y \in Y$, identifying $f^{-1}(y) \times_X T^*(X/Y)$ with $T^*(f^{-1}(y))$,*

$$SS(F|_{f^{-1}(y)}) \subset f^{-1}(y) \times_X S.$$

Proof. (i)$\Rightarrow$(ii) Since $SS(F \otimes \mathbb{C}_{f^{-1}(y)}) = h^{-1}(SS(F|_{f^{-1}(y)}))$, it is enough to show that

$$SS(F \otimes \mathbb{C}_{f^{-1}(y)}) \subset h^{-1}S.$$

Since $SS(F \otimes \mathbb{C}_{f^{-1}(y)}) \subset SS(F)\widehat{+}T^*_{f^{-1}(y)}X$, we may reduce the assertion to

$$h^{-1}S\widehat{+}T^*_{f^{-1}(y)}X) \subset h^{-1}S. \tag{5}$$

Let $N = X \times_Y T^*Y \subset T^*X$. Then $T^*(X/Y) = T^*X/N$, and $T^*_{f^{-1}(y)}X \subset N$. Hence (5) is a consequence of $h^{-1}S\widehat{+}N = \overline{h^{-1}S + N}$ (Lemma 4.1 (i)).
(ii)$\Rightarrow$(i) Let us take a coordinate system $x = (x_1, x_2)$ on X such that f is given by $x \mapsto x_1$. Assume that $(x_0, \xi_0) \in T^*X \setminus h^{-1}S$. Set $L(x, \delta, \varepsilon) = \{z \in X; \varepsilon > \langle z - \xi_0, \xi_0\rangle \geq \delta|z - \xi_0|\}$. It is enough to show that

$$R\Gamma(X; F \otimes \mathbb{C}_{L(x,\delta,\varepsilon)}) = 0$$

for x sufficiently close to x_0 and $0 < \varepsilon, \delta \ll 1$. For any $y \in Y$, $R\Gamma(f^{-1}(y); F \otimes \mathbb{C}_{L(x,\delta,\varepsilon)}|_{f^{-1}(y)}) = 0$ by the assumption. Hence $Rf_*(F \otimes \mathbb{C}_{L(x,\delta,\varepsilon)}) = 0$, which implies $R\Gamma(X; F \otimes \mathbb{C}_{L(x,\delta,\varepsilon)}) = 0$. q.e.d.

3 Regularity for $\mathcal{D}$-Modules

The results contained in this section are extracted or adapted from [2] and [5].

Recall that a good filtration on a $\mathcal{D}_X$-module $\mathcal{M}$ is a sequence of coherent $\mathcal{O}_X$-submodules $\{\mathcal{M}_k\}_{k\in\mathbb{Z}}$ satisfying:

(i) $\mathcal{D}_X(l)\mathcal{M}_k \subset \mathcal{M}_{k+l}$ for any l, $k \in \mathbb{Z}$,

(ii) $\mathcal{M} = \bigcup_k \mathcal{M}_k$,

(iii) locally on X, $\mathcal{M}_k = 0$ for $k \ll 0$,

(iv) locally on X, $\mathcal{D}_X(l)\mathcal{M}_k = \mathcal{M}_{l+k}$ for $k \gg 0$ and any $l \geq 0$.

We shall use the following notations. For a coherent $\mathcal{D}_X$-module $\mathcal{M}$, we denote by $\mathcal{E}_X\mathcal{M}$ the coherent $\mathcal{E}_X$-module $\mathcal{E}_X \underset{\pi^{-1}\mathcal{D}_X}{\otimes} \pi^{-1}\mathcal{M}$. For a coherent $\mathcal{E}_X$-module $\mathcal{M}$ and a coherent $\mathcal{E}_X(0)$-submodule $\mathcal{N}_0$ of $\mathcal{M}$, we set

$$\mathcal{N}_0(m) := \mathcal{E}_X(m)\mathcal{N}_0 \text{ for } m \in \mathbb{Z}.$$

Lemma 3.1 ([6]). *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module and let $\mathcal{N}_0$ be a coherent $\mathcal{E}_X(0)$-submodule of $\mathcal{E}_X\mathcal{M}|_{\dot{T}^*X}$ such that $\mathcal{N}_0$ generates $\mathcal{E}_X\mathcal{M}$ on $\dot{T}^*X$. Set*

$$\mathcal{M}_k = \{u \in \mathcal{M}; (1 \otimes u)|_{\dot{T}^*X} \in \mathcal{N}_0(k)\} \quad \text{for } k \geq 0,$$
$$\mathcal{M}_k = 0 \quad \text{for } k < 0.$$

Then $\mathcal{M}_k$ defines a good filtration on $\mathcal{M}$.

We shall use the notion of regularity along a closed analytic subset V of $\dot{T}^*X$ due to [7]. Let us denote by $\mathcal{J}_V$ the subsheaf of $\mathcal{E}_X$ of microdifferential operators of order at most 1 whose symbol of order 1 vanishes on V. In particular $\mathcal{J}_V$ contains $\mathcal{E}_X(0)$. Then $\mathcal{E}_V$ denotes the sub-sheaf of rings of $\mathcal{E}_X$ generated by $\mathcal{J}_V$. More precisely,

$$\mathcal{E}_V = \cup_{m \geq 0} \mathcal{J}_V^m.$$

Definition 3.2. [7] Let $\mathcal{M}$ be a coherent $\mathcal{E}_X$-module. An $\mathcal{E}_V$-lattice in $\mathcal{M}$ is an $\mathcal{E}_V$-submodule $\mathcal{N}_0$ of $\mathcal{M}$ such that $\mathcal{N}_0$ is $\mathcal{E}_X(0)$-coherent and generates $\mathcal{M}$ over $\mathcal{E}_X$.

Definition 3.3. [7] A coherent $\mathcal{E}_X$-module $\mathcal{M}$ is called regular along V if $\mathcal{M}$ has an $\mathcal{E}_V$-lattice locally on $\dot{T}^*X$.

Note that if a coherent $\mathcal{E}_X$-module $\mathcal{M}$ is regular along V, its support is contained in V.

If $\mathcal{M} \in D^b_{\mathrm{coh}}(\mathcal{E}_X)$, one says that $\mathcal{M}$ is regular along V if $H^k(\mathcal{M})$ is regular along V for every $k \in \mathbb{Z}$.

Definition 3.4. Let V be a closed analytic subset of T^*X. A coherent $\mathcal{D}_X$-module $\mathcal{M}$ is called regular along V if the characteristic variety of $\mathcal{M}$ is contained in V and if $\mathcal{E}_X\mathcal{M}$ is regular along $\dot{V} := V \setminus T^*_X X$.

One extends this definition to $D^b_{\mathrm{coh}}(\mathcal{D}_X)$ as in the $\mathcal{E}_X$-module case.

The next result follows from [7].

Lemma 3.5. *Consider a distinguished triangle $\mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \underset{1}{\to}$ in $D^b_{\mathrm{coh}}(\mathcal{D}_X)$ and assume that two of these three objects are regular along V. Then so is the third one.*

The next lemma gives a characterization of the $\mathcal{D}_X$-modules which are regular along V when V has a special form. If V is an involutive vector subbundle of T^*X, then one can find locally on X a smooth morphism $f\colon X \to Y$ such that $V = X \times_Y T^*Y$

Lemma 3.6. *Let $f\colon X \to Y$ be a smooth morphism and let $V := X \times_Y T^*Y$. Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module regular along V. Then, locally on X, $\mathcal{M}$ is the pull-back of a coherent $\mathcal{D}_Y$-module by f. In particular $\mathcal{M}$ admits a finite resolution locally on X*

$$0 \to \mathcal{L}^N \to \mathcal{L}^{N-1} \to \cdots \to \mathcal{L}^0 \to \mathcal{M} \to 0,$$

each $\mathcal{D}_X$-module $\mathcal{L}^j$ being isomorphic to a finite direct sum of the $\mathcal{D}_X$-module $\mathcal{D}_{X\to Y}$.

Proof. Since the assertion is local on X we may assume that $X = Z \times Y$ and f is the projection $Z \times Y \to Y$. Hence $\dot{V} = Z \times \dot{T}^*Y$. Let $\Theta_f \subset \Theta_X$ denote the $\mathcal{O}_X$-module of vector fields tangent to the fibers of f. With the above definition, an $\mathcal{E}_{\dot{V}}$-lattice for $\mathcal{E}_X\mathcal{M}|_{\dot{T}^*X}$ is a coherent $\mathcal{E}_X(0)$-submodule $\mathcal{N}_0$ of $\mathcal{E}_X\mathcal{M}$ such that $\mathcal{N}_0$ generates $\mathcal{E}_X\mathcal{M}$ on $\dot{T}^*X$ and such that $\Theta_f\mathcal{N}_0 \subset \mathcal{N}_0$.

Locally on X, there exists a coherent $\mathcal{O}_X$-submodule $\mathcal{M}_0$ of $\mathcal{M}$ such that $\mathcal{M} \simeq \mathcal{D}_X\mathcal{M}_0$. Let us prove that the coherent $\mathcal{E}_{\dot{V}}$-module $\mathcal{N}_0 := \mathcal{E}_{\dot{V}}\mathcal{M}_0$ is an $\mathcal{E}_{\dot{V}}$-lattice in $\mathcal{E}_X\mathcal{M}|_{\dot{T}^*X}$. Since $\mathcal{M}_0$ generates $\mathcal{E}_X\mathcal{M}$ it is sufficient to prove that $\mathcal{N}_0$ is $\mathcal{E}_X(0)$-coherent.

Locally on X, there exists a finite covering of $\dot{T}^*X$ by $\mathbb{C}^\times$-conic open subsets U_j and $\mathcal{E}_{\dot{V}}$-modules $\mathcal{N}_j$ such that $\mathcal{N}_j$ is an $\mathcal{E}_{\dot{V}}$-lattice in $\mathcal{M}$ on U_j. Hence, for each j, there exists $m_j \in \mathbb{Z}$ such that $\mathcal{N}_0$ is contained in $\mathcal{N}_j(m_j)$ on U_j.

Consider the increasing sequence $\{\mathcal{J}_V^k\mathcal{M}_0\}_{k\geq 0}$ of coherent $\mathcal{E}_X(0)$-submodules of $\mathcal{N}_0$. For each j, the restriction of this sequence to U_j is contained in $\mathcal{N}_j(m_j)$, which is $\mathcal{E}_X(0)$-coherent, hence it is locally stationary. Therefore $\mathcal{N}_0$ is $\mathcal{E}_X(0)$-coherent.

To summarize, we have constructed an $\mathcal{E}_{\dot{V}}$-module $\mathcal{N}_0$, coherent over $\mathcal{E}_X(0)$-module and which generates $\mathcal{M}$. We may now apply Lemma 3.1 and consider the good filtration $\mathcal{M}_k$ associated to $\mathcal{N}_0$.

By the construction, $\Theta_f\mathcal{M}_k \subset \mathcal{M}_k$. Setting $\mathcal{L} = \mathcal{M}_k$ for $k \gg 0$, we find a coherent $\mathcal{O}_X$-submodule $\mathcal{L}$ of $\mathcal{M}$ such that $\Theta_f\mathcal{L} \subset \mathcal{L}$ and $\mathcal{M} = \mathcal{D}_X\mathcal{L}$. Hence, locally on X, there is a $\mathcal{D}_X$-linear epimorphism $\varphi_0\colon (\mathcal{D}_{X\to Y})^{N_0} \to \mathcal{M}$. Repeating the same construction with $\mathcal{M}$ replaced by $\ker(\varphi_0)$, we construct an exact sequence

$$\mathcal{D}_{X\to Y}^{N_1} \underset{\varphi_1}{\longrightarrow} \mathcal{D}_{X\to Y}^{N_0} \underset{\varphi_0}{\longrightarrow} \mathcal{M} \to 0.$$

Since

$$\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_{X\to Y}, \mathcal{D}_{X\to Y}) \simeq f^{-1}\mathcal{D}_Y,$$

there is a $\mathcal{D}_Y$-linear morphism $\psi\colon \mathcal{D}_Y^{N_1} \to \mathcal{D}_Y^{N_0}$ such that $\varphi_1 = f^{-1}\psi$. Set $\mathcal{N} = \operatorname{coker}\psi$. This is a coherent $\mathcal{D}_Y$-module and we have an isomorphism

$$\mathcal{M} \simeq \mathcal{D}_{X\to Y} \otimes_{f^{-1}\mathcal{D}_Y} f^{-1}\mathcal{N}.$$

To conclude, we choose a finite resolution of $\mathcal{N}$ by finitely free $\mathcal{D}_Y$-modules, and tensorize over $f^{-1}\mathcal{D}_Y$ by the flat $f^{-1}\mathcal{D}_Y$-module $\mathcal{D}_{X\to Y}$.

q.e.d.

4 Review on normal cones

We shall now recall some constructions of [8] which will be useful for the next steps. To start with, we shall assume that X is a real manifold. Let S_1 and S_2 be two subsets of X. The normal cone $C(S_1, S_2)$ is a closed conic subset of TX which can be described as follows:

Let (x) be a system of local coordinates on X. Then $(x_0; v_0) \in C(S_1, S_2)$ if and only if there exists a sequence $\{(x_n, y_n, c_n)\}$ in $S_1 \times S_2 \times \mathbb{R}^+$ such that

$$x_n \underset{n}{\to} x_0,\ y_n \underset{n}{\to} x_0,\ c_n(x_n - y_n) \underset{n}{\to} v_0 \tag{6}$$

If A is a conic subset of T^*X, we denote by A° its polar which is a conic subset in TX. Let A and B be two conic subsets of T^*X. One defines the sum

$$A + B = \{(x;\xi) \in T^*X; \xi = \xi_1 + \xi_2 \text{ for some } (x;\xi_1) \in A \text{ and } (x;\xi_2) \in B\}.$$

If A and B are two closed conic subsets of T^*X, one also defines $A \widehat{+} B$, a closed conic set containing $A + B$, which may be described in a local canonical coordinate system $(x;\xi)$ as follows: $(x_0;\xi_0)$ belongs to $A\widehat{+}B$ if and only if there exist a sequence $\{(x_n;\xi_n)\}_n$ in A, a sequence $\{(y_n;\eta_n)\}_n$ in B, such that

$$x_n \underset{n}{\to} x_0, y_n \underset{n}{\to} x_0,\ (\xi_n + \eta_n) \underset{n}{\to} \xi_0, |x_n - y_n||\xi_n| \underset{n}{\to} 0. \tag{7}$$

If $A \cap B^a \subset T^*_X X$, we have

$$A\widehat{+}B = A + B.$$

Let now Y be a closed submanifold of X and let $i : Y \to X$ be the inclusion morphism. Using the Hamiltonian isomorphism, we get an embedding of T^*Y into $T_{T^*_Y X}(T^*X)$. Let A be a conic subset of T^*X. One sets

$$i^\sharp(A) := T^*Y \cap C_{T^*_Y X}(A).$$

This set can be described explicitly by local coordinate systems as follows. Let (x, y) be a local coordinate system on X such that $Y = \{x = 0\}$. Then $(y_0; \eta_0) \in i^\sharp(A) \Leftrightarrow$ there exists a sequence $\{(x_n, y_n; \xi_n, \eta_n)\}_n$ in A such that

$$x_n \underset{n}{\to} 0, y_n \underset{n}{\to} y_0, \eta_n \underset{n}{\to} \eta_0 \text{ and } |x_n||\xi_n| \underset{n}{\to} 0. \tag{8}$$

Lemma 4.1. *Let X be an open subset in a finite dimensional real vector space E with $0 \in E$. Let Λ be closed conic subset of T^*X. Let L be a vector subspace of the dual vector space E^* and let $V = X \times L$. Then we have*

(i) $V \widehat{+} \Lambda = \overline{V + \Lambda}$.

(ii) *For any $\theta \in E^*$ we have*

$$\overline{V + (\Lambda \widehat{+} X \times \mathbb{R}^{\leq 0}\theta)} \cap X \times \mathbb{R}^{\geq 0}\theta \subset \overline{V + \Lambda}.$$

(iii) *Let $i\colon E^* \to T^*X$ be the map $\theta \mapsto (0;\theta)$ and assume $(0;0) \in \Lambda$. Then, for any vector subspace N of E^*,*

$$(\{0\} \times N) \cap (V \widehat{+} \Lambda) \subset T^*_X X$$

if and only if

$$(N + i^{-1}(\Lambda)) \cap L = \{0\}.$$

Proof. (i) The inclusion $V \widehat{+} \Lambda \subset \overline{V + \Lambda}$ is clear. Conversely, assume that there are sequences $\{(x_n; \zeta_n)\}_n \subset V$ and $\{(y_n; \eta_n)\}_n \subset \Lambda$ such that

$$x_n \underset{n}{\to} x,\ y_n \underset{n}{\to} x,\ \zeta_n + \eta_n \underset{n}{\to} \xi.$$

Then the sequence $\{(y_n; \zeta_n)\}_n$ is contained in V. Therefore, $\{(y_n; \zeta_n)\}_n$ and $\{(y_n; \eta_n)\}_n$ satisfy (2).

(ii) Suppose that $(z;\theta) \in \overline{V + (\Lambda \widehat{+} X \times \mathbb{R}^{\leq 0}\theta)}$. Then by (i) there exist sequences $(\zeta_n)_n$ in L, $\{(y_n; \eta_n)\}_n$ in Λ, $\lambda_n \geq 0$, such that $y_n \underset{n}{\to} z$ and $(\zeta_n + \eta_n - \lambda_n\theta) \underset{n}{\to} \theta$. Therefore

$$\zeta_n + \eta_n - (\lambda_n + 1)\theta \underset{n}{\to} 0.$$

Since $\lambda_n \geq 0$, we get

$$\frac{\zeta_n + \eta_n}{\lambda_n + 1} \underset{n}{\to} \theta,$$

hence $(z;\theta) \in \overline{V + \Lambda}$.

(iii) The condition is obviously necessary. Let us now assume that there exists $\theta \in N$, $\theta \neq 0$, such that $(0;\theta) \in V \widehat{+} \Lambda$. Then there exist sequences $(\zeta_n)_n$ in L and $(y_n;\eta_n)_n$ in Λ such that $y_n \underset{n}{\to} 0$ and $\zeta_n + \eta_n \underset{n}{\to} \theta$. Taking suitable subsequences, we may assume that $\zeta_n/|\zeta_n|$ converges to $l \in L, l \neq 0$. Suppose that $\zeta_n \underset{n}{\to} 0$. Then $(0;\theta) \in \Lambda$, a contradiction. If $|\zeta_n|$ is unbounded we get $\zeta_n/|\zeta_n| + \eta_n/|\zeta_n| \underset{n}{\to} 0$ hence $(0;-l) \in L \cap i^{-1}(\Lambda)$, a contradiction. In the other case, we may assume that $\zeta_n \underset{n}{\to} l$ and setting $(0;\eta) = (0;\theta - l) \in (N+L) \cap i^{-1}(\Lambda)$, we get $\eta + l = \theta$, a contradiction. q.e.d.

5 Proof of Theorem 2.1

Let X be an open subset of a finite-dimensional real vector space E. For $(x_0;\xi_0) \in T^*X$, $\varepsilon > 0$, $\delta > 0$, an open convex proper cone γ of E and $v \in \gamma$, we introduce the following notation:

$$\begin{aligned} L_{\varepsilon,\xi_0} &= \{y \in E; \langle y - x_0, \xi_0\rangle > -\varepsilon\}, \\ Z(x,\gamma,\varepsilon,\xi_0) &= (x + \overline{\gamma}) \cap L_{\varepsilon,\xi_0}. \end{aligned}$$

Here $\overline{\gamma}$ denotes the closure of γ, The following result is proved in [8].

Lemma 5.1. *Let $F \in D^b(\mathbb{C}_X)$ and let $p = (x_0;\xi_0) \in T^*X$. The conditions below are equivalent:*

(i) $p \notin SS(F)$.

(ii) *There exists an open neighborhood U of x_0 and an open convex proper subanalytic cone $\gamma \subset E$ such that $\xi_0 \in \mathrm{Int}(\gamma^\circ)^a$, satisfying:*
for any $x \in U$ and sufficiently small $\varepsilon > 0$, $Z(x,\gamma,\varepsilon,\xi_0)$ is contained in X and

$$\mathrm{R}\Gamma_c(Z(x,\gamma,\varepsilon,\xi_0);F) = 0. \tag{9}$$

We can now embark into the proof of Theorem 2.1

We may assume $X = Z \times Y$, f is the projection, and X, Z, Y are open subsets of affine complex spaces. Moreover, using the results of § 3, we may assume that $\mathcal{M} = \mathcal{D}_{X \to Y}$. We shall set for short:

$$\mathcal{H}(F) := R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{T}hom(F, \mathcal{O}_X)).$$

Let $\theta = (x_0;\xi_0) \notin V \widehat{+} (SS(F))^a$. Let us take an open convex proper cone γ and an open neighborhood U of x_0 such that $\xi_0 \in \mathrm{Int}(\gamma^\circ)^a$ and

$U \times \gamma^\circ \cap \big(V \widehat{+} (SS(F))^a\big) \subset T^*_X X$. By Lemma 5.1 and keeping its notations, it is enough to prove

$$\mathrm{R}\Gamma_c\big(Z(x,\gamma,\varepsilon,\xi_0); \mathcal{H}(F)\big) = 0 \ \text{ for any } x \in U \text{ and sufficiently small } \varepsilon > 0.$$

Taking $v \in \gamma$, set $Z_\delta = Z(x - \delta v, \gamma, \varepsilon - \delta, \xi_0)$ for $0 < \delta \ll \varepsilon$. Then we have

$$\mathbb{C}_{Z(x,\gamma,\varepsilon,\xi_0)} = \varinjlim_{\delta > 0} \mathbb{C}_{Z_\delta},$$

and hence we obtain

$$H^j_c(Z(x,\gamma,\varepsilon,\xi_0); \mathcal{H}(F)) \simeq \varinjlim_{\delta>0} H^j_c\big(X; \mathcal{H}(F) \otimes \mathbb{C}_{Z_\delta}\big).$$

Set

$$Z'_\delta = (x - \delta v + \gamma) \cap \{y \in X; \langle y - x_0, \xi_0 \rangle \geq -\varepsilon + \delta\}.$$

Then for $0 < \delta' < \delta$, there is a chain of morphisms

$$\mathcal{H}(F) \otimes \mathbb{C}_{Z_\delta} \to \mathcal{H}(F \otimes \mathbb{C}_{Z'_\delta}) \to \mathcal{H}(F) \otimes \mathbb{C}_{Z_{\delta'}} \to \mathcal{H}(F \otimes \mathbb{C}_{Z'_{\delta'}}).$$

Therefore we have

$$\varinjlim_{\delta>0} H^j_c\big(X; \mathcal{H}(F) \otimes \mathbb{C}_{Z_\delta}\big) \simeq \varinjlim_{\delta>0} H^j_c\big(X; \mathcal{H}(F \otimes \mathbb{C}_{Z'_\delta})\big).$$

Since f is proper over the support of $\mathbb{C}_{Z'_\delta}$, we may apply Theorem 7.2 of [9] and obtain

$$H^j_c\big(Z(x,\gamma,\varepsilon,\xi_0); \mathcal{H}(F)\big) \simeq \varinjlim_{\delta>0} H^j_c\Big(Y; \mathcal{T}hom\big(Rf_!(F \otimes \mathbb{C}_{Z'_\delta}), \mathcal{O}_Y\big)\Big).$$

Hence, we are reduced to prove

$$Rf_!(F \otimes \mathbb{C}_{Z'(x,\gamma,\varepsilon,\xi_0)}) = 0, \tag{10}$$

where $Z'(x,\gamma,\varepsilon,\xi_0) = (x+\gamma) \cap \{y \in X; \langle y - x_0, \xi_0 \rangle \geq -\varepsilon\}$. In order to prove this, we shall apply [8, Proposition 5.4.17]. Set $X_t = \{y \in X; \langle y - x_0, \xi_0 \rangle \geq t\}$. Then if we prove

$$(y; -\xi_0) \notin \Big(SS(F \otimes \mathbb{C}_{(x+\gamma)}) + V\Big), \tag{11}$$

for $y \in U$, we have

$$Rf_!(F \otimes \mathbb{C}_{Z'(x,\gamma,\varepsilon,\xi_0)}) = Rf_!(F \otimes \mathbb{C}_{(x+\gamma) \cap X_t}).$$

Hence taking $t > 0$, we obtain the desired result (10).

Thus the proof is reduced to (11). We have

$$SS(\mathbb{C}_{(x+\gamma)}) \subset X \times \gamma^{\circ a}.$$

Hence we have

$$SS(F \otimes \mathbb{C}_{(x+\gamma)}) \subset SS(F) \widehat{+} \big(X \times \gamma^{\circ a}\big).$$

Since $SS(F) \cap (X \times \gamma^\circ) \subset T^*_X X$, we get

$$SS(F) + (X \times \gamma^\circ) = SS(F) \widehat{+} (X \times \gamma^\circ)$$

and

$$SS(F \otimes \mathbb{C}_{(x+\gamma)}) \subset SS(F) + \big(X \times \gamma^{\circ a}\big).$$

On the other hand, by the choice of γ, we have

$$(X \times \gamma^\circ) \cap (SS(F) + V) \subset T^*_X X.$$

Hence

$$(X \times \gamma^\circ) \cap \Big(SS(F) + (X \times \gamma^{\circ a}) + V\Big) \subset T^*_X X$$

and we obtain

$$(X \times \mathrm{Int}(\gamma^\circ)) \cap \Big(\big(SS(F) \widehat{+} (X \times \gamma^{\circ a})\big) + V\Big) = \emptyset.$$

Then the desired result follows from $-\xi_0 \in \mathrm{Int}(\gamma^\circ)$.

6 Proof of Theorem 2.2

We shall now embark in the proof of Theorem 2.2. Since the question is local on X, by Lemma 3.6 we may assume that $\mathcal{M}$ is isomorphic to $\mathcal{D}_{X \to Y}$. By (7.5) of [9], if d denotes the codimension of Z, we have a natural isomorphism

(12)

$$R\mathcal{H}om_{\mathcal{D}_Z}(\mathcal{M}_Z, \mathcal{T}hom(F_Z, \mathcal{O}_Z)) \simeq R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{T}hom(F_Z, \mathcal{O}_X))|_Z[2d]$$

With this isomorphism in hand, Theorem 2.2 will be a consequence of the next Lemma for a real submanifold.

If Z is a real submanifold of a complex manifold X, we still denote by $T^*_Z X$ the conormal bundle to Z of the real underlying manifold $X^{\mathbb{R}}$.

Lemma 6.1. *Let Z be a real analytic closed submanifold of X of codimension $d \geq 1$ and assume that*

$$T_Z^*X \cap (V \widehat{+} SS(F)^a) \subset T_X^*X.$$

Let $\mathcal{M}$ be a regular $\mathcal{D}_X$-module along V. Then the following natural morphism is an isomorphism:
(13)
$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{T}hom(F, \mathcal{O}_X))|_Z \to R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{T}hom(F_Z, \mathcal{O}_X))|_Z[d]$$

Proof. Since $T_Z^*X \cap V \subset T_X^*X$, there exists a coordinate system $(y_1, \cdots, y_N)$ on Y as a real analytic manifold and a coordinate system $(f_1, \cdots, f_m, y_1 \circ f, \cdots, y_N \circ f)$ of X such that Z is defined by the equations $f_1 = \cdots = f_d = 0$. We shall argue by induction on d.
(i) Let us prove the result for $d = 1$. Assume that Z is a real analytic hypersurface defined by the equation $f(z) = 0$, and set

$$Z^- = \{z \in X; f(z) \leq 0\}, \ \ Z^+ = \{z \in X; f(z) \geq 0\}.$$

Assume that $df, -df \notin V \widehat{+} SS(F)^a$. We shall show that the morphism
(14)
$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{T}hom(F, \mathcal{O}_X))|_Z \to R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{T}hom(F_Z, \mathcal{O}_X))|_Z[1].$$

is an isomorphism. The morphism (14) is given by

$$F \to F_{Z^+} \oplus F_{Z^-} \to F_Z \overset{+1}{\to}$$

Therefore to obtain (14) it is enough to prove that

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{T}hom(F_{Z^\pm}, \mathcal{O}_X))|_Z = 0.$$

Since
$$SS(F_{Z^\pm}) \subset SS(F) \widehat{+} (X \times \mathbb{R}^{\geq 0}(\pm df)),$$
$\pm df \in V \hat{+} SS(F_{Z^\pm})^a$ implies $\pm df \in V \hat{+} SS(F)^a$ by Lemma 4.1 (ii), which contradicts the assumption. Hence

$$\pm df \notin V \hat{+} SS(F_{Z^\pm})^a.$$

Denoting
$$\mathcal{S}_\pm = \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{T}hom(F_{Z^\pm}, \mathcal{O}_X)),$$
then, by Theorem 2.1, $\pm df \notin SS(\mathcal{S}_\pm)$ hence $\mathcal{S}_\pm|_Z = R\Gamma_{Z^\pm}(\mathcal{S}_\pm|_Z) = 0$.

(*ii*) Let us set $Z_1 = \{f_1 = 0\}$ and $Z_2 = \{f_2 = \cdots = f_d = 0\}$. Since

$$T^*_{Z_2}X \cap (V \widehat{+} (SS(F)^a)) \subset T^*_X X$$

in a neighborhood of Z, the hypothesis of induction implies

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{T}hom(F, \mathcal{O}_X))|_Z \simeq R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{T}hom(F_{Z_2}, \mathcal{O}_X))|_Z[d-1].$$

On the other hand, $SS(F_{Z_2}) \subset SS(F) \widehat{+} T^*_{Z_2}X$. Hence by (*i*)

$$\begin{aligned} R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{T}hom(F, \mathcal{O}_X))|_Z &\simeq R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{T}hom(F_{Z_2}, \mathcal{O}_X))|_Z[d-1] \\ &\simeq R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{T}hom(F_Z, \mathcal{O}_X))|_Z[d]. \end{aligned}$$

This ends the proof of (13).

q.e.d.

7 Proof of Theorem 2.3

We start by recalling the functor of tempered microlocalization.

7.1 Review on $\mathcal{T}\mu hom$

We shall recall the construction of the functor $\mathcal{T}\nu hom(\cdot, \mathcal{O}_X)$ of tempered specialization of ([1]).

Let $\tilde{X}^{\mathbb{C}}$ be the complex normal deformation of $X \times X$ along the diagonal Δ which we identify with X by the first projection p_1. We may then identify TX with the normal bundle $T_\Delta(X \times X)$. Let $t : \tilde{X}^{\mathbb{C}} \to \mathbb{C}$ and $p : \tilde{X}^{\mathbb{C}} \to X \times X$ be the canonical maps, let $\tilde{\Omega}$ be $t^{-1}(\mathbb{C} - \{0\})$ and $\Omega = t^{-1}(\mathbb{R}^+) \subset \tilde{\Omega}$. Let $p_2 : X \times X \to X$ be the second projection.

Consider the following diagram of morphisms:

$$TX \simeq T_\Delta(X \times X) \overset{i}{\hookrightarrow} \tilde{X}^{\mathbb{C}} \overset{j}{\hookleftarrow} \Omega = t^{-1}(\mathbb{R}^+) \tag{15}$$

Let $\tilde{p} : \tilde{\Omega} \to X \times X$, be the restriction of p. Finally denote by $\overline{p}_1$ the composition $p_1 \circ p$ and by $\overline{p}_2$ the composition $p_2 \circ p$.

Under these notations, $\mathcal{T}\nu hom(F, \mathcal{O}_X)$ is defined by

$$\mathcal{T}\nu hom(F, \mathcal{O}_X) = i^{-1} R\mathcal{H}om_{\mathcal{D}_{\tilde{X}^{\mathbb{C}}}}(\mathcal{D}_{\tilde{X}^{\mathbb{C}} \underset{\overline{p}_1}{\to} X}, \mathcal{T}hom(\overline{p}_2^{-1}F \otimes \mathbb{C}_\Omega, \mathcal{O}_{\tilde{X}^{\mathbb{C}}})).$$

Let $D^b_{\mathbb{R}^+}(\mathbb{C}_{TX})$ (resp. $D^b_{\mathbb{R}^+}(\mathbb{C}_{T^*X})$) be the derived category of complexes of sheaves on TX (resp. T^*X) with conic cohomologies. We denote by the symbol $\widehat{\ }$ the Fourier-Sato Transform from $D^b_{\mathbb{R}+}(\mathbb{C}_{TX})$ to $D^b_{\mathbb{R}+}(\mathbb{C}_{T^*X})$. Then by definition, $\mathcal{T}\mu hom(F,\mathcal{O}_X) = \mathcal{T}\nu hom(F,\mathcal{O}_X)\widehat{\ }$. Let us recall that under the identification of $T^*(TX)$ with $T^*(T^*X)$ by the Hamiltonian isomorphism we have $SS(F) = SS(F)\widehat{\ }$ for any $F \in D^b_{\mathbb{R}+}(\mathbb{C}_{TX})$.

Remark that for any coherent $\mathcal{D}_X$-module $\mathcal{M}$, one has

$$\begin{aligned} &R\mathcal{H}om_{\pi^{-1}\mathcal{D}_X}(\pi^{-1}\mathcal{M}, \mathcal{T}\mu hom(F,\mathcal{O}_X)) \\ &\simeq R\mathcal{H}om_{\tau^{-1}\mathcal{D}_X}(\tau^{-1}\mathcal{M}, \mathcal{T}\nu hom(F,\mathcal{O}_X))\widehat{\ }. \end{aligned} \tag{16}$$

The proof of Theorem 2.3 will be performed in two steps.

7.2 First reduction

First of all remark that the statement of Theorem 2.3 is local on T^*X and invariant by local canonical transformation as proved in [1], 5.5. Therefore, since V is regular involutive, locally in $\dot{T}^*X$, we may choose a canonical coordinate system $(x;\xi)$, $x = (x_1, ..., x_n)$, $\xi = (\xi_1, ..., \xi_n)$, such that $V = \{(x;\xi) \in \dot{T}^*X; \xi_1 = 0, ..., \xi_d = 0\}$, in other words, locally on X, we have $X \simeq Z \times Y$ where Y is an open subset of $\mathbb{C}^{n-d}$ and $V = X \underset{Y}{\times} T^*Y$, is associated to the projection $f : X \to Y$. By the results of [7], we have an exact sequence of coherent $\mathcal{E}_X$-modules

$$0 \to \mathcal{N} \to (\mathcal{E}_X/(\mathcal{E}_X D_{x_1} + \mathcal{E}_X D_{x_2} + ... + \mathcal{E}_X D_{x_d}))^N \to \mathcal{M} \to 0 \tag{17}$$

where $\mathcal{N}$ is still regular along V.

By "devissage" thanks to (17), we may then assume that $\mathcal{M} = \mathcal{E}_X/(\mathcal{E}_X D_{x_1} + \mathcal{E}_X D_{x_2} + ... + \mathcal{E}_X D_{x_d}) = \mathcal{E}_{X\to Y}$.

Hence from now on we will assume $\mathcal{M} = \mathcal{E}_{X\to Y}$. Of course, in that case, $\mathcal{M} \simeq \mathcal{E}_X \otimes_{\pi^{-1}\mathcal{D}_X} \pi^{-1}\mathcal{D}_{X\to Y}$.

7.3 End of the proof

Having (16) in mind, we are bound to prove the analogue of (4) with $\mathcal{T}\mu hom(F,\mathcal{O}_X)$ replaced by $\mathcal{T}\nu hom(F,\mathcal{O}_X)$. We have

$$\begin{aligned} &R\mathcal{H}om_{\tau^{-1}\mathcal{D}_X}(\tau^{-1}\mathcal{M}, \mathcal{T}\nu hom(F,\mathcal{O}_X)) \\ &\simeq i^{-1}R\mathcal{H}om_{\mathcal{D}_{\tilde{X}}}(\mathcal{D}_{\tilde{X}^{\mathbb{C}} \underset{h}{\to} Y}, \mathcal{T}hom(\overline{p}_2^{-1}F \otimes \mathbb{C}_\Omega, \mathcal{O}_{\tilde{X}^{\mathbb{C}}})) \end{aligned}$$

where $h = f \circ \overline{p}_1$ is a smooth morphism since f and $\overline{p}_1$ are smooth.

By Proposition 6.6.2 of [8], we have an inclusion

$$\begin{aligned} (18) \qquad & SS(\, i^{-1} R\mathcal{H}om_{\mathcal{D}^{\mathbb{C}}_{\tilde{X}}}(\mathcal{D}_{\tilde{X}^{\mathbb{C}} \underset{h}{\to} Y}, \mathcal{T}hom(\overline{p}_2^{-1} F \otimes \mathbb{C}_\Omega, \mathcal{O}_{\tilde{X}^{\mathbb{C}}}))) \\ & \subset i^\sharp(SS(R\mathcal{H}om_{\mathcal{D}_{\tilde{X}^{\mathbb{C}}}}(\mathcal{D}_{\tilde{X}^{\mathbb{C}} \underset{h}{\to} Y}, \mathcal{T}hom(\overline{p}_2^{-1} F \otimes \mathbb{C}_\Omega, \mathcal{O}_{\tilde{X}^{\mathbb{C}}})))). \end{aligned}$$

Therefore it is enough to consider the case of the partial De Rham system $\mathcal{D}_{\tilde{X}^{\mathbb{C}} \underset{h}{\to} Y}$.

Let us take a local coordinate system $(x) = (x_1, x_2)$ on X such that $f\colon X \to Y$ is by $(x_1, x_2) \mapsto x_1$. Then $V = X \times_Y T^*Y$ is given by $V = \{(x_1, x_2; \xi_1, \xi_2); \xi_2 = 0\}$. Endow $X \times X$ with the system of local coordinates (x, x'), so that $\Delta \subset X \times X$ is defined by $x = x'$. Under the change of coordinates : x, $y = x - x'$, Δ will be defined by $y = 0$. Using (x, y), $\tilde{X}^{\mathbb{C}}$ is endowed with the coordinates (t, x, y), and

$$(19) \qquad p(t,x,y) = (x, x - ty),\ \overline{p}_1(t,x,y) = x, \text{ and } \overline{p}_2(t,x,y) = x - ty$$

Let $(t, x, y; \tau, \xi, \eta)$ be the associated coordinates of $T^*(\tilde{X}^{\mathbb{C}})$.

Let $\tilde{V}$ be the submanifold $\overline{p}_{1d}(\tilde{X}^{\mathbb{C}} \underset{Y}{\times} T^*Y)$ of $T^*\tilde{X}^{\mathbb{C}}$, which is explicitly given by

$$\tilde{V} = \{(t, x, y; \tau, \xi, \eta); \tau = 0, \eta = 0,\ (x; \xi) \in V\}.$$

By Theorem 2.1 we have the following estimate:

(20)

$$SS(R\mathcal{H}om_{\mathcal{D}_{\tilde{X}^{\mathbb{C}}}}(\mathcal{D}_{\tilde{X}^{\mathbb{C}} \underset{h}{\to} Y}, \mathcal{T}hom(\overline{p}_2^{-1} F \otimes \mathbb{C}_\Omega, \mathcal{O}_{\tilde{X}^{\mathbb{C}}}))) \subset \tilde{V} \widehat{+} SS(\, \overline{p}_2^{-1} F \otimes \mathbb{C}_\Omega)^a.$$

By (18) and (20) we get

(21)

$$SS(R\mathcal{H}om_{\tau^{-1}\mathcal{D}_X}(\tau^{-1}\mathcal{M}, \mathcal{T}\nu hom(F, \mathcal{O}_X))) \subset i^\sharp(\tilde{V} \widehat{+} SS(\overline{p}_2^{-1} F \otimes \mathbb{C}_\Omega)^a).$$

Therefore it is enough to prove the inclusion

$$i^\sharp(\tilde{V} \widehat{+} SS(\, \overline{p}_2^{-1} F \otimes \mathbb{C}_\Omega)^a) \subset C(V, SS(F)).$$

More precisely, since $SS(\overline{p}_2^{-1} F \otimes \mathbb{C}_\Omega) \subset SS(\overline{p}_2^{-1} F) \widehat{+} SS(\mathbb{C}_\Omega)$ we shall prove the inclusion

$$i^\sharp(\tilde{V} \widehat{+} (SS(\, \overline{p}_2^{-1} F)^a \widehat{+} SS(\mathbb{C}_\Omega)^a)) \subset C(V, SS(F)).$$

We have

$$\begin{aligned} (22) \quad SS(\mathbb{C}_\Omega) = & \{(t, x, y; \tau, \xi, \eta); \xi = 0, \eta = 0, \mathrm{Im}\, t = 0, \mathrm{Re}\, t \geq 0, \mathrm{Re}\, \tau = 0\} \\ & \cup \{(t, x, y; \tau, \xi, \eta); \xi = 0, \eta = 0, t = 0, \mathrm{Re}\, \tau \leq 0\}, \end{aligned}$$

and since $\overline{p}_2$ is smooth,

(23)
$$\begin{aligned} SS(\overline{p}_2^{-1}F) &= \overline{p}_{2,d}\overline{p}_{2,\pi}^{-1}(SS(F)) \\ &= \{(t,x,y;\tau,\xi,\eta);(x-ty;\xi)\in SS(F),\quad \eta=-t\xi,\ \tau=-\langle\xi,y\rangle\}. \end{aligned}$$

Hence $SS(\overline{p}_2^{-1}F)^a\cap SS(\mathbb{C}_\Omega)\subset T^*_XX$, which implies

$$SS(\ \overline{p}_2^{-1}F)^a\widehat{+}SS(\mathbb{C}_\Omega)^a = SS(\ \overline{p}_2^{-1}F)^a+SS(\mathbb{C}_\Omega)^a.$$

Remark that the identification $T^*(TX)$ with $T(T^*X)$ is described by

$$T^*(TX)\ni(x,y;\xi,\eta)\longleftrightarrow(x,\eta;y,\xi)\in T(T^*X).$$

Let $(x_0,y_0;\xi_0,\eta_0)\in T^*(TX)$ and assume that

$$(x_0,y_0;\xi_0,\eta_0)\in i^\sharp(\tilde{V}\widehat{+}(SS(\ \overline{p}_2^{-1}F)^a+SS(\mathbb{C}_\Omega)^a)).$$

Then there exist sequences
$\{(t_n,x_n,y_n;0,\xi_n,0)\}_n$ in $\tilde{V}$, $\{(t'_n,x'_n,y'_n;\tau'_n,\xi'_n,\eta'_n)\}_n$ in $SS(\overline{p}_2^{-1}F)^a$
and $\{(t''_n,x''_n,y''_n;\tau''_n,0,0)\}_n$ in $SS(\mathbb{C}_\Omega)^a$ such that

(i) $t_n\underset{n}{\to}0$, $t'_n\underset{n}{\to}0$, $t''_n\underset{n}{\to}0$, $t'_n,t''_n\geq 0$.

(ii) $x_n\underset{n}{\to}x_0$, $x'_n\underset{n}{\to}x_0$, $x''_n\underset{n}{\to}x_0$.

(iii) $y_n\underset{n}{\to}y_0$, $y'_n\underset{n}{\to}y_0$, $y''_n\underset{n}{\to}y_0$.

(iv) $\tau'_n+\tau''_n\underset{n}{\to}0$.

(v) $\xi_n+\xi'_n\underset{n}{\to}\xi_0$.

(vi) $\eta'_n\underset{n}{\to}\eta_0$ (hence $t'_n\xi'_n\underset{n}{\to}-\eta_0$, and $t'_n\xi_n\underset{n}{\to}\eta_0$ by (v)).

By (vi), there exists a sequence of positive numbers $(a_n)_n$ such that $a_n\underset{n}{\to}0$ and $a_n\xi_n\underset{n}{\to}\eta_0$. Consider the sequence $\{(x'_n-t'_ny'_n;-a_n\xi'_n)\}_n$ in $SS(F)$ and $\{(x'_n+(a_n-t'_n)y'_n;a_n\xi_n)\}_n$ in V. Then

$$(x'_n-t'_ny'_n;-a_n\xi'_n)\underset{n}{\to}(x_0;\eta_0),\ \{(x'_n+(a_n-t'_n)y'_n;a_n\xi_n)\underset{n}{\to}(x_0;\eta_0)$$

and

$$a_n{}^{-1}\big((x'_n+(a_n-t'_n)y'_n,a_n\xi_n)-(x'_n-t'_ny'_n,-a_n\xi'_n)\big)=(y'_n,\xi_n+\xi'_n)\underset{n}{\to}(y_0,\xi_0).$$

Hence one has $(x_0,\eta_0;y_0,\xi_0)\in C(V,SS(F))$.

Since $C(V,SS(F))=\rho_V{}^{-1}(C_V(SS(F))^a)$, we finally obtain

$$SS(\ R\mathcal{H}om_{\pi^{-1}\mathcal{D}_X}(\pi^{-1}\mathcal{M},\mathcal{T}\mu hom(F,\mathcal{O}_X)))\subset\rho_V{}^{-1}(C_V(SS(F))^a)$$

as asserted. $\square$

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