

# Sheaves for analysis: an overview

for Hikosaburo Komatsu 80's and Takahiro Kawai 70's

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## Abstract

This is an expository paper, without proofs, extracted from the paper [GS16] with Stéphane Guillermou. We show how the use of Grothendieck topologies on a real analytic manifold allows one to recover classical spaces of analysis which are not of local nature for the usual topology. We apply these results to endow the subanalytic sheaf of holomorphic functions with a filtration, then to endow regular holonomic D-modules with a functorial filtration (in the derived sense).

## Introduction

Let  $M$  be a real analytic manifold. Denote by  $\text{Op}_{M_{\text{sa}}}$  the family of relatively compact subanalytic open subsets of  $M$ , ordered by inclusion. The Grothendieck subanalytic topology on  $M$ , denoted  $M_{\text{sa}}$ , was first introduced in [KS01]. Its objects are those of  $\text{Op}_{M_{\text{sa}}}$  and the coverings are, roughly speaking, the finite coverings. One denotes by  $\rho_{\text{sa}}: M \rightarrow M_{\text{sa}}$  the natural morphism of sites.

In [GS16], we introduce another Grothendieck topology that we call the linear subanalytic topology on  $M$  and denote by  $M_{\text{sal}}$ . The objects are the same, but there are much less coverings: these are those satisfying some linear condition given in (1.4). Hence, there is a natural morphism of sites  $\rho_{\text{sal}}: M_{\text{sa}} \rightarrow M_{\text{sal}}$ .

Choose a field  $\mathbf{k}$  and denote by  $\text{D}(\mathbf{k}_{\mathcal{F}})$  ( $\tau = M, M_{\text{sa}}, M_{\text{sal}}$ ) the derived category of sheaves on  $M, M_{\text{sa}}, M_{\text{sal}}$ . An important result of this paper is that the direct image functor  $\text{R}\rho_{\text{sal}*}: \text{D}^+(\mathbf{k}_{M_{\text{sa}}}) \rightarrow \text{D}^+(\mathbf{k}_{M_{\text{sal}}})$  admits a left adjoint functor  $\rho_{\text{sal}}^!$  and that, if  $U \in \text{Op}_{M_{\text{sa}}}$  has Lipschitz boundary, one has for  $F \in \text{D}^+(\mathbf{k}_{M_{\text{sal}}})$

$$(0.1) \quad \text{R}\Gamma(U; \rho_{\text{sal}}^! F) \simeq \text{R}\Gamma(U; F).$$

It follows that if a presheaf  $F$  on  $M_{\text{sa}}$  has the property that the Mayer-Vietoris sequences

$$(0.2) \quad 0 \rightarrow F(U \cup V) \rightarrow F(U) \oplus F(V) \rightarrow F(U \cap V) \rightarrow 0$$

are exact, as soon as  $\{U, V\}$  is a covering of  $U \cup V$  for the linear subanalytic topology, then  $F$  is a sheaf on  $M_{\text{sal}}$  and  $\text{R}\Gamma(U; \rho_{\text{sal}}^! F)$  is concentrated in degree 0 and is isomorphic to  $F(U)$  for any  $U$  with Lipschitz boundary. In other words, to a presheaf on  $M_{\text{sa}}$  satisfying a natural condition, we are able to associate an object of the derived category of sheaves on  $M_{\text{sa}}$  which has the same sections as  $F$  on any Lipschitz open set. This construction is in particular used by Gilles Lebeau [Leb16] who obtains for  $s \leq 0$  the ‘‘Sobolev sheaves  $\mathcal{H}_{M_{\text{sa}}}^s$ ’’, objects of  $\text{D}^+(\mathbb{C}_{M_{\text{sa}}})$  with the property that if  $U \in \text{Op}_{M_{\text{sa}}}$  has a Lipschitz boundary, then  $\text{R}\Gamma(U; \mathcal{H}_{M_{\text{sa}}}^s)$  is concentrated in degree 0 and coincides with the classical Sobolev space  $H^s(U)$ .

The subanalytic topology and its refinement, the linear subanalytic topology, thus allow one to construct new sheaves which would have no meaning on the usual topology. On  $M_{\text{sa}}$  we shall construct the sheaf  $\mathcal{C}_{M_{\text{sa}}}^{\infty, \text{tp}}$  of  $\mathcal{C}^\infty$ -functions with temperate growth and the sheaf  $\mathcal{D}b_{M_{\text{sa}}}^{\text{tp}}$  of temperate distributions (see [KS01]). On  $M_{\text{sal}}$  we shall construct the sheaf  $\mathcal{C}_{M_{\text{sal}}}^{\infty, t}$  of functions of temperate growth of order  $t \geq 0$ , and the sheaves  $\mathcal{C}_{M_{\text{sal}}}^{\infty, \text{gev}(s)}$  and  $\mathcal{C}_{M_{\text{sal}}}^{\infty, \text{gev}\{s\}}$  of functions with Gevrey growth of order  $s > 1$ . By applying the functor  $\rho_{\text{sal}}^!$  we obtain new sheaves (in the derived sense) on  $M_{\text{sa}}$ .

On a complex manifold  $X$ , we can take the Dolbeault complexes with coefficients in such sheaves and obtain their holomorphic counterparts. We construct in particular on  $X_{\text{sa}}$  the sheaves (in the derived sense)  $\mathcal{O}_{X_{\text{sa}}}^{\text{gev}(s)}$  and  $\mathcal{O}_{X_{\text{sa}}}^{\text{gev}\{s\}}$  of holomorphic functions of Gevrey growth of type  $s > 1$ .

By considering the family of sheaves  $\{\mathcal{C}_{M_{\text{sal}}}^{\infty, t}\}_{t \geq 0}$  as a filtration on a sheaf  $\mathcal{C}_{M_{\text{sal}}}^{\infty, \text{tp } st}$  and taking their Dolbeault complexes, we can endow the sheaf  $\mathcal{O}_{X_{\text{sa}}}^{\text{tp}}$  with a filtration  $F_\infty \mathcal{O}_{X_{\text{sa}}}^{\text{tp}}$ . For that purpose, we need to study first filtered objects in tensor categories, following [SS16]. Then the Riemann-Hilbert correspondence (Kashiwara’s theorem of [Kas84]) allows us to endow *functorially* regular holonomic D-modules with filtrations (in the derived sense).

## 1 Subanalytic topologies

### Notations and conventions

We shall mainly follow the notations of [KS90], [KS01] and [KS06].

In this paper, unless otherwise specified, a manifold means a real analytic manifold. We shall freely use the theory of subanalytic sets, due to Gabrielov

and Hironaka, after the pioneering work of Lojasiewicz. A short presentation of this theory may be found in [BM88].

For a subset  $A$  in a topological space  $X$ ,  $\overline{A}$  denotes its closure,  $\text{Int } A$  its interior and  $\partial A$  its boundary,  $\partial A = \overline{A} \setminus \text{Int } A$ .

Recall that given two metric spaces  $(X, d_X)$  and  $(Y, d_Y)$ , a function  $f: X \rightarrow Y$  is Lipschitz if there exists a constant  $C \geq 0$  such that  $d_Y(f(x), f(x')) \leq C \cdot d_X(x, x')$  for all  $x, x' \in X$ .

(1.1)  $\left\{ \begin{array}{l} \text{All along this paper, if } M \text{ is a real analytic manifold, we} \\ \text{choose a distance } d_M \text{ on } M \text{ such that, for any } x \in M \text{ and} \\ \text{any local chart } (U, \varphi: U \hookrightarrow \mathbb{R}^n) \text{ around } x, \text{ there exists a} \\ \text{neighborhood of } x \text{ over which } d_M \text{ is Lipschitz equivalent to} \\ \text{the pull-back of the Euclidean distance by } \varphi. \text{ If there is no} \\ \text{risk of confusion, we write } d \text{ instead of } d_M. \end{array} \right.$

In the following, we will adopt the convention

$$(1.2) \quad d(x, \emptyset) = D_M + 1, \quad \text{for all } x \in M,$$

where  $D_M = \sup\{d(y, z); y, z \in M\}$ . In this way we avoid distinguishing the special case where  $M = \bigcup_{i \in I} U_i$  in (1.4) below (which can happen if  $M$  is compact).

### The sites $M_{\text{sa}}$ and $M_{\text{sal}}$

The subanalytic topology was introduced in [KS01] in the general framework of ind-sheaves. A more direct and elementary treatment of subanalytic sheaves may be found in [Pre08]. The linear subanalytic topology was introduced in [GS16].

Let  $M$  be a real analytic manifold and denote by  $\text{Op}_{M_{\text{sa}}}$  the category of relatively compact subanalytic open subsets of  $M$ , the morphisms being the inclusion morphisms.

**Definition 1.1.** (a) The subanalytic site  $M_{\text{sa}}$  is the presite  $M_{\text{sa}}$  endowed with the Grothendieck topology for which the coverings are defined as follows. A family  $\{U_i\}_{i \in I}$  of objects of  $\text{Op}_{M_{\text{sa}}}$  is a covering of  $U \in \text{Op}_{M_{\text{sa}}}$  if  $U_i \subset U$  for all  $i \in I$  and there exists a finite subset  $J \subset I$  such that  $\bigcup_{j \in J} U_j = U$ .

(b) We denote by  $\rho_{\text{sa}}: M \rightarrow M_{\text{sa}}$  the natural morphism of sites.

It follows from the theory of subanalytic sets that in this situation there exist a constant  $C > 0$  and a positive integer  $N$  such that

$$(1.3) \quad d(x, M \setminus U)^N \leq C \cdot (\max_{j \in J} d(x, M \setminus U_j)).$$

**Definition 1.2.** Let  $\{U_j\}_{j \in J}$  be a finite family in  $\text{Op}_{M_{\text{sa}}}$ . We say that this family is 1-regularly situated if one can choose  $N = 1$  in (1.3), that is, if there is a constant  $C$  such that for any  $x \in M$

$$(1.4) \quad d(x, M \setminus \bigcup_{j \in J} U_j) \leq C \cdot \max_{j \in J} d(x, M \setminus U_j).$$

Of course, this definition does not depend on the choice of the distance  $d$ .

**Example 1.3.** On  $\mathbb{R}^2$  with coordinates  $(x_1, x_2)$  consider the open sets:

$$\begin{aligned} U_1 &= \{(x_1, x_2); x_2 > -x_1^2, x_1 > 0\}, \\ U_2 &= \{(x_1, x_2); x_2 < x_1^2, x_1 > 0\}, \\ U_3 &= \{(x_1, x_2); x_1 > -x_2^2, x_2 > 0\}. \end{aligned}$$

Then  $\{U_1, U_2\}$  is not 1-regularly situated. Indeed, set  $W := U_1 \cup U_2 = \{x_1 > 0\}$ . Then, if  $x = (x_1, 0)$ ,  $x_1 > 0$ ,  $d(x, \mathbb{R}^2 \setminus W) = x_1$  and  $d(x, \mathbb{R}^2 \setminus U_i)$  ( $i = 1, 2$ ) is less than  $x_1^2$ .

On the other hand  $\{U_1, U_3\}$  is 1-regularly situated. Indeed,

$$d(x, \mathbb{R}^2 \setminus (U_1 \cup U_3)) \leq \sqrt{2} \max(d(x, \mathbb{R}^2 \setminus U_1), d(x, \mathbb{R}^2 \setminus U_3)).$$

**Definition 1.4.** A linear covering of  $U$  is a small family  $\{U_i\}_{i \in I}$  of objects of  $\text{Op}_{M_{\text{sa}}}$  such that  $U_i \subset U$  for all  $i \in I$  and

$$(1.5) \quad \begin{cases} \text{there exists a finite subset } J \subset I \text{ such that the family } \{U_j\}_{j \in J} \\ \text{is 1-regularly situated and } \bigcup_{j \in J} U_j = U. \end{cases}$$

**Proposition 1.5.** *The family of linear coverings satisfies the axioms of Grothendieck topologies (see [KS06, § 16.1]).*

**Definition 1.6.** (a) The linear subanalytic site  $M_{\text{sal}}$  is the presite  $M_{\text{sa}}$  endowed with the Grothendieck topology for which the coverings are the linear coverings given by Definition 1.4.

(b) We denote by  $\rho_{\text{sal}}: M_{\text{sa}} \rightarrow M_{\text{sal}}$  and by  $\rho_{\text{sl}}: M \rightarrow M_{\text{sal}}$  the natural morphisms of sites.

The morphisms of sites constructed above are summarized by the diagram

$$\begin{array}{ccc} M & \xrightarrow{\rho_{\text{sa}}} & M_{\text{sa}} \\ & \searrow \rho_{\text{sl}} & \downarrow \rho_{\text{sal}} \\ & & M_{\text{sal}}. \end{array}$$

## 2 Sheaves

We shall mainly follow the notations of [KS90, KS01] and [KS06].

In this paper, we denote by  $\mathbf{k}$  a field. Unless otherwise specified, a manifold means a real analytic manifold.

If  $\mathcal{C}$  is an additive category, we denote by  $C(\mathcal{C})$  the additive category of complexes in  $\mathcal{C}$ . For  $* = +, -, b$  we also consider the full additive subcategory  $C^*(\mathcal{C})$  of  $C(\mathcal{C})$  consisting of complexes bounded from below (resp. from above, resp. bounded) and  $C^{\text{ub}}(\mathcal{C})$  means  $C(\mathcal{C})$  (“ub” stands for “unbounded”). If  $\mathcal{C}$  is an abelian category, we denote by  $D(\mathcal{C})$  its derived category and similarly with  $D^*(\mathcal{C})$  for  $* = +, -, b, \text{ub}$ .

For a site  $\mathcal{T}$ , we denote by  $\text{PSh}(\mathbf{k}_{\mathcal{T}})$  and  $\text{Mod}(\mathbf{k}_{\mathcal{T}})$  the abelian categories of presheaves and sheaves of  $\mathbf{k}$ -modules on  $\mathcal{T}$ . We denote by  $\iota: \text{Mod}(\mathbf{k}_{\mathcal{T}}) \rightarrow \text{PSh}(\mathbf{k}_{\mathcal{T}})$  the forgetful functor and by  $(\cdot)^a$  its left adjoint, the functor which associates a sheaf to a presheaf. Note that in practice we shall often not write  $\iota$ . Recall that  $\text{Mod}(\mathbf{k}_{\mathcal{T}})$  is a Grothendieck category and, in particular, has enough injectives. We write  $D^*(\mathbf{k}_{\mathcal{T}})$  instead of  $D^*(\text{Mod}(\mathbf{k}_{\mathcal{T}}))$  ( $* = +, -, b, \text{ub}$ ).

For an object  $U$  of  $\mathcal{T}$ , recall that there is a sheaf naturally attached to  $U$  (see *e.g.* [KS06, § 17.6]). We shall denote it here by  $\mathbf{k}_{U, \mathcal{T}}$  or simply  $\mathbf{k}_U$  if there is no risk of confusion. If  $V \rightarrow U$  is a monomorphism in  $\mathcal{T}$ , then the natural morphism  $\mathbf{k}_V \rightarrow \mathbf{k}_U$  also is a monomorphism.

### Sheaves on $M$ , $M_{\text{sa}}$ and $M_{\text{sal}}$

The direct image functors  $\rho_{\text{sa}*}$  and  $\rho_{\text{sal}*}$  are left exact and their left adjoint functors  $\rho_{\text{sa}}^{-1}$  and  $\rho_{\text{sal}}^{-1}$  are exact. Hence, we have the pairs of adjoint functors

$$(2.1) \quad \text{Mod}(\mathbf{k}_M) \begin{array}{c} \xleftarrow{\rho_{\text{sa}*}} \\ \xrightarrow{\rho_{\text{sa}}^{-1}} \end{array} \text{Mod}(\mathbf{k}_{M_{\text{sa}}}), \quad \text{D}^b(\mathbf{k}_M) \begin{array}{c} \xleftarrow{\text{R}\rho_{\text{sa}*}} \\ \xrightarrow{\rho_{\text{sa}}^{-1}} \end{array} \text{D}^b(\mathbf{k}_{M_{\text{sa}}}).$$

$$(2.2) \quad \text{Mod}(\mathbf{k}_{M_{\text{sa}}}) \begin{array}{c} \xleftarrow{\rho_{\text{sal}*}} \\ \xrightarrow{\rho_{\text{sal}}^{-1}} \end{array} \text{Mod}(\mathbf{k}_{M_{\text{sal}}}), \quad \text{D}^b(\mathbf{k}_{M_{\text{sa}}}) \begin{array}{c} \xleftarrow{\text{R}\rho_{\text{sal}*}} \\ \xrightarrow{\rho_{\text{sal}}^{-1}} \end{array} \text{D}^b(\mathbf{k}_{M_{\text{sal}}}).$$

The functor  $\rho_{\text{sa}*}$  is fully faithful and  $\rho_{\text{sa}}^{-1}\rho_{\text{sa}*} \simeq \text{id}$ . Moreover,  $\rho_{\text{sa}}^{-1}\text{R}\rho_{\text{sa}*} \simeq \text{id}$  and  $\text{R}\rho_{\text{sa}*}$  in (2.1) is fully faithful.

The same results hold with  $\rho_{\text{sal}*}$

The functors  $\rho_{\text{sa}}^{-1}$  and  $\rho_{\text{sl}}^{-1}$  also admit left adjoint functors  $\rho_{\text{sa}!}$  and  $\rho_{\text{sl}!}$ , respectively. For  $F \in \text{Mod}(\mathbf{k}_M)$ ,  $\rho_{\text{sa}!}F$  (resp.  $\rho_{\text{sl}!}F$ ) is the sheaf on  $M_{\text{sa}}$  (resp.  $M_{\text{sal}}$ ) associated with the presheaf  $U \mapsto F(\overline{U})$ . The functors  $\rho_{\text{sa}!}$  and  $\rho_{\text{sl}!}$  are exact, fully faithful and commute with tensor products.

One denotes by  $\text{Mod}_{\mathbb{R}\text{-c}}(\mathbf{k}_M)$  the category of  $\mathbb{R}$ -constructible sheaves on  $M$ . One denotes by  $\mathbf{D}_{\mathbb{R}\text{-c}}^{\text{b}}(\mathbf{k}_M)$  the full triangulated subcategory of  $\mathbf{D}^{\text{b}}(\mathbf{k}_M)$  consisting of objects with  $\mathbb{R}$ -constructible cohomologies.

The functor  $\rho_{\text{sa}*}$  is exact when restricted to the subcategory  $\text{Mod}_{\mathbb{R}\text{-c}}(\mathbf{k}_M)$ . Hence we shall consider this last category both as a full subcategory of  $\text{Mod}(\mathbf{k}_M)$  and a full subcategory of  $\text{Mod}(\mathbf{k}_{M_{\text{sa}}})$ .

For  $U \in \text{Op}_{M_{\text{sa}}}$  we shall simply denote by  $\mathbf{k}_U$  the sheaf  $\mathbf{k}_{U_{\mathcal{T}}}$  for  $\mathcal{T} = M, M_{\text{sa}}$  or  $M_{\text{sal}}$ .

**Proposition 2.1.** *Let  $\mathcal{T}$  be either the site  $M_{\text{sa}}$  or the site  $M_{\text{sal}}$ . Then a presheaf  $F$  is a sheaf if and only if it satisfies:*

- (i)  $F(\emptyset) = 0$ ,
- (ii) *for any  $U_1, U_2 \in \text{Op}_{M_{\text{sa}}}$  such that  $\{U_1, U_2\}$  is a covering of  $U_1 \cup U_2$ , the sequence  $0 \rightarrow F(U_1 \cup U_2) \rightarrow F(U_1) \oplus F(U_2) \rightarrow F(U_1 \cap U_2)$  is exact.*

Of course, if  $\mathcal{T} = M_{\text{sa}}$ ,  $\{U_1, U_2\}$  is always a covering of  $U_1 \cup U_2$ .

**Lemma 2.2.** *Let  $\mathcal{T}$  be either the site  $M_{\text{sa}}$  or the site  $M_{\text{sal}}$ . Let  $U \in \text{Op}_{M_{\text{sa}}}$  and let  $\{F_i\}_{i \in I}$  be an inductive system in  $\text{Mod}(\mathbf{k}_{\mathcal{T}})$  indexed by a small filtrant category  $I$ . Then*

$$(2.3) \quad \varinjlim_i \Gamma(U; F_i) \xrightarrow{\simeq} \Gamma(U; \varinjlim_i F_i).$$

This kind of results is well-known from the specialists (see *e.g.* [KS01, EP]).

### $\Gamma$ -acyclic sheaves

In this subsection,  $\mathcal{T}$  denotes either the site  $M_{\text{sa}}$  or the site  $M_{\text{sal}}$ . In the literature, one often encounters sheaves which are  $\Gamma(U; \bullet)$ -acyclic for a given  $U \in \mathcal{T}$  but the next definition does not seem to be frequently used.

**Definition 2.3.** Let  $F \in \text{Mod}(\mathbf{k}_{\mathcal{T}})$ . We say that  $F$  is  $\Gamma$ -acyclic if we have  $H^k(U; F) \simeq 0$  for all  $k > 0$  and all  $U \in \mathcal{T}$ .

We shall give criteria in order that a sheaf  $F$  on the site  $\mathcal{T}$  be  $\Gamma$ -acyclic.

Let  $U \in \text{Op}_{M_{\text{sa}}}$  and let  $\mathcal{U} := \{U_i\}_i \in I$  be a finite covering of  $U$  in  $\mathcal{T}$  (a regular covering in case  $\mathcal{T} = M_{\text{sal}}$ ). We denote by  $C^\bullet(\mathcal{U}; F)$  the associated Čech complex:

$$(2.4) \quad C^\bullet(\mathcal{U}; F) := \text{Hom}_{\mathbf{k}_{M_{\text{sal}}}}(\mathbf{k}_{\mathcal{U}}^\bullet, F).$$

One can write more explicitly this complex as the complex:

$$(2.5) \quad 0 \rightarrow \bigoplus_{J \subset I, |J|=1} F(U_J) \otimes e_J \xrightarrow{d} \cdots \xrightarrow{d} \bigoplus_{J \subset I, |J|=N} F(U_J) \otimes e_J \rightarrow 0$$

where the differential  $d$  is obtained by sending  $F(U_J) \otimes e_J$  to  $\bigoplus_{i \in I} F(U_J \cap U_i) \otimes e_i \wedge e_J$ .

**Proposition 2.4.** *Let  $\mathcal{T}$  be either the site  $M_{\text{sa}}$  or the site  $M_{\text{sal}}$  and let  $F \in \text{Mod}(\mathbf{k}_{\mathcal{T}})$ . The conditions below are equivalent.*

- (i) *For any  $\{U_1, U_2\}$  which is a covering of  $U_1 \cup U_2$ , the sequence  $0 \rightarrow F(U_1 \cup U_2) \rightarrow F(U_1) \oplus F(U_2) \rightarrow F(U_1 \cap U_2) \rightarrow 0$  is exact.*
- (ii) *The sheaf  $F$  is  $\Gamma$ -acyclic.*
- (iii) *For any exact sequence in  $\text{Mod}(\mathbf{k}_{\mathcal{T}})$*

$$(2.6) \quad G^\bullet := 0 \rightarrow \bigoplus_{i_0 \in A_0} \mathbf{k}_{U_{i_0}} \rightarrow \cdots \rightarrow \bigoplus_{i_N \in A_N} \mathbf{k}_{U_{i_N}} \rightarrow 0,$$

*in which the  $U_{i_j}$  belong to  $\text{Op}_{M_{\text{sa}}}$  and the sets  $A_j$  ( $0 \leq j \leq N$ ) are finite, the sequence  $\text{Hom}_{\mathbf{k}_{\mathcal{T}}}(G^\bullet, F)$  is exact.*

- (iv) *For any finite covering  $\mathcal{U}$  of  $U$  (regular covering in case  $\mathcal{T} = M_{\text{sal}}$ ), the morphism  $F(U) \rightarrow C^\bullet(\mathcal{U}; F)$  is a quasi-isomorphism.*

**The functor  $\rho_{\text{sal}}^!$**

**Theorem 2.5.** (i) *The functor  $R\rho_{\text{sal}*} : D(\mathbf{k}_{M_{\text{sa}}}) \rightarrow D(\mathbf{k}_{M_{\text{sal}}})$  admits a right adjoint  $\rho_{\text{sal}}^! : D(\mathbf{k}_{M_{\text{sal}}}) \rightarrow D(\mathbf{k}_{M_{\text{sa}}})$ .*

- (ii) *The functor  $\rho_{\text{sal}}^!$  induces a functor  $\rho_{\text{sal}}^! : D^+(\mathbf{k}_{M_{\text{sal}}}) \rightarrow D^+(\mathbf{k}_{M_{\text{sa}}})$ .*

The proof is based on the Brown representability theorem (see for example [KS06, Th 14.3.1]) which essentially asserts that it is enough to check that the functor  $R\rho_{\text{sal}*}$  commutes with small direct sums.

## Open sets with Lipschitz boundaries

**Definition 2.6.** We say that  $U \in \text{Op}_{M_{\text{sa}}}$  has Lipschitz boundary or simply that  $U$  is Lipschitz if, for any  $x \in \partial U$ , there exist an open neighborhood  $V$  of  $x$  and a bi-Lipschitz subanalytic homeomorphism  $\psi : V \xrightarrow{\sim} W$  with  $W$  an open subset of  $\mathbb{R}^n$  such that  $\psi(V \cap U) = W \cap \{x_n > 0\}$ .

**Remark 2.7.** (i) The property of being Lipschitz is local and thus the preceding definition extends to subanalytic but not necessarily relatively compact open subsets of  $M$ .

(ii) If  $U_i$  is Lipschitz in  $M_i$  ( $i = 1, 2$ ) then  $U_1 \times U_2$  is Lipschitz in  $M_1 \times M_2$ .

(iii) If  $U$  is Lipschitz and  $x \in \partial U$ , there exist a constant  $C > 0$  and a sequence  $\{y_n\}_{n \in \mathbb{N}}$ ,  $y_n \in U$ , such that  $d(y_n, x) \rightarrow 0$  and  $d(y_n, x) \leq Cd(y_n, \partial U)$ , for all  $n \in \mathbb{N}$  (in the notations of the definition, assume  $\psi(x) = (x', 0)$  and set  $y_n = \psi^{-1}(x', 1/n)$ ).

**Example 2.8.** (i) Proposition 2.11 below will provide many examples of Lipschitz open sets.

(ii) Let  $(x, y)$  denotes the coordinates on  $\mathbb{R}^2$ . Using (iii) of Remark 2.7 we see that the open set  $U = \{(x, y); 0 < y < x^2\}$  is not Lipschitz.

**Lemma 2.9.** *Let  $U \in \text{Op}_{M_{\text{sa}}}$ . We assume that, for any  $x \in \partial U$ , there exist an open neighborhood  $V$  of  $x$  and a bi-analytic isomorphism  $\psi: V \xrightarrow{\sim} W$  with  $W$  an open subset of  $\mathbb{R}^n$  such that  $\psi(V \cap U) = W \cap \{(x', x_n); x_n > \varphi(x')\}$  for a Lipschitz subanalytic function  $\varphi$ . Then  $U$  is Lipschitz.*

We refer to [KS90, Def 4.1.1] for the definition of the normal cone  $C(A, B)$  associated with two subsets  $A$  and  $B$  of  $M$ .

Also recall [KS90, § 5.3] that for  $S \subset M$ , the strict normal cone  $N_x(S)$  is given by

$$N_x(S) = T_x M \setminus C(M \setminus S, S), \text{ an open cone in } T_x M.$$

**Definition 2.10.** We shall say that an open subset  $U$  of  $M$  satisfies a cone condition if for any  $x \in \partial U$ ,  $N_x(U)$  is non empty.

**Proposition 2.11.** *Let  $U \in \text{Op}_{M_{\text{sa}}}$ . If  $U$  satisfies a cone condition, then  $U$  is Lipschitz.*

## A vanishing theorem

The next theorem is a key result for this paper and its proof is due to A. Parusinski [Par16].

**Theorem 2.12.** (A. Parusinski) *Let  $V \in \text{Op}_{M_{\text{sa}}}$ . Then there exists a finite covering  $V = \bigcup_{j \in J} V_j$  with  $V_j \in \text{Op}_{M_{\text{sa}}}$  such that the family  $\{V_j\}_{j \in J}$  is a covering of  $V$  in  $M_{\text{sal}}$  and moreover  $H^k(V_j; \mathbf{k}_M) \simeq 0$  for all  $k > 0$  and all  $j \in J$ .*

We need to extend Definition 2.6.



**Definition 2.13.** We say that  $U \in \text{Op}_{M_{\text{sa}}}$  is weakly Lipschitz if for each  $x \in M$  there exist a neighborhood  $V \in \text{Op}_{M_{\text{sa}}}$  of  $x$ , a finite set  $I$  and  $U_i \in \text{Op}_{M_{\text{sa}}}$  such that  $U \cap V = \bigcup_i U_i$  and

$$(2.7) \quad \left\{ \begin{array}{l} \text{for all } \emptyset \neq J \subset I, \text{ the set } U_J = \bigcap_{j \in J} U_j \text{ is a disjoint union of} \\ \text{Lipschitz open sets.} \end{array} \right.$$

By its definition, the property of being weakly Lipschitz is local on  $M$ .

**Proposition 2.14.** *Let  $U \in \text{Op}_{M_{\text{sa}}}$  and consider a finite family of smooth submanifolds  $\{Z_i\}_{i \in I}$ , closed in a neighborhood of  $\bar{U}$ . Set  $Z = \bigcup_{i \in I} Z_i$ . Assume that*

- (a)  $U$  is Lipschitz,
- (b)  $Z_i \cap Z_j \cap \partial U = \emptyset$  for  $i \neq j$ ,  $\partial U$  is smooth in a neighborhood of  $Z \cap \partial U$  and the intersection is transversal,
- (c) in a neighborhood of each point of  $Z \cap U$  there exist a local coordinate system  $(x_1, \dots, x_n)$  and for each  $i \in I$ , a subset  $I_i$  of  $\{1, \dots, n\}$  such that  $Z_i = \bigcap_{j \in I_i} \{x_j = 0\}$ .

Then  $U \setminus Z$  is weakly Lipschitz.

**Theorem 2.15.** *Let  $U \in \text{Op}_{M_{\text{sa}}}$  and assume that  $U$  is weakly Lipschitz. Then*

- (i)  $R\rho_{\text{sal}*} \mathbf{k}_{UM_{\text{sa}}} \simeq \rho_{\text{sal}*} \mathbf{k}_{UM_{\text{sa}}} \simeq \mathbf{k}_{UM_{\text{sal}}}$  is concentrated in degree zero.
- (ii) For  $F \in \mathbf{D}^b(\mathbf{k}_{M_{\text{sal}}})$ , one has  $\text{R}\Gamma(U; \rho_{\text{sal}}^! F) \simeq \text{R}\Gamma(U; F)$ .
- (iii) Let  $F \in \text{Mod}(\mathbf{k}_{M_{\text{sal}}})$  and assume that  $F$  is  $\Gamma$ -acyclic. Then  $\text{R}\Gamma(U; \rho_{\text{sal}}^! F)$  is concentrated in degree 0 and is isomorphic to  $F(U)$ .

Note that the result in (i) is local and it is not necessary to assume here that  $U$  is relatively compact.

### 3 Construction of sheaves

On the site  $M_{\text{sa}}$ , the sheaves  $\mathcal{C}_{M_{\text{sa}}}^{\infty, \text{tp}}$  and  $\mathcal{D}b_{M_{\text{sa}}}^{\text{tp}}$  below have been constructed in [KS96, KS01]. By using the linear topology we shall construct sheaves on  $M_{\text{sal}}$  associated with more precise growth conditions.

We follow Convention 1.1.

As usual, we denote by  $\mathcal{C}_M^\infty$  (resp.  $\mathcal{A}_M$ ) the sheaf of complex valued functions of class  $\mathcal{C}^\infty$  (resp. real analytic), by  $\mathcal{D}b_M$  (resp.  $\mathcal{B}_M$ ) the sheaf of Schwartz's distributions (resp. Sato's hyperfunctions) and by  $\mathcal{D}_M$  the sheaf of finite-order differential operators with coefficients in  $\mathcal{A}_M$ . References for the theory of D-modules is made to [Kas03].

### Temperate growth on $M_{\text{sa}}$

**Definition 3.1.** Let  $U \in \text{Op}_{M_{\text{sa}}}$  and let  $f \in \mathcal{C}_M^\infty(U)$ . One says that  $f$  has *polynomial growth* at  $p \in M \setminus U$  if it satisfies the following condition. For a local coordinate system  $(x_1, \dots, x_n)$  around  $p$ , there exist a sufficiently small compact neighborhood  $K$  of  $p$  and a positive integer  $N$  such that

$$(3.1) \quad \sup_{x \in K \cap U} (d(x, K \setminus U))^N |f(x)| < \infty.$$

We say that  $f$  is temperate at  $p$  if all its derivatives have polynomial growth at  $p$ . We say that  $f$  is temperate if it is temperate at any point  $p \in M \setminus U$ .

For  $U \in \text{Op}_{M_{\text{sa}}}$ , we shall denote by  $\mathcal{C}_M^{\infty, \text{tp}}(U)$  the subspace of  $\mathcal{C}_M^\infty(U)$  consisting of temperate functions.

For  $U \in \text{Op}_{M_{\text{sa}}}$ , we shall denote by  $\mathcal{D}b_M^{\text{tp}}(U)$  the space of temperate distributions on  $U$ , defined by the exact sequence

$$0 \rightarrow \Gamma_{M \setminus U}(M; \mathcal{D}b_M) \rightarrow \Gamma(M; \mathcal{D}b_M) \rightarrow \mathcal{D}b_M^{\text{tp}}(U) \rightarrow 0.$$

It follows from (1.3) that  $U \mapsto \mathcal{C}_M^{\infty, \text{tp}}(U)$  is a sheaf and it follows from the work of Lojasiewicz [Loj59] that  $U \mapsto \mathcal{D}b_M^{\text{tp}}(U)$  is also a sheaf. We denote by  $\mathcal{C}_{M_{\text{sa}}}^{\infty, \text{tp}}$  and  $\mathcal{D}b_{M_{\text{sa}}}^{\text{tp}}$  these sheaves on  $M_{\text{sa}}$ . The first one is called the sheaf of  $\mathcal{C}^\infty$ -functions with temperate growth and the second the sheaf of temperate distributions. Note that both sheaves are  $\Gamma$ -acyclic (see [KS01, Lem 7.2.4]).

We denote as usual by  $\mathcal{D}_M$  the sheaf of rings of finite order differential operators on the real analytic manifold  $M$ . If  $\iota_M: M \hookrightarrow X$  is a complexification of  $M$ , then  $\mathcal{D}_M \simeq \iota_M^{-1} \mathcal{D}_X$ . We set, following [KS01]:

$$(3.2) \quad \mathcal{D}_{M_{\text{sa}}} := \rho_{\text{sa}!} \mathcal{D}_M.$$

The sheaves  $\mathcal{C}_{M_{\text{sa}}}^{\infty, \text{tp}}$  and  $\mathcal{D}b_{M_{\text{sa}}}^{\text{tp}}$  are  $\mathcal{D}_{M_{\text{sa}}}$ -modules.

### Temperate growth of a given order on $M_{\text{sal}}$

If a sheaf  $\mathcal{F}$  on  $M_{\text{sa}}$  is  $\Gamma$ -acyclic, then  $R\rho_{\text{sal}*} \mathcal{F}$  is concentrated in degree 0. This applies in particular to the sheaves  $\mathcal{C}_{M_{\text{sa}}}^{\infty, \text{tp}}$  and  $\mathcal{D}b_{M_{\text{sa}}}^{\text{tp}}$ . We set

$$\mathcal{C}_{M_{\text{sal}}}^{\infty, \text{tp}} := \rho_{\text{sal}*} \mathcal{C}_{M_{\text{sa}}}^{\infty, \text{tp}}, \quad \mathcal{D}b_{M_{\text{sal}}}^{\text{tp}} := \rho_{\text{sal}*} \mathcal{D}b_{M_{\text{sa}}}^{\text{tp}}.$$

**Definition 3.2.** Let  $U \in \text{Op}_{M_{\text{sa}}}$ , let  $f \in \mathcal{C}_M^\infty(U)$  and let  $t \in \mathbb{R}_{\geq 0}$ . We say that  $f$  has *polynomial growth of order  $\leq t$*  at  $p \in M \setminus U$  if it satisfies the following condition. For a local coordinate system  $(x_1, \dots, x_n)$  around  $p$ , there exists a sufficiently small compact neighborhood  $K$  of  $p$  such that

$$(3.3) \quad \sup_{x \in K \cap U} (d(x, K \setminus U))^t |f(x)| < \infty.$$

We say that  $f$  is temperate of order  $t$  at  $p$  if, for each  $m \in \mathbb{N}$ , all its derivatives of order  $\leq m$  have polynomial growth of order  $\leq t + m$  at  $p$ . We say that  $f$  is temperate of order  $t$  if it is temperate of order  $t$  at any point  $p \in M \setminus U$ .

For  $U \in \text{Op}_{M_{\text{sa}}}$ , we denote by  $\mathcal{C}_M^{\infty, t}(U)$  the subspace of  $\mathcal{C}_M^\infty(U)$  consisting of functions temperate of order  $t$  and we denote by  $\mathcal{C}_{M_{\text{sal}}}^{\infty, t}$  the presheaf on  $M_{\text{sal}}$  so obtained.

The next result is clear by Proposition 2.1.

**Proposition 3.3.** (i) *The presheaves  $\mathcal{C}_{M_{\text{sal}}}^{\infty, t}$  ( $t \geq 0$ ) are sheaves on  $M_{\text{sal}}$ ,*

(ii) *the sheaf  $\mathcal{C}_{M_{\text{sal}}}^{\infty, 0}$  is a sheaf of rings,*

(iii) *for  $t \geq 0$ ,  $\mathcal{C}_{M_{\text{sal}}}^{\infty, t}$  is a  $\mathcal{C}_{M_{\text{sal}}}^{\infty, 0}$ -module and there are natural morphisms*  

$$\mathcal{C}_{M_{\text{sal}}}^{\infty, t} \otimes_{\mathcal{C}_{M_{\text{sal}}}^{\infty, 0}} \mathcal{C}_{M_{\text{sal}}}^{\infty, t'} \rightarrow \mathcal{C}_{M_{\text{sal}}}^{\infty, t+t'}.$$

We also introduce the sheaf

$$\mathcal{C}_{M_{\text{sal}}}^{\infty, \text{tp } st} := \varinjlim_t \mathcal{C}_{M_{\text{sal}}}^{\infty, t}.$$

(Of course, the limit is taken in the category of sheaves on  $M_{\text{sal}}$ .) Then, for  $0 \leq t \leq t'$ , there are natural monomorphisms of sheaves on  $M_{\text{sal}}$  :

$$(3.4) \quad \mathcal{C}_{M_{\text{sal}}}^{\infty, 0} \hookrightarrow \mathcal{C}_{M_{\text{sal}}}^{\infty, t} \hookrightarrow \mathcal{C}_{M_{\text{sal}}}^{\infty, t'} \hookrightarrow \mathcal{C}_{M_{\text{sal}}}^{\infty, \text{tp } st} \hookrightarrow \mathcal{C}_{M_{\text{sal}}}^{\infty, \text{tp}}.$$

Note that the inclusion  $\mathcal{C}_{M_{\text{sal}}}^{\infty, \text{tp } st} \hookrightarrow \mathcal{C}_{M_{\text{sal}}}^{\infty, \text{tp}}$  is strict since there exists a function  $f$  (say on an open subset  $U$  of  $\mathbb{R}$ ) with polynomial growth of order  $\leq t$  and such that its derivative does not have polynomial growth of order  $\leq t + 1$ .

### Gevrey growth on $M_{\text{sal}}$

The definition below is inspired by the definition of the sheaves of  $\mathcal{C}^\infty$ -functions of Gevrey classes, but is completely different from the classical one. Here we are interested in the growth of functions at the boundary contrarily to the classical setting where one is interested in the Taylor expansion

of the function. As usual, there are two kinds of regularity which can be interesting: regularity at the interior or at the boundary. Since we shall soon consider the Dolbeault complexes of our new sheaves, the interior regularity is irrelevant and we are only interested in the growth at the boundary.

We refer to [Kom73] for an exposition on classical Gevrey functions or distributions and their link with Sato's theory of boundary values of holomorphic functions. Note that there is also a recent study by [HM11] of these sheaves using the tools of subanalytic geometry.

**Definition 3.4.** Let  $U \in \text{Op}_{M_{\text{sa}}}$ , let  $(s, h) \in ]1, +\infty[ \times ]0, +\infty[$  and let  $f \in \mathcal{C}_M^\infty(U)$ . We say that  $f$  has *0-Gevrey growth of type  $(s, h)$*  at  $p \in M$  if it satisfies the following condition. For a local coordinate system  $(x_1, \dots, x_n)$  around  $p$ , there exists a sufficiently small compact neighborhood  $K$  of  $p$  such that

$$(3.5) \quad \sup_{x \in K \cap U} (\exp(-h \cdot d(x, K \setminus U)^{1-s})) |f(x)| < \infty,$$

with the convention that, if  $K \cap U = \emptyset$  or  $K \subset U$ , the left-hand side of (3.5) is 0. It is obvious that  $f$  has 0-Gevrey growth of type  $(s, h)$  at any point of  $U$ . We say that  $f$  has Gevrey growth of type  $(s, h)$  at  $p$  if all its derivatives have 0-Gevrey growth of type  $(s, h)$  at  $p$ . We say that  $f$  has Gevrey growth of type  $(s, h)$  if it has such a growth at any point.

We denote by  $G_M^{s,h}(U)$  the subspace of  $\mathcal{C}_M^\infty(U)$  consisting of functions with Gevrey growth of type  $(s, h)$ .

**Definition 3.5.** For  $U \in \text{Op}_{M_{\text{sa}}}$  and  $s \in ]1, +\infty[$ , we set:

$$G_M^{(s)}(U) := \varprojlim_h G_M^{s,h}(U), \quad G_M^{\{s\}}(U) := \varinjlim_h G_M^{s,h}(U)$$

and we denote by  $\mathcal{C}_{M_{\text{sal}}}^{\infty, \text{gev}(s)}$  and  $\mathcal{C}_{M_{\text{sal}}}^{\infty, \text{gev}\{s\}}$  the presheaves on  $M_{\text{sal}}$  so obtained.

Clearly, the presheaves  $\mathcal{C}_{M_{\text{sal}}}^{\infty, \text{gev}(s)}$  and  $\mathcal{C}_{M_{\text{sal}}}^{\infty, \text{gev}\{s\}}$  do not depend on the choice of the distance.

**Proposition 3.6.** (i) *The presheaves  $\mathcal{C}_{M_{\text{sal}}}^{\infty, \text{gev}(s)}$  and  $\mathcal{C}_{M_{\text{sal}}}^{\infty, \text{gev}\{s\}}$  are sheaves on  $M_{\text{sal}}$ ,*

(ii) *the sheaves  $\mathcal{C}_{M_{\text{sal}}}^{\infty, \text{gev}(s)}$  and  $\mathcal{C}_{M_{\text{sal}}}^{\infty, \text{gev}\{s\}}$  are  $\mathcal{C}_{M_{\text{sal}}}^{\infty, \text{tp}}$ -modules,*

(iii) *the presheaves  $\mathcal{C}_{M_{\text{sal}}}^{\infty, \text{gev}(s)}$  and  $\mathcal{C}_{M_{\text{sal}}}^{\infty, \text{gev}\{s\}}$  are  $\Gamma$ -acyclic,*

(iv) we have natural monomorphisms of sheaves on  $M_{\text{sal}}$  for  $1 < s < s'$

$$\mathcal{C}_{M_{\text{sal}}}^{\infty, \text{gev}(s)} \hookrightarrow \mathcal{C}_{M_{\text{sal}}}^{\infty, \text{gev}\{s\}} \hookrightarrow \mathcal{C}_{M_{\text{sal}}}^{\infty, \text{gev}(s')} \hookrightarrow \mathcal{C}_{M_{\text{sal}}}^{\infty, \text{gev}\{s'\}}.$$

We set

$$\mathcal{C}_{M_{\text{sal}}}^{\infty, \text{gev st}} := \varinjlim_{s>1} \mathcal{C}_{M_{\text{sal}}}^{\infty, \text{gev}\{s\}}.$$

Hence, we have monomorphisms of sheaves on  $M_{\text{sal}}$  for  $0 \leq t$  and  $1 < s$

$$\begin{aligned} \mathcal{C}_{M_{\text{sal}}}^{\infty, 0} &\hookrightarrow \mathcal{C}_{M_{\text{sal}}}^{\infty, t} \hookrightarrow \mathcal{C}_{M_{\text{sal}}}^{\infty, \text{tp st}} \hookrightarrow \mathcal{C}_{M_{\text{sal}}}^{\infty, \text{tp}} \\ &\hookrightarrow \mathcal{C}_{M_{\text{sal}}}^{\infty, \text{gev}(s)} \hookrightarrow \mathcal{C}_{M_{\text{sal}}}^{\infty, \text{gev}\{s\}} \hookrightarrow \mathcal{C}_{M_{\text{sal}}}^{\infty, \text{gev st}}. \end{aligned}$$

**Definition 3.7.** If  $\mathcal{F}_{M_{\text{sal}}}$  is one of the sheaves  $\mathcal{C}_{M_{\text{sal}}}^{\infty, t}$ ,  $\mathcal{C}_{M_{\text{sal}}}^{\infty, \text{tp st}}$ ,  $\mathcal{C}_{M_{\text{sal}}}^{\infty, \text{gev}(s)}$ ,  $\mathcal{C}_{M_{\text{sal}}}^{\infty, \text{gev}\{s\}}$  or  $\mathcal{C}_{M_{\text{sal}}}^{\infty, \text{gev st}}$ , we set  $\mathcal{F}_{M_{\text{sa}}} := \rho_{\text{sal}}^! \mathcal{F}$ .

Let us apply Theorem 2.15 and Corollary 3.10. We get that if  $U \in \text{Op}_{M_{\text{sa}}}$  is weakly Lipschitz and if  $\mathcal{F}_{M_{\text{sal}}}$  denotes one of the sheaves above, then

$$\text{R}\Gamma(U; \mathcal{F}_{M_{\text{sa}}}) \simeq \Gamma(U; \mathcal{F}_{M_{\text{sal}}}).$$

We call  $\mathcal{C}_{M_{\text{sa}}}^{\infty, t}$ ,  $\mathcal{C}_{M_{\text{sa}}}^{\infty, \text{tp st}}$ ,  $\mathcal{C}_{M_{\text{sa}}}^{\infty, \text{gev}(s)}$ ,  $\mathcal{C}_{M_{\text{sa}}}^{\infty, \text{gev}\{s\}}$  and  $\mathcal{C}_{M_{\text{sa}}}^{\infty, \text{gev st}}$  the sheaves on  $M_{\text{sa}}$  of  $\mathcal{C}^\infty$ -functions of growth  $t$ , strictly temperate growth, Gevrey growth of type  $(s)$  and  $\{s\}$  and strictly Gevrey growth, respectively. Recall that on  $M_{\text{sa}}$ , we also have the sheaf  $\mathcal{C}_{M_{\text{sa}}}^{\infty, \text{tp}}$  of  $\mathcal{C}^\infty$ -functions of temperate growth and the sheaf  $\mathcal{D}b_{M_{\text{sa}}}^{\text{tp}}$  of temperate distributions.

### A refined cutoff lemma

Lemma 3.8 below follows from Hörmander [Hör83, Cor.1.4.11]. Note that this result was already used in [KS96, Prop. 10.2]. Hörmander's result is stated for  $M = \mathbb{R}^n$  but it can be extended to an arbitrary manifold.

**Lemma 3.8.** *Let  $M$  be a manifold. Let  $Z_1$  and  $Z_2$  be two closed subsets of  $M$  such that  $M \setminus (Z_1 \cap Z_2)$  is relatively compact and such there exists  $C > 0$  with*

$$(3.6) \quad d(x, Z_1 \cap Z_2) \leq C(d(x, Z_1) + d(x, Z_2)) \text{ for any } x \in M.$$

*Then there exists  $\psi \in \mathcal{C}_M^{\infty, 0}(M \setminus (Z_1 \cap Z_2))$  such that  $\psi = 0$  on a neighborhood of  $Z_1 \setminus Z_2$  and  $\psi = 1$  on a neighborhood of  $Z_2 \setminus Z_1$ .*

**Corollary 3.9.** *Let  $\mathcal{F}$  be a sheaf of  $\mathcal{C}_{M_{\text{sal}}}^{\infty,0}$ -modules on  $M_{\text{sal}}$ . Then  $\mathcal{F}$  is  $\Gamma$ -acyclic.*

Let  $\mathcal{F}_{M_{\text{sal}}}$  denote one of the sheaves  $\mathcal{C}_{M_{\text{sal}}}^{\infty,\text{tp } st}$ ,  $\mathcal{C}_{M_{\text{sal}}}^{\infty,\text{tp}}$ ,  $\mathcal{D}b_{M_{\text{sal}}}^{\text{tp}}$ ,  $\mathcal{C}_{M_{\text{sal}}}^{\infty,t}$  ( $t \in \mathbb{R}_{\geq 0}$ ),  $\mathcal{C}_{M_{\text{sal}}}^{\infty,\text{gev}(s)}$  and  $\mathcal{C}_{M_{\text{sal}}}^{\infty,\text{gev}\{s\}}$  ( $s > 1$ ) and  $\mathcal{C}_{M_{\text{sal}}}^{\infty,\text{gev } st}$  and set  $\mathcal{F}_{M_{\text{sa}}} := \rho_{\text{sal}}^! \mathcal{F}_{M_{\text{sal}}}$ , an object of  $D^+(\mathcal{D}_{M_{\text{sa}}})$ .

**Corollary 3.10.** *Let  $\mathcal{F}_{M_{\text{sal}}}$  and  $\mathcal{F}_{M_{\text{sa}}}$  be as above.*

- (a)  $\mathcal{F}_{M_{\text{sal}}}$  is  $\Gamma$ -acyclic,
- (b) if  $U$  is weakly Lipschitz, then  $\text{R}\Gamma(U; \mathcal{F}_{M_{\text{sa}}})$  is concentrated in degree 0 and coincides with  $\mathcal{F}_{M_{\text{sal}}}(U)$ .

## 4 Sheaves on complex manifolds

Let  $X$  be a complex manifold of complex dimension  $d_X$  and denote by  $X_{\mathbb{R}}$  the real analytic underlying manifold. Denote by  $\overline{X}$  the complex manifold conjugate to  $X$ . (The holomorphic functions on  $\overline{X}$  are the anti-holomorphic functions on  $X$ .) Then  $X \times \overline{X}$  is a complexification of  $X_{\mathbb{R}}$  and  $\mathcal{O}_{\overline{X}}$  is a  $\mathcal{D}_{X \times \overline{X}}$ -module which plays the role of the Dolbeault complex. In the sequel, when there is no risk of confusion, we write for short  $X$  instead of  $X_{\mathbb{R}}$ .

**Notation 4.1.** In the sequel, we will often have to consider the composition  $\text{R}\rho_{\text{sal}*} \circ \rho_{\text{sa}}^!$ . For convenience, we introduce a notation. We set

$$(4.1) \quad \rho_{\text{sl}*!} := \rho_{\text{sal}*} \circ \rho_{\text{sa}}^!.$$

By applying the Dolbeault functor  $\text{R}\mathcal{H}om_{\mathcal{D}_{\overline{X}}_{\text{sal}}}(\rho_{\text{sl}*!} \mathcal{O}_{\overline{X}}, \bullet)$  to one of the sheaves

$$\mathcal{C}_{X_{\text{sal}}}^{\infty,\text{tp } st}, \quad \mathcal{C}_{X_{\text{sal}}}^{\infty,\text{tp}}, \quad \mathcal{C}_{X_{\text{sal}}}^{\infty,\text{gev}(s)}, \quad \mathcal{C}_{X_{\text{sal}}}^{\infty,\text{gev}\{s\}}, \quad \mathcal{C}_{X_{\text{sal}}}^{\infty,\text{gev } st}$$

we obtain respectively the sheaves

$$\mathcal{O}_{X_{\text{sal}}}^{\text{tp } st}, \quad \mathcal{O}_{X_{\text{sal}}}^{\text{tp}}, \quad \mathcal{O}_{X_{\text{sal}}}^{\text{gev}(s)}, \quad \mathcal{O}_{X_{\text{sal}}}^{\text{gev}\{s\}}, \quad \mathcal{O}_{X_{\text{sal}}}^{\text{gev } st}.$$

All these objects belong to  $D^+(\mathcal{D}_{X_{\text{sal}}})$ . Then we can apply the functor  $\rho_{\text{sal}}^!$  and we obtain the sheaves

$$\mathcal{O}_{X_{\text{sa}}}^{\text{tp } st}, \quad \mathcal{O}_{X_{\text{sa}}}^{\text{tp}}, \quad \mathcal{O}_{X_{\text{sa}}}^{\text{gev}(s)}, \quad \mathcal{O}_{X_{\text{sa}}}^{\text{gev}\{s\}}, \quad \mathcal{O}_{X_{\text{sa}}}^{\text{gev } st}.$$

Recall the natural isomorphism [KS96, Th. 10.5]

$$\mathcal{O}_{X_{\text{sa}}}^{\text{tp}} \xrightarrow{\simeq} \text{R}\mathcal{H}om_{\mathcal{D}_{\overline{X}}_{\text{sa}}}(\rho_{\text{sa}}^! \mathcal{O}_{\overline{X}}, \mathcal{D}b_{X_{\text{sa}}}^{\text{tp}}).$$

A similar proof also gives the natural isomorphism

$$\mathcal{O}_{X_{\text{sal}}}^{\text{tp } st} \xrightarrow{\simeq} \mathcal{O}_{X_{\text{sal}}}^{\text{tp}}.$$

## 5 Filtrations on $\mathcal{O}_{X_{\text{sa}}}^{\text{tp}}$

### Filtered objects

Let us recall some results of [SS16] generalizing previous results of [Sch99].

We consider

$$(5.1) \quad \begin{cases} \text{a Grothendieck tensor category } \mathcal{C} \text{ (with unit) in which small} \\ \text{inductive limits commute with } \otimes, \\ \text{a filtrant preordered additive monoid } \Lambda \text{ (viewed as a tensor} \\ \text{category with unit).} \end{cases}$$

Denote by  $\text{Fct}(\Lambda, \mathcal{C})$  the abelian category of functors from  $\Lambda$  to  $\mathcal{C}$ . It is naturally endowed with a structure of a tensor category with unit by setting for  $M_1, M_2 \in \text{Fct}(\Lambda, \mathcal{C})$ ,

$$(M_1 \otimes M_2)(\lambda) = \varinjlim_{\lambda_1 + \lambda_2 \leq \lambda} M_1(\lambda_1) \otimes M_2(\lambda_2).$$

A  $\Lambda$ -ring  $A$  of  $\mathcal{C}$  is a ring with unit of the tensor category  $\text{Fct}(\Lambda, \mathcal{C})$  and we denote by  $\text{Mod}(A)$  the abelian category of  $A$ -modules.

We denote by  $F_\Lambda \mathcal{C}$  the full subcategory of  $\text{Fct}(\Lambda, \mathcal{C})$  consisting of functors  $M$  such that for each morphism  $\lambda \rightarrow \lambda'$  in  $\Lambda$ , the morphism  $M(\lambda) \rightarrow M(\lambda')$  is a monomorphism. This is a quasi-abelian category. Let

$$\iota: F_\Lambda \mathcal{C} \rightarrow \text{Fct}(\Lambda, \mathcal{C})$$

denote the inclusion functor. This functor admits a left adjoint  $\kappa$  and the category  $F_\Lambda \mathcal{C}$  is again a tensor category by setting

$$M_1 \otimes_F M_2 = \kappa(\iota(M_1) \otimes \iota(M_2)).$$

A ring object in the tensor category  $F_\Lambda \mathcal{C}$  will be called a  $\Lambda$ -filtered ring in  $\mathcal{C}$  and usually denoted  $FA$ . An  $FA$ -module  $FM$  is then simply a module over  $FA$  in  $F_\Lambda \mathcal{C}$  and we denote by  $\text{Mod}(FA)$  the quasi-abelian category of  $FA$ -modules.

**Notation 5.1.** In the sequel, for a ring object  $B$  in a tensor category, we shall write  $D^*(B)$  instead of  $D^*(\text{Mod}(B))$ ,  $*$  = +, -, b, ub.

The next theorem is due to [SS16] and generalizes previous results of [Sch99].

**Theorem 5.2.** *Let  $FA$  be a  $\Lambda$ -filtered ring in  $\mathcal{C}$ . Then the category  $\text{Mod}(FA)$  is quasi-abelian, the functor  $\iota: \text{Mod}(FA) \rightarrow \text{Mod}(\iota FA)$  is strictly exact and induces an equivalence of categories for  $*$  = ub, +, -, b:*

$$(5.2) \quad \iota: D^*(FA) \xrightarrow{\simeq} D^*(\iota FA).$$

**Notation 5.3.** Let  $\Lambda$  and  $\mathcal{C}$  be as above. The functor  $\varinjlim : \text{Fct}(\Lambda, \mathcal{C}) \rightarrow \mathcal{C}$  is exact. Let  $FA$  be a  $\Lambda$ -filtered ring in  $F_\Lambda \mathcal{C}$  and set

$$(5.3) \quad A := \varinjlim_{\lambda} A(\lambda).$$

The functor  $\varinjlim$  induces an exact functor

$$(5.4) \quad \varinjlim : \text{Mod}(FA) \rightarrow \text{Mod}(A),$$

thus, using Theorem 5.2, for  $*$  = ub, +, -, b, a functor

$$(5.5) \quad \varinjlim : D^*(FA) \rightarrow D^*(A).$$

Since one often considers  $FA$  as a filtration on the ring  $A$ , we shall denote by *for* (forgetful) the functor  $\varinjlim$ :

$$(5.6) \quad \text{for} : D^*(FA) \rightarrow D^*(A), \quad \text{for} := \varinjlim.$$

## 6 Filtrations on $\mathcal{O}_{X_{\text{sal}}}$

### The filtered ring of differential operators

**Definition 6.1.** Let  $\mathcal{T}$  be the site  $M$  or  $M_{\text{sa}}$  or  $M_{\text{sal}}$ . We define the filtered sheaf  $F\mathcal{D}_{\mathcal{T}}$  over  $\Lambda = \mathbb{R}$  by setting:

$$F^s \mathcal{D}_{\mathcal{T}} = \mathcal{D}_{\mathcal{T}}([s])$$

where  $[s]$  is the integral part of  $s$  and  $\mathcal{D}_{\mathcal{T}}([s])$  is the sheaf of differential operators of order  $\leq [s]$ . In particular,  $F^s \mathcal{D}_{\mathcal{T}} = 0$  for  $s < 0$ . We denote by  $\text{Mod}(F\mathcal{D}_{\mathcal{T}})$  the category of filtered modules over  $\mathcal{D}_{\mathcal{T}}$ .

Let  $M_{\mathcal{T}}$  be either  $M$ ,  $M_{\text{sa}}$  or  $M_{\text{sal}}$ . In the sequel, we look at  $\text{Mod}(\mathbb{C}_{M_{\mathcal{T}}})$  as an abelian Grothendieck tensor category with unit and at  $F\mathcal{D}_{M_{\mathcal{T}}}$  as a  $\Lambda$ -ring object in  $F_\Lambda \mathcal{C}$  (with  $\Lambda = \mathbb{R}$ ) and  $\mathcal{C} = \text{Mod}(\mathbb{C}_{M_{\mathcal{T}}})$ . In the sequel, if  $F\mathcal{M}$  is a filtered object in  $\mathcal{C}$  over the ordered additive monoid  $\mathbb{R}$ , we shall write  $F^s \mathcal{M}$  instead of  $(F\mathcal{M})(s)$  to denote the image of the functor  $F\mathcal{M}$  at  $s \in \mathbb{R}$ . This induces a functor  $D(F_{\mathbb{R}} \mathcal{C}) \rightarrow D(\mathcal{C})$  denoted in the same way  $F\mathcal{M} \mapsto F^s \mathcal{M}$ .

Since  $\rho_{\text{sa}}^{-1}(\mathcal{D}_{M_{\text{sa}}}(m)) \simeq \mathcal{D}_M(m)$  and  $\rho_{\text{sal}}^{-1}(\mathcal{D}_{M_{\text{sal}}}(m)) \simeq \mathcal{D}_{M_{\text{sa}}}(m)$  get the functors

$$(6.1) \quad \begin{aligned} \rho_{\text{sa}}^{-1} : \text{Mod}(F\mathcal{D}_{M_{\text{sa}}}) &\rightarrow \text{Mod}(F\mathcal{D}_M), \\ \rho_{\text{sal}}^{-1} : \text{Mod}(F\mathcal{D}_{M_{\text{sal}}}) &\rightarrow \text{Mod}(F\mathcal{D}_{M_{\text{sa}}}). \end{aligned}$$

We will also use the fully faithful right adjoint of  $\rho_{\text{sal}}^{-1}$

$$(6.2) \quad \rho_{\text{sal}*} : \text{Mod}(F\mathcal{D}_{M_{\text{sa}}}) \rightarrow \text{Mod}(F\mathcal{D}_{M_{\text{sal}}}).$$



**Theorem 6.2.** (i) *The functor  $\rho_{\text{sal}*}$  in (6.2) admits a right derived functor  $R\rho_{\text{sal}*}: D^*(F\mathcal{D}_{M_{\text{sa}}}) \rightarrow D^*(F\mathcal{D}_{M_{\text{sa}}})$  ( $*$  = ub, +) which is fully faithful and admits a left adjoint functor  $\rho_{\text{sal}}^{-1}: D^*(F\mathcal{D}_{M_{\text{sa}}}) \rightarrow D^*(F\mathcal{D}_{M_{\text{sa}}})$  ( $*$  = ub, +).*

(ii) *The functor  $R\rho_{\text{sal}*}$  ( $*$  = ub, +) commutes with small direct sums and admits a right adjoint  $\rho_{\text{sal}}^!: D^*(F\mathcal{D}_{M_{\text{sa}}}) \rightarrow D^*(F\mathcal{D}_{M_{\text{sa}}})$  ( $*$  = ub, +).*

(iii) *The functor  $\rho_{\text{sa}}^{-1}: D^+(F\mathcal{D}_{M_{\text{sa}}}) \rightarrow D^+(F\mathcal{D}_M)$  has a fully faithful right adjoint  $R\rho_{\text{sa}}!: D^+(F\mathcal{D}_M) \rightarrow D^+(F\mathcal{D}_{M_{\text{sa}}})$ .*

We define a functor

$$F\mathcal{H}om: \text{Mod}_{\mathbb{R}\text{-c}}(\mathbb{C}_M) \times \text{Mod}(F\mathcal{D}_{M_{\text{sa}}}) \rightarrow \text{Mod}(F\mathcal{D}_{M_{\text{sa}}})$$

by setting for  $G \in \text{Mod}_{\mathbb{R}\text{-c}}(\mathbb{C}_M)$  and  $F\mathcal{M} \in \text{Mod}(F\mathcal{D}_{M_{\text{sa}}})$

$$\mathcal{H}om(G, F\mathcal{M})(\lambda) = \mathcal{H}om(G, \mathcal{M}(\lambda)).$$

Using Theorem 5.2, this functor admits a derived functor

$$FR\mathcal{H}om: D_{\mathbb{R}\text{-c}}^b(\mathbb{C}_M) \times D^+(F\mathcal{D}_{M_{\text{sa}}}) \rightarrow D^+(F\mathcal{D}_{M_{\text{sa}}}).$$

On a complex manifold  $X$ , we endow the  $\mathcal{D}_X$ -module  $\mathcal{O}_X$  with the filtration  $F\mathcal{O}_X$  given by

$$(6.3) \quad F^s \mathcal{O}_X = \begin{cases} 0 & \text{if } s < 0, \\ \mathcal{O}_X & \text{if } s \geq 0. \end{cases}$$

By applying the functors  $\rho_{\text{sa}}^!$  and  $\rho_{\text{sal}*}$ , we get the objects  $\rho_{\text{sa}}^! \mathcal{O}_X$  and  $\rho_{\text{sl}*} \mathcal{O}_X$  of  $\text{Mod}(F\mathcal{D}_{X_{\text{sa}}})$  and  $\text{Mod}(F\mathcal{D}_{X_{\text{sal}}})$ , respectively. One shall be aware that these objects are in degree 0 contrarily to the sheaf  $\mathcal{O}_{X_{\text{sa}}}$  (when  $d_X > 1$ ).

### The $L^\infty$ -filtration on $\mathcal{C}_{M_{\text{sal}}}^{\infty, \text{tp}}$

Recall that on the site  $M_{\text{sal}}$ , the sheaf  $\mathcal{C}_{M_{\text{sal}}}^{\infty, \text{tp } st}$  is endowed with a filtration, given by the sheaves  $\mathcal{C}_{M_{\text{sal}}}^{\infty, t}$  ( $t \in \mathbb{R}_{\geq 0}$ ). We also set  $\mathcal{C}_{M_{\text{sal}}}^{\infty, t} = 0$  for  $t < 0$ .

**Definition 6.3.** (a) We denote by  $F_\infty \mathcal{C}_{M_{\text{sal}}}^{\infty, \text{tp}}$  the object of  $\text{Mod}(F\mathcal{D}_{M_{\text{sal}}})$  given by the sheaves  $\mathcal{C}_{M_{\text{sal}}}^{\infty, t}$  ( $t \in \mathbb{R}$ ).

(b) We set  $F_\infty \mathcal{C}_{M_{\text{sa}}}^{\infty, \text{tp}} := \rho_{\text{sal}}^! F_\infty \mathcal{C}_{M_{\text{sal}}}^{\infty, \text{tp}}$ , an object of  $D^+(F\mathcal{D}_{M_{\text{sa}}})$ .

We call these filtrations the  $L^\infty$ -filtration on  $\mathcal{C}_{M_{\text{sal}}}^{\infty, \text{tp}}$  and  $\mathcal{C}_{M_{\text{sa}}}^{\infty, \text{tp}}$ , respectively.

If  $U \in \text{Op}_{M_{\text{sa}}}$  is weakly Lipschitz, we thus have for  $s \geq 0$ :

$$(6.4) \quad \text{R}\Gamma(U; \mathbb{F}_\infty^s \mathcal{C}_{M_{\text{sa}}}^{\infty, \text{tp}}) \simeq \mathcal{C}_M^{\infty, s}(U).$$

**Remark 6.4.** One could have also endowed  $\mathcal{C}_{M_{\text{sa}}}^{\infty, \text{tp}}$  with the  $L^2$ -filtration constructed similarly as the  $L^\infty$ -filtration, associated with the  $L^2$ -norm:

$$(6.5) \quad \|\varphi\|_2 = \left( \int_U |\varphi(x)|^2 dx \right)^{1/2}, \quad \|\varphi\|_2^s = \|d(x)^s \varphi(x)\|_2.$$

One gets the filtered sheaves  $\mathbb{F}_2 \mathcal{C}_{M_{\text{sa}}}^{\infty, \text{tp}}$  and  $\mathbb{F}_2 \mathcal{C}_{M_{\text{sa}}}^{\infty, \text{tp}}$ .

**The  $L^\infty$ -filtration on  $\mathcal{O}_{X_{\text{sa}}}^{\text{tp}}$**

On a complex manifold  $X$ , we set:

$$(6.6) \quad \mathbb{F}_\infty \mathcal{O}_{X_{\text{sa}}}^{\text{tp}} := \text{R}\mathcal{H}om_{\mathbb{F}\mathcal{D}_{\overline{X}_{\text{sa}}}}(\rho_{\text{sl}*}! \mathcal{O}_{\overline{X}}, \mathbb{F}_\infty \mathcal{C}_{X_{\text{sa}}}^{\infty, \text{tp}}) \in \text{D}^+(\mathbb{F}\mathcal{D}_{X_{\text{sa}}}),$$

$$(6.7) \quad \begin{aligned} \mathbb{F}_\infty \mathcal{O}_{X_{\text{sa}}}^{\text{tp}} &:= \text{R}\mathcal{H}om_{\mathbb{F}\mathcal{D}_{\overline{X}_{\text{sa}}}}(\rho_{\text{sa}*}! \mathcal{O}_{\overline{X}}, \mathbb{F}_\infty \mathcal{C}_{X_{\text{sa}}}^{\infty, \text{tp}}) \\ &\simeq \rho_{\text{sa}}^! \mathbb{F}_\infty \mathcal{O}_{X_{\text{sa}}}^{\text{tp}} \in \text{D}^+(\mathbb{F}\mathcal{D}_{X_{\text{sa}}}). \end{aligned}$$

One proves:

**Proposition 6.5.** *Let  $U \subset X$  be an open relatively compact subanalytic subset. Assume that  $U$  is weakly Lipschitz. Then the object  $\text{R}\Gamma(U; \mathbb{F}_\infty^s \mathcal{O}_{X_{\text{sa}}}^{\text{tp}})$  is represented by the complex*

$$(6.8) \quad 0 \rightarrow \mathcal{C}_X^{\infty, s, (0,0)}(U) \xrightarrow{\bar{\partial}} \mathcal{C}_X^{\infty, s+1, (0,1)}(U) \rightarrow \dots \rightarrow \mathcal{C}_X^{\infty, s+d_X, (0, d_X)}(U) \rightarrow 0.$$

Applying the functor  $\rho_{\text{sa}}^{-1}$ , one recovers the filtration introduced in (6.3):

$$(6.9) \quad \rho_{\text{sa}}^{-1} \mathbb{F}_\infty \mathcal{O}_{X_{\text{sa}}}^{\text{tp}} \simeq \mathbb{F} \mathcal{O}_X.$$

### A functorial filtration on regular holonomic modules

Good filtrations on holonomic modules already exist in the literature, in the regular case (see [KK81, BK86]) and also in the irregular case (see [Mal96]). But these filtrations are constructed on each holonomic module and are by no means functorial. Here we directly construct objects of  $\text{D}^+(\mathbb{F}\mathcal{D}_X)$ , the derived category of filtered  $\mathcal{D}$ -modules.

Denote by  $D_{\text{holreg}}^b(\mathcal{D}_X)$  the full triangulated subcategory of  $D^b(\mathcal{D}_X)$  consisting of objects with regular holonomic cohomology. To  $\mathcal{M} \in D_{\text{holreg}}^b(\mathcal{D}_X)$ , one associates

$$\text{Sol}(\mathcal{M}) := R\mathcal{H}om_{\mathcal{D}}(\mathcal{M}, \mathcal{O}_X).$$

We know by [Kas75] that  $\text{Sol}(\mathcal{M})$  belongs to  $D_{\mathbb{C}\text{-c}}^b(\mathbb{C}_X)$ , that is,  $\text{Sol}(\mathcal{M})$  has  $\mathbb{C}$ -constructible cohomology. Moreover, one can recover  $\mathcal{M}$  from  $\text{Sol}(\mathcal{M})$  by the formula:

$$(6.10) \quad \mathcal{M} \simeq \rho_{\text{sa}}^{-1} R\mathcal{H}om(\text{Sol}(\mathcal{M}), \mathcal{O}_{X_{\text{sa}}}^{\text{tp}}).$$

This is the Riemann-Hilbert correspondence obtained by Kashiwara in [Kas80, Kas84].

Using the filtration  $F_{\infty} \mathcal{O}_{X_{\text{sa}}}^{\text{tp}}$  on  $\mathcal{O}_{X_{\text{sa}}}$  we can set:

**Definition 6.6.** Let  $\mathcal{M}$  be a regular holonomic module. We define the filtered Riemann-Hilbert functors  $\text{RHF}_{\infty, \text{sa}}$  and  $\text{RHF}_{\infty}$  by the formulas

$$\begin{aligned} \text{RHF}_{\infty, \text{sa}} : D_{\text{holreg}}^+(\mathcal{D}_X) &\rightarrow D^+(F\mathcal{D}_{X_{\text{sa}}}), \\ \mathcal{M} &\mapsto \text{FR}\mathcal{H}om(\text{Sol}(\mathcal{M}), F_{\infty} \mathcal{O}_{X_{\text{sa}}}^{\text{tp}}), \\ \text{RHF}_{\infty} = \rho_{\text{sa}}^{-1} \text{RHF}_{\infty, \text{sa}} : D_{\text{holreg}}^+(\mathcal{D}_X) &\rightarrow D^+(F\mathcal{D}_X). \end{aligned}$$

Note that  $\text{RHF}_{\infty, \text{sa}}$  and  $\text{RHF}_{\infty}$  are triangulated functors.

Recall Notation 5.3 and the functor *for*.

**Proposition 6.7.** *In the diagram below*

$$D_{\text{holreg}}^b(\mathcal{D}_X) \xrightarrow{\text{RHF}_{\infty}} D^+(F\mathcal{D}_X) \xrightarrow{\text{for}} D^+(\mathcal{D}_X)$$

*the composition is isomorphic to the natural inclusion functor.*

*Proof.* Since  $\rho_{\text{sa}}^{-1}$  commutes with inductive limits, the diagram below commutes:

$$\begin{array}{ccccc} D_{\text{holreg}}^b(\mathcal{D}_X) & \xrightarrow{\text{RHF}_{\infty, \text{sa}}} & D^+(F\mathcal{D}_{X_{\text{sa}}}) & \xrightarrow{\text{for}} & D^+(\mathcal{D}_{X_{\text{sa}}}) \\ & & \rho_{\text{sa}}^{-1} \downarrow & & \rho_{\text{sa}}^{-1} \downarrow \\ & & D^+(F\mathcal{D}_X) & \xrightarrow{\text{for}} & D^+(\mathcal{D}_X). \end{array}$$

Now let  $\mathcal{M} \in D_{\text{holreg}}^b(\mathcal{D}_X)$  and set for short  $G = \text{Sol}_X(\mathcal{M})$ . Then

$$\begin{aligned} \text{forFR}\mathcal{H}om(G, F_{\infty} \mathcal{O}_{X_{\text{sa}}}^{\text{tp}}) &\simeq R\mathcal{H}om(G, \text{for}F_{\infty} \mathcal{O}_{X_{\text{sa}}}^{\text{tp}}) \\ &\simeq R\mathcal{H}om(G, \mathcal{O}_{X_{\text{sa}}}^{\text{tp}}) \end{aligned}$$

and we conclude with (6.10).

Q.E.D.

**Example 6.8.** Let  $D$  be a normal crossing divisor in  $X$  and let  $\mathcal{M}$  be a regular holonomic module such that  $\text{Sol}(\mathcal{M}) \simeq \mathbb{C}_{X \setminus D}$ . Let  $W \in \text{Op}_{X_{\text{sa}}}$  with smooth boundary transversal to the strata of  $D$  so that  $W \setminus D$  is weakly Lipschitz. Set  $U := W \setminus D$ . Then  $\text{R}\Gamma(W; \text{F}_{\infty, \text{sa}}^s \mathcal{M}) \simeq \text{R}\Gamma(U; \text{F}_{\infty}^s \mathcal{O}_{X_{\text{sa}}}^{\text{tp}})$  and therefore the object  $\text{R}\Gamma(W; \text{F}_{\infty}^s \mathcal{M})$  is represented by the complex (6.8).

**Remark 6.9.** By using the filtration  $\text{F}_2$  on  $\mathcal{C}_{X_{\text{sal}}}^{\infty, \text{tp}}$  (see Remark 6.4), one can also endow  $\mathcal{O}_{X_{\text{sal}}}^{\text{tp}}$  with an  $L^2$ -filtration and define similarly  $\text{F}_2 \mathcal{O}_{\text{sal}}^{\text{tp}}$ . Unfortunately, Hörmander's theory does not apply immediately to this situation.

Given a regular holonomic  $\mathcal{D}_X$ -module  $\mathcal{M}$ , natural questions arise.

- (i) Does there exist an integer  $r$  such that  $H^j(\text{F}_{\infty}^s \mathcal{M}) \rightarrow H^j(\text{F}_{\infty}^{s+r} \mathcal{M})$  is the zero morphism for  $s \gg 0$  and  $j \neq 0$ .
- (ii) Is the filtration  $H^0(\text{F}_{\infty} \mathcal{M})$  a good filtration?
- (iii) Does there exist a discrete set  $Z \subset \mathbb{R}_{\geq 0}$  such that the morphisms  $\text{F}_{\infty}^s \mathcal{M} \rightarrow \text{F}_{\infty}^t \mathcal{M}$  ( $s \leq t$ ) are isomorphisms for  $[s, t]$  contained in a connected component of  $\mathbb{R}_{\geq 0} \setminus Z$ ?

Note that it may be convenient to use better the  $L^2$ -filtration (see Remark 6.9).

One can also ask the question of comparing these filtrations with other filtrations already existing in the literature.

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