Reprinted from ADVANCES IN MATHEMATICS
All Rights Reserved by Academic Press, New York and London

Vol. 93, No. 1, May 1992
Printed in Belgium

Morse Inequalities for R-constructible Sheaves

P. SCHAPIRA

Département de Mathématiques et Informatiques, Université de Paris Nord, 93430 Villetaneuse Cedex, France

AND

N. TOSE

Department of Mathematics, Faculty of Sciences, Hokkaido University, Sapporo, 060, Japan

This note aims at generalizing of classical Morse inequalities for Betti numbers of compact manifolds (cf. [10, 1, 2]). In this paper, we work with **R**-constructible sheaves instead and encounter the tight relation between Morse theory and microlocal analysis of sheaves. See Witten [11], Helffer and Sjöstrand [5, 6], and Henniart [7] for another approach to Morse inequalities via microlocal analysis and also Goresky and MacPherson [3, 4] who introduced the "stratified Morse theory". This paper may be considered as a variation on Kashiwara's index theorem [8], and in fact our proof is a slight modification of his.

1. STATEMENT OF THE MAIN THEOREM

Let k be a commutative field. We denote by $\mathrm{Mod}^f(k)$ the category of finite dimensional k-vector spaces and by $D^b(\mathrm{Mod}^f(k))$ its derived category with bounded cohomologies.

Let $b = \{b_l\}_{l \in \mathbf{Z}}$ be a sequence of integers with $b_l = 0$ for $|l| \gg 0$. We define

$$b_l^* = (-1)^l \sum_{j \leq l} (-1)^j b_j, \qquad b_\infty^* = \sum_j (-1)^j b_j. \tag{1.1}$$

If $V \in \mathrm{ob}(D^b(\mathrm{Mod}^f(k)))$, we set

$$b_l(V) = \dim H^l(V)$$

(and we define $b_l^*(V)$ and $\chi(V) = b_\infty^*(V)$ as in (1.1)).

0001-8708/92 $9.00

Let X be a C^∞ manifold, and $\pi: T^*X \to X$ its cotangent bundle. We denote by $D^b(X)$ the derived category with bounded cohomologies of sheaves of k vector spaces on X.

If $F \in \mathrm{ob}(D^b(X))$, we denote by $\mathrm{SS}(F)$ its microsupport. This is an $\mathbf{R}^+$-conic closed subset in T^*X. Refer to [9] for information about $\mathrm{SS}(F)$. Now let $F \in \mathrm{ob}(D^b(X))$, and let $\phi: X \to \mathbf{R}$ be a real valued C^2 function on X. We set

$$\Lambda = \mathrm{SS}(F),$$
$$\Lambda_\phi = \{(x, d\phi(x)) \in T^*X;\, x \in X\}.$$

We assume

$$\phi^{-1}(]-\infty, t]) \cap \mathrm{supp}(F) \text{ is compact for any } t \in \mathbf{R}, \tag{H.1}$$

(in particular ϕ is proper on $\mathrm{supp}(F)$),

there is a finite family $\{K_j\}_{1 \leqslant j \leqslant N}$ of disjoint compact subsets in X and real numbers $\{s_j\}_{1 \leqslant j \leqslant N}$ such that

$$\pi(\Lambda_\phi \cap \Lambda) = \bigcup_j K_j, \qquad \phi|_{K_j} = s_j, \tag{H.2}$$

$$V_j \overset{\text{def}}{=} \mathbf{R}\Gamma(K_j; \mathbf{R}\Gamma_{\{x;\phi(x) \geqslant s_j\}}(F)) \text{ belongs to } D^b(\mathrm{Mod}^f(k)). \tag{H.3}$$

Set

$$n_l = \sum_j \dim H^l(V_j). \tag{1.2}$$

Then we have

THEOREM 1.1 (Generalized Morse Inequalities). *Assume* (H.1), (H.2), *and* (H.3). *Then*

(i) $\mathbf{R}\Gamma(X; F) \in \mathrm{ob}(D^b(\mathrm{Mod}^f(k)))$,

(ii) *setting* $b_l(X, F) = b_l(\mathbf{R}\Gamma(X; F))$, *we have the inequalities*

$$b_l^*(X, F) \leqslant n_l^*$$

where n_l *is given by* (1.2).

Remark 1.2. The conclusion (i) has already been obtained in [8]. Moreover since $b_l(X, F) = n_l = 0$ for $l \gg 0$, we have

$$\chi(X; F) = n_\infty^*.$$

Here $\chi(X; F)$ is the Euler–Poincaré index of F on X. This is an obious version of Kashiwara's index theorem (cf. [8]).

We can recover with Theorem 4.1 the "Morse–Bott inequalities" (cf. [1, 6]). Let ϕ be a real valued C^∞ function on X, and let $\{M_j\}_{j=1}^N$ (resp. $\{s_j\}_{j=1}^N$) be a family of connected compact manifolds (resp. of real numbers). Assume

$$\phi^{-1}(]-\infty, t]) \text{ is compact for any } t \in \mathbf{R}, \tag{B.1}$$

$$\phi|_{M_j} = s_j, \qquad (d\phi)|_{M_j} = 0 \text{ for any } j,$$
$$d\phi(x) \neq 0 \text{ if } x \notin \cup_j M_j, \tag{B.2}$$

for any j, there exists a linear decomposition $T_{M_j}X = T^-_{M_j}X \oplus T^+_{M_j}X$ for which the hessian $H(\phi)$ of ϕ satisfies (B.3)

$$\pm H(\phi)|_{T^\pm_{M_j}X} \text{ is positive definite.}$$

Let σ_j denote the fiber dimension of $T^-_{M_j}X$, and $\mathrm{or}_{M_j^-}$ the orientation sheaf associated to $T_{M_j}X$:

$$\mathrm{or}_{M_j^-} = H^{\sigma_j}(\mathbf{R}\Gamma_{M_j}(k_{T^-_{M_j}X})). \tag{1.3}$$

Set

$$n_l = \sum_j \dim H^{l-\sigma_j}(M_j; \mathrm{or}_{M_j^-}). \tag{1.4}$$

Corollary 1.3 [1]. *Assume* (B.1), (B.2), (B.3). *Then*:

$$b_l^*(X; k_X) \leqslant n_l^*,$$

where the sequence $\{n_l\}$ *is given by* (1.4).

Proof. By Theorem 1.1, it is enough to check the isomorphism

$$\mathbf{R}\Gamma(M_j; \mathbf{R}\Gamma_{\{\phi \geqslant s_j\}}(k_X)) \simeq \mathbf{R}\Gamma(M_j; \mathbf{R}\Gamma_{M_j}(k_{T^-_{M_j}X})),$$

hence the isomorphism

$$\mathbf{R}\Gamma_{\{\phi \geqslant s_j\}}(k_X)|_{M_j} \simeq \mathbf{R}\Gamma_{M_j}(k_{T^-_{M_j}X})|_{M_j}.$$

But it is easily checked (cf. Bott [1]). Q.E.D.

2. Proof of the Main Theorem

In order to prove the theorem, we note

Lemma 2.1. *Consider a distinguished triangle in* $D^b(\mathrm{Mod}^f(k))$

$$V' \to V \to V'' \xrightarrow{+1}.$$

Then for any $l \in \mathbf{Z}$ we have an inequality

$$b_l^*(V) \leqslant b_l^*(V') + b_l^*(V'').$$

Proof. We may assume that V, V', and V'' are concentrated in degree $\geqslant 0$. Then we have a long exact sequence

$$0 \to H^0(V') \to H^0(V) \to \cdots \to H^l(V') \to H^l(V) \to B^l(V'') \to 0,$$

where

$$B^l(V'') = \mathrm{Im}(H^l(V) \to H^l(V'')).$$

Then setting

$$\tilde{b}_l(V'') = \dim B^l(V'') \; (j = l)$$

and

$$\tilde{b}_j(V'') = b_j(V'') \; (j < l),$$

we get

$$b_l^*(V) = b_l^*(V') + (-)^l \sum_{l' \leqslant l} (-)^{l'} \tilde{b}_{l'}(V'').$$

Since $\tilde{b}_l(V'') \leqslant \dim H^l(V'')$, the proof follows. Q.E.D.

Proof of Theorem 1.1. We shall reduce the problem to the 1 dimensional case. For this purpose, we put

$$G = \mathbf{R}\phi_* F$$

and write

$$\phi|_{K_j} = s_j, \; \{s_1, ..., s_N\} = \{t_1, ..., t_L\}$$

with $t_j < t_{j+1}$. We set

$$\Lambda_t = \{(t; dt) \in T^*\mathbf{R}; t \in \mathbf{R}\}.$$

First we remark that the hypothesis (H.1) is trivially satisfied by $(\mathbf{R}, t, G)$:

$$]-\infty, t] \cap \mathrm{supp}(G) \text{ is compact for any } t \in \mathbf{R}. \qquad \text{(H}'\text{.1)}$$

Next, applying Proposition 4.1 of [9] we get

$$\mathrm{SS}(G) \cap \Lambda_t \subset \{(t_i; dt) \in T^*\mathbf{R}; i = 1, ..., L\}. \qquad \text{(H}'\text{.2)}$$

Finally we have

$$(\mathbf{R}\Gamma_{\{t\geq t_l\}}G)_{t_l} \simeq \bigoplus_{s_j=t_l} V_j. \tag{H'.3}$$

In fact

$$(\mathbf{R}\Gamma_{\{t\geq t_l\}}G)_{t_l} \simeq \mathbf{R}\Gamma(\phi^{-1}(t_l); \mathbf{R}\Gamma_{\{x;\,\phi(x)\geq t_l\}}(F))$$
$$\simeq \bigoplus_{s_j=t_l} \mathbf{R}\Gamma(K_j; \mathbf{R}\Gamma_{\{x;\phi(x)\geq s_j\}}(F))$$

by the definition of the micro-support. Thus the triple $(\mathbf{R}, t, G)$ satisfies the same hypotheses as (X, ϕ, F). Since

$$\mathbf{R}\Gamma(X; F) = \mathbf{R}\Gamma(\mathbf{R}; G),$$

we have

$$b_l(X, F) = b_l(\mathbf{R}, G),$$

and it is enough to prove the theorem for $X=\mathbf{R}$, $\phi(t)=t$.

We set $X=\mathbf{R}$. Put $t_0=-\infty$, $t_{L+1}=+\infty$, and define

$$I_t =]-\infty, t[,\ Z_t =]-\infty, t], \qquad I_j = I_{t_j}, \qquad Z_j = Z_{t_j}.$$

Introduce

$$b_l^*(Z_j, F) = b_l^*(\mathbf{R}\Gamma(Z_j; F)),$$
$$b_l^*(I_j, F) = b_l^*(\mathbf{R}\Gamma(I_j, F)).$$

Then by Theorem 1.4.3 of [9], we have the isomorphism

$$H^k(I_{j+1}; F) \simeq H^k(I_t; F)\ (t_j < t \leq t_{j+1}).$$

By taking the inductive limit of the right hand side, we derive

$$H^k(I_{j+1}; F) \simeq H^k(Z_j; F). \tag{2.1}$$

Consider the distinguished triangle

$$(\mathbf{R}\Gamma_{\{t\geq t_j\}}(F))_{t_j} \to \mathbf{R}\Gamma(Z_j; F) \to \mathbf{R}\Gamma(I_j; F) \xrightarrow{+1}.$$

Since $\mathbf{R}\Gamma(I_1; F)=0$, we find by induction from (2.1) that both $\mathbf{R}\Gamma(Z_j; F)$ and $\mathbf{R}\Gamma(I_j; F)$ belong to $D^b(\mathrm{Mod}^f(k))$.

Moreover (2.1) gives

$$\dim H^k(X; F) = \sum_{1\leq j\leq L} \{\dim H^k(Z_j; F) - \dim H^k(I_j; F)\}.$$

Hence

$$b_l^*(X, F) = \sum_{1 \leqslant j \leqslant L} \{b_l^*(Z_j, F) - b_l^*(I_j, F)\}.$$

On the other hand, we get by Lemma 2.1 from (2.2)

$$b_l^*(Z_j, F) - b_l^*(I_j, F) \leqslant b_l^*(\mathbf{R}\Gamma_{\{t \geqslant t_j\}}(F))_{t_j}).$$

Hence we have

$$b_l^*(X, F) \leqslant \sum_{1 \leqslant j \leqslant L} b_l^*((\mathbf{R}\Gamma_{\{t \geqslant t_j\}}(F)_{t_j})) = n_l^*. \qquad \text{Q.E.D.}$$

3. Application to Pure Sheaves

Let X be a real analytic manifold, and let $D^b_{\mathbf{R}-c}(X)$ denote the subcategory of $D^b(X)$ consisting of objects with $\mathbf{R}$-constructible cohomologies (cf. [9]).

Let $F \in \mathrm{ob}(D^b_{\mathbf{R}-c}(X))$. Then $\Lambda = \mathrm{SS}(F)$ is a Lagrangean subanalytic subset of T^*X. Take a real valued C^2 function ϕ on X and suppose

$$\phi^{-1}(]-\infty, t]) \cap \mathrm{supp}(F) \text{ is compact for any } t \in \mathbf{R}, \tag{3.1}$$

$$\Lambda_\phi \cap \Lambda = \Lambda_\phi \cap \Lambda_{\mathrm{reg}} = \{p_1, ..., p_N\}, \tag{3.2}$$

$$\Lambda_\phi \text{ and } \Lambda_{\mathrm{reg}} \text{ intersect transversally at each point } p_l, \tag{3.3}$$

F is pure at each p_l with multiplicity m_l and shift d_l along Λ in the sense of [9]. (3.4)

Recall that (3.4) is equivalent to

$$(\mathbf{R}\Gamma_{\{\phi(x) \geqslant \phi(x_l)\}}(F))_{x_l} = k^{m_l}[\delta^l], \tag{3.5}$$

where $x_l = \pi(p_l)$, and

$$\delta^l = d_l - \tfrac{1}{2}\dim X - \tfrac{1}{2}\tau(\lambda_0(p_l), \lambda_\Lambda(p_l), \lambda_\phi(p_l)). \tag{3.6}$$

See Chapter 7 of [9] for the definition of Maslov index $\tau(\cdot, \cdot, \cdot)$. Under the above conditions, we get by Theorem 1.1

$$\mathbf{R}\Gamma(X; F) \in \mathrm{ob}(D^b(\mathrm{Mod}^f(k))). \tag{3.7}$$

We set

$$n_l = \sum_{\delta^l = -l} m_l, \qquad n_l^* = (-)^l \sum_{j \leqslant l} (-)^j n_j. \tag{3.8}$$

Then we have, by applying Theorem 1.1,

THEOREM 3.1. *For any $l \in \mathbf{Z}$, we have the inequality*

$$b_l^*(X, F) \leqslant n_l^*. \tag{3.9}$$

Remark 3.2. Assume moreover

$$\Lambda = T^*_{S_i} X \qquad \text{in a neighborhood of } p_i, \tag{3.10}$$

where S_i is a real analytic submanifold of X. By (3.3), $\phi|_{S_i}$ is a Morse function at $x_i = \pi(p_i)$. Let $s^{\pm}(x_i)$ be the number of positive or negative eigenvalues of the Hessian of $\phi|_{S_i}$ at x_i. Then under the notation (3.6), we have

$$\delta^i = d_i - \tfrac{1}{2}\dim X + \tfrac{1}{2}(s^+(x_i) - s^-(x_i)). \tag{3.11}$$

Remark moreover that if X is a complex manifold, F has **C**-constructible cohomologies, and F is perverse, then we have

$$d_i = 0 \qquad \text{for all } i. \tag{3.12}$$

Hence in this situation, we can deduce the Morse inequalities from the multiplicity of F at generic points of Λ.

EXAMPLE 3.3. Let X be $\mathbf{C}^N$ $(N > 2)$ with coordinates $z = (z_1, ..., z_N)$, and define a hypersurface S in X by

$$S = \left\{ z \in X : \sum_{1 \leqslant j \leqslant N} z_j^2 = 0 \right\}.$$

We study $F \in \mathrm{ob}(D^b_{\mathbf{C}-c}(X))$(i.e., F has **C**-constructible cohomologies) satisfying

$$\Lambda = \mathrm{SS}(F) \subset T^*_{S_{\mathrm{reg}}} X \cup T^*_{\{0\}} X \cup T^*_X X, \tag{3.13}$$

$$F \text{ is perverse.} \tag{3.14}$$

We put

$$\Lambda_1 = T^*_{S_{\mathrm{reg}}} X, \qquad \Lambda_N = T^*_{\{0\}} X, \qquad \Lambda_0 = T^*_X X,$$

and let m_j denote the multiplicity of F along Λ_j $(j = 0, 1, N)$. Choosing for example $\phi(z) = |z - a|^2$ with $a = (1, 2\sqrt{-1}, 0, ..., 0)$, we find for any l

$$b_l^*(X, F) \leqslant n_l^*$$

and

$$\chi(X; F) = m_0 - m_1 + (-1)^{N-1} m_1 + (-1)^N m_N$$

with

$$n_0 = m_0, \quad n_1 = m_1, \quad n_{N-1} = m_1, \quad n_N = m_N,$$
$$\text{and} \quad n_j = 0 \quad \text{for } j \notin \{0, 1, N-1, N\}.$$

Remark 3.4. In [4], Goresky and MacPherson also obtained "Morse inequalities" for the intersection cohomology of compact complex spaces [4, p. 174], but their formulation appears rather different from ours.

References

1. R. Bott, Nondegenerate critical manifolds, *Ann. of Math.* **60**, No. 2 (1954), 248–261.
2. R. Bott, Lectures on Morse theory, old and new, *Bull. Amer. Math. Soc.* **7** (1982), 331–358.
3. M. Goresky and R. MacPherson, "Stratifield Morse Theory," Ergebnisse der Mathematik und ihrer Grenzgebiete (3), Vol. 14, Springer-Verlag, Berlin/New York, 1988.
4. M. Goresky and R. MacPherson, Morse theory and intersection homology theory, *Astérisque* **101–102** (1981), 135–192.
5. B. Helffer and J. Sjöstrand, Puits multiples en mécanique semi-classique, IV, étude du complexe de Witten, *Comm. Partial Differential Equations* **10**, No 3 (1985), 245–380.
6. B. Helffer and J. Sjöstrand, A proof of the Bott inequalities, *in* "Algebraic Analysis," Academic Press, New York, (1988), 171–183.
7. G. Henniart, Les inégalités de Morse [d'après E. Witten], *Sém. Bourbaki* **617**, 1983–1984.
8. M. Kashiwara, Index theorem for constructible sheaves, *Astérisque* **130** (1985), 193–209.
9. M. Kashiwara and P. Schapira, Microlocal study of sheaves, *Astérisque* **128** (1985).
10. J. M. Milnor, "Morse Theory," Annals of Mathematics Studies, Vol. 51, Princeton Univ. Press, Princeton, NJ, 1963.
11. E. Witten, Super symmetry and Morse theory, *J. Differential Geom.* **17** (1982), 661–692.

Printed by Catherine Press, Ltd., Tempelhof 41, B-8000 Brugge, Belgium