Leray's quantization of projective duality

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Contents

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1 Introduction

Let $\mathbb{P}$ be a complex n-dimensional projective space, $\mathbb{P}^*$ the dual projective space, and $\mathbb{A}$ the hypersurface of $\mathbb{P} \times \mathbb{P}^*$ given by the incidence relation $\mathbb{A} = \{(z, \zeta) \in \mathbb{P} \times \mathbb{P}^*;\ \langle z, \zeta \rangle = 0\}$. We shall consider the correspondence $\mathbb{P} \underset{f}{\longleftarrow} \mathbb{A} \underset{g}{\longrightarrow} \mathbb{P}^*$, where f and g are the natural projections.

It is well-known that the conormal bundle to $\mathbb{A}$ in $\mathbb{P} \times \mathbb{P}^*$ is the Lagrangian manifold associated to a contact transformation between $\dot{T}^*\mathbb{P}$ and $\dot{T}^*\mathbb{P}^*$, the cotangent bundles to $\mathbb{P}$ and $\mathbb{P}^*$ respectively, with the zero-section removed. This contact transformation induces an equivalence of categories between constructible sheaves on $\mathbb{P}$ modulo locally constant sheaves and the similar category on $\mathbb{P}^*$ (cf. Brylinski [5]), or between coherent $\mathcal{D}$-modules on $\mathbb{P}$ modulo flat connections and the similar category on $\mathbb{P}^*$ (cf. [6] in which we treat the case of general correspondences, not necessarily associated to contact transformations. Cf. also [3] for an interesting study of flag correspondences in the language of representation theory).

Our aim here is to show that a kernel introduced by J. Leray [21] allows us to quantize this transformation, and for an appropriate twist by a line bundle to extend this quantification across the zero-section. More precisely, for $k \in \mathbb{Z}$, we denote by $\mathcal{O}_{\mathbb{P}}(k)$ the $-k$-th tensor power of the tautological line bundle, and we set $\mathcal{D}_{\mathbb{P}}(k) = \mathcal{D}_{\mathbb{P}} \otimes_{\mathcal{O}_{\mathbb{P}}} \mathcal{O}_{\mathbb{P}}(k)$. If F is a sheaf on $\mathbb{P}$, and $\mathcal{M}$ is a coherent $\mathcal{D}_{\mathbb{P}}$-module, we set

$$\Phi_{\mathbb{A}}\, F = Rg_! f^{-1} F[n-1], \quad \underline{\Phi}_{\mathbb{A}}\, \mathcal{M} = \underline{g}_* \underline{f}^{-1} \mathcal{M}.$$

The main result of this paper is that the projective duality can be "quantized" to give an isomorphism of $\mathcal{D}_{\mathbb{P}^*}$-modules for $-n-1 < k < 0$:

$$\mathcal{D}_{\mathbb{P}^*}(-k^*) \xrightarrow{\sim} \underline{\Phi}_{\mathbb{A}}(\mathcal{D}_{\mathbb{P}}(-k)), \tag{1.1}$$

where $k^* = -n-1-k$. In particular, taking holomorphic solutions we get an isomorphism (of sheaves):

$$\Phi_{\mathbb{A}}\, \mathcal{O}_{\mathbb{P}}(k) \xrightarrow{\sim} \mathcal{O}_{\mathbb{P}^*}(k^*). \tag{1.2}$$

We shall give two different proofs of (1.1). The first one is rather abstract and deals with general contact transformations globally defined outside of the zero-section of complex manifolds, relying on the work of Sato-Kawai-Kashiwara [24]. The second one is more computational and is an adaptation in the language of $\mathcal{D}$-modules of classical results which go back to Leray loc. cit.

Let F be a sheaf on $\mathbb{P}$. By (1.2), and using classical adjunction formulas, we get the isomorphisms:

$$\left\{ \begin{array}{rcl} \mathrm{R}\Gamma(\mathbb{P}; F \otimes \mathcal{O}_{\mathbb{P}}(k)) & \simeq & \mathrm{R}\Gamma(\mathbb{P}^*; \Phi_{\mathbb{A}}\, F \otimes \mathcal{O}_{\mathbb{P}^*}(k^*)), \\ \mathrm{R}\Gamma(\mathbb{P}; R\mathcal{H}om(F, \mathcal{O}_{\mathbb{P}}(k))) & \simeq & \mathrm{R}\Gamma(\mathbb{P}^*; R\mathcal{H}om(\Phi_{\mathbb{A}}\, F, \mathcal{O}_{\mathbb{P}^*}(k^*))). \end{array} \right. \tag{1.3}$$

Moreover, assuming F is $\mathbb{R}$-constructible and using adjunction formulas of [19], we get the isomorphisms:

$$\left\{\begin{array}{rcl} \mathrm{R}\Gamma(\mathbb{P}; F \overset{\mathrm{w}}{\otimes} \mathcal{O}_{\mathbb{P}}(k)) & \simeq & \mathrm{R}\Gamma(\mathbb{P}^*; \Phi_{\mathbb{A}}\, F \overset{\mathrm{w}}{\otimes} \mathcal{O}_{\mathbb{P}^*}(k^*)), \\ \mathrm{R}\Gamma(\mathbb{P}; \mathcal{T}hom(F, \mathcal{O}_{\mathbb{P}}(k))) & \simeq & \mathrm{R}\Gamma(\mathbb{P}^*; \mathcal{T}hom(\Phi_{\mathbb{A}}\, F, \mathcal{O}_{\mathbb{P}^*}(k^*))), \end{array}\right. \tag{1.4}$$

where $\overset{\mathrm{w}}{\otimes}$ and $\mathcal{T}hom$ are the functors of formal and moderate cohomology introduced in [14] and [19].

Formulas (1.3), (1.4) allow us to recover many classical integral formulas for various choices of F. We only treat here a few examples.

As a first application, we generalize Martineau's isomorphism (cf. [22]) as follows. Let $U \subset \mathbb{P}$ be an open neighborhood of the origin in an affine chart $E \subset \mathbb{P}$, and assume that its hyperplane sections are cohomologically trivial. Denote by $K \subset E^*$ the set of complex hyperplanes which do not intersect U. Then formulas (1.3) entail the isomorphisms:

$$\begin{array}{rcl} \mathrm{R}\Gamma_c(U; \mathcal{O}_E)[n] & \simeq & \mathrm{R}\Gamma(K; \mathcal{O}_{E^*}), \\ \mathrm{R}\Gamma(U; \mathcal{O}_E) & \simeq & \mathrm{R}\Gamma_K(E^*; \mathcal{O}_{E^*})[n], \end{array}$$

which show in particular that all these complexes are concentrated in degree zero. Using (1.4), we also obtain similar isomorphisms using the functors of temperate and formal cohomology mentioned above. Moreover, from these isomorphisms we recover in the lines of [12] or [29] various vanishing theorems for local cohomology.

Another example is real projective duality. Denote by P and P^* a real projective space of dimension $n > 1$ and its dual, and consider $\mathbb{P}$ and $\mathbb{P}^*$ as complexifications of P and P^*. Denote by $\mathcal{A}_{\mathrm{P}}$ (resp. $\mathcal{C}^\infty_{\mathrm{P}}$, $\mathcal{D}b_{\mathrm{P}}$, $\mathcal{B}_{\mathrm{P}}$) the sheaf of real analytic functions on P (resp. $\mathcal{C}^\infty$ functions, distributions, hyperfunctions). For $k \in \mathbb{Z}$, $\varepsilon \in \mathbb{Z}_2$, we denote by $\mathcal{C}^\infty_{\mathrm{P}}(k, \varepsilon)$ the locally constant sheaf of rank one over $\mathcal{C}^\infty_{\mathrm{P}}$ whose global sections are represented by $\mathcal{C}^\infty$-functions f on $\mathbb{R}^{n+1} \setminus \{0\}$ satisfying the homogeneity condition:

$$f(\lambda x) = (\operatorname{sgn} \lambda)^\varepsilon \lambda^k f(x), \quad \text{for } \lambda \in \mathbb{R} \setminus \{0\}.$$

Using explicit integral formulas, Gelfand et al. [9] proved the isomorphisms for $-n-1 < k < 0$, and $\varepsilon^* = -n - 1 - \varepsilon \bmod 2$:

$$\Gamma(\mathrm{P}; \mathcal{C}^\infty_{\mathrm{P}}(k, \varepsilon)) \simeq \Gamma(\mathrm{P}^*; \mathcal{C}^\infty_{\mathrm{P}^*}(k^*, \varepsilon^*)).$$

We recover here these isomorphisms (and the similar ones with $\mathcal{C}^\infty$ replaced by $\mathcal{A}$, $\mathcal{D}b$ or $\mathcal{B}$) by applying (1.3) (or (1.4)) to the case where F is either the constant sheaf $\mathbb{C}_{\mathrm{P}}$ or the canonical line bundle K_{P}. In fact, $\mathcal{C}^\infty_{\mathrm{P}}(k, 0) \simeq \mathbb{C}_{\mathrm{P}} \overset{\mathrm{w}}{\otimes} \mathcal{O}_{\mathbb{P}}(k)$ and $\mathcal{C}^\infty_{\mathrm{P}}(k, 1) \simeq K_{\mathrm{P}} \overset{\mathrm{w}}{\otimes} \mathcal{O}_{\mathbb{P}}(k)$.

These two examples (Martineau's isomorphism and the Gelfand-Radon transform) show that isomorphism (1.1), and its corollaries (1.2), (1.3) and (1.4), allow us to reduce many problems of integral geometry to purely topological problems, namely the computation of $\Phi_{\mathbb{A}}(F)$ for various constructible sheaves F on $\mathbb{P}$.

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2 Review on correspondences for sheaves and $\mathcal{D}$-modules

Here, we recall some results of [6] on correspondences for sheaves and $\mathcal{D}$-modules in the particular case where the manifolds X and Y (see below) have the same dimension.

2.1 Notations

References are made to [18] for the theory of sheaves, and to [24] and [13] for the theory of $\mathcal{D}$ and $\mathcal{E}$-modules (see [25] and [27] for a detailed exposition).

2.1.1 Geometry

Let X, Y be real analytic manifolds. We denote by a_X the map from X to the set consisting of a single element, by $r : X \times Y \longrightarrow Y \times X$ the map $r(x, y) = (y, x)$, and by q_1, q_2 the first and second projection from $X \times Y$ to the corresponding factor.

We denote by $\pi_X : T^*X \longrightarrow X$ the cotangent bundle to X, by $\dot{\pi}_X : \dot{T}^*X \longrightarrow X$ the cotangent bundle with the zero-section removed, and by $T^*_M X$ the conormal bundle to a submanifold $M \subset X$. We denote by p_1 and p_2 the projections from $T^*(X \times Y) \simeq T^*X \times T^*Y$, and by p_2^a the composite of p_2 with the antipodal map of T^*Y.

2.1.2 Sheaves

We denote by $\mathbf{D}^{\mathrm{b}}(\mathbb{C}_X)$ the derived category of the category of bounded complexes of sheaves of $\mathbb{C}$-vector spaces on a topological space X. If $A \subset X$ is a locally closed subset, we denote by $\mathbb{C}_A$ the sheaf on X which is the constant sheaf on A with stalk $\mathbb{C}$, and zero on $X \setminus A$. We consider the "six operations" of sheaf theory $R\mathcal{H}om(\cdot,\cdot)$, $\cdot \otimes \cdot$, $Rf_!$, Rf_*, f^{-1}, $f^!$, and we denote by $\boxtimes$ the exterior tensor product. Recall that $\mathrm{RHom}(\cdot,\cdot) = Ra_{X*}R\mathcal{H}om(\cdot,\cdot)$. For $F \in \mathbf{D}^{\mathrm{b}}(\mathbb{C}_X)$ we set $D'F = R\mathcal{H}om(F, \mathbb{C}_X)$.

If X is a real analytic manifold, we denote by $SS(F)$ the micro-support of F, a closed conic involutive subset of T^*X. We denote by $\mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_X)$ the full triangulated subcategory of $\mathbf{D}^{\mathrm{b}}(\mathbb{C}_X)$ of objects with $\mathbb{R}$-constructible cohomology. If X is a complex manifold, one defines similarly the category $\mathbf{D}^{\mathrm{b}}_{\mathbb{C}-c}(\mathbb{C}_X)$ of $\mathbb{C}$-constructible objects.

2.1.3 $\mathcal{D}$-modules

In the rest of this paper, unless otherwise stated, all manifolds and morphisms of manifolds will be complex analytic.

Let X be a complex manifold. We denote by $\mathcal{O}_X$ the structural sheaf, by Ω_X the sheaf of holomorphic forms of maximal degree, and by $\mathcal{D}_X$ the sheaf of rings of linear differential operators. We denote by $\mathrm{Mod}(\mathcal{D}_X)$ the category of left $\mathcal{D}_X$-modules, and by $\mathrm{Mod}_{\mathrm{coh}}(\mathcal{D}_X)$ the thick abelian subcategory of coherent $\mathcal{D}_X$-modules. Following [26], we say that a coherent $\mathcal{D}_X$-module $\mathcal{M}$ is good if, in a neighborhood of any compact subset of X, $\mathcal{M}$ admits a finite filtration by coherent $\mathcal{D}_X$-submodules $\mathcal{M}_k$ $(k = 1, \ldots, l)$ such that each quotient $\mathcal{M}_k/\mathcal{M}_{k-1}$ can be endowed with a good filtration. We denote by $\mathrm{Mod}_{\mathrm{good}}(\mathcal{D}_X)$ the full subcategory of $\mathrm{Mod}_{\mathrm{coh}}(\mathcal{D}_X)$ consisting of good $\mathcal{D}_X$-modules. This definition ensures that $\mathrm{Mod}_{\mathrm{good}}(\mathcal{D}_X)$ is the smallest thick subcategory of $\mathrm{Mod}(\mathcal{D}_X)$ containing the modules which can be endowed with good filtrations on a neighborhood of any compact subset of X. Note that in the algebraic case, coherent $\mathcal{D}$-modules are good. We denote by $\mathbf{D}^{\mathrm{b}}(\mathcal{D}_X)$ the derived category of the category of bounded complexes of left $\mathcal{D}_X$-modules, and by $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X)$ (resp. by $\mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_X)$) its full triangulated subcategory whose objects have cohomology groups belonging to $\mathrm{Mod}_{\mathrm{coh}}(\mathcal{D}_X)$ (resp. to $\mathrm{Mod}_{\mathrm{good}}(\mathcal{D}_X)$). We consider the operations in the derived category of (left or right) $\mathcal{D}$-modules: $R\mathcal{H}om_{\mathcal{D}_X}(\cdot,\cdot)$, $\cdot \otimes^L_{\mathcal{D}_X} \cdot$, $\underline{f}^{-1}$, $\underline{f}_*$. In particular, if $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}(\mathcal{D}_X)$ and $\mathcal{N} \in \mathbf{D}^{\mathrm{b}}(\mathcal{D}_Y)$:

$$\underline{f}^{-1}\mathcal{M} = \mathcal{D}_{Y\to X} \otimes^L_{f^{-1}\mathcal{D}_X} f^{-1}\mathcal{M}, \quad \underline{f}_*\mathcal{N} = Rf_*(\mathcal{D}_{X\leftarrow Y} \otimes^L_{\mathcal{D}_Y} \mathcal{N}),$$

where $\mathcal{D}_{Y\to X}$ and $\mathcal{D}_{X\leftarrow Y}$ are the transfer bimodules associated to $f : Y \longrightarrow X$. We denote by $\underline{\boxtimes}$ the exterior tensor product, and we also use the notation:

$$\underline{D}_X\mathcal{M} = R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{K}_X),$$

where $\mathcal{K}_X$ denotes the dualizing complex for left $\mathcal{D}_X$-modules, defined by $\mathcal{K}_X = \mathcal{D}_X \otimes_{\mathcal{O}_X} \Omega_X^{\otimes -1}[\dim X]$.

If $\mathcal{F}$ is a holomorphic vector bundle on X, we set:

$$\mathcal{F}^* = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X), \quad \mathcal{D}\mathcal{F} = \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{F}.$$

Note that $\mathcal{D}\mathcal{F}$ is a left $\mathcal{D}_X$-module. We shall also need the right $\mathcal{D}_X$-module:

$$\mathcal{F}\mathcal{D} = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{D}_X.$$

If Z is a closed complex submanifold of codimension d of X, we shall consider the holonomic left $\mathcal{D}_X$-module $\mathcal{B}_{Z|X}$ of [24]. Recall that $\mathcal{B}_{Z|X} = H^d_{[Z]}(\mathcal{O}_X)$ (algebraic cohomology) is a subsheaf of $\mathcal{B}^\infty_{Z|X} = H^d_Z(\mathcal{O}_X)$.

2.1.4 $\mathcal{E}$-modules

Since the proof of Theorem 3.3 below will make use of the theory of microdifferential operators, we recall here some definitions. We refer to [24] for the theory of $\mathcal{E}$-modules, and to [25] for an exposition.

Let $\mathcal{E}_X$ denote the sheaf of microdifferential operators of finite order on T^*X, and let U be a subset of T^*X. We denote by $\mathrm{Mod}(\mathcal{E}_X|_U)$ the category of left $\mathcal{E}_X|_U$-modules, and by $\mathrm{Mod}_{\mathrm{coh}}(\mathcal{E}_X|_U)$ the thick abelian subcategory of coherent $\mathcal{E}_X|_U$-modules. We denote by $\mathbf{D}^{\mathrm{b}}(\mathcal{E}_X|_U)$ the derived category of the category of bounded complexes of left $\mathcal{E}_X|_U$-modules, and by $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{E}_X|_U)$ its full triangulated subcategory whose objects have cohomology groups belonging to $\mathrm{Mod}_{\mathrm{coh}}(\mathcal{E}_X|_U)$.

If $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}(\mathcal{D}_X)$, we set:

$$\mathcal{E}\mathcal{M} = \mathcal{E}_X \otimes_{\pi_X^{-1}\mathcal{D}_X} \pi_X^{-1}\mathcal{M},$$

an object of $\mathbf{D}^{\mathrm{b}}(\mathcal{E}_X)$, and if $\mathcal{F}$ is a holomorphic vector bundle on X, we set:

$$\mathcal{E}\mathcal{F} = \mathcal{E}(\mathcal{D}\mathcal{F}).$$

If Z is a closed complex submanifold of codimension d of X, we shall consider the holonomic left $\mathcal{E}_X$-module $\mathcal{C}_{Z|X} = \mathcal{E}\mathcal{B}_{Z|X}$.

2.2 Correspondences for sheaves and $\mathcal{D}$-modules

We recall here some results of [6] that we shall use in this paper.

Let X and Y be two complex manifolds of dimension n, and let $S \subset X \times Y$ be a closed submanifold of codimension c. Let $\widetilde{S} = r(S) \subset Y \times X$. Set

$$\Lambda = T^*_S(X \times Y) \cap (\dot{T}^*X \times \dot{T}^*Y),$$

and consider the correspondences:

$$\begin{tikzcd} & S \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y, \end{tikzcd} \qquad \begin{tikzcd} & \Lambda \arrow[dl, "p_1|_\Lambda"'] \arrow[dr, "p_2^a|_\Lambda"] & \\ \dot{T}^*X & & \dot{T}^*Y, \end{tikzcd} \tag{2.1}$$

where we set $f = q_1|_S$, $g = q_2|_S$. We shall consider the hypotheses:

$$\begin{cases} (a) & f \text{ and } g \text{ are smooth and proper,} \\ (b) & p_1|_\Lambda \text{ and } p_2^a|_\Lambda \text{ are isomorphisms.} \end{cases} \tag{2.2}$$

Note that (2.2) implies that $\Lambda = \dot{T}^*_S(X\times Y)$ is the Lagrangian manifold associated to the graph of a contact transformation globally defined outside of the zero-sections:

$$\left\{ \begin{array}{rccl} \chi: & \dot{T}^*X & \longrightarrow & \dot{T}^*Y \\ & p & \mapsto & p_2^a|_\Lambda(p_1|_\Lambda^{-1}(p)). \end{array} \right. \tag{2.3}$$

Definition 2.1 For $A\subset X$, $F\in\mathbf{D}^{\mathrm{b}}(\mathbb{C}_X)$, $\mathcal{M}\in\mathbf{D}^{\mathrm{b}}(\mathcal{D}_X)$, and $B\subset Y$, $G\in\mathbf{D}^{\mathrm{b}}(\mathbb{C}_Y)$, $\mathcal{N}\in\mathbf{D}^{\mathrm{b}}(\mathcal{D}_Y)$, we set:

$$\begin{array}{ll} \widehat{A} = g(f^{-1}(A)), & \widehat{B} = f(g^{-1}(B)), \\ \Phi_S\, F = Rg_! f^{-1}F[n-c], & \Phi_{\widetilde{S}}\, G = Rf_! g^{-1}G[n-c], \\ \underline{\Phi}_S\, \mathcal{M} = \underline{g}_*\underline{f}^{-1}\mathcal{M}, & \underline{\Phi}_{\widetilde{S}}\, \mathcal{N} = \underline{f}_*\underline{g}^{-1}\mathcal{N}. \end{array}$$

If $B=\{y\}$, we write $\widehat{y}$ instead of $\widehat{\{y\}}$.

Remark 2.2 With the notations introduced in Appendix A.2, Φ_S is the integral transform associated to the kernel $\mathbb{C}_S[n-c]$, and $\underline{\Phi}_S$ to the kernel $\mathcal{B}_{S|X\times Y}$.

One easily deduces the next proposition from classical formulas of sheaf and $\mathcal{D}$-module theory (see [6] for details).

Proposition 2.3 *Assume (2.2)-(a). For $F\in\mathbf{D}^{\mathrm{b}}(\mathbb{C}_X)$ and $G\in\mathbf{D}^{\mathrm{b}}(\mathbb{C}_Y)$ there are natural isomorphisms:*

$$\begin{aligned} \Phi_S(D'_X F) &\simeq D'_Y\,\Phi_S(F), && (2.4)\\ \mathrm{R}\Gamma(X; F\otimes\Phi_{\widetilde{S}}G) &\simeq \mathrm{R}\Gamma(Y;\Phi_S\,F\otimes G), && (2.5)\\ \mathrm{R}\Gamma(X; R\mathcal{H}om(F,\Phi_{\widetilde{S}}G)) &\simeq \mathrm{R}\Gamma(Y; R\mathcal{H}om(\Phi_S\,F,G)). && (2.6) \end{aligned}$$

For $\mathcal{M}\in\mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_X)$ there are natural isomorphisms:

$$\begin{aligned} \underline{\Phi}_S(\underline{D}_X\mathcal{M}) &\simeq \underline{D}_Y(\underline{\Phi}_S\,\mathcal{M}), && (2.7)\\ \Phi_S\, R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{O}_X) &\simeq R\mathcal{H}om_{\mathcal{D}_Y}(\underline{\Phi}_S\,\mathcal{M},\mathcal{O}_Y). && (2.8) \end{aligned}$$

For $G\in\mathbf{D}^{\mathrm{b}}(\mathbb{C}_Y)$ and $\mathcal{M}\in\mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_X)$, there are natural isomorphisms:

$$\begin{aligned} \mathrm{R}\Gamma(X; R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\Phi_{\widetilde{S}}G\otimes\mathcal{O}_X)) &\simeq \mathrm{R}\Gamma(Y; R\mathcal{H}om_{\mathcal{D}_Y}(\underline{\Phi}_S\,\mathcal{M}, G\otimes\mathcal{O}_Y)), && (2.9)\\ \mathrm{R}\Gamma(X; R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}\otimes\Phi_{\widetilde{S}}G,\mathcal{O}_X)) &\simeq \mathrm{R}\Gamma(Y; R\mathcal{H}om_{\mathcal{D}_Y}(\underline{\Phi}_S\,\mathcal{M}\otimes G,\mathcal{O}_Y)). && (2.10) \end{aligned}$$

Moreover, in (2.5), (2.6), (2.9), or (2.10) we may replace $\mathrm{R}\Gamma$ *by* $\mathrm{R}\Gamma_c$.

As a particular case of (2.9), if $\mathcal{F}$ is a holomorphic vector bundle on X and $y\in Y$, one deduces the germ formula:

$$\mathrm{R}\Gamma(\widehat{y};\mathcal{F}) \simeq R\mathcal{H}om_{\mathcal{D}_Y}(\underline{\Phi}_S\,\mathcal{D}\mathcal{F}^*,\mathcal{O}_Y)_y[c-n]. \tag{2.11}$$

In Appendix A.1 we describe the functors $\overset{\mathrm{w}}{\otimes}$ and $\mathcal{T}hom$ of formal and moderate cohomology, and recall the corresponding formulas to (2.9) and (2.10) for these functors assuming G is $\mathbb{R}$-constructible (see [19]).

Set

$$\mathcal{B}^{(n,0)}_{S|X\times Y} = q_1^{-1}\Omega_X \otimes_{q_1^{-1}\mathcal{O}_X} \mathcal{B}_{S|X\times Y}.$$

Proposition 2.4 *There is a natural isomorphism of $(\mathcal{D}_Y,\mathcal{D}_X)$-bimodules on S:*

$$\mathcal{D}_{Y\leftarrow S}\otimes^L_{\mathcal{D}_S}\mathcal{D}_{S\to X} \xrightarrow{\sim} \mathcal{B}^{(n,0)}_{S|X\times Y}. \tag{2.12}$$

In particular, $\mathcal{D}_{Y\leftarrow S}\otimes^L_{\mathcal{D}_S}\mathcal{D}_{S\to X}$ is concentrated in degree zero, and for $\mathcal{M}\in\mathbf{D}^{\mathrm{b}}(\mathcal{D}_X)$ we have the isomorphism:

$$\underline{\Phi}_S\,\mathcal{M} \simeq Rq_{2!}(\mathcal{B}^{(n,0)}_{S|X\times Y}\otimes^L_{q_1^{-1}\mathcal{D}_X} q_1^{-1}\mathcal{M}).$$

Since the analogous of (2.12) holds for $\mathcal{E}$-modules, we give the following definition.

Definition 2.5 For $\mathcal{M}\in\mathbf{D}^{\mathrm{b}}(\mathcal{E}_X)$ we set:

$$\underline{\Phi}^\mu_S\,\mathcal{M} = Rp^a_{2!}(\mathcal{C}^{(n,0)}_{S|X\times Y}\otimes^L_{p_1^{-1}\mathcal{E}_X} p_1^{-1}\mathcal{M}).$$

This is an object of $\mathbf{D}^{\mathrm{b}}(\mathcal{E}_Y)$.

An important tool is given by the isomorphism below (see [26, Corollary 7.6]).

Proposition 2.6 *Assume (2.2). For $\mathcal{M}\in\mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_X)$, we have the following isomorphism in $\mathbf{D}^{\mathrm{b}}(\mathcal{E}_Y)$:*

$$\mathcal{E}(\underline{\Phi}_S\,\mathcal{M}) \simeq \underline{\Phi}^\mu_S(\mathcal{E}\mathcal{M}).$$

In [6], using the germ formula (2.11) and Proposition 2.6, we established the following properties of $\underline{\Phi}_S\,\mathcal{M}$. (Notice that the hypotheses of [6] are actually weaker than (2.2).)

Proposition 2.7 *Let $\mathcal{M}\in\mathrm{Mod}_{\mathrm{good}}(\mathcal{D}_X)$, and assume (2.2). Then, for $j\neq 0$, the module $H^j\,\underline{\Phi}_S\,\mathcal{M}$ is associated to a flat connection (i.e., its characteristic variety is contained in the zero-section).*

Proof: Since $\mathrm{char}(\underline{\Phi}_S\,\mathcal{M}) = \mathrm{supp}(\mathcal{E}\,\underline{\Phi}_S\,\mathcal{M})$, by Proposition 2.6 it is enough to prove that:

$$H^j(\underline{\Phi}^\mu_S\,\mathcal{E}\mathcal{M})|_{\dot{T}^*Y} = 0 \quad \text{for } j\neq 0.$$

Since $\Lambda = \dot{T}^*_S(X\times Y)$ is the Lagrangian manifold associated to a contact transformation, $\mathcal{C}^{(n,0)}_{S|X\times Y}$ is flat over $p_1^{-1}\mathcal{E}_X$, and $p^a_2|_\Lambda$ is finite. The statement follows. q.e.d.

Proposition 2.8 *Let $\mathcal{F}$ be a holomorphic vector bundle on X, and assume (2.2). Then:*

(i) for every $j < n - c$, there exists a locally constant sheaf of finite rank G^j on Y such that

$$G^j_y \simeq H^j(\widehat{y}; \mathcal{F}) \quad \forall y \in Y;$$

(ii) the complex $\underline{\Phi}_S\, \mathcal{D}\mathcal{F}$ is concentrated in degree ≥ 0;

(iii) assuming that Y is connected, the complex $\underline{\Phi}_S\, \mathcal{D}\mathcal{F}$ is concentrated in degree zero if and only if there exists $y \in Y$ such that $H^j(\widehat{y}; \mathcal{F}^*) = 0$ for every $j < n - c$;

(iv) assuming that Y is connected, the complex $\underline{D}_Y\, \underline{\Phi}_S\, \mathcal{D}\mathcal{F}[-n]$ is concentrated in degree zero if and only if there exists $y \in Y$ such that $H^j(\widehat{y}; \mathcal{F} \otimes_{\mathcal{O}_X} \Omega_X) = 0$ for every $j < n - c$.

Proof: (i) follows from Proposition 2.7, and from the germ formula (2.11).

(ii) follows from Proposition 2.4, since $\mathcal{D}\mathcal{F}$ is flat over $\mathcal{D}_X$.

(iii) Recall that Y is connected, and let $y \in Y$. Set $\mathcal{N} = \underline{\Phi}_S\, \mathcal{D}\mathcal{F}$ for short, and consider the distinguished triangle

$$H^0(\mathcal{N}) \longrightarrow \mathcal{N} \longrightarrow \tau^{>0}\mathcal{N} \xrightarrow[+1]{},$$

which gives rise to the distinguished triangle

$$R\mathcal{H}om_{\mathcal{D}_Y}(\tau^{>0}\mathcal{N}, \mathcal{O}_Y) \longrightarrow R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{N}, \mathcal{O}_Y) \longrightarrow R\mathcal{H}om_{\mathcal{D}_Y}(H^0(\mathcal{N}), \mathcal{O}_Y) \xrightarrow[+1]{}. \tag{2.13}$$

Notice that since $\mathrm{char}(\tau^{>0}\mathcal{N}) \subset T^*_Y Y$, one has

$$H^j R\mathcal{H}om_{\mathcal{D}_Y}(\tau^{>0}\mathcal{N}, \mathcal{O}_Y) = 0 \quad \forall j \geq 0, \tag{2.14}$$

and

$$\begin{aligned} \tau^{>0}\mathcal{N} = 0 \;\; &\Leftrightarrow \;\; R\mathcal{H}om_{\mathcal{D}_Y}(\tau^{>0}\mathcal{N}, \mathcal{O}_Y) = 0 \\ &\Leftrightarrow \;\; H^j R\mathcal{H}om_{\mathcal{D}_Y}(\tau^{>0}\mathcal{N}, \mathcal{O}_Y) = 0 \quad \forall j < 0 \\ &\Leftrightarrow \;\; H^j R\mathcal{H}om_{\mathcal{D}_Y}(\tau^{>0}\mathcal{N}, \mathcal{O}_Y)_y = 0 \quad \forall j < 0 \\ &\Leftrightarrow \;\; H^j R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{N}, \mathcal{O}_Y)_y = 0 \quad \forall j < 0, \end{aligned}$$

where the last equivalence comes from the distinguished triangle (2.13), and the fact that $H^j R\mathcal{H}om_{\mathcal{D}_Y}(H^0(\mathcal{N}), \mathcal{O}_Y) = 0$ for $j < 0$. To conclude, it remains to apply the germ formula

$$H^j R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{N}, \mathcal{O}_Y)_y \simeq H^{n-c+j}(\widehat{y}; \mathcal{F}^*).$$

(iv) Note that $\mathcal{D}(\mathcal{F}^* \otimes_{\mathcal{O}_X} \Omega_X^{\otimes -1}) \simeq \underline{D}_X \mathcal{D}\mathcal{F}[-n]$. Using (iii), we have:

$$\begin{aligned}
& H^j(\widehat{y}; \mathcal{F} \otimes_{\mathcal{O}_X} \Omega_X) = 0 \quad \forall j < n - c \\
& \quad \Leftrightarrow \quad H^j(\underline{\Phi}_S\, \mathcal{D}(\mathcal{F}^* \otimes_{\mathcal{O}_X} \Omega_X^{\otimes -1})) = 0 \quad \forall j \neq 0 \\
& \quad \Leftrightarrow \quad H^j(\underline{\Phi}_S\, \underline{D}_X \mathcal{D}\mathcal{F}[-n]) = 0 \quad \forall j \neq 0 \\
& \quad \Leftrightarrow \quad H^j(\underline{D}_Y\, \underline{\Phi}_S\, \mathcal{D}\mathcal{F}[-n]) = 0 \quad \forall j \neq 0.
\end{aligned}$$

q.e.d.

Remark 2.9 In [6] we study general correspondences, without assuming $\dim X = \dim Y$. Assuming $p_1|_\Lambda$ is surjective, $p_2^a|_\Lambda$ is a closed embedding on a smooth regular involutive submanifold $V \subset \dot{T}^*Y$, and the fibers of f are connected and simply connected, we prove that there is an equivalence of categories:

$$\mathrm{Mod}_{\mathrm{good}}(\mathcal{D}_X; \dot{T}^*X) \underset{H^0\, \underline{\Phi}_{\widetilde{S}}}{\overset{H^0\, \underline{\Phi}_S}{\rightleftarrows}} \mathrm{Mod}_{RS(V)}(\mathcal{D}_Y; \dot{T}^*Y),$$

where $\mathrm{Mod}_{\mathrm{good}}(\mathcal{D}_X; \dot{T}^*X)$ denotes the localization of $\mathrm{Mod}_{\mathrm{good}}(\mathcal{D}_X)$ by the subcategory of flat connections, and $\mathrm{Mod}_{RS(V)}(\mathcal{D}_Y; \dot{T}^*Y)$ denotes the localization of the category of good $\mathcal{D}_Y$-modules with regular singularities along V by the subcategory of flat connections. As a particular case, assuming (2.2), we recover a result of [5], namely, the equivalence:

$$\mathrm{Mod}_{\mathrm{good}}(\mathcal{D}_X; \dot{T}^*X) \underset{H^0\, \underline{\Phi}_{\widetilde{S}}}{\overset{H^0\, \underline{\Phi}_S}{\rightleftarrows}} \mathrm{Mod}_{\mathrm{good}}(\mathcal{D}_Y; \dot{T}^*Y).$$

3 Globally defined contact transformations

3.1 Main theorem

Consider the correspondences (2.1). Let $\mathcal{F}$ and $\mathcal{G}$ be holomorphic line bundles on X and Y respectively, let $\mathcal{L}$ be a $\mathcal{D}_{X\times Y}$-module (e.g., $\mathcal{L} = \mathcal{B}_{S|X\times Y}$), and set:

$$\mathcal{L}^{(n,0)}(\mathcal{F}, \mathcal{G}) = q_2^{-1}\mathcal{G}\mathcal{D} \otimes_{q_2^{-1}\mathcal{D}_Y} \mathcal{L} \otimes_{q_1^{-1}\mathcal{D}_X} q_1^{-1}\mathcal{D}(\mathcal{F} \otimes_{\mathcal{O}_X} \Omega_X).$$

Lemma 3.1 *Assuming g is proper, there is a natural isomorphism:*

$$\alpha : \Gamma(X \times Y; \mathcal{B}^{(n,0)}_{S|X\times Y}(\mathcal{F}, \mathcal{G}^*)) \simeq \mathrm{Hom}_{\mathbf{D}^b(\mathcal{D}_Y)}(\mathcal{D}\mathcal{G}, \underline{\Phi}_S\, \mathcal{D}\mathcal{F}). \tag{3.1}$$

Proof: It is enough to apply $H^0(\cdot)$ to the following chain of isomorphisms:

$$\begin{aligned}
& Ra_{X\times Y *}(\mathcal{B}^{(n,0)}_{S|X\times Y}(\mathcal{F}, \mathcal{G}^*)) \simeq \\
& \quad \simeq Ra_{Y*} Rq_{2*} R\mathcal{H}om_{q_2^{-1}\mathcal{D}_Y}(q_2^{-1}\mathcal{D}\mathcal{G}, \mathcal{B}^{(n,0)}_{S|X\times Y} \otimes_{q_1^{-1}\mathcal{O}_X} q_1^{-1}\mathcal{F}) \\
& \quad \simeq Ra_{Y*} R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{D}\mathcal{G}, Rq_{2*}(\mathcal{B}^{(n,0)}_{S|X\times Y} \otimes_{q_1^{-1}\mathcal{O}_X} q_1^{-1}\mathcal{F})) \\
& \quad \simeq Ra_{Y*} R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{D}\mathcal{G}, \underline{\Phi}_S\, \mathcal{D}\mathcal{F}).
\end{aligned}$$

Here, in the first isomorphism we used the fact that $\mathcal{DG}$ is $\mathcal{D}_Y$-coherent, and in the last one we used the fact that q_2 is proper on S. q.e.d.

Definition 3.2 (i) Assuming g is proper, and using Lemma 3.1, we associate to $s \in \Gamma(X \times Y; \mathcal{B}^{(n,0)}_{S|X\times Y}(\mathcal{F}, \mathcal{G}^*))$, the $\mathcal{D}_Y$-linear morphism:

$$\alpha(s) : \mathcal{DG} \longrightarrow \underline{\Phi}_S\, \mathcal{DF}.$$

(ii) We refer to [24] for the notion of a non-degenerate section of $\mathcal{C}_{S|X\times Y}$ on an open subset of $\Lambda = \dot{T}^*_S(X \times Y)$. We shall say that a section of $\mathcal{B}_{S|X\times Y}$ is non-degenerate in a neighborhood of $p \in \Lambda$ if the section of the sheaf of microfunctions $\mathcal{C}_{S|X\times Y}$ that it defines is non-degenerate. This definition extends to $\mathcal{B}^{(n,0)}_{S|X\times Y}(\mathcal{F}, \mathcal{G}^*)$, since this sheaf is locally isomorphic to $\mathcal{B}_{S|X\times Y}$.

Theorem 3.3 *Let $\mathcal{F}$ and $\mathcal{G}$ be holomorphic line bundles on X and Y respectively, and choose a section $s \in \Gamma(X \times Y; \mathcal{B}^{(n,0)}_{S|X\times Y}(\mathcal{F}, \mathcal{G}^*))$. Assume (2.2), Y is connected, and:*

(a) s is non-degenerate on Λ.

Then:

(i) the induced morphism $H^0(\alpha(s)) : \mathcal{DG} \longrightarrow H^0(\underline{\Phi}_S\, \mathcal{DF})$ is a $\mathcal{D}_Y$-linear isomorphism.

Assume moreover:

(b) there exists $y \in Y$ such that $H^j(\widehat{y}; \mathcal{F}^*) = 0$ for every $j < n - c$.

Then, $\underline{\Phi}_S\, \mathcal{DF}$ is concentrated in degree zero, and:

(ii) $\alpha(s) : \mathcal{DG} \longrightarrow \underline{\Phi}_S\, \mathcal{DF}$ is an isomorphism in $\mathbf{D}^{\mathrm{b}}(\mathcal{D}_Y)$,

(iii) $\alpha(s)$ induces an isomorphism $\Phi_S\, \mathcal{F}^* \longrightarrow \mathcal{G}^*$ in $\mathbf{D}^{\mathrm{b}}(\mathbb{C}_Y)$.

Proof: Hypothesis (a) and the theory of [24] ensure that $\alpha(s)$ is an isomorphism "outside of the zero-section", and we shall show that this isomorphism extends to the whole space. More precisely, it follows from Proposition 2.6 that $\alpha(s)$ induces a morphism:

$$\mathcal{E}\alpha(s) : \mathcal{EG} \longrightarrow \underline{\Phi}^{\mu}_S(\mathcal{EF}).$$

For conic objects of $\mathbf{D}^{\mathrm{b}}(\mathbb{C}_{T^*Y})$, we have Sato's distinguished triangle

$$R\pi_{Y!}(\cdot) \longrightarrow R\pi_{Y*}(\cdot) \longrightarrow R\dot{\pi}_{Y*}(\cdot) \xrightarrow[+1]{} .$$

Applying it to the morphism $\mathcal{E}\alpha(s)$, we get the morphism of distinguished triangles:

$$\begin{tikzcd} R\pi_{Y\,!}(\mathcal{E}\mathcal{G}) \arrow[r] \arrow[d, "\widetilde{\alpha}(s)"'] & R\pi_{Y\,*}(\mathcal{E}\mathcal{G}) \arrow[r] \arrow[d, "\alpha(s)"'] & R\dot{\pi}_{Y\,*}(\mathcal{E}\mathcal{G}) \arrow[r, "+1"'] \arrow[d, "\dot{\alpha}(s)"'] & {} \\ R\pi_{Y\,!}(\underline{\Phi}^{\mu}_{S}(\mathcal{E}\mathcal{F})) \arrow[r] & R\pi_{Y\,*}(\underline{\Phi}^{\mu}_{S}(\mathcal{E}\mathcal{F})) \arrow[r] & R\dot{\pi}_{Y\,*}(\underline{\Phi}^{\mu}_{S}(\mathcal{E}\mathcal{F})) \arrow[r, "+1"'] & {} \end{tikzcd} \tag{3.2}$$

Notice that:

(1) there are natural isomorphisms:

$$R\pi_{Y\,*}\mathcal{E}\mathcal{G} \simeq \mathcal{D}\mathcal{G}, \quad R\pi_{Y\,*}\,\underline{\Phi}^{\mu}_{S}\,\mathcal{E}\mathcal{F} \simeq \underline{\Phi}_{S}\,\mathcal{D}\mathcal{F}.$$

In particular, $R\pi_{Y\,*}\mathcal{E}\mathcal{G}$ is concentrated in degree zero.

(2) $\mathcal{E}\alpha(s)$ is an isomorphism all over $\dot{T}^*Y$ by [24], and hence $\dot{\alpha}(s)$ is an isomorphism.

(3) Considering the diagram:

$$\begin{tikzcd} T^*X \arrow[d, "\pi_X"'] & T^*_S(X\times Y) \arrow[l, "p_1"'] \arrow[r, "p_2^a"] \arrow[d, "\pi"'] & T^*Y \arrow[d, "\pi_Y"'] \\ X & S \arrow[l, "f"'] \arrow[r, "g"] & Y, \end{tikzcd}$$

and using the isomorphism $R\pi_!\mathcal{C}^{(n,0)}_{S|X\times Y} \simeq \mathcal{O}^{(n,0)}_{X\times Y}|_S[-c]$, we have:

$$\begin{aligned} R\pi_{Y\,!}(\underline{\Phi}^{\mu}_{S}\,\mathcal{E}\mathcal{F}) &\simeq R\pi_{Y\,!}Rp^a_{2\,*}(\mathcal{C}^{(n,0)}_{S|X\times Y} \otimes_{p_1^{-1}\pi_X^{-1}\mathcal{O}_X} p_1^{-1}\pi_X^{-1}\mathcal{F}) \\ &\simeq Rg_!R\pi_!(\mathcal{C}^{(n,0)}_{S|X\times Y} \otimes_{\pi^{-1}f^{-1}\mathcal{O}_X} \pi^{-1}f^{-1}\mathcal{F}) \\ &\simeq Rg_!(\mathcal{O}^{(n,0)}_{X\times Y}|_S \otimes_{f^{-1}\mathcal{O}_X} f^{-1}\mathcal{F})[-c], \end{aligned}$$

and similarly:

$$R\pi_{Y\,!}(\mathcal{E}\mathcal{G}) \simeq \mathcal{O}^{(n,0)}_{Y\times Y}|_Y \otimes_{\mathcal{O}_Y} \mathcal{G}[-n].$$

Summarizing, we have the following morphism of distinguished triangles:

$$\begin{tikzcd} \mathcal{O}^{(n,0)}_{Y\times Y}|_Y \otimes_{\mathcal{O}_Y} \mathcal{G}[-n] \arrow[r] \arrow[d, "\widetilde{\alpha}(s)"'] & \mathcal{D}\mathcal{G} \arrow[r] \arrow[d, "\alpha(s)"'] & R\dot{\pi}_{Y\,*}(\mathcal{E}\mathcal{G}) \arrow[r, "+1"'] \arrow[d, "\wr"'] & {} \\ Rg_!(\mathcal{O}^{(n,0)}_{X\times Y}|_S \otimes_{f^{-1}\mathcal{O}_X} f^{-1}\mathcal{F})[-c] \arrow[r] & \underline{\Phi}_S(\mathcal{D}\mathcal{F}) \arrow[r] & R\dot{\pi}_{Y\,*}(\underline{\Phi}^{\mu}_{S}(\mathcal{E}\mathcal{F})) \arrow[r, "+1"'] & {} \end{tikzcd} \tag{3.3}$$

Since $c, n > 0$, $H^0(\tilde{\alpha}(s))$ is the morphism $0 \longrightarrow 0$. Taking the cohomology groups in (3.3), we get the commutative diagram in which the horizontal lines are exact:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \mathcal{DG} \arrow[r] \arrow[d, "H^0(\alpha(s))"'] & \dot{\pi}_{Y*}(\mathcal{EG}) \arrow[r] \arrow[d, "\wr"'] & H^{-n+1}(\mathcal{O}^{(n,0)}_{Y\times Y}|_Y \otimes_{\mathcal{O}_Y} \mathcal{G}) \arrow[d, "H^1(\tilde{\alpha}(s))"'] \\
0 \arrow[r] & H^0(\underline{\Phi}_S(\mathcal{DF})) \arrow[r] & \dot{\pi}_{Y*}(\underline{\Phi}^\mu_S(\mathcal{EF})) \arrow[r, "\beta"] & H^{-c+1}(Rg_!(\mathcal{O}^{(n,0)}_{X\times Y}|_S \otimes_{f^{-1}\mathcal{O}_X} f^{-1}\mathcal{F})).
\end{tikzcd}$$

First, assume $n, c > 1$. Then $H^1(\tilde{\alpha}(s))$ is the morphism $0 \longrightarrow 0$.

Next, assume $n = 1$. In such a case $c = 1$ and the contact transform (2.3) comes from an isomorphism $X \simeq Y$, since $P^*X \simeq X$, $P^*_S(X \times Y) \simeq S$, and $P^*Y \simeq Y$. Moreover, the non-degenerate section $s \in H^1_S(X \times Y; \mathcal{O}^{(1,0)}_{X\times Y})$ is locally a multiple of the fundamental class of S in $X \times Y$. Then $H^1(\tilde{\alpha}(s))$ is clearly an isomorphism.

Finally, assume $n > 1$, $c = 1$. In this case, $H^{-c+1}(Rg_!(\mathcal{O}^{(n,0)}_{X\times Y}|_S \otimes_{f^{-1}\mathcal{O}_X} f^{-1}\mathcal{F})) = 0$, and hence, the diagram being commutative, $\beta = 0$. This proves (i).

As for (ii), by Proposition 2.8-(ii) and hypothesis (b), the complex $\underline{\Phi}_S\,\mathcal{DF}$ is concentrated in degree zero. The claim follows.

(iii) follows by applying the functor $R\mathcal{H}om_{\mathcal{D}_Y}(\cdot, \mathcal{O}_Y)$ to (ii), and by using isomorphism (2.8) for $\mathcal{M} = \mathcal{DF}$. q.e.d.

Remark 3.4 Hypotheses (a) and (b) in the theorem above imply that

$$H^j(\hat{y}; \mathcal{F} \otimes_{\mathcal{O}_X} \Omega_X) = 0 \quad \forall j < n - c.$$

In fact, by Proposition 2.8 (iv) this is equivalent to saying that $\underline{D}_Y\,\underline{\Phi}_S\,\mathcal{DF}[-n]$ is concentrated in degree zero. This is the case since $\underline{\Phi}_S\,\mathcal{DF} \simeq \mathcal{DG}$.

Combining the above theorem and Proposition 2.3 or Proposition A.1, we get the following corollary.

Corollary 3.5 *Let $G \in \mathbf{D}^{\mathrm{b}}(\mathbb{C}_Y)$, let $\mathcal{F}$ and $\mathcal{G}$ be holomorphic line bundles on X and Y respectively, and take a section $s \in \Gamma(X \times Y; \mathcal{B}^{(n,0)}_{S|X\times Y}(\mathcal{F}, \mathcal{G}^*))$. Assume (2.2), Y is connected, and assume that hypotheses (a) and (b) of Theorem 3.3 are verified. Then, $\alpha(s)$ induces the isomorphisms:*

$$\begin{aligned}
\mathrm{R}\Gamma(X; \Phi_{\widetilde{S}}\, G \otimes \mathcal{F}^*) &\simeq \mathrm{R}\Gamma(Y; G \otimes \mathcal{G}^*), \\
\mathrm{R}\Gamma(X; R\mathcal{H}om(\Phi_{\widetilde{S}}\, G, \mathcal{F}^*)) &\simeq \mathrm{R}\Gamma(Y; R\mathcal{H}om(G, \mathcal{G}^*)),
\end{aligned}$$

and similarly with $\mathrm{R}\Gamma$ replaced by $\mathrm{R}\Gamma_c$, or $\otimes$ replaced by $\overset{\mathrm{w}}{\otimes}$, or $R\mathcal{H}om$ replaced by $\mathcal{T}hom$.

3.2 Another approach, using kernels

In this section we will make use of Kashiwara's functor $\mathcal{T}hom$ (see Appendix A.1), and of the notations and results of Appendix A.2.

Consider two correspondences

$$\begin{tikzcd} & S \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y, \end{tikzcd} \qquad \begin{tikzcd} & T \arrow[dl, "h"'] \arrow[dr, "k"] & \\ Y & & Z, \end{tikzcd}$$

satisfying (2.2)-(a) (i.e., f, g, h, k are smooth and proper), and set $d_X = \dim^{\mathbb{C}} X$, $c_S = \operatorname{codim}^{\mathbb{C}}_{X\times Y} S$, and similarly for d_Y, d_Z, c_T.

Our first aim is to discuss the compatibility of the isomorphism in Lemma 3.1 with the composition:

$$\begin{aligned} \circ : \operatorname{Hom}_{\mathbf{D}^{\mathrm{b}}(\mathcal{D}_Z)}(\mathcal{D}\mathcal{H}, \underline{\Phi}_T\,\mathcal{D}\mathcal{G}) \otimes \operatorname{Hom}_{\mathbf{D}^{\mathrm{b}}(\mathcal{D}_Z)}(\underline{\Phi}_T\,\mathcal{D}\mathcal{G}, \underline{\Phi}_T\,\underline{\Phi}_S\,\mathcal{D}\mathcal{F}) \longrightarrow \\ \longrightarrow \operatorname{Hom}_{\mathbf{D}^{\mathrm{b}}(\mathcal{D}_Z)}(\mathcal{D}\mathcal{H}, \underline{\Phi}_T\,\underline{\Phi}_S\,\mathcal{D}\mathcal{F}), \end{aligned}$$

where $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$, are holomorphic line bundles on X, Y, Z respectively.

Recall that $\mathcal{B}_{S|X\times Y} \simeq \mathcal{T}hom(\mathbb{C}_S[-c_S], \mathcal{O}_{X\times Y})$, and consider the morphisms:

$$\begin{aligned} \underline{\Phi}_T : \quad & \operatorname{Hom}_{\mathbf{D}^{\mathrm{b}}(\mathcal{D}_Y)}(\mathcal{D}\mathcal{G}, \underline{\Phi}_S\,\mathcal{D}\mathcal{F}) \longrightarrow \operatorname{Hom}_{\mathbf{D}^{\mathrm{b}}(\mathcal{D}_Z)}(\underline{\Phi}_T\,\mathcal{D}\mathcal{G}, \underline{\Phi}_T\,\underline{\Phi}_S\,\mathcal{D}\mathcal{F}), \\ \alpha_S : \quad & H^0\operatorname{THom}(\mathbb{C}_S[-c_S], \mathcal{O}^{(n,0)}_{X\times Y}(\mathcal{F}, \mathcal{G}^*)) \xrightarrow{\sim} \operatorname{Hom}_{\mathbf{D}^{\mathrm{b}}(\mathcal{D}_Y)}(\mathcal{D}\mathcal{G}, \underline{\Phi}_S\,\mathcal{D}\mathcal{F}), \\ \alpha_T : \quad & H^0\operatorname{THom}(\mathbb{C}_T[-c_T], \mathcal{O}^{(n,0)}_{Y\times Z}(\mathcal{G}, \mathcal{H}^*)) \xrightarrow{\sim} \operatorname{Hom}_{\mathbf{D}^{\mathrm{b}}(\mathcal{D}_Z)}(\mathcal{D}\mathcal{H}, \underline{\Phi}_T\,\mathcal{D}\mathcal{G}), \\ \alpha_{ST} : \quad & H^0\operatorname{THom}(\mathbb{C}_S \circ \mathbb{C}_T[d_Y - c_S - c_T], \mathcal{O}^{(n,0)}_{X\times Z}(\mathcal{F}, \mathcal{H}^*)) \\ & \xrightarrow{\sim} \operatorname{Hom}_{\mathbf{D}^{\mathrm{b}}(\mathcal{D}_Z)}(\mathcal{D}\mathcal{H}, \underline{\Phi}_T\,\underline{\Phi}_S\,\mathcal{D}\mathcal{F}), \\ \circ : \quad & H^0\operatorname{THom}(\mathbb{C}_S[-c_S], \mathcal{O}^{(n,0)}_{X\times Y}(\mathcal{F}, \mathcal{G}^*)) \otimes H^0\operatorname{THom}(\mathbb{C}_T[-c_T], \mathcal{O}^{(n,0)}_{Y\times Z}(\mathcal{G}, \mathcal{H}^*)) \\ & \longrightarrow H^0\operatorname{THom}(\mathbb{C}_S \circ \mathbb{C}_T[d_Y - c_S - c_T], \mathcal{O}^{(n,0)}_{X\times Z}(\mathcal{F}, \mathcal{H}^*)). \end{aligned}$$

The first morphism expresses the functoriality of $\underline{\Phi}_T$. The isomorphisms α_S and α_T are those introduced in Lemma 3.1. The isomorphism α_{ST} is similarly obtained using Proposition A.7. The last morphism is constructed using Proposition A.7 and the integration morphism

$$Rq_{13!}\mathcal{O}^{(0,n,0)}_{X\times Y\times Z} \longrightarrow \mathcal{O}_{X\times Z}[-d_Y].$$

For a proof of the next proposition, we refer to [18, Proposition 11.4.7], where a similar result is obtained for $H^0\operatorname{THom}$ replaced by Hom.

Proposition 3.6 *With the same notations as above, let:*

$$s_1 \in H^0 \operatorname{THom}(\mathbb{C}_S[-c_S], \mathcal{O}^{(n,0)}_{X\times Y}(\mathcal{F},\mathcal{G}^*)), \quad s_2 \in H^0 \operatorname{THom}(\mathbb{C}_T[-c_T], \mathcal{O}^{(n,0)}_{Y\times Z}(\mathcal{G},\mathcal{H}^*)).$$

Then we have the equality in $\operatorname{Hom}_{\mathbf{D}^{\mathrm{b}}(\mathcal{D}_Z)}(\mathcal{D}\mathcal{H}, \underline{\Phi}_T\,\underline{\Phi}_S\,\mathcal{D}\mathcal{F})$*:*

$$\underline{\Phi}_T(\alpha_T(s_2)) \circ \alpha_S(s_1) = \alpha_{ST}(s_2 \circ s_1).$$

Consider now the correspondences

$$\begin{tikzcd}[column sep=small] & S \arrow[dl] \arrow[dr] & & & \widetilde{S} \arrow[dl] \arrow[dr] & \\ X & & Y, & Y & & X, \end{tikzcd}$$

and denote by Δ_X the diagonal of $X \times X$.

Definition 3.7 We denote by $\delta_{\mathcal{F}}$ the global section of $\mathcal{B}^{(d_X,0)}_{\Delta_X|X\times X}(\mathcal{F},\mathcal{F}^*)$ corresponding to the identity of $\mathcal{D}\mathcal{F}$ via the isomorphism:

$$\Gamma(X\times X; \mathcal{B}^{(d_X,0)}_{\Delta_X|X\times X}(\mathcal{F},\mathcal{F}^*)) \simeq H^0\mathrm{R}\Gamma(X; R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}\mathcal{F},\mathcal{D}\mathcal{F})).$$

Theorem 3.8 *(i) Assume (2.2)-(a). Let*

$$s_1 \in \Gamma(X\times Y; \mathcal{B}^{(n,0)}_{S|X\times Y}(\mathcal{F},\mathcal{G}^*)), \quad s_2 \in \Gamma(Y\times X; \mathcal{B}^{(n,0)}_{\widetilde{S}|Y\times X}(\mathcal{G},\mathcal{F}^*)),$$

and assume that there is a distinguished triangle:

$$\mathbb{C}_S[-c_S] \circ \mathbb{C}_{\widetilde{S}}[-c_S] \longrightarrow \mathbb{C}_{\Delta_X}[-d_X - d_Y] \longrightarrow M_{X\times X}[-d_Y] \xrightarrow{+1} \tag{3.4}$$

for an object $M \in \mathbf{D}^{\mathrm{b}}(\mathrm{Mod}(\mathbb{C}))$ *satisfying:*

$$H^j \operatorname{THom}(M_{X\times X}, \mathcal{O}^{(d_X,0)}_{X\times X}(\mathcal{F},\mathcal{F}^*)) = 0, \quad \text{for } j = 0,1.$$

Then:

$$s_1 \circ s_2 \in \Gamma(X\times X; \mathcal{B}^{(d_X,0)}_{\Delta_X|X\times X}(\mathcal{F},\mathcal{F}^*)). \tag{3.5}$$

(ii) Assume moreover that there is a distinguished triangle:

$$\mathbb{C}_{\widetilde{S}}[-c_S] \circ \mathbb{C}_S[-c_S] \longrightarrow \mathbb{C}_{\Delta_Y}[-d_X - d_Y] \longrightarrow N_{Y\times Y}[-d_X] \xrightarrow{+1}$$

for an object $N \in \mathbf{D}^{\mathrm{b}}(\mathrm{Mod}(\mathbb{C}))$ *satisfying:*

$$H^j \operatorname{THom}(N_{Y\times Y}, \mathcal{O}^{(d_Y,0)}_{Y\times Y}(\mathcal{G},\mathcal{G}^*)) = 0, \quad \text{for } j = 0,1,$$

and that the equalities $s_1\circ s_2 = \delta_{\mathcal{F}}$*, and* $s_2 \circ s_1 = \delta_{\mathcal{G}}$ *hold. Then, the morphisms:*

$$\begin{aligned} \alpha(s_1): &\quad \mathcal{D}\mathcal{G} \longrightarrow \underline{\Phi}_S\,\mathcal{D}\mathcal{F}, \\ \alpha(s_2): &\quad \mathcal{D}\mathcal{F} \longrightarrow \underline{\Phi}_{\widetilde{S}}\,\mathcal{D}\mathcal{G}. \end{aligned}$$

are isomorphisms.

Proof: One has

$$\begin{aligned} s_1 \circ s_2 \;\in\; & H^0\,\mathrm{THom}(\mathbb{C}_S \circ \mathbb{C}_{\widetilde{S}}[d_Y - 2c_S], \mathcal{O}^{(d_X,0)}_{X\times X}(\mathcal{F},\mathcal{F}^*)) \\ & \simeq H^0\,\mathrm{THom}(\mathbb{C}_{\Delta_X}[-d_X], \mathcal{O}^{(d_X,0)}_{X\times X}(\mathcal{F},\mathcal{F}^*)) \\ & \simeq \Gamma(X\times X; \mathcal{B}^{(d_X,0)}_{\Delta_X|X\times X}(\mathcal{F},\mathcal{F}^*)), \end{aligned}$$

where the first isomorphism follows by applying the functor $\mathrm{THom}(\cdot, \mathcal{O}^{(d_X,0)}_{X\times X}(\mathcal{F},\mathcal{F}^*))$ to the distinguished triangle (3.4). This proves the first statement.

The second statement follows from Proposition 3.6. q.e.d.

Theorem 3.9 *Let $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_X)$. Assume (2.2), assume that X is connected and simply connected, and assume that $\mathrm{R}\Gamma(X; R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{O}_X)) = 0$. Then, the adjunction morphism:*

$$\mathcal{M} \longrightarrow \underline{\Phi}_{\widetilde{S}}(\underline{\Phi}_S(\mathcal{M}))$$

is an isomorphism.

Proof: Combining Proposition A.7 and Corollary A.4, we obtain that the adjunction morphism is induced by a natural morphism $\mathbb{C}_{\widetilde{S}}[-c_S] \circ \mathbb{C}_S[-c_S] \longrightarrow \mathbb{C}_{\Delta_X}[-d_X]$. By hypothesis (2.2), the third term N of a distinguished triangle

$$\mathbb{C}_{\widetilde{S}}[-c_S] \circ \mathbb{C}_S[-c_S] \longrightarrow \mathbb{C}_{\Delta_X}[-d_X] \longrightarrow N \xrightarrow[+1]{}$$

satisfies $SS(N) \subset T^*_{X\times X}(X\times X)$. Applying the functor $\mathcal{T}hom(\cdot, \mathcal{O}_{X\times X}) \underline{\circ} \mathcal{M}$, we get a distinguished triangle:

$$\mathcal{T}hom(N, \mathcal{O}_{X\times X}) \underline{\circ} \mathcal{M} \longrightarrow \mathcal{M} \longrightarrow \underline{\Phi}_{\widetilde{S}}(\underline{\Phi}_S(\mathcal{M})) \xrightarrow[+1]{}$$

If $N = 0$ the claim follows. If $N \neq 0$, then for all j, $H^j(N)$ is a constant sheaf, since it is locally constant and X is simply connected. Moreover, X is compact. In fact, consider the commutative diagram

$$\begin{tikzcd} S \times_Y \widetilde{S} \arrow[r, "h"] \arrow[d, "\gamma"'] & \widetilde{S} \arrow[d, "f"] \\ X \times X \arrow[r, "q_2"] & X, \end{tikzcd}$$

where the arrows are the natural ones. Since there exists j with $H^j(N)$ constant and non zero, γ is surjective. Since h and f are proper, $q_2 \circ \gamma$ is proper. It follows that $X \times X = \gamma(\gamma^{-1}(q_2^{-1}(X)))$, is compact. Hence X is compact.

We shall prove that $\underline{\Phi}_N(\mathcal{M})_x = 0$. Arguing by induction on the amplitude of N, we may reduce to $N = \mathbb{C}_{X\times X}$. In such a case

$$\begin{aligned}
(\mathcal{T}hom(N, \mathcal{O}_{X\times X}) \underline{\circ} \mathcal{M})_x &\simeq \underline{q_{2!}}(\mathcal{M} \boxtimes \mathcal{O}_X)_x \\
&\simeq Rq_{2!}(q_1^{-1}\Omega_X \otimes^L_{q_1^{-1}\mathcal{D}_X} (\mathcal{M} \boxtimes \mathcal{O}_X)) \\
&\simeq \mathrm{R}\Gamma(X; \Omega_X \otimes^L_{\mathcal{D}_X} \mathcal{M}) \otimes \mathcal{O}_{X,x}.
\end{aligned}$$

In the last isomorphism we used the Künneth formula (cf. [26]), which holds since X is compact and $\mathcal{M}$ is good. Since $\mathrm{R}\Gamma(X; R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)) = 0$, we have by duality $\mathrm{R}\Gamma(X; \Omega_X \otimes^L_{\mathcal{D}_X} \mathcal{M}) = 0$, which completes the proof. q.e.d.

4 Projective duality

4.1 Main theorem

Let $\mathbb{P}$ be a complex n-dimensional projective space, $\mathbb{P}^*$ the dual projective space, and $\mathbb{A}$ the hypersurface of $\mathbb{P} \times \mathbb{P}^*$ given by the incidence relation

$$\mathbb{A} = \{(z, \zeta) \in \mathbb{P} \times \mathbb{P}^*;\ \langle z, \zeta\rangle = 0\}.$$

Let us consider the correspondence:

$$\begin{tikzcd} & \mathbb{A} \arrow[dl, "f"'] \arrow[dr, "g"] & \\ \mathbb{P} & & \mathbb{P}^*. \end{tikzcd} \tag{4.1}$$

Denote by $P^*\mathbb{P} = \dot{T}^*\mathbb{P}/\mathbb{C}^\times$ the projective cotangent bundle to $\mathbb{P}$, and notice that $P^*\mathbb{P} \simeq \mathbb{A} \subset \mathbb{P} \times \mathbb{P}^*$. Since $\mathbb{A}$ is a hypersurface, $\mathbb{A} \simeq P^*_{\mathbb{A}}(\mathbb{P} \times \mathbb{P}^*)$, and we have the diagram:

$$\begin{tikzcd} P^*\mathbb{P} \arrow[d] & P^*_{\mathbb{A}}(\mathbb{P} \times \mathbb{P}^*) \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[d, "\wr"'] & P^*\mathbb{P}^* \arrow[d] \\ \mathbb{P} & \mathbb{A} \arrow[l, "f"'] \arrow[r, "g"] & \mathbb{P}^*. \end{tikzcd}$$

In particular, setting $\Lambda = \dot{T}^*_{\mathbb{A}}(\mathbb{P} \times \mathbb{P}^*)$, the associated microlocal correspondence:

$$\begin{tikzcd} & \Lambda \arrow[dl, "p_1|_\Lambda"'] \arrow[dr, "p_2^a|_\Lambda"] & \\ \dot{T}^*\mathbb{P} & & \dot{T}^*\mathbb{P}^* \end{tikzcd} \tag{4.2}$$

induces a globally defined contact transformation

$$\chi : \dot{T}^*\mathbb{P} \xrightarrow{\sim} \dot{T}^*\mathbb{P}^*. \tag{4.3}$$

Hypothesis (2.2) is thus satisfied for the correspondence (4.1).

For $k, k' \in \mathbb{Z}$, denote by $\mathcal{O}_\mathbb{P}(k)$ the $-k$-th tensor power of the tautological line bundle, and set for short:

$$\begin{aligned} \mathcal{D}_\mathbb{P}(k) &= \mathcal{D}_\mathbb{P} \otimes_{\mathcal{O}_\mathbb{P}} \mathcal{O}_\mathbb{P}(k), \\ \mathcal{B}^{(n,0)}_\mathbb{A}(k,k') &= \mathcal{B}^{(n,0)}_{\mathbb{A}|\mathbb{P}\times\mathbb{P}^*}(\mathcal{O}_\mathbb{P}(k), \mathcal{O}_{\mathbb{P}^*}(k')). \end{aligned}$$

Set

$$k^* = -n-1-k,$$

and recall that $\Omega_\mathbb{P} \simeq \mathcal{O}_\mathbb{P}(-n-1)$, so that $\Omega_\mathbb{P} \otimes_{\mathcal{O}_\mathbb{P}} (\mathcal{O}_\mathbb{P}(k))^* \simeq \mathcal{O}_\mathbb{P}(k^*)$.

Notation 4.1 Let $[z] = [z_0, z'] = [z_0, \ldots, z_n]$ be a system of homogeneous coordinates on $\mathbb{P}$. Let $E \simeq \mathbb{C}^n$ be the affine chart of $\mathbb{P}$ defined by $z_0 \neq 0$, endowed with the system of coordinates $(t) = (z_1/z_0, \ldots, z_n/z_0)$. Set $O = [1, 0, \ldots, 0] \in E$. Let $[\zeta]$ be the dual system of homogeneous coordinates in $\mathbb{P}^*$. Let $E^* \subset \mathbb{P}^*$ be the affine chart given by $\zeta_0 \neq 0$, endowed with the system of coordinates $(\tau) = (\zeta_1/\zeta_0, \ldots, \zeta_n/\zeta_0)$. Set $O^* = [1, 0, \ldots, 0] \in E^*$.

Remark 4.2 Note that, using the identification $T^*E \simeq E \times E^*$, the restriction of the contact transformation (4.3) to the affine chart E is the Legendre transform, defined for $\langle t, \tau \rangle \neq 0$:

$$\begin{array}{rccc} \chi : & T^*E & \longrightarrow & T^*(E^*) \\ & (t;\tau) & \mapsto & (\tau/\langle t,\tau\rangle; -\langle t,\tau\rangle t). \end{array}$$

Following Leray [21, p. 94], we set:

$$\begin{aligned} \omega(t) &= dt_1 \wedge \ldots \wedge dt_n, \\ \omega^*(z) &= \sum_{i=0}^n (-1)^i z_i \, dz_0 \wedge \ldots \wedge \widehat{dz_i} \wedge \ldots \wedge dz_n. \end{aligned}$$

Notice that, in the affine chart E, one has:

$$\omega^*(z) = z_0^{n+1}\, \omega(z'/z_0).$$

The form

$$s_k = s_k(z,\zeta) = \frac{\omega^*(z)}{\langle z, \zeta\rangle^{n+1+k}} \tag{4.4}$$

is thus a well defined section of $\mathcal{O}^{(n,0)}_{\mathbb{P}\times\mathbb{P}^*}(-k,k^*)$ on $\mathbb{P}\times\mathbb{P}^*\setminus\mathbb{A}$.

If $n+1+k>0$ (i.e., if $k^*<0$), s_k has meromorphic singularities on $\mathbb{A}$, and its image via the natural morphism

$$\Gamma(\mathbb{P}\times\mathbb{P}^*\setminus\mathbb{A};\mathcal{O}^{(n,0)}_{\mathbb{P}\times\mathbb{P}^*}(-k,k^*))\longrightarrow H^1_{\mathbb{A}}(\mathbb{P}\times\mathbb{P}^*;\mathcal{O}^{(n,0)}_{\mathbb{P}\times\mathbb{P}^*}(-k,k^*))$$

defines a section (that we denote by the same symbol):

$$s_k\in\Gamma(\mathbb{P}\times\mathbb{P}^*;\mathcal{B}^{(n,0)}_{\mathbb{A}}(-k,k^*)),$$

which is non-degenerate.

Theorem 4.3 *(i) Assume $k>-n-1$. Then*

$$H^0\alpha(s_k):\mathcal{D}_{\mathbb{P}^*}(-k^*)\longrightarrow H^0\,\underline{\Phi}_{\mathbb{A}}(\mathcal{D}_{\mathbb{P}}(-k))$$

is an isomorphism in $\mathrm{Mod}(\mathcal{D}_{\mathbb{P}^*})$.

(ii) Assume $-n-1<k<0$. Then

$$\alpha(s_k):\mathcal{D}_{\mathbb{P}^*}(-k^*)\longrightarrow\underline{\Phi}_{\mathbb{A}}(\mathcal{D}_{\mathbb{P}}(-k))$$

is an isomorphism in $\mathbf{D}^{\mathrm{b}}(\mathcal{D}_{\mathbb{P}^*})$, and it induces an isomorphism in $\mathbf{D}^{\mathrm{b}}(\mathbb{C}_{\mathbb{P}^*})$:

$$\Phi_{\mathbb{A}}\,\mathcal{O}_{\mathbb{P}}(k)\xrightarrow{\sim}\mathcal{O}_{\mathbb{P}^*}(k^*).$$

Proof: We shall apply Theorem 3.3 with $\mathcal{F}=\mathcal{O}_{\mathbb{P}}(-k)$, $\mathcal{G}=\mathcal{O}_{\mathbb{P}}(-k^*)$, $s=s_k$. Since s_k is non-degenerate for $k^*<0$, it remains to check hypothesis (b). By the following Lemma 4.4, hypothesis (b) is verified if and only if $k<0$. q.e.d.

Lemma 4.4 *Let $\zeta\in\mathbb{P}^*$, and consider $\widehat{\zeta}\subset\mathbb{P}$. Then:*

$$\begin{cases}\Gamma(\widehat{\zeta};\mathcal{O}_{\mathbb{P}}(k))=\begin{cases}0 & \text{for } k<0,\ n\geq 1\\ \text{non zero and finite dimensional for } k\geq 0,\ n>1,\end{cases}\\ H^{n-1}(\widehat{\zeta};\mathcal{O}_{\mathbb{P}}(k)) \text{ is infinite dimensional for every } k, \text{ and for } n>1,\\ H^j(\widehat{\zeta};\mathcal{O}_{\mathbb{P}}(k))=0 \quad \text{for every } k \text{ and for } j\neq 0,n-1.\end{cases}$$

Proof: First, recall the following result of Serre:

$$\begin{cases}\oplus_k\Gamma(\mathbb{P};\mathcal{O}_{\mathbb{P}}(k))\simeq\mathbb{C}[z_0,\dots,z_n] \quad \text{as graded rings,}\\ H^j(\mathbb{P};\mathcal{O}_{\mathbb{P}}(k))=0 \quad \text{for } 0<j<n \text{ and for every } k,\\ H^n(\mathbb{P};\mathcal{O}_{\mathbb{P}}(k))'\simeq\Gamma(\mathbb{P};\mathcal{O}_{\mathbb{P}}(k^*)),\end{cases}\tag{4.5}$$

where $(\cdot)'$ denotes the dual of a finite dimensional vector space.

We have a distinguished triangle:

$$\mathrm{R}\Gamma_c(\mathbb{P}\setminus\widehat{\zeta};\mathcal{O}_{\mathbb{P}}(k)) \longrightarrow \mathrm{R}\Gamma(\mathbb{P};\mathcal{O}_{\mathbb{P}}(k)) \longrightarrow \mathrm{R}\Gamma(\widehat{\zeta};\mathcal{O}_{\mathbb{P}}(k)) \xrightarrow[+1]{} .$$

The subset $\widehat{\zeta}$ is a hyperplane of $\mathbb{P}$. Identifying $\mathbb{P}\setminus\widehat{\zeta}$ to an affine chart $E \simeq \mathbb{C}^n$, we thus have an isomorphism:

$$\begin{aligned}\mathrm{R}\Gamma_c(\mathbb{P}\setminus\widehat{\zeta};\mathcal{O}_{\mathbb{P}}(k))[n] &\simeq \mathrm{R}\Gamma_c(E;\mathcal{O}_E)[n]\\ &\simeq H^n_c(E;\mathcal{O}_E).\end{aligned}$$

Recall that $H^n_c(E;\mathcal{O}_E)$ is isomorphic to the space $\Gamma(E;\Omega_E)'$ of analytic functionals of Martineau. Using (4.5), we are reduced to prove the surjectivity of the map:

$$\Gamma(E;\Omega_E)' \longrightarrow H^n(\mathbb{P};\mathcal{O}_{\mathbb{P}}(k)).$$

Since the right hand side is finite dimensional, by duality it is equivalent to show the injectivity of the natural restriction map:

$$\Gamma(\mathbb{P};\mathcal{O}_{\mathbb{P}}(k^*)) \longrightarrow \Gamma(E;\mathcal{O}_E),$$

which is obvious. q.e.d.

Applying Corollary 3.5 with $\mathcal{F}=\mathcal{O}_{\mathbb{P}}(-k)$, $\mathcal{G}=\mathcal{O}_{\mathbb{P}^*}(-k^*)$, we get:

Corollary 4.5 *Assume $-n-1<k<0$, and let $F\in\mathbf{D}^{\mathrm{b}}(\mathbb{C}_{\mathbb{P}})$. Then, $\alpha(s_k)$ induces isomorphisms:*

$$\begin{aligned}\mathrm{R}\Gamma(\mathbb{P};F\otimes\mathcal{O}_{\mathbb{P}}(k)) &\simeq \mathrm{R}\Gamma(\mathbb{P}^*;\Phi_{\mathbb{A}}F\otimes\mathcal{O}_{\mathbb{P}^*}(k^*)),\\ \mathrm{R}\Gamma(\mathbb{P};R\mathcal{H}om(F,\mathcal{O}_{\mathbb{P}}(k))) &\simeq \mathrm{R}\Gamma(\mathbb{P}^*;R\mathcal{H}om(\Phi_{\mathbb{A}}F,\mathcal{O}_{\mathbb{P}^*}(k^*))).\end{aligned}$$

Moreover, assuming F is $\mathbb{R}$-constructible, similar isomorphisms hold with $\otimes$ replaced by $\overset{\mathrm{w}}{\otimes}$ and $R\mathcal{H}om$ replaced by $\mathcal{T}hom$.

4.2 Another approach, using kernels

In this section, we will use Theorem 3.8 to give an alternative proof of Theorem 4.3. In particular, we shall explicitly construct an inverse (up to a non zero constant) for the isomorphism in Theorem 4.3-(ii). The arguments that we shall use here are very classical, and go back to Leray [21] (see also [28]).

We begin with a geometric lemma.

Lemma 4.6 *Let $\Delta_{\mathbb{P}\times\mathbb{P}}$ be the diagonal of $\mathbb{P}\times\mathbb{P}$. Then, there is a natural isomorphism in $\mathbf{D}^{\mathrm{b}}(\mathbb{C}_{\mathbb{P}\times\mathbb{P}})$:*

$$H^j(\mathbb{C}_{\mathbb{A}}\circ\mathbb{C}_{\widetilde{\mathbb{A}}}) \simeq \begin{cases}\mathbb{C}_{\Delta_{\mathbb{P}}} & \text{for } j=2n-2,\\ \mathbb{C}_{\mathbb{P}\times\mathbb{P}} & \text{for } j=0,2,\ldots,2n-4,\\ 0 & \text{otherwise.}\end{cases}$$

Proof: By definition,

$$\mathbb{C}_{\mathbb{A}} \circ \mathbb{C}_{\widetilde{\mathbb{A}}} \simeq Rq_{13!}(\mathbb{C}_{\mathbb{A}\times_{\mathbb{P}^*}\widetilde{\mathbb{A}}}).$$

Recall that for $z \in \mathbb{P}$, the set $\widehat{z}$ is a hyperplane of $\mathbb{P}^*$. Since

$$\mathbb{A} \times_{\mathbb{P}^*} \widetilde{\mathbb{A}} = \{(z, \zeta, \widetilde{z}) \in \mathbb{P} \times \mathbb{P}^* \times \mathbb{P}; \zeta \in z \cap \widetilde{z}\},$$

it follows that

$$q_{13}^{-1}(z, \widetilde{z}) \simeq \begin{cases} \mathbb{P}^{n-1}, & \text{if } z = \widetilde{z}, \\ \mathbb{P}^{n-2}, & \text{otherwise,} \end{cases}$$

and hence:

$$Rq_{13!}(\mathbb{C}_{\mathbb{A}\times_{\mathbb{P}^*}\widetilde{\mathbb{A}}})_{(z,\widetilde{z})} \simeq \begin{cases} \bigoplus\limits_{j=0}^{n-1} \mathbb{C}[-2j], & \text{if } z = \widetilde{z}, \\ \bigoplus\limits_{j=0}^{n-2} \mathbb{C}[-2j], & \text{otherwise.} \end{cases}$$

Moreover, the locally constant sheaves $H^j(Rq_{13!}(\mathbb{C}_{\mathbb{A}\times_{\mathbb{P}^*}\widetilde{\mathbb{A}}}))$ are constant since $\mathbb{P} \times \mathbb{P}$ and $\Delta_{\mathbb{P}}$ are simply connected. q.e.d.

Remark 4.7 One could prove that in fact $\mathbb{C}_{\mathbb{A}} \circ \mathbb{C}_{\widetilde{\mathbb{A}}}$ splits as the direct sum of its cohomology groups, but we shall not need this stronger result here. (This is also a very special case of the results of Beilinson-Bernstein-Deligne-Gabber [4].)

Theorem 4.8 *Assume $-n-1 < k < 0$. Then $\alpha(s_k)$ is an isomorphism, the inverse being given by $\alpha(s_{k^*})$, up to a non zero constant.*

Proof: The fact that $\alpha(s_k)$ is an isomorphism was proven in Theorem 4.3-(ii). In order to show that an inverse of $\alpha(s_k)$ is given by $\alpha(s_{k^*})$ (up to a non zero constant), we will apply Theorem 3.8 to the present situation, using the results of Appendix A.2.

Using Lemma 4.6 and Theorem 3.8, to complete the proof it remains to show that $s_k \circ s_{k^*}$ and $s_{k^*} \circ s_k$ are non zero multiples of the canonical sections $\delta_k = \delta_{\mathcal{O}_{\mathbb{P}}(k)}$ and $\delta_{k^*} = \delta_{\mathcal{O}_{\mathbb{P}^*}(k^*)}$ of $\mathcal{B}^{(n,0)}_{\Delta_{\mathbb{P}}|\mathbb{P}\times\mathbb{P}}(-k, k)$ and $\mathcal{B}^{(n,0)}_{\Delta_{\mathbb{P}^*}|\mathbb{P}^*\times\mathbb{P}^*}(-k^*, k^*)$ respectively, introduced in Definition 3.7. Since the arguments are identical, we will just treat the first composite.

Consider the projections:

$$\begin{tikzcd} & \mathbb{P} \times \mathbb{P}^* \times \mathbb{P} \arrow[dl, "q_{12}"'] \arrow[d, "q_{13}"] \arrow[dr, "q_{23}"] & \\ \mathbb{P} \times \mathbb{P}^* & \mathbb{P} \times \mathbb{P} & \mathbb{P}^* \times \mathbb{P}. \end{tikzcd}$$

We have the morphisms:

$$\begin{aligned}
&\mathcal{T}hom(\mathbb{C}_{\mathbb{A}}, \mathcal{O}^{(n,0)}_{\mathbb{P}\times\mathbb{P}^*}(-k,k^*)) \circ \mathcal{T}hom(\mathbb{C}_{\widetilde{\mathbb{A}}}, \mathcal{O}^{(n,0)}_{\mathbb{P}^*\times\mathbb{P}}(-k^*,k)) \\
&\qquad \longrightarrow Rq_{13!}\,\mathcal{T}hom(\mathbb{C}_{\mathbb{A}\times_{\mathbb{P}^*}\widetilde{\mathbb{A}}}, \mathcal{O}^{(n,n,0)}_{\mathbb{P}\times\mathbb{P}^*\times\mathbb{P}}(-k,0,k)) \\
&\qquad \xrightarrow{\sim} \mathcal{T}hom(\mathbb{C}_{\mathbb{A}} \circ \mathbb{C}_{\widetilde{\mathbb{A}}}, \mathcal{O}^{(n,0)}_{\mathbb{P}\times\mathbb{P}}(-k,k))[-n],
\end{aligned}$$

where the last isomorphism follows from a theorem of [14], and is induced by the residue map:

$$Rq_{13!}\mathcal{O}^{(n,n,0)}_{\mathbb{P}\times\mathbb{P}^*\times\mathbb{P}}[n] \longrightarrow \mathcal{O}^{(n,0)}_{\mathbb{P}\times\mathbb{P}}.$$

Applying the functor $\mathrm{R}\Gamma(\mathbb{P}\times\mathbb{P};\cdot)$, and using Lemma 4.6, we see that $s_k \circ s_{k^*}$ is the image of $s_k \otimes s_{k^*}$ by the maps:

$$\begin{aligned}
&\mathrm{THom}(\mathbb{C}_{\mathbb{A}}, \mathcal{O}^{(n,0)}_{\mathbb{P}\times\mathbb{P}^*}(-k,k^*))[1] \otimes \mathrm{THom}(\mathbb{C}_{\widetilde{\mathbb{A}}}, \mathcal{O}^{(n,0)}_{\mathbb{P}^*\times\mathbb{P}}(-k^*,k))[1] \\
&\qquad \longrightarrow \mathrm{THom}(\mathbb{C}_{\mathbb{A}\times_{\mathbb{P}^*}\widetilde{\mathbb{A}}}, \mathcal{O}^{(n,n,0)}_{\mathbb{P}\times\mathbb{P}^*\times\mathbb{P}}(-k,0,k))[2] \\
&\qquad \longrightarrow \mathrm{THom}(\mathbb{C}_{\Delta_{\mathbb{P}}}, \mathcal{O}^{(n,0)}_{\mathbb{P}\times\mathbb{P}}(-k,k))[n].
\end{aligned} \tag{4.6}$$

Denote by $\mathbb{A}^c$ the complementary set of $\mathbb{A}$ in $\mathbb{P}\times\mathbb{P}^*$, and set for short:

$$\begin{aligned}
\Gamma[\mathbb{A}^c] &= H^0\,\mathrm{THom}(\mathbb{C}_{\mathbb{A}^c}, \mathcal{O}^{(n,0)}_{\mathbb{P}\times\mathbb{P}^*}(-k,k^*)), \\
\Gamma[(\mathbb{A}^c\times\mathbb{P})\cap(\mathbb{P}\times\widetilde{\mathbb{A}}^c)] &= H^0\,\mathrm{THom}(\mathbb{C}_{(\mathbb{A}^c\times\mathbb{P})\cap(\mathbb{P}\times\widetilde{\mathbb{A}}^c)}, \mathcal{O}^{(n,n,0)}_{\mathbb{P}\times\mathbb{P}^*\times\mathbb{P}}(-k,0,k)), \\
H^1_{[\mathbb{A}]}(\mathbb{P}\times\mathbb{P}^*) &= H^1\,\mathrm{THom}(\mathbb{C}_{\mathbb{A}}, \mathcal{O}^{(n,0)}_{\mathbb{P}\times\mathbb{P}^*}(-k,k^*)), \\
H^1[(\mathbb{A}\times_{\mathbb{P}^*}\widetilde{\mathbb{A}})^c] &= H^1\,\mathrm{THom}(\mathbb{C}_{(\mathbb{A}\times_{\mathbb{P}^*}\widetilde{\mathbb{A}})^c}, \mathcal{O}^{(n,n,0)}_{\mathbb{P}\times\mathbb{P}^*\times\mathbb{P}}(-k,0,k)), \\
H^2_{[\mathbb{A}\times_{\mathbb{P}^*}\widetilde{\mathbb{A}}]}(\mathbb{P}\times\mathbb{P}^*\times\mathbb{P}) &= H^2\,\mathrm{THom}(\mathbb{C}_{\mathbb{A}\times_{\mathbb{P}^*}\widetilde{\mathbb{A}}}, \mathcal{O}^{(n,n,0)}_{\mathbb{P}\times\mathbb{P}^*\times\mathbb{P}}(-k,0,k)), \\
H^n_{[\Delta_{\mathbb{P}}]}(\mathbb{P}\times\mathbb{P}) &= H^n\,\mathrm{THom}(\mathbb{C}_{\Delta_{\mathbb{P}}}, \mathcal{O}^{(n,0)}_{\mathbb{P}\times\mathbb{P}}(-k,k)),
\end{aligned}$$

and define similarly $\Gamma[\widetilde{\mathbb{A}}^c]$ and $H^1_{[\widetilde{\mathbb{A}}]}(\mathbb{P}^*\times\mathbb{P})$. Note that, for example, $\Gamma[\mathbb{A}^c]$ is the set of those sections of $\mathcal{O}^{(n,0)}_{\mathbb{P}\times\mathbb{P}^*}(-k,k^*)$ on $\mathbb{A}^c$ which extend as distributions to the whole space $\mathbb{P}\times\mathbb{P}^*$. Taking the zero-th cohomology of (4.6), we get the commutative diagram:

$$\begin{tikzcd}
\Gamma[\mathbb{A}^c]\otimes\Gamma[\widetilde{\mathbb{A}}^c] \arrow[r, "\alpha"] \arrow[d, "\otimes"'] & H^1_{[\mathbb{A}]}(\mathbb{P}\times\mathbb{P}^*)\otimes H^1_{[\widetilde{\mathbb{A}}]}(\mathbb{P}^*\times\mathbb{P}) \arrow[d, "\otimes"'] \\
\Gamma[(\mathbb{A}^c\times\mathbb{P})\cap(\mathbb{P}\times\widetilde{\mathbb{A}}^c)] \arrow[r, "\beta"] \arrow[dr, "\int_{q_{13}}"'] & H^2_{[\mathbb{A}\times_{\mathbb{P}^*}\widetilde{\mathbb{A}}]}(\mathbb{P}\times\mathbb{P}^*\times\mathbb{P}) \arrow[d, "\int_{q_{13}}"'] \\
& H^n_{[\Delta_{\mathbb{P}}]}(\mathbb{P}\times\mathbb{P}),
\end{tikzcd} \tag{4.7}$$

where α is the coboundary map, and β is the composite of the coboundary maps γ and δ below:

$$\begin{aligned}\beta : \Gamma[(\mathbb{A}^c \times \mathbb{P}) \cap (\mathbb{P} \times \widetilde{\mathbb{A}}^c)] &\xrightarrow{\gamma} H^1[(\mathbb{A} \times_{\mathbb{P}^*} \widetilde{\mathbb{A}})^c] \\ &\xrightarrow{\delta} H^2_{[\mathbb{A} \times_{\mathbb{P}^*} \widetilde{\mathbb{A}}]}(\mathbb{P} \times \mathbb{P}^* \times \mathbb{P}).\end{aligned}$$

Recall Notations 4.1, and denote by $([z], [\zeta], [\widetilde{z}])$ the coordinates on $\mathbb{P} \times \mathbb{P}^* \times \mathbb{P}$. We have

$$\begin{aligned} s_k \circ s_{k^*} &= \int_{q_{13}} s_k \otimes s_{k^*} \\ &= \int_{q_{13}} \frac{\omega^*(z)\omega^*(\zeta)}{\langle z, \zeta\rangle^{n+1+k} \langle \widetilde{z}, \zeta\rangle^{-k}} \end{aligned}$$

in $H^n_{[\Delta_{\mathbb{P}}]}(\mathbb{P} \times \mathbb{P})$. Since the above equalities are local in $\mathbb{P} \times \mathbb{P}$, in order to calculate explicitly the last integral we will restrict to the affine chart $E \times E$, endowed with the system of coordinates $(t, \widetilde{t})$.

Set

$$\mathbb{P}'^* = (\mathbb{P}^{n-1})^*, \quad \mathbb{P}^*_\circ = \mathbb{P}^* \setminus \{O^*\},$$

and notice that $\widehat{E} = \mathbb{P}^*_\circ$. Consider the maps

$$\begin{tikzcd} & E \times \mathbb{P}^*_\circ \times E \arrow[dl, "q_{12}"'] \arrow[d, "q"] \arrow[dr, "q_{23}"] & \\ E \times \mathbb{P}^*_\circ & E \times \mathbb{P}'^* \times E \arrow[d, "q'_{13}"] & \mathbb{P}^*_\circ \times E \\ & E \times E & \end{tikzcd}$$

where q is induced by the map $\mathbb{P}^*_\circ \longrightarrow \mathbb{P}'^*$ given by $[\zeta_0, \zeta'] \mapsto [\zeta']$, and q'_{13} is the natural projection. Hence, $q_{13} = q'_{13} \circ q$. Let

$$\begin{aligned} B &= (\mathbb{A} \times_{\mathbb{P}^*} \widetilde{\mathbb{A}}) \cap (E \times \mathbb{P}^* \times E) \\ &= \{(t, [\zeta], \widetilde{t}) \in E \times \mathbb{P}^* \times E;\ \zeta_0 - \langle t, \zeta'\rangle = \zeta_0 - \langle \widetilde{t}, \zeta'\rangle = 0\}, \\ B' &= \{(t, [\zeta'], \widetilde{t}) \in E \times \mathbb{P}'^* \times E;\ \langle t - \widetilde{t}, \zeta'\rangle = 0\}. \end{aligned}$$

Notice that $B \subset E \times \mathbb{P}^*_\circ \times E$ (and hence q_{13} is proper on B over $E \times E$), and that q induces a bijection between B and B'. Hence:

$$Rq_! \mathbb{C}_B \simeq \mathbb{C}_{B'}.$$

Set for short:

$$\begin{aligned}
\Gamma[(\mathbb{A}^c\times E)\cap(E\times\widetilde{\mathbb{A}}^c)] &= H^0\,\mathrm{THom}(\mathbb{C}_{(\mathbb{A}^c\times E)\cap(E\times\widetilde{\mathbb{A}}^c)},\mathcal{O}^{(n,n,0)}_{E\times\mathbb{P}^*_\circ\times E}),\\
\Gamma[(B')^c] &= H^0\,\mathrm{THom}(\mathbb{C}_{(B')^c},\mathcal{O}^{(n,n-1,0)}_{E\times\mathbb{P}'^*\times E}),\\
H^2_{[B]}(E\times\mathbb{P}^*_\circ\times E) &= H^2\,\mathrm{THom}(\mathbb{C}_B,\mathcal{O}^{(n,n-1,0)}_{E\times\mathbb{P}^*_\circ\times E}),\\
H^1_{[B']}(E\times\mathbb{P}'^*\times E) &= H^1\,\mathrm{THom}(\mathbb{C}_{B'},\mathcal{O}^{(n,n-1,0)}_{E\times\mathbb{P}'^*\times E}),\\
H^n_{[\Delta_E]}(E\times E) &= H^n\,\mathrm{THom}(\mathbb{C}_{\Delta_E},\mathcal{O}^{(n,0)}_{E\times E}).
\end{aligned}$$

The bottom diagram in (4.7) factorizes in the commutative diagrams:

$$\begin{tikzcd}
\Gamma[(\mathbb{A}^c\times E)\cap(E\times\widetilde{\mathbb{A}}^c)] \arrow[r] \arrow[d, "\int_q"'] & H^2_{[B]}(E\times\mathbb{P}^*_\circ\times E) \arrow[d, "\int_q"'] \\
\Gamma[(B')^c] \arrow[r] \arrow[dr, "\int_{q'_{13}}"'] & H^1_{[B']}(E\times\mathbb{P}'^*\times E) \arrow[d, "\int_{q'_{13}}"'] \\
& H^n_{[\Delta_E]}(E\times E).
\end{tikzcd} \tag{4.8}$$

It follows that:

$$\int_{q_{13}} s_k\otimes s_{k^*}|_{E\times E} = \int_{q'_{13}}\int_q \frac{\omega(t)\omega^*(\zeta)}{(\zeta_0+\langle t,\zeta'\rangle)^{n+1+k}(\zeta_0+\langle\tilde t,\zeta'\rangle)^{-k}}$$

in $H^n_{[\Delta_E]}(E\times E)$. As explained in [15, §3.1], the arrow $\int_q$ is a contour integral on a loop surrounding the only singular point of $s_k\otimes s_{k^*}$ in the complex line $q^{-1}(t,[\zeta'],\tilde t)$. To explicitly compute it, let $[\zeta]=[\zeta_0,\zeta'',\zeta_n]$ be an homogeneous coordinate system on $\mathbb{P}^*$, and consider the affine charts $F\subset\mathbb{P}^*$, $F'\subset\mathbb{P}'^*$ defined by $\zeta_n\neq 0$, endowed with the systems of coordinates $(\sigma)=(\sigma_0,\sigma'')=(\zeta_0/\zeta_n,\zeta''/\zeta_n)$, (σ'') respectively. For $(t)=(t'',t_n)$, $(\tilde t)=(\tilde t'',\tilde t_n)$, we have

$$\begin{aligned}
&\int_q \frac{\omega(t)\omega^*(\zeta)}{(\zeta_0+\langle t,\zeta'\rangle)^{n+1+k}(\zeta_0+\langle\tilde t,\zeta'\rangle)^{-k}}|_{E\times F\times E}\\
&\quad=\int_q \frac{\omega(t)\omega(\sigma)}{(\sigma_0+\langle t',\sigma'\rangle+t_n)^{n+1+k}(\sigma_0+\langle\tilde t',\sigma'\rangle+\tilde t_n)^{-k}}\\
&\quad=(-1)^{n+k}\begin{pmatrix} n-1\\ n+k\end{pmatrix}\frac{\omega(t)\omega(\sigma')}{(\langle t'-\tilde t',\sigma'\rangle+t_n-\tilde t_n)^n}\\
&\quad=(-1)^{n+k}\begin{pmatrix} n-1\\ n+k\end{pmatrix}\frac{\omega(t)\omega^*(\zeta')}{\langle t-\tilde t,\zeta'\rangle^n}
\end{aligned}$$

in $\Gamma[(B')^c]$. To complete the proof, we have the following proposition, analogous to the classical plane wave decomposition of the delta function in the framework of hyperfunctions. q.e.d.

Proposition 4.9 *With the same notations as above, we have for a non zero constant C:*

$$\int_{q'_{13}} \frac{\omega(t)\omega^*(\zeta')}{\langle t-\widetilde{t},\zeta'\rangle^n} = C\,\delta_E,$$

where δ_E denotes the fundamental class of $H^n_{[\Delta_E]}(E\times E;\mathcal{O}^{(n,0)}_{E\times E})$.

Proof: This is just a reformulation of the first Cauchy-Fantappiè formula of [21, formula (56.1)]. In fact,

$$\frac{\omega(t)\omega^*(\zeta')}{\langle t-\widetilde{t},\zeta'\rangle^n}$$

is precisely the kernel of the Cauchy-Fantappiè transform. q.e.d.

5 Applications

5.1 On a theorem of Martineau

In this section we will show how the previous results allow us to recover and precise Martineau's isomorphism, as well as some related results by Henkin, Leiterer, Trépreau, and others.

Consider the correspondence (4.1). We begin with a geometrical lemma.

Definition 5.1 Let D be a locally closed subset of $\mathbb{P}$. We say that D is $\mathbb{A}$-trivial if for any $\zeta\in\widehat{D}$, one has:

$$\mathrm{R}\Gamma(\widehat{\zeta}\cap D;\mathbb{C}_D)\simeq\mathbb{C}.$$

Lemma 5.2 *Assume D is compact (resp. open) and $\mathbb{A}$-trivial. Then:*

$$\begin{aligned} &\Phi_{\mathbb{A}}\,\mathbb{C}_D \simeq \mathbb{C}_{\widehat{D}}[n-1]\\ (\textit{resp.}\quad &\Phi_{\mathbb{A}}\,\mathbb{C}_D \simeq \mathbb{C}_{\widehat{D}}[1-n]).\end{aligned}$$

Proof: (i) Assume D is compact. Set $\widetilde{g}=g|_{f^{-1}(D)}$. The natural morphism $id\longrightarrow R\widetilde{g}_*\widetilde{g}^{-1}$ defines the morphism

$$\mathbb{C}_{\widehat{D}}\longrightarrow Rg_*\mathbb{C}_{f^{-1}(D)},$$

and this morphism is an isomorphism by the hypothesis. Hence $\mathbb{C}_{\widehat{D}}\simeq\Phi_{\mathbb{A}}\,\mathbb{C}_D[1-n]$.

(ii) Assume D is open. Then $\widetilde{g}: f^{-1}(D)\longrightarrow\widehat{D}$ is smooth, and by [18, Remark 3.3.10], we obtain that if $\widehat{\zeta}\cap D\neq\emptyset$, then:

$$\mathrm{R}\Gamma_c(\widehat{\zeta}\cap D;\mathbb{C}_D)\simeq\mathbb{C}[2-2n], \tag{5.1}$$

since $\tilde{g}^!(\cdot) \simeq \tilde{g}^{-1}(\cdot)[2n-2]$. The natural morphism $R\tilde{g}_!\tilde{g}^! \longrightarrow id$ defines the morphism:

$$Rg_!\mathbb{C}_{f^{-1}(D)}[2n-2] \longrightarrow \mathbb{C}_{\widehat{D}},$$

which is an isomorphism by (5.1). q.e.d.

We say that a subset $D \subset \mathbb{P}$ is affine if it is contained in an affine chart. We introduce the notation:

$$D^{\#} = \mathbb{P}^* \setminus \widehat{D},$$

and we keep these notations for subsets of $\mathbb{P}^*$. Note that

$$D^{\#} = \{\zeta \in \mathbb{P}^*;\ \widehat{\zeta} \cap D = \emptyset\},$$

and that

$$\begin{aligned} D \neq \emptyset \quad &\Rightarrow \quad D^{\#} \text{ is affine,} \\ D \text{ affine} \quad &\Rightarrow \quad D^{\#} \neq \emptyset. \end{aligned}$$

Using Notations 4.1, let $K \subset E \subset \mathbb{P}$ be a compact convex subset. Then, it is easy to check that $K^{\#\#} = K$. More generally, recall that Martineau [22] calls "linéellement convexe" those subsets $D \subset E$ such that $D^{\#\#} = D$. We have a criterion for such a convexity.

Proposition 5.3 *Let $D \subset \mathbb{P}$ be either compact or open. Assume that D is non-empty, affine, and $\mathbb{A}$-trivial. Then $D^{\#}$ is non-empty, affine, $\widetilde{\mathbb{A}}$-trivial, and $D^{\#\#} = D$. Moreover, if $n > 1$, we have $\mathrm{R}\Gamma(D; \mathbb{C}_D) \simeq \mathbb{C}$.*

Proof: Assume D is compact (the proof when D is open is similar). We have already noticed that $D^{\#}$ is non-empty and affine. Applying Lemma 4.6, we find:

$$\Phi_{\widetilde{\mathbb{A}}}(\Phi_{\mathbb{A}}(\mathbb{C}_D)) \simeq \mathbb{C}_D \oplus (N \otimes \mathrm{R}\Gamma(D; \mathbb{C}_D)),$$

where $N = \oplus_{j=1}^{n-1}\mathbb{C}_{\mathbb{P}}[-2j]$. Applying Lemma 5.2, we obtain:

$$\Phi_{\widetilde{\mathbb{A}}}(\mathbb{C}_{\widehat{D}}) = \mathbb{C}_D[1-n] \oplus (N' \otimes \mathrm{R}\Gamma(D; \mathbb{C}_D)),$$

where $N' = N[1-n]$. On the other hand:

$$\Phi_{\widetilde{\mathbb{A}}}(\mathbb{C}_{\mathbb{P}^*}) \simeq \mathbb{C}_{\mathbb{P}}[1-n] \oplus N'.$$

Writing $\mathbb{C}_{D^{\#}}$ as the complex $[\mathbb{C}_{\mathbb{P}^*} \longrightarrow \mathbb{C}_{\widehat{D}}]$, we find:

$$\Phi_{\widetilde{\mathbb{A}}}(\mathbb{C}_{D^{\#}}) \simeq \mathbb{C}_{\mathbb{P}\setminus D}[1-n] \oplus M,$$

where M is the complex $[N' \longrightarrow N' \otimes \mathrm{R}\Gamma(D;\mathbb{C}_D)]$. Since

$$\mathrm{supp}\, M \subset \mathrm{supp}\, \Phi_{\widetilde{\mathbb{A}}}(\mathbb{C}_{D^\#}) \subset D^{\#\wedge} \neq \mathbb{P},$$

we find $M = 0$, and if $n > 1$, this implies $\mathrm{R}\Gamma(D;\mathbb{C}_D) \simeq \mathbb{C}$. Hence, we get

$$\Phi_{\widetilde{\mathbb{A}}}(\mathbb{C}_{D^\#}) \simeq \mathbb{C}_{\mathbb{P}\setminus D}[1-n].$$

For $z \in \mathbb{P}$, set $D_z = D^\# \cap \widehat{z} = \widetilde{\mathbb{A}} \cap (\{z\} \times D^\#)$. This is an open subset of $\widetilde{\mathbb{A}}$, and:

$$\mathrm{R}\Gamma_c(D_z;\mathbb{C}_{D_z})[2n-2] = \begin{cases} 0, & \text{if } z \notin \mathbb{P}\setminus D, \\ \mathbb{C}, & \text{if } z \in \mathbb{P}\setminus D. \end{cases}$$

Since $(\mathrm{R}\Gamma_c(D_z;\mathbb{C}_{D_z})[2n-2])' \simeq \mathrm{R}\Gamma(D_z;\mathbb{C}_{D_z})$, we find that $D^{\#\wedge} = \mathbb{P}\setminus D$, and hence $D^{\#\#} = D$, and $D^\#$ is $\widetilde{\mathbb{A}}$-trivial. q.e.d.

Remark 5.4 Much work has been done on the geometry of "linéellement convexe" sets. Beside Martineau (loc. cit.), let us quote in particular [1] and [30].

Theorem 5.5 *Let $U \subset \mathbb{P}$ be open, $K \subset \mathbb{P}^*$ be compact, and assume one of the equivalent conditions below:*

(a) U is affine, non-empty, $\mathbb{A}$-trivial, and $K = U^\#$,

(b) K is affine, non-empty, $\widetilde{\mathbb{A}}$-trivial, and $U = K^\#$.

Then, assuming $O \in U \subset E$ for simplicity (hence $K \subset E^*$), there are isomorphisms:

$$\mathrm{R}\Gamma_c(U;\mathcal{O}_E)[n] \simeq \mathrm{R}\Gamma(K;\mathcal{O}_{E^*}), \tag{5.2}$$
$$\mathrm{R}\Gamma(U;\mathcal{O}_E) \simeq \mathrm{R}\Gamma_K(E^*;\mathcal{O}_{E^*})[n], \tag{5.3}$$

and these four complexes are concentrated in degree zero.

Proof: By Proposition 5.3, conditions (a) and (b) are equivalent. Let us apply Lemma 5.2 together with Corollary 4.5 to the sheaf $F = \mathbb{C}_U$. For $-n-1 < k < 0$, we find:

$$\mathrm{R}\Gamma_c(U;\mathcal{O}_E) \simeq \mathrm{R}\Gamma_c(\mathbb{P}^*\setminus K;\mathcal{O}_{\mathbb{P}^*}(k))[1-n],$$
$$\mathrm{R}\Gamma(U;\mathcal{O}_E) \simeq \mathrm{R}\Gamma(\mathbb{P}^*\setminus K;\mathcal{O}_{\mathbb{P}^*}(k))[n-1].$$

Using the fact that $\mathrm{R}\Gamma(\mathbb{P}^*;\mathcal{O}_{\mathbb{P}^*}(k)) = 0$ for $-n-1 < k < 0$, the isomorphisms (5.3), (5.2) follow.

Finally, since the complex $\mathrm{R}\Gamma(U;\mathcal{O}_E)$ is concentrated in degree ≥ 0 as well as $\mathrm{R}\Gamma(K;\mathcal{O}_{E^*})$, and $\mathrm{R}\Gamma_K(\mathbb{P}^*;\mathcal{O}_{E^*})[n]$ is concentrated in degree ≤ 0 as well as $\mathrm{R}\Gamma_c(U;\mathcal{O}_E)[n]$, all these complexes are in degree zero. q.e.d.

Remark 5.6 It is important to distinguish between $\mathbb{P}$ and $\mathbb{P}^*$. For example, when $n = 1$ one may identify $\mathbb{P}$ to $\mathbb{P}^*$ and $\mathbb{P} \setminus K$ to $K^\#$, but (5.3) cannot be identified to the morphism:

$$\mathrm{R}\Gamma(\mathbb{P} \setminus K; \mathcal{O}_{\mathbb{P}}) \longrightarrow \mathrm{R}\Gamma_K(\mathbb{P}; \mathcal{O}_{\mathbb{P}})[1]$$

associated to the inclusion $K \hookrightarrow \mathbb{P}$, since the zero-th cohomology of this last morphism has non zero kernel.

Remark 5.7 If K is a compact convex affine subset of $\mathbb{P}$, then $\Phi_{\mathbb{A}}(\mathbb{C}_K) = \mathbb{C}_{\widehat{K}}[n-1]$. However, one shall take care that if K_1 and K_2 are two such sets, then in general

$$(K_1 \cap K_2)^\wedge \subsetneqq \widehat{K}_1 \cap \widehat{K}_2.$$

In particular, the distinguished triangle

$$\Phi_{\mathbb{A}}(\mathbb{C}_{K_1 \cup K_2})[1-n] \longrightarrow \mathbb{C}_{\widehat{K}_1} \oplus \mathbb{C}_{\widehat{K}_2} \longrightarrow \mathbb{C}_{(K_1 \cap K_2)^\wedge} \xrightarrow{+1}$$

is not the Mayer-Vietoris sequence associated to $\widehat{K}_1$ and $\widehat{K}_2$.

Remark 5.8 With the notations of Theorem 5.5, the isomorphism

$$H^0(K^\#; \mathcal{O}_{\mathbb{P}^*}) \simeq H^n_K(E^*; \mathcal{O}_{E^*})$$

is a theorem of Martineau [22]. Note that Martineau's proof was essentially not different from ours, since it is based on Leray's Cauchy-Fantappiè formula.

Corollary 5.9 *For $i = 1, 2$, let $U_i \subset \mathbb{P}$ be open (resp. $K_i \subset \mathbb{P}^*$ be compact) and assume that they satisfy condition (a) (resp. (b)) in Theorem 5.5, and that $U_1 \subset U_2$ (resp. $K_2 \subset K_1$). Then, assuming $U_2 \subset E$ (resp. $K_1 \subset E^*$), the natural morphisms:*

$$\begin{array}{rcl} H^n_c(U_1; \mathcal{O}_E) & \longrightarrow & H^n_c(U_2; \mathcal{O}_E), \\ H^n_{K_2}(E^*; \mathcal{O}_{E^*}) & \longrightarrow & H^n_{K_1}(E^*; \mathcal{O}_{E^*}), \end{array}$$

are injective.

Proof: We may assume $K_i = U_i^\#$. Then Theorem 5.5 interchanges the morphisms in the statement with the morphisms:

$$\begin{array}{rcl} \Gamma(K_1; \mathcal{O}_{E^*}) & \longrightarrow & \Gamma(K_2; \mathcal{O}_{E^*}), \\ \Gamma(U_2; \mathcal{O}_E) & \longrightarrow & \Gamma(U_1; \mathcal{O}_E), \end{array}$$

which are injective by analytic continuation. q.e.d.

Using the formalism of formal and temperate cohomology introduced in [14], [19] (and reviewed in Appendix A.1), we obtain the following results, analogous to Theorem 5.5 and Corollary 5.9.

Recall first that if $U \subset \mathbb{P}$ is open and subanalytic, and $K \subset \mathbb{P}^*$ is compact and subanalytic, then $\mathbb{C}_U \overset{\mathrm{w}}{\otimes} \mathcal{O}_{\mathbb{P}}$ is the Dolbeault complex with coefficients in the sheaf of $\mathcal{C}^\infty$ functions on U vanishing up to infinite order at the boundary of U, $\mathbb{C}_K \overset{\mathrm{w}}{\otimes} \mathcal{O}_{E^*}$ is the Dolbeault complex with coefficients in the sheaf of Whitney functions on K, $\mathcal{T}hom(\mathbb{C}_U, \mathcal{O}_{\mathbb{P}})$ is the Dolbeault complex with coefficients in the sheaf of temperate distributions on U, and $\mathrm{R}\Gamma_{[K]}(E^*; \mathcal{O}_{E^*})$ is the Dolbeault complex with coefficients in the sheaf of distributions supported by K.

Theorem 5.10 *Let $U \subset \mathbb{P}$ be open and subanalytic, and let $K \subset \mathbb{P}^*$ be compact and subanalytic. Assume one of the equivalent conditions (a), (b) in Theorem 5.5. Then, assuming $O \in U \subset E$ for simplicity (hence $K \subset E^*$), there are isomorphisms:*

$$
\begin{aligned}
\mathrm{R}\Gamma(\mathbb{P}; \mathbb{C}_U \overset{\mathrm{w}}{\otimes} \mathcal{O}_{\mathbb{P}})[n] &\simeq \mathrm{R}\Gamma(E^*; \mathbb{C}_K \overset{\mathrm{w}}{\otimes} \mathcal{O}_{E^*}), &\quad (5.4)\\
\mathrm{R}\Gamma(\mathbb{P}; \mathcal{T}hom(\mathbb{C}_U, \mathcal{O}_{\mathbb{P}})) &\simeq \mathrm{R}\Gamma_{[K]}(E^*; \mathcal{O}_{E^*})[n], &\quad (5.5)
\end{aligned}
$$

and these four complexes are concentrated in degree zero.

Proof: The proof is the same as that of Theorem 5.5, replacing Proposition 2.3 by Proposition A.1, and noticing that $\mathrm{R}\Gamma(\mathbb{P}; \mathbb{C}_U \overset{\mathrm{w}}{\otimes} \mathcal{O}_{\mathbb{P}}(k)) \simeq \mathrm{R}\Gamma(\mathbb{P}; \mathbb{C}_U \overset{\mathrm{w}}{\otimes} \mathcal{O}_{\mathbb{P}})$. In fact, $\mathrm{R}\Gamma(\mathbb{P}; \mathbb{C}_U \overset{\mathrm{w}}{\otimes} \mathcal{O}_{\mathbb{P}}(k))$ is the Dolbeault complex with coefficients in $\Gamma(\mathbb{P}; \mathbb{C}_U \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_{\mathbb{P}}(k))$, and this space is the completion of $\Gamma_c(U; \mathcal{C}^\infty_{\mathbb{P}}(k)) \simeq \Gamma_c(U; \mathcal{C}^\infty_{\mathbb{P}})$ for the topology induced by $\Gamma(\mathbb{P}; \mathcal{C}^\infty_{\mathbb{P}}(k))$. This topology is the same as that induced by $\Gamma(\mathbb{P}; \mathcal{C}^\infty_{\mathbb{P}})$.

By duality, one also has $\mathrm{R}\Gamma(\mathbb{P}; \mathcal{T}hom(\mathbb{C}_U, \mathcal{O}_{\mathbb{P}}(k))) \simeq \mathrm{R}\Gamma(\mathbb{P}; \mathcal{T}hom(\mathbb{C}_U, \mathcal{O}_{\mathbb{P}}))$. q.e.d.

Corollary 5.11 *Let $U_i \subset \mathbb{P}$ and $K_i \subset \mathbb{P}^*$ $(i = 1, 2)$ be as in Corollary 5.9. Assume that they are subanalytic subsets of $\mathbb{P}$ and $\mathbb{P}^*$ respectively. Then, the natural morphisms:*

$$
\begin{aligned}
H^n(\mathbb{P}; \mathbb{C}_{U_1} \overset{\mathrm{w}}{\otimes} \mathcal{O}_{\mathbb{P}}) &\longrightarrow H^n(\mathbb{P}; \mathbb{C}_{U_2} \overset{\mathrm{w}}{\otimes} \mathcal{O}_{\mathbb{P}}),\\
H^n_{[K_2]}(E^*; \mathcal{O}_{E^*}) &\longrightarrow H^n_{[K_1]}(E^*; \mathcal{O}_{E^*}),
\end{aligned}
$$

are injective.

Concerning local cohomology, we have the following result.

Corollary 5.12 *Let $U \subset E$ be an open subset with real analytic boundary, let $t_\circ \in \partial U$, and assume that U is strictly pseudo-convex at $t_\circ$. Then:*

$$
\begin{aligned}
H^j(\mathbb{C}_U \overset{\mathrm{w}}{\otimes} \mathcal{O}_E)_{t_\circ} &= 0 \qquad \text{for } j \neq n, && (5.6)\\
H^j(\mathrm{R}\Gamma_{\overline{U}}\mathcal{O}_E)_{t_\circ} &= 0 \qquad \text{for } j \neq n, && (5.7)\\
H^j(\mathrm{R}\Gamma_{[\overline{U}]}\mathcal{O}_E)_{t_\circ} &= 0 \qquad \text{for } j \neq n. && (5.8)
\end{aligned}
$$

Proof: Since the proofs of these formulas are similar, we will only show (5.6).

The problem being local at $t_\circ$, it is not restrictive to assume that U is strictly convex. Let $E^{\mathbb{R}}$ be the real underlying affine space to E, and denote by $\langle\cdot,\cdot\rangle^{\mathbb{R}}$ the scalar product on $E^{\mathbb{R}}$. In the identification $T^*E \simeq E \times E^*$, let $(t_\circ, \tau_\circ)$ be an exterior conormal to U at $t_\circ$. For $\varepsilon > 0$, let:

$$U_\varepsilon = \{t \in U;\ \langle t - t_\circ, \tau_\circ\rangle^{\mathbb{R}} < -\varepsilon\},$$

and let V_ε be a fundamental system of neighborhoods of $t_\circ$ such that $V_\varepsilon \cap U = U_\varepsilon$.

Consider the complex $L_\varepsilon = \Gamma(V_\varepsilon; \mathbb{C}_U \overset{\mathrm{w}}{\otimes} \mathcal{C}^{\infty,(0,\cdot)}_{E^{\mathbb{R}}})$. We have:

$$H^j(\mathbb{C}_U \overset{\mathrm{w}}{\otimes} \mathcal{O}_E)_{t_\circ} \simeq \varinjlim_{\varepsilon} H^j(L_\varepsilon).$$

On the other hand, consider the complex $L'_\varepsilon = \Gamma(E; \mathbb{C}_{U\setminus U_\varepsilon} \overset{\mathrm{w}}{\otimes} \mathcal{C}^{\infty,(0,\cdot)}_{E^{\mathbb{R}}})$. It enters the short exact sequence of complexes:

$$0 \longrightarrow \Gamma(E; \mathbb{C}_{U_\varepsilon} \overset{\mathrm{w}}{\otimes} \mathcal{C}^{\infty,(0,\cdot)}_{E^{\mathbb{R}}}) \longrightarrow \Gamma(E; \mathbb{C}_U \overset{\mathrm{w}}{\otimes} \mathcal{C}^{\infty,(0,\cdot)}_{E^{\mathbb{R}}}) \longrightarrow L'_\varepsilon \longrightarrow 0,$$

or, equivalently, it is the third term of a distinguished triangle

$$\mathrm{R}\Gamma(E; \mathbb{C}_{U_\varepsilon} \overset{\mathrm{w}}{\otimes} \mathcal{O}_E) \longrightarrow \mathrm{R}\Gamma(E; \mathbb{C}_U \overset{\mathrm{w}}{\otimes} \mathcal{O}_E) \longrightarrow L'_\varepsilon \underset{+1}{\longrightarrow}.$$

Using Theorem 5.10 and Corollary 5.11, one gets that $H^j(L'_\varepsilon) = 0$ for $j \neq n$. To conclude, one notices that the two complexes L'_ε and L_ε are cofinal. q.e.d.

Remark 5.13 (i) Corollary 5.12 was obtained in [12] (in a different language) by explicit integral formulas.

(ii) Let M be a real C^2 hypersurface of a complex manifold X of dimension n, let $p \in \dot{T}^*_M X$, and let M^+ be the germ of closed half space at $t_\circ = \pi(p)$, with interior conormal p. Assume the Levi form of M has exactly q negative eigenvalues in a neighborhood of p. A theorem of [17] asserts that

$$\varinjlim_{\Omega} H^j_{M^+\cap\Omega}(\Omega; \mathcal{O}_X) = 0, \quad \text{for } j \neq q+1. \tag{5.9}$$

Trépreau [29] has noticed that there exists a holomorphic chart in a neighborhood of $t_\circ$ in which a partial Legendre transform interchanges $X \setminus M^+$ with a pseudoconvex open subset. He has then obtained a more direct proof of (5.9) using the analogous of the result of [12] with parameters. It would be possible to adapt Trépreau's method, using a generalization of Theorem 4.3 to a situation with parameters.

5.2 Real projective duality

In this section we shall apply Corollary 4.5 to give an alternative approach to the results of Gelfand, Gindikin and Graev [9] on the (real) Radon transform.

Denote by P a real projective space of dimension n, and assume for simplicity that $n > 1$. Recall that there is a natural embedding $\mathrm{P} \longrightarrow \mathbb{P}$ (compatible with the embedding of affine charts $\mathbb{R}^n \longrightarrow \mathbb{C}^n$) by which $\mathbb{P}$ is a complexification of P. Set $\mathrm{A} = \mathbb{A} \cap (\mathrm{P} \times \mathrm{P}^*)$, and consider the embedding of the real projective correspondence in its complexification:

$$\begin{array}{ccc} \begin{array}{ccc} & \mathrm{A} & \\ {}^{\tilde f}\swarrow & & \searrow^{\tilde g} \\ \mathrm{P} & & \mathrm{P}^* \end{array} & \xrightarrow{i} & \begin{array}{ccc} & \mathbb{A} & \\ {}^{f}\swarrow & & \searrow^{g} \\ \mathbb{P} & & \mathbb{P}^*. \end{array} \end{array}$$

First we need to recall some basic facts on the topology of P and A (references are made to Ehresmann [8]).

Since $\pi_1(\mathrm{P}) = \mathbb{Z}_2$, there are essentially two locally constant sheaves of rank one on P: the constant sheaf $\mathbb{C}_P$, and the canonical line bundle that we denote by K_P. Consider the universal covering of P:

$$q : \mathrm{S} \longrightarrow \mathrm{P}, \tag{5.10}$$

where S denotes a real n-dimensional sphere. There is a split exact sequence:

$$0 \longrightarrow K_\mathrm{P} \longrightarrow q_!\mathbb{C}_\mathrm{S} \xrightarrow{\mathrm{tr}} \mathbb{C}_\mathrm{P} \longrightarrow 0 \tag{5.11}$$

(where $\mathrm{tr} : q_!\mathbb{C}_\mathrm{S} \simeq q_!q^!\mathbb{C}_\mathrm{P} \longrightarrow \mathbb{C}_\mathrm{P}$ is the natural trace morphism), from which one deduces:

$$\mathrm{R}\Gamma(\mathrm{P};\mathbb{C}_\mathrm{P}) \simeq \begin{cases} \mathbb{C}, & \text{for } n \text{ even}, \\ \mathbb{C}\oplus\mathbb{C}[-n], & \text{for } n \text{ odd}, \end{cases} \tag{5.12}$$

$$\mathrm{R}\Gamma(\mathrm{P};K_\mathrm{P}) \simeq \begin{cases} \mathbb{C}[-n], & \text{for } n \text{ even}, \\ 0, & \text{for } n \text{ odd}. \end{cases} \tag{5.13}$$

Moreover one has $K_\mathrm{P} \simeq D'_\mathrm{P} K_\mathrm{P}$, $K_\mathrm{P} \simeq or_\mathrm{P}$ if and only if n is even, $\mathbb{C}_\mathrm{P} \simeq or_\mathrm{P}$ if and only if n is odd.

Concerning the topology of A, one knows that there are essentially four locally constant sheaves of rank one on A: $\mathbb{C}_{\mathrm{P}} \boxtimes \mathbb{C}_{\mathrm{P}^*}|_{\mathrm{A}}$, $K_{\mathrm{P}} \boxtimes \mathbb{C}_{\mathrm{P}^*}|_{\mathrm{A}}$, $\mathbb{C}_{\mathrm{P}} \boxtimes K_{\mathrm{P}^*}|_{\mathrm{A}}$, and $K_{\mathrm{P}} \boxtimes K_{\mathrm{P}^*}|_{\mathrm{A}}$. Moreover:

$$or_{\mathrm{A}/\mathrm{P}\times\mathrm{P}^*} \simeq K_{\mathrm{P}} \boxtimes K_{\mathrm{P}^*}|_{\mathrm{A}}. \tag{5.14}$$

In order to calculate the Radon transform (i.e., $\Phi_{\mathbb{A}}(i_!F)$) of a sheaf F on P, we need some preliminary results.

For $\varepsilon \in \mathbb{Z}_2$, set

$$\mathbb{C}_{\mathrm{P}}(\varepsilon) = \begin{cases} \mathbb{C}_{\mathrm{P}}, & \text{for } \varepsilon = 0, \\ K_{\mathrm{P}}, & \text{for } \varepsilon = 1. \end{cases}$$

We also set

$$\varepsilon^* = -n - 1 - \varepsilon \bmod 2.$$

Proposition 5.14 *Let $\varepsilon_1, \varepsilon_2 \in \mathbb{Z}_2$. Then:*

$$\mathrm{RHom}(\Phi_{\mathbb{A}}(\mathbb{C}_{\mathrm{P}}(\varepsilon_1)), \mathbb{C}_{\mathrm{P}^*}(\varepsilon_2)) \simeq \mathrm{R}\Gamma(\mathrm{A}; [\mathbb{C}_{\mathrm{P}}(\varepsilon_1^* + 1) \boxtimes \mathbb{C}_{\mathrm{P}^*}(\varepsilon_2 + 1)]|_{\mathrm{A}}).$$

Proof: Let F and G be locally constant sheaves of rank one on P and P^* respectively (so that $F \simeq \mathbb{C}_{\mathrm{P}}$ or $F \simeq K_{\mathrm{P}}$, and similarly for G). One has the following chain of isomorphisms:

$$\begin{aligned} \mathrm{RHom}(\Phi_{\mathbb{A}}(i_!F), i_!G) &\simeq \mathrm{RHom}((i_!F \boxtimes \mathbb{C}_{\mathbb{P}^*}) \otimes \mathbb{C}_{\mathbb{A}}[n-1], \mathbb{C}_{\mathbb{P}} \boxtimes i_!G[2n]) \\ &\simeq \mathrm{RHom}(i_!((F \boxtimes \mathbb{C}_{\mathrm{P}^*}) \otimes \mathbb{C}_{\mathrm{A}}), \mathbb{C}_{\mathbb{P}} \boxtimes i_!G)[n+1] \\ &\simeq \mathrm{RHom}(i_!\mathbb{C}_{\mathrm{A}}, i_!((D'_{\mathrm{P}}F \otimes or_{\mathrm{P}}) \boxtimes G))[1] \\ &\simeq \mathrm{R}\Gamma(\mathrm{A}; or_{\mathrm{A}/\mathrm{P}\times\mathrm{P}^*} \otimes [(D'_{\mathrm{P}}F \otimes or_{\mathrm{P}}) \boxtimes G]|_{\mathrm{A}}). \end{aligned}$$

This completes the proof by (5.14), noticing that $D'_{\mathrm{P}}(\mathbb{C}_{\mathrm{P}}(\varepsilon_1)) \otimes or_{\mathrm{P}} \simeq \mathbb{C}_{\mathrm{P}}(\varepsilon_1^*)$. q.e.d.

Corollary 5.15 *For $\varepsilon_1, \varepsilon_2 \in \mathbb{Z}_2$, one has the following isomorphisms*

$$\mathrm{Hom}(\Phi_{\mathbb{A}}(\mathbb{C}_{\mathrm{P}}(\varepsilon_1)), \mathbb{C}_{\mathrm{P}^*}(\varepsilon_2)) \simeq \begin{cases} \mathbb{C}, & \text{for } \varepsilon_1^* = \varepsilon_2 = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 5.16

$$H^j\,\Phi_{\mathbb{A}}(\mathbb{C}_{\mathrm{P}}) \simeq \begin{cases} \mathbb{C}_{\mathbb{P}^*}, & \text{for } j = 1-n, \\ K_{\mathrm{P}^*}, & \text{for } j = 0, \quad n \text{ even}, \\ \mathbb{C}_{\mathbb{P}^*\setminus\mathrm{P}^*}, & \text{for } j = -1, \quad n \text{ odd}, \\ 0, & \text{otherwise,} \end{cases}$$

$$\Phi_{\mathbb{A}}(K_{\mathrm{P}}) = \begin{cases} \mathbb{C}_{\mathbb{P}^*\setminus\mathrm{P}^*}[1], & \text{for } n \text{ even}, \\ K_{\mathrm{P}^*}, & \text{for } n \text{ odd}. \end{cases}$$

Proof: (a) First we calculate the stalks of $\Phi_{\mathbb{A}}(\mathbb{C}_{\mathrm{P}})$, $\Phi_{\mathbb{A}}(K_{\mathrm{P}})$.

For $\zeta \in \mathbb{P}^*$,

$$H^j(\Phi_{\mathbb{A}}(\mathbb{C}_{\mathrm{P}})[1-n])_\zeta \simeq H^j(\widehat{\zeta} \cap \mathrm{P}; \mathbb{C}_{\widehat{\zeta}\cap \mathrm{P}}).$$

Note that every $z \in \mathrm{P}$ may be written as $z = [x_0, \dots, x_n]$ with $x_i \in \mathbb{R}$, and that it is not restrictive to assume $\zeta = [1, \zeta']$. It follows that:

$$\begin{aligned} \widehat{\zeta} \cap \mathrm{P} &= \{[x] \in \mathrm{P};\ x_0 + \langle x', \zeta' \rangle = 0\} \\ &= \{[x] \in \mathrm{P};\ x_0 + \langle x', \operatorname{Re}\zeta' \rangle = 0,\ \langle x', \operatorname{Im}\zeta' \rangle = 0\} \\ &\simeq \begin{cases} \mathrm{P}_{n-1} & \text{for } \operatorname{Im}\zeta' = 0, \\ \mathrm{P}_{n-2} & \text{for } \operatorname{Im}\zeta' \neq 0, \end{cases} \end{aligned}$$

where P_k denotes a k-dimensional real projective space. Using (5.12), we then get

$$H^j\, \Phi_{\mathbb{A}}(\mathbb{C}_{\mathrm{P}})_\zeta \simeq \begin{cases} \mathbb{C}, & \text{for } j = 1-n, \\ \mathbb{C}, & \text{for } j = 0, \quad n \text{ even}, \quad \zeta \in \mathrm{P}^*, \\ \mathbb{C}, & \text{for } j = -1, \quad n \text{ odd}, \quad \zeta \in \mathbb{P}^* \setminus \mathrm{P}^*, \\ 0, & \text{otherwise}. \end{cases} \tag{5.15}$$

In order to compute $\Phi_{\mathbb{A}}(K_{\mathrm{P}})_\zeta$, consider the distinguished triangle

$$\Phi_{\mathbb{A}}(K_{\mathrm{P}})_\zeta \longrightarrow \Phi_{\mathbb{A}}(q_!\mathbb{C}_{\mathrm{S}})_\zeta \xrightarrow{\mathrm{tr}} \Phi_{\mathbb{A}}(\mathbb{C}_{\mathrm{P}})_\zeta \xrightarrow{+1} .$$

One easily checks that tr corresponds to the natural map:

$$\mathrm{R}\Gamma(q^{-1}(\widehat{\zeta}) \cap \mathrm{S}; \mathbb{C}_{q^{-1}(\widehat{\zeta})\cap \mathrm{S}}) \longrightarrow \mathrm{R}\Gamma(\widehat{\zeta} \cap \mathrm{P}; \mathbb{C}_{\widehat{\zeta}\cap \mathrm{P}}).$$

One has

$$\begin{aligned} q^{-1}(\widehat{\zeta}) \cap \mathrm{S} &= \{(x) \in \mathrm{S};\ x_0 + \langle x', \zeta' \rangle = 0\} \\ &= \{(x) \in \mathrm{S};\ x_0 + \langle x', \operatorname{Re}\zeta' \rangle = 0,\ \langle x', \operatorname{Im}\zeta' \rangle = 0\} \\ &\simeq \begin{cases} \mathrm{S}^{n-1} & \text{for } \operatorname{Im}\zeta' = 0, \\ \mathrm{S}^{n-2} & \text{for } \operatorname{Im}\zeta' \neq 0, \end{cases} \end{aligned}$$

where S^k denotes a real k-dimensional sphere. Recalling that

$$H^j(\mathrm{S}; \mathbb{C}_{\mathrm{S}}) \simeq \begin{cases} \mathbb{C}, & \text{for } j = 0, \\ \mathbb{C}, & \text{for } j = n, \\ 0, & \text{otherwise}, \end{cases}$$

we get

$$H^j\, \Phi_{\mathbb{A}}(q_!\mathbb{C}_{\mathrm{S}})_\zeta \simeq \begin{cases} \mathbb{C}, & \text{for } j = 1-n \\ \mathbb{C}, & \text{for } j = 0, \quad \zeta \in \mathrm{P}^*, \\ \mathbb{C}, & \text{for } j = -1, \quad \zeta \in \mathbb{P}^* \setminus \mathrm{P}^*, \\ 0, & \text{otherwise}. \end{cases}$$

It follows that:

$$H^j(\Phi_{\mathbb{A}}(K_{\mathrm{P}})_\zeta) \simeq \begin{cases} \mathbb{C}, & \text{for } j = 0, \quad \zeta \in \mathrm{P}^*, \quad n \text{ odd}, \\ \mathbb{C}, & \text{for } j = -1, \quad \zeta \in \mathbb{P}^* \setminus \mathrm{P}^*, \quad n \text{ even}, \\ 0, & \text{otherwise}. \end{cases} \tag{5.16}$$

(b) The sheaves $H^j\,\Phi_{\mathbb{A}}(\mathbb{C}_{\mathrm{P}})$, $H^j\,\Phi_{\mathbb{A}}(K_{\mathrm{P}})$ are clearly locally free of rank one or zero on $\mathbb{P}^* \setminus \mathrm{P}^*$ and on P^*. Moreover, $\mathbb{P}^*$ is simply connected as well as $\mathbb{P}^* \setminus \mathrm{P}^*$ for $n > 2$. Thus, by Corollary 5.15 it only remains to prove that

$$\Phi_{\mathbb{A}}(K_{\mathrm{P}}) \simeq \mathbb{C}_{\mathbb{P}^*\setminus\mathrm{P}^*}[1] \quad \text{for } n = 2.$$

Let $j : \mathbb{P}^* \setminus \mathrm{P}^* \longrightarrow \mathbb{P}^*$ be the open embedding, and set $L = H^{-1}(\Phi_{\mathbb{A}}(K_{\mathrm{P}}))|_{\mathbb{P}^*\setminus\mathrm{P}^*}$, so that $\Phi_{\mathbb{A}}(K_{\mathrm{P}}) \simeq j_!L[1]$. Noticing that

$$\Phi_{\tilde{\mathbb{A}}}(\mathbb{C}_{\mathbb{P}^*}) \simeq \bigoplus_{j=0}^{n-1} \mathbb{C}_{\mathbb{P}}[n-1-2j],$$

and using (5.13), we have (for $n = 2$):

$$\begin{aligned} \mathrm{RHom}(\mathbb{C}_{\mathbb{P}^*}, \Phi_{\mathbb{A}}(K_{\mathrm{P}})) &\simeq \mathrm{RHom}(\Phi_{\tilde{\mathbb{A}}}(\mathbb{C}_{\mathbb{P}^*}), K_{\mathrm{P}}) \\ &\simeq \bigoplus_{j=0}^{1} \mathrm{RHom}(\mathbb{C}_{\mathbb{P}}, K_{\mathrm{P}})[2j-1] \\ &\simeq \mathbb{C}[-1] \oplus \mathbb{C}[-3]. \end{aligned}$$

Applying the functor $H^0\mathrm{RHom}(\cdot, \Phi_{\mathbb{A}}(K_{\mathrm{P}}))$ to the exact sequence

$$0 \longrightarrow \mathbb{C}_{\mathbb{P}^*\setminus\mathrm{P}^*} \longrightarrow \mathbb{C}_{\mathbb{P}^*} \longrightarrow \mathbb{C}_{\mathrm{P}^*} \longrightarrow 0,$$

we get:

$$\begin{aligned} \mathrm{Hom}(\mathbb{C}_{\mathbb{P}^*\setminus\mathrm{P}^*}, j_!L) &\simeq H^{-1}\mathrm{RHom}(\mathbb{C}_{\mathbb{P}^*\setminus\mathrm{P}^*}, \Phi_{\mathbb{A}}(K_{\mathrm{P}})) \\ &\simeq H^0\mathrm{RHom}(\mathbb{C}_{\mathbb{P}^*}, \Phi_{\mathbb{A}}(K_{\mathrm{P}})) \\ &\simeq \mathbb{C}, \end{aligned}$$

where the last isomorphism is obtained by adjunction from the second isomorphism of Corollary 5.15. Since L is locally constant of rank one, this implies $L \simeq \mathbb{C}_{\mathbb{P}^*\setminus\mathrm{P}^*}$. q.e.d.

For $k \in \mathbb{Z}$, $\varepsilon \in \mathbb{Z}_2$, we consider the sheaves of twisted analytic functions, $\mathcal{C}^\infty$-functions, distributions and hyperfunctions given by:

$$\begin{aligned} \mathcal{A}_{\mathrm{P}}(k,\varepsilon) &\simeq \mathbb{C}_{\mathrm{P}}(\varepsilon) \otimes \mathcal{O}_{\mathbb{P}}(k), \\ \mathcal{C}^\infty_{\mathrm{P}}(k,\varepsilon) &\simeq \mathbb{C}_{\mathrm{P}}(\varepsilon) \overset{\mathrm{w}}{\otimes} \mathcal{O}_{\mathbb{P}}(k), \\ \mathcal{D}b_{\mathrm{P}}(k,\varepsilon) &\simeq \mathcal{T}hom(D'\mathbb{C}_{\mathrm{P}}(\varepsilon), \mathcal{O}_{\mathbb{P}}(k)), \\ \mathcal{B}_{\mathrm{P}}(k,\varepsilon) &\simeq R\mathcal{H}om(D'\mathbb{C}_{\mathrm{P}}(\varepsilon), \mathcal{O}_{\mathbb{P}}(k)). \end{aligned}$$

Notice that, for example, the global sections of $\mathcal{C}^\infty_{\mathrm{P}}(k,\varepsilon)$ are those $\mathcal{C}^\infty$-functions f on $\mathbb{R}^{n+1}\setminus\{0\}$ satisfying the homogeneity condition:

$$f(\lambda x) = (\operatorname{sgn}\lambda)^\varepsilon \lambda^k f(x), \quad \text{for } \lambda \in \mathbb{R}\setminus\{0\}.$$

Applying Corollary 4.5 and Proposition 5.16, we get the following result.

Theorem 5.17 *For $-n-1 < k < 0$, $\varepsilon \in \mathbb{Z}_2$, the section s_k introduced in (4.4) induces the following isomorphism:*

$$\Gamma(\mathrm{P};\mathcal{C}^\infty_{\mathrm{P}}(k,\varepsilon)) \simeq \Gamma(\mathrm{P}^*;\mathcal{C}^\infty_{\mathrm{P}^*}(k^*,\varepsilon^*)). \tag{5.17}$$

and similarly with $\mathcal{C}^\infty$ replaced by $\mathcal{A}$, $\mathcal{D}b$ or $\mathcal{B}$.

Notice that this theorem was already obtained (in the $\mathcal{C}^\infty$-case) by explicit computations in [9]. The case of hyperfunctions is treated in [16, Proposition 4.1.3] who also generalize it to the case of arbitrary homogeneity (i.e., $k \in \mathbb{C}$).

5.3 Other applications

(a) It is well known (cf. [5]) that the transform $\Phi_{\mathbb{A}}$ interchanges $\mathbf{D}^{\mathrm{b}}_{\mathbb{C}-c}(\mathbb{P})$ and $\mathbf{D}^{\mathrm{b}}_{\mathbb{C}-c}(\mathbb{P}^*)$, and interchanges perverse objects with perverse objects modulo constant sheaves (this is in fact an immediate consequence of the microlocal characterization of perversity in [18, Ch. 10]).

Let us call generalized holomorphic function a section of $H^j R\mathcal{H}om(K,\mathcal{L})$ for $K \in \mathbf{D}^{\mathrm{b}}_{\mathbb{C}-c}(\mathbb{P}^*)$ and $\mathcal{L}$ a holomorphic line bundle. Corollary 4.5 shows that $\Phi_{\mathbb{A}}$ interchanges generalized holomorphic functions on $\mathbb{P}^*$ and $\mathbb{P}$.

It would be interesting to study more precisely the transform of the sheaf of ramified holomorphic functions on a hypersurface of $\mathbb{P}^*$. Related results are obtained in [28].

(b) Let $Z \subset \mathbb{P}$ be a complete intersection subvariety of codimension d, and denote by $\mathcal{O}_Z$ the quotient of $\mathcal{O}_{\mathbb{P}}$ by the defining ideal $\mathcal{I}_Z$. Note that $\mathcal{O}_Z^* \simeq \mathcal{O}_Z[-d]$.

A result of Henkin [11] asserts that the transform $\Phi_{\mathbb{A}}$ interchanges holomorphic functions on Z with holomorphic functions on $\mathbb{P}^*$ satisfying a system of constant coefficient differential equations associated to Z. This can be rephrased by the isomorphisms:

$$\begin{aligned}\Phi_{\mathbb{A}}(\mathcal{O}_Z) &\simeq \Phi_{\mathbb{A}}(R\mathcal{H}om_{\mathcal{D}_{\mathbb{P}}}(\mathcal{D}\mathcal{O}_Z,\mathcal{O}_{\mathbb{P}})[d]) \\ &\simeq R\mathcal{H}om_{\mathcal{D}_{\mathbb{P}}}(\underline{\Phi}_{\mathbb{A}}(\mathcal{D}\mathcal{O}_Z),\mathcal{O}_{\mathbb{P}^*})[d].\end{aligned}$$

Moreover, let $f_1 = \cdots = f_d = 0$ be a system of globally defined equations for Z with f_j homogeneous of degree m_j. For $k >> 0$, the sheaf $\mathcal{O}_Z(k)$ is quasi-isomorphic to the complex

$$0 \longrightarrow \mathcal{L}^d \longrightarrow \cdots \longrightarrow \mathcal{L}^2 \longrightarrow \bigoplus_j \mathcal{O}_{\mathbb{P}}(k+m_j) \xrightarrow{(f_j)} \mathcal{O}_{\mathbb{P}}(k) \longrightarrow 0$$

where the $\mathcal{L}^j$ are locally free. Applying the functor $\mathcal{D}_{\mathbb{P}} \otimes_{\mathcal{O}_{\mathbb{P}}} (\cdot)$, we get a complex quasi-isomorphic to $\mathcal{D}\mathcal{O}_Z(k)$. Next, applying the functor $\underline{\Phi}_{\mathbb{A}}$ we find a complex $\mathcal{N}^{\cdot}_Z(k)$ quasi-isomorphic to $\underline{\Phi}_{\mathbb{A}}(\mathcal{D}\mathcal{O}_Z(k))$.

One knows that $H^j(\mathcal{N}^{\cdot}_Z(k))$ is a flat connection for $j \neq 0$. Moreover, by this construction we find that for $k > -n-1$, $H^0(\mathcal{N}^{\cdot}_Z(k))$ is isomorphic to the cokernel of

$$\bigoplus_j \mathcal{D}_{\mathbb{P}^*}(k^* - m_j) \xrightarrow{(\widehat{f_j})} \mathcal{D}_{\mathbb{P}^*}(k^*),$$

where $\widehat{f_j}$ is the constant coefficient differential operator homogeneous of degree d_j, the Fourier-Radon transform of f_j.

A Appendices and comments

A.1 The functors of temperate and formal cohomology

Let us briefly recall some constructions of [14] and [19].

First, assume $X^{\mathbb{R}}$ is a real analytic manifold, and let $\mathbb{R}-Cons(X^{\mathbb{R}})$ denote the abelian category of $\mathbb{R}$-constructible sheaves on $X^{\mathbb{R}}$. Denote by $\mathcal{D}b_{X^{\mathbb{R}}}$ the sheaf of Schwartz's distributions on $X^{\mathbb{R}}$, and by $\mathcal{C}^{\infty}_{X^{\mathbb{R}}}$ the sheaf of functions of class $\mathcal{C}^{\infty}$. There exists a unique contravariant exact functor:

$$\mathcal{T}hom(\cdot, \mathcal{D}b_{X^{\mathbb{R}}}) : \mathbb{R}-Cons(X^{\mathbb{R}})^{\mathrm{op}} \longrightarrow \mathrm{Mod}(\mathcal{D}_{X^{\mathbb{R}}})$$

such that if Z is a closed subanalytic subset of $X^{\mathbb{R}}$, then

$$\mathcal{T}hom(\mathbb{C}_Z, \mathcal{D}b_{X^{\mathbb{R}}}) = \Gamma_Z \mathcal{D}b_{X^{\mathbb{R}}}.$$

Similarly, there exists a unique exact functor

$$\cdot \overset{\mathrm{w}}{\otimes} \mathcal{C}^{\infty}_{X^{\mathbb{R}}} : \mathbb{R}-Cons(X^{\mathbb{R}}) \longrightarrow \mathrm{Mod}(\mathcal{D}_{X^{\mathbb{R}}})$$

such that for Z as above

$$\mathbb{C}_{X^{\mathbb{R}} \setminus Z} \overset{\mathrm{w}}{\otimes} \mathcal{C}^{\infty}_{X^{\mathbb{R}}} = \mathcal{I}^{\infty}_{Z,X^{\mathbb{R}}},$$

where $\mathcal{I}^{\infty}_{Z,X^{\mathbb{R}}}$ denotes the ideal of $\mathcal{C}^{\infty}_{X^{\mathbb{R}}}$ of functions vanishing to infinite order on Z. These functors being exact, they naturally extend as functors

$$\begin{aligned} \mathcal{T}hom(\cdot, \mathcal{D}b_{X^{\mathbb{R}}}) : \quad & \mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_{X^{\mathbb{R}}})^{\mathrm{op}} \longrightarrow \mathbf{D}^{\mathrm{b}}(\mathcal{D}_{X^{\mathbb{R}}}), \\ \cdot \overset{\mathrm{w}}{\otimes} \mathcal{C}^{\infty}_{X^{\mathbb{R}}} : \quad & \mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_{X^{\mathbb{R}}}) \longrightarrow \mathbf{D}^{\mathrm{b}}(\mathcal{D}_{X^{\mathbb{R}}}). \end{aligned}$$

Let X be a complex manifold. Denote by $\overline{X}$ the associated anti-holomorphic manifold, by $X^{\mathbb{R}}$ the underlying real analytic manifold, and identify $X^{\mathbb{R}}$ to the diagonal of $X \times \overline{X}$. For $F \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_X)$ one sets:

$$\begin{aligned}
\mathcal{T}hom(F, \mathcal{O}_X) &= R\mathcal{H}om_{\mathcal{D}_{\overline{X}}}(\mathcal{O}_{\overline{X}}, \mathcal{T}hom(F, \mathcal{D}b_{X^{\mathbb{R}}})), \\
F \overset{\mathrm{w}}{\otimes} \mathcal{O}_X &= R\mathcal{H}om_{\mathcal{D}_{\overline{X}}}(\mathcal{O}_{\overline{X}}, F \overset{\mathrm{w}}{\otimes} \mathcal{C}^{\infty}_{X^{\mathbb{R}}}).
\end{aligned}$$

In other words, one defines $\mathcal{T}hom(F, \mathcal{O}_X)$ as the Dolbeault complex with coefficients in $\mathcal{T}hom(F, \mathcal{D}b_{X^{\mathbb{R}}})$, and $F \overset{\mathrm{w}}{\otimes} \mathcal{O}_X$ as the Dolbeault complex with coefficients in $F \overset{\mathrm{w}}{\otimes} \mathcal{C}^{\infty}_{X^{\mathbb{R}}}$.

For $\mathcal{F}$ a locally free $\mathcal{O}_X$-module of finite rank, one sets for short:

$$\begin{aligned}
\mathcal{T}hom(\cdot, \mathcal{F}) &= \mathcal{T}hom(\cdot, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{F}, \\
\mathrm{THom}(\cdot, \mathcal{F}) &= \mathrm{R}\Gamma(X, \mathcal{T}hom(\cdot, \mathcal{F})), \\
\cdot \overset{\mathrm{w}}{\otimes} \mathcal{F} &= (\cdot \overset{\mathrm{w}}{\otimes} \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{F}.
\end{aligned}$$

If $K \subset X$ is a compact subanalytic subset, one sets:

$$\mathrm{R}\Gamma_{[K]}(\mathcal{O}_X) = \mathcal{T}hom(\mathbb{C}_K, \mathcal{O}_X).$$

In [19], the following adjunction formulas are established in a general setting which apply in particular to our case.

Proposition A.1 *Consider the correspondence (2.1), and assume (2.2). Let* $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_X)$, $G \in \mathbf{D}^{\mathrm{b}}(\mathbb{C}_Y)$. *Then:*

$$\begin{aligned}
\mathrm{R}\Gamma(X; R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \Phi_{\widetilde{S}} G \overset{\mathrm{w}}{\otimes} \mathcal{O}_X)) &\simeq \mathrm{R}\Gamma(Y; R\mathcal{H}om_{\mathcal{D}_Y}(\underline{\Phi}_S \mathcal{M}, G \overset{\mathrm{w}}{\otimes} \mathcal{O}_Y)), \\
\mathrm{R}\Gamma(X; R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{T}hom(\Phi_S G, \mathcal{O}_X))) &\simeq \mathrm{R}\Gamma(Y; R\mathcal{H}om_{\mathcal{D}_Y}(\underline{\Phi}_S \mathcal{M}, \mathcal{T}hom(G, \mathcal{O}_Y)),
\end{aligned}$$

and there are similar formulas by replacing everywhere $\mathrm{R}\Gamma$ *by* $\mathrm{R}\Gamma_c$.

A.2 Kernels

Most of the results of this section are well-known from the specialists, and we shall present them here without proof. In particular, the formalism of kernels for sheaves is discussed in [18, §3.6].

As general notations, we will consider three complex manifolds X, Y, and Z of dimension d_X, d_Y, and d_Z respectively. For $i, j = 1, 2, 3$, we denote by q_{ij} the projections from $X \times Y \times Z$ to the corresponding factor (e.g., $q_{23} : X \times Y \times Z \longrightarrow Y \times Z$).

Definition A.2 For $K \in \mathbf{D}^{\mathrm{b}}(\mathbb{C}_{X\times Y})$ and $L \in \mathbf{D}^{\mathrm{b}}(\mathbb{C}_{Y\times Z})$, we set:

$$\begin{aligned} K \circ L &= Rq_{13!}(q_{12}^{-1}K \otimes q_{23}^{-1}L), \\ {}^tK &= D'(r_*K)[2d_X], \end{aligned}$$

where $r : X \times Y \longrightarrow Y \times X$ is the natural map.

For $K \in \mathbf{D}^{\mathrm{b}}(\mathbb{C}_{X\times Y})$, we shall consider the following hypothesis:

$$\text{the projection } p_2 : T^*(X \times Y) \longrightarrow T^*Y \text{ is proper on } SS(K). \tag{A.1}$$

Proposition A.3 *Let $K \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_{X\times Y})$, $F \in \mathbf{D}^{\mathrm{b}}(\mathbb{C}_{Z\times X})$, $G \in \mathbf{D}^{\mathrm{b}}(\mathbb{C}_{Z\times Y})$. Assume (A.1). Then:*

$$\mathrm{RHom}(F \circ K, G) \simeq \mathrm{RHom}(F, G \circ {}^tK).$$

Corollary A.4 *Let $K \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_{X\times Y})$, and assume (A.1). Then there are natural morphisms:*

$$\begin{aligned} \mathbb{C}_{\Delta_X} &\longrightarrow K \circ {}^tK, \\ {}^tK \circ K &\longrightarrow \mathbb{C}_{\Delta_Y}. \end{aligned}$$

Proof: The first morphism is the one associated to $id_{F\circ K}$ by the isomorphism of Proposition A.3, applied with $Z = X$, $F = \mathbb{C}_{\Delta_X}$, and $G = F \circ K$. The second morphism is similarly constructed. q.e.d.

Definition A.5 For $\mathcal{K} \in \mathbf{D}^{\mathrm{b}}(\mathcal{D}_{X\times Y})$ and $\mathcal{L} \in \mathbf{D}^{\mathrm{b}}(\mathcal{D}_{Y\times Z})$, we set:

$$\mathcal{K} \underline{\circ} \mathcal{L} = \underline{q_{13}}_!(\underline{q_{12}}^{-1}\mathcal{K} \otimes^L_{\mathcal{O}_{X\times Y\times Z}} \underline{q_{23}}^{-1}\mathcal{L}).$$

Proposition A.6 *Let $\mathcal{K} \in \mathbf{D}^{\mathrm{b}}(\mathcal{D}_{X\times Y})$ and $\mathcal{L} \in \mathbf{D}^{\mathrm{b}}(\mathcal{D}_{Y\times Z})$. Then, there is a natural isomorphism in $\mathbf{D}^{\mathrm{b}}(\mathcal{D}_{X\times Z})$:*

$$\mathcal{K} \underline{\circ} \mathcal{L} \simeq Rq_{13!}(q_{12}^{-1}\mathcal{K}^{(0,d_Y)} \otimes^L_{q_2^{-1}\mathcal{D}_Y} q_{23}^{-1}\mathcal{L}),$$

where $\mathcal{K}^{(0,d_Y)}$ is endowed with its natural $(q_1^{-1}\mathcal{D}_X, q_2^{-1}\mathcal{D}_Y)$-bimodule structure.

Proof: By definition,

$$\mathcal{K} \underline{\circ} \mathcal{L} \simeq Rq_{13!}((\mathcal{D}_X \boxtimes \Omega_Y \boxtimes \mathcal{D}_Z) \otimes^L_{\mathcal{D}_{X\times Y\times Z}} (\mathcal{K} \boxtimes \mathcal{O}_Z \otimes^L_{\mathcal{O}_{X\times Y\times Z}} \mathcal{O}_X \boxtimes \mathcal{L}))$$

Moreover, one has the following chain of isomorphisms:

$$\begin{aligned} &(\mathcal{D}_X \boxtimes \Omega_Y \boxtimes \mathcal{D}_Z) \otimes^L_{\mathcal{D}_{X\times Y\times Z}} (\mathcal{K} \boxtimes \mathcal{O}_Z \otimes^L_{\mathcal{O}_{X\times Y\times Z}} \mathcal{O}_X \boxtimes \mathcal{L}) \\ \simeq\ & q_2^{-1}\Omega_Y \otimes^L_{q_2^{-1}\mathcal{D}_Y} (q_{12}^{-1}\mathcal{K} \otimes^L_{q_2^{-1}\mathcal{O}_Y} q_{23}^{-1}\mathcal{L}) \\ \simeq\ & (q_2^{-1}\Omega_Y \otimes^L_{q_2^{-1}\mathcal{O}_Y} q_{12}^{-1}\mathcal{K}) \otimes^L_{q_2^{-1}\mathcal{D}_Y} q_{23}^{-1}\mathcal{L} \\ \simeq\ & q_{12}^{-1}\mathcal{K}^{(0,d_Y)} \otimes^L_{q_2^{-1}\mathcal{D}_Y} q_{23}^{-1}\mathcal{L}. \end{aligned}$$

q.e.d.

For the following result, we refer for example to [2, §1].

Proposition A.7 *Let* $K \in \mathbf{D}^{\mathrm{b}}_{\mathbb{C}-c}(\mathbb{C}_{X\times Y})$, $L \in \mathbf{D}^{\mathrm{b}}_{\mathbb{C}-c}(\mathbb{C}_{Y\times Z})$. *Assume that* q_2 *is proper on* $\operatorname{supp} K$. *Then:*

$$\mathcal{T}hom(K, \mathcal{O}_{X\times Y}) \,\underline{\circ}\, \mathcal{T}hom(L, \mathcal{O}_{Y\times Z}) \xrightarrow{\sim} \mathcal{T}hom(K \circ L, \mathcal{O}_{X\times Z}).$$

Remark A.8 Consider the correspondence (2.1). It is then immediate to check that for $F \in \mathbf{D}^{\mathrm{b}}(\mathbb{C}_X)$, one has:

$$\Phi_S(F) \simeq F \circ \mathbb{C}_S[n-c].$$

Moreover, using Proposition A.6, it is easy to check that for $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}(\mathcal{D}_X)$:

$$\underline{\Phi}_S(\mathcal{M}) \simeq \mathcal{M} \,\underline{\circ}\, \mathcal{B}_{S|X\times Y}.$$

A.3 Final comments

In this paper, we have used adjunction formulas associated to a submanifold S of $X \times Y$, and this was enough for the applications we had in mind. However, more general kernels associated to perverse sheaves on $X \times Y$ may be of interest.

For $\mathcal{K}$ a regular holonomic $\mathcal{D}_{X\times Y}$-module, set $K = R\mathcal{H}om_{\mathcal{D}_{X\times Y}}(\mathcal{K}, \mathcal{O}_{X\times Y})$. Then, using the notations of Appendix A.2, formulas (A.4), (A.5) below are well-known from the specialists: let us mention in particular M. Kashiwara and also J.-P. Schneiders, with whom we had many discussions on this subject.

Assume that:

$$(\operatorname{supp}(\mathcal{M}) \times Y) \cap \operatorname{supp}(K) \text{ is proper over } Y, \tag{A.2}$$
$$(\operatorname{char}(\mathcal{M}) \times T^*_Y Y) \cap \operatorname{char}(\mathcal{K}) \subset T^*_{X\times Y}(X \times Y). \tag{A.3}$$

Then, we have isomorphisms:

$$\begin{aligned} &\mathrm{R}\Gamma_c(X; R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, (K \circ G) \otimes \mathcal{O}_X))[d_X] && \text{(A.4)}\\ &\quad\simeq\ \mathrm{R}\Gamma_c(Y; R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M} \,\underline{\circ}\, \mathcal{K}, G \otimes \mathcal{O}_Y)),\\ &\mathrm{R}\Gamma(X; R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, R\mathcal{H}om(K \circ G, \mathcal{O}_X)))[d_X] && \text{(A.5)}\\ &\quad\simeq\ \mathrm{R}\Gamma(Y; R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M} \,\underline{\circ}\, \mathcal{K}, R\mathcal{H}om(G, \mathcal{O}_Y)))[2d_Y]. \end{aligned}$$

These formulas were recently extended to formal and moderate cohomology in [19, Theorem 10.8] as follows:

Assume (A.2) and:

$$(\mathrm{supp}(G) \times Y) \cap \mathrm{supp}(K) \text{ is proper over } X. \tag{A.6}$$

Then, we have an isomorphism:

$$\begin{aligned} &\mathrm{R}\Gamma(X; R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, (K \circ G) \overset{\mathrm{w}}{\otimes} \mathcal{O}_X))[d_X] \\ &\quad\simeq \mathrm{R}\Gamma(Y; R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M} \underline{\circ} \mathcal{K}, G \overset{\mathrm{w}}{\otimes} \mathcal{O}_Y)). \end{aligned} \tag{A.7}$$

If moreover (A.3) holds, then we have an isomorphism:

$$\begin{aligned} &\mathrm{R}\Gamma_c(X; R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{T}hom(K \circ G, \mathcal{O}_X)))[d_X] \\ &\quad\simeq \mathrm{R}\Gamma_c(Y; R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M} \underline{\circ} \mathcal{K}, \mathcal{T}hom(G, \mathcal{O}_Y)))[2d_Y]. \end{aligned} \tag{A.8}$$

In the case of projective duality —the subject we are concerned with in this paper— the following kernels are remarkable. Let $\Omega = \mathbb{P} \times \mathbb{P}^* \setminus \mathbb{A}$, and let $j : \Omega \hookrightarrow \mathbb{P} \times \mathbb{P}^*$ be the embedding. Set

$$K = \mathbb{C}_\Omega, \quad \mathcal{K} = \mathcal{T}hom(K, \mathcal{O}_{\mathbb{P}\times\mathbb{P}^*}).$$

Then ${}^tK = Rj_*j^{-1}\mathbb{C}_\Omega$, and Kashiwara (see [20]) has noticed that

$$\begin{aligned} K \circ {}^tK &\simeq \mathbb{C}_{\Delta_{\mathbb{P}}}, \\ {}^tK \circ K &\simeq \mathbb{C}_{\Delta_{\mathbb{P}^*}}. \end{aligned}$$

Hence, $\mathcal{K}$ gives an equivalence of categories between $\mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_{\mathbb{P}})$ and $\mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_{\mathbb{P}^*})$. (Compare with Lemma 4.6 and Remark 2.9.)

Notes added in proofs See also A. Goncharov "Integral Geometry and $\mathcal{D}$-modules", Math. Research Letters, 2 (1995) 915-935, for another approach to integral geometry via $\mathcal{D}$-modules theory.

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B Errata

Errata to the paper:
Leray's quantization of projective duality
A. D'Agnolo and P. Schapira
Duke Math. J. **84** (1996), 453–496

1) To get the injectivity in the first morphism of Corollary 5.11 one should also assume that $U_2^\#$ has non-empty interior.

2) Proposition B.5 holds only in the algebraic case. In the analytic case, it holds only when Z is reduced to a point; i.e., it should be replaced by:

Let $\mathcal{K} \in \mathbf{D}^{\mathrm{b}}(\mathcal{D}_{X\times Y})$ and $\mathcal{L} \in \mathbf{D}^{\mathrm{b}}(\mathcal{D}_Y)$. Then, there is a natural isomorphism in $\mathbf{D}^{\mathrm{b}}(\mathcal{D}_X)$:

$$\mathcal{K} \underline{\circ} \mathcal{L} \simeq Rq_{1!}(\mathcal{K}^{(0,d_Y)} \otimes^L [q_2^{-1}\mathcal{D}_Y] q_2^{-1}\mathcal{L}),$$

where $\mathcal{K}^{(0,d_Y)}$ is endowed with its natural $(q_1^{-1}\mathcal{D}_X, q_2^{-1}\mathcal{D}_Y)$-bimodule structure.

3) In formulas (C.4) and (C.7), $K \circ G$ should be replaced by $G \circ {}^tK$. In other words, formula (C.4) should read:

$$\begin{aligned} &\mathrm{R}\Gamma(X; R\mathcal{H}om[\mathcal{D}_X](\mathcal{M}, R\mathcal{H}om(G \circ {}^tK, \mathcal{O}_X)))[d_X] \\ &\quad \simeq \ \mathrm{R}\Gamma(Y; R\mathcal{H}om[\mathcal{D}_Y](\mathcal{M} \underline{\circ} \mathcal{K}, R\mathcal{H}om(G, \mathcal{O}_Y)))[2d_Y], \end{aligned}$$

and formula (C.7) should read:

$$\begin{aligned} &\mathrm{R}\Gamma_c(X; R\mathcal{H}om[\mathcal{D}_X](\mathcal{M}, \mathcal{T}hom(G \circ {}^tK, \mathcal{O}_X)))[d_X] \\ &\quad \simeq \ \mathrm{R}\Gamma_c(Y; R\mathcal{H}om[\mathcal{D}_Y](\mathcal{M} \underline{\circ} \mathcal{K}, \mathcal{T}hom(G, \mathcal{O}_Y)))[2d_Y]. \end{aligned}$$