Radon-Penrose transform for $\mathcal{D}$-modules

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Abstract

Let $X \underset{f}{\longleftarrow} S \underset{g}{\longrightarrow} Y$ be a correspondence of complex analytic manifolds, F be a sheaf on X, and $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. Consider the associated sheaf theoretical and $\mathcal{D}$-module integral transforms given by $\Phi_S F = Rg_! f^{-1} F[d]$ and $\underline{\Phi}_S \mathcal{M} = \underline{g}_! \underline{f}^{-1} \mathcal{M}$, where $Rg_!$ and f^{-1} (resp. $\underline{g}_!$ and $\underline{f}^{-1}$) denote the direct and inverse image functors for sheaves (resp. for $\mathcal{D}$-modules), and $d = d_S - d_Y$ is the difference of dimension between S and Y. In this paper, assuming that f is smooth, g is proper, and (f, g) is a closed embedding, we prove some general adjunction formulas for the functors Φ_S and $\underline{\Phi}_S$. Moreover, under an additional geometrical hypothesis, we show that the transformation $\underline{\Phi}_S$ establishes an equivalence of categories between coherent $\mathcal{D}_X$-modules, modulo flat connections, and coherent $\mathcal{D}_Y$-modules with regular singularities along an involutive manifold V, modulo flat connections (here V is determined by the geometry of the correspondence). Applications are given to the case of Penrose's twistor correspondence.

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Contents

1 Introduction

The Penrose correspondence is an integral transformation which interchanges global sections of line bundles on some flag manifolds, with holomorphic solutions of partial differential equations on other flag manifolds (see [6], [1]). For example, consider the twistor correspondence:

$$
(1.1) \qquad \begin{tikzcd} & \mathbb{F} \arrow[dl, "f"'] \arrow[dr, "g"] & \\ \mathbb{P} & & \mathbb{M}, \end{tikzcd}
$$

where $\mathbb{F} = \mathbb{F}_{1,2}(\mathbb{T})$ is the flag manifold of type $(1,2)$ associated to a four-dimensional complex vector space $\mathbb{T}$, $\mathbb{P} = \mathbb{F}_1(\mathbb{T})$ is a projective three-space, and $\mathbb{M} = \mathbb{F}_2(\mathbb{T})$. The projections are given by $f(L_1, L_2) = L_1$, $g(L_1, L_2) = L_2$, where $L_1 \subset L_2 \subset \mathbb{T}$ are complex subspaces of dimension one and two respectively, defining an element (L_1, L_2) of $\mathbb{F}$. Since $\mathbb{M}$ is identified with the four-dimensional compactified complexified Minkowski space, the family of massless field equations on the Minkowski space gives rise to a family of differential operators acting between sections of holomorphic bundles on $\mathbb{M}$. This is a family, denoted here by $\square_h$, which is parameterized by a half-integer h called helicity, and which includes Maxwell's wave equation, Dirac-Weyl neutrino equations and Einstein linearized vacuum equations.

The Penrose transform associated to the correspondence (1.1) allows to represent the holomorphic solutions of the equation $\square_h \phi = 0$ on some open subsets $U \subset \mathbb{M}$ in terms of cohomology classes of line bundles on $\widehat{U} = f(g^{-1}(U)) \subset \mathbb{P}$. More precisely, recall that the line bundles on $\mathbb{P}$ are given, for $k \in \mathbb{Z}$, by the $-k$-th tensor powers $\mathcal{O}_{\mathbb{P}}(k)$ of the tautological bundle. Set $h(k) = -(1 + k/2)$, and for $x \in \mathbb{P}$, set $\widehat{x} = g(f^{-1}(x))$. We then have the result of Eastwood, Penrose and Wells [6] below.

Theorem 1.1 *Let $U \subset \mathbb{M}$ be an open subset such that:*

(1.2) $\qquad$ *$U \cap \widehat{x}$ is connected and simply connected for every $x \in \widehat{U}$.*

Then, for $k < 0$, the natural morphism associated to (1.1), which maps a one-form on $\widehat{U}$ to the integral along the fibers of g of its inverse image by f, induces an isomorphism:

$$H^1(\widehat{U}; \mathcal{O}_{\mathbb{P}}(k)) \xrightarrow{\sim} \ker(U; \square_{h(k)}).$$

In this paper we shall formulate the Penrose correspondence in the language of sheaves and $\mathcal{D}$-modules. First of all, we can rephrase the above construction in a more general setting as follows. Consider a correspondence:

$$\begin{tikzcd} & S \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y, \end{tikzcd}$$

where all manifolds are complex analytic, f is smooth, g is proper, and where (f, g) induces a closed embedding $S \hookrightarrow X \times Y$. Set $d_S = \dim^{\mathbb{C}} S$, $d_{S/Y} = d_S - d_Y$.

Let us define the transform of a sheaf F on X (more generally, of an object of the derived category of sheaves) as $\Phi_S F = Rg_! f^{-1} F[d_{S/Y}]$, and define the transform of a coherent $\mathcal{D}_X$-module $\mathcal{M}$ as $\underline{\Phi}_S \mathcal{M} = \underline{g}_* \underline{f}^{-1} \mathcal{M}$, where $\underline{g}_*$ and $\underline{f}^{-1}$ denote the direct and inverse images in the sense of $\mathcal{D}$-module theory. We also consider $\Phi_{\widetilde{S}} G = Rf_! g^{-1} G[d_{S/X}]$, for a sheaf G on Y. One then proves the formula:

$$\Phi_S R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X) \simeq R\mathcal{H}om_{\mathcal{D}_Y}(\underline{\Phi}_S \mathcal{M}, \mathcal{O}_Y), \tag{1.3}$$

from which one deduces the following formula, where G denotes a sheaf on Y:

$$\mathrm{R}\Gamma(X; R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M} \otimes \Phi_{\widetilde{S}} G, \mathcal{O}_X))[d_X] \simeq \mathrm{R}\Gamma(Y; R\mathcal{H}om_{\mathcal{D}_Y}(\underline{\Phi}_S \mathcal{M} \otimes G, \mathcal{O}_Y))[d_Y]. \tag{1.4}$$

Let $\mathcal{F}$ be a holomorphic vector bundle on X, denote by $\mathcal{F}^*$ its dual, and set $\mathcal{D}\mathcal{F}^* = \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{F}^*$. When applying (1.4) to the case of $\mathcal{M} = \mathcal{D}\mathcal{F}^*$ and $G = \mathbb{C}_U$, the constant sheaf on an open subset $U \subset Y$ satisfying suitable hypotheses, one gets the formula:

$$\mathrm{R}\Gamma(\widehat{U}; \mathcal{F}) \simeq \mathrm{R}\Gamma(U; R\mathcal{H}om_{\mathcal{D}_Y}(\underline{\Phi}_S \mathcal{D}\mathcal{F}^*, \mathcal{O}_Y))[-d_{S/Y}]. \tag{1.5}$$

In other words, the cohomology of $\mathcal{F}$ on $\widehat{U}$ is isomorphic to the holomorphic solutions on U of some complex of coherent $\mathcal{D}_Y$-modules, namely the complex $\underline{\Phi}_S \mathcal{D}\mathcal{F}^*$.

In the particular case of the twistor correspondence, the above results show that Theorem 1.1 is better understood by saying that the $\mathcal{D}$-module transform of $\mathcal{D}\mathcal{O}_{\mathbb{P}}(-k)$ (for $k < 0$) is the $\mathcal{D}_{\mathbb{M}}$-module associated to the differential operator $\square_{h(k)}$. Moreover, formula (1.4) shows that each of the many problems encountered in literature can be split into two different ones:

(i) to calculate the sheaf theoretical transform $\Phi_{\widetilde{S}} G$ of G,

(ii) to calculate the $\mathcal{D}$-module transform $\underline{\Phi}_S \mathcal{M}$ of $\mathcal{M}$.

The calculation of $\Phi_{\widetilde{S}} G$ relies on the particular geometry considered (see section 5.2 for an example, where we easily recover Wells's result on hyperfunction solutions).

The calculation of $\underline{\Phi}_S\mathcal{M}$ leads to more difficult problems. For instance, notice that in general $\underline{\Phi}_S\mathcal{M}$ is a complex, not necessarily concentrated in degree zero. This implies many technical difficulties when interpreting the cohomology groups of the right hand side of (1.4). In this paper we give several properties of $\underline{\Phi}_S\mathcal{M}$, which hold under geometrical hypotheses that we will formulate later:

(i) $H^0(\underline{\Phi}_S\mathcal{M})$ is a coherent $\mathcal{D}_Y$-module with regular singularities along an involutive manifold V of the cotangent bundle T^*Y given by the geometry,

(ii) for $j \neq 0$, $H^j(\underline{\Phi}_S\mathcal{M})$ is a locally free $\mathcal{O}_Y$-module of finite rank endowed with a flat connection,

(iii) in the case $\mathcal{M} = \mathcal{D}\mathcal{F}^*$, for a complex vector bundle $\mathcal{F}$, we give several formulas similar to (1.5), and in particular we prove the germ formula (where $\widehat{y} = f(g^{-1}(y))$):

$$\mathrm{R}\Gamma(\widehat{y}; \mathcal{F}) \simeq R\mathcal{H}om_{\mathcal{D}_Y}(\underline{\Phi}_S\mathcal{D}\mathcal{F}^*, \mathcal{O}_Y)_y[-d_{S/Y}],$$

from which we deduce that $H^j(\underline{\Phi}_S\mathcal{D}\mathcal{F}^*) = 0$ for $j \neq 0$ if and only if (Y being connected) there exists $y \in Y$ such that $H^j(\widehat{y}; \mathcal{F}) = 0$ for $j < d_{S/Y}$.

Then, and it is our main result, we prove that (under suitable hypotheses which are satisfied in the twistor case) the transform $\underline{\Phi}_S$ induces an equivalence of categories between coherent $\mathcal{D}$-modules on X modulo flat connections, and coherent $\mathcal{D}$-modules on Y with regular singularities along the involutive submanifold V of (i), modulo flat connections. When applied to the twistor case, our results show in particular that any $\mathcal{D}$-module on the Minkowski space with regular singularities along the characteristic variety of the wave equation, may be obtained (up to flat connections) as the image of a coherent $\mathcal{D}$-module on $\mathbb{P}$.

The results of this paper were announced in [3]. When writing this paper we benefitted from many classical works on the Penrose correspondence. In particular, let us mention the books [19], [1], [27] and the papers [6], [5], [28], [29]. Note that a microlocal approach in the study of correspondences was initiated in the paper [7] of Guillemin and Sternberg.

Finally, we would like to thank Jean-Pierre Schneiders for fruitful discussions.

2 Adjunction formulas

2.1 Sheaves

Let X be a real analytic manifold. We denote by $\mathbf{D}(X)$ the derived category of the category of complexes of sheaves of $\mathbb{C}$-vector spaces on X, by $\mathbf{D}^{\mathrm{b}}(X)$ the full triangulated subcategory of $\mathbf{D}(X)$ whose objects have bounded cohomology, and we refer to [17] for a detailed exposition on sheaves, in the framework of derived categories.

If $A \subset X$ is a locally closed subset, we denote by $\mathbb{C}_A$ the sheaf on X which is the constant sheaf on A with stalk $\mathbb{C}$, and zero on $X \setminus A$. We will consider the classical six operations in the derived category of sheaves of $\mathbb{C}$-vector spaces f^{-1}, $Rf_!$, $\otimes$, Rf_*, $f^!$, $R\mathcal{H}om$. We denote by $\omega_{Y/X}$ the relative dualizing complex $\omega_{Y/X} = f^!\mathbb{C}_X$. Recall that $\omega_{Y/X} \simeq or_{Y/X}[d]$, where $or_{Y/X}$ is the relative orientation sheaf on Y, and $d = \dim^{\mathbb{R}} Y - \dim^{\mathbb{R}} X$, $\dim^{\mathbb{R}} X$ denoting the dimension of X. We use the notations $D'_X(\cdot) = R\mathcal{H}om(\cdot, \mathbb{C}_X)$ and $D_X(\cdot) = R\mathcal{H}om(\cdot, \omega_X)$, where $\omega_X = \omega_{X/\{\mathrm{pt}\}}$. We denote by $a_X : X \longrightarrow \{\mathrm{pt}\}$ the map from X to the set consisting of a single element.

In the rest of this section, all manifolds and morphisms of manifolds will be complex analytic. We denote by d_S the complex dimension of a manifold S. Given a morphism $f : S \longrightarrow X$ of complex manifolds, we set for short $d_{S/X} = d_S - d_X$.

Consider a correspondence of complex analytic manifolds:

$$\begin{tikzcd} & S \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y. \end{tikzcd} \tag{2.1}$$

Definition 2.1 For $F \in \mathbf{D}^{\mathrm{b}}(X)$, we set:

$$\Phi_S F = Rg_! f^{-1}(F)[d_{S/Y}], \quad \Psi_S F = Rg_* f^!(F)[-d_{S/X}].$$

For $G \in \mathbf{D}^{\mathrm{b}}(Y)$, we similarly define:

$$\Phi_{\tilde{S}} G = Rf_! g^{-1}(G)[d_{S/X}], \quad \Psi_{\tilde{S}} G = Rf_* g^!(G)[-d_{S/Y}].$$

In other words, we denote by

$$\begin{tikzcd} & \tilde{S} \arrow[dl, "g"'] \arrow[dr, "f"] & \\ Y & & X \end{tikzcd} \tag{2.2}$$

the correspondence deduced from (2.1) by interchanging X and Y.

Lemma 2.2 *Let $F \in \mathbf{D}^{\mathrm{b}}(X)$ and $G \in \mathbf{D}^{\mathrm{b}}(Y)$. Then we have the isomorphisms:*

$$Ra_{X*} R\mathcal{H}om(\Phi_{\tilde{S}} G, F) \simeq Ra_{Y*} R\mathcal{H}om(G, \Psi_S F), \tag{2.3}$$

$$Ra_{X!}(\Phi_{\tilde{S}} G \otimes F)[d_X] \simeq Ra_{Y!}(G \otimes \Phi_S F)[d_Y], \tag{2.4}$$

$$\Psi_S(D_X F)[d_Y] \simeq D_Y(\Phi_S F)[d_X]. \tag{2.5}$$

Proof: All the above isomorphisms are easy consequences of classical adjunction formulas, such as the Poincaré-Verdier duality formula (see e.g. [17, chapters II and III]). For example, in order to prove (2.4) one considers the sequence of isomorphisms:

$$\begin{aligned} Ra_{X!}(\Phi_{\tilde{S}}G \otimes F)[d_X] &= Ra_{X!}(Rf_!g^{-1}G \otimes F)[d_S] \\ &\simeq Ra_{X!}Rf_!(g^{-1}G \otimes f^{-1}F)[d_S] \\ &\simeq Ra_{Y!}Rg_!(g^{-1}G \otimes f^{-1}F)[d_S] \\ &\simeq Ra_{Y!}(G \otimes Rg_!f^{-1}F)[d_S] \\ &= Ra_{Y!}(G \otimes \Phi_S F)[d_Y]. \end{aligned}$$

□

2.2 $\mathcal{D}$-modules

Let $\mathcal{O}_X$ denote the sheaf of holomorphic functions on a complex manifold X, Ω_X the sheaf of holomorphic forms of maximal degree, and $\mathcal{D}_X$ the sheaf of rings of holomorphic linear differential operators. We refer to [15], [23] for the theory of $\mathcal{D}$-modules (see [24] for a detailed exposition).

Denote by $\mathrm{Mod}(\mathcal{D}_X)$ the category of left $\mathcal{D}_X$-modules, and by $\mathrm{Mod}_{\mathrm{coh}}(\mathcal{D}_X)$ the thick abelian subcategory of coherent $\mathcal{D}_X$-modules. Following [25], we say that a coherent $\mathcal{D}_X$-module $\mathcal{M}$ is good if, in a neighborhood of any compact subset of X, $\mathcal{M}$ admits a finite filtration by coherent $\mathcal{D}_X$-submodules $\mathcal{M}_k$ ($k = 1, \ldots, l$) such that each quotient $\mathcal{M}_k/\mathcal{M}_{k-1}$ can be endowed with a good filtration. We denote by $\mathrm{Mod}_{\mathrm{good}}(\mathcal{D}_X)$ the full subcategory of $\mathrm{Mod}_{\mathrm{coh}}(\mathcal{D}_X)$ consisting of good $\mathcal{D}_X$-modules. This definition ensures that $\mathrm{Mod}_{\mathrm{good}}(\mathcal{D}_X)$ is the smallest thick subcategory of $\mathrm{Mod}(\mathcal{D}_X)$ containing the modules which can be endowed with good filtrations on a neighborhood of any compact subset of X. Note that in the algebraic case, coherent $\mathcal{D}$-modules are good.

Denote by $\mathbf{D}^{\mathrm{b}}(\mathcal{D}_X)$ the derived category of the category of bounded complexes of left $\mathcal{D}_X$-modules, and by $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X)$ (resp. by $\mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_X)$) its full triangulated subcategory whose objects have cohomology groups belonging to $\mathrm{Mod}_{\mathrm{coh}}(\mathcal{D}_X)$ (resp. to $\mathrm{Mod}_{\mathrm{good}}(\mathcal{D}_X)$).

Let $f : Y \longrightarrow X$ be a morphism of complex manifolds. We denote by $\underline{f}^{-1}$ and $\underline{f}_*$ the inverse and direct images in the sense of $\mathcal{D}$-modules. Hence, for $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}(\mathcal{D}_X)$ and $\mathcal{N} \in \mathbf{D}^{\mathrm{b}}(\mathcal{D}_Y)$:

$$\underline{f}^{-1}\mathcal{M} = \mathcal{D}_{Y\to X} \otimes^L_{f^{-1}\mathcal{D}_X} f^{-1}\mathcal{M}, \quad \underline{f}_*\mathcal{N} = Rf_*(\mathcal{D}_{X\leftarrow Y} \otimes^L_{\mathcal{D}_Y} \mathcal{N}),$$

where $\mathcal{D}_{Y\to X}$ and $\mathcal{D}_{X\leftarrow Y}$ are the transfer bimodules. We denote by $\boxtimes$ the exterior tensor product, and we also use the notation:

$$\underline{D}_X\mathcal{M} = R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{K}_X),$$

where $\mathcal{K}_X$ is the dualizing complex for left $\mathcal{D}_X$-modules, $\mathcal{K}_X = \mathcal{D}_X \otimes_{\mathcal{O}_X} \Omega_X^{\otimes -1}[d_X]$.

Proposition 2.3 *Let $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_X)$, $\mathcal{N} \in \mathbf{D}^{\mathrm{b}}(\mathcal{D}_Y)$, and $G \in \mathbf{D}^{\mathrm{b}}(Y)$. Assume that f is non-characteristic for $\mathcal{M}$. Then $\underline{f}^{-1}\mathcal{M}$ is good, and we have the isomorphisms:*

$$\underline{D}\underline{f}^{-1}\mathcal{M} \simeq \underline{f}^{-1}\underline{D}\mathcal{M}, \tag{2.6}$$

$$Rf_*R\mathcal{H}om_{\mathcal{D}_Y}(\underline{f}^{-1}\mathcal{M}, \mathcal{N}[d_{Y/X}]) \simeq R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \underline{f}_*\mathcal{N}), \tag{2.7}$$

$$f^{-1}R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X) \simeq R\mathcal{H}om_{\mathcal{D}_Y}(\underline{f}^{-1}\mathcal{M}, \mathcal{O}_Y), \tag{2.8}$$

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, Rf_!G \otimes \mathcal{O}_X) \simeq Rf_!R\mathcal{H}om_{\mathcal{D}_Y}(\underline{f}^{-1}\mathcal{M}, G \otimes \mathcal{O}_Y), \tag{2.9}$$

$$\begin{aligned} &R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, R\mathcal{H}om(Rf_!G, \mathcal{O}_X)) \\ &\quad \simeq Rf_*R\mathcal{H}om_{\mathcal{D}_Y}(\underline{f}^{-1}\mathcal{M}, R\mathcal{H}om(G, \mathcal{O}_Y))[2d_{Y/X}]. \end{aligned} \tag{2.10}$$

Proof: The fact that $\underline{f}^{-1}\mathcal{M}$ is good and the first isomorphism are results of [23]. The second isomorphism is easily deduced from the first one, and the third isomorphism is the Cauchy-Kowalevski-Kashiwara theorem. Let us prove (2.9). Let us set for short $\mathcal{S}ol(\mathcal{M}) = R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$. Then we have the chain of isomorphisms:

$$\begin{aligned} R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, Rf_!G \otimes \mathcal{O}_X) &\simeq Rf_!G \otimes \mathcal{S}ol(\mathcal{M}) \\ &\simeq Rf_!(G \otimes f^{-1}\mathcal{S}ol(\mathcal{M})) \\ &\simeq Rf_!(G \otimes \mathcal{S}ol(\underline{f}^{-1}\mathcal{M})), \end{aligned}$$

where the last isomorphism follows from (2.8). To prove (2.10), consider the chain of isomorphisms:

$$\begin{aligned} R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, R\mathcal{H}om(Rf_!G, \mathcal{O}_X)) &\simeq R\mathcal{H}om(Rf_!G, \mathcal{S}ol(\mathcal{M})) \\ &\simeq Rf_*R\mathcal{H}om(G, f^!\mathcal{S}ol(\mathcal{M})) \\ &\simeq Rf_*R\mathcal{H}om(G, f^{-1}\mathcal{S}ol(\mathcal{M}))[2d_{Y/X}] \\ &\simeq Rf_*R\mathcal{H}om(G, \mathcal{S}ol(\underline{f}^{-1}\mathcal{M}))[2d_{Y/X}], \end{aligned}$$

where, in order to prove the third isomorphism, we have used Proposition 5.4.13 and Theorem 11.3.3 of [17]. □

Proposition 2.4 *Let $\mathcal{N} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_Y)$, $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}(\mathcal{D}_X)$, and $F \in \mathbf{D}^{\mathrm{b}}(X)$. Assume f is proper on* $\mathrm{supp}\,\mathcal{N}$. *Then $\underline{f}_*\mathcal{N}$ is good, and:*

$$\underline{D}\underline{f}_*\mathcal{N} \simeq \underline{f}_*\underline{D}\mathcal{N}, \tag{2.11}$$

$$Rf_*R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{N}, \underline{f}^{-1}\mathcal{M}[d_{Y/X}]) \simeq R\mathcal{H}om_{\mathcal{D}_X}(\underline{f}_*\mathcal{N}, \mathcal{M}), \tag{2.12}$$

$$Rf_*R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{N}, \mathcal{O}_Y)[d_{Y/X}] \simeq R\mathcal{H}om_{\mathcal{D}_X}(\underline{f}_*\mathcal{N}, \mathcal{O}_X), \tag{2.13}$$

$$Rf_!R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{N}, f^{-1}F \otimes \mathcal{O}_Y)[d_{Y/X}] \simeq R\mathcal{H}om_{\mathcal{D}_X}(\underline{f}_*\mathcal{N}, F \otimes \mathcal{O}_X), \tag{2.14}$$

$$Rf_* R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{N}, R\mathcal{H}om(f^{-1}F, \mathcal{O}_Y))[d_{Y/X}] \simeq R\mathcal{H}om_{\mathcal{D}_X}(\underline{f}_*\mathcal{N}, R\mathcal{H}om(F, \mathcal{O}_X)). \tag{2.15}$$

Proof: The fact that $\underline{f}_*\mathcal{N}$ is good, and the first isomorphism are results of [14], [12], [26], and [25]. The second and third isomorphisms follow from the first one. To prove (2.14), consider the chain of isomorphisms:

$$\begin{aligned} Rf_!(f^{-1}F \otimes \mathcal{S}ol(\mathcal{N}))[d_{Y/X}] &\simeq F \otimes Rf_!\mathcal{S}ol(\mathcal{N})[d_{Y/X}] \\ &\simeq F \otimes \mathcal{S}ol(\underline{f}_*\mathcal{N}). \end{aligned}$$

To prove (2.14), consider the chain of isomorphisms:

$$\begin{aligned} Rf_* R\mathcal{H}om(f^{-1}F, \mathcal{S}ol(\mathcal{N}))[d_{Y/X}] &\simeq R\mathcal{H}om(F, Rf_*\mathcal{S}ol(\mathcal{N}))[d_{Y/X}] \\ &\simeq R\mathcal{H}om(F, \mathcal{S}ol(\underline{f}_*\mathcal{N})). \end{aligned}$$

□

2.3 Correspondences

Instead of considering morphisms, we shall now consider correspondences of complex analytic manifolds:

$$\begin{tikzcd} & S \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y. \end{tikzcd} \tag{2.16}$$

Definition 2.5 For $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}(\mathcal{D}_X)$, we set:

$$\underline{\Phi}_S\mathcal{M} = \underline{g}_*\underline{f}^{-1}\mathcal{M}, \quad \underline{\Psi}_S\mathcal{M} = \underline{\Phi}_S\mathcal{M}[d_Y - d_X].$$

For $\mathcal{N} \in \mathbf{D}^{\mathrm{b}}(\mathcal{D}_Y)$ we similarly define:

$$\underline{\Phi}_{\tilde{S}}\mathcal{N} = \underline{f}_*\underline{g}^{-1}\mathcal{N}, \quad \underline{\Psi}_{\tilde{S}}\mathcal{N} = \underline{\Phi}_{\tilde{S}}\mathcal{N}[d_X - d_Y].$$

As an immediate consequence of Propositions 2.3 and 2.4, we get

Proposition 2.6 *Let $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_X)$, $\mathcal{N} \in \mathbf{D}^{\mathrm{b}}(\mathcal{D}_Y)$, and $G \in \mathbf{D}^{\mathrm{b}}(Y)$. Assume that f is non-characteristic for $\mathcal{M}$, and that g is proper on $f^{-1}\operatorname{supp}\mathcal{M}$. Then $\underline{\Phi}_S\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_Y)$, and:*

$$\underline{\Phi}_S(\underline{D}_X\mathcal{M}) \simeq \underline{D}_Y(\underline{\Phi}_S\mathcal{M}), \tag{2.17}$$

$$Ra_{X*}R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \underline{\Psi}_{\tilde{S}}\mathcal{N}) \simeq Ra_{Y*}R\mathcal{H}om_{\mathcal{D}_Y}(\underline{\Phi}_S\mathcal{M}, \mathcal{N}), \tag{2.18}$$

$$\Phi_S R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X) \simeq R\mathcal{H}om_{\mathcal{D}_Y}(\underline{\Phi}_S\mathcal{M}, \mathcal{O}_Y), \tag{2.19}$$

$$
\begin{aligned}
(2.20)\qquad & Ra_{X!}R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \Phi_{\widetilde{S}}G \otimes \mathcal{O}_X)[d_X] \\
& \quad \simeq \; Ra_{Y!}R\mathcal{H}om_{\mathcal{D}_Y}(\underline{\Phi}_S\mathcal{M}, G \otimes \mathcal{O}_Y)[d_Y], \\
(2.21)\qquad & Ra_{X*}R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M} \otimes \Phi_{\widetilde{S}}G, \mathcal{O}_X)[d_X] \\
& \quad \simeq \; Ra_{Y*}R\mathcal{H}om_{\mathcal{D}_Y}(\underline{\Phi}_S\mathcal{M} \otimes G, \mathcal{O}_Y)[d_Y].
\end{aligned}
$$

As already mentioned in the introduction, this result allows us to distinguish between two kind of problems arising in the Penrose transform:

(i) to compute the sheaf theoretical transform of G,

(ii) to compute the $\mathcal{D}$-module transform of $\mathcal{M}$.

The first problem is of a topological nature, and under reasonable hypotheses is not very difficult (a first example appears in Corollary 2.9 below). The study of $\underline{\Phi}_S\mathcal{M}$ is, in general, a more difficult problem. For instance, $\underline{\Phi}_S\mathcal{M}$ is a complex of $\mathcal{D}$-modules, and is not necessarily concentrated in degree zero. This does not affect the formulas as long as we use derived categories, but things may become rather complicated when computing explicitly cohomology groups.

In the next sections we will study the transform $\underline{\Phi}_S$. We begin here with some easy corollaries of Proposition 2.6.

For $x \in X$, $y \in Y$, $A \subset X$ and $B \subset Y$, we set for short:

$$
\begin{aligned}
\widehat{x} &= g(f^{-1}(x)), \qquad & \widehat{A} = g(f^{-1}(A)), \\
\widehat{y} &= f(g^{-1}(y)), \qquad & \widehat{B} = f(g^{-1}(B)).
\end{aligned}
$$

Definition 2.7 (i) We say that a topological space A is globally cohomologically trivial (g.c.t. for short) if the natural morphism:

$$
\mathbb{C} \longrightarrow \mathrm{R}\Gamma(A; \mathbb{C}_A)
$$

is an isomorphism.

(ii) We say that a locally closed subset $A \subset X$ is S-trivial if $A \cap \widehat{y}$ is g.c.t. for every $y \in \widehat{A}$.

Notice that contractible spaces are g.c.t. Moreover, a C^0-manifold A is g.c.t. if and only if the natural morphism

$$
\mathrm{R}\Gamma_c(A; \omega_A) \longrightarrow \mathbb{C} \tag{2.22}
$$

is an isomorphism (see [17, Remark 3.3.10]).

Recall that one says that a morphism $f : S \longrightarrow X$ of real analytic manifolds is smooth at $s \in S$ if the tangent map $f'(s)$ is surjective, that f is an immersion if $f'(s)$ is injective, and that f is an embedding if it is both injective and an immersion.

In the following, we will make some of the hypotheses:

(2.23) f is smooth and g is proper,

(2.24) $(f,g): S \longrightarrow X \times Y$ is a closed embedding.

Lemma 2.8 *Assume (2.23) and (2.24).*

(i) Let $U \subset Y$ be a $\widetilde{S}$-trivial open subset. Then, there is a natural isomorphism:

$$\Phi_{\widetilde{S}}(\mathbb{C}_U) \simeq \mathbb{C}_{\widehat{U}}[-d_{S/X}].$$

(ii) Let $K \subset Y$ be a $\widetilde{S}$-trivial compact subset. Then, there is a natural isomorphism:

$$\Phi_{\widetilde{S}}(\mathbb{C}_K) \simeq \mathbb{C}_{\widehat{K}}[d_{S/X}]$$

Proof: (i) One has $\Phi_{\widetilde{S}}(\mathbb{C}_U) \simeq Rf_!\mathbb{C}_{g^{-1}(U)}[d_{S/X}]$. Setting $f_U = f|_{g^{-1}(U)}$, it remains to check that $Rf_{U!}\mathbb{C}_{g^{-1}(U)} \simeq \mathbb{C}_{\widehat{U}}[-2d_{S/X}]$. Since f_U is smooth, one has

$$\begin{aligned} Rf_{U!}\mathbb{C}_{g^{-1}(U)} &\simeq Rf_{U!}f_U^{-1}\mathbb{C}_{\widehat{U}} \\ &\simeq Rf_{U!}f_U^!\mathbb{C}_{\widehat{U}}[-2d_{S/X}]. \end{aligned}$$

By hypothesis (2.24), g induces an isomorphism from $g^{-1}(U) \cap f^{-1}(x)$ to $U \cap \widehat{x}$. Hence, $\widetilde{S}$-triviality of U implies that the fibers of f_U are g.c.t. Then, the natural morphism

$$Rf_{U!}f_U^!\mathbb{C}_{\widehat{U}} \longrightarrow \mathbb{C}_{\widehat{U}}$$

is an isomorphism by (2.22).

(ii) One has $\Phi_{\widetilde{S}}(\mathbb{C}_K) \simeq Rf_!\mathbb{C}_{g^{-1}(K)}[d_{S/X}]$. Setting $f_K = f|_{g^{-1}(K)}$, it remains to check that $Rf_{K!}\mathbb{C}_{g^{-1}(K)} \simeq \mathbb{C}_{\widehat{K}}$. One has

$$Rf_{K!}\mathbb{C}_{g^{-1}(K)} \simeq Rf_{K*}f_K^{-1}\mathbb{C}_{\widehat{K}}.$$

Moreover, the natural morphism $\mathbb{C}_{\widehat{K}} \longrightarrow Rf_{K*}f_K^{-1}\mathbb{C}_{\widehat{K}}$ is an isomorphism by the hypothesis that K is $\widetilde{S}$-trivial. □

By Proposition 2.6 and Lemma 2.8, we get the corollary below.

Corollary 2.9 *Assume (2.23) and (2.24). Let $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_X)$.*

(i) Let $U \subset Y$ be a $\widetilde{S}$-trivial open subset. Then, there are natural isomorphisms:

$$\begin{aligned} \mathrm{R}\Gamma_c(\widehat{U}; R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{O}_X)) &\simeq \mathrm{R}\Gamma_c(U; R\mathcal{H}om_{\mathcal{D}_Y}(\underline{\Phi}_S\mathcal{M},\mathcal{O}_Y))[-d_{S/Y}+2d_{S/X}], \\ \mathrm{R}\Gamma(\widehat{U}; R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{O}_X)) &\simeq \mathrm{R}\Gamma(U; R\mathcal{H}om_{\mathcal{D}_Y}(\underline{\Phi}_S\mathcal{M},\mathcal{O}_Y))[-d_{S/Y}]. \end{aligned}$$

(ii) *Let $K \subset Y$ be a $\widetilde{S}$-trivial compact subset. Then, there are natural isomorphisms:*

$$\begin{aligned} \mathrm{R}\Gamma(\widehat{K}; R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)) &\simeq \mathrm{R}\Gamma(K; R\mathcal{H}om_{\mathcal{D}_Y}(\underline{\Phi}_S\mathcal{M}, \mathcal{O}_Y))[-d_{S/Y}], \\ \mathrm{R}\Gamma_{\widehat{K}}(X; R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)) &\simeq \mathrm{R}\Gamma_K(Y; R\mathcal{H}om_{\mathcal{D}_Y}(\underline{\Phi}_S\mathcal{M}, \mathcal{O}_Y))[2d_{S/X} - d_{S/Y}]. \end{aligned}$$

(iii) *For $y \in Y$ we have the germ formula:*

$$\mathrm{R}\Gamma(\widehat{y}; R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)) \simeq R\mathcal{H}om_{\mathcal{D}_Y}(\underline{\Phi}_S\mathcal{M}, \mathcal{O}_Y)_y[-d_{S/Y}].$$

Notation 2.10 Let $\mathcal{F}$ be a locally free $\mathcal{O}_X$-module of finite rank (a complex vector bundle). We set:

$$\begin{aligned} \mathcal{F}^* &= \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X), \\ \mathcal{D}\mathcal{F}^* &= \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{F}^*. \end{aligned}$$

Notice that $\mathcal{D}\mathcal{F}^*$ is a locally free $\mathcal{D}_X$-module whose holomorphic solutions are given by:

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}\mathcal{F}^*, \mathcal{O}_X) \simeq \mathcal{F},$$

where the last isomorphism is $\mathbb{C}$-linear, but not $\mathcal{O}$-linear.

Replacing $\mathcal{M}$ by $\mathcal{D}\mathcal{F}^*$ in Corollary 2.9, and with the notations and hypotheses of this corollary, we get in particular the isomorphisms:

$$\mathrm{R}\Gamma(\widehat{U}; \mathcal{F}) \simeq \mathrm{R}\Gamma(U; R\mathcal{H}om_{\mathcal{D}_Y}(\underline{\Phi}_S(\mathcal{D}\mathcal{F}^*), \mathcal{O}_Y))[-d_{S/Y}], \tag{2.25}$$

$$\mathrm{R}\Gamma(\widehat{y}; \mathcal{F}) \simeq R\mathcal{H}om_{\mathcal{D}_Y}(\underline{\Phi}_S(\mathcal{D}\mathcal{F}^*), \mathcal{O}_Y)_y[-d_{S/Y}]. \tag{2.26}$$

Remark 2.11 There are many other interesting applications of the results in this section that we will not develop here. Among others, let us mention the following:

(i) Several authors have considered the formal completion of a locally free sheaf $\mathcal{F}$ along $\widehat{y}$, or more generally along a complex submanifold of X (see [21], [13], [8], [9]). One could also incorporate this point of view in our formalism, and, for example, prove the formula below, similar to (2.26):

$$\mathrm{R}\Gamma(\widehat{y}; \mathcal{F}\hat{|}\widehat{y}) \simeq R\mathcal{H}om_{\mathcal{D}_Y}(\underline{\Phi}_S(\mathcal{D}\mathcal{F}^*), \widehat{\mathcal{O}}_{Y,y})[-d_{S/Y}],$$

where $\mathcal{F}\hat{|}\widehat{y}$ is the formal completion of $\mathcal{F}$ along the submanifold $\widehat{y}$ of X, and $\widehat{\mathcal{O}}_{Y,y}$ the formal completion of $\mathcal{O}_Y$ at y.

(ii) In (2.17) we have given a duality formula that we shall not develop here. We refer to [6], in which the duality is used to treat the positive helicity case as potentials modulo gauges.

(iii) The results of [10] concerning the group $SO(8)$ could be reformulated in our language.

(iv) In our paper [4], we treat along these lines the classical projective duality, in which $Y = \mathbb{P}^n$, $X = (\mathbb{P}^n)^*$ and $S \subset X \times Y$ is given by the incidence relation $S = \{(x,\xi) \in \mathbb{P}^n \times (\mathbb{P}^n)^*;\ \langle x,\xi\rangle = 0\}$.

2.4 Kernels

In this section, we shall describe an equivalent construction of the transforms Φ_S, $\underline{\Phi}_S$ and their generalizations.

On a product space $X \times Y$, we denote by q_1 and q_2 the projections on X and Y respectively, and by $r : X \times Y \longrightarrow Y \times X$ the map $r(x,y) = (y,x)$.

Consider the correspondence (2.16). Assuming (2.24), we identify S to a closed submanifold of $X \times Y$, and we set $\widetilde{S} = r(S) \subset Y \times X$. We denote by $\mathcal{B}_{S|X\times Y}$ the holonomic $\mathcal{D}_{X\times Y}$-module associated to S:

$$\mathcal{B}_{S|X\times Y} = H^{d_X+d_Y-d_S}_{[S]}(\mathcal{O}_{X\times Y}),$$

and we also consider the associated $(\mathcal{D}_Y, \mathcal{D}_X)$-bimodule:

$$\mathcal{B}^{(d_X,0)}_{S|X\times Y} = q_1^{-1}\Omega_X \otimes_{q_1^{-1}\mathcal{O}_X} \mathcal{B}_{S|X\times Y}.$$

Proposition 2.12 *Assume (2.24). Then:*

(i) There is a natural isomorphism of $(\mathcal{D}_Y, \mathcal{D}_X)$-bimodules on S:

$$\mathcal{D}_{Y\leftarrow S} \otimes^L_{\mathcal{D}_S} \mathcal{D}_{S\to X} \xrightarrow{\sim} \mathcal{B}^{(d_X,0)}_{S|X\times Y}. \tag{2.27}$$

In particular $\mathcal{D}_{Y\leftarrow S} \otimes^L_{\mathcal{D}_S} \mathcal{D}_{S\to X}$ is concentrated in degree zero.

(ii) For $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}(\mathcal{D}_X)$, the following isomorphism holds:

$$\underline{\Phi}_S\mathcal{M} \simeq Rq_{2!}(\mathcal{B}^{(d_X,0)}_{S|X\times Y} \otimes^L_{q_1^{-1}\mathcal{D}_X} q_1^{-1}\mathcal{M}). \tag{2.28}$$

Proof: (i) Denote by Δ_f the graph of f in $S \times X$, and set $\widetilde{\Delta_f} = r(\Delta_f) \subset X \times S$. Then $\mathcal{D}_{S\to X} \simeq \mathcal{B}^{(d_X,0)}_{\widetilde{\Delta_f}|X\times S}$. Consider the diagram, where $\tilde{g} = id \times g$

$$\begin{tikzcd} X \times S \arrow[r, "\tilde{g}"] & X \times Y \\ \widetilde{\Delta_f} \arrow[u, "i"] \arrow[r, "\sim"] & S. \arrow[u] \end{tikzcd} \tag{2.29}$$

Then

$$\begin{aligned} \mathcal{D}_{Y\leftarrow S}\otimes^L_{\mathcal{D}_S}\mathcal{D}_{S\to X}\otimes_{\mathcal{O}_X}\Omega_X^{\otimes -1} &\simeq \mathcal{D}_{X\times Y\leftarrow X\times S}\otimes^L_{\mathcal{D}_{X\times S}}\mathcal{B}_{\widetilde{\Delta_f}|X\times S} \\ &\simeq \underline{\tilde{g}}_*\mathcal{B}_{\widetilde{\Delta_f}|X\times S} \\ &\xrightarrow{\sim} \mathcal{B}_{S|X\times Y}, \end{aligned}$$

where the last isomorphism comes from the fact that $\tilde{g}$ induces an isomorphism $\widetilde{\Delta_f}\simeq S$, and that $\mathcal{B}_{\widetilde{\Delta_f}|X\times S}=\underline{i}_*\mathcal{O}_{\widetilde{\Delta_f}}$.

(ii) is an immediate consequence of (i). □

According to Proposition 2.12, a natural generalization of the transform $\underline{\Phi}_S$ is obtained if one replaces $\mathcal{B}^{(d_X,0)}_{S|X\times Y}$ by $\mathcal{K}^{(d_X,0)}$ in formula (2.28), where $\mathcal{K}$ is a holonomic module on $X\times Y$. Set:

$$K=R\mathcal{H}om_{\mathcal{D}_{X\times Y}}(\mathcal{K},\mathcal{O}_{X\times Y}).$$

Definition 2.13 For $G\in\mathbf{D}^{\mathrm{b}}(Y)$ and $\mathcal{M}\in\mathbf{D}^{\mathrm{b}}(\mathcal{D}_X)$, set:

$$\begin{aligned} \Phi_K G &= Rq_{1!}(K\otimes q_2^{-1}G)[d_Y], \\ \underline{\Phi}_{\mathcal{K}}\mathcal{M} &= Rq_{2!}(\mathcal{K}^{(d_X,0)}\otimes^L_{q_1^{-1}\mathcal{D}_X}q_1^{-1}\mathcal{M}). \end{aligned}$$

Let $\Lambda=\mathrm{char}(\mathcal{K})$ be the characteristic variety of $\mathcal{K}$. In order to deal with these generalized transforms, one has to replace the assumptions (2.23) and (2.24) by the following:

$$\text{(2.30)}\qquad \begin{aligned} &\text{the projection } q_2:\pi(\Lambda)\longrightarrow Y \text{ is proper,} \\ &\text{and } \Lambda\cap(T^*X\times T^*_YY)\subset T^*_{X\times Y}(X\times Y). \end{aligned}$$

It is possible to state our results in this more general framework, but we will not develop this approach here.

3 Vanishing theorems

3.1 $\mathcal{E}$-modules

Let X be a complex manifold. We denote by $\pi:T^*X\longrightarrow X$ its cotangent bundle, by T^*_XX the zero section of T^*X, and we set:

$$\dot{T}^*X=T^*X\setminus T^*_XX.$$

If $M\subset X$ is a closed submanifold, we denote by T^*_MX its conormal bundle.

We refer to [23], [15] (see [24] for a detailed exposition) for the theory of modules over the ring $\mathcal{E}_X$ of finite order microdifferential operators on T^*X.

If $\mathcal{M}$ is a $\mathcal{D}_X$-module, we set:

$$\mathcal{E}\mathcal{M} = \mathcal{E}_X \otimes_{\pi^{-1}\mathcal{D}_X} \pi^{-1}\mathcal{M}.$$

Recall that $\mathcal{M} \simeq \mathcal{E}\mathcal{M}|_{T^*_X X}$, and that if $\mathcal{M}$ is coherent, its characteristic variety, denoted $\mathrm{char}(\mathcal{M})$, is the support of $\mathcal{E}\mathcal{M}$.

Let U be a subset of T^*X. We denote by $\mathrm{Mod}_{\mathrm{coh}}(\mathcal{E}_X|_U)$ the category of coherent $\mathcal{E}_X$-modules on U, by $\mathbf{D}^{\mathrm{b}}(\mathcal{E}_X|_U)$ the full triangulated subcategory of the derived category of $\mathcal{E}_X|_U$-modules whose objects have bounded cohomology, by $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{E}_X|_U)$ the full triangulated subcategory of $\mathbf{D}^{\mathrm{b}}(\mathcal{E}_X|_U)$ whose objects have coherent cohomology.

To $f : S \longrightarrow X$ one associates the maps

$$T^*S \underset{{}^tf'}{\longleftarrow} S \times_X T^*X \underset{f_\pi}{\longrightarrow} T^*X.$$

We will denote by $\underline{f}_{\mathcal{E}}^{-1}$ and $\underline{f}_*^{\mathcal{E}}$ the inverse and direct images in the sense of $\mathcal{E}$-modules. Hence, for $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}(\mathcal{E}_X)$ and $\mathcal{P} \in \mathbf{D}^{\mathrm{b}}(\mathcal{E}_S)$:

$$\underline{f}_{\mathcal{E}}^{-1}\mathcal{M} = R^t f'_*(\mathcal{E}_{S\to X} \otimes^L_{f_\pi^{-1}\mathcal{E}_X} f_\pi^{-1}\mathcal{M}), \quad \underline{f}_*^{\mathcal{E}}\mathcal{P} = Rf_{\pi *}(\mathcal{E}_{X\leftarrow S} \otimes^L_{{}^tf'^{-1}\mathcal{E}_S} {}^tf'^{-1}\mathcal{P}),$$

where $\mathcal{E}_{S\to X}$ and $\mathcal{E}_{X\leftarrow S}$ are the transfer bimodules.

Recall the correspondence (2.16), and assume (2.23), (2.24). Setting

$$\Lambda = T^*_S(X \times Y) \cap (\dot{T}^*X \times \dot{T}^*Y),$$

we consider the associated "microlocal correspondences":

(3.1)
$$\begin{tikzcd}[column sep=small] & T^*_S(X\times Y) \arrow[dl, "p_1|_{T^*_S(X\times Y)}"'] \arrow[dr, "p_2^a|_{T^*_S(X\times Y)}"] & \\ T^*X & & T^*Y, \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & \Lambda \arrow[dl, "p_1|_\Lambda"'] \arrow[dr, "p_2^a|_\Lambda"] & \\ \dot{T}^*X & & \dot{T}^*Y, \end{tikzcd}$$

where p_1 and p_2 denote the projections on $T^*(X\times Y) \simeq T^*X \times T^*Y$, and $p_2^a = a \circ p_2$, where a denotes the antipodal map. Note that $\Lambda = \dot{T}^*_S(X \times Y)$ if and only if f and g are smooth.

If A is a conic subset of T^*X, we set

(3.2)
$$\Phi^\mu_S(A) = p_2^a(T^*_S(X \times Y) \cap p_1^{-1}(A)).$$

We denote by $\mathcal{C}_{S|X\times Y}$ the holonomic $\mathcal{E}_{X\times Y}$-module associated to S:

$$\mathcal{C}_{S|X\times Y} = \mathcal{E}\mathcal{B}_{S|X\times Y},$$

and we consider the associated $(\mathcal{E}_Y, \mathcal{E}_X)$-bimodule:

(3.3)
$$\mathcal{C}^{(d_X,0)}_{S|X\times Y} = \pi^{-1}q_1^{-1}\Omega_X \otimes_{\pi^{-1}q_1^{-1}\mathcal{O}_X} \mathcal{C}_{S|X\times Y}.$$

Definition 3.1 We define the functors from $\mathbf{D}^{\mathrm{b}}(\mathcal{E}_X)$ to $\mathbf{D}^{\mathrm{b}}(\mathcal{E}_Y)$:

$$\underline{\Phi}^{\mu}_S(\mathcal{M}) = \underline{g}^{\mathcal{E}}_* \underline{f}^{-1}_{\mathcal{E}} \mathcal{M}, \quad \underline{\Psi}^{\mu}_S(\mathcal{M}) = \underline{\Phi}^{\mu}_S(\mathcal{M})[d_Y - d_X].$$

Hence, we get the functors from $\mathbf{D}^{\mathrm{b}}(\mathcal{E}_Y)$ to $\mathbf{D}^{\mathrm{b}}(\mathcal{E}_X)$:

$$\underline{\Phi}^{\mu}_{\widetilde{S}}(\mathcal{N}) = \underline{f}^{\mathcal{E}}_* \underline{g}^{-1}_{\mathcal{E}} \mathcal{N}, \quad \underline{\Psi}^{\mu}_{\widetilde{S}}(\mathcal{N}) = \underline{\Phi}^{\mu}_{\widetilde{S}}(\mathcal{N})[d_X - d_Y].$$

We identify $S \times_X T^*X$ to $T^*_{\widetilde{\Delta}_f}(X \times S)$ and $S \times_Y T^*Y$ to $T^*_{\Delta_g}(S \times Y)$, and we consider the diagram

$$\begin{tikzcd}
T^*(X \times S \times Y) \arrow[r, "p_{13}"] & T^*(X \times Y) \\
T^*_{\widetilde{\Delta}_f}(X \times S) \times_{T^*S} T^*_{\Delta_g}(S \times Y) \arrow[u] \arrow[r, "\sim"] & T^*_S(X \times Y), \arrow[u]
\end{tikzcd}$$

where p_{13} denotes as usual the natural projection defined on a product of three factors.

In the next proposition we shall write for example $\mathcal{E}_{Y \leftarrow S}$ instead of $p_{23}^{-1} \mathcal{E}_{Y \leftarrow S}$, for short.

Proposition 3.2 *Assume (2.24). Then:*

(i) There is a natural isomorphism of $(\mathcal{E}_Y, \mathcal{E}_X)$-bimodules on $T^*_S(X \times Y)$:

$$\mathcal{E}_{Y \leftarrow S} \otimes^L_{\mathcal{E}_S} \mathcal{E}_{S \to X} \xrightarrow{\sim} \mathcal{C}^{(d_X, 0)}_{S|X \times Y}.$$

(ii) For $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}(\mathcal{E}_X)$, the following isomorphism holds:

$$\underline{\Phi}^{\mu}_S(\mathcal{M}) \simeq Rp^a_{2*}(\mathcal{C}^{(d_X,0)}_{S|X \times Y} \otimes^L_{p_1^{-1}\mathcal{E}_X} p_1^{-1}\mathcal{M}).$$

Proof: (i) We shall follow the notations of diagram (2.29). We have the chain of isomorphisms:

$$\begin{aligned}
\mathcal{E}_{Y \leftarrow S} \otimes^L_{\mathcal{E}_S} \mathcal{E}_{S \to X} \otimes_{\pi^{-1}\mathcal{O}_X} \pi^{-1}\Omega_X^{\otimes -1} &\simeq \mathcal{E}_{X \times Y \leftarrow X \times S} \otimes^L_{\mathcal{E}_{X \times S}} \mathcal{C}_{\widetilde{\Delta_f}|X \times S} \\
&= \underline{\tilde{g}}^{\mathcal{E}}_* \mathcal{C}_{\widetilde{\Delta_f}|X \times S} \\
&\xrightarrow{\sim} \mathcal{C}_{S|X \times Y},
\end{aligned}$$

where we have used the isomorphism $\mathcal{E}_{S \to X} \simeq \mathcal{C}^{(d_X,0)}_{\widetilde{\Delta_f}|X \times S}$, and where the last isomorphism follows from the fact that $\tilde{g}$ induces an isomorphism between $\widetilde{\Delta_f}$ and S (see [23]).

(ii) follows from (i). $\square$

The next result will play a crucial role in the rest of the paper.

Proposition 3.3 *(i) Let $f : S \longrightarrow X$, and let $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X)$. Assume that f is non-characteristic for $\mathcal{M}$. Then*

$$\mathcal{E}\underline{f}^{-1}\mathcal{M} \simeq \underline{f}_{\mathcal{E}}^{-1}\mathcal{E}\mathcal{M}.$$

(ii) Let $g : S \longrightarrow Y$, and let $\mathcal{P} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_S)$. Assume that g is proper on $\operatorname{supp}\mathcal{P}$. Then

$$\mathcal{E}\underline{g}_*\mathcal{P} \simeq \underline{g}_*^{\mathcal{E}}\mathcal{E}\mathcal{P}.$$

Proof: (i) is proved in [23].

(ii) was obtained in [14] in the projective case, then extended to the general case in [12], [25]. □

Corollary 3.4 *Assume (2.23), (2.24), and let $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_X)$. Then*

$$\mathcal{E}(\underline{\Phi}_S\mathcal{M}) \simeq \underline{\Phi}_S^{\mu}(\mathcal{E}\mathcal{M}).$$

3.2 Vanishing theorems

In this section, we shall state some vanishing theorems for the cohomology of $\underline{\Phi}_S(\mathcal{M})$, $\mathcal{M}$ being a good $\mathcal{D}_X$-module, making the following hypothesis:

$$\text{the map } p_2^a|_\Lambda : \Lambda \longrightarrow \dot{T}^*Y \text{ is finite.} \tag{3.4}$$

Note that hypotheses (2.23), (2.24), and (3.4) imply that g is open.

Proposition 3.5 *Assume (2.23), (2.24), and (3.4).*

(i) Let $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_X)$. Then $\mathrm{char}(\underline{\Phi}_S(\mathcal{M})) \subset \Phi_S^{\mu}(\mathrm{char}(\mathcal{M}))$.

(ii) Assume (3.4), and $\mathcal{M} \in \mathrm{Mod}_{\mathrm{good}}(\mathcal{D}_X)$ (i.e. $\mathcal{M}$ is in degree 0). Then for $j \neq 0$, $H^j(\underline{\Phi}_S\mathcal{M})$ is a holomorphic vector bundle endowed with a flat connection.

Proof: (i) is an obvious consequence of classical results on the operations on $\mathcal{D}$-modules (see [15], [24]).

(ii) It is well known that a $\mathcal{D}_Y$-module whose characteristic variety is contained in the zero section is a locally free $\mathcal{O}_Y$-module of finite rank endowed with a flat connection. Hence, by Corollary 3.4 it is enough to prove that:

$$H^j(\underline{\Phi}_S^{\mu}\mathcal{E}\mathcal{M})|_{\dot{T}^*Y} = 0 \quad \text{for } j \neq 0.$$

This is clear since f being smooth $\underline{f}_{\mathcal{E}}^{-1}$ is exact, and $p_2^a|_\Lambda$ being finite $\underline{g}_*^{\mathcal{E}}$ is exact (see [23] or [24, Ch. II, Theorem 3.4.4]). □

Proposition 3.6 *Assume (2.23), (2.24), (3.4), and that Y is connected. Let $\mathcal{F}$ be a holomorphic vector bundle on X, and recall that we set $\mathcal{DF} = \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{F}$. Then:*

(i) $H^j(\underline{\Phi}_S \mathcal{DF}) = 0$ *for* $j < 0$,

(ii) $\underline{\Phi}_S \mathcal{DF}$ *is concentrated in degree zero if and only if there exists* $y \in Y$ *such that* $H^j(\widehat{y}; \mathcal{F}^*) = 0$ *for every* $j < d_{S/Y}$,

(iii) we have the isomorphism:

$$H^j(R\mathcal{H}om_{\mathcal{D}_Y}(\underline{\Phi}_S \mathcal{DF}, \mathcal{O}_Y)) \simeq \mathcal{H}om_{\mathcal{D}_Y}(H^{-j}(\underline{\Phi}_S \mathcal{DF}), \mathcal{O}_Y) \quad \text{for } j \leq 0,$$

(iv) if g is smooth, we have:

$$H^j(R\mathcal{H}om_{\mathcal{D}_Y}(\underline{\Phi}_S \mathcal{DF}, \mathcal{O}_Y)) = 0 \quad \text{for } j > 0.$$

Proof: (i) follows from Proposition 2.12 (ii), since $\mathcal{DF}$ is flat over $\mathcal{D}_X$.

(ii) Recall that Y is connected, and let $y \in Y$. Set $\mathcal{N} = \underline{\Phi}_S \mathcal{DF}$ for short, and consider the distinguished triangle

$$H^0(\mathcal{N}) \longrightarrow \mathcal{N} \longrightarrow \tau^{>0}\mathcal{N} \underset{+1}{\longrightarrow},$$

which gives rise to the distinguished triangle

$$\mathcal{S}ol(\tau^{>0}\mathcal{N}) \longrightarrow \mathcal{S}ol(\mathcal{N}) \longrightarrow \mathcal{S}ol(H^0(\mathcal{N})) \underset{+1}{\longrightarrow} \tag{3.5}$$

(where, as usual, $\mathcal{S}ol(\mathcal{N}) = R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{N}, \mathcal{O}_Y)$). Notice that since $\operatorname{char}(\tau^{>0}\mathcal{N}) \subset T^*_Y Y$, one has

$$H^j \mathcal{S}ol(\tau^{>0}\mathcal{N}) = 0 \quad \forall j \geq 0, \tag{3.6}$$

and

$$\begin{aligned} \tau^{>0}\mathcal{N} = 0 \quad &\Leftrightarrow \quad \mathcal{S}ol(\tau^{>0}\mathcal{N}) = 0 \\ &\Leftrightarrow \quad H^j \mathcal{S}ol(\tau^{>0}\mathcal{N}) = 0 \quad \forall j < 0 \\ &\Leftrightarrow \quad H^j \mathcal{S}ol(\tau^{>0}\mathcal{N})_y = 0 \quad \forall j < 0 \\ &\Leftrightarrow \quad H^j \mathcal{S}ol(\mathcal{N})_y = 0 \quad \forall j < 0, \end{aligned}$$

where the last equivalence comes from the distinguished triangle (3.5), and the fact that $H^j \mathcal{S}ol(H^0(\mathcal{N})) = 0$ for $j < 0$. To conclude, it remains to apply the germ formula

$$H^j \mathcal{S}ol(\mathcal{N})_y \simeq H^{d_{S/Y}+j}(\widehat{y}; \mathcal{F}^*).$$

(iii) For $j < 0$ we have the sequence of isomorphisms:

$$\begin{aligned} H^j \mathcal{S}ol(\mathcal{N}) &\simeq H^j \mathcal{S}ol(\tau^{>0}\mathcal{N}) \\ &\simeq \mathcal{S}ol(H^{-j}(\tau^{>0}\mathcal{N})) \\ &\simeq \mathcal{S}ol(H^{-j}(\mathcal{N})). \end{aligned}$$

For $j = 0$ the result follows from (3.5), (3.6).

(iv) One has $H^j(R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{N}, \mathcal{O}_Y))_y \simeq H^{d_{S/Y}+j}(\widehat{y}; \mathcal{F}^*)$ and this group is zero since $\widehat{y} \simeq g^{-1}(y)$ is a compact smooth submanifold of X of dimension $d_{S/Y}$. In fact, if Z is a compact smooth submanifold of dimension d of the complex manifold X, and $\mathcal{F}$ is a locally free $\mathcal{O}_X$-module of finite rank on X, the vanishing of $H^{d+j}(Z; \mathcal{F})$ for $j > 0$ follows by duality from the fact that $H^k_Z(X; \mathcal{F}^* \otimes_{\mathcal{O}_Y} \Omega_Y)$ is zero for $k < d_X - d$, and for $k = d_X - d$ this space is isomorphic to $\Gamma(X; \mathcal{H}^{d_X-d}_Z(\mathcal{F}^* \otimes_{\mathcal{O}_Y} \Omega_Y))$, hence has a natural topology of a (separated) Fréchet space. □

4 A regularity theorem and an equivalence of categories

4.1 Modules with regular singularities

We review some notions and results from Kashiwara and Oshima's work [16].

The ring $\mathcal{E}_X$ is naturally endowed with a $\mathbb{Z}$-filtration by the degree, and we denote by $\mathcal{E}_X(k)$ the sheaf of operators of degree at most k. Denote by $\mathcal{O}_{T^*X}(k)$ the sheaf of holomorphic functions on T^*X, homogeneous of degree k.

Let $V \subset \dot{T}^*X$ be a conic regular involutive submanifold of codimension c_V. Recall that one says that a smooth conic involutive manifold V is regular if the canonical one-form on T^*X does not vanish on V. Denote by $\mathcal{I}_V(k)$ the sheaf ideal of sections of $\mathcal{O}_{T^*X}(k)$ vanishing on V. Let $\mathcal{E}_V$ be the subalgebra of $\mathcal{E}_X$ generated over $\mathcal{E}_X(0)$ by the sections P of $\mathcal{E}_X(1)$ such that $\sigma_1(P)$ belongs to $\mathcal{I}_V(1)$ (here $\sigma_1(\cdot)$ denotes the symbol of order 1).

Example 4.1 Let $X = W \times Z$, $V = U_W \times T^*_Z Z$ for an open subset $U_W \subset \dot{T}^*W$. Then $\mathcal{E}_V$ is the subalgebra $\mathcal{D}_Z\mathcal{E}_X(0)$ of $\mathcal{E}_X$ generated over $\mathcal{E}_X(0)$ by the differential operators of Z.

Definition 4.2 (cf [16]). Let $\mathcal{M}$ be a coherent $\mathcal{E}_X$-module. One says that $\mathcal{M}$ has *regular singularities along* V if locally there exists a coherent sub-$\mathcal{E}_X(0)$-module $\mathcal{M}_0$ of $\mathcal{M}$ which generates it over $\mathcal{E}_X$, and such that $\mathcal{E}_V\mathcal{M}_0 \subset \mathcal{M}_0$. One says that $\mathcal{M}$ is *simple along* V if locally there exists an $\mathcal{E}_X(0)$-module $\mathcal{M}_0$ as above such that $\mathcal{M}_0/\mathcal{E}_X(-1)\mathcal{M}_0$ is a locally free $\mathcal{O}_V(0)$-module of rank one.

Notice that the above definitions are invariant by quantized contact transformations, and that a system with regular singularities along V is supported by V (cf. [16, Lemma 1.13]).

We will denote by $\mathrm{Mod}_{RS(V)}(\mathcal{E}_X)$ the thick abelian subcategory of $\mathrm{Mod}_{\mathrm{coh}}(\mathcal{E}_X)$ whose objects have regular singularities along V. We denote by $\mathbf{D}^{\mathrm{b}}_{RS(V)}(\mathcal{E}_X)$ the full triangulated subcategory of $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{E}_X)$ whose objects $\mathcal{M}$ have cohomology groups with regular singularities along V. This category is invariant by quantized contact transformations.

Example 4.3 Let $V \subset \dot{T}^*X$ be a regular involutive submanifold, and let $\mathcal{S}_V$ be simple along V. We may locally assume, after a quantized contact transformation, that $X = W \times Z$, $V = U_W \times T^*_Z Z$. In this case $\mathcal{S}_V$ is isomorphic to the partial de Rham system $\mathcal{E}_W \boxtimes \mathcal{O}_Z$ (cf. [16, Theorem 1.9]). Denoting by $\rho : V \longrightarrow U_W$ the natural projection, we notice that $\mathcal{E}nd_{\mathcal{E}_X}(\mathcal{S}_V) \simeq \rho^{-1}\mathcal{E}_W$, which shows in particular that the ring $\mathcal{E}nd_{\mathcal{E}_X}(\mathcal{S}_V)$ of $\mathcal{E}_X$-linear endomorphisms of $\mathcal{S}_V$ is coherent.

There are useful criterion to ensure that an $\mathcal{E}_X$-module $\mathcal{M}$ has regular singularities on V.

Proposition 4.4 *Let $V \subset \dot{T}^*X$ be a regular involutive submanifold, let $\mathcal{M}$ be a coherent $\mathcal{E}_X$-module, and let $\mathcal{S}_V$ be simple along V. Then $\mathcal{M}$ has regular singularities along V if and only if for any $d > 0$ there locally exists an exact sequence of $\mathcal{E}_X$-modules:*

$$\mathcal{S}_V^{N_d} \longrightarrow \cdots \longrightarrow \mathcal{S}_V^{N_0} \longrightarrow \mathcal{M} \longrightarrow 0. \tag{4.1}$$

Proof: We may assume that $X = W \times Z$, with $\dim Z = d_Z = c_V$, and $V = U_W \times T^*_Z Z$, U_W being open in $\dot{T}^*W$.

Assume that $\mathcal{M}$ has regular singularities. Let $\mathcal{M}_0$ be a coherent $\mathcal{E}_X(0)$-module which generates $\mathcal{M}$, such that $\mathcal{E}_V\mathcal{M}_0 \subset \mathcal{M}_0$, and let $u_1, \ldots, u_r$ be a system of generators of $\mathcal{M}_0$. Let $(y_1, \ldots, y_{d_Z})$ be a local coordinate system on Z. Then there exist $r \times r$ matrices A_j, $j = 1, \ldots, c_V$, with entries in $\mathcal{E}_X(0)$, such that:

$$D_{y_j}\begin{pmatrix} u_1 \\ \vdots \\ u_r \end{pmatrix} = A_j \begin{pmatrix} u_1 \\ \vdots \\ u_r \end{pmatrix} \tag{4.2}$$

Denote by $\mathcal{M}'$ the $\mathcal{E}_X$-module with generators $v_1, \ldots, v_r$, and relations (4.2). Then the map $v_j \mapsto u_j$ defines the $\mathcal{E}_X$-linear exact sequence:

$$\mathcal{M}' \underset{\psi}{\longrightarrow} \mathcal{M} \longrightarrow 0.$$

Set $\mathcal{M}'_0 = \mathcal{E}_X(0)v_1 + \cdots + \mathcal{E}_X(0)v_r$. By the above relations, $\mathcal{M}'_0/\mathcal{E}_X(-1)\mathcal{M}'_0$ is locally free over $\mathcal{O}_V(0)$. Hence $\mathcal{M}'$ is locally isomorphic to $\mathcal{S}_V^{N_0}$ for some N_0 (see [16, Theorem 1.9]). Let $\mathcal{M}_1$ be the kernel of ψ. Then $\mathcal{M}_1$ has regular singularities on V, and the induction proceeds.

Conversely, assume (4.1). Then the fact that $\mathcal{M}$ has regular singularities is a consequence of the fact that $\mathrm{Mod}_{RS(V)}(\mathcal{E}_X)$ is a thick subcategory of $\mathrm{Mod}_{\mathrm{coh}}(\mathcal{E}_X)$. $\square$

Remark 4.5 Let us denote by $\mathrm{Car}^1_V(\mathcal{M})$ the 1-micro-characteristic variety of a coherent $\mathcal{E}_X$-module, introduced by Y. Laurent [18] and T. Monteiro-Fernandes [20] (see also [24, p. 123]). Then one can show that $\mathcal{M}$ has regular singularities along V if and only if $\mathrm{Car}^1_V(\mathcal{M}) \subset V$, the zero section of $T_V T^*X$.

Proposition 4.6 *Let $V \subset \dot{T}^*X$ be a regular involutive submanifold, and let $\mathcal{S}_V$ be simple along V. Set $\mathcal{A}_V = \mathcal{E}nd_{\mathcal{E}_X}(\mathcal{S}_V)$. Then the two functors:*

$$\mathrm{Mod}_{RS(V)}(\mathcal{E}_X) \underset{\alpha}{\overset{\beta}{\rightleftarrows}} \mathrm{Mod}_{\mathrm{coh}}(\mathcal{A}_V)$$

given by:

$$\alpha(\mathcal{M}) = \mathcal{H}om_{\mathcal{E}_X}(\mathcal{S}_V, \mathcal{M}), \quad \beta(\mathcal{R}) = \mathcal{S}_V \otimes_{\mathcal{A}_V} \mathcal{R},$$

are well-defined and quasi-inverse to each other. Moreover, $\mathcal{M} \in \mathrm{Mod}_{RS(V)}(\mathcal{E}_X)$ is simple along V if and only if $\alpha(\mathcal{M})$ is a locally free $\mathcal{A}_V$-module of rank one.

Proof: (i) Let us show that α is well-defined. First, notice that for any coherent $\mathcal{E}_X$-module $\mathcal{M}$:

$$\mathcal{E}xt^j_{\mathcal{E}_X}(\mathcal{S}_V, \mathcal{M}) = 0, \quad \text{for } j > c_V.$$

For $\mathcal{M}$ in $\mathrm{Mod}_{RS(V)}(\mathcal{E}_X)$, consider an $\mathcal{E}_X$-linear exact sequence:

$$(4.3) \qquad 0 \longrightarrow \mathcal{Z}^d \longrightarrow \mathcal{S}_V^{N_d} \longrightarrow \cdots \longrightarrow \mathcal{S}_V^{N_0} \longrightarrow \mathcal{M} \longrightarrow 0.$$

Arguing by induction, we find that $\mathcal{Z}^d$ has regular singularities on V. Moreover, by standard arguments, we get that

$$\mathcal{E}xt^j_{\mathcal{E}_X}(\mathcal{S}_V, \mathcal{M}) = 0, \quad \text{for } j > 0.$$

Hence $\mathcal{H}om_{\mathcal{E}_X}(\mathcal{S}_V, \cdot)$ is exact on $\mathrm{Mod}_{RS(V)}(\mathcal{E}_X)$. Applying $\mathcal{H}om_{\mathcal{E}_X}(\mathcal{S}_V, \cdot)$ to (4.3), we thus get an $\mathcal{A}_V$-linear resolution:

$$\mathcal{A}_V^{N_1} \longrightarrow \mathcal{A}_V^{N_0} \longrightarrow \alpha(\mathcal{M}) \longrightarrow 0.$$

(ii) β is well defined. In fact, it is enough to check it for $\mathcal{R} = \mathcal{A}_V$, that is, to check that $\mathcal{S}_V$ has regular singularities along V, which is clear.

(iii) $id \xrightarrow{\sim} \alpha\circ\beta$. In fact, let $\mathcal{R}\in \mathrm{Mod}_{\mathrm{coh}}(\mathcal{A}_V)$. Then:

$$\begin{aligned}\mathcal{R} &\xrightarrow{\sim} \mathcal{H}om_{\mathcal{E}_X}(\mathcal{S}_V,\mathcal{S}_V)\otimes_{\mathcal{A}_V}\mathcal{R}\\ &\xrightarrow{\sim} \mathcal{H}om_{\mathcal{E}_X}(\mathcal{S}_V,\mathcal{S}_V\otimes_{\mathcal{A}_V}\mathcal{R})\\ &= \alpha\circ\beta(\mathcal{R}).\end{aligned}$$

(iv) $\beta\circ\alpha \xrightarrow{\sim} id$. In fact, let $\mathcal{M}\in\mathrm{Mod}_{RS(V)}(\mathcal{E}_X)$. To check that the natural morphism:

$$\mathcal{S}_V\otimes_{\mathcal{A}_V}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{S}_V,\mathcal{M})\longrightarrow\mathcal{M} \tag{4.4}$$

is an isomorphism, we may proceed locally and use Proposition 4.4. Consider a resolution:

$$\mathcal{S}_V^{N_1}\longrightarrow\mathcal{S}_V^{N_0}\longrightarrow\mathcal{M}\longrightarrow 0. \tag{4.5}$$

We have already noticed that the sequence:

$$\alpha(\mathcal{S}_V^{N_1})\longrightarrow\alpha(\mathcal{S}_V^{N_0})\longrightarrow\alpha(\mathcal{M})\longrightarrow 0$$

remains exact. The functor β being right exact, the sequence:

$$\beta(\alpha(\mathcal{S}_V^{N_1}))\longrightarrow\beta(\alpha(\mathcal{S}_V^{N_0}))\longrightarrow\beta(\alpha(\mathcal{M}))\longrightarrow 0$$

is exact. Since $\beta(\alpha(\mathcal{S}_V))\xrightarrow{\sim}\mathcal{S}_V$ this last sequence is but the exact sequence of $\mathcal{E}_X$-modules:

$$\mathcal{S}_V^{N_1}\longrightarrow\mathcal{S}_V^{N_0}\longrightarrow\beta(\alpha(\mathcal{M}))\longrightarrow 0. \tag{4.6}$$

Comparing (4.5) and (4.6) we get (4.4) by the five lemma. □

4.2 A regularity theorem

Let $V\subset\dot{T}^*X$ be a conic regular involutive submanifold. We say that a coherent $\mathcal{D}_X$-module $\mathcal{M}$ has regular singularities on V, if so has $\mathcal{E}\mathcal{M}$. We denote by $\mathrm{Mod}_{RS(V)}(\mathcal{D}_X)$ the thick subcategory of $\mathrm{Mod}_{\mathrm{good}}(\mathcal{D}_X)$ whose objects have regular singularities on V, and by $\mathbf{D}^{\mathrm{b}}_{RS(V)}(\mathcal{D}_X)$ the full triangulated subcategory of $\mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_X)$ whose objects have cohomology groups belonging to $\mathrm{Mod}_{RS(V)}(\mathcal{D}_X)$.

Recall the correspondence (2.16), and the associated microlocal correspondence (3.1):

$$\begin{tikzcd} & \Lambda \arrow[dl, "p_1|_\Lambda"'] \arrow[dr, "p_2^a|_\Lambda"] & \\ \dot{T}^*X & & \dot{T}^*Y. \end{tikzcd}$$

The manifold Λ being Lagrangian, it is well known that $p_1|_\Lambda$ is smooth if and only if $p_2^a|_\Lambda$ is an immersion. We will assume:

$$(4.7)\qquad \begin{cases} p_2^a|_\Lambda \text{ is a closed embedding identifying } \Lambda \text{ to a} \\ \text{closed regular involutive submanifold } V \subset \dot{T}^*Y, \\ \text{and } p_1|_\Lambda \text{ is smooth and surjective on } \dot{T}^*X. \end{cases}$$

Let us denote by c_V the complex codimension of V in T^*Y. We have the following local model for the correspondence (3.1).

Lemma 4.7 *Assume (2.24), (4.7). Then, for every $(p, q^a) \in \Lambda$ there exist open subsets $U_X, U'_X \subset \dot{T}^*X$, $U_Y \subset \dot{T}^*Y$, with $p \in U_X$ and $q \in U_Y$, a complex manifold Z of dimension c_V, and a contact transformation $\psi : U_Y \xrightarrow{\sim} U'_X \times T^*Z$, such that $id_{U_X} \times \psi$ induces an isomorphism of correspondences:*

$$\begin{tikzcd}[column sep=small] & \Lambda \cap (U_X \times U_Y^a) \arrow[dl, "p_1"'] \arrow[dr, "p_2^a"] & & \xrightarrow{\sim} & & \Lambda_\chi \times T_Z^*Z \arrow[dl, "p_1"'] \arrow[dr, "p_{23}^a"] & \\ U_X & & V \cap U_Y & & U_X & & U'_X \times T_Z^*Z, \end{tikzcd}$$

where $\Lambda_\chi \subset U_X \times {U'}_X^a$ is the graph of a contact transformation $\chi : U_X \xrightarrow{\sim} U'_X$, and p_{23}^a denotes the projection $U_X \times {U'}_X^a \times T^*Z \longrightarrow U'_X \times T^*Z$.

Proof: Since V is regular involutive, there exist complex manifolds X' and Z of dimension $d_Y - c_V$ and c_V respectively, open subsets $U'_X \subset \dot{T}^*X'$, $U_Y \subset \dot{T}^*Y$ with $p \in U_Y$, and a contact transformation $\psi : U_Y \xrightarrow{\sim} U'_X \times T^*Z$ such that

$$\psi(V \cap U_Y) = U'_X \times T_Z^*Z.$$

By hypothesis (4.7), one has $\Lambda \subset \dot{T}^*X \times V$. In particular, Λ is invariant by the bicharacteristic flow of $\dot{T}^*X \times V$, and hence $id_{\dot{T}^*X} \times \psi$ interchanges $\Lambda \cap (\dot{T}^*X \times U_Y)$ with $\Lambda' \times T_Z^*Z$, for a Lagrangian manifold Λ' of $\dot{T}^*X \times U'_X$. Since $p_2^a|_\Lambda$ gives an isomorphism $\Lambda \xrightarrow{\sim} V$, we also have $\Lambda' \xrightarrow{\sim} U'_X$. The manifold Λ' being Lagrangian, there exists an open subset $U_X \subset \dot{T}^*X$ such that $\Lambda' = \Lambda_\chi$ is the graph of a contact transformation $\chi : U_X \xrightarrow{\sim} U'_X$. In particular, $d_Y - c_V = d_X$, and it is not restrictive to assume that $X' = X$. □

Theorem 4.8 *Assume (2.23), (2.24) and (4.7).*

(i) If $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_X)$, then $\underline{\Phi}_S(\mathcal{M})$ belongs to $\mathbf{D}^{\mathrm{b}}_{RS(V)}(\mathcal{D}_Y)$.

(ii) If $\mathcal{M} \in \mathrm{Mod}_{\mathrm{good}}(\mathcal{D}_X)$, and $\mathcal{M}$ is locally free of rank one, then $H^0(\underline{\Phi}_S(\mathcal{M}))$ is simple along V on $\dot{T}^*Y$.

Proof: By (4.7), the problem is local on Λ. Hence by "dévissage", we may assume $\mathcal{M} = \mathcal{D}_X$. Since the transform $\underline{\Phi}_S$ "commutes to microlocalization" (Corollary 3.4), it is enough to show that $p^a_{2*}\mathcal{C}^{(d_X,0)}_{S|X\times Y}$ has regular singularities on V. This statement is invariant by quantized contact transformations on Y, and we may apply Lemma 4.7. Hence, it is enough to show that if $\mathcal{S}$ is a simple holonomic $\mathcal{E}_{X\times Y}$-module on $\Lambda = \Lambda_\chi \times T^*_Z Z$, where $\Lambda_\chi \subset U_X \times U'^a_X$ is the graph of a contact transformation, then $p^a_{23*}\mathcal{S}^{(d_X,0)}$ is a simple system on $U'_X \times T^*_Z Z$. This is obvious, since locally $\mathcal{S}^{(d_X,0)} \simeq \mathcal{S}^{(d_X,0)}_\chi \underline{\boxtimes} \mathcal{O}_Z$, $\mathcal{S}_\chi$ being a simple holonomic system on Λ_χ, and $\mathcal{S}^{(d_X,0)}_\chi$ is isomorphic (as an $\mathcal{E}_X$-module) to $\mathcal{E}_X$. □

4.3 An equivalence of categories

In this paragraph we shall prove that the transform $\underline{\Phi}_S$ "almost" induces an equivalence between the category of coherent $\mathcal{D}$-modules on X, and the category of coherent $\mathcal{D}$-modules on Y with regular singularities on V.

Here, we will make the assumptions below:

(4.8) $\quad f$ and g are smooth and proper,

(4.9) $\quad (f,g) : S \hookrightarrow X \times Y$ is a closed embedding,

(4.10) $\quad f$ has connected and simply connected fibers,

$$(4.11)\qquad \begin{cases} p^a_2|_\Lambda \text{ is a closed embedding identifying } \Lambda \text{ to a} \\ \text{closed regular involutive submanifold } V \subset \dot{T}^*Y, \\ \text{and } p_1|_\Lambda \text{ is smooth and surjective on } \dot{T}^*X. \end{cases}$$

We define the category $\mathrm{Mod}_{\mathrm{good}}(\mathcal{D}_X;\dot{T}^*X)$ as the localization of $\mathrm{Mod}_{\mathrm{good}}(\mathcal{D}_X)$ by the thick subcategory of holomorphic vector bundles endowed with a flat connection:

$$N_X = \{\mathcal{M} \in \mathrm{Mod}_{\mathrm{good}}(\mathcal{D}_X);\ \mathrm{char}(\mathcal{M}) \subset T^*_X X\}.$$

In particular, the objects of $\mathrm{Mod}_{\mathrm{good}}(\mathcal{D}_X;\dot{T}^*X)$ are the same as the objects of $\mathrm{Mod}_{\mathrm{good}}(\mathcal{D}_X)$, and a morphism $u : \mathcal{M} \longrightarrow \mathcal{M}'$ in $\mathrm{Mod}_{\mathrm{good}}(\mathcal{D}_X)$ becomes an isomorphism in $\mathrm{Mod}_{\mathrm{good}}(\mathcal{D}_X;\dot{T}^*X)$ if $\ker u$ and $\mathrm{coker}\, u$ belong to N_X. This is equivalent to say that $\mathcal{E}u : \mathcal{E}\mathcal{M} \longrightarrow \mathcal{E}\mathcal{M}'$ is an isomorphism on $\dot{T}^*X$. We similarly define N_Y, and the category $\mathrm{Mod}_{RS(V)}(\mathcal{D}_Y;\dot{T}^*Y)$ obtained by localizing $\mathrm{Mod}_{RS(V)}(\mathcal{D}_Y)$ with respect to N_Y.

Taking the zero-th cohomology groups of the functors $\underline{\Phi}_S$ and $\underline{\Psi}_{\widetilde{S}}$, we get functors that we will denote by $\underline{\Phi}^0_S$ and $\underline{\Psi}^0_{\widetilde{S}}$. In other words, we set:

$$\underline{\Phi}^0_S = H^0 \circ \underline{\Phi}_S, \quad \underline{\Psi}^0_{\widetilde{S}} = H^0 \circ \underline{\Psi}_{\widetilde{S}}.$$

Hence (using Theorem 4.8) we get the functors:

$$\mathrm{Mod}_{\mathrm{good}}(\mathcal{D}_X) \underset{\underline{\Phi}^0_S}{\overset{\underline{\Psi}^0_{\widetilde{S}}}{\rightleftarrows}} \mathrm{Mod}_{RS(V)}(\mathcal{D}_Y).$$

These functors interchange N_X and N_Y, and hence induce the functors, that we denote by $\underline{\Phi}^{0\cdot}_S$ and $\underline{\Psi}^{0\cdot}_{\widetilde{S}}$:

$$\mathrm{Mod}_{\mathrm{good}}(\mathcal{D}_X;\dot{T}^*X) \underset{\underline{\Phi}^{0\cdot}_S}{\overset{\underline{\Psi}^{0\cdot}_{\widetilde{S}}}{\rightleftarrows}} \mathrm{Mod}_{RS(V)}(\mathcal{D}_Y;\dot{T}^*Y). \tag{4.12}$$

Theorem 4.9 *Assume (4.8)–(4.11). Then the functors $\underline{\Phi}^{0\cdot}_S$ and $\underline{\Psi}^{0\cdot}_{\widetilde{S}}$ are quasi-inverse to each other, hence define an equivalence of categories.*

Recall that by Proposition 2.6 the functors $\underline{\Phi}_S$ and $\underline{\Psi}_{\widetilde{S}}$:

$$\mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_X) \underset{\underline{\Phi}_S}{\overset{\underline{\Psi}_{\widetilde{S}}}{\rightleftarrows}} \mathbf{D}^{\mathrm{b}}_{RS(V)}(\mathcal{D}_Y),$$

are adjoint to each other. Hence we have natural morphisms:

$$id \longrightarrow \underline{\Psi}_{\widetilde{S}} \circ \underline{\Phi}_S \quad \text{in } \mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_X), \tag{4.13}$$

and:

$$\underline{\Phi}_S \circ \underline{\Psi}_{\widetilde{S}} \longrightarrow id \quad \text{in } \mathbf{D}^{\mathrm{b}}_{RS(V)}(\mathcal{D}_Y). \tag{4.14}$$

Let $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_X)$, $\mathcal{N} \in \mathbf{D}^{\mathrm{b}}_{RS(V)}(\mathcal{D}_Y)$. By Corollary 3.4, we get morphisms:

$$\mathcal{E}\mathcal{M} \longrightarrow \underline{\Psi}^{\mu}_{\widetilde{S}}(\underline{\Phi}^{\mu}_S(\mathcal{E}\mathcal{M})) \quad \text{in } \mathbf{D}^{\mathrm{b}}(\mathcal{E}_X|_{\dot{T}^*X}), \tag{4.15}$$

$$\underline{\Phi}^{\mu}_S(\underline{\Psi}^{\mu}_{\widetilde{S}}(\mathcal{E}\mathcal{N})) \longrightarrow \mathcal{E}\mathcal{N} \quad \text{in } \mathbf{D}^{\mathrm{b}}(\mathcal{E}_Y|_{\dot{T}^*Y}). \tag{4.16}$$

Recall that

$$\begin{aligned}
\underline{\Phi}^{\mu}_S(\mathcal{M}) &\simeq Rp^a_{2*}(\mathcal{C}^{(d_X,0)}_{S|X\times Y} \otimes^L_{p_1^{-1}\mathcal{E}_X} p_1^{-1}\mathcal{M}) \quad \text{for } \mathcal{M} \in \mathbf{D}^{\mathrm{b}}(\mathcal{E}_X|_{\dot{T}^*X}),\\
\underline{\Psi}^{\mu}_{\widetilde{S}}(\mathcal{N}) &\simeq Rp_{1*}(\mathcal{C}^{(d_Y,0)}_{\widetilde{S}|Y\times X} \otimes^L_{p_2^{a-1}\mathcal{E}_Y} p_2^{a-1}\mathcal{N})[d_X-d_Y] \quad \text{for } \mathcal{N} \in \mathbf{D}^{\mathrm{b}}(\mathcal{E}_Y|_{\dot{T}^*Y}),
\end{aligned}$$

where $\mathcal{C}^{(d_X,0)}_{S|X\times Y}$ is the module defined in (3.3).

In order to prove Theorem 4.9, we will need a few lemmas.

Lemma 4.10 *On $\Lambda = \dot{T}^*_S(X\times Y)$, the module $\mathcal{C}^{(d_X,0)}_{S|X\times Y}$ is flat over $p_1^{-1}\mathcal{E}_X$, and has $\mathcal{T}or$-dimension $d_Y - d_X$ over $p_2^{a-1}\mathcal{E}_Y$. In particular, the functor $\underline{\Psi}^{\mu}_{\widetilde{S}}$ is left exact for the natural t-structures of $\mathbf{D}^{\mathrm{b}}(\mathcal{E}_Y|_{\dot{T}^*Y})$ and $\mathbf{D}^{\mathrm{b}}(\mathcal{E}_X|_{\dot{T}^*X})$.*

Proof: The problem is local on Λ. Applying Lemma 4.7, we may assume $Y = X\times Z$, $\Lambda = \Lambda_\chi \times T^*_Z Z$, $\mathcal{C}_{S|X\times Y} \simeq \mathcal{C}_{\Lambda_\chi} \boxtimes \mathcal{O}_Z$, where $\mathcal{C}_{\Lambda_\chi}$ is a simple $\mathcal{E}_{X\times X}$-module on Λ_χ. The result is then clear, since $d_Z = d_Y - d_X$. □

Lemma 4.11 *The functor:*

$$\underline{\Phi}_S^\mu : \mathrm{Mod}_{\mathrm{coh}}(\mathcal{E}_X|_{\dot{T}^*X}) \longrightarrow \mathrm{Mod}_{\mathrm{coh}}(\mathcal{E}_Y|_{\dot{T}^*Y}),$$

is well defined and exact.

Proof: We have:

$$\underline{\Phi}_S^\mu(\cdot) = Rp^a_{2*}(\mathcal{C}^{(d_X,0)}_{S|X\times Y} \otimes^L_{p_1^{-1}\mathcal{E}_X} p_1^{-1}(\cdot)).$$

Since the map p_2^a is finite on $\dot{T}^*_S(X\times Y)$, and $\mathcal{C}^{(d_X,0)}_{S|X\times Y}$ is flat over $p_1^{-1}\mathcal{E}_X$, the statement follows, using Proposition 3.2. □

We set:

$$\underline{\Phi}_S^{\mu 0} = H^0 \circ \underline{\Phi}_S^\mu, \quad \underline{\Psi}_{\widetilde{S}}^{\mu 0} = H^0 \circ \underline{\Psi}_{\widetilde{S}}^\mu.$$

Let $\mathcal{M} \in \mathrm{Mod}_{\mathrm{good}}(\mathcal{D}_X)$ and $\mathcal{N} \in \mathrm{Mod}_{RS(V)}(\mathcal{D}_Y)$. Applying the functor $H^0(\cdot)$ to the morphisms (4.15) and (4.16), we get the following morphisms, in view of Lemmas 4.10 and 4.11:

$$\mathcal{E}\mathcal{M} \longrightarrow \underline{\Psi}_{\widetilde{S}}^{\mu 0}(\underline{\Phi}_S^{\mu 0}(\mathcal{E}\mathcal{M})) \quad \text{in } \mathrm{Mod}(\mathcal{E}_X|_{\dot{T}^*X}), \tag{4.17}$$

$$\underline{\Phi}_S^{\mu 0}(\underline{\Psi}_{\widetilde{S}}^{\mu 0}(\mathcal{E}\mathcal{N})) \longrightarrow \mathcal{E}\mathcal{N} \quad \text{in } \mathrm{Mod}(\mathcal{E}_Y|_{\dot{T}^*Y}). \tag{4.18}$$

Lemma 4.12 *The morphisms (4.17) and (4.18) are isomorphisms on $\dot{T}^*X$ and $\dot{T}^*Y$ respectively.*

Proof: For $\mathcal{M} \in \mathrm{Mod}_{\mathrm{coh}}(\mathcal{E}_X|_{\dot{T}^*X})$ and $\mathcal{N} \in \mathrm{Mod}_{RS(V)}(\mathcal{E}_Y|_{\dot{T}^*Y})$, we have:

$$\begin{aligned}
\underline{\Phi}_S^{\mu 0}(\mathcal{M}) &= p^a_{2*}(\mathcal{C}^{(d_X,0)}_{S|X\times Y} \otimes_{p_1^{-1}\mathcal{E}_X} p_1^{-1}\mathcal{M}), \\
\underline{\Psi}_{\widetilde{S}}^{\mu 0}(\mathcal{E}\mathcal{N}) &= p_{1*}\mathcal{T}or^{p_2^{a-1}\mathcal{E}_Y}_{d_Y-d_X}(\mathcal{C}^{(d_Y,0)}_{\widetilde{S}|Y\times X}, p_2^{a-1}\mathcal{N}).
\end{aligned}$$

From now on, we shall not write p_2^a (which is an isomorphism) for short.

(i) Let us prove that (4.17) is an isomorphism on $\dot{T}^*X$.

Since p_1 has connected and simply connected fibers by (4.10), $\mathcal{M}$ is isomorphic to $p_{1*}p_1^{-1}\mathcal{M}$, and hence it is enough to prove the isomorphism:

$$p_1^{-1}\mathcal{M} \simeq \mathcal{T}or^{\mathcal{E}_Y}_{d_Y-d_X}(\mathcal{C}^{(d_Y,0)}_{\widetilde{S}|Y\times X}, \mathcal{C}^{(d_X,0)}_{S|X\times Y} \otimes_{p_1^{-1}\mathcal{E}_X} p_1^{-1}\mathcal{M}).$$

This is a local problem on Λ. Applying Lemma 4.7, the correspondence:

$$\begin{tikzcd}
 & \Lambda \arrow[dl, "p_1"'] \arrow[dr, "p_2^a"] & \\
\dot{T}^*X & & V,
\end{tikzcd}$$

is locally isomorphic to the correspondence:

$$\begin{tikzcd} & \Lambda_\chi \times T^*_Z Z \arrow[dl, "p_1"'] \arrow[dr, "p^a_{23}"] & \\ U_X & & U'_X \times T^*_Z Z. \end{tikzcd}$$

Let $\mathcal{C}_{\Lambda_\chi}$ be a simple holonomic $\mathcal{E}_{X\times X}$-module on Λ_χ. Then, locally on Λ one has an isomorphism:

$$\mathcal{C}_{\widetilde{S}|Y\times X} \simeq \mathcal{C}_{\Lambda_\chi} \underline{\boxtimes} \mathcal{O}_Z$$

as an $\mathcal{E}_{Y\times X}$-module, and:

$$\begin{aligned} &\mathcal{C}^{(d_Y,0)}_{\widetilde{S}|Y\times X} \otimes^L_{\mathcal{E}_Y} (\mathcal{C}^{(d_X,0)}_{S|X\times Y} \otimes_{p_1^{-1}\mathcal{E}_X} p_1^{-1}\mathcal{M})[d_X - d_Y] \\ &\qquad \simeq \ (\mathcal{C}^{(d_X,0)}_{\Lambda_\chi} \underline{\boxtimes} \Omega_Z) \otimes^L_{\mathcal{E}_{X\times Z}} (\mathcal{C}^{(d_X,0)}_{\widetilde{\Lambda}_\chi} \underline{\boxtimes} \mathcal{O}_Z) \otimes_{p_1^{-1}\mathcal{E}_X} p_1^{-1}\mathcal{M}[d_X - d_Y] \\ &\qquad \simeq \ (\mathcal{C}^{(d_X,0)}_{\Lambda_\chi} \otimes^L_{\mathcal{E}_X} \mathcal{C}^{(d_X,0)}_{\widetilde{\Lambda}_\chi}) \otimes_{p_1^{-1}\mathcal{E}_X} p_1^{-1}\mathcal{M} \\ &\qquad \simeq \ p_1^{-1}\mathcal{M}, \end{aligned}$$

where we set $\mathcal{C}^{(d_X,0)}_{\Lambda_\chi} = \mathcal{C}_{\Lambda_\chi} \otimes_{q_2^{-1}\mathcal{O}_X} q_2^{-1}\Omega_X$, and we similarly define $\mathcal{C}^{(d_X,0)}_{\widetilde{\Lambda}_\chi}$ for $\widetilde{\Lambda}_\chi = r(\Lambda_\chi)$ (here q_2 is the second projection $X \times X \longrightarrow X$). To prove the above isomorphisms we have used the fact that:

$$\Omega_Z \otimes^L_{\mathcal{D}_Z} \mathcal{O}_Z[d_X - d_Y] \simeq \mathbb{C}_Z$$

(recall that $d_Y - d_X = d_Z$), and:

$$\mathcal{C}^{(d_X,0)}_{\Lambda_\chi} \otimes^L_{\mathcal{E}_X} \mathcal{C}^{(d_X,0)}_{\widetilde{\Lambda}_\chi} \simeq \mathcal{E}_X,$$

which holds since Λ_χ is the graph of a contact transformation on $\dot{T}^*X$.

(ii) Let us prove that (4.18) is an isomorphism.

One has:

$$\underline{\Phi}^{\mu 0}_S(\underline{\Psi}^{\mu 0}_{\widetilde{S}}(\mathcal{E}\mathcal{N})) \simeq \mathcal{C}^{(d_X,0)}_{S|X\times Y} \otimes_{p_1^{-1}\mathcal{E}_X} p_1^{-1} p_{1*} \mathcal{T}or^{p_2^{a-1}\mathcal{E}_Y}_{d_Y - d_X}(\mathcal{C}^{(d_Y,0)}_{\widetilde{S}|Y\times X}, \mathcal{N}).$$

Set:

$$\mathcal{Q} = \mathcal{T}or^{p_2^{a-1}\mathcal{E}_Y}_{d_Y - d_X}(\mathcal{C}^{(d_Y,0)}_{\widetilde{S}|Y\times X}, \mathcal{N}).$$

Let us show that $\mathcal{Q}$ is locally constant along the fibers of p_1. With the same notations as in (i), one may assume $\mathcal{N} \simeq \mathcal{N}' \underline{\boxtimes} \mathcal{O}_Z$, $\mathcal{C}^{(d_Y,0)}_{\widetilde{S}|Y\times X} \simeq \mathcal{C}^{(d_X,0)}_{\Lambda_\chi} \underline{\boxtimes} \mathcal{O}_Z$. Hence

$$\begin{aligned} \mathcal{Q} \ &\simeq \ H^0((\mathcal{C}^{(d_X,0)}_{\Lambda_\chi} \underline{\boxtimes} \Omega_Z) \otimes^L_{\mathcal{E}_{X\times Z}} (\mathcal{N}' \underline{\boxtimes} \mathcal{O}_Z)[d_X - d_Y]) \\ &\simeq \ H^0(\mathcal{C}^{(d_X,0)}_{\Lambda_\chi} \otimes^L_{\mathcal{E}_X} \mathcal{N}') \boxtimes \mathbb{C}_Z \\ &\simeq \ (\mathcal{C}^{(d_X,0)}_{\Lambda_\chi} \otimes_{\mathcal{E}_X} \mathcal{N}') \boxtimes \mathbb{C}_Z \end{aligned}$$

(note that $\mathcal{C}^{(d_X,0)}_{\Lambda_X}$ is flat over $\mathcal{E}_X$).

Since the fibers of p_1 are connected and simply connected, and $\mathcal{Q}$ is locally constant along these fibers, one has $p_1^{-1}p_{1*}\mathcal{Q} \xrightarrow{\sim} \mathcal{Q}$. Whence:

$$\underline{\Phi}^{\mu 0}_S(\underline{\Psi}^{\mu 0}_{\widetilde{S}}(\mathcal{E}\mathcal{N})) \simeq \mathcal{C}^{(d_X,0)}_{S|X\times Y} \otimes_{p_1^{-1}\mathcal{E}_X} \mathcal{Q},$$

and to check that it is isomorphic to $\mathcal{N}$ is now a local problem on V, i.e. on Λ. Then the proof goes as in (i): we may assume $\mathcal{C}^{(d_Y,0)}_{\widetilde{S}|Y\times X} \simeq \mathcal{C}^{(d_Y,0)}_{\Lambda_X} \underline{\boxtimes}\, \mathcal{O}_Z$, $\mathcal{Q} \simeq (\mathcal{C}^{(d_X,0)}_{\Lambda_X} \otimes^L_{\mathcal{E}_X} \mathcal{N}') \boxtimes \mathbb{C}_Z$, $\mathcal{N} \simeq \mathcal{N}' \underline{\boxtimes}\, \mathcal{O}_Z$, and we get:

$$\begin{aligned} \mathcal{C}^{(d_X,0)}_{S|X\times Y} \otimes_{p_1^{-1}\mathcal{E}_X} \mathcal{Q} &\simeq \mathcal{N}' \underline{\boxtimes}\, \mathcal{O}_Z \\ &\simeq \mathcal{N}. \end{aligned}$$

□

Proof of of Theorem 4.9: Denote by $\mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_X; \dot{T}^*X)$ the localization of $\mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_X)$ by the null system:

$$\{\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_X);\ \mathrm{char}(\mathcal{M}) \subset T^*_X X\}.$$

This is a triangulated category which inherits of the natural t-structure of $\mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_X)$ and its heart for this t-structure is the category $\mathrm{Mod}_{\mathrm{good}}(\mathcal{D}_X; \dot{T}^*X)$. One has a similar construction for $\mathbf{D}^{\mathrm{b}}_{RS(V)}(\mathcal{D}_X; \dot{T}^*X)$.

Denote by $\underline{\Phi}^{\cdot}_S$ and $\underline{\Psi}^{\cdot}_{\widetilde{S}}$ the image of $\underline{\Phi}_S$ and $\underline{\Psi}_{\widetilde{S}}$ in these localized categories. The morphisms (4.13) and (4.14) define the morphisms:

$$\begin{aligned} id &\longrightarrow \underline{\Psi}^{\cdot}_{\widetilde{S}} \circ \underline{\Phi}^{\cdot}_S, \\ \underline{\Phi}^{\cdot}_S \circ \underline{\Psi}^{\cdot}_{\widetilde{S}} &\longrightarrow id. \end{aligned}$$

Moreover, $\underline{\Phi}^{\cdot}_S$ is t-exact in view of Proposition 3.5 (ii). Hence we obtain the morphisms:

$$id \longrightarrow \underline{\Psi}^{0\cdot}_{\widetilde{S}} \circ \underline{\Phi}^{0\cdot}_S \quad \text{in } \mathrm{Mod}_{\mathrm{good}}(\mathcal{D}_X; \dot{T}^*X), \tag{4.19}$$

$$\underline{\Phi}^{0\cdot}_S \circ \underline{\Psi}^{0\cdot}_{\widetilde{S}} \longrightarrow id \quad \text{in } \mathrm{Mod}_{RS(V)}(\mathcal{D}_Y; \dot{T}^*Y), \tag{4.20}$$

and we have to show that these are isomorphisms.

Let $\mathcal{M} \in \mathrm{Mod}_{\mathrm{good}}(\mathcal{D}_X; \dot{T}^*X)$. Since $\mathcal{E}\mathcal{M} \simeq 0$ implies $\mathcal{M} \simeq 0$, (4.19) and (4.20) are isomorphisms in view of Lemma 4.12. □

Remark 4.13 A related result to Theorem 4.9 is obtained by Brylinski [2] in the framework of perverse sheaves.

An equivalence of categories for modules with regular singularities along the manifold $X \times_Y T^*Y$ associated to a smooth map $Y \longrightarrow X$ is obtained by Honda [11].

5 Applications

In this section we will apply our general results to the classical case of the twistor correspondence considered in the introduction. In particular, we discuss to what extent we can recover, without any explicit calculation, the results of [6], [29].

5.1 The twistor correspondence (holomorphic solutions)

Let us consider the twistor correspondence

$$
\begin{tikzcd}
 & \mathbb{F} \arrow[dl, "f"'] \arrow[dr, "g"] & \\
\mathbb{P} & & \mathbb{M},
\end{tikzcd}
\tag{5.1}
$$

where $\mathbb{F} = \mathbb{F}_{1,2}(\mathbb{T})$ is the flag manifold of type $(1, 2)$ of a four-dimensional complex vector space $\mathbb{T}$ (called twistor space), $\mathbb{P} = \mathbb{F}_1(\mathbb{T})$ is a projective three-space, and $\mathbb{M} = \mathbb{F}_2(\mathbb{T})$ is identified with the four-dimensional compactified complexified Minkowski space. The projections here are given by $f(L_1, L_2) = L_1$, $g(L_1, L_2) = L_2$, where $L_1 \subset L_2 \subset \mathbb{T}$ are complex subspaces of dimension one and two respectively, defining an element (L_1, L_2) of $\mathbb{F}$.

As we said in the introduction, we denote by $\Box_h$ the massless field equation of helicity h, and, for $k \in \mathbb{Z}$, we denote by $\mathcal{O}_{\mathbb{P}}(k)$ the $-k$-th tensor power of the tautological bundle on $\mathbb{P}$.

In order to apply to this situation our previous results, let us begin by verifying that hypotheses (4.8)–(4.11) are satisfied.

Clearly f and g are smooth and proper.

Choose local coordinates (x_1, x_2, x_3), (y_1, y_2, y_3, y_4) on affine charts of $\mathbb{P}$ and $\mathbb{M}$ respectively and denote by $(x; \xi)$, $(y; \eta)$ the associated coordinates on $T^*\mathbb{P}$ and $T^*\mathbb{M}$ respectively. Here (x_1, x_2, x_3) corresponds to the line generated by $(1, x_1, x_2, x_3) \in \mathbb{T}$ and (y_1, y_2, y_3, y_4) corresponds to the two-plane of $\mathbb{T}$ generated by the vectors $(y_2, y_4, 0, 1)$ and $(y_1, y_3, 1, 0)$. The submanifold $\mathbb{F}$ of $\mathbb{P} \times \mathbb{M}$ is given by the system of equations

$$
\begin{pmatrix} 1 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \wedge \begin{pmatrix} y_1 \\ y_3 \\ 1 \\ 0 \end{pmatrix} \wedge \begin{pmatrix} y_2 \\ y_4 \\ 0 \\ 1 \end{pmatrix} = 0.
$$

On the open set $y_1 \neq 0$ we find the independent equations

$$
\begin{cases} x_1 - x_2 y_3 - x_3 y_4 = 0, \\ 1 - x_2 y_1 - x_3 y_2 = 0. \end{cases}
$$

The fiber of $\Lambda = T^*_{\mathbb{F}}(\mathbb{P} \times \mathbb{M}) \cap (\dot{T}^*\mathbb{P} \times \dot{T}^*\mathbb{M})$ at (x, y) is given by:

$$\begin{aligned}&\lambda dx_1 + (-\lambda y_3 - \mu y_1)dx_2 + (-\lambda y_4 - \mu y_2)dx_3 \\ &-\mu x_2 dy_1 - \mu x_3 dy_2 - \lambda x_2 dy_3 - \lambda x_3 dy_4\end{aligned}$$

for $\lambda, \mu \in \mathbb{C}^\times$. Then one checks that $p_2^a|_\Lambda$ is an embedding. Set:

$$V = p_2^a(\Lambda).$$

Then $V \cap \pi_{\mathbb{M}}^{-1}(\{y;\, y_1 \neq 0\})$ is given by the equation $\eta_1\eta_4 = \eta_2\eta_3$, i.e. it is the characteristic variety of the wave equation (up to a $\mathbb{C}$-linear change of coordinates). It is then easy to check that (4.9), (4.10) and (4.11) are verified.

Lemma 5.1 *For $y \in \mathbb{M}$ one has*

(i) $H^0(\widehat{y}; \mathcal{O}_{\mathbb{P}}(k)) = \begin{cases} 0 & \text{for } k < 0, \\ \neq 0 & \text{and finite dimensional for } k \geq 0, \end{cases}$

(ii) $H^1(\widehat{y}; \mathcal{O}_{\mathbb{P}}(k))$ *is infinite dimensional for any* k*,*

(iii) $H^j(\widehat{y}; \mathcal{O}_{\mathbb{P}}(k)) = 0$ *for* $j \neq 0, 1$ *and for any* k*.*

Since this result is well known from the specialists, we do not write down the proof, which is an explicit calculation using Čech cohomology. (A detailed proof appears in [3, Correspondences for $\mathcal{D}$-modules and Penrose transform].)

Hence, as a particular case of the preceding results, we obtain:

Proposition 5.2 *(i) Let $\mathcal{M}$ be a good $\mathcal{D}_{\mathbb{P}}$-module. Then $H^0(\underline{\Phi}_{\mathbb{F}}(\mathcal{M}))$ is a good $\mathcal{D}_{\mathbb{M}}$-module with regular singularities on V, and $H^j(\underline{\Phi}_{\mathbb{F}}(\mathcal{M}))$ is a flat holomorphic connection for $j \neq 0$. Moreover, any good $\mathcal{D}_{\mathbb{M}}$-module with regular singularities on V is, up to a flat connection, the unique image by $\underline{\Phi}_{\mathbb{F}}$ of a good $\mathcal{D}_{\mathbb{P}}$-module (see Theorem 4.9 for a precise statement).*

(ii) Let $k \in \mathbb{Z}$, and set $\mathcal{D}_{\mathbb{P}}(-k) = \mathcal{D}_{\mathbb{P}} \otimes_{\mathcal{O}_{\mathbb{P}}} \mathcal{O}_{\mathbb{P}}(-k)$. Then $H^j(\underline{\Phi}_{\mathbb{F}}(\mathcal{D}_{\mathbb{P}}(-k))) = 0$ for $j < 0$, for $j = 0$ this module is simple along V, and it is zero for $j > 0$ if and only if $k < 0$.

(iii) For $y \in Y$ one has:

$$\mathrm{R}\Gamma(\widehat{y}; \mathcal{O}_{\mathbb{P}}(k)) \simeq R\mathcal{H}om_{\mathcal{D}_Y}(\underline{\Phi}_{\mathbb{F}}\mathcal{D}_{\mathbb{P}}(-k), \mathcal{O}_Y)_y[-1].$$

(iv) Let $U \subset \mathbb{M}$ be a $\widetilde{\mathbb{F}}$-trivial open subset. Then for $k < 0$:

$$H^{j+1}(\widehat{U}; \mathcal{O}_{\mathbb{P}}(k)) \simeq \mathrm{Ext}^j_{\mathcal{D}_{\mathbb{M}}}(U; \underline{\Phi}_{\mathbb{F}}\mathcal{D}_{\mathbb{P}}(-k), \mathcal{O}_{\mathbb{M}}).$$

Remark 5.3 (i) Theorem 1.1 may now be stated as follows: for $k < 0$, the $\mathcal{D}$-module transform of $\mathcal{D}_{\mathbb{P}}(-k)$ is the $\mathcal{D}_{\mathbb{M}}$-module associated to the equation $\square_{h(k)}$, where $h(k) = -(1 + k/2)$. We refer to [6] for the calculations implicitly leading to this conclusion.

(ii) Eastwood et al. [6] deals with complexes of locally free $\mathcal{O}_{\mathbb{M}}$-modules and $\mathcal{D}_{\mathbb{M}}$-linear morphisms. This category is equivalent to that of filtered $\mathcal{D}_{\mathbb{M}}$-modules as proved by Saito [22]. This shows that these authors indeed use $\mathcal{D}$-module theory.

(iii) In order to get the isomorphism of Proposition 5.2 (iv), we assumed $\widetilde{\mathbb{F}}$-triviality of U. Since in [6] the authors were not interested in computing all cohomology groups, they could slightly weaken the topological hypothesis and only assume that $U \cap \widehat{x}$ is connected and simply connected for every $x \in \widehat{U}$.

5.2 The twistor correspondence (hyperfunction solutions)

Now we shall apply Proposition 2.6 to the study of hyperfunction solutions, and show how to easily recover the results of Wells [29].

Let ϕ be a Hermitian form on $\mathbb{T}$ of signature $(+, +, -, -)$. Let us choose a basis for $\mathbb{T}$ such that

$$\phi = \begin{pmatrix} 0 & iI_2 \\ -iI_2 & 0 \end{pmatrix}$$

where $I_2 \in M_2(\mathbb{C})$ denotes the identity matrix. For $A \in M_2(\mathbb{C})$, we have:

$$(A^*, I_2)\ \phi \begin{pmatrix} A \\ I_2 \end{pmatrix} = 0 \quad \text{iff} \quad A \text{ is Hermitian.}$$

In other words, the local chart

$$\begin{array}{ccc} \mathbb{C}^4 & \longrightarrow & \mathbb{M} \\ (y_1, y_2, y_3, y_4) & \mapsto & \begin{pmatrix} y_3 + y_4 & y_1 + iy_2 \\ y_1 - iy_2 & y_3 - y_4 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array}$$

identifies the Minkowski space $M^4 = (\mathbb{R}^4, \phi)$ to an open subset of the completely real compact submanifold M of $\mathbb{M}$ defined by:

$$M = \{L_2 \in \mathbb{M};\ \phi(v) = 0\ \forall v \in L_2\}.$$

Note that M is a conformal compactification of the Minkowski space M^4. Let us consider

$$\begin{aligned} F &= \{(L_1, L_2) \in \mathbb{F};\ \phi(v) = 0\ \forall v \in L_2\}, \\ P &= \{L_1 \in \mathbb{P};\ \phi(v) = 0\ \forall v \in L_1\}, \end{aligned}$$

and the induced correspondence

$$\begin{array}{ccccccc} & F & & & & \mathbb{F} & \\ {\scriptstyle \tilde{f}} \swarrow & & \searrow {\scriptstyle \tilde{g}} & \hookrightarrow & {\scriptstyle f} \swarrow & & \searrow {\scriptstyle g} \\ P & & M & & \mathbb{P} & & \mathbb{M}. \end{array}$$

Recall that $\mathbb{M}$ is a complexification of M, that P is a real hypersurface of $\mathbb{P}$ topologically isomorphic to $S^2 \times S^3$, and that $\tilde{f}$ is locally isomorphic to a projection $P \times S^1 \longrightarrow P$ (cf [28]).

The sheaves $\mathcal{A}_M$ and $\mathcal{B}_M$ of analytic functions and Sato hyperfunctions respectively are given by

$$\begin{aligned} \mathcal{A}_M &= \mathbb{C}_M \otimes \mathcal{O}_{\mathbb{M}}, \\ \mathcal{B}_M &= R\mathcal{H}om(D'_{\mathbb{M}}\mathbb{C}_M, \mathcal{O}_{\mathbb{M}}). \end{aligned}$$

In order to apply formulas (2.20) and (2.21), let us calculate $\Phi_{\tilde{\mathbb{F}}}G$ for $G = \mathbb{C}_M[-2]$.

Since $g^{-1}(M) = F$, we have $g^{-1}\mathbb{C}_M = \mathbb{C}_F$. Moreover, since $\tilde{f}$ is locally isomorphic to $P \times S^1 \longrightarrow P$, we find that $\Phi_{\tilde{\mathbb{F}}}G = Rf_*\mathbb{C}_F$ is concentrated in degree $0, 1$ and $H^j(\Phi_{\tilde{\mathbb{F}}}G)$ is locally free of rank one for $j = 0$ or 1. Finally, since $P \simeq S^2 \times S^3$ is connected and simply connected, we have $H^0(\Phi_{\tilde{\mathbb{F}}}G) \simeq \mathbb{C}_P$, $H^1(\Phi_{\tilde{\mathbb{F}}}G) \simeq \mathbb{C}_P$.

Hence, the distinguished triangle:

$$\tau^{\leq 0}\Phi_{\tilde{\mathbb{F}}}G \longrightarrow \Phi_{\tilde{\mathbb{F}}}G \longrightarrow \tau^{\geq 1}\Phi_{\tilde{\mathbb{F}}}G \underset{+1}{\longrightarrow}$$

is isomorphic to the distinguished triangle

$$\mathbb{C}_P \longrightarrow \Phi_{\tilde{\mathbb{F}}}G \longrightarrow \mathbb{C}_P[-1] \underset{+1}{\longrightarrow}. \tag{5.2}$$

Applying the functor $D'_{\mathbb{P}}(\cdot)$ and using the isomorphism $D'_{\mathbb{P}}(\mathbb{C}_P) \simeq or_P[-1] \simeq \mathbb{C}_P[-1]$, we get the distinguished triangle

$$\mathbb{C}_P \longrightarrow D'_{\mathbb{P}}\Phi_{\tilde{\mathbb{F}}}G \longrightarrow \mathbb{C}_P[-1] \underset{+1}{\longrightarrow}. \tag{5.3}$$

Note that since $H^2(P;\mathbb{C}_P)$ is different from 0, we do not know whether these triangles split or not (i.e. whether $\Phi_{\widetilde{\mathbb{F}}}G \simeq \mathbb{C}_P \oplus \mathbb{C}_P[-1]$).

Now we assume

$$k < 0 \tag{5.4}$$

so that $\underline{\Phi}_{\mathbb{F}}\mathcal{D}_{\mathbb{P}}(-k)$ is a coherent $\mathcal{D}_{\mathbb{M}}$-module (concentrated in degree zero) by Proposition 5.2.

Applying the functors $\mathrm{R}\Gamma(\mathbb{P};\cdot\otimes\mathcal{O}_{\mathbb{P}}(k))$ and $\mathrm{R}\Gamma(\mathbb{P};R\mathcal{H}om(\cdot,\mathcal{O}_{\mathbb{P}}(k)))$ to (5.2) and (5.3) respectively, we get the long exact sequences:

$$\begin{array}{rl} \cdots \longrightarrow & H^{j+1}(P;\mathcal{O}_{\mathbb{P}}(k)) \longrightarrow H^{j+1}\mathrm{R}\Gamma(\mathbb{P};\Phi_{\widetilde{\mathbb{F}}}G\otimes\mathcal{O}_{\mathbb{P}}(k)) \longrightarrow \\ \longrightarrow & H^{j}(P;\mathcal{O}_{\mathbb{P}}(k)) \longrightarrow H^{j+2}(P;\mathcal{O}_{\mathbb{P}}(k)) \longrightarrow \cdots \end{array} \tag{5.5}$$

and

$$\begin{array}{rl} \cdots \longrightarrow & H^{j+1}_P(\mathbb{P};\mathcal{O}_{\mathbb{P}}(k)) \longrightarrow H^{j}\mathrm{R}\Gamma(\mathbb{P};R\mathcal{H}om(D'_{\mathbb{P}}\Phi_{\widetilde{\mathbb{F}}}G,\mathcal{O}_{\mathbb{P}}(k))) \longrightarrow \\ \longrightarrow & H^{j}_P(\mathbb{P};\mathcal{O}_{\mathbb{P}}(k)) \longrightarrow H^{j+2}_P(\mathbb{P};\mathcal{O}_{\mathbb{P}}(k)) \longrightarrow \cdots \end{array} \tag{5.6}$$

Taking the cohomology of degree -3 in (2.20), (2.21), we get:

$$H^0(\mathrm{R}\Gamma(\mathbb{P};\Phi_{\widetilde{\mathbb{F}}}G\otimes\mathcal{O}_{\mathbb{P}}(k))) \simeq H^{-1}\mathrm{R}\Gamma(\mathbb{M};R\mathcal{H}om_{\mathcal{D}_{\mathbb{M}}}(\underline{\Phi}_{\mathbb{F}}\mathcal{D}_{\mathbb{P}}(-k),\mathcal{A}_M)),$$

$$H^0(\mathrm{R}\Gamma(\mathbb{P};R\mathcal{H}om(D'\Phi_{\widetilde{\mathbb{F}}}G,\mathcal{O}_{\mathbb{P}}(k)))) \simeq H^{-1}\mathrm{R}\Gamma(\mathbb{M};R\mathcal{H}om_{\mathcal{D}_{\mathbb{M}}}(\underline{\Phi}_{\mathbb{F}}\mathcal{D}_{\mathbb{P}}(-k),\mathcal{B}_M)),$$

and the terms on the right hand side both vanish by hypothesis (5.4) and Proposition 5.2. Hence we get from (5.5) and (5.6)

$$\begin{cases} H^0(P;\mathcal{O}_{\mathbb{P}}(k)) = 0 \\ H^1_P(\mathbb{P};\mathcal{O}_{\mathbb{P}}(k)) = 0 \end{cases}$$

Taking the zero-th cohomology group in (2.20), (2.21), we find the isomorphisms:

$$\begin{array}{rcl} H^1(P;\mathcal{O}_{\mathbb{P}}(k)) & \simeq & \mathrm{Hom}_{\mathcal{D}_{\mathbb{M}}}(\underline{\Phi}_{\mathbb{F}}\mathcal{D}_{\mathbb{P}}(-k),\mathcal{A}_M), \\ H^2_P(\mathbb{P};\mathcal{O}_{\mathbb{P}}(k)) & \simeq & \mathrm{Hom}_{\mathcal{D}_{\mathbb{M}}}(\underline{\Phi}_{\mathbb{F}}\mathcal{D}_{\mathbb{P}}(-k),\mathcal{B}_M). \end{array}$$

This is Theorem 6.1 of [29].

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