

The Radon-Penrose correspondence II: Line bundles and simple \mathcal{D} -modules

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Abstract

On a complex manifold X of dimension ≥ 3 , we show that coherent \mathcal{D}_X -modules which are “simple” all over P^*X are classified by $\text{Pic}(X)$. As a corollary, using the Penrose transform, we obtain that on the complex Minkowski space \mathbb{M} , simple $\mathcal{D}_{\mathbb{M}}$ -modules along the characteristic variety of the wave equation are classified by $\mathbb{Z}/2\mathbb{Z}$ (the so-called helicity).

Introduction

Consider a complex manifold X of dimension ≥ 3 and a regular involutive submanifold V of P^*X , the projective cotangent bundle to X . A natural problem is to classify all systems of linear PDE (i.e., coherent \mathcal{D}_X -modules) with simple characteristics along V . In the extreme case where $V = P^*X$, we solve this problem by showing that such modules are classified —modulo flat connections— by $\text{Pic}(X)$. Starting from this result, the theory of integral transformations for \mathcal{D} -modules allows us to treat the case of other involutive manifolds, such as the characteristic variety of the wave equation in the conformally compactified Minkowski space or, more generally, the case where X is a Grassmann manifold $G_p(\mathbb{C}^n)$, and V is the image of the conormal bundle to the incidence relation in $G_1(\mathbb{C}^n) \times G_p(\mathbb{C}^n)$. We show in particular that on such a space, simple modules along V are determined, up to flat connections, by an integer corresponding to the so-called helicity. In [5] it was shown that the Penrose transform allows one to obtain the whole family of massless field equations. By the results of this paper, one gets that there are no other simple modules than those of this family.

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1 Simple \mathcal{D} -modules

We shall use the classical notations concerning \mathcal{D} -modules. References are made to [6], [9], [10].

Let X be a complex manifold. We denote by d_X its dimension, by $\pi : T^*X \rightarrow X$ its cotangent bundle, and by \mathcal{O}_X its structural sheaf. The sheaf \mathcal{D}_X of linear holomorphic partial differential operators on X is naturally endowed with a structure of filtered ring, the filtration being given by the subsheaves $\mathcal{D}_X(k)$ of operators of degree at most k . The associated graded ring $G\mathcal{D}_X$ is identified to $\bigoplus_{k \in \mathbb{N}} \pi_* \mathcal{O}_{T^*X}(k)$, where $\mathcal{O}_{T^*X}(k)$ denotes the subsheaf of \mathcal{O}_{T^*X} whose sections are homogeneous of degree k in the fiber variables.

Let \mathcal{M} be a coherent \mathcal{D}_X -module. A filtration on \mathcal{M} is an increasing sequence $\{\mathcal{M}_k\}_{k \in \mathbb{Z}}$ of \mathcal{O}_X -submodules of \mathcal{M} , such that $\mathcal{M} = \cup_k \mathcal{M}_k$, and $\mathcal{D}_X(l)\mathcal{M}_k \subset \mathcal{M}_{k+l}$. A filtration $\{\mathcal{M}_k\}_{k \in \mathbb{Z}}$ is called good if the \mathcal{M}_k 's are \mathcal{O}_X -coherent and, locally on X , $\mathcal{M}_k = 0$ for $k \ll 0$, and $\mathcal{D}_X(l)\mathcal{M}_k = \mathcal{M}_{k+l}$ for any $l \geq 0$ and for $k \gg 0$. If \mathcal{M} is endowed with a good filtration, we set:

$$\overline{G}\mathcal{M} = \mathcal{O}_{T^*X} \otimes_{\pi^{-1}G\mathcal{D}_X} \pi^{-1}G\mathcal{M},$$

where $G\mathcal{M} = \bigoplus_k \mathcal{M}_k / \mathcal{M}_{k-1}$ is the associated graded module. This is a coherent \mathcal{O}_{T^*X} -module. Recall that any coherent \mathcal{D}_X -module locally admits a good filtration, that $\overline{G}\mathcal{M}$ does not depend on the choice of the good filtration, and that $\text{char}(\mathcal{M})$, the characteristic variety of \mathcal{M} , is defined as the support of $\overline{G}\mathcal{M}$.

Let us denote by $\text{Mod}(\mathcal{D}_X)$ the abelian category of \mathcal{D}_X -modules, and by $\text{Mod}_{\text{coh}}(\mathcal{D}_X)$ its full abelian subcategory of coherent \mathcal{D}_X -modules. We denote by $\mathbf{D}^b(\mathcal{D}_X)$ the bounded derived category of $\text{Mod}(\mathcal{D}_X)$, and by $\mathbf{D}_{\text{coh}}^b(\mathcal{D}_X)$ the full triangulated subcategory of $\mathbf{D}^b(\mathcal{D}_X)$ whose objects have coherent cohomology groups.

Definition 1.1. Let $\varphi : \mathcal{M} \rightarrow \mathcal{N}$ be a morphism of coherent \mathcal{D}_X -modules. We shall say that φ is an isomorphism modulo flat connections (an m-f-c isomorphism for short) if $\ker \varphi$ and $\text{coker} \varphi$ are flat connections (i.e., coherent \mathcal{D}_X -modules whose characteristic varieties are contained in the zero-section).

We denote by $\dot{\pi} : \dot{T}^*X \rightarrow X$ the cotangent bundle with the zero-section removed, by $\tau : P^*X \rightarrow X$ the projective cotangent bundle, and by $\gamma : \dot{T}^*X \rightarrow P^*X$ the natural projection. For $V \subset T^*X$, we set $\dot{V} = V \cap \dot{T}^*X$.

Denote by $\widehat{\mathcal{E}}_X$ the sheaf of formal pseudo-differential operators on P^*X . If \mathcal{M} is a coherent \mathcal{D}_X -module, we set:

$$\widehat{\mathcal{M}} = \widehat{\mathcal{E}}_X \otimes_{\tau^{-1}\mathcal{D}_X} \tau^{-1}\mathcal{M}.$$

If $\varphi : \mathcal{M} \rightarrow \mathcal{N}$ is a morphism of coherent \mathcal{D}_X -modules, we denote by $\widehat{\varphi} : \widehat{\mathcal{M}} \rightarrow \widehat{\mathcal{N}}$ the associated morphism of $\widehat{\mathcal{E}}_X$ -modules.

Lemma 1.2. (i) *A morphism $\varphi : \mathcal{M} \rightarrow \mathcal{N}$ is an m-f-c isomorphism if and only if $\widehat{\varphi}$ is an isomorphism.*

(ii) The set of m-f-c isomorphisms is a multiplicative system (as defined e.g. in [7, Definition 1.6.1]) in $\text{Mod}_{\text{coh}}(\mathcal{D}_X)$.

Proof. Set $\mathcal{K} = \ker \varphi$, $\mathcal{L} = \text{coker } \varphi$. Since $\widehat{\mathcal{E}}_X$ is flat over $\tau^{-1}\mathcal{D}_X$, $\widehat{\mathcal{K}} = \ker \widehat{\varphi}$, $\widehat{\mathcal{L}} = \text{coker } \widehat{\varphi}$. Recalling that $T^*X \cap \text{char}(\mathcal{M}) = \gamma^{-1} \text{supp}(\widehat{\mathcal{M}})$, it is clear that \mathcal{K} and \mathcal{L} are flat connections if and only if $\widehat{\varphi}$ is an isomorphism. This proves (i). To prove (ii), we have to check that properties (S 1)–(S 4) of [7, Definition 1.6.1] are satisfied. (S 1), asserting that the identity morphisms are m-f-c isomorphisms, is obvious. (S 2) requires that a composition of two m-f-c isomorphisms be again an m-f-c isomorphism, and follows from (i). A simple proof of (S 3) and (S 4) is obtained by working in the derived category $\mathbf{D}_{\text{coh}}^b(\mathcal{D}_X)$, along the lines of Proposition 1.6.7 of loc. cit. Let us prove for example that any diagram

$$\begin{array}{ccc} & & \mathcal{P} \\ & & \downarrow \varphi \\ \mathcal{M} & \xrightarrow{f} & \mathcal{N}, \end{array}$$

where φ is an m-f-c isomorphism, can be completed into a commutative diagram

$$\begin{array}{ccc} \mathcal{Q} & \longrightarrow & \mathcal{P} \\ \downarrow \psi & & \downarrow \varphi \\ \mathcal{M} & \xrightarrow{f} & \mathcal{N}, \end{array} \quad (1.1)$$

with ψ an m-f-c isomorphism. (We shall leave the other verifications to the reader.) Embed φ in a distinguished triangle $\mathcal{P} \xrightarrow{\varphi} \mathcal{N} \xrightarrow{g} \mathcal{R} \xrightarrow{+1}$, and $g \circ f$ in a distinguished triangle $\widetilde{\mathcal{Q}} \rightarrow \mathcal{M} \xrightarrow{h} \mathcal{R} \xrightarrow{+1}$. The diagram

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{h} & \mathcal{R} \\ \downarrow f & & \downarrow \text{id} \\ \mathcal{N} & \xrightarrow{g} & \mathcal{R} \end{array}$$

gives rise to a morphism of distinguished triangles

$$\begin{array}{ccccccc} \widetilde{\mathcal{Q}} & \longrightarrow & \mathcal{M} & \longrightarrow & \mathcal{R} & \longrightarrow & \\ \downarrow \widetilde{\psi} & & \downarrow & & \downarrow & & \\ \mathcal{P} & \longrightarrow & \mathcal{N} & \longrightarrow & \mathcal{R} & \longrightarrow & \end{array} \begin{array}{c} +1 \\ \\ +1 \end{array}$$

Since $\text{char}(\mathcal{R}) \subset T_X^*X$, $\psi = H^0(\widetilde{\psi})$ is an m-f-c isomorphism. Setting $\mathcal{Q} = H^0(\widetilde{\mathcal{Q}})$, we thus get (1.1). \square

Definition 1.3. We denote by $\text{Mod}_{\text{coh}}(\mathcal{D}_X; \mathcal{O}_X)$ the quotient category of $\text{Mod}_{\text{coh}}(\mathcal{D}_X)$ by the multiplicative system of m-f-c isomorphisms.

Recall that an m -f-c isomorphism of $\text{Mod}_{\text{coh}}(\mathcal{D}_X)$ becomes an isomorphism in $\text{Mod}_{\text{coh}}(\mathcal{D}_X; \mathcal{O}_X)$.

Definition 1.4. Let V be a closed conic involutive submanifold of \dot{T}^*X , and let \mathcal{M} be a coherent \mathcal{D}_X -module. We shall say that \mathcal{M} is simple along V if \mathcal{M} can be endowed with a good filtration $\{\mathcal{M}_k\}$ such that $\overline{GM}|_{\dot{T}^*X}$ is locally isomorphic to \mathcal{O}_V as an $\mathcal{O}_{\dot{T}^*X}$ -module.

If \mathcal{L} is a locally free \mathcal{O}_X -module, we set:

$$\mathcal{D}\mathcal{L} = \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{L}.$$

The \mathcal{D}_X -module $\mathcal{D}\mathcal{L}$ has a natural good filtration $\mathcal{D}\mathcal{L}_k = \mathcal{D}_X(k) \otimes_{\mathcal{O}_X} \mathcal{L}$, and is thus an example of simple module along \dot{T}^*X . We show in the next theorem that, essentially, any simple module along \dot{T}^*X is of this form.

For $m \in \mathbb{Z}$, denote by $\mathcal{O}_{P^*X}(m)$ the $-m$ -th tensor power of the canonical line bundle on P^*X .

Lemma 1.5. *If $d_X \geq 2$, there is a natural isomorphism:*

$$\begin{aligned} \text{Pic}(X) \times \mathbb{Z} &\xrightarrow{\sim} \text{Pic}(P^*X) \\ (\mathcal{F}, m) &\mapsto \tau^{-1}\mathcal{F} \otimes_{\tau^{-1}\mathcal{O}_X} \mathcal{O}_{P^*X}(m). \end{aligned} \quad (1.2)$$

Proof. We may assume X is connected.

(o) If $x \in X$ and \mathcal{L} is a line bundle on P^*X , denote by \mathcal{L}_x its holomorphic restriction to P_x^*X . First, notice that if U is an open ball in \mathbb{C}^n , then $\text{Pic}(P^*U) \simeq \mathbb{Z}$. Hence, the Chern class of \mathcal{L}_x is locally constant w.r.t. x .

(i) The map is injective. In fact, if $\tau^{-1}\mathcal{F} \otimes_{\tau^{-1}\mathcal{O}_X} \mathcal{O}_{P^*X}(m)$ is trivial, by restriction to P_x^*X we find that $m = 0$. Next, by taking the direct image on X , we find that \mathcal{F} is trivial.

(ii) The map is surjective. In fact, if \mathcal{L} is a line bundle on P^*X , then $\mathcal{L}_x \simeq \mathcal{O}_{P_x^*X}(m)$, and m does not depend on $x \in X$ by (o). Let $\mathcal{L}' = \mathcal{L} \otimes_{\mathcal{O}_{P^*X}} \mathcal{O}_{P^*X}(-m)$. Then \mathcal{L}'_x is trivial for all $x \in X$, and hence the natural morphism $\mathcal{L}' \rightarrow \tau^*\tau_*\mathcal{L}'$ is an isomorphism. One concludes, since then \mathcal{L} is the image of $(\tau_*\mathcal{L}', m)$ by (1.2). \square

Theorem 1.6. *Let \mathcal{M} be a coherent \mathcal{D}_X -module. Assume \mathcal{M} is simple along \dot{T}^*X , and assume $d_X \geq 3$. Then, there exists a line bundle \mathcal{L} on X and an m -f-c isomorphism $\mathcal{M} \rightarrow \mathcal{D}\mathcal{L}$.*

Proof. If $\{\mathcal{M}_k\}$ is a good filtration of \mathcal{M} , $\widehat{\mathcal{M}}$ has a natural filtration given by

$$\widehat{\mathcal{M}}_k = \sum_{l \in \mathbb{Z}} \widehat{\mathcal{E}}_X(k-l) \tau^{-1}\mathcal{M}_l,$$

where $\widehat{\mathcal{E}}_X(k)$ is the subsheaf of $\widehat{\mathcal{E}}_X$ of operators of degree at most k . Since $\{\mathcal{M}_k\}$ is a good filtration, the above sum is finite, and hence the $\widehat{\mathcal{M}}_k$'s are $\widehat{\mathcal{E}}_X(0)$ -coherent. Moreover,

$$\widehat{\mathcal{M}}_k = \widehat{\mathcal{E}}_X(k) \widehat{\mathcal{M}}_0$$

for all $k \in \mathbb{Z}$. Since \mathcal{M} is simple along T^*X , $\widehat{\mathcal{M}}_0/\widehat{\mathcal{M}}_{-1}$ is a line bundle on P^*X , and by Lemma 1.5, there exists a line bundle \mathcal{F} on X and $m \in \mathbb{Z}$, such that

$$\widehat{\mathcal{M}}_0/\widehat{\mathcal{M}}_{-1} \simeq \tau^{-1}\mathcal{F} \otimes_{\tau^{-1}\mathcal{O}_X} \mathcal{O}_{P^*X}(m).$$

By shifting the filtration, we may assume $m = 0$. Let us cover X by Stein open affine charts U . Then, $\widehat{\mathcal{M}}_0/\widehat{\mathcal{M}}_{-1}|_{P^*U} \simeq \mathcal{O}_{P^*U}$. In particular, this implies:

$$\widehat{\mathcal{M}}_k/\widehat{\mathcal{M}}_{k-1}|_{P^*U} \simeq \mathcal{O}_{P^*U}(k). \quad (1.3)$$

Since U is affine, $P^*U \simeq U \times P$, where P is a $d_X - 1$ -dimensional complex projective space. Since $d_X - 1 > 1$ and U is Stein, $H^1(P^*U; \mathcal{O}_{P^*U}(k)) \simeq \Gamma(U; \mathcal{O}_U) \widehat{\otimes} H^1(P; \mathcal{O}_P) = 0$ for $k < 0$. Applying the functor $\mathrm{R}\Gamma(P^*U; \cdot)$ to the exact sequence:

$$0 \rightarrow \widehat{\mathcal{M}}_k/\widehat{\mathcal{M}}_{k-1} \rightarrow \widehat{\mathcal{M}}_0/\widehat{\mathcal{M}}_{k-1} \rightarrow \widehat{\mathcal{M}}_0/\widehat{\mathcal{M}}_k \rightarrow 0,$$

we thus get, for $k < 0$, the surjectivity of the morphism:

$$\Gamma(P^*U; \widehat{\mathcal{M}}_0/\widehat{\mathcal{M}}_{k-1}) \rightarrow \Gamma(P^*U; \widehat{\mathcal{M}}_0/\widehat{\mathcal{M}}_k).$$

Let \bar{s}_U be a free generator of $\widehat{\mathcal{M}}_0/\widehat{\mathcal{M}}_{-1}$ on P^*U . By induction on k , using the above surjection we get a section

$$\begin{aligned} s_U \in \varprojlim_k \Gamma(P^*U; \widehat{\mathcal{M}}_0/\widehat{\mathcal{M}}_k) &\simeq \Gamma(P^*U; \varprojlim_k \widehat{\mathcal{M}}_0/\widehat{\mathcal{M}}_k) \\ &\simeq \Gamma(P^*U; \widehat{\mathcal{M}}_0) \end{aligned}$$

whose class modulo $\widehat{\mathcal{M}}_{-1}$ is \bar{s}_U (here, the last isomorphism follows from [9, Proposition II 3.2.5]). Consider the morphism of $\widehat{\mathcal{E}}_U(0)$ -modules:

$$\widehat{\mathcal{E}}_U(0) \xrightarrow{\varphi_U} \widehat{\mathcal{M}}_0|_{P^*U},$$

given by $\varphi_U(P) = P s_U$, and set $\mathcal{K} = \ker \varphi_U$, $\mathcal{L} = \mathrm{coker} \varphi_U$. By construction, φ_U induces an isomorphism $\widehat{\mathcal{E}}_U(0)/\widehat{\mathcal{E}}_U(-1) \xrightarrow{\sim} \widehat{\mathcal{M}}_0/\widehat{\mathcal{M}}_{-1}|_{P^*U}$. It follows that $\mathcal{K}/\widehat{\mathcal{E}}_U(-1)\mathcal{K} \simeq \mathcal{L}/\widehat{\mathcal{E}}_U(-1)\mathcal{L} \simeq 0$, and hence, again by loc. cit., $\mathcal{K} = \mathcal{L} = 0$.

Summarizing up, we have shown that $\widehat{\mathcal{M}}_0|_{P^*U}$ is a free $\widehat{\mathcal{E}}_U(0)$ -module of rank one. This implies that $\widehat{\mathcal{M}}|_{P^*U}$ is a free $\widehat{\mathcal{E}}_U$ -module of rank one. Hence, $\tau_*\widehat{\mathcal{M}}$ is a locally free \mathcal{D}_X -module of rank one. Since the only invertible differential operators are of degree zero, $\tau_*\widehat{\mathcal{M}} \simeq \mathcal{D}\mathcal{L}$ for a line bundle \mathcal{L} on X .

Let $\varphi: \mathcal{M} \rightarrow \tau_*\widehat{\mathcal{M}}$ be the natural adjunction morphism. To conclude, by Lemma 1.2 (i), it is enough to check that $\widehat{\varphi}$ is an isomorphism. Since $\widehat{\mathcal{M}}$ is free of rank one on P^*U , we are reduced to prove that the natural morphism:

$$\mathcal{E}_U \rightarrow \mathcal{E}_U \otimes_{\tau^{-1}\mathcal{D}_X} \tau^{-1}\tau_*\mathcal{E}_U$$

is an isomorphism, which is obvious. \square

2 Integral transforms

Let X and Y be complex analytic manifolds of dimension d_X and d_Y , respectively, let $\Lambda \subset \dot{T}^*(X \times Y)$ be a closed smooth Lagrangian submanifold, and consider the natural projections:

$$X \xleftarrow{q_1} X \times Y \xrightarrow{q_2} Y, \quad \dot{T}^*X \xleftarrow{p_1} \Lambda \xrightarrow{p_2^a} \dot{T}^*Y, \quad (2.1)$$

where we denote by p_2^a the composition of p_2 with the antipodal map. Here, we will make the following assumptions:

$$\left\{ \begin{array}{l} \text{(i)} \quad q_1 \text{ and } q_2 \text{ are proper on } \dot{\pi}(\Lambda), \\ \text{(ii)} \quad \Lambda \cap (\dot{T}^*X \times T_Y^*Y) = \Lambda \cap (T_X^*X \times \dot{T}^*Y) = \emptyset, \\ \text{(iii)} \quad p_1 \text{ is smooth and surjective on } \dot{T}^*X, \text{ and has} \\ \quad \text{connected and simply connected fibers,} \\ \text{(iv)} \quad p_2^a \text{ is a closed embedding identifying } \Lambda \text{ to a} \\ \quad \text{closed regular involutive submanifold } V \text{ of } \dot{T}^*Y. \end{array} \right. \quad (2.2)$$

(Recall that a conic involutive submanifold V of T^*X is called regular if the restriction to V of the canonical one form never vanishes.)

If $f: S \rightarrow X$ is a morphism, we denote by $f_!$ and f^{-1} the proper direct image and inverse image for \mathcal{D} -modules, and we denote by \boxtimes the exterior tensor product. To $\mathcal{M} \in \mathbf{D}^b(\mathcal{D}_X)$ we associate its dual

$$\underline{D}'\mathcal{M} = R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X \otimes_{\mathcal{O}_X} \Omega_X^{\otimes -1}),$$

where Ω_X is the sheaf of holomorphic forms of maximal degree. We also set $\underline{D}\mathcal{M} = \underline{D}'\mathcal{M}[d_X]$. Thus, $\underline{D}'\mathcal{M}$ and $\underline{D}\mathcal{M}$ belong to $\mathbf{D}^b(\mathcal{D}_X)$.

Let \mathcal{K} be a simple $\mathcal{D}_{X \times Y}$ -module along Λ . In particular, \mathcal{K} is regular holonomic, and hence $\underline{D}\mathcal{K}$ is concentrated in degree zero. For $\mathcal{M} \in \mathbf{D}^b(\mathcal{D}_X)$, $\mathcal{N} \in \mathbf{D}^b(\mathcal{D}_Y)$, we set:

$$\begin{aligned} \underline{\Phi}_{\mathcal{K}}\mathcal{M} &= \underline{q}_{2!}(\mathcal{K} \otimes_{\mathcal{O}_{X \times Y}}^L \underline{q}_1^{-1}\mathcal{M}), & \underline{\Phi}_{\mathcal{K}}^j\mathcal{M} &= H^j \underline{\Phi}_{\mathcal{K}}\mathcal{M}, \\ \underline{\Psi}_{\mathcal{K}}\mathcal{N} &= \underline{q}_{1!}(\underline{D}\mathcal{K} \otimes_{\mathcal{O}_{X \times Y}}^L \underline{q}_2^{-1}\mathcal{N})[d_X - d_Y], & \underline{\Psi}_{\mathcal{K}}^j\mathcal{N} &= H^j \underline{\Psi}_{\mathcal{K}}\mathcal{N}. \end{aligned}$$

Theorem 2.1. *Assume (2.2). Let \mathcal{M} be a simple \mathcal{D}_X -module along \dot{T}^*X , and let \mathcal{N} be a simple \mathcal{D}_Y -module along V . Then:*

- (o) $\underline{\Phi}_{\mathcal{K}}^0$ and $\underline{\Psi}_{\mathcal{K}}^0$ send m - f - c isomorphisms to m - f - c isomorphisms.
- (i) $\underline{\Phi}_{\mathcal{K}}^0\mathcal{M}$ is simple along V , and $\underline{\Psi}_{\mathcal{K}}^0\mathcal{N}$ is simple along \dot{T}^*X . Moreover, $\underline{\Phi}_{\mathcal{K}}^j\mathcal{M}$ and $\underline{\Psi}_{\mathcal{K}}^j\mathcal{N}$ are flat connections for $j \neq 0$.
- (ii) The natural adjunction morphisms $\mathcal{M} \rightarrow \underline{\Psi}_{\mathcal{K}}^0 \underline{\Phi}_{\mathcal{K}}^0 \mathcal{M}$ and $\underline{\Phi}_{\mathcal{K}}^0 \underline{\Psi}_{\mathcal{K}}^0 \mathcal{N} \rightarrow \mathcal{N}$ are m - f - c isomorphisms.

Proof. The above theorem has been proved in [3] in the case where $\Lambda = \dot{T}_S^*(X \times Y)$, for a smooth submanifold $S \subset X \times Y$, and $\mathcal{K} = \mathcal{B}_S$, the sheaf of holomorphic hyperfunctions along S . There, we also show that this statement is of a microlocal nature, i.e., local on Λ . The general case follows since, in a neighborhood of any point of Λ , one may find a quantized contact transformation interchanging the pair (Λ, \mathcal{K}) with the pair $(\dot{T}_S^*(X \times Y), \mathcal{B}_S)$. \square

Theorem 2.2. *Assume (2.2), and assume $d_X \geq 3$. Let \mathcal{N} be a simple \mathcal{D}_Y -module along V . Then \mathcal{N} is isomorphic to $\underline{\Phi}_{\mathcal{K}}^0 \mathcal{D}\mathcal{L}$ in $\text{Mod}_{\text{coh}}(\mathcal{D}_Y; \mathcal{O}_Y)$ for a line bundle \mathcal{L} on X , and such an \mathcal{L} is unique up to \mathcal{O}_X -linear isomorphisms. In particular, simple \mathcal{D}_Y -modules along V are classified, up to flat connections, by $\text{Pic}(X)$.*

Proof. By Theorem 2.1 (i), $\underline{\Psi}_{\mathcal{K}}^0(\mathcal{N})$ is simple along \dot{T}^*X . Hence, by Theorem 1.6, there exists a line bundle \mathcal{L} on X and an m-f-c isomorphism $\mathcal{D}\mathcal{L} \rightarrow \underline{\Psi}_{\mathcal{K}}^0(\mathcal{N})$. By Theorem 2.1 (o), we get an m-f-c isomorphism $\underline{\Phi}_{\mathcal{K}}^0(\underline{\Psi}_{\mathcal{K}}^0(\mathcal{N})) \rightarrow \underline{\Phi}_{\mathcal{K}}^0 \mathcal{D}\mathcal{L}$. We conclude by recalling that Theorem 2.1 (ii) gives an m-f-c isomorphism $\underline{\Phi}_{\mathcal{K}}^0(\underline{\Psi}_{\mathcal{K}}^0(\mathcal{N})) \rightarrow \mathcal{N}$. \square

Theorem 2.1 gives an equivalence of categories between simple \mathcal{D}_X -modules on \dot{T}^*X , and simple \mathcal{D}_Y -modules on V , modulo flat connections. However, if one is interested in calculating explicitly the image of a \mathcal{D}_X -module associated to a line bundle, one way to do it consists in “quantizing” this equivalence. This is the purpose of the next result.

With the same notations as in Theorem 2.2, let \mathcal{M} be a simple \mathcal{D}_X -module along \dot{T}^*X , and let \mathcal{N} be a simple \mathcal{D}_Y -module along V . Then $\underline{D}'\mathcal{M} \boxtimes \mathcal{N}$ is a simple $\mathcal{D}_{X \times Y}$ -module along $\dot{T}^*X \times V$. Let $p \in \Lambda$, and let u be a generator of $(\underline{D}'\mathcal{M} \boxtimes \mathcal{N})^\wedge$ (the $\widehat{\mathcal{E}}_{X \times Y}$ -module associated to $\underline{D}'\mathcal{M} \boxtimes \mathcal{N}$) in a neighborhood of p . We say that a section

$$s \in \text{Hom}_{\mathcal{D}_{X \times Y}}(\underline{D}'\mathcal{M} \boxtimes \mathcal{N}, \mathcal{K}) \quad (2.3)$$

is non degenerate at p if $s(u)$ is a non degenerate section of \mathcal{K} in the sense of [9]. (One checks immediately that this definition does not depend on u .) We say that s is non degenerate on Λ if s is non degenerate at any $p \in \Lambda$. There is a natural isomorphism (see e.g., [4, Lemma 3.1]):

$$\alpha: \text{Hom}_{\mathcal{D}_{X \times Y}}(\underline{D}'\mathcal{M} \boxtimes \mathcal{N}, \mathcal{K}) \xrightarrow{\sim} \text{Hom}_{\mathcal{D}_Y}(\mathcal{N}, \underline{\Phi}_{\mathcal{K}}(\mathcal{M})).$$

Hence, a section s as in (2.3) defines a \mathcal{D}_Y -linear morphism $\alpha(s): \mathcal{N} \rightarrow \underline{\Phi}_{\mathcal{K}}(\mathcal{M})$.

Theorem 2.3. *With the above notations, if s is non degenerate on Λ , then $\alpha(s)$ is an m-f-c isomorphism.*

Proof. Let $\widehat{\alpha}(s): \widehat{\mathcal{N}} \rightarrow \underline{\Phi}_{\mathcal{K}}(\mathcal{M})^\wedge$ denote the associated $\widehat{\mathcal{E}}_{X \times Y}$ -linear morphism. It is enough to check that $\widehat{\alpha}(s)$ is an isomorphism at each $p \in V$. This can be done as in [4, Theorem 3.3], using [3, Lemma 4.7]. \square

3 Application 1: projective duality

By the methods above, we will recall here some results of [4], [8] on the complex projective Radon transform.

Let us begin with an obvious remark: let $S \subset X \times Y$ be a smooth hypersurface, and set $\Omega = (X \times Y) \setminus S$. Let $\mathcal{B}_S = T\mathcal{H}om(\mathbb{C}_S[-1], \mathcal{O}_{X \times Y})$, $\mathcal{B}_\Omega = T\mathcal{H}om(\mathbb{C}_\Omega, \mathcal{O}_{X \times Y})$, $\mathcal{B}_\Omega^* = \underline{D}\mathcal{B}_\Omega$. The short exact sequence:

$$0 \rightarrow \mathbb{C}_\Omega \rightarrow \mathbb{C}_{X \times Y} \rightarrow \mathbb{C}_S \rightarrow 0$$

induces the distinguished triangles:

$$\mathcal{O}_{X \times Y} \rightarrow \mathcal{B}_\Omega \rightarrow \mathcal{B}_S \xrightarrow{+1}, \quad \mathcal{B}_S \rightarrow \underline{D}\mathcal{B}_\Omega^* \rightarrow \mathcal{O}_{X \times Y} \xrightarrow{+1}.$$

In particular, this implies that for any coherent \mathcal{D}_X -module \mathcal{M} , the modules $\underline{\Phi}_{\mathcal{B}_S}^0(\mathcal{M})$, $\underline{\Phi}_{\mathcal{B}_\Omega}^0(\mathcal{M})$ and $\underline{\Phi}_{\mathcal{B}_\Omega^*}^0(\mathcal{M})$ are isomorphic in $\text{Mod}_{\text{coh}}(\mathcal{D}_Y; \mathcal{O}_Y)$.

Let \mathbb{P} be a complex n -dimensional projective space, \mathbb{P}^* the dual projective space, $\mathbb{A} = \{(z, \zeta); \langle z, \zeta \rangle = 0\}$ the incidence relation, and set $\Omega = (\mathbb{P} \times \mathbb{P}^*) \setminus \mathbb{A}$. Note that $\Lambda = \dot{T}_{\mathbb{A}}^*(\mathbb{P} \times \mathbb{P}^*)$ is the graph of a globally defined contact transformation (the Legendre transform):

$$\dot{T}^*\mathbb{P} \xleftarrow[p_1]{\sim} \Lambda \xrightarrow[p_2^*]{\sim} \dot{T}^*\mathbb{P}^*.$$

Consider the kernel:

$$\mathcal{K} = T\mathcal{H}om(\mathbb{C}_\Omega, \mathcal{O}_{\mathbb{P} \times \mathbb{P}^*}) \simeq \mathcal{O}_{\mathbb{P} \times \mathbb{P}^*}(*\mathbb{A}),$$

the sheaf of meromorphic functions with poles on \mathbb{A} . Denote by $\mathcal{O}_{\mathbb{P}}(k)$ the $-k$ -th tensor power of the tautological line bundle, and set $\mathcal{D}_{\mathbb{P}}(k) = \mathcal{D}_{\mathbb{P}} \otimes_{\mathcal{O}_{\mathbb{P}}} \mathcal{O}_{\mathbb{P}}(k)$,

$$\mathcal{K}^{(n,0)}(k, l) = q_1^{-1}(\Omega_{\mathbb{P}} \otimes_{\mathcal{O}_{\mathbb{P}}} \mathcal{O}_{\mathbb{P}}(k)) \otimes_{q_1^{-1}\mathcal{O}_{\mathbb{P}}} \mathcal{K} \otimes_{q_2^{-1}\mathcal{O}_{\mathbb{P}^*}} q_2^{-1}\mathcal{O}_{\mathbb{P}^*}(l).$$

Let

$$\omega(z) = \sum_{j=0}^n (-1)^j z_j dz_0 \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge dz_n$$

be the Leray form, and set:

$$s_k(z, \zeta) = \frac{\omega(z)}{\langle z, \zeta \rangle^{n+1+k}}, \quad k^* = -n - 1 - k.$$

Then:

$$\begin{aligned} s_k(z, \zeta) &\in \Gamma(\mathbb{P} \times \mathbb{P}^*; \mathcal{K}^{(n,0)}(-k, k^*)) \\ &\simeq \text{Hom}_{\mathcal{D}_{X \times Y}}(\mathcal{D}_{\mathbb{P}}(k) \boxtimes \mathcal{D}_{\mathbb{P}^*}(-k^*), \mathcal{K}). \end{aligned}$$

If $k^* < 0$, s_k is a non degenerate section on Λ . Similarly, denote by $Y(t)$ the canonical generator of $\underline{D}(\mathcal{T}\mathcal{H}om(\mathbb{C}_{\mathbb{C}\setminus\{0\}}, \mathcal{O}_{\mathbb{C}})) \simeq \mathcal{D}_{\mathbb{C}}/\mathcal{D}_{\mathbb{C}}(t\partial_t - 1)$, and consider

$$\tilde{s}_k(z, \zeta) = \langle z, \zeta \rangle^{k^*} Y(\langle z, \zeta \rangle) \omega(z).$$

Then

$$\begin{aligned} \tilde{s}_k(z, \zeta) &\in \Gamma(\mathbb{P} \times \mathbb{P}^*; \underline{D}\mathcal{K}^{(n,0)}(-k, k^*)) \\ &\simeq \text{Hom}_{\mathcal{D}_{X \times Y}}(\mathcal{D}_{\mathbb{P}}(k) \boxtimes \mathcal{D}_{\mathbb{P}^*}(-k^*), \underline{D}\mathcal{K}). \end{aligned}$$

If $k^* \geq 0$, s_k is a non degenerate section on Λ .

By Theorem 2.3, we thus get:

Theorem 3.1. *With the above notations, one has*

- (i) *If $k^* < 0$, then $\alpha(s_k): \mathcal{D}_{\mathbb{P}^*}(-k^*) \rightarrow \underline{\Phi}_{\mathcal{K}}^0 \mathcal{D}_{\mathbb{P}}(-k)$ is an m-f-c isomorphism.*
- (ii) *If $k^* \geq 0$, then $\alpha(\tilde{s}_k): \mathcal{D}_{\mathbb{P}^*}(-k^*) \rightarrow \underline{\Phi}_{\underline{D}\mathcal{K}}^0 \mathcal{D}_{\mathbb{P}}(-k)$ is an m-f-c isomorphism.*

Remark 3.2. Theorem 3.1 was obtained in [4] for $-n - 1 < k < 0$, in which case it was also shown that $\underline{\Phi}_{\mathcal{B}_{\mathbb{A}}}^j(\mathcal{D}_{\mathbb{P}}(-k)) = 0$ for $j \neq 0$. The kernels \mathcal{K} and $\underline{D}\mathcal{K}$ were introduced in [8].

The results of section 1 imply that simple $\mathcal{D}_{\mathbb{P}}$ -modules along $\dot{T}^*\mathbb{P}$ are classified, up to flat connections, by $\text{Pic}(\mathbb{P}) \simeq \mathbb{Z}$. If \mathcal{M} is simple along $\dot{T}^*\mathbb{P}$, we denote by $ch(\mathcal{M})$ the Chern class of the line bundle \mathcal{L} such that \mathcal{M} is m-f-c isomorphic to $\mathcal{D}\mathcal{L}$.

Corollary 3.3. *Let \mathcal{M} be simple along $\dot{T}^*\mathbb{P}$. Then:*

$$ch(\underline{\Phi}_{\mathcal{B}_{\mathbb{A}}}^0(\mathcal{M})) = -n - 1 - ch(\mathcal{M})$$

4 Application 2: twistor correspondence

For $1 \leq q \leq p \leq n$, denote by $F(q, p)$ the flag manifold of type (q, p) in an $(n + 1)$ dimensional complex vector space V , and denote by $G(p) = F(p, p)$ the Grassmannian manifold of p -planes in V . Let $\mathbb{P} = G(1)$, a complex n -dimensional projective space, $\mathbb{M} = G(p)$, and denote by $\mathbb{F} = F(1, p)$ the incidence relation. To $\Lambda = \dot{T}_{\mathbb{F}}^*(\mathbb{P} \times \mathbb{M})$ is associated a diagram:

$$\dot{T}^*\mathbb{P} \xleftarrow[p_1]{} \Lambda \xrightarrow[p_2^*]{} V \subset \dot{T}^*\mathbb{M},$$

which satisfies hypotheses (2.2). Consider the kernel $\mathcal{K} = \mathcal{B}_{\mathbb{F}}$. Theorem 2.2 thus implies:

Theorem 4.1. *Simple $\mathcal{D}_{\mathbb{M}}$ -modules along $V \subset \dot{T}^*\mathbb{M}$ are classified, up to flat connections, by \mathbb{Z} .*

Let \mathcal{N} be a simple $\mathcal{D}_{\mathbb{M}}$ -modules along V , and let k be the unique integer such that \mathcal{N} is isomorphic to $\Phi_{\mathcal{B}^{\mathbb{A}}}^0(\mathcal{D}_{\mathbb{P}}(-k))$ (up to flat connections). Following Penrose and the physics literature, one sets:

$$h(\mathcal{N}) = -(1 + k/2),$$

and calls it the “helicity” of \mathcal{N} .

Remark 4.2. Consider the case $n = 4$, $p = 2$, as in [5]. Theorem 4.1 shows in particular that there are no other simple $\mathcal{D}_{\mathbb{M}}$ -modules along V than those corresponding to the massless field equations in the conformally compactified Minkowski space \mathbb{M} (see [5] and [1] for related results).

5 Comments

The classification of locally free \mathcal{D} -modules (see [2]) or, more generally, simple $\widehat{\mathcal{E}}$ -modules globally defined on an involutive manifold V , would be, in our opinion, a very interesting task. This paper should be considered as a first step in this direction.

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