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MICROLOCAL EULER CLASSES AND HOCHSCHILD HOMOLOGY

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Abstract We define the notion of a trace kernel on a manifold M. Roughly speaking, it is a sheaf on $M \times M$ for which the formalism of Hochschild homology applies. We associate a microlocal Euler class with such a kernel, a cohomology class with values in the relative dualizing complex of the cotangent bundle T^*M over M, and we prove that this class is functorial with respect to the composition of kernels.

This generalizes, unifies and simplifies various results from (relative) index theorems for constructible sheaves, $\mathscr{D}$-modules and elliptic pairs.

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1. Introduction

Our constructions mainly concern real manifolds, but in order to introduce the subject we first consider a complex manifold $(X, \mathscr{O}_X)$. Denote by ω_X^{hol} the dualizing complex in the category of $\mathscr{O}_X$-modules, that is, $\omega_X^{\mathrm{hol}} = \Omega_X\,[d_X]$, where d_X is the complex dimension of X and Ω_X is the sheaf of holomorphic forms of degree d_X. Denote by $\mathscr{O}_{\Delta_X}$ and $\omega_{\Delta_X}^{\mathrm{hol}}$ the direct images of $\mathscr{O}_X$ and ω_X^{hol} respectively under the diagonal embedding $\delta\colon X \hookrightarrow X \times X$. It is well-known (see in particular [3, 4]) that the Hochschild homology of $\mathscr{O}_X$ may be defined by using the isomorphism

$$\delta_*\mathscr{H}\mathscr{H}(\mathscr{O}_X) \simeq \mathrm{R}\mathscr{H}om_{\mathscr{O}_{X\times X}}(\mathscr{O}_{\Delta_X}, \omega_{\Delta_X}^{\mathrm{hol}}). \tag{1.1}$$

Moreover, if $\mathscr{F}$ is a coherent $\mathscr{O}_X$-module and $\mathrm{D}_{\mathscr{O}}\mathscr{F} := \mathrm{R}\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \omega_X^{\mathrm{hol}})$ denotes its dual, there are natural morphisms

$$\mathscr{O}_{\Delta_X} \to \mathscr{F} \boxtimes \mathrm{D}_{\mathscr{O}}\mathscr{F} \to \omega_{\Delta_X}^{\mathrm{hol}} \tag{1.2}$$

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whose composition defines the Hochschild class of $\mathscr{F}$:

$$\mathrm{hh}_{\mathscr{O}}(\mathscr{F}) \in H^0_{\mathrm{Supp}(\mathscr{F})}(X; \mathscr{H}\mathscr{H}(\mathscr{O}_X)).$$

These constructions have been extended when replacing $\mathscr{O}_X$ with a so-called DQ-algebroid stack $\mathscr{A}_X$ in [15] (DQ stands for "deformation quantization"). One of the main results of this reference is that Hochschild classes are functorial with respect to the composition of kernels, a kind of (relative) index theorem for coherent DQ-modules.

On the other hand, the notion of Lagrangian cycles of constructible sheaves on real analytic manifolds has been introduced by the first-named author (see [9]) in order to prove an index theorem for such sheaves, after they first appeared in the complex case (see [8, 19]). We refer the reader to [13, Chapter 9] for a systematic study of Lagrangian cycles and for historical comments. Let us briefly recall the construction.

Consider a real analytic manifold M and let $\mathbf{k}$ be a unital commutative ring with finite global dimension. Denote by ω_M the (topological) dualizing complex of M, that is, $\omega_M = \mathrm{or}_M[\dim M]$ where or_M is the orientation sheaf of M and $\dim M$ is the dimension. Finally, denote by $\pi_M \colon T^*M \to M$ the cotangent bundle of M. Let Λ be a conic subanalytic Lagrangian subset of T^*M. The group of Lagrangian cycles supported by Λ is given by $H^0_\Lambda(T^*M; \pi_M^{-1}\omega_M)$. Denote by $\mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbf{k}_M)$ the bounded derived category of $\mathbb{R}$-constructible sheaves on M. With an object F of this category, one associates a Lagrangian cycle supported by $\mathrm{SS}(F)$, the microsupport of F. This cycle is called the characteristic cycle, or the Lagrangian cycle or else the *microlocal Euler class* of F and is denoted here by $\mu\mathrm{eu}_M(F)$.

In fact, it is possible to treat the microlocal Euler classes of $\mathbb{R}$-constructible sheaves on real manifolds like Hochschild classes of coherent sheaves on complex manifolds. Denote as above by $\mathbf{k}_{\Delta_M}$ and ω_{Δ_M} the direct image of $\mathbf{k}_M$ and ω_M under the diagonal embedding $\delta_M \colon M \hookrightarrow M \times M$. Then we have an isomorphism

$$H^0_\Lambda(T^*M; \pi_M^{-1}\omega_M) \simeq H^0_\Lambda\big(T^*M; \mu hom(\mathbf{k}_{\Delta_M}, \omega_{\Delta_M})\big), \tag{1.3}$$

where μhom is the microlocalization of the functor $\mathrm{R}\mathscr{H}om$. Then $\mu\mathrm{eu}_M(F)$ is obtained as follows. Denote by $\mathrm{D}_M F := \mathrm{R}\mathscr{H}om(F, \omega_M)$ the dual of F. There are natural morphisms

$$\mathbf{k}_{\Delta_M} \to F \boxtimes \mathrm{D}_M F \to \omega_{\Delta_M}, \tag{1.4}$$

whose composition gives the microlocal Euler class of F.

In this paper, we construct the microlocal Euler class for a wide class of sheaves, including of course the constructible sheaves but also the sheaves of holomorphic solutions of coherent $\mathscr{D}$-modules and, more generally, of elliptic pairs in the sense of [23]. To treat such situations, we are led to introduce the notion of a trace kernel.

On a real manifold M (say of class C^∞), a *trace kernel* is the data of a triplet (K, u, v) where K is an object of the derived category of sheaves $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M\times M})$ and u, v are morphisms

$$u \colon \mathbf{k}_{\Delta_M} \to K, \quad v \colon K \to \omega_{\Delta_M}. \tag{1.5}$$

One then naturally defines the microlocal Euler class $\mu\mathrm{eu}_M(K,u,v)$ of such a kernel, an element of $H^0_\Lambda(T^*M;\mu hom(\mathbf{k}_{\Delta_M},\omega_{\Delta_M}))$ where $\Lambda = \mathrm{SS}(K)\cap T^*_{\Delta_M}(M\times M)$. By (1.4), a constructible sheaf gives rise to a trace kernel.

If X is a complex manifold and $\mathscr{M}$ is a coherent $\mathscr{D}_X$-module, we construct natural morphisms (over the base ring $\mathbf{k}=\mathbb{C}$)

$$\mathbb{C}_{\Delta_X} \to \Omega_{X\times X} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_{X\times X}} (\mathscr{M}\boxtimes \mathrm{D}_D\mathscr{M}) \to \omega_{\Delta_X}, \tag{1.6}$$

where $\mathrm{D}_D\mathscr{M}$ denotes the dual of $\mathscr{M}$ as a $\mathscr{D}$-module. In other words, one naturally associates a trace kernel on X with a coherent $\mathscr{D}_X$-module. Moreover, we prove that under suitable microlocal conditions, the tensor product of two trace kernels is again a trace kernel, and it follows that one can associate a trace kernel with an elliptic pair.

We study trace kernels and their microlocal Euler classes, showing that some proofs of [15] can be easily adapted to this situation. One of our main results is the functoriality of the microlocal Euler classes: the microlocal Euler class of the composition $K_1\circ K_2$ of two trace kernels is the composition of the microlocal Euler classes of K_1 and K_2 (see Theorem 6.3 for a precise statement). Another essential result is that the composition of classes coincides with the composition for $\pi_M^{-1}\omega_M$ constructed in [13] via the isomorphism between $\mu hom(\mathbf{k}_{\Delta_M},\omega_{\Delta_M})$ and $\pi_M^{-1}\omega_M$.

As an application, we recover in a single proof the classical results on the index theorem for constructible sheaves (see [13, § 9.5]) as well as the index theorem for elliptic pairs of [23], that is, sheaves of generalized holomorphic solutions of coherent $\mathscr{D}$-modules. We also briefly explain how to adapt trace kernels to the formalism of the Lefschetz trace formula.

We call here $\mu hom(\mathbf{k}_{\Delta_M},\omega_{\Delta_M})$ the *microlocal homology of M*, and this paper shows that, in some sense, the microlocal homology of real manifolds plays the same role as the Hochschild homology of complex manifolds.

To conclude this introduction, let us make a general remark. The category $\mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbf{k}_M)$ of constructible sheaves on a compact real analytic manifold M is "proper" in the sense of Kontsevich (that is, Ext finite) but it does not admit a Serre functor (in the sense of Bondal and Kapranov) and it is not clear whether it is smooth (again in the sense of Kontsevich). However this category naturally appears in mirror symmetry (see [5]) and it would be a natural aim to try to understand its Hochschild homology in the sense of [17, 16]. We do not know how to compute it, but the above construction, with the use of $\mu hom(\mathbf{k}_{\Delta_M},\omega_{\Delta_M})$, provides an alternative approach to the Hochschild homology of this category. This result is not totally surprising if one recalls the formula (see [13, Proposition 8.4.14])

$$\mathrm{D}_{T^*M}(\mu hom(F,G)) \simeq \mu hom(G,F)\otimes \pi_M^{-1}\omega_M.$$

Hence, in some sense, $\pi_M^{-1}\omega_M$ plays the role of a microlocal Serre functor. Note that thanks to Nadler and Zaslow [18], we have that the category $\mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbf{k}_M)$ is equivalent to the Fukaya category of the symplectic manifold T^*M, and this is another argument for treating sheaves from a microlocal point of view.

2. A short review on sheaves

Throughout this paper, a manifold means a real manifold of class C^∞. We shall mainly follow the notation of [13] and use some of the main notions introduced there, in particular that of microsupport and the functor μhom.

Let M be a manifold. We denote by $\pi_M \colon T^*M \to M$ its cotangent bundle. For a submanifold N of M, we denote by $T^*_N M$ the conormal bundle to N. In particular, $T^*_M M$ denotes the zero-section. We set $\dot{T}^*M := T^*M \setminus T^*_M M$ and we denote by $\dot{\pi}_M$ the restriction of π_M to $\dot{T}^*M$. If there is no risk of confusion, we write simply π and $\dot{\pi}$ instead of π_M and $\dot{\pi}_M$. One denotes by $a \colon T^*M \to T^*M$ the antipodal map, $(x;\xi) \mapsto (x;-\xi)$, and for a subset S of T^*M, one denotes by S^a its image under this map. A set $A \subset T^*M$ is conic if it is invariant under the action of $\mathbb{R}^+$ on T^*M.

Let $f \colon M \to N$ be a morphism of manifolds. With f one associates as usual the maps

$$\begin{array}{ccccc} T^*M & \xleftarrow{\;f_d\;} & M \times_N T^*N & \xrightarrow{\;f_\pi\;} & T^*N \\ & \searrow^{\pi_M} & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_N} \\ & & M & \xrightarrow{\;f\;} & N. \end{array} \tag{2.1}$$

(Note that in the above citation the map f_d is denoted by ${}^t f'^{-1}$.)

Let Λ be a closed conic subset of T^*N. One says that f is *non-characteristic* for Λ if the map f_d is proper on $f_\pi^{-1}\Lambda$ or, equivalently, $f_\pi^{-1}\Lambda \cap f_d^{-1}(T^*_M M) \subset M \times_N T^*_N N$.

Let $\mathbf{k}$ be a commutative unital ring with finite global homological dimension. One denotes by $\mathbf{k}_M$ the constant sheaf on M with stalk $\mathbf{k}$ and by $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$ the bounded derived category of sheaves of $\mathbf{k}$-modules on M. When M is a real analytic manifold, one denotes by $\mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbf{k}_M)$ the full triangulated subcategory of $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$ consisting of $\mathbb{R}$-constructible objects.

One denotes by ω_M the dualizing complex on M and by $\omega_M^{\otimes -1}$ its dual, that is, $\omega_M^{\otimes -1} = \mathrm{R}\mathscr{H}om(\omega_M, \mathbf{k}_M)$. More generally, for a morphism $f \colon M \to N$, one denotes by $\omega_{M/N} := f^{!}\mathbf{k}_N \simeq \omega_M \otimes f^{-1}(\omega_N^{\otimes -1})$ the relative dualizing complex. Recall that $\omega_M \simeq \mathrm{or}_M[\dim M]$ where or_M is the orientation sheaf and $\dim M$ is the dimension of M. Also recall the natural morphism of functors

$$\omega_{M/N} \otimes f^{-1} \to f^{!}. \tag{2.2}$$

We have the duality functors

$$\mathrm{D}'_M F = \mathrm{R}\mathscr{H}om(F, \mathbf{k}_M), \quad \mathrm{D}_M F = \mathrm{R}\mathscr{H}om(F, \omega_M).$$

For $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$, one denotes by $\mathrm{Supp}(F)$ the support of F and by $\mathrm{SS}(F)$ its microsupport, a closed $\mathbb{R}^+$-conic co-isotropic subset of T^*M. For a morphism $f \colon M \to N$ and $G \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_N)$, one says that f is non-characteristic for G if f is non-characteristic for $\mathrm{SS}(G)$.

We shall use systematically the functor μhom, a variant of Sato's microlocalization functor. Recall that for a closed submanifold N of M, there is a functor $\mu_N \colon \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M) \to \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{T^*_N M})$ constructed by Sato (see [22]) and for $F_1, F_2 \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$, one defines in [13] the functor

$$\mu hom\colon \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)^{\mathrm{op}} \times \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M) \to \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{T^*M}),$$
$$\mu hom(F_1, F_2) := \mu_\Delta \mathrm{R}\mathscr{H}om(q_2^{-1}F_1, q_1^! F_2)$$

where q_1 and q_2 are the first and second projections defined on $M \times M$ and Δ is the diagonal. This sheaf is supported by $T^*_\Delta(M \times M)$ that we identify with T^*M via the first projection $T^*(M \times M) \simeq T^*M \times T^*M \to T^*M$. Note that

$$\mathrm{Supp}(\mu hom(F_1, F_2)) \subset \mathrm{SS}(F_1) \cap \mathrm{SS}(F_2) \tag{2.3}$$

and we have Sato's distinguished triangle, functorial in F_1 and F_2:

$$\mathrm{R}\pi_!\mu hom(F_1, F_2) \to \mathrm{R}\pi_*\mu hom(F_1, F_2) \to \mathrm{R}\dot{\pi}_*\big(\mu hom(F_1, F_2)|_{\dot{T}^*M}\big) \xrightarrow{+1} . \tag{2.4}$$

Moreover, we have the isomorphism

$$\mathrm{R}\pi_*\mu hom(F_1, F_2) \simeq \mathrm{R}\mathscr{H}om(F_1, F_2), \tag{2.5}$$

and, assuming that M is real analytic and F_1 is $\mathbb{R}$-constructible, the isomorphism

$$\mathrm{R}\pi_!\mu hom(F_1, F_2) \simeq \mathrm{D}'_M F_1 \overset{\mathrm{L}}{\otimes} F_2. \tag{2.6}$$

In particular, assuming that F_1 is $\mathbb{R}$-constructible and $\mathrm{SS}(F_1) \cap \mathrm{SS}(F_2) \subset T^*_M M$, we have the natural isomorphism (see [13, Corollary 6.4.3])

$$\mathrm{D}'_M F_1 \overset{\mathrm{L}}{\otimes} F_2 \xrightarrow{\sim} \mathrm{R}\mathscr{H}om(F_1, F_2). \tag{2.7}$$

As recalled in the Introduction, assuming that M is real analytic and the sheaves are constructible, we have the formula (see [13, Proposition 8.4.14])

$$\mathrm{D}_{T^*M}(\mu hom(F_1, F_2)) \simeq \mu hom(F_2, F_1) \otimes \pi_M^{-1}\omega_M \quad \text{for } F_1, F_2 \in \mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbf{k}_M). \tag{2.8}$$

3. Compositions of kernels

Notation 3.1. (i) For a manifold M, let $\delta_M\colon M \to M \times M$ denote the diagonal embedding, and Δ_M the diagonal set of $M \times M$.

(ii) Let M_i $(i = 1, 2, 3)$ be manifolds. For short, we write $M_{ij} := M_i \times M_j$ $(1 \leqslant i, j \leqslant 3)$, $M_{123} = M_1 \times M_2 \times M_3$, $M_{1223} = M_1 \times M_2 \times M_2 \times M_3$, etc.

(iii) We will often write for short $\mathbf{k}_i$ instead of $\mathbf{k}_{M_i}$ and $\mathbf{k}_{\Delta_i}$ instead of $\mathbf{k}_{\Delta_{M_i}}$, and similarly with ω_{M_i}, etc., and with the index i replaced with several indices ij, etc.

(iv) We denote by π_i, π_{ij}, etc. the projection $T^*M_i \to M_i$, $T^*M_{ij} \to M_{ij}$, etc.

(v) We denote by q_i the projection $M_{ij} \to M_i$ or the projection $M_{123} \to M_i$ and by q_{ij} the projection $M_{123} \to M_{ij}$. Similarly, we denote by p_i the projection $T^*M_{ij} \to T^*M_i$ or the projection $T^*M_{123} \to T^*M_i$ and by p_{ij} the projection $T^*M_{123} \to T^*M_{ij}$.

(vi) We also need to introduce the maps p_{j^a} or p_{ij^a}, the composition of p_j or p_{ij} and the antipodal map on T^*M_j. For example,

$$p_{12^a}((x_1,x_2,x_3;\xi_1,\xi_2,\xi_3)) = (x_1,x_2;\xi_1,-\xi_2).$$

(vii) We let $\delta_2\colon M_{123}\to M_{1223}$ be the natural diagonal embedding.

We consider the operation of composition of kernels:

$$\begin{aligned}\underset{2}{\circ}\colon \mathsf{D}^{\mathrm b}(\mathbf{k}_{M_{12}})\times\mathsf{D}^{\mathrm b}(\mathbf{k}_{M_{23}}) &\to \mathsf{D}^{\mathrm b}(\mathbf{k}_{M_{13}})\\ (K_1,K_2)\mapsto K_1\underset{2}{\circ}K_2 &:= \mathrm{R}q_{13!}(q_{12}^{-1}K_1\overset{\mathrm L}{\otimes} q_{23}^{-1}K_2)\\ &\simeq \mathrm{R}q_{13!}\delta_2^{-1}(K_1\overset{\mathrm L}{\boxtimes}K_2).\end{aligned}\tag{3.1}$$

We will use a variant of $\circ$:

$$\begin{aligned}\underset{2}{*}\colon \mathsf{D}^{\mathrm b}(\mathbf{k}_{M_{12}})\times\mathsf{D}^{\mathrm b}(\mathbf{k}_{M_{23}}) &\to \mathsf{D}^{\mathrm b}(\mathbf{k}_{M_{13}})\\ (K_1,K_2)\mapsto K_1\underset{2}{*}K_2 &:= \mathrm{R}q_{13*}\big(q_2^{-1}\omega_2\otimes\delta_2^!(K_1\overset{\mathrm L}{\boxtimes}K_2)\big).\end{aligned}\tag{3.2}$$

We also have $\omega_{M_{123}/M_{1223}}\simeq q_2^{-1}\omega_{M_2}^{\otimes -1}$ and we deduce from (2.2) a morphism $\delta_2^{-1}\to q_2^{-1}\omega_{M_2}\otimes\delta_2^!$. Using the morphism $\mathrm{R}p_{13!}\to\mathrm{R}p_{13*}$ we obtain a natural morphism for $K_1\in\mathsf{D}^{\mathrm b}(\mathbf{k}_{M_{12}})$ and $K_2\in\mathsf{D}^{\mathrm b}(\mathbf{k}_{M_{23}})$:

$$K_1\circ K_2\to K_1 * K_2.\tag{3.3}$$

It is an isomorphism if $p_{12^a}^{-1}\mathrm{SS}(K_1)\cap p_{23^a}^{-1}\mathrm{SS}(K_2)\to T^*M_{13}$ is proper.

We define the composition of kernels on cotangent bundles (see [13, Proposition 4.4.11]):

$$\begin{aligned}\underset{2}{\overset{a}{\circ}}\colon \mathsf{D}^{\mathrm b}(\mathbf{k}_{T^*M_{12}})\times\mathsf{D}^{\mathrm b}(\mathbf{k}_{T^*M_{23}}) &\to \mathsf{D}^{\mathrm b}(\mathbf{k}_{T^*M_{13}})\\ (K_1,K_2)\mapsto K_1\underset{2}{\overset{a}{\circ}}K_2 &:= \mathrm{R}p_{13!}(p_{12^a}^{-1}K_1\overset{\mathrm L}{\otimes}p_{23}^{-1}K_2)\\ &\simeq \mathrm{R}p_{13^a!}(p_{12^a}^{-1}K_1\overset{\mathrm L}{\otimes}p_{23^a}^{-1}K_2).\end{aligned}\tag{3.4}$$

We also define the corresponding operations for subsets of cotangent bundles. Let $A\subset T^*M_{12}$ and $B\subset T^*M_{23}$. We set

$$\begin{aligned}A\underset{2}{\overset{a}{\times}}B &= p_{12^a}^{-1}(A)\cap p_{23}^{-1}(B),\\ A\underset{2}{\overset{a}{\circ}}B &= p_{13}(A\underset{2}{\overset{a}{\times}}B)\\ &= \left\{\begin{matrix}(x_1,x_3;\xi_1,\xi_3)\in T^*M_{13};\text{ there exists }(x_2;\xi_2)\in T^*M_2\\ \text{such that }(x_1,x_2;\xi_1,-\xi_2)\in A,\ (x_2,x_3;\xi_2,\xi_3)\in B\end{matrix}\right\}.\end{aligned}\tag{3.5}$$

We have the following result which slightly strengthens Proposition 4.4.11 of [13] in which the composition $*$ is not used.

Proposition 3.2. *For $G_1, F_1 \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M_{12}})$ and $G_2, F_2 \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M_{23}})$ there exists a canonical morphism (whose construction is similar to that of [13, Proposition 4.4.11]):*

$$\mu hom(G_1, F_1) \overset{a}{\underset{2}{\circ}} \mu hom(G_2, F_2) \to \mu hom(G_1 \underset{2}{*} G_2, F_1 \underset{2}{\circ} F_2).$$

Proof. In Proposition 4.4.8(i) of the earlier citation, one may replace $F_2 \overset{\mathrm{L}}{\boxtimes}_S G_2$ with $j^!(F_2 \overset{\mathrm{L}}{\boxtimes} G_2) \otimes \omega^{\otimes -1}_{X\times_S Y/X\times Y}$. Then the proof goes exactly like that of Proposition 4.4.11 in the earlier citation. □

Let $\Lambda_{ij} \subset T^*M_{ij}$ $(i = 1, 2, j = i + 1)$ be closed conic subsets and consider the condition

$$\text{the projection } p_{13} \colon \Lambda_{12} \overset{a}{\underset{2}{\times}} \Lambda_{23} \longrightarrow T^*M_{13} \text{ is proper.} \tag{3.6}$$

We set

$$\Lambda_{13} = \Lambda_{12} \overset{a}{\underset{2}{\circ}} \Lambda_{23}. \tag{3.7}$$

Corollary 3.3. *Assume that Λ_{ij} $(i = 1, 2, j = i + 1)$ satisfy (3.6). We have a composition morphism*

$$\mathrm{R}\Gamma_{\Lambda_{12}} \mu hom(G_1, F_1) \overset{a}{\underset{2}{\circ}} \mathrm{R}\Gamma_{\Lambda_{23}} \mu hom(G_2, F_2) \to \mathrm{R}\Gamma_{\Lambda_{13}} \mu hom(G_1 \underset{2}{*} G_2, F_1 \underset{2}{\circ} F_2).$$

Convention 3.4. *In (3.1), we have introduced the composition $\underset{2}{\circ}$ of kernels $K_1 \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M_{12}})$ and $K_2 \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M_{23}})$. However we shall also use the notation $M_{22} = M_2 \times M_2$ and consider for example kernels $L_1 \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M_{122}})$ and $L_2 \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M_{223}})$. Then when writing $L_1 \underset{2}{\circ} L_2$ we mean that the composition is taken with respect to the last variable of M_{22} for L_1 and the first variable for L_2. In other words, set $M_4 = M_2$ and consider L_1 and L_2 as objects of $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M_{142}})$ and $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M_{243}})$ respectively, in which case the composition $L_1 \underset{2}{\circ} L_2$ is unambiguously defined.*

4. Microlocal homology

Let M be a real manifold. Recall that $\delta_M \colon M \hookrightarrow M \times M$ denotes the diagonal embedding. We shall identify M with the diagonal Δ_M of $M \times M$ and we sometimes write Δ instead of Δ_M if there is no risk of confusion. We shall identify T^*M with $T^*_\Delta(M \times M)$ via the map

$$\delta^a_{T^*M} \colon T^*M \hookrightarrow T^*(M \times M), \quad (x; \xi) \mapsto (x, x; \xi, -\xi).$$

We denote by $\mathbf{k}_{\Delta_M}$, ω_{Δ_M} and $\omega^{\otimes -1}_{\Delta_M}$ the direct image under δ_M of $\mathbf{k}_M$, ω_M and $\omega^{\otimes -1}_M := \mathrm{R}\mathscr{H}om(\omega_M, \mathbf{k}_M)$, respectively.

The next definition is inspired by that of Hochschild homology on complex manifolds (see the Introduction).

Definition 4.1. Let Λ be a closed conic subset of T^*M. We set

$$\begin{aligned}\mathscr{MH}_\Lambda(\mathbf{k}_M) &:= \mathrm{R}\Gamma_\Lambda(\delta^a_{T^*M})^{-1}\mu hom(\mathbf{k}_{\Delta_M},\omega_{\Delta_M}),\\ \mathbb{MH}_\Lambda(\mathbf{k}_M) &:= \mathrm{R}\Gamma(T^*M;\mathscr{MH}_\Lambda(\mathbf{k}_M)),\\ \mathbb{MH}^k_\Lambda(\mathbf{k}_M) &:= H^k(\mathbb{MH}_\Lambda(\mathbf{k}_M)) = H^k(T^*M;\mathscr{MH}_\Lambda(\mathbf{k}_M)).\end{aligned} \tag{4.1}$$

We call $\mathscr{MH}_\Lambda(\mathbf{k}_M)$ the *microlocal homology* of M with support in Λ.

We also write $\mathscr{MH}(\mathbf{k}_M)$ instead of $\mathscr{MH}_{T^*M}(\mathbf{k}_M)$.

Remark 4.2. (i) We have $\mu hom(\mathbf{k}_{\Delta_M},\omega_{\Delta_M}) \simeq (\delta^a_{T^*M})_*\pi_M^{-1}\omega_M$. In particular, we have $\mathbb{MH}_\Lambda(\mathbf{k}_M) \simeq \mathrm{R}\Gamma_\Lambda(T^*M;\pi_M^{-1}\omega_M)$ and $\mathbb{MH}(\mathbf{k}_M) \simeq \mathrm{R}\Gamma(M;\omega_M)$. Assuming that M is real analytic and Λ is a closed conic subanalytic Lagrangian subset of T^*M, we recover the space of Lagrangian cycles with support in Λ as defined in [13, § 9.3].

(ii) The support of $\mu hom(\mathbf{k}_{\Delta_M},\omega_{\Delta_M})$ is $T^*_{\Delta_M}(M\times M)$. Hence, we have $\mathrm{R}\Gamma_{\delta^a_{T^*M}\Lambda}\,\mu hom(\mathbf{k}_{\Delta_M},\omega_{\Delta_M}) \simeq (\delta^a_{T^*M})_*\mathscr{MH}_\Lambda(\mathbf{k}_M)$.

(iii) If M is real analytic and Λ is a Lagrangian subanalytic closed conic subset, then we have $H^k(\mathscr{MH}_\Lambda(\mathbf{k}_M)) = 0$ for $k<0$ (see [13, Proposition 9.2.2]).

In the sequel, we denote by Δ_i (resp. Δ_{ij}) the diagonal subset $\Delta_{M_i} \subset M_{ii}$ (resp. $\Delta_{M_{ij}} \subset M_{iijj}$).

Lemma 4.3. *We have natural morphisms:*

(i) $\omega_{\Delta_{12}} \underset{22}{\circ} (\mathbf{k}_{\Delta_2} \overset{\mathrm{L}}{\boxtimes} \omega_{\Delta_3}) \to \omega_{\Delta_{13}}$,

(ii) $\mathbf{k}_{\Delta_{13}} \to \mathbf{k}_{\Delta_{12}} \underset{22}{*} (\omega_{\Delta_2}^{\otimes -1} \overset{\mathrm{L}}{\boxtimes} \mathbf{k}_{\Delta_3})$.

Proof. Denote by δ_{22} the diagonal embedding $M_{112233} \hookrightarrow M_{11222233}$.
(i) We have the morphisms

$$\begin{aligned}\omega_{\Delta_{12}} \underset{22}{\circ} (\mathbf{k}_{\Delta_2} \overset{\mathrm{L}}{\boxtimes} \omega_{\Delta_3}) &= \mathrm{R}q_{1133!}\delta_{22}^{-1}(\omega_{\Delta_{12}} \overset{\mathrm{L}}{\boxtimes} \mathbf{k}_{\Delta_2} \overset{\mathrm{L}}{\boxtimes} \omega_{\Delta_3})\\ &\simeq \mathrm{R}q_{1133!}\omega_{\Delta_{123}}\\ &\to \omega_{\Delta_{13}}.\end{aligned}$$

(ii) The isomorphism

$$\delta_{22}^!(\mathbf{k}_{\Delta_2} \boxtimes \omega_{\Delta_2}) \simeq \mathbf{k}_{\Delta_2}$$

gives rise to the isomorphisms

$$\begin{aligned}\mathbf{k}_{\Delta_{12}} \underset{22}{*} (\omega_{\Delta_2}^{\otimes -1} \overset{\mathrm{L}}{\boxtimes} \mathbf{k}_{\Delta_3}) &= \mathrm{R}q_{1133*}\big(q_{1133}^{-1}\omega_{22} \otimes \delta_{22}^!(\mathbf{k}_{\Delta_{12}} \overset{\mathrm{L}}{\boxtimes} \omega_{\Delta_2}^{\otimes -1} \overset{\mathrm{L}}{\boxtimes} \mathbf{k}_{\Delta_3})\big)\\ &\simeq \mathrm{R}q_{1133*}\delta_{22}^!(\mathbf{k}_{\Delta_1} \overset{\mathrm{L}}{\boxtimes} \omega_{\Delta_2} \overset{\mathrm{L}}{\boxtimes} \mathbf{k}_{\Delta_{23}})\\ &\simeq \mathrm{R}q_{1133*}\mathbf{k}_{\Delta_{123}}\end{aligned}$$

and the result follows by adjunction from the morphism

$$q_{1133}^{-1}\mathbf{k}_{\Delta_{13}} \simeq \mathbf{k}_{\Delta_1} \overset{\mathrm{L}}{\boxtimes} \mathbf{k}_{22} \overset{\mathrm{L}}{\boxtimes} \mathbf{k}_{\Delta_3} \to \mathbf{k}_{\Delta_1} \overset{\mathrm{L}}{\boxtimes} \mathbf{k}_{\Delta_2} \overset{\mathrm{L}}{\boxtimes} \mathbf{k}_{\Delta_3} = \mathbf{k}_{\Delta_{123}}. \qquad \square$$

Proposition 4.4. *Let M_i $(i=1,2,3)$ be manifolds. We have a natural composition morphism (whose construction will be given in the course of the proof):*

$$\mu hom(\mathbf{k}_{\Delta_{12}}, \omega_{\Delta_{12}}) \underset{22}{\overset{a}{\circ}} \mu hom(\mathbf{k}_{\Delta_{23}}, \omega_{\Delta_{23}}) \to \mu hom(\mathbf{k}_{\Delta_{13}}, \omega_{\Delta_{13}}). \tag{4.2}$$

In particular, let Λ_{ij} be a closed conic subset of T^*M_{ij} $(ij = 12, 13, 23)$. If $\Lambda_{12} \underset{2}{\overset{a}{\circ}} \Lambda_{23} \subset \Lambda_{13}$, then we have a morphism

$$\mathscr{M}\mathscr{H}_{\Lambda_{12}}(\mathbf{k}_{12}) \underset{2}{\overset{a}{\circ}} \mathscr{M}\mathscr{H}_{\Lambda_{23}}(\mathbf{k}_{23}) \to \mathscr{M}\mathscr{H}_{\Lambda_{13}}(\mathbf{k}_{13}). \tag{4.3}$$

Proof. Consider the morphism (see Proposition 3.2 and Convention 3.4)

$$\begin{aligned}\mu hom(\omega_{\Delta_2}^{\otimes -1}, \omega_{\Delta_2}^{\otimes -1}) \underset{2}{\overset{a}{\circ}} \mu hom(\mathbf{k}_{\Delta_{23}}, \omega_{\Delta_{23}}) &\to \mu hom(\omega_{\Delta_2}^{\otimes -1} \underset{2}{\ast} \mathbf{k}_{\Delta_{23}}, \omega_{\Delta_2}^{\otimes -1} \underset{2}{\circ} \omega_{\Delta_{23}}) \\ &\simeq \mu hom(\omega_{\Delta_2}^{\otimes -1} \overset{\mathrm{L}}{\boxtimes} \mathbf{k}_{\Delta_3}, \mathbf{k}_{\Delta_2} \overset{\mathrm{L}}{\boxtimes} \omega_{\Delta_3}).\end{aligned}$$

It induces an isomorphism

$$\mu hom(\mathbf{k}_{\Delta_{23}}, \omega_{\Delta_{23}}) \simeq \mu hom(\omega_{\Delta_2}^{\otimes -1} \overset{\mathrm{L}}{\boxtimes} \mathbf{k}_{\Delta_3}, \mathbf{k}_{\Delta_2} \overset{\mathrm{L}}{\boxtimes} \omega_{\Delta_3}). \tag{4.4}$$

Note that this isomorphism is also obtained from

$$\begin{aligned}\mu hom(\mathbf{k}_{\Delta_{23}}, \omega_{\Delta_{23}}) &\simeq \mu hom\big((\omega_2^{\otimes -1} \overset{\mathrm{L}}{\boxtimes} \mathbf{k}_{233}) \overset{\mathrm{L}}{\otimes} \mathbf{k}_{\Delta_{23}}, (\omega_2^{\otimes -1} \overset{\mathrm{L}}{\boxtimes} \mathbf{k}_{233}) \overset{\mathrm{L}}{\otimes} \omega_{\Delta_{23}}\big) \\ &\simeq \mu hom(\omega_{\Delta_2}^{\otimes -1} \overset{\mathrm{L}}{\boxtimes} \mathbf{k}_{\Delta_3}, \mathbf{k}_{\Delta_2} \overset{\mathrm{L}}{\boxtimes} \omega_{\Delta_3}).\end{aligned}$$

Applying Proposition 3.2, we get a morphism:

$$\begin{aligned}&\mu hom(\mathbf{k}_{\Delta_{12}}, \omega_{\Delta_{12}}) \underset{22}{\overset{a}{\circ}} \mu hom(\mathbf{k}_{\Delta_{23}}, \omega_{\Delta_{23}}) \\ &\quad \to \mu hom(\mathbf{k}_{\Delta_{12}} \underset{22}{\ast} (\omega_{\Delta_2}^{\otimes -1} \overset{\mathrm{L}}{\boxtimes} \mathbf{k}_{\Delta_3}), \omega_{\Delta_{12}} \underset{22}{\circ} (\mathbf{k}_{\Delta_2} \overset{\mathrm{L}}{\boxtimes} \omega_{\Delta_3})).\end{aligned} \tag{4.5}$$

It remains to apply Lemma 4.3. $\square$

Corollary 4.5. *Let Λ_{ij} $(i = 1,2, j = i+1)$ satisfying (3.6) and let $\Lambda_{13} = \Lambda_{12} \underset{2}{\overset{a}{\circ}} \Lambda_{23}$. The composition of kernels in (4.3) induces a morphism*

$$\underset{2}{\overset{a}{\circ}} : \mathbb{MH}_{\Lambda_{12}}(\mathbf{k}_{12}) \overset{\mathrm{L}}{\otimes} \mathbb{MH}_{\Lambda_{23}}(\mathbf{k}_{23}) \to \mathbb{MH}_{\Lambda_{13}}(\mathbf{k}_{13}). \tag{4.6}$$

In particular, each $\lambda \in \mathbb{MH}^0_{\Lambda_{12}}(\mathbf{k}_{12})$ defines a morphism

$$\lambda \underset{2}{\overset{a}{\circ}} : \mathbb{MH}_{\Lambda_{23}}(\mathbf{k}_{23}) \to \mathbb{MH}_{\Lambda_{13}}(\mathbf{k}_{13}). \tag{4.7}$$

Proof. These morphisms follow from (4.3). The second assertion follows from the isomorphism $H^0(X) \simeq \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathbf{k})}(\mathbf{k}, X)$ in the category $\mathsf{D}^{\mathrm{b}}(\mathbf{k})$. $\square$

Theorem 4.6. (i) *We have the isomorphisms*

$$\begin{aligned}\mu hom(\mathbf{k}_{\Delta_M},\omega_{\Delta_M}) &\simeq (\delta^a_{T^*M})_*\pi_M^{-1}\mathrm{R}\mathscr{H}om(\mathbf{k}_M,\omega_M)\\ &\simeq (\delta^a_{T^*M})_*\pi_M^{-1}\omega_M.\end{aligned}$$

(ii) *We have a commutative diagram*

$$\begin{array}{ccc} \mu hom(\mathbf{k}_{\Delta_{12}},\omega_{\Delta_{12}}) \overset{a}{\underset{22}{\circ}} \mu hom(\mathbf{k}_{\Delta_{23}},\omega_{\Delta_{23}}) & \longrightarrow & \mu hom(\mathbf{k}_{\Delta_{13}},\omega_{\Delta_{13}}) \\ \downarrow\wr & & \downarrow\wr \\ (\delta^a_{T^*M_{13}})_*\bigl(\pi^{-1}_{M_{12}}\omega_{M_{12}} \overset{a}{\underset{2}{\circ}} \pi^{-1}_{M_{23}}\omega_{M_{23}}\bigr) & \longrightarrow & (\delta^a_{T^*M_{13}})_*\pi^{-1}_{M_{13}}\omega_{M_{13}}. \end{array} \tag{4.8}$$

Here the top horizontal arrow of (4.8) is given in Proposition 4.4, and the bottom horizontal arrow is induced by

$$\begin{aligned}&p^{-1}_{12^a}\pi^{-1}_{M_{12}}\omega_{M_{12}} \overset{\mathrm{L}}{\otimes} p^{-1}_{23}\pi^{-1}_{M_{23^a}}\omega_{M_{23}} \simeq \pi^{-1}_{M_1}\omega_{M_1} \overset{\mathrm{L}}{\boxtimes} \pi^{-1}_{M_2}(\omega_{M_2} \overset{\mathrm{L}}{\otimes} \omega_{M_2}) \overset{\mathrm{L}}{\boxtimes} \pi^{-1}_{M_3}\omega_{M_3},\\ &\pi^{-1}_{M_2}(\omega_{M_2} \overset{\mathrm{L}}{\otimes} \omega_{M_2}) \simeq \omega_{T^*M_2},\\ &\mathrm{R}p_{13!}\bigl(\pi^{-1}_{M_1}\omega_{M_1} \overset{\mathrm{L}}{\boxtimes} \omega_{T^*M_2} \overset{\mathrm{L}}{\boxtimes} \pi^{-1}_{M_3}\omega_{M_3}\bigr) \longrightarrow \pi^{-1}_{M_1}\omega_{M_1} \overset{\mathrm{L}}{\boxtimes} \pi^{-1}_{M_3}\omega_{M_3}.\end{aligned}$$

Proof. (i) is obvious.

(ii)-(a) By [13, Proposition 4.4.8], we have natural morphisms for $(i,j)=(1,2)$ or $(i,j)=(2,3)$:

$$\mu hom(\mathbf{k}_{\Delta_i},\omega_{\Delta_i}) \overset{\mathrm{L}}{\boxtimes} \mu hom(\mathbf{k}_{\Delta_j},\omega_{\Delta_j}) \to \mu hom(\mathbf{k}_{\Delta_{ij}},\omega_{\Delta_{ij}})$$

and it follows from (i) that these morphisms are isomorphisms. These isomorphisms give rise to the isomorphism

$$\begin{aligned}&\mu hom(\mathbf{k}_{\Delta_{12}},\omega_{\Delta_{12}}) \overset{a}{\underset{22}{\circ}} \mu hom(\mathbf{k}_{\Delta_{23}},\omega_{\Delta_{23}})\\ &\quad\simeq \mu hom(\mathbf{k}_{\Delta_1},\omega_{\Delta_1}) \overset{\mathrm{L}}{\boxtimes} \bigl(\mu hom(\mathbf{k}_{\Delta_2},\omega_{\Delta_2}) \overset{a}{\underset{22}{\circ}} \mu hom(\mathbf{k}_{\Delta_2},\omega_{\Delta_2})\bigr) \overset{\mathrm{L}}{\boxtimes} \mu hom(\mathbf{k}_{\Delta_3},\omega_{\Delta_3}).\end{aligned}$$

Similarly, we have an isomorphism

$$\pi^{-1}_{M_{12}}\omega_{M_{12}} \overset{a}{\underset{2}{\circ}} \pi^{-1}_{M_{23}}\omega_{M_{23}} \simeq \pi^{-1}_{M_1}\omega_{M_1} \boxtimes \bigl(\pi^{-1}_{M_2}\omega_{M_2} \overset{a}{\underset{2}{\circ}} \pi^{-1}_{M_2}\omega_{M_2}\bigr) \boxtimes \pi^{-1}_{M_3}\omega_{M_3}.$$

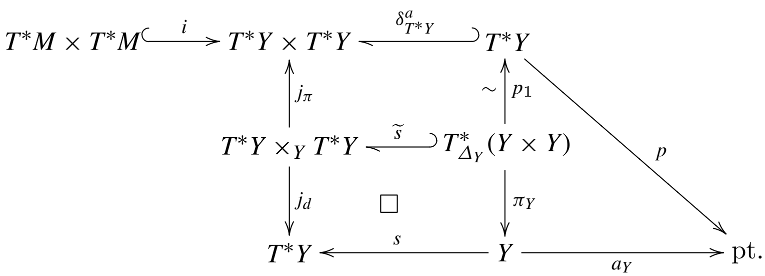
Hence, we are reduced to the case where $M_1 = M_3 = \mathrm{pt}$, which we shall assume now.

(ii)-(b) We change our notation and set

$$
\begin{aligned}
&M := M_2, \quad Y := M \times M,\\
&\delta_M\colon M \hookrightarrow Y \text{ the diagonal embedding, } \Delta_M = \delta_M(M),\\
&j\colon Y \hookrightarrow Y \times Y \text{ the diagonal embedding, } \Delta_Y = j(Y),\\
&\delta^a_{T^*M}\colon T^*M \hookrightarrow T^*Y,\ (x;\xi) \mapsto (x,x;\xi,-\xi),\\
&\delta^a_{T^*Y}\colon T^*Y \hookrightarrow T^*Y \times T^*Y,\\
&p\colon T^*Y \to \mathrm{pt} \text{ the projection},\\
&a_Y\colon Y \to \mathrm{pt} \text{ the projection}.
\end{aligned}
$$

With this new notation, the composition $\overset{a}{\underset{22}{\circ}}$ will be denoted by $\overset{a}{\underset{T^*Y}{\circ}}$.

Consider the diagram (4.9) similar to Diagram (4.4.15) of [13]:



(4.9)

Here, i is the canonical embedding induced by $\delta^a_{T^*M}$, p_1 is induced by the first projection $T^*Y \times T^*Y \to T^*Y$, $s\colon Y \hookrightarrow T^*Y$ is the zero-section embedding and $\widetilde{s}$ is the natural embedding. Note that the square labeled by $\square$ is Cartesian. We have

$$
\begin{aligned}
\mathrm{R}p_! \circ (\delta^a_{T^*Y})^{-1} &\simeq \mathrm{R}a_{Y!} \circ \mathrm{R}\pi_{Y!} \circ p_1^{-1} \circ (\delta^a_{T^*Y})^{-1}\\
&\simeq \mathrm{R}a_{Y!} \circ \mathrm{R}\pi_{Y!} \circ \widetilde{s}^{-1} \circ j_\pi^{-1}\\
&\simeq \mathrm{R}a_{Y!} \circ s^{-1} \circ \mathrm{R}j_{d!} \circ j_\pi^{-1}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\mu hom(\mathbf{k}_{\Delta_M}, \omega_{\Delta_M}) \overset{a}{\underset{T^*Y}{\circ}} \mu hom(\mathbf{k}_{\Delta_M}, \omega_{\Delta_M})\\
&\simeq \mathrm{R}p_!(\delta^a_{T^*Y})^{-1}\big(\mu hom(\mathbf{k}_{\Delta_M}, \omega_{\Delta_M}) \overset{\mathrm{L}}{\boxtimes} \mu hom(\mathbf{k}_{\Delta_M}, \omega_{\Delta_M})\big)\\
&\simeq \mathrm{R}a_{Y!}s^{-1}\mathrm{R}j_{d!}j_\pi^{-1}\mu hom(\mathbf{k}_{\Delta_M} \overset{\mathrm{L}}{\boxtimes} \mathbf{k}_{\Delta_M}, \omega_{\Delta_M} \overset{\mathrm{L}}{\boxtimes} \omega_{\Delta_M}).
\end{aligned}
$$

Hence, by adjunction, giving a morphism

$$\mu hom(\mathbf{k}_{\Delta_M}, \omega_{\Delta_M}) \overset{a}{\underset{T^*Y}{\circ}} \mu hom(\mathbf{k}_{\Delta_M}, \omega_{\Delta_M}) \to \mathbf{k}$$

is equivalent to giving a morphism in $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_Y)$

$$s^{-1}\mathrm{R}j_{d!}j_\pi^{-1}\mu hom(\mathbf{k}_{\Delta_M} \overset{\mathrm{L}}{\boxtimes} \mathbf{k}_{\Delta_M}, \omega_{\Delta_M} \overset{\mathrm{L}}{\boxtimes} \omega_{\Delta_M}) \to a_Y^!\mathbf{k}_{\mathrm{pt}}. \tag{4.10}$$

Note that the left hand side of (4.10) is supported on Δ_M. Hence in order to give a morphism (4.10), it is necessary and sufficient to give a morphism in $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$

$$\delta_M^{-1}s^{-1}\mathrm{R}j_{d!}j_\pi^{-1}\mu hom(\mathbf{k}_{\Delta_M}\overset{\mathrm{L}}{\boxtimes}\mathbf{k}_{\Delta_M},\omega_{\Delta_M}\overset{\mathrm{L}}{\boxtimes}\omega_{\Delta_M})\to\delta_M^{!}a_Y^{!}\mathbf{k}_{\mathrm{pt}}. \tag{4.11}$$

Hence, it is enough to check the commutativity of the upper square in the following diagram in $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$:

$$\begin{array}{ccc}
\delta_M^{-1}s^{-1}\mathrm{R}j_{d!}j_\pi^{-1}\mu hom(\mathbf{k}_{\Delta_M}\overset{\mathrm{L}}{\boxtimes}\mathbf{k}_{\Delta_M},\omega_{\Delta_M}\overset{\mathrm{L}}{\boxtimes}\omega_{\Delta_M}) & \longrightarrow & \delta_M^{!}a_Y^{!}\mathbf{k}_{\mathrm{pt}} \\
\downarrow\wr & & \downarrow\mathrm{id} \\
\delta_M^{-1}s^{-1}\mathrm{R}j_{d!}j_\pi^{-1}i_*\big(\pi_M^{-1}\omega_M\overset{\mathrm{L}}{\boxtimes}\pi_M^{-1}\omega_M\big) & \longrightarrow & \delta_M^{!}a_Y^{!}\mathbf{k}_{\mathrm{pt}} \\
\downarrow\wr & & \downarrow\wr \\
\omega_M & \xrightarrow{\mathrm{id}} & \omega_M.
\end{array} \tag{4.12}$$

The top horizontal arrow is constructed from a chain of morphisms (see [13, § 4.4]):

$$\begin{aligned}
&\mathrm{R}j_{d!}j_\pi^{-1}\mu hom(\mathbf{k}_{\Delta_M}\overset{\mathrm{L}}{\boxtimes}\mathbf{k}_{\Delta_M},\omega_{\Delta_M}\overset{\mathrm{L}}{\boxtimes}\omega_{\Delta_M})\\
&\quad\to\mu hom(j^{!}(\mathbf{k}_{\Delta_M}\overset{\mathrm{L}}{\boxtimes}\mathbf{k}_{\Delta_M})\overset{\mathrm{L}}{\otimes}\omega_Y,j^{-1}(\omega_{\Delta_M}\overset{\mathrm{L}}{\boxtimes}\omega_{\Delta_M}))\\
&\quad\simeq\mu hom(\omega_{\Delta_M},\omega_{\Delta_M}\otimes\omega_{\Delta_M})\simeq(\delta^a_{T^*M})_*\pi_M^{-1}\omega_M
\end{aligned}$$

and

$$\delta_M^{-1}s^{-1}\mathrm{R}j_{d!}j_\pi^{-1}\mu hom(\mathbf{k}_{\Delta_M}\overset{\mathrm{L}}{\boxtimes}\mathbf{k}_{\Delta_M},\omega_{\Delta_M}\overset{\mathrm{L}}{\boxtimes}\omega_{\Delta_M})\to\delta_M^{-1}s^{-1}(\delta^a_{T^*M})_*\pi_M^{-1}\omega_M\simeq\omega_M. \tag{4.13}$$

Hence, the commutativity of the diagram (4.12) is reduced to the commutativity of the diagram below:

$$\begin{array}{ccc}
\delta_M^{-1}s^{-1}\mathrm{R}j_{d!}j_\pi^{-1}\mu hom(\mathbf{k}_{\Delta_M}\overset{\mathrm{L}}{\boxtimes}\mathbf{k}_{\Delta_M},\omega_{\Delta_M}\overset{\mathrm{L}}{\boxtimes}\omega_{\Delta_M}) & & \\
\downarrow & \searrow^{\lambda} & \\
\delta_M^{-1}s^{-1}\mathrm{R}j_{d!}j_\pi^{-1}i_*\big(\pi_M^{-1}\omega_M\overset{\mathrm{L}}{\boxtimes}\pi_M^{-1}\omega_M\big) & \xrightarrow{\sim} & \omega_M
\end{array} \tag{4.14}$$

where the morphism λ is given by the morphisms in (4.13). All terms of (4.14) are concentrated at the degree $-\dim M$. Hence the commutativity of (4.14) is a local problem in M and we can assume that M is a Euclidean space. We can check directly in this case. □

Remark 4.7. Theorem 4.6 may be applied as follows. Let Λ_{ij} be a closed conic subset of T^*M_{ij} ($i=1.2, j=i+1$). Assume (3.6), that is, the projection $p_{13}\colon\Lambda_{12}\overset{a}{\underset{2}{\times}}\Lambda_{23}\longrightarrow T^*M_{13}$

is proper, and set $\Lambda_{13} = \Lambda_{12} \overset{a}{\underset{2}{\circ}} \Lambda_{23}$. Let $\lambda_{ij} \in \mathbb{MH}^0_{\Lambda_{ij}}(\mathbf{k}_{M_{ij}}) \simeq H^0_{\Lambda_{ij}}(T^*M_{ij}; \pi^{-1}\omega_{ij})$. Then

$$\lambda_{12} \overset{a}{\underset{2}{\circ}} \lambda_{23} = \int_{T^*M_2} \lambda_{12} \cup \lambda_{23} \tag{4.15}$$

where the right hand side is obtained as follows. Set $\Lambda := \Lambda_{12} \overset{a}{\underset{2}{\times}} \Lambda_{23}$ and consider the morphisms

$$\begin{aligned}
&H^0_{\Lambda_{12}}(T^*M_{12}; \pi^{-1}\omega_{12}) \times H^0_{\Lambda_{23}}(T^*M_{23}; \pi^{-1}\omega_{23})\\
&\to H^0_{\Lambda}(T^*M_{123}; \pi^{-1}\omega_1 \overset{\mathrm{L}}{\boxtimes} \omega_{T^*M_2} \overset{\mathrm{L}}{\boxtimes} \pi^{-1}\omega_3)\\
&\to H^0_{\Lambda_{13}}(T^*M_{13}; \pi^{-1}\omega_{13}).
\end{aligned}$$

The first morphism is the cup product and the second one is the integration morphism with respect to T^*M_2.

5. Microlocal Euler classes of trace kernels

In this section, we often write Δ instead of Δ_M.

Definition 5.1. A *trace kernel* (K, u, v) on M is the data of $K \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M\times M})$ together with morphisms

$$\mathbf{k}_\Delta \xrightarrow{u} K \quad \text{and} \quad K \xrightarrow{v} \omega_\Delta. \tag{5.1}$$

In the sequel, as long as there is no risk of confusion, we simply write K instead of (K, u, v).

For a trace kernel K as above, we set

$$\mathrm{SS}_\Delta(K) := \mathrm{SS}(K) \cap T^*_\Delta(M \times M) = (\delta^a_{T^*M})^{-1}\mathrm{SS}(K). \tag{5.2}$$

(Recall that one often identifies T^*M and $T^*_\Delta(M\times M)$ through $\delta^a_{T^*M} \colon T^*M \hookrightarrow T^*M\times T^*M$.)

Definition 5.2. Let (K, u, v) be a trace kernel.

(a) The morphism u defines an element $\tilde{u}$ in $H^0_{\mathrm{SS}_\Delta(K)}(T^*M; \mu hom(\mathbf{k}_\Delta, K))$ and the *microlocal Euler class* $\mu\mathrm{eu}_M(K)$ of K is the image of $\tilde{u}$ under the morphism $\mu hom(\mathbf{k}_\Delta, K) \to \mu hom(\mathbf{k}_\Delta, \omega_\Delta)$ associated with the morphism v.

(b) Let Λ be a closed conic subset of T^*M containing $\mathrm{SS}_\Delta(K)$. One denotes by $\mu\mathrm{eu}_\Lambda(K)$ the image of $\tilde{u}$ in $H^0_\Lambda\big(T^*M; \mu hom(\mathbf{k}_\Delta, \omega_\Delta)\big)$.

Hence,

$$\mu\mathrm{eu}_\Lambda(K) \in \mathbb{MH}^0_\Lambda(\mathbf{k}_M) \simeq H^0_\Lambda(T^*M; \pi^{-1}\omega_M). \tag{5.3}$$

Let $\tilde{v}$ be the element of $H^0_{\mathrm{SS}_\Delta(K)}(T^*M; \mu hom(K, \omega_\Delta))$ induced by v. Then the microlocal Euler class $\mu\mathrm{eu}_M(K)$ of K coincides with the image of $\tilde{v}$ under the morphism $\mu hom(K, \omega_{\Delta_M}) \to \mu hom(\mathbf{k}_\Delta, \omega_\Delta)$ associated with the morphism u, which can be easily seen from the following commutative diagram:

$$\begin{array}{ccc}
(\delta^a_{T^*M})^{-1}\mu hom(K,K) & \xrightarrow{\ \nu\ } & (\delta^a_{T^*M})^{-1}\mu hom(K,\omega_\Delta) \\
\downarrow{\scriptstyle u} & & \downarrow{\scriptstyle u} \\
(\delta^a_{T^*M})^{-1}\mu hom(\mathbf{k}_\Delta,K) & \xrightarrow{\ \nu\ } & (\delta^a_{T^*M})^{-1}\mu hom(\mathbf{k}_\Delta,\omega_\Delta).
\end{array}$$

One denotes by $\mathrm{eu}(K)$ the restriction of $\mu\mathrm{eu}(K)$ to the zero-section M of T^*M and calls it the *Euler class* of K. Hence

$$\mathrm{eu}_M(K) \in H^0_{\mathrm{Supp}(K)\cap\Delta}(M;\omega_M). \tag{5.4}$$

It is nothing but the class induced by the composition $\mathbf{k}_{\Delta_M} \to K \to \omega_{\Delta_M}$.

We say that $L \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$ is *invertible* if L is locally isomorphic to $\mathbf{k}_M[d]$ for some $d \in \mathbb{Z}$. Then, $L^{\otimes -1} := \mathrm{R}\mathscr{H}om(L,\mathbf{k}_M)$ is also invertible and $L \overset{\mathrm{L}}{\otimes} L^{\otimes -1} \simeq \mathbf{k}_M$.

Proposition 5.3. *Let L be an invertible object in $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$ and K a trace kernel. Then $K \overset{\mathrm{L}}{\otimes} (L \overset{\mathrm{L}}{\boxtimes} L^{\otimes -1})$ is a trace kernel and $\mu\mathrm{eu}\big(K \overset{\mathrm{L}}{\otimes} (L \overset{\mathrm{L}}{\boxtimes} L^{\otimes -1})\big) = \mu\mathrm{eu}(K)$.*

Proof. $L \overset{\mathrm{L}}{\boxtimes} L^{\otimes -1}$ is canonically isomorphic to $\mathbf{k}_{M\times M}$ on a neighborhood of the diagonal set Δ_M of $M \times M$. □

Remark 5.4. Of course, we could also have defined a trace kernel as a sequence of morphisms

$$\omega^{\otimes -1}_{\Delta_M} \to \widetilde{K} \to \mathbf{k}_{\Delta_M}. \tag{5.5}$$

When treating sheaves, the two definitions would give the same microlocal Euler class on taking $K = \widetilde{K} \otimes (\mathbf{k}_M \overset{\mathrm{L}}{\boxtimes} \omega_M)$. However, when working with $\mathscr{O}$-modules or with DQ-modules as in [15], the two constructions give different classes. Note that we have chosen an analogue of (5.5) in [15].

Trace kernels for constructible sheaves

Let us denote by $\mathsf{D}^{\mathrm{b}}_{\mathrm{cc}}(\mathbf{k}_M)$ the full triangulated subcategory of $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$ consisting of cohomologically constructible sheaves (see [13, § 3.4]).

Lemma 5.5. *Let $F \in \mathsf{D}^{\mathrm{b}}_{cc}(\mathbf{k}_M)$. There are natural morphisms in $\mathsf{D}^{\mathrm{b}}_{cc}(\mathbf{k}_{M\times M})$:*

$$\mathbf{k}_{\Delta_M} \to F \overset{\mathrm{L}}{\boxtimes} \mathrm{D}_M F, \tag{5.6}$$

$$F \overset{\mathrm{L}}{\boxtimes} \mathrm{D}_M F \to \omega_{\Delta_M}. \tag{5.7}$$

In other words, an object $F \in \mathsf{D}^{\mathrm{b}}_{\mathrm{cc}}(\mathbf{k}_M)$ defines naturally a trace kernel on M.

Proof. (i) We have

$$\mathbf{k}_M \to \mathrm{R}\mathscr{H}om(F,F) \simeq \delta^{\,!}(F \overset{\mathrm{L}}{\boxtimes} \mathrm{D}_M F).$$

Hence, the result follows by adjunction.

(ii) The morphism (5.7) may be deduced from (5.6) by duality, or by adjunction from the morphism

$$\delta^{-1}(F \overset{\mathrm{L}}{\boxtimes} \mathrm{D}_M F) \to \omega_M. \qquad \square$$

Notation 5.6. We shall denote by $\mathrm{TK}(F)$ the trace kernel associated with $F \in \mathsf{D}^{\mathrm{b}}_{\mathrm{cc}}(\mathbf{k}_M)$, that is the data of $F \overset{\mathrm{L}}{\boxtimes} \mathrm{D}_M F$ and the morphisms (5.6), (5.7). Note that we always have $\mathrm{SS}_\Delta(\mathrm{TK}(F)) \subset \mathrm{SS}(F)$ and the equality holds if M is real analytic and F is $\mathbb{R}$-constructible.

We have the chain of morphisms

$$\begin{aligned} \mu hom(F,F) &\simeq (\delta^a_{T^*M})^{-1}\mu hom(\mathbf{k}_\Delta, F \overset{\mathrm{L}}{\boxtimes} \mathrm{D}F) \\ &\to (\delta^a_{T^*M})^{-1}\mu hom(\mathbf{k}_\Delta, \omega_\Delta). \end{aligned}$$

We deduce the map

$$H^0_{\mathrm{SS}(F)}(T^*M; \mu hom(F,F)) \to \mathbb{MH}^0_{\mathrm{SS}(F)}(\mathbf{k}_M). \tag{5.8}$$

Definition 5.7. Let $F \in \mathsf{D}^{\mathrm{b}}_{\mathrm{cc}}(\mathbf{k}_M)$. The image of id_F under the map (5.8) is called the microlocal Euler class of F and is denoted by $\mu\mathrm{eu}_M(F)$.

Clearly, one has

$$\mu\mathrm{eu}_M(F) = \mu\mathrm{eu}_M(\mathrm{TK}(F)). \tag{5.9}$$

Assume that M is real analytic and denote by $\mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbf{k}_M)$ the full triangulated subcategory of $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$ consisting of $\mathbb{R}$-constructible complexes. Of course, $\mathbb{R}$-constructible complexes are cohomologically constructible. In [13, § 9.4] the microlocal Euler class of an object $F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbf{k}_M)$ is constructed as above and this class is also called the characteristic cycle, or else, the Lagrangian cycle, of F.

Remark 5.8. Let (K,u,v) be a trace kernel on M. Let $\delta\colon M \to M \times M$ be the diagonal embedding. Then u and v decompose as

$$\mathbf{k}_{\Delta_M} \to \delta_*\delta^{\,!}K \to K \to \delta_*\delta^{-1}K \to \omega_{\Delta_M}.$$

Hence $\delta_*\delta^{\,!}K$ and $\delta_*\delta^{-1}K$ are also trace kernels. We have evidently

$$\mu\mathrm{eu}_M\big(\delta_*\delta^{\,!}K\big) = \mu\mathrm{eu}_M\big(\delta_*\delta^{-1}K\big) = \mu\mathrm{eu}_M(K) \quad \text{as elements in } \mathbb{MH}^0_{T^*M}(\mathbf{k}_M).$$

Trace kernels over one point

Let us consider the particular case where M is a single point, $M = \mathrm{pt}$, and let us identify a sheaf over pt with a $\mathbf{k}$-module. In this situation, a trace kernel (K,u,v) is the data of $K \in \mathsf{D}^{\mathrm{b}}(\mathbf{k})$ together with linear maps

$$\mathbf{k} \xrightarrow{u} K \xrightarrow{v} \mathbf{k}.$$

The (microlocal) Euler class $\mathrm{eu}_{\mathrm{pt}}(K)$ of this kernel is the image of $1 \in \mathbf{k}$ under $v \circ u$.

Assume now that $\mathbf{k}$ is a field and denote by $\mathsf{D}^{\mathrm{b}}_f(\mathbf{k})$ the full triangulated subcategory of $\mathsf{D}^{\mathrm{b}}(\mathbf{k})$ consisting of objects with finite-dimensional cohomologies. Let $V \in \mathsf{D}^{\mathrm{b}}_f(\mathbf{k})$ and set $V^* = \mathrm{RHom}(V, \mathbf{k})$. Let $K = \mathrm{TK}(V) = V \otimes V^*$, and let v be the trace morphism and u its dual. Then

$$\begin{aligned}&\text{(a) } \mathrm{eu}_{\mathrm{pt}}(V \otimes V^*) = \mathrm{tr}(\mathrm{id}_V), \text{ the trace of the identity of } V.\\ &\text{(b) If } \mathbf{k} \text{ has characteristic } 0, \text{ then}\\ &\qquad \mathrm{eu}_{\mathrm{pt}}(V \otimes V^*) = \chi(V), \quad \text{the Euler–Poincaré index of } V.\end{aligned} \tag{5.10}$$

Trace kernels for $\mathscr{D}$-modules

In this subsection, we denote by X a complex manifold of complex dimension d_X and the base ring $\mathbf{k}$ is the field $\mathbb{C}$. We denote by $\mathscr{O}_X$ the structure sheaf and by Ω_X the sheaf of holomorphic forms of maximal degree. We still denote by ω_X the topological dualizing complex and recall the isomorphism $\omega_X \simeq \mathbb{C}_X\,[2d_X]$.

One denotes by $\mathscr{D}_X$ the sheaf of $\mathbb{C}_X$-algebras of (finite-order) holomorphic differential operators on X and we refer the reader to [11] for a detailed exposition of the theory of $\mathscr{D}$-modules. We denote by $\mathrm{Mod}(\mathscr{D}_X)$ the category of left $\mathscr{D}_X$-modules and by $\mathsf{D}^{\mathrm{b}}(\mathscr{D}_X)$ its bounded derived category. We also denote by $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X)$ the abelian category of coherent $\mathscr{D}_X$-modules and by $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$ the full triangulated subcategory of $\mathsf{D}^{\mathrm{b}}(\mathscr{D}_X)$ consisting of objects with coherent cohomologies.

We denote by $\mathrm{D}_{\mathscr{D}} \colon \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X)^{\mathrm{op}} \to \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X)$ the duality functor for left $\mathscr{D}$-modules:

$$\mathrm{D}_{\mathscr{D}}\mathscr{M} := \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}_X) \otimes_{\mathscr{O}_X} \Omega_X^{\otimes -1}\,[d_X].$$

We denote by $\cdot \underline{\boxtimes} \cdot$ the external product for $\mathscr{D}$-modules:

$$\mathscr{M} \underline{\boxtimes} \mathscr{N} := \mathscr{D}_{X\times X} \otimes_{\mathscr{D}_X \boxtimes \mathscr{D}_X} (\mathscr{M} \overset{\mathrm{L}}{\boxtimes} \mathscr{N}).$$

Let Δ be the diagonal of $X \times X$. The left $\mathscr{D}_{X\times X}$-module $H^{d_X}_{[\Delta]}(\mathscr{O}_{X\times X})$(the algebraic cohomology with support in Δ) is denoted as usual by $\mathscr{B}_\Delta$. Note that

$$\mathrm{D}_{\mathscr{D}}\mathscr{B}_\Delta \simeq \mathscr{B}_\Delta.$$

One should be aware that here, the dual is taken over $X \times X$. We also introduce

$$\mathscr{B}_\Delta^\vee := \mathscr{B}_\Delta\,[2d_X].$$

For $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$, we have the isomorphism

$$\mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{M}) \simeq \mathrm{R}\mathscr{H}om_{\mathscr{D}_{X\times X}}(\mathscr{B}_\Delta, \mathscr{M} \underline{\boxtimes} \mathrm{D}_{\mathscr{D}}\mathscr{M})\,[d_X].$$

We deduce the morphism in $\mathsf{D}^{\mathrm{b}}(\mathscr{D}_{X\times X})$

$$\mathscr{B}_\Delta \to \mathscr{M} \underline{\boxtimes} \mathrm{D}_{\mathscr{D}}\mathscr{M}\,[d_X] \tag{5.11}$$

and by duality, the morphism in $\mathsf{D}^{\mathrm{b}}(\mathscr{D}_{X\times X})$

$$\mathscr{M} \underline{\boxtimes} \mathrm{D}_{\mathscr{D}}\mathscr{M}\,[d_X] \to \mathscr{B}_\Delta^\vee. \tag{5.12}$$

Denote by $\mathscr{E}_X$ the sheaf on T^*X of microdifferential operators of [22]. For a coherent $\mathscr{D}_X$-module $\mathscr{M}$ set

$$\mathscr{M}^E := \mathscr{E}_X \otimes_{\pi^{-1}\mathscr{D}_X} \pi^{-1}\mathscr{M}$$

and recall that, denoting by $\mathrm{char}(\mathscr{M})$ the characteristic variety of $\mathscr{M}$, we have $\mathrm{char}(\mathscr{M}) = \mathrm{Supp}(\mathscr{M}^E)$. One also sets

$$\mathscr{C}_\Delta := \mathscr{B}_\Delta^E, \quad \mathscr{C}_\Delta^\vee := \left(\mathscr{B}_\Delta^\vee\right)^E.$$

We denote by $\mathrm{D}_\mathscr{E} \colon \mathsf{D}^{\mathrm{b}}(\mathscr{E}_X)^{\mathrm{op}} \to \mathsf{D}^{\mathrm{b}}(\mathscr{E}_X)$ the duality functor for left $\mathscr{E}$-modules:

$$\mathrm{D}_\mathscr{E}\mathscr{M} := \mathrm{R}\mathscr{H}om_{\mathscr{E}_X}(\mathscr{M}, \mathscr{E}_X) \otimes_{\pi^{-1}\mathscr{O}_X} \pi^{-1}\Omega_X^{\otimes -1}[d_X]$$

and we denote by $\cdot \underline{\boxtimes} \cdot$ the external product for $\mathscr{E}$-modules:

$$\mathscr{M} \underline{\boxtimes} \mathscr{N} := \mathscr{E}_{X\times X} \otimes_{\mathscr{E}_X \boxtimes \mathscr{E}_X} (\mathscr{M} \overset{\mathrm{L}}{\boxtimes} \mathscr{N}).$$

The morphisms (5.11) and (5.12) give rise to the morphisms

$$\mathscr{C}_\Delta \to \mathscr{M}^E \underline{\boxtimes} \mathrm{D}_\mathscr{E}\mathscr{M}^E[d_X] \to \mathscr{C}_\Delta^\vee. \tag{5.13}$$

Let Λ be a closed conic subset of T^*X. One sets

$$\begin{aligned}
&\mathscr{H}\mathscr{H}(\mathscr{E}_X) = (\delta^a_{T^*X})^{-1}\mathrm{R}\mathscr{H}om_{\mathscr{E}_{X\times X}}(\mathscr{C}_\Delta, \mathscr{C}_\Delta^\vee),\\
&\mathbb{HH}_\Lambda(\mathscr{E}_X) = \mathrm{R}\Gamma_\Lambda(T^*X; \mathscr{H}\mathscr{H}(\mathscr{E}_X)),\\
&\mathbb{HH}^k_\Lambda(\mathscr{E}_X) = H^k(\mathbb{HH}_\Lambda(\mathscr{E}_X)) = H^k_\Lambda(T^*X; \mathscr{H}\mathscr{H}(\mathscr{E}_X)).
\end{aligned}$$

We call $\mathbb{HH}_\Lambda(\mathscr{E}_X)$, the *Hochschild homology* of $\mathscr{E}_X$ with support in Λ.

The morphisms in (5.13) define a class

$$\mathrm{hh}_\mathscr{E}(\mathscr{M}) \in \mathbb{HH}^0_{\mathrm{char}(\mathscr{M})}(\mathscr{E}_X) \tag{5.14}$$

that we call the *Hochschild class* of $\mathscr{M}$.

Let S be a closed subset of X. By restricting the above construction to the zero-section X of T^*X, we obtain the Hochschild homology of $\mathscr{D}_X$:

$$\begin{aligned}
&\mathscr{H}\mathscr{H}(\mathscr{D}_X) = (\delta_X)^{-1}\mathrm{R}\mathscr{H}om_{\mathscr{D}_{X\times X}}(\mathscr{B}_\Delta, \mathscr{B}_\Delta^\vee) \simeq \mathscr{H}\mathscr{H}(\mathscr{E}_X)|_X,\\
&\mathbb{HH}_S(\mathscr{D}_X) = \mathrm{R}\Gamma_S(X; \mathscr{H}\mathscr{H}(\mathscr{D}_X)),\\
&\mathbb{HH}^k_S(\mathscr{D}_X) = H^k(\mathbb{HH}_S(\mathscr{D}_X)) = H^k_S(X; \mathscr{H}\mathscr{H}(\mathscr{D}_X)).
\end{aligned}$$

Then, for $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$ one obtains

$$\mathrm{hh}_\mathscr{D}(\mathscr{M}) := \mathrm{hh}_\mathscr{E}(\mathscr{M})|_X \in \mathbb{HH}^0_{\mathrm{Supp}(\mathscr{M})}(\mathscr{D}_X).$$

We shall make a link between the Hochschild class of $\mathscr{M}$ and the microlocal Euler class of a trace kernel attached to the sheaves of holomorphic solutions of $\mathscr{M}$. We need a lemma.

Lemma 5.9. *For $\mathscr{N}_1$ and $\mathscr{N}_2$ in $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$, there exists a natural morphism*

$$\mathrm{R}\mathscr{H}om_\mathscr{E}(\mathscr{N}_1^E, \mathscr{N}_2^E) \to \mu hom(\Omega_X \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{N}_1, \Omega_X \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{N}_2). \tag{5.15}$$

Moreover, this morphism is compatible with the composition

$$\mathrm{R}\mathscr{H}om_{\mathscr{E}}(\mathscr{N}_1^E, \mathscr{N}_2^E) \otimes \mathrm{R}\mathscr{H}om_{\mathscr{E}}(\mathscr{N}_2^E, \mathscr{N}_3^E) \to \mathrm{R}\mathscr{H}om_{\mathscr{E}}(\mathscr{N}_1^E, \mathscr{N}_3^E),$$
$$\mu hom(F_1, F_2) \otimes \mu hom(F_2, F_3) \to \mu hom(F_1, F_3).$$

Proof. We have the natural morphism in $\mathsf{D}^{\mathrm{b}}(\pi^{-1}\mathscr{D}_X \otimes \pi^{-1}\mathscr{D}_X^{\mathrm{op}})$ (see [12, Proposition 10.6.2])

$$\mathscr{E}_X \to \mu hom(\Omega_X, \Omega_X).$$

This gives rise to the morphisms

$$\begin{aligned}
&\mathrm{R}\mathscr{H}om_{\pi^{-1}\mathscr{D}_X}(\pi^{-1}\mathscr{N}_1, \mathscr{E}_X \otimes_{\pi^{-1}\mathscr{D}_X} \pi^{-1}\mathscr{N}_2)\\
&\quad \to \mathrm{R}\mathscr{H}om_{\pi^{-1}\mathscr{D}_X}(\pi^{-1}\mathscr{N}_1, \mu hom(\Omega_X, \Omega_X)) \otimes_{\pi^{-1}\mathscr{D}_X} \pi^{-1}\mathscr{N}_2\\
&\quad \simeq \mu hom(\Omega_X \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{N}_1, \Omega_X \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{N}_2).
\end{aligned}$$

□

We have

$$\Omega_{X\times X}[-d_X] \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_{X\times X}} \mathscr{B}_\Delta \simeq \mathbb{C}_\Delta,$$
$$\Omega_{X\times X}[-d_X] \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_{X\times X}} \mathscr{B}_\Delta^\vee \simeq \omega_\Delta.$$

Applying Lemma 5.9, one deduces the morphisms

$$\begin{aligned}
\mathrm{R}\mathscr{H}om_{\mathscr{E}_{X\times X}}(\mathscr{C}_\Delta, \mathscr{C}_\Delta^\vee) &\to \mu hom(\Omega_{X\times X} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_{X\times X}} \mathscr{B}_\Delta, \Omega_{X\times X} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_{X\times X}} \mathscr{B}_\Delta^\vee)\\
&\simeq \mu hom(\mathbb{C}_\Delta, \omega_\Delta).
\end{aligned}$$

An easy calculation shows that the first arrow is also an isomorphism. Therefore, we get the isomorphism

$$\mathscr{H}\mathscr{H}(\mathscr{E}_X) \xrightarrow{\sim} \mathscr{M}\mathscr{H}(\mathbb{C}_X). \tag{5.16}$$

Recall that the Hochschild homology of $\mathscr{E}_X$ has already been calculated in [2].

Applying the functor $\Omega_{X\times X}[-d_X] \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_{X\times X}} \cdot$ to (5.11) and (5.12) we get the morphisms

$$\mathbb{C}_\Delta \to \Omega_{X\times X} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_{X\times X}} (\mathscr{M} \boxtimes \mathrm{D}_{\mathscr{D}}\mathscr{M}) \to \omega_\Delta. \tag{5.17}$$

Notation 5.10. For $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$, we denote by $\mathrm{TK}(\mathscr{M})$ the trace kernel given by (5.17).

Since $\mathrm{char}(\mathscr{M}) = \mathrm{SS}(\mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X))$ by [13, Theorem 11.3.3], we get that $\mu\mathrm{eu}_M(\mathrm{TK}(\mathscr{M}))$ is supported by $\mathrm{char}(\mathscr{M})$, the characteristic variety of $\mathscr{M}$.

Proposition 5.11. *After identifying $\mathscr{H}\mathscr{H}(\mathscr{E}_X)$ and $\mathscr{M}\mathscr{H}(\mathbb{C}_X)$ through the isomorphism (5.16), we have $\mathrm{hh}_{\mathscr{E}}(\mathscr{M}) = \mu\mathrm{eu}_X(\mathrm{TK}(\mathscr{M}))$ in $\mathbb{HH}^0_{\mathrm{char}(\mathscr{M})}(\mathbb{C}_X)$.*

Proof. This follows from Lemma 5.9 applied to (5.13). □

Note that the class $\mu\mathrm{eu}_X(\mathrm{TK}(\mathscr{M}))$ coincides with the microlocal Euler class of $\mathscr{M}$ already introduced by Schapira and Schneiders in [23].

6. Operations on microlocal Euler classes I

In this section, we shall adapt to trace kernels the constructions of [15, Chapter 4 § 3] and we shall show that under natural microlocal conditions of properness, the microlocal Euler class of the composition of two kernels is the composition of the classes.

We use Notation 3.1 and we consider a trace kernel (K,u,v) on M_{12}.

Lemma 6.1. *Let K be a trace kernel on M_{12}. There are natural morphisms in $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M_{11}})$:*

$$\mathbf{k}_{\Delta_{13}} \to K \underset{22}{*} (\omega_{\Delta_2}^{\otimes -1} \overset{\mathrm{L}}{\boxtimes} \mathbf{k}_{\Delta_3}), \tag{6.1}$$

$$K \underset{22}{\circ} (\mathbf{k}_{\Delta_2} \overset{\mathrm{L}}{\boxtimes} \omega_{\Delta_3}) \to \omega_{\Delta_{13}}. \tag{6.2}$$

Proof. (i) By Lemma 4.3(ii) we have a morphism $\mathbf{k}_{\Delta_{13}} \to \mathbf{k}_{\Delta_{12}} \underset{22}{*} (\omega_{\Delta_2}^{\otimes -1} \overset{\mathrm{L}}{\boxtimes} \mathbf{k}_{\Delta_3})$. By composing this morphism with $\mathbf{k}_{\Delta_{12}} \to K$, we get (6.1).
(ii) By Lemma 4.3(i) we have a morphism $\omega_{\Delta_{12}} \underset{22}{\circ} (\mathbf{k}_{\Delta_2} \overset{\mathrm{L}}{\boxtimes} \omega_{\Delta_3}) \to \omega_{\Delta_{13}}$. By composing this morphism with $K \to \omega_{\Delta_{12}}$ we get (6.2). □

Let K be a trace kernel on M_{12} with microsupport $\mathrm{SS}(K)$ contained in a closed conic subset Λ_{1122} of T^*M_{1122} and let Λ_{23} a closed conic subset of T^*M_{23}. We assume

$$\Lambda_{1122} \overset{a}{\underset{22}{\times}} \delta^a_{T^*M_{23}} \Lambda_{23} \text{ is proper over } T^*M_{1133}. \tag{6.3}$$

We set

$$\begin{cases} \Lambda_{12} := \Lambda_{1122} \cap T^*_{\Delta_{12}} M_{1122}, \\ \Lambda_{1133} := \Lambda_{1122} \overset{a}{\underset{22}{\circ}} \delta^a_{T^*M_{23}} \Lambda_{23}, \\ \Lambda_{13} := \Lambda_{1133} \cap T^*_{\Delta_{13}} M_{1133} = \Lambda_{12} \overset{a}{\underset{2}{\circ}} \Lambda_{23}. \end{cases} \tag{6.4}$$

We define a map

$$\Phi_K \colon \mathbb{MH}_{\Lambda_{23}}(\mathbf{k}_{23}) \longrightarrow \mathbb{MH}_{\Lambda_{13}}(\mathbf{k}_{13}) \tag{6.5}$$

by the sequence of morphisms

$$\begin{aligned} \mathbb{MH}_{\Lambda_{23}}(\mathbf{k}_{23}) &\simeq \mathrm{R}\Gamma_{\delta^a_{T^*M_{23}}\Lambda_{23}}(T^*M_{2233}; \mu hom(\mathbf{k}_{\Delta_{23}}, \omega_{\Delta_{23}})) \\ &\simeq \mathrm{R}\Gamma_{\delta^a_{T^*M_{23}}\Lambda_{23}}\big(T^*M_{2233}; \mu hom(\omega_{\Delta_2}^{\otimes -1} \overset{\mathrm{L}}{\boxtimes} \mathbf{k}_{\Delta_3}, \mathbf{k}_{\Delta_2} \overset{\mathrm{L}}{\boxtimes} \omega_{\Delta_3})\big) \\ &\to \mathrm{R}\Gamma_{\Lambda_{1133}}\big(T^*M_{1133}; \mu hom(K,K) \overset{a}{\underset{22}{\circ}} \mu hom(\omega_{\Delta_2}^{\otimes -1} \overset{\mathrm{L}}{\boxtimes} \mathbf{k}_{\Delta_3}, \mathbf{k}_{\Delta_2} \overset{\mathrm{L}}{\boxtimes} \omega_{\Delta_3})\big) \\ &\to \mathrm{R}\Gamma_{\Lambda_{1133}}\big(T^*M_{1133}; \mu hom(K \underset{22}{*} (\omega_{\Delta_2}^{\otimes -1} \overset{\mathrm{L}}{\boxtimes} \mathbf{k}_{\Delta_3}), K \underset{22}{\circ} (\mathbf{k}_{\Delta_2} \overset{\mathrm{L}}{\boxtimes} \omega_{\Delta_3}))\big) \\ &\to \Gamma\big(T^*M_{1133}; \mu hom(\mathbf{k}_{\Delta_{13}}, \omega_{\Delta_{13}})\big) \simeq \mathbb{MH}_{\Lambda_{13}}(\mathbf{k}_{13}). \end{aligned}$$

Here the first arrow is given by id_K, the second is given by Proposition 3.2, and the last arrow is induced by the morphisms in Lemma 6.1.

The next result is similar to [15, Theorem 4.3.5].

Proposition 6.2. *Let $\Lambda_{1122} \subset T^*M_{1122}$ and $\Lambda_{23} \subset T^*M_{23}$ be closed conic subsets satisfying (6.3) and recall the notation (6.4). Let K be a trace kernel on M_{12} with microsupport contained in Λ_{1122}. Then the map Φ_K in (6.5) is the map $\mu\mathrm{eu}_{M_{12}}(K) \overset{a}{\underset{12}{\circ}}$ given by Corollary 4.5.*

Proof. By using the morphism $\mathbf{k}_{\Delta_{12}} \to K$, we find the commutative diagram below:

$$\begin{array}{ccc}
\mathrm{R}\Gamma_{\Lambda_{23}}\big(T^*M_{2233}; \mu hom(\mathbf{k}_{\Delta_{23}}, \omega_{\Delta_{23}})\big) & \longrightarrow & \mathrm{R}\Gamma_{\Lambda_{13}}\big(T^*M_{1133}; \mu hom(\mathbf{k}_{\Delta_{12}} \underset{22}{\ast} \mathbf{k}_{\Delta_{23}}, \mathbf{k}_{\Delta_{12}} \underset{22}{\circ} \omega_{\Delta_{23}})\big) \\
\downarrow & & \downarrow \\
\mathrm{R}\Gamma_{\Lambda_{1133}}\big(T^*M_{1133}; \mu hom(K \underset{22}{\ast} \mathbf{k}_{\Delta_{23}}, K \underset{22}{\circ} \omega_{\Delta_{23}})\big) & \longrightarrow & \mathrm{R}\Gamma_{\Lambda_{13}}\big(T^*M_{1133}; \mu hom(\mathbf{k}_{\Delta_{12}} \underset{22}{\ast} \mathbf{k}_{\Delta_{23}}, K \underset{22}{\circ} \omega_{\Delta_{23}})\big).
\end{array}$$

By using the morphism $K \to \omega_{\Delta_{12}}$, we get the commutative diagram

$$\begin{array}{ccc}
\mathrm{R}\Gamma_{\Lambda_{23}}\big(T^*M_{2233}; \mu hom(\mathbf{k}_{\Delta_{23}}, \omega_{\Delta_{23}})\big) & \longrightarrow & \mathrm{R}\Gamma_{\Lambda_{13}}\big(T^*M_{1133}; \mu hom(\mathbf{k}_{\Delta_{12}} \underset{22}{\ast} \mathbf{k}_{\Delta_{23}}, \omega_{\Delta_{12}} \underset{22}{\circ} \omega_{\Delta_{23}})\big). \\
\searrow & & \nearrow \\
& \mathrm{R}\Gamma_{\Lambda_{1133}}\big(T^*M_{1133}; \mu hom(K \underset{22}{\ast} \mathbf{k}_{\Delta_{23}}, K \underset{22}{\circ} \omega_{\Delta_{23}})\big) &
\end{array} \tag{6.6}$$

Recall the morphisms in Lemma 4.3:

$$\omega_{\Delta_{12}} \underset{22}{\circ} (\mathbf{k}_{\Delta_2} \overset{\mathrm{L}}{\boxtimes} \omega_{\Delta_3}) \to \omega_{\Delta_{13}}, \quad \mathbf{k}_{\Delta_{13}} \to \mathbf{k}_{\Delta_{12}} \underset{22}{\ast} (\omega_{\Delta_2}^{\otimes -1} \overset{\mathrm{L}}{\boxtimes} \mathbf{k}_{\Delta_3}). \tag{6.7}$$

We get the morphisms

$$\begin{aligned}
w\colon & \mathrm{R}\Gamma_{\delta^a_{T^*M_{13}}\Lambda_{13}}\big(T^*M_{1133}; \mu hom(\mathbf{k}_{\Delta_{12}} \underset{22}{\ast} \mathbf{k}_{\Delta_{23}}, \omega_{\Delta_{12}} \underset{22}{\circ} \omega_{\Delta_{23}}\big) \\
& \simeq \mathrm{R}\Gamma_{\delta^a_{T^*M_{13}}\Lambda_{13}}\big(T^*M_{1133}; \mu hom(\mathbf{k}_{\Delta_{12}} \underset{22}{\ast} (\omega_{\Delta_2}^{\otimes -1} \overset{\mathrm{L}}{\boxtimes} \mathbf{k}_{\Delta_3}), \omega_{\Delta_{12}} \underset{22}{\circ} (\mathbf{k}_{\Delta_2} \overset{\mathrm{L}}{\boxtimes} \omega_{\Delta_3}))\big) \\
& \to \mathrm{R}\Gamma_{\delta^a_{T^*M_{13}}\Lambda_{13}}\big(T^*M_{1133}; \mu hom(\mathbf{k}_{\Delta_{13}}, \omega_{\Delta_{13}})\big).
\end{aligned}$$

By its construction, the morphism $\mu\mathrm{eu}_{M_{12}}(K) \circ$ is obtained as the composition with the map w of the top row of the diagram (6.6). Since the composition with w of the two other arrows is the morphism Φ_K, the proof is complete. □

The next result is similar to [15, Theorem 4.3.6].

Let $i = 1, 2$, $j = i+1$ and let Λ_{iijj} be a closed conic subset of T^*M_{iijj}. Assume that

$$\Lambda_{1122} \overset{a}{\underset{22}{\times}} \Lambda_{2233} \text{ is proper over } T^*M_{1133}. \tag{6.8}$$

Set $\Lambda_{1133} = \Lambda_{1122} \overset{a}{\underset{22}{\circ}} \Lambda_{2233}$ and $\Lambda_{ij} = \Lambda_{iijj} \cap T^*_{\Delta_{ij}} M_{iijj}$.

Theorem 6.3. *Let K_{ij} be a trace kernel on M_{ij} with $\mathrm{SS}(K_{ij}) \subset \Lambda_{iijj}$. Assume (6.8), set $\widetilde{K}_{23} = \omega_{\Delta_2}^{\otimes -1} \underset{2}{\circ} K_{23} \simeq (\omega_2^{\otimes -1} \overset{\mathrm{L}}{\boxtimes} \mathbf{k}_{233}) \overset{\mathrm{L}}{\otimes} K$ and set $K_{13} = K_{12} \underset{22}{\circ} \widetilde{K}_{23}$. Then*

(a) *K_{13} is a trace kernel on M_{13},*

(b) $\mu\mathrm{eu}_{M_{13}}(K_{13}) = \mu\mathrm{eu}_{M_{12}}(K_{12}) \underset{2}{\overset{a}{\circ}} \mu\mathrm{eu}_{M_{23}}(K_{23})$ *as elements of* $\mathbb{MH}^0_{\Lambda_{13}}(\mathbf{k}_{13})$.

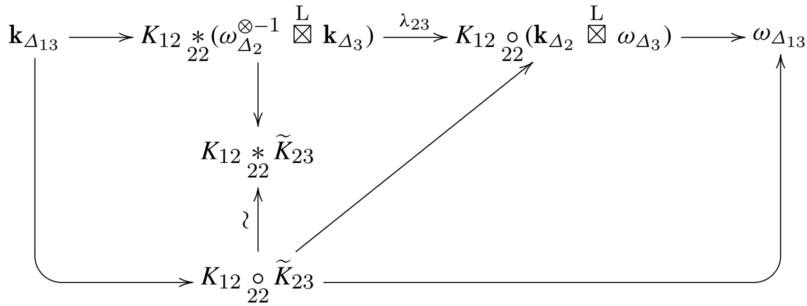
(c) *In particular, we have* $\Phi_{K_{12}} \circ \Phi_{K_{23}} \simeq \Phi_{K_{13}}$.

Proof. (a) The trace kernel K_{23} defines morphisms

$$\omega_{\Delta_2}^{\otimes -1} \overset{\mathrm{L}}{\boxtimes} \mathbf{k}_{\Delta_3} \to \widetilde{K}_{23} \to \mathbf{k}_{\Delta_2} \overset{\mathrm{L}}{\boxtimes} \omega_{\Delta_3}.$$

Assuming (6.8) and using (6.1) and (6.2), we get that $K_{13} = K_{12} \underset{22}{\circ} \widetilde{K}_{23}$ is a trace kernel on M_{13}.

(b) We get a commutative diagram in which we set $\lambda_{23} = \mu\mathrm{eu}_{M_{23}}(K_{23}) \in \mathbb{MH}^0(\mathbf{k}_{23}) \simeq \mathrm{Hom}(\omega_{\Delta_2}^{\otimes -1} \overset{\mathrm{L}}{\boxtimes} \mathbf{k}_{\Delta_3}, \mathbf{k}_{\Delta_2} \overset{\mathrm{L}}{\boxtimes} \omega_{\Delta_3})$:



The composition of the arrows at the bottom is $\mu\mathrm{eu}_{M_{13}}(K_{13})$ and the composition of the arrows at the top is $\Phi_{K_{12}}(\mu\mathrm{eu}_{M_{23}}(K_{23}))$. Hence, the assertion follows from the commutativity of the diagram by Proposition 6.2.

(c) follows from (b) and Proposition 6.2. □

7. Operations on microlocal Euler classes II

We shall combine Theorems 4.6 and 6.3 and make more explicit the operations on microlocal Euler classes for direct or inverse images. In particular, applying our results to the case of constructible sheaves, we shall recover the results of [13, Chapter IX § 5].

Let M be a manifold and let $\iota\colon N \hookrightarrow M$ be a closed embedding of a smooth submanifold N. If there is no risk of confusion, we shall still denote by $\mathbf{k}_N$ and ω_N the sheaves $\iota_*\mathbf{k}_N$ and $\iota_*\omega_N$ on M. Then $\mathbf{k}_N$ is cohomologically constructible and moreover

$$\mathrm{D}_M\mathbf{k}_N = \mathrm{R}\mathscr{H}om(\mathbf{k}_N, \omega_M) \simeq \omega_N.$$

Hence, $\mathrm{TK}(\mathbf{k}_N) = \mathbf{k}_N \overset{\mathrm{L}}{\boxtimes} \omega_N$ is a trace kernel on M.

Let M_i be a manifold ($i = 1, 2$), let K_i be a trace kernel on M_i and let Λ_{ii} be a closed conic subset of T^*M_{ii} with $\mathrm{SS}(K_i) \subset \Lambda_{ii}$. We set

$$\Lambda_i = \Lambda_{ii} \cap T^*_{\Delta_i}M_{ii}.$$

For a morphism of manifolds $f\colon M_1 \to M_2$, we denote by Γ_f its graph, a smooth closed submanifold of M_{12}, and we set for short

$$\Lambda_f := T^*_{\Gamma_f}(M_{12}), \quad \widetilde{f} = (f,f)\colon M_{11} \to M_{22}.$$

Recall the diagram (2.1)

$$\begin{array}{ccccc}
T^*M_1 & \xleftarrow{f_d} & M_1 \times_{M_2} T^*M_2 & \xrightarrow{f_\pi} & T^*M_2 \\
 & \searrow^{\pi_{M_1}} & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_{M_2}} \\
 & & M_1 & \xrightarrow{f} & M_2.
\end{array}$$

Note that

$$\Lambda_{11} \overset{a}{\underset{11}{\circ}} \Lambda_{\widetilde{f}} = \widetilde{f}_\pi \widetilde{f}_d^{-1} \Lambda_{11}, \quad \Lambda_{\widetilde{f}} \overset{a}{\underset{22}{\circ}} \Lambda_{22} = \widetilde{f}_d \widetilde{f}_\pi^{-1} \Lambda_{22}.$$

In the sequel, we shall identify M_{1212} with M_{1122}. We take as kernel the sheaf $\mathrm{TK}(\mathbf{k}_{\Gamma_f})$. Then

$$\begin{aligned}
\mathrm{TK}(\mathbf{k}_{\Gamma_f}) &= \mathbf{k}_{\Gamma_f} \overset{\mathrm{L}}{\boxtimes} \omega_{\Gamma_f} \simeq \mathbf{k}_{\Gamma_{\widetilde{f}}} \otimes (\mathbf{k}_1 \overset{\mathrm{L}}{\boxtimes} \omega_1 \overset{\mathrm{L}}{\boxtimes} \mathbf{k}_{22}) \\
&\simeq \omega_{\Delta_1} \underset{11}{\circ} \big((\omega_1^{\otimes -1} \overset{\mathrm{L}}{\boxtimes} \omega_1 \overset{\mathrm{L}}{\boxtimes} \mathbf{k}_{22}) \overset{\mathrm{L}}{\otimes} \mathbf{k}_{\Gamma_{\widetilde{f}}}\big). \qquad (7.1)
\end{aligned}$$

Moreover, we have (see (5.9))

$$\mu\mathrm{eu}_{M_{12}}(\mathrm{TK}(\mathbf{k}_{\Gamma_f})) = \mu\mathrm{eu}_{M_{12}}(\mathbf{k}_{\Gamma_f}).$$

Also note that

$$\mathrm{R}\widetilde{f}_! K_1 \simeq K_1 \underset{11}{\circ} \mathbf{k}_{\Gamma_{\widetilde{f}}}, \quad \widetilde{f}^{-1} K_2 \simeq \mathbf{k}_{\Gamma_{\widetilde{f}}} \underset{22}{\circ} K_2.$$

External product

Applying Theorem 4.6 with $M_2 = \mathrm{pt}$ and M_3 being here M_2, we get the commutative diagram

$$\begin{array}{ccc}
\mathscr{MH}_{\Lambda_1}(\mathbf{k}_{M_1}) \overset{\mathrm{L}}{\boxtimes} \mathscr{MH}_{\Lambda_2}(\mathbf{k}_{M_2}) & \xrightarrow{\ \circ\ } & \mathscr{MH}_{\Lambda_1\times\Lambda_2}(\mathbf{k}_{M_{12}}) \\
\downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle \sim} \\
\mathrm{R}\Gamma_{\Lambda_1}(\pi_{M_1}^{-1}\omega_{M_1}) \overset{\mathrm{L}}{\boxtimes} \mathrm{R}\Gamma_{\Lambda_2}(\pi_{M_2}^{-1}\omega_{M_2}) & \xrightarrow{\overset{\mathrm{L}}{\boxtimes}} & \mathrm{R}\Gamma_{\Lambda_1\times\Lambda_2}(\pi_{M_{12}}^{-1}\omega_{M_{12}})
\end{array}$$

and taking the global sections and the zeroth cohomology,

$$\begin{array}{ccc}
\mathbb{MH}^0_{\Lambda_1}(\mathbf{k}_{M_1}) \otimes \mathbb{MH}^0_{\Lambda_2}(\mathbf{k}_{M_2}) & \xrightarrow{\ \circ\ } & \mathbb{MH}^0_{\Lambda_1\times\Lambda_2}(\mathbf{k}_{M_{12}}) \\
\downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle \sim} \\
H^0_{\Lambda_1}(T^*M_1; \pi_{M_1}^{-1}\omega_{M_1}) \otimes H^0_{\Lambda_2}(T^*M_2; \pi_{M_2}^{-1}\omega_{M_2}) & \xrightarrow{\overset{\mathrm{L}}{\boxtimes}} & H^0_{\Lambda_1\times\Lambda_2}(T^*M_{12}; \pi_{M_{12}}^{-1}\omega_{M_{12}}).
\end{array}$$

Applying Theorem 6.3, we obtain

Proposition 7.1. *The object $K_1 \overset{\mathrm{L}}{\boxtimes} K_2$ is a trace kernel on M_{12} and*

$$\mu\mathrm{eu}_{M_{12}}(K_1 \overset{\mathrm{L}}{\boxtimes} K_2) = \mu\mathrm{eu}_{M_1}(K_1) \overset{\mathrm{L}}{\boxtimes} \mu\mathrm{eu}_{M_2}(K_2).$$

Direct image

Let $f\colon M_1 \to M_2$ and Γ_f be as above. Applying Theorem 4.6 with $M_1 = \mathrm{pt}$ and M_2, M_3 being the current M_1, M_2, we get the commutative diagram

$$\begin{array}{ccc}
\mathscr{MH}(\mathbf{k}_{M_1}) \underset{1}{\overset{a}{\circ}} \mathscr{MH}(\mathbf{k}_{M_{12}}) & \longrightarrow & \mathscr{MH}(\mathbf{k}_{M_2}) \\
\downarrow\wr & & \downarrow\wr \\
\pi_{M_1}^{-1}\omega_{M_1} \underset{1}{\overset{a}{\circ}} \pi_{M_{12}}^{-1}\omega_{M_{12}} & \longrightarrow & \pi_{M_2}^{-1}\omega_{M_2}.
\end{array}$$

Now we assume

$$f \text{ is proper on } \Lambda_1 \cap T^*_{M_1}M_1, \text{ or, equivalently, } f_\pi \text{ is proper on } f_d^{-1}\Lambda_1. \tag{7.2}$$

We set

$$f_\mu(\Lambda_1) = \Lambda_1 \circ \Lambda_f = f_\pi(f_d^{-1}(\Lambda_1)).$$

Taking the global sections and the zeroth cohomology of the diagram above, we obtain the commutative diagram

$$\begin{array}{ccc}
\mathbb{MH}^0_{\Lambda_1}(\mathbf{k}_{M_1}) & \xrightarrow{\circ\mu\mathrm{eu}(\mathbf{k}_{\Gamma_f})} & \mathbb{MH}^0_{f_\mu\Lambda_1}(\mathbf{k}_{M_2}) \\
\downarrow\wr & & \downarrow\wr \\
H^0_{\Lambda_1}(T^*M_1; \pi_{M_1}^{-1}\omega_{M_1}) & \xrightarrow{\circ\mu\mathrm{eu}(\mathbf{k}_{\Gamma_f})} & H^0_{f_\mu\Lambda_1}(T^*M_2; \pi_{M_2}^{-1}\omega_{M_2}).
\end{array}$$

We have the natural morphism and isomorphisms, already constructed in [13]:

$$\begin{aligned}
f_{\pi\,!}f_d^{-1}\pi_{M_1}^{-1}\omega_{M_1} &\simeq f_{\pi\,!}\pi^{-1}\omega_{M_1} \simeq \pi_{M_2}^{-1}f_!\omega_{M_1} \\
&\to \pi_{M_2}^{-1}\omega_{M_2}.
\end{aligned}$$

These induce a morphism:

$$f_\mu \colon \mathrm{R}\Gamma_{\Lambda_1}(\pi_{M_1}^{-1}\omega_{M_1}) \to \mathrm{R}\Gamma_{f_\mu\Lambda_1}(\pi_{M_2}^{-1}\omega_{M_2}).$$

Lemma 7.2. *Let $\lambda \in H^0_{\Lambda_1}(T^*M_1; \pi_{M_1}^{-1}\omega_{M_1})$. Then $\lambda \circ \mu\mathrm{eu}_{M_{12}}(\mathbf{k}_{\Gamma_f}) = f_\mu(\lambda)$.*

Proposition 7.3. *Assume that $\widetilde{f}$ is proper on $\Lambda_{11} \cap T^*_{M_{11}}M_{11}$. Then the object $\mathrm{R}\widetilde{f}_!K_1$ is a trace kernel on M_2 and*

$$\begin{aligned}
\mu\mathrm{eu}_{M_2}(\mathrm{R}\widetilde{f}_!K_1) &= \mu\mathrm{eu}_{M_1}(K_1) \underset{1}{\overset{a}{\circ}} \mu\mathrm{eu}_{M_{12}}(\mathbf{k}_{\Gamma_f}) \\
&= f_\mu(\mu\mathrm{eu}_{M_1}(K_1)).
\end{aligned}$$

Proof.
Note that $\mu\mathrm{eu}_{M_{12}}(\mathbf{k}_{\Gamma_f}) = \mu\mathrm{eu}_{M_{12}}\big((\omega_1^{\otimes -1} \overset{\mathrm{L}}{\boxtimes} \omega_1 \overset{\mathrm{L}}{\boxtimes} \mathbf{k}_{22}) \overset{\mathrm{L}}{\otimes} \mathrm{TK}(\mathbf{k}_{\Gamma_f})\big)$ by Proposition 5.3. We have $\mathrm{R}\widetilde{f}_!K_1 \simeq K_1 \underset{11}{\circ} \Big(\omega_{\Delta_1}^{\otimes -1} \underset{1}{\circ} \big((\omega_1^{\otimes -1} \overset{\mathrm{L}}{\boxtimes} \omega_1 \overset{\mathrm{L}}{\boxtimes} \mathbf{k}_{22}) \overset{\mathrm{L}}{\otimes} \mathrm{TK}(\mathbf{k}_{\Gamma_f})\big)\Big)$. It remains to apply Theorem 6.3 in which one replaces M_1, M_2, M_3 with pt, M_1, M_2, respectively. □

Inverse image

Let $f\colon M_1 \to M_2$ and Γ_f be as above. Applying Theorem 4.6 with $M_3 = \mathrm{pt}$, we get the commutative diagram

$$\begin{array}{ccc}
\mathscr{MH}(\mathbf{k}_{M_{12}}) \overset{a}{\underset{2}{\circ}} \mathscr{MH}(\mathbf{k}_{M_2}) & \longrightarrow & \mathscr{MH}(\mathbf{k}_{M_1}) \\
\big\downarrow \wr & & \big\downarrow \wr \\
\pi_{M_{12}}^{-1}\omega_{M_{12}} \overset{a}{\underset{2}{\circ}} \pi_{M_2}^{-1}\omega_{M_2} & \longrightarrow & \pi_{M_1}^{-1}\omega_{M_1}.
\end{array}$$

Now we assume

$$f \text{ is non-characteristic for } \Lambda_2, \text{ or, equivalently, } f_d \text{ is proper on } f_\pi^{-1}\Lambda_2. \tag{7.3}$$

We set

$$f^\mu(\Lambda_2) = \Lambda_f \circ \Lambda_1 = f_d(f_\pi^{-1}(\Lambda_2)).$$

Taking the global sections and the zeroth cohomology of the diagram above, we obtain the commutative diagram

$$\begin{array}{ccc}
\mathrm{MH}^0_{\Lambda_2}(\mathbf{k}_{M_2}) & \xrightarrow{\ \mu\mathrm{eu}(\mathbf{k}_{\Gamma_f})\circ\ } & \mathrm{MH}^0_{f^\mu\Lambda_2}(\mathbf{k}_{M_1}) \\
\big\downarrow \wr & & \big\downarrow \wr \\
H^0_{\Lambda_2}(T^*M_2; \pi_{M_2}^{-1}\omega_{M_2}) & \xrightarrow{\ \mu\mathrm{eu}(\mathbf{k}_{\Gamma_f})\circ\ } & H^0_{f^\mu\Lambda_2}(T^*M_1; \pi_{M_1}^{-1}\omega_{M_1}).
\end{array}$$

We have a natural morphism constructed in the proof of [13, Proposition 9.3.2]:

$$f^\mu\colon f_{d!}f_\pi^{-1}\pi_{M_2}^{-1}\omega_{M_2} \to \pi_{M_1}^{-1}\omega_{M_1}.$$

Hence, we get a map:

$$f^\mu\colon \mathrm{R}\Gamma_{\Lambda_2}(\pi_{M_2}^{-1}\omega_{M_2}) \to \mathrm{R}\Gamma_{f^\mu\Lambda_2}(\pi_{M_1}^{-1}\omega_{M_1}).$$

Lemma 7.4. *Let $\lambda \in H^0_{\Lambda_1}(T^*M_2; \pi_{M_2}^{-1}\omega_{M_2})$. Then $\mu\mathrm{eu}_{M_{12}}(\mathbf{k}_{\Gamma_f}) \circ \lambda = f^\mu(\lambda)$.*

Proposition 7.5. *Assume that $\widetilde{f}$ is non-characteristic with respect to Λ_{22}. Then the object $(\mathbf{k}_1 \overset{\mathrm{L}}{\boxtimes} \omega_{M_1/M_2}) \overset{\mathrm{L}}{\otimes} \widetilde{f}^{-1}K_2$ is a trace kernel on M_1 and*

$$\begin{aligned}
\mu\mathrm{eu}_{M_1}\big(\omega_{\Delta_1} \underset{1}{\circ} \widetilde{f}^{-1}(\omega_{\Delta_2}^{\otimes -1} \underset{2}{\circ} K_2)\big) &= \mu\mathrm{eu}_{M_{12}}(\mathbf{k}_{\Gamma_f}) \overset{a}{\underset{2}{\circ}} \mu\mathrm{eu}_{M_2}(K_2) \\
&= f^\mu(\mu\mathrm{eu}_{M_2}(K_2)).
\end{aligned}$$

Proof. Applying Theorem 6.3 with $M_3 = \mathrm{pt}$, we get that

$$(\mathbf{k}_1 \overset{\mathrm{L}}{\boxtimes} \omega_{M_1/M_2}) \overset{\mathrm{L}}{\otimes} \widetilde{f}^{-1}K_2 \simeq \mathrm{TK}(\mathbf{k}_f) \underset{22}{\circ} (\omega_{\Delta_2}^{\otimes -1} \underset{2}{\circ} (\omega_2 \overset{\mathrm{L}}{\boxtimes} \omega_2^{\otimes -1}) \overset{\mathrm{L}}{\otimes} K_2)$$

is a trace kernel. Since $\mu\mathrm{eu}_{M_2}\big((\omega_2 \overset{\mathrm{L}}{\boxtimes} \omega_2^{\otimes -1}) \overset{\mathrm{L}}{\otimes} K_2)\big) = \mu\mathrm{eu}_{M_2}(K_2)$ by Proposition 5.3, we obtain the result. □

Tensor product

Consider now the case where $M_1 = M_2 = M$ and the Λ_{ii} satisfy the transversality condition

$$\Lambda_{11} \cap \Lambda_{22}^a \subset T^*_{M\times M}(M \times M). \tag{7.4}$$

Then by composing the external product with the restriction to the diagonal, we get a convolution map

$$\star\colon \mathbb{MH}_{\Lambda_1}(\mathbf{k}_M) \times \mathbb{MH}_{\Lambda_2}(\mathbf{k}_M) \to \mathbb{MH}_{\Lambda_1+\Lambda_2}(\mathbf{k}_M). \tag{7.5}$$

Applying Propositions 7.1 and 7.5, we get

Proposition 7.6. *Assume (7.4). Then the object* $K_1 \overset{\mathrm{L}}{\otimes} (\mathbf{k}_M \overset{\mathrm{L}}{\boxtimes} \omega_M^{\otimes -1}) \overset{\mathrm{L}}{\otimes} K_2$ *is a trace kernel on* M *and*

$$\mu\mathrm{eu}_M(K_1 \overset{\mathrm{L}}{\otimes} (\mathbf{k}_M \overset{\mathrm{L}}{\boxtimes} \omega_M^{\otimes -1}) \overset{\mathrm{L}}{\otimes} K_2) = \mu\mathrm{eu}_M(K_1) \star \mu\mathrm{eu}_M(K_2).$$

Following [23, II, Corollary 5.6], we shall recall the link between the product $\star$ and the cup product.

Proposition 7.7. *Let* $\lambda_i \in H^0_{\Lambda_i}(T^*M_i; \pi_M^{-1}\omega_M)$ $(i = 1, 2)$*, and assume that* $\Lambda_1 \cap \Lambda_2^a \subset T^*_M M$*. Then*

$$(\lambda_1 \star \lambda_2)|_M = \int_{\pi_M} (\lambda_1 \cup \lambda_2) \tag{7.6}$$

as elements of $H^0_{\pi(\Lambda_1\cap\Lambda_2)}(M; \omega_M)$.

Proof. Denote by $\delta\colon \Delta \hookrightarrow M_{12} = M \times M$ the diagonal embedding and let us identify M with Δ. Consider the diagram

$$\begin{array}{ccc} T^*_\Delta M_{12} & \xrightarrow{\;f\;} & \Delta \times_{M_{12}} T^*M_{12} \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \delta_d} \\ \Delta & \xrightarrow{\;s\;} & T^*\Delta \end{array} \tag{7.7}$$

where π is the projection, δ_d is the map associated with δ, s is the zero-section embedding and f is the restriction to $\Delta \times_M T^*M_{12}$ of the embedding $T^*_\Delta M_{12} \hookrightarrow T^*M_{12}$. Since this diagram is Cartesian, we have

$$s^{-1}\delta_{d!} \simeq \pi_! f^{-1}.$$

Now let $\lambda_1 \times \lambda_2 \in H^0_{\Lambda_1\times\Lambda_2}(T^*M_{12};\pi^{-1}\omega_{M_{12}})$ and denote by $\lambda_1 {\times_M} \lambda_2$ its image under the map

$$H^0_{\Lambda_1\times\Lambda_2}(T^*M_{12};\pi^{-1}\omega_{M_{12}}) \to H^0_{\Lambda_1\times_M\Lambda_2}(\Delta\times_{M_{12}}T^*M_{12};\pi^{-1}\omega_{M_{12}}).$$

(Here, on the right hand side, we still denote by π the restriction of the projection $\pi_{M_{12}}$ to $\Delta\times_{M_{12}}T^*M_{12}$.) Then

$$\int_\pi(\lambda_1\cup\lambda_2) = \pi_! f^{-1}(\lambda_1\times_M\lambda_2),$$
$$(\lambda_1\star\lambda_2)|_M = s^{-1}\delta_{d!}(\lambda_1\times_M\lambda_2). \qquad \square$$

Corollary 7.8. *Let K_1 and K_2 be two trace kernels on M with* $\mathrm{SS}(K_i)\subset\Lambda_{ii}$. *Assume (7.4) and assume moreover that* $\mathrm{Supp}(K_1)\cap\mathrm{Supp}(K_2)$ *is compact. Then the object* $\mathrm{R}\Gamma\big(M\times M; K_1 \overset{\mathrm{L}}{\otimes} (\mathbf{k}_M \overset{\mathrm{L}}{\boxtimes} \omega_M^{\otimes-1}) \overset{\mathrm{L}}{\otimes} K_2\big)$ *is a trace kernel on* pt *and*

$$\mathrm{eu}_{\mathrm{pt}}\big(\mathrm{R}\Gamma(M;K_1 \overset{\mathrm{L}}{\otimes} (\mathbf{k}_M \overset{\mathrm{L}}{\boxtimes} \omega_M^{\otimes-1}) \overset{\mathrm{L}}{\otimes} K_2)\big) = \int_{T^*M}\mu\mathrm{eu}(K_1)\cup\mu\mathrm{eu}(K_2).$$

Remark 7.9. Let M be a real analytic manifold and let $F\in\mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbf{k}_M)$. Recall that one associates with F the trace kernel $\mathrm{TK}(F) = F \overset{\mathrm{L}}{\boxtimes} \mathrm{D}_M F$ and that $\mu\mathrm{eu}_M(F) = \mu\mathrm{eu}_M(\mathrm{TK}(F))$. Assume now that $f\colon M_1\to M_2$ is a morphism of real analytic manifolds.

Let $F_1\in\mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbf{k}_{M_1})$ and assume that f is proper on $\mathrm{Supp}(F_1)$. Applying Proposition 7.3 and noticing that

$$\mathrm{R}\widetilde{f_!}\,\mathrm{TK}(F_1)\simeq\mathrm{TK}(\mathrm{R}f_!F_1), \tag{7.8}$$

we find that $\mu\mathrm{eu}(\mathrm{R}f_!F_1) = f_\mu(\mu\mathrm{eu}(F_1))$. This is nothing but [13, Proposition 9.4.2].

Let $F_2\in\mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbf{k}_{M_2})$ and assume that f is non-characteristic with respect to F_2. Applying Proposition 7.5 and noticing that

$$\mathrm{TK}(f^{-1}F_2)\simeq(\mathbf{k}_1 \overset{\mathrm{L}}{\boxtimes} \omega_{M_1/M_2}) \overset{\mathrm{L}}{\otimes} \widetilde{f}^{-1}\mathrm{TK}(F_2),$$

we find that $\mu\mathrm{eu}(f^{-1}F_2) = f^\mu(\mu\mathrm{eu}(F_2))$. Hence, we recover [13, Proposition 9.4.3].

8. Applications: $\mathscr{D}$-modules and elliptic pairs

We shall, as an application of Theorem 6.3, recover the theorem of [23] on the index of elliptic pairs. In this section, X is a complex manifold, $\mathbf{k}=\mathbb{C}$, $\mathscr{M}$ is an object of $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$ and F is an object of $\mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$.

Recall that we have denoted by $\mathrm{TK}(F)$ and $\mathrm{TK}(\mathscr{M})$ (see Notation 5.10) the trace kernels associated with F and with $\mathscr{M}$, respectively:

$$\mathrm{TK}(F) := F \overset{\mathrm{L}}{\boxtimes} \mathrm{D}_X F,$$
$$\mathrm{TK}(\mathscr{M}) := \Omega_{X\times X} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_{X\times X}} (\mathscr{M}\boxtimes\mathrm{D}_{\mathscr{D}}\mathscr{M}).$$

The pair $(\mathscr{M},F)$ is called an *elliptic pair* in the earlier citation if $\mathrm{char}(\mathscr{M})\cap\mathrm{SS}(F)\subset T^*_XX$. From now on, we assume that $(\mathscr{M},F)$ is an elliptic pair.

It follows from Proposition 7.6 that the tensor product of $\mathrm{TK}(F)$ and $\mathrm{TK}(\mathscr{M})$ shifted by $-2d_X$ is again a trace kernel. We denote it by $\mathrm{TK}(\mathscr{M}, F)$. Hence

$$\mathrm{TK}(\mathscr{M}, F) \simeq \Omega_{X\times X} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_{X\times X}} (\mathscr{M} \underline{\boxtimes} \mathrm{D}_{\mathscr{D}}\mathscr{M}) \otimes (F \overset{\mathrm{L}}{\boxtimes} \mathrm{D}'_X F). \tag{8.1}$$

Moreover the same statement gives

$$\mu\mathrm{eu}_X\big(\mathrm{TK}(\mathscr{M}, F)\big) = \mu\mathrm{eu}_X(\mathscr{M}) \star \mu\mathrm{eu}_X(F). \tag{8.2}$$

We set

$$\mathrm{Sol}(\mathscr{M}, F) := \mathrm{RHom}_{\mathscr{D}_X}(\mathscr{M} \otimes F, \mathscr{O}_X), \tag{8.3}$$

$$\mathrm{DR}(\mathscr{M}, F) := \mathrm{R}\Gamma(X; \Omega_X \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{M} \otimes F)\,[d_X]. \tag{8.4}$$

As explained in [23], [13, Theorem 11.3.3] and isomorphism (2.7) provide a generalization of the classical Petrovsky regularity theorem, namely, the natural isomorphisms

$$\mathrm{R}\mathscr{H}\!om_{\mathscr{D}_X}(\mathscr{M}, \mathrm{D}'_X F \otimes \mathscr{O}_X) \xrightarrow{\sim} \mathrm{R}\mathscr{H}\!om_{\mathscr{D}_X}(\mathscr{M} \otimes F, \mathscr{O}_X). \tag{8.5}$$

Now assume that $\mathrm{Supp}(\mathscr{M}) \cap \mathrm{Supp}(F)$ is compact and let us take the global sections of the isomorphism (8.5). We find the isomorphism

$$\mathrm{RHom}_{\mathscr{D}_X}(\mathscr{M}, \mathrm{D}'_X F \otimes \mathscr{O}_X) \xrightarrow{\sim} \mathrm{RHom}_{\mathscr{D}_X}(\mathscr{M} \otimes F, \mathscr{O}_X). \tag{8.6}$$

It is proved in [23] (assuming $\mathscr{M}$ has a good filtration) that one can represent the left hand side of (8.6) by a complex of topological vector spaces of type DFN and the right hand side of (8.6) by a complex of topological vector spaces of type FN. It follows that the complexes $\mathrm{Sol}(\mathscr{M}, F)$ and $\mathrm{DR}(\mathscr{M}, F)$ have finite-dimensional cohomology and are dual to each other. More precisely, denoting by $(\bullet)^*$ the duality functor in $\mathsf{D}^{\mathrm{b}}_f(\mathbb{C})$, we have

$$\big(\mathrm{Sol}(\mathscr{M}, F)\big)^* \simeq \mathrm{DR}(\mathscr{M}, F).$$

It follows from the finiteness of the cohomology of the complexes $\mathrm{Sol}(\mathscr{M}, F)$ and $\mathrm{DR}(\mathscr{M}, F)$ that

$$\mathrm{R}\Gamma(X \times X; \mathrm{TK}(\mathscr{M}, F)) \simeq \mathrm{Sol}(\mathscr{M}, F) \otimes \mathrm{DR}(\mathscr{M}, F).$$

One checks that this isomorphism commutes with the composition of the morphisms $\mathbb{C} \to \mathrm{R}\Gamma(X \times X; \mathrm{TK}(\mathscr{M}, F)) \to \mathbb{C}$ and $\mathbb{C} \to \mathrm{Sol}(\mathscr{M}, F) \otimes \mathrm{DR}(\mathscr{M}, F) \to \mathbb{C}$, which implies

$$\mathrm{eu}_{\mathrm{pt}}\big(\mathrm{R}\Gamma(X \times X; \mathrm{TK}(\mathscr{M}, F))\big) = \chi\big(\mathrm{Sol}(\mathscr{M}, F)\big). \tag{8.7}$$

Therefore, one recovers the index formula of the earlier citation:

$$\begin{aligned} \chi\big(\mathrm{RHom}_{\mathscr{D}_X}(\mathscr{M} \otimes F, \mathscr{O}_X)\big) &= \int_X (\mu\mathrm{eu}_X(\mathscr{M}) \star \mu\mathrm{eu}_X(F))|_X \\ &= \int_{T^*X} \mu\mathrm{eu}_X(\mathscr{M}) \cup \mu\mathrm{eu}_X(F). \end{aligned} \tag{8.8}$$

Remark 8.1. In general the direct image of an elliptic pair is no longer an elliptic pair. However, it remains a trace kernel.

Remark 8.2. As already mentioned in [23], formula (8.8) has many applications, as long as one is able to calculate $\mu\mathrm{eu}_X(\mathscr{M})$ (see the final remarks below). For example, if M is a compact real analytic manifold and X is a complexification of M, one recovers the Atiyah–Singer theorem by choosing $F = \mathrm{D}'\mathbb{C}_M$. If X is a complex compact manifold, one recovers the Riemann–Roch theorem: one takes $F = \mathbb{C}_X$ and if $\mathscr{F}$ is a coherent $\mathscr{O}_X$-module, one sets $\mathscr{M} = \mathscr{D}_X \otimes_{\mathscr{O}_X} \mathscr{F}$.

9. The Lefschetz fixed point formula

In this section, we shall briefly show how to adapt the formalism of trace kernels to the Lefschetz trace formula as treated in [13, § 9.6]. Here we assume that $\mathbf{k}$ is a field.

Assume that we are given two maps $f, g\colon N \to M$ of real analytic manifolds, an object $F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbf{k}_M)$ and a morphism

$$\varphi\colon f^{-1}F \to g^{\,!}F. \tag{9.1}$$

Set

$$\begin{aligned}
&h = (g, f)\colon N \times N \to M \times M,\\
&S = \mathrm{Supp}(F), \quad L = h^{-1}(\Delta_M) = \{(x, y) \in N \times N;\, g(x) = f(y)\},\\
&i\colon L \hookrightarrow N \times N,\\
&T = f^{-1}(S) \cap g^{-1}(S).
\end{aligned}$$

One makes the following assumption:

$$\text{The set } T \text{ is compact.} \tag{9.2}$$

Then we have the maps

$$\mathrm{R}\Gamma(M; F) \to \mathrm{R}\Gamma_{f^{-1}S}(N; f^{-1}F) \xrightarrow{\varphi} \mathrm{R}\Gamma_T(N; g^{\,!}F) \to \mathrm{R}\Gamma(M; F).$$

The composition gives a map

$$\int \varphi\colon \mathrm{R}\Gamma(M; F) \to \mathrm{R}\Gamma(M; F), \tag{9.3}$$

and this map factorizes through $\mathrm{R}\Gamma_T(N; g^{\,!}F)$ which has finite-dimensional cohomologies. Hence, we can define the trace $\mathrm{tr}(\int \varphi)$.

We have the chain of morphisms

$$\begin{aligned}
\mathbf{k}_N &\to \mathrm{R}\mathscr{H}om(g^{\,!}F, g^{\,!}F)\\
&\xrightarrow{\varphi} \mathrm{R}\mathscr{H}om(f^{-1}F, g^{\,!}F) \simeq \delta_N^{\,!}(g^{\,!}F \overset{\mathrm{L}}{\boxtimes} \mathrm{D}_N f^{-1}F)\\
&\simeq \delta_N^{\,!}(g^{\,!}F \overset{\mathrm{L}}{\boxtimes} f^{\,!}\mathrm{D}_M F) \simeq \delta_N^{\,!} h^{\,!}(F \overset{\mathrm{L}}{\boxtimes} \mathrm{D}_M F).
\end{aligned}$$

We have thus constructed the morphism

$$\mathbf{k}_{\Delta_N} \to h^{\,!}(F \overset{\mathrm{L}}{\boxtimes} \mathrm{D}_M F).$$

By using the morphism $F \overset{\mathrm{L}}{\boxtimes} \mathrm{D}_M F \to \omega_{\Delta_M}$ and the isomorphism $h^! \omega_{\Delta_M} \simeq i_* \omega_L$, we get the morphisms

$$\mathbf{k}_{\Delta_N} \to h^!(F \overset{\mathrm{L}}{\boxtimes} \mathrm{D}_M F) \to i_* \omega_L \tag{9.4}$$

in $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_{N \times N})$. The support of the composition is contained in $\delta_N(T) \cap L$.

Theorem 9.1 ([13, Proposition 9.6.2]). *The trace* $\mathrm{tr}(\int \varphi)$ *coincides with the image of* $1 \in \mathbf{k}$ *under the composition of the morphisms*

$$\mathbf{k} \to \mathrm{R}\Gamma(N, \mathbf{k}_N) \to \mathrm{R}\Gamma_c(L, \omega_L) \to \mathbf{k}.$$

Here the middle arrow is derived from (9.4).

Although (9.4) is not a trace kernel in the sense of Definition 5.1, it should be possible to adapt the previous constructions to the case of $\mathscr{D}$-modules and to elliptic pairs, and then to recover a theorem of [7], but we do not develop this point here (see [21] for related results).

Final remarks

The microlocal Euler class of constructible sheaves is easy to compute since it is enough to calculate some multiplicities at generic points. We refer the reader to [13] for examples.

On the other hand, there is no direct method for calculating the microlocal Euler class of a coherent $\mathscr{D}$-module $\mathscr{M}$ (except in the holonomic case). In [23], the authors made a precise conjecture relying on $\mu\mathrm{eu}_X(\mathscr{M})$ and the Chern character of the associated graded module (an $\mathscr{O}_{T^*X}$-module), and this conjecture has been proved by Bressler, Nest and Tsygan [1].

Similarly, the Hochschild class of coherent $\mathscr{O}_X$-modules is usually calculated through the so-called Hochschild–Kostant–Rosenberg isomorphism, but this isomorphism does not commute with proper direct images, and a precise conjecture (involving the Todd class) has been made by Kashiwara in [10] and this conjecture has recently been proved in the algebraic case by Ramadoss [20] and in the general case by Grivaux [6].

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