Sheaves for spacetimes: a microlocal approach*

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Abstract

On a spacetime endowed with a time function (such as a globally hyperbolic manifold) we solve the global Cauchy problem for hyperbolic $\mathscr{D}$-modules in the framework of Sato's hyperfunctions, illustrating the fact that the tools of microlocal sheaf theory are extremely efficient when treating analytic questions.

Contents

Sections 2–4 are joint work with Masaki Kashaiwara (see in particular [KS90]), sections 5–6 with Benoît Jubin (see [JS16] and also [Sch26] for complements) and section 7 is extracted from [GKS12].

1 Introduction and history

A *spacetime* is a connected manifold M endowed with a non-degenerate quadratic form of signature $(+, -, \dots, -)$.

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This form defines a *cone* in the tangent space TM; the affine model is the Minkowski space.

Two classical questions: (i) To define the *future* of $x \in M$ — the light cone issued from x. The classical approach uses piecewise smooth paths whose derivative belongs to the cone. *We shall propose a different candidate.*
(ii) To solve the *Cauchy problem* (in spaces of distributions or analogues) for the wave equations associated with the form. *We shall treat hyperbolic $\mathscr{D}$-modules in the space of Sato's hyperfunctions.*

There is a huge literature concerned with globally hyperbolic spacetimes and the Cauchy problem in the framework of distributions. Let us quote, among many others, [BGP07, BS05, BF00, FS11, Ger70, HE73, MS08]:

Brunetti, Fredenhagen, Bär, Ginoux, Pfäffle,
Bernal, Fathi, Siconolfi, Geroch,
Hawking, Minguzzi, Sánchez, ...

The aim of this paper is to show that, in the framework of *Sato's hyperfunctions* [Sat60] and *microlocal sheaf theory* [KS90], the global Cauchy problem for hyperbolic $\mathscr{D}$-modules is well-posed and the proof is *particularly simple*.

The classical Cauchy problem for hyperbolic equations is associated with the names of **Hadamard**, **Leray** [Ler53], **Gårding**, **Hörmander** and many others.

- **Hadamard:** the operator $\partial_t^2 - \partial_x$ on $\mathbb{R}^2$ — The Cauchy problem with data on $\{t = 0\}$ is *not* well-posed in spaces of $\mathscr{C}^\infty$-functions or distributions.
- **Hörmander:** has characterised the hyperbolic differential operators with *constant* coefficients.
- With *variable* coefficients, very little is known — except for operators "with simple characteristics" (microlocally equivalent to ∂_t).

The situation is *radically different* when working with Sato's hyperfunctions [Sat60, SKK73] instead of distributions:

- Bony–Schapira [BS73] solve the local Cauchy problem for weakly hyperbolic operators.
- Kashiwara–Schapira [KS79]: treat the case of general systems ($\mathscr{D}$-modules) and give an essentially topological proof (as we shall see here) in [KS90].
- Jubin–Schapira [JS16] solve the *global* Cauchy problem for hyperbolic $\mathscr{D}$-modules by using a time function which allows one to reduce to the local Cauchy problem, as we shall see here.

2 Microlocal sheaf theory and the micro-support

Some notations

Here, $\mathbf{k}$ is a unital commutative ring with finite homological dimension. For a locally compact topological space M, $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_M)$ denotes the bounded derived category of sheaves of $\mathbf{k}$-modules on M. We shall mainly follow the notations of [KS90] and freely use the 6 Grothendieck operations:

internal: $\cdot \otimes \cdot$ and $\mathrm{R}\mathscr{H}om\,(\cdot,\cdot)$ with adjunction $(\otimes, \mathrm{R}\mathscr{H}om\,)$,

for a morphism $f\colon M \to N$: inverse images f^{-1}, $f^{!}$ and direct images $\mathrm{R}f_*$, $\mathrm{R}f_!$, with adjunctions $(f^{-1}, \mathrm{R}f_*)$, $(\mathrm{R}f_!, f^!)$.

For $A \subset M$ locally closed, $\mathbf{k}_A$ denotes the constant sheaf on A with stalk $\mathbf{k}$ extended by zero on $M \setminus A$.

Here, ω_M denotes the dualizing complex on M, $\omega_M := a_M^! \mathbf{k}_{\mathrm{pt}}$, where $a_M\colon M \to \mathrm{pt}$. If M is a $\mathscr{C}^0$-manifold, ω_M is isomorphic to $\mathrm{or}_M\,[\dim M]$, or_M being the orientation sheaf. Moreover, for $f\colon M \to N$ a continuous map, one sets $\omega_{M/N} := \omega_M \otimes f^{-1}\omega_N^{\otimes -1}$.

Micro-support

Microlocal analysis was created by **Mikio Sato** [Sat70]. It was adapted to sheaf theory by **Kashiwara–Schapira** [KS82], who introduced the *micro-support* of sheaves.

From now on, M is a real $\mathscr{C}^\infty$-manifold, $\pi\colon T^*M \to M$ denotes its cotangent bundle,

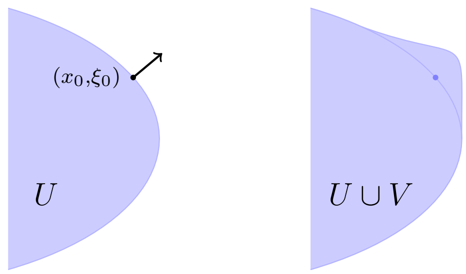
General idea. For $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_M)$, the micro-support $\mathrm{SS}(F)$ is a closed subset of T^*M recording in *which co-directions F do not propagate.*

The micro-support has connections, and similar functorial properties, with the wave-front set of distributions and with the characteristic variety of $\mathscr{D}$-modules. However, the wave front set lives on $\sqrt{-1}T^*M$, not on T^*M, and the characteristic variety of $\mathscr{D}$-modules lives on T^*X for X a complex manifold. See § 3 for a precise statement.

Definition 2.1. *Let $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_M)$. The micro-support $\mathrm{SS}(F) \subset T^*M$ is the closed subset defined as follows. For $W \subset T^*M$ open, $W \cap \mathrm{SS}(F) = \emptyset$ if and only if, for any x_0 and any real $\mathscr{C}^1$-function φ with $p = (x_0; d\varphi(x_0)) \in W$,*

$$\big(\mathrm{R}\Gamma_{\{x\,;\,\varphi(x) \geq \varphi(x_0)\}} F\big)_{x_0} \simeq 0.$$

In other words, $p \notin \mathrm{SS}(F)$ if F has no local cohomology supported by the closed "half-spaces" whose conormal lies in a neighbourhood of p. If U is the complementary of this "half-spaces" and u is a section of F which vanishes on U, then u vanishes on $U \cup V$ for an open neighborhood V of x_0.



$$(x_0,\xi_0) \notin \mathrm{SS}(F) \implies R\Gamma(U\cup V;\, F) \xrightarrow{\sim} R\Gamma(U;\, F)$$

Properties of the micro-support

- $\mathrm{SS}(F)$ is $\mathbb{R}^+$-conic (invariant under the $\mathbb{R}^+$-action on T^*M).
- $\mathrm{SS}(F) \cap T^*_M M = \pi(\mathrm{SS}(F)) = \mathrm{supp}(F)$.
- **Triangular inequality:** for a distinguished triangle $F_1 \to F_2 \to F_3 \xrightarrow{+1}$,
$$\mathrm{SS}(F_i) \subset \mathrm{SS}(F_j) \cup \mathrm{SS}(F_k) \qquad (j \neq k).$$
- $\mathrm{SS}(F)$ is co-isotropic (also called "involutive"), see [KS90, Def. 6.4.1].

Co-isotropy is defined via the Whitney normal cone of $\mathrm{SS}(F)$ (see below);

Definition 2.2 (co-isotropic)**.** *a subset $S \subset T^*M$ is co-isotropic at $p \in T^*M$ if $C_p(S,S)^\perp \subset C_p(S)$, where $C_p(S,S)^\perp \subset T^*_pT^*M$ is identified with a subset of T_pT^*M via the Hamiltonian isomorphism.*

If $\mathrm{SS}(F)$ is smooth in a neighborhood of p, one recovers the usual notion: $T_p\,\mathrm{SS}(F)$ contains its symplectic orthogonal. In particular, $\dim T_p\,\mathrm{SS}(F) \geq \dim M$.

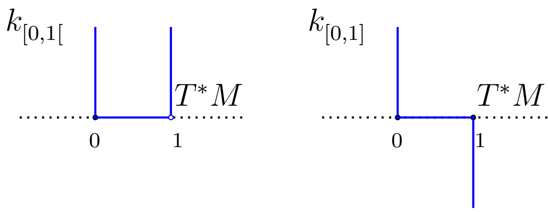
Examples

(i) Let F be a non-zero local system on $M$and assume that M is connected. Then $\mathrm{SS}(F) = T^*_MM$, the zero section of T^*M.

(ii) Let $N \subset M$ be a closed submanifold. Then $\mathrm{SS}(\mathbf{k}_N) = T^*_NM$, the conormal bundle of N in M. Note that one can now define the conormal bundle of any locally closed subset $A \subset M$ by setting $T^*_AM := \mathrm{SS}(\mathbf{k}_A)$.

(iii) Let X be a complex manifod and let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module. Set $F = \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X)$. Then $\mathrm{SS}(F) = \mathrm{char}(\mathscr{M})$, the characteristic variety — see § 3).

(iv) $M = \mathbb{R}$ (hence, $T^*M = \mathbb{R}^2$), $F = \mathbf{k}_I$ with I an interval. The case $I = [0,1)$ and $I = [0,1]$ are shown on the picture.



Functoriality

Let $f\colon M \to N$ be a morphism of manifolds. We get the maps

$$\begin{tikzcd}
TM \arrow[r, "f'"] \arrow[dr, "\tau"'] & M \times_N TN \arrow[r, "f_\tau"] \arrow[d, "\tau"] & TN \arrow[d, "\tau"] \\
& M \arrow[r, "f"] & N,
\end{tikzcd}
\qquad
\begin{tikzcd}
T^*M \arrow[dr, "\pi"'] & M \times_N T^*N \arrow[l, "f_d"'] \arrow[r, "f_\pi"] \arrow[d, "\pi"] & T^*N \arrow[d, "\pi"] \\
& M \arrow[r, "f"] & N.
\end{tikzcd}$$

Definition 2.3. *Let $\Lambda \subset T^*N$ be a closed $\mathbb{R}^+$-conic subset. One says that f is* non-characteristic *for Λ if*

$$f_\pi^{-1}\Lambda \cap f_d^{-1}T^*_M M \subset M \times_N T^*_N N.$$

Since Λ is closed and $\mathbb{R}^+$-conic, this is equivalent to:

$$f_d \text{ is proper on } f_\pi^{-1}\Lambda.$$

Theorem 2.4. *Let $F \in \mathrm{D^b}(\mathbf{k}_M)$, $G \in \mathrm{D^b}(\mathbf{k}_N)$.*

(a) If f is proper *on* $\mathrm{supp}(F)$, *then* $\mathrm{SS}(\mathrm{R}f_*F) \subset f_\pi\, f_d^{-1}\ \mathrm{SS}(F)$.

(b) If f is non-characteristic *for* $\mathrm{SS}(G)$, *then* $\mathrm{SS}(f^{-1}G) \subset f_d\, f_\pi^{-1}\ \mathrm{SS}(G)$ *and one has the natural isomorphism $f^{-1}G \otimes \omega_{M/N} \xrightarrow{\sim} f^!G$.*

Micro-support along a submanifold

When f is *characteristic*, the inverse-image statement becomes delicate. We need to define the micro-support along a submanifold.

Definition 2.5 (Whitney normal cone)**.** *For $S \subset M$ locally closed and $N \subset M$ a submanifold, the Whitney normal cone of S along N is the closed cone of the normal bundle T_NM defined in a coordinate system (x', x'') with $N = \{x' = 0\}$:*

$$(x''_0; v_0) \in C_N(S) \subset T_NM$$

iff there is a sequence $(x_n, c_n) \in S \times \mathbb{R}^+$ with $x'_n \to 0$, $x''_n \to x''_0$, $c_n x'_n \to v_0$.

For $S_1, S_2 \subset M$, $C(S_1, S_2)$ is the normal cone of $S_1 \times S_2$ along the diagonal, $TM \simeq T_\Delta(M \times M)$.

Let $j_M\colon M \hookrightarrow X$ be a closed embedding. The projection $\pi\colon T^*_MX \to M$ defines

$$\begin{tikzcd}
T^*T^*_MX & T^*_MX \times_M T^*M \arrow[l, hook', "\pi_d"'] \arrow[r, "\pi_\pi"] & T^*M\ .
\end{tikzcd}$$

Restricting the embedding π_d to the zero-section of $T^*_M X$ and using the Hamiltonian isomorphism, one obtains the embedding

$$T^*M \hookrightarrow T^*T^*_M X \simeq T_{T^*_M X}T^*X. \tag{2.1}$$

In coordinates $(x, y; \xi, \eta)$ on T^*X with $M = \{y = 0\}$ and $T^*_M X = \{y = \xi = 0\}$, this map is given by $(x; \xi) \mapsto (x, 0; \xi, 0)$.

Definition 2.6. *For $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$, set*

$$\mathrm{SS}_M(F) = T^*M \cap C_{T^*_M X}(\mathrm{SS}(F)),$$

where $C_{T^*_M X}(\mathrm{SS}(F)) \subset T_{T^*_M X}T^*X$ is the Whitney normal cone of $\mathrm{SS}(F)$ along $T^*_M X$ and T^*M is embedded in $T_{T^*_M X}T^*X$ by (2.1).

Theorem 2.7 ([KS90, Cor. 6.4.4, Prop. 6.2.4]). *Let X be a manifold, M a closed submanifold, $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$. Then*

$$\mathrm{SS}(\mathrm{R}\Gamma_M F) \subset \mathrm{SS}_M(F).$$

3 Links with $\mathscr{D}$-modules and the characteristic variety

Let $(X, \mathscr{O}_X)$ be a complex manifold and $\mathscr{D}_X$ the sheaf of finite-order holomorphic differential operators on X. We refer to [Kas03] for an exposition of $\mathscr{D}$-module theory.

A coherent $\mathscr{D}_X$-module $\mathscr{M}$ admits locally a finite free resolution

$$0 \to \mathscr{D}_X^{N_n} \xrightarrow{\cdot P_{n-1}} \cdots \to \mathscr{D}_X^{N_1} \xrightarrow{\cdot P_0} \mathscr{D}_X^{N_0} \to \mathscr{M} \to 0.$$

The complex of its holomorphic solutions $\mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X)$ is then represented by the complex

$$0 \to \mathscr{O}_X^{N_0} \xrightarrow{P_0 \cdot} \mathscr{O}_X^{N_1} \xrightarrow{P_1 \cdot} \cdots \xrightarrow{P_{n-1} \cdot} \mathscr{O}_X^{N_n} \to 0.$$

The characteristic variety

To $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$ is associated its *characteristic variety* $\mathrm{char}(\mathscr{M}) \subset T^*X$, a closed $\mathbb{C}^\times$-conic analytic variety.

If $\mathscr{M} = \mathscr{D}_X / \mathscr{D}_X \cdot P$ with P of order m,

$$\mathrm{char}(\mathscr{M}) = \{(x; \xi) \in T^*X\,;\, \sigma_m(P)(x; \xi) = 0\},$$

the zero-set of the *principal symbol* of P.

Theorem 3.1 (Sato–Kawai–Kashiwara [SKK73]). *Let $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$. Then $\mathrm{char}(\mathscr{M})$ is co-isotropic in T^*X.*

A purely algebraic proof of this result has been given later by O. Gabber [Gab81]; partial result were first obtained by Guillemin–Quillen–Sternberg [GQS70].

Cauchy–Kowalevska–Kashiwara

Let $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$ and let $f\colon Y \to X$ be a morphism of complex manifolds. One denotes by $\mathscr{M}_Y := {}^D f^{-1}\mathscr{M}$ the induced $\mathscr{D}_Y$-module on Y (see [Kas03]).

One says that Y is *non-characteristic* for $\mathscr{M}$ if f is non-characteristic for $\mathrm{char}(\mathscr{M})$:

$$f_\pi^{-1}\,\mathrm{char}(\mathscr{M}) \cap f_d^{-1} T^*_Y Y \subset Y \times_X T^*_X X.$$

Theorem 3.2 (Kashiwara, master's thesis [Kas70])**.** *If Y is non-characteristic for $\mathscr{M}$, then $\mathscr{M}_Y$ is $\mathscr{D}_Y$-coherent and there is a canonical isomorphism*

$$f^{-1}\mathrm{R}\mathscr{H}\!om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X) \xrightarrow{\sim} \mathrm{R}\mathscr{H}\!om_{\mathscr{D}_Y}(\mathscr{M}_Y, \mathscr{O}_Y).$$

Micro-support and characteristic variety

Theorem 3.3 ([KS90, Th. 11.3.3])**.** *Let $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$ and let $F = \mathrm{R}\mathscr{H}\!om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X)$ be its complex of holomorphic solutions. Then*

$$\mathrm{SS}(F) = \mathrm{char}(\mathscr{M}).$$

Sketch of proof. (i) "$\subset$". One reduces to the case where $\mathscr{M} = \mathscr{D}_X/\mathscr{D}_X \cdot P$ for a differential operator P and one proves that if $U \subset X$ is an open subset with smooth boundary in a neighborhood of $x_0 \in \partial U$, non-charateristic for P, then any holomorphic solution u of $Pu = 0$ defined on U extends holomorphically in a neighborhood of x_0. As remarked by Zerner [Zer71], this follows from the *the refined* Cauchy–Kowalevska theorem proved by Petrowsky and implicitly used by Leray (see [Ler57]). One can find a detailed proof in [Hör83, Th. 9.4.7].
(ii) "$\supset$". The sheaf $\mathscr{E}^{\mathbb{R}}_X$ of microlocal operators of Sato–Kawai–Kashiwara [SKK73] is defined by

$$\mathscr{E}^{\mathbb{R}}_X = \mu_\Delta(\mathscr{O}^{(0,d_X)}_{X\times X})\,[d_X].$$

Here, μ_Δ is the Sato's microlocalization functor along the diagonal. The sheaf $\mathscr{E}^{\mathbb{R}}_X$ is faithfully flat over the sheaf $\mathscr{E}_X$ of micro-differential operators and one proves that

$$\mathrm{char}(\mathscr{M}) = \mathrm{supp}(\mathscr{E}_X \otimes_{\pi^{-1}\mathscr{D}_X} \pi^{-1}\mathscr{M}).$$

Therefore

$$\begin{aligned} \mathrm{char}(\mathscr{M}) &= \mathrm{supp}(\mathscr{E}^{\mathbb{R}}_X \otimes_{\pi^{-1}\mathscr{D}_X} \pi^{-1}\mathscr{M}) \\ &= \mathrm{supp}\,\mu_\Delta \mathrm{R}\mathscr{H}\!om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_{X\times X}). \end{aligned}$$

Finally, $\mathrm{supp}\,\mu_\Delta F \subset \mathrm{SS}(F) \cap T^*_\Delta(X\times X)$ by [KS90, Cor. 5.4.10]. Q.E.D.

As a corollary one gets another proof of the involutivity of $\mathrm{char}(\mathscr{M})$ – at the opposite from Gabber's algebraic argument.

4 Hyperfunctions and the local Cauchy problem

Hyperfunctions

From now on, M is a real analytic manifold ($\dim M = n$) and X is a complexification of M. Define

$$\mathscr{A}_M := \mathscr{O}_X \otimes \mathbb{C}_M, \qquad \mathscr{B}_M := \mathrm{R}\mathscr{H}om\,(\mathrm{D}'_X\mathbb{C}_M, \mathscr{O}_X),$$

the sheaves of real analytic functions and of *Sato's hyperfunctions*. Here, $\mathrm{D}'_X\mathbb{C}_M = \mathrm{R}\mathscr{H}om\,(\mathbb{C}_M, \mathbb{C}_X) \simeq \mathrm{or}_M[-n]$.
It is proved by Sato that $\mathscr{B}_M$ is concentrated in degree 0. Then, $\mathscr{B}_M \simeq H^n_M(\mathscr{O}_X) \otimes \mathrm{or}_M$. Roughly speaking, up to shift and orientation, $\mathscr{B}_M = \mathrm{R}\Gamma_M \mathscr{O}_X$.

Hyperfunctions may be represented as boundary values of holomorphic functions. The sheaf $\mathscr{B}_M$ is a flabby sheaf, much bigger than the sheaf of distributions.

The hyperbolic characteristic variety

Definition 4.1. *For $\mathscr{M}$ a coherent left $\mathscr{D}_X$-module, the* hyperbolic characteristic variety *of $\mathscr{M}$ along M is*

$$\mathrm{char}_M(\mathscr{M}) = T^*M \cap C_{T^*_M X}(\mathrm{char}(\mathscr{M})).$$

Remark 4.2. *In case where $\mathscr{M} = \mathscr{D}_X/\mathscr{D}_X \cdot P$ for a differential operator P, what we call "hyperbolic" corresponds to what was classically called "weakly hyperbolic".*

Moreover, one sometimes says that $\theta \in T^*M$ is hyperbolic if θ does not belong to the hyperbolic characteristic variety.

Combining the preceding results, we get

Theorem 4.3. *Let $F = \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{B}_M)$. Then*

$$\mathrm{SS}(F) \subset \mathrm{char}_M(\mathscr{M}).$$

Hyperfunction solutions of a $\mathscr{D}$-module *propagate* in the hyperbolic co-directions, that is, outside the hyperbolic characteristic variety.

Corollary 4.4. *Let $N \subset M$ be a closed analytic submanifold, Y a complexification of N in X, $f\colon Y \hookrightarrow X$. Assume the* hyperbolicity condition

$$T^*_N M \cap \mathrm{char}_M(\mathscr{M}) \subset T^*_M M.$$

Then

$$\mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{B}_M)|_N \xrightarrow{\sim} \mathrm{R}\mathscr{H}om_{\mathscr{D}_Y}(\mathscr{M}_Y, \mathscr{B}_N).$$

In other words, the Cauchy problem for hyperfunctions is well-posed for non hyperbolic characteristic submanifolds.

Proof. Set $n = \dim M$, $d = n - \dim N$. The proof reduces to a chain of isomorphisms:

$$\begin{aligned}
\mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{B}_M)|_N &\simeq \mathrm{R}\Gamma_N \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{B}_M) \otimes \mathrm{or}_{N/M}[d] \\
&\simeq \mathrm{R}\Gamma_N \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathrm{R}\Gamma_M \mathscr{O}_X) \otimes \mathrm{or}_N[n+d] \\
&\simeq \mathrm{R}\Gamma_N \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathrm{R}\Gamma_Y \mathscr{O}_X) \otimes \mathrm{or}_N[n+d] \\
&\simeq \mathrm{R}\Gamma_N \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X)|_Y \otimes \mathrm{or}_N[n-d] \\
&\simeq \mathrm{R}\Gamma_N \mathrm{R}\mathscr{H}om_{\mathscr{D}_Y}(\mathscr{M}_Y, \mathscr{O}_Y) \otimes \mathrm{or}_N[n-d] \\
&\simeq \mathrm{R}\mathscr{H}om_{\mathscr{D}_Y}(\mathscr{M}_Y, \mathscr{B}_N).
\end{aligned}$$

Here, the first isomorphism follows from the fact that N is non-characteristic for $\mathrm{char}_M(\mathscr{M})$, the second from the definition of $\mathscr{B}_M$, the third from the fact that $N = M \cap Y$, the fourth from the fact that Y is non-characteristic for $\mathrm{char}(\mathscr{M})$, the fifth from the Cauchy-Kowalevska-Kashiwara theorem and the las one from the definition of $\mathscr{B}_N$.
Q.E.D.

5 Causal manifolds and the λ-topology

Causal manifolds

In a Lorentzian spacetime, the quadratic form gives a *light cone* in the tangent space. For our purposes we don't need the metric: only the cone, and we prefer to work in the cotangent space.

Definition 5.1. (i) *A* causal manifold (M, λ) *is a manifold M together with λ, a closed convex proper cone of T^*M satisfying $T^*_M M \subset \lambda$.*

(ii) *A morphism $f\colon (M, \lambda_M) \to (N, \lambda_N)$ is* causal *if $f_d f_\pi^{-1} \lambda_N \subset \lambda_M$.*

(iii) $(\mathbb{R}, +)$ *denotes the causal manifold $\mathbb{R}$ with $\Lambda_{\mathbb{R}} = \{(t; \tau)\,;\, \tau \geq 0\}$.*

Note that the cone λ satisfies

$$\lambda \cap \lambda^a = T^*_M M, \quad \lambda + \lambda = \lambda. \tag{5.1}$$

Example 5.2. *Let $\mathbb{V}$ be a finite-dimensional real vector space, θ a closed proper convex cone with non-empty interior, $V \subset \mathbb{V}$ open convex, $\lambda = V \times \theta^{\circ a}$. Then (V, λ) is a causal manifold.*

The λ-topology

Definition 5.3. (a) *A locally closed set $A \subset M$ is a* λ-set *if* $\mathrm{SS}(\mathbf{k}_A) \subset \lambda$.

(b) *A* λ-open *(resp.* λ-closed*) set is an open (resp. closed) λ-set.*

Proposition 5.4. *The family of λ-open sets defines a topology on M, called the λ-topology, the same one given by the λ-closed sets.*

Proof. Recall that an open set U is λ-open if and only if $M \setminus U$ is λ-closed.
(i) If U_1 and U_2 are λ-open, so is $U_1 \cap U_2$ since $\mathbf{k}_{U_1\cap U_2} \simeq \mathbf{k}_{U_1} \otimes \mathbf{k}_{U_2}$ and $\mathrm{SS}(\mathbf{k}_{U_1} \otimes \mathbf{k}_{U_2}) \subset \lambda$ by (5.1).
(ii) If U_1 and U_2 are λ-open, so is $U_1 \cup U_2$. This follows from (i) and the distinguished triangle $\mathbf{k}_{U_1\cap U_2} \to \mathbf{k}_{U_1} \oplus \mathbf{k}_{U_2} \to \mathbf{k}_{U_1\cup U_2} \xrightarrow{+1}$.
(iii) Let $\{U_i\}_{i\in I}$ be a family of λ-open subsets. Let us order I by the relation $i \leq j$ if $U_i \subset U_j$. We may assume that I is non-empty and by (ii) we may assume that $(I, \leq)$ is directed. It then follows from [KS90, Exe. 5.7] that, setting $U = \bigcup_{i\in I} U_i$, $\mathrm{SS}(\mathbf{k}_U) \subset \lambda$.
(iv) Since $\mathrm{SS}(\mathbf{k}_M) = T^*_M M \subset \lambda$ and $\mathrm{SS}(\mathbf{k}_\varnothing) = \varnothing$, both M and $\varnothing$ are λ-open and the proof is complete. Q.E.D.

Definition 5.5. *One denotes by M_λ the space M endowed with the λ^a-topology defined in Proposition 5.4 and by $\rho_\lambda\colon M \to M_\lambda$ the continuous map associated with the identity of the set M.*

Example 5.6. (a) *If $M = \mathbb{V}$ is a vector space and $\lambda^a = \mathbb{V} \times \gamma^\circ$ for a closed convex proper cone with non-empty interior $\gamma \subset \mathbb{V}$, then an open subset $U \subset \mathbb{V}$ is λ^a-open if and only if $U = U + \gamma$.*

(b) *If $\lambda = T^*_M M$, the only λ-open subsets are the connected components of M.*

(c) *If $\lambda = T^*M$ (in which case, λ is not proper), the λ-topology is the usual one.*

Remark 5.7. *In* [Sch26], *one proves that, under suitable hypotheses, the functor* $\mathrm{R}\rho_{\lambda *}\colon \mathrm{D}^{\mathrm{b}}_\lambda(\mathbf{k}_M) \to \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{M_{\lambda^a}})$ *is an equivalence of categories with quasi-inverse ρ_λ^{-1}, generalizing a result of* [KS90] *which treated the case of Example 5.6 (i).*

Futures and preorders

Definition 5.8. *Let (M, λ) be a causal manifold.*

(a) *For $A \subset M$, the* future *of A, denoted $J^+_\lambda(A)$, is the closure of A for the λ-topology.*

(b) *The* causal preorder *on (M, λ), denoted $\preceq_\lambda$ is defined by*

$$x \preceq_\lambda y \quad \Longleftrightarrow \quad y \in J^+_\lambda(x).$$

(c) *A diamond in M is a set $J^+_\lambda(K) \cap J^+_{\lambda^a}(L)$ for K and L compact in M.*

Note that the order on $(\mathbb{R}, +)$ (see Definition 5.1 (iii)) is the usual order. Indeed, the intervals $[a, +\infty)$ are λ-closed for $\lambda = \{(t;\tau); \tau \geq 0\}$.

Classically one considers the ps-preorder: $x \underset{ps}{\preceq} y$ if there exists a piecewise smooth causal path $c\colon I \to M$ with $c(0) = x$, $c(1) = y$. Then, denoting by $J^+_{\mathrm{ps}}(A)$ the future set of A for the ps-preorder, one can show that $J^+_{\lambda^a}(A)$ is the closure of $J^+_{\mathrm{ps}}(A)$.

Example 5.9. *Let (x_1, x_2) be the coordinates on $\mathbb{R}^2$, $Z = \{(x_1, x_2); x_1 \le 0, x_2 = 0\}$, $M = \mathbb{R}^2 \setminus Z$. We identify $\mathbb{R}^2$ and $(\mathbb{R}^2)^*$ by using the Euclidian structure of $\mathbb{R}^2$. Set*

$$\theta = \{(x_1, x_2) \in \mathbb{R}^2; x_2 \ge |x_1|\}, \quad \lambda = M \times \theta.$$

Let $x_0 = (-1, -1)$. One gets

$$J^+_\lambda(x_0) = (x_0 + \theta) \cap \{x_2 < 0\}.$$

Let $x_1 = (1, 1)$. One gets

$$J^+_{\lambda^a}(x_1) = (x_1 + \theta^a) \cap M.$$

Note that $x_0 \in J^+_{\lambda^a}(x_1)$ but $x_1 \notin J^+_\lambda(x_0)$.

Proposition 5.10. *Let $f \colon (M, \lambda_M) \to (N, \lambda_N)$ be a morphism of causal manifolds. Assume that f is non-characteristic with respect to λ_N, that is:*

$$f_\pi^{-1}(\lambda_N) \cap f_d^{-1}(T^*_M M) \subset M \times_N T^*_N N. \tag{5.2}$$

Then the map f induces a continuous map $M_{\lambda_M} \to N_{\lambda_N}$.

Proof. It follows from the hypothesis and [KS90, Prop. 5.4.13] or Theorem 2.4 (ii) that the inverse image functor f^{-1} induces a functor

$$f^{-1} \colon \mathrm{D}^{\mathrm{b}}_{\lambda_N}(\mathbf{k}_N) \to \mathrm{D}^{\mathrm{b}}_{\lambda_M}(\mathbf{k}_M). \tag{5.3}$$

Now let V be a λ_N-open subset of N, that is, V is open and $\mathrm{SS}(\mathbf{k}_V) \subset \lambda_N$. Then $\mathrm{SS}(f^{-1}\mathbf{k}_V) \subset \lambda_M$ by (5.3) and the result follows since $f^{-1}\mathbf{k}_V \simeq \mathbf{k}_{f^{-1}V}$. Q.E.D.

Note that f is non-characteristic with respect to λ_N if and only if f is non-characteristic with respect to λ_N^a.

Corollary 5.11. *Let $f \colon (M, \lambda_M) \to (N, \lambda_N)$ be a morphism of causal manifolds and assume that f is non-characteristic with respect to λ_N. Then f preserves the causal preorders, that is,*

$$x \preceq_{\lambda_M} y \text{ implies } f(x) \preceq_{\lambda_N} f(y).$$

Proof. If $f \colon X \to Y$ is continuous and $A \subset X$, then $f(\overline{A}) \subset \overline{f(A)}$. Therefore, $y \in J^+_{\lambda_M}(x)$ implies $f(y) \in J^+_{\lambda_N}(f(x))$. Q.E.D.

6 Time functions and the global Cauchy problem

Time functions

Definition 6.1. (a) *A* time function (*also called a* Cauchy time function) *on (M, λ) is a submersive causal morphism $q \colon (M, \lambda) \to (\mathbb{R}, +)$, proper on $J^+_\lambda(K)$ and $J^+_{\lambda^a}(K)$ for every compact $K \subset M$.*

(b) *A* G-causal manifold (M, λ, q) *is a causal manifold endowed with a time function.*

- Since q is submersive, it follows from Corollary 5.11 that q preserves the causal preorder: if $x \preceq_\lambda y$, then $q(x) \leq q(y)$.
- Since q is causal, $dq(x) \in \lambda$, for any $x \in M$.

Proposition 6.2. *Let (M, λ, q) be a G-causal manifold. Then diamonds are compact.*

Proof. Let K and L be two compact subsets of M. It follows from Corollary 5.11 that $J^+_\lambda(K)$ is contained in $q^{-1}([t_0, +\infty))$ for some t_0 and similarly, $J^+_{\lambda^a}(L)$ is contained in $q^{-1}((-\infty, t_1])$ for some t_1. Then $J^+_\lambda(K) \cap J^+_{\lambda^a}(L)$ is contained in $q^{-1}([t_0, t_1])$. Since q is proper on $J^+_\lambda(K)$, the result follows. Q.E.D.

Classically, a space time is globally hyperbolic if diamonds are compact and there are no causal curves. However, the causal order used for this definition is not the same as the one we have used, using the λ-topology. This point is discussed in [Sch26], based on [JS16].

Theorems of Geroch, Bernal–Sánchez, Fathi–Siconolfi, etc. assert that a globally hyperbolic Lorentzian spacetime admits a Cauchy time function.

The key propagation lemma

Lemma 6.3. *Let (M, λ, q) be G-causal, $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_M)$, and assume*

$$\mathrm{SS}(F) \cap \lambda \subset T^*_M M.$$

Then $\mathrm{SS}(\mathrm{R}q_*F) \subset \{(t; \tau) \in T^*\mathbb{R}\,;\, \tau \leq 0\}$.

Sketch of proof. (i) If q is proper on $\mathrm{supp}(F)$, this is immediate from the functoriality of the micro-support.

(ii) In general, one considers an increasing exhaustive family $\{K_n\}_n$ of compact subsets and one sets $Z_n = J^+_{\lambda^a}(K_n)$. Then $\mathrm{SS}(\mathbf{k}_{Z_n}) \subset \lambda^a$, so that $\mathrm{SS}(F) \cap \mathrm{SS}(\mathbf{k}_{Z_n})^a \subset T^*_M M$ and thus $\mathrm{SS}(F_{Z_n}) \subset \mathrm{SS}(F) + \lambda^a$ and .$\mathrm{SS}(F_{Z_n}) \cap \lambda \subset T^*_M M$. Then the result follows for the sheaves F_{Z_n}.

(iii) To conclude, apply [KS90, exe. 5.7]. Q.E.D.

The global Cauchy problem — abstract form

Theorem 6.4. *Let (M, λ, q) be a G-causal manifold, $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_M)$ with*

$$\mathrm{SS}(F) \cap (\lambda \cup \lambda^a) \subset T^*_M M.$$

For $t_0 \in q(M)$, set $N_0 = q^{-1}(t_0)$. Then N_0 is non-characteristic for F and there is a natural isomorphism

$$\mathrm{R}\Gamma(M; F) \xrightarrow{\sim} \mathrm{R}\Gamma(N_0; F|_{N_0}).$$

Sketch of proof. Since $dq(x) \in \lambda$, $\pm dq(x) \notin \mathrm{SS}(F)$ and N_0 is non-characteristic for F.

Lemma 6.3 applied both to λ and λ^a gives $\mathrm{SS}(\mathrm{R}q_*F) \subset T^*_{\mathbb{R}}\mathbb{R}$, so $\mathrm{R}q_*F$ is constant on $\mathbb{R}$. Hence $\mathrm{R}\Gamma(M; F) \simeq \mathrm{R}\Gamma(\mathbb{R}; \mathrm{R}q_*F) \simeq (\mathrm{R}q_*F)_{t_0} \simeq \mathrm{R}\Gamma(N_0; F|_{N_0})$. Q.E.D.

The global Cauchy problem for hyperfunctions

Applying Theorems 6.4 and Corollary 4.4, we get:

Corollary 6.5. *Let (M, λ, q) be a real analytic G-causal manifold (meaning that both M and q are real analytic), X a complexification of M, $\mathscr{M}$ a coherent $\mathscr{D}_X$-module. Let $N = q^{-1}(t_0)$, $Y \subset X$ a complexification of N. Assume*

$$\mathrm{char}_M(\mathscr{M}) \cap \lambda \subset T^*_M M.$$

Then

$$\mathrm{R}\Gamma\big(M; \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{B}_M)\big) \xrightarrow{\sim} \mathrm{R}\Gamma\big(N; \mathrm{R}\mathscr{H}om_{\mathscr{D}_Y}(\mathscr{M}_Y, \mathscr{B}_N)\big).$$

In other words, the Cauchy problem for hyperfunctions and hyperbolic $\mathscr{D}$-modules is *globally* well-posed.

Remark 6.6. *In [JS16], one proves a generalization of Corollary 6.5 to higher codimensional submanifold N.*

Examples (wave-type operators)

Example 6.7. *Let (M, λ, q) be a real analytic G-causal manifold and assume that $M = N \times \mathbb{R}$ and q is the projection on $\mathbb{R}$. As usual, denote by $(t; \tau)$ the coordinates on $T^*\mathbb{R}$. Let R be a differential operator on M of order ≤ 2 whose principal symbol of order 2 is ≥ 0 on $T^*_M X$ and independent of τ. Then*

$$P = \partial_t^2 - R$$

is hyperbolic in the codirections $(x, t; \pm dt)$ and Corollary 6.5 applies.

Indeed, let $\sigma(P)$ be the principal symbol of P and let (x) be a local coordinate system on N, $(x, \sqrt{-1}\eta)$ the associated coordinates on $T^*_N Y \simeq \sqrt{-1}T^*N$. Let $(t; \theta + \sqrt{-1}\tau)$ denote the coordinate on $\mathbb{C} \otimes T^*\mathbb{R}$. Then

$$\sigma(P)(x, t; \sqrt{-1}\eta, \sqrt{-1}\tau + \theta) = \theta^2 - \tau^2 + \sigma_2(R)(x, t; \eta) + 2\sqrt{-1}\,\theta\tau.$$

Therefore, $\sigma(P)(x, t; \sqrt{-1}\eta, \sqrt{-1}\tau + \theta) \neq 0$ for $\theta \neq 0$. This implies

$$(x, t; \theta) \notin C_{T^*_M X}(\sigma(P)^{-1}(0)).$$

This last point is not obvious but follows from the local Bochner tube theorem as remarked by Kashiwara (see [BS73]).

Hence, we may apply Corollary 4.4 with $N \times \{t_0\}$ as a Cauchy hypersurface.

Example 6.8. *In particular, if $(g_t)_{t \in \mathbb{R}}$ is an analytic family of Riemannian metrics on N with Laplace–Beltrami operators Δ_t and $a(t) \geq 0$, then*

$$P = \partial_t^2 - a(t)\Delta_t + \textit{lower order terms}$$

satisfies the hypotheses of Example 6.7.

Example 6.9. *Assume as above that $M = N \times \mathbb{R}$ and assume moreover that N compact and connected. We choose the cone $\lambda \subset T^*M$ as*

$$\lambda = T^*_N N \times \{(t;\tau); \tau \geq 0\}.$$

Note that this cone is closed and proper but with empty interior.

Let $x_0 = (y_0, t_0) \in N \times \mathbb{R}$. Then $J^+_\lambda(x_0) = N \times [t_0, +\infty)$. Since N is compact, the map q is a time function.

In this situation Corollary 6.5 applies and we may solve the global Cauchy problem for hyperfunction solutions.

Remark 6.10. *Hyperbolicity in a codirection depends only on the top-order part of the operator. For all such hyperbolic operators, the global Cauchy problem for hyperfunctions is well-posed.*

The analogous results fail in general for distributions or $\mathscr{C}^\infty$-functions. Indeed, ***Hadamard*** *has shown that the (local) Cauchy problem for $\partial_t^2 - \partial_x$ on $\mathbb{R}^2$ with data on $\{t = 0\}$ is not well-posed for $\mathscr{C}^\infty$-functions or distributions, contrarily to hyperfunctions.*

7 Before the Big Bang

A mathematical candidate for the Big Bang

Suppose $q\colon M \to \mathbb{R}_{>0}$ is a time function and each $N_t = q^{-1}(t)$ is a compact Riemannian manifold.

Question. What happens for $t < 0$ if the diameter of N_t goes to 0 as $t \to 0^+$?

Mathematicians (Manin–Marcolli [MM14]) and physicists (Penrose [Pen12]) have proposed interpretations of the Big Bang.

Here is a *purely mathematical* candidate — in fact, a model for any phase transition.

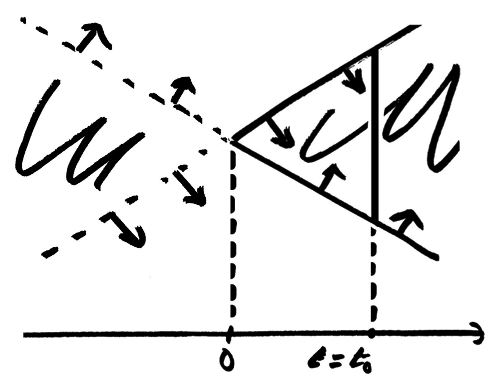
No physics is assumed — only sheaves and their micro-supports.

Extending the universe to $t < 0$

Represent the universe as a closed ball in $\mathbb{R}^n$ (a manifold with boundary) whose radius grows linearly with t: the sheaf $\mathbf{k}_{\{|x| \leq t\}}$, defined for $t \geq 0$.

(i) The micro-support at the boundary is the *interior conormal*,

(ii) extending naturally to $t < 0$ gives the *exterior* conormal — the micro-support of the constant sheaf on the open cone.



Before the Big Bang

A distinguished triangle

Set $X = \mathbb{R}^n_x \times \mathbb{R}_t$. How does one glue the sheaf $\mathbf{k}_{\{|x|\le t\}}$ (defined for $t \ge 0$) and the sheaf $\mathbf{k}_{\{|x|<-t\}}$ (defined for $t < 0$)?

With S. Guillermou and M. Kashiwara, we constructed a distinguished triangle

$$\mathbf{k}_{\{|x|<-t\}}[n] \to K \to \mathbf{k}_{\{|x|\le t\}} \xrightarrow{\psi} [+1].$$

In other words, the natural extension of the constant sheaf on the closed cone for $t \ge 0$ is the constant sheaf on the *open* cone, *shifted* by the dimension.

The Hamiltonian isotopy

As we shall see, K is the *quantization of a Hamiltonian isotopy.*

On $T^*\mathbb{R}^n$ with homogeneous symplectic coordinates $(x;\xi)$, consider

$$\varphi_t(x;\xi) = \Big(x - t\frac{\xi}{|\xi|}\,;\,\xi\Big), \qquad t \in \mathbb{R}.$$

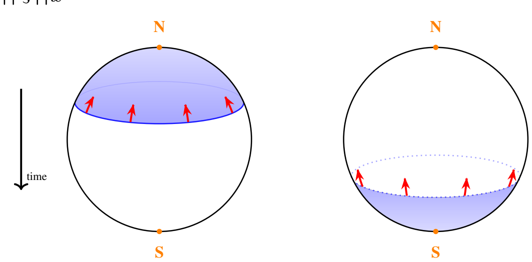
Outside the zero-section, $\mathrm{SS}(K)$ is the smooth Lagrangian manifold obtained as the image of $T^*_{\{0\}}\mathbb{R}^n$ under this isotopy.

A phase transition (such as a Big Bang) is the instant at which the *rank* of the projection $\pi\colon \Lambda \to M$ of a smooth Lagrangian submanifold $\Lambda \subset T^*M$ *drops.*

A periodic Big Bang on the sphere

Replace $\mathbb{R}^n_x$ by a Riemannian manifold (positive convexity radius and injectivity radius), using the Hamiltonian isotopy associated with $||\xi||_x$.

For the Euclidean n-sphere $M = \mathbb{S}^n$ (with $n \ge 2$): the sheaf obtained has a *shift that jumps by the dimension at each pole.* Here the time is the vertical line from the top (past) to the bottom (future).



Periodic Big Bang

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