Triangulated categories for the analysts

Pierre Schapira

September 11, 2007

Abstract

This paper aims at showing how the tools of Algebraic Geometry apply to Analysis. We will review various classical constructions, including Sato's hyperfunctions, Fourier-Sato transform and microlocalization, the microlocal theory of sheaves (with some applications to PDE) and explain the necessity of Grothendieck topologies to treat algebraically generalized functions with growth conditions.

Mathematics Subject Classification: 58G07, 32A45, 32C38, 35A27

Introduction

In this paper, we will show how the tools of Algebraic Geometry– sheaves, triangulated and derived categories, Grothendieck topologies and stacks– play (or should play) a crucial role in Analysis.

Note that, conversely, some problems of Analysis led to new algebraic concepts. For example, one of the deepest notion related to triangulated categories is that of t-structure, and as a particular case, that of perverse sheaves, and these notions emerged with the study of the Riemann Hilbert correspondence, a problem dealing with differential equations.

Another example is the Fourier transform, clearly one of the most essential tools of the analysts, until it was categorified by Sato and applied to algebraic analysis, next transposed to algebraic geometry (the Fourier-Mukai transform).

The classical analysts are used to work in various functional spaces constructed with the machinery of functional analysis and Fourier transform, but Sato's construction of hyperfunctions [28] in the 60's does not use any of these tools. It is a radically new approach which indeed has entirely modified the mathematical landscape in this area. The functional spaces are now replaced by "functorial spaces", that is, sheaves of generalized holomorphic functions on a complex manifold X or, more precisely, complexes of sheaves $R\mathcal{H}om\,(G, \mathcal{O}_X)$, where G is an object of the derived category of $\mathbb{R}$-constructible sheaves on the real underlying manifold to X. Putting general systems of linear partial differential equations, *i.e.,* $\mathcal{D}_X$-modules, in the machinery, one is led to study complexes $R\mathcal{H}om\,(G, F)$, where $F = \mathrm{Sol}(\mathcal{M}) := R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$ is the complex of holomorphic solutions of a coherent $\mathcal{D}_X$-module $\mathcal{M}$.

The main invariant attached to a coherent $\mathcal{D}_X$-module $\mathcal{M}$ is its characteristic variety $\mathrm{char}(\mathcal{M})$, a closed conic involutive subset of the cotangent bundle T^*X. For a sheaf G on a *real* manifold there is a similar invariant, the microsupport $SS(G)$ which describes the directions of non propagation of G, and this set is again a closed conic involutive subset of the cotangent bundle. In case of $F = \mathrm{Sol}(\mathcal{M})$, it follows from the Cauchy-Kowalevsky theorem that its microsupport is nothing but $\mathrm{char}(\mathcal{M})$. Therefore, in order to study $R\mathcal{H}om\,(G, \mathrm{Sol}(\mathcal{M}))$, one can forget that one is working on a complex manifold X and dealing with $\mathcal{D}_X$-modules, keeping only in mind two geometrical informations, the microsupport of G and that of F (see [16]).

The study of the characteristic variety of $\mathcal{D}_X$-modules naturally leads to introduce the ring $\mathcal{E}_X$ of microdifferential operators, a kind of localization of $\mathcal{D}_X$ in T^*X. The ring $\mathcal{E}_X$ was first constructed in [29] using the Sato's microlocalization functor, the Fourier-Sato transform of the specialization functor. We shall briefly recall here the main steps of these constructions.

Finally, although classical sheaf theory does not allow one to treat usual spaces of analysis involving growth conditions, these conditions being not of local nature, we shall show here how it is possible to overcome this difficulty by using Grothendieck topologies (see [17]).

0.1 Notations

In this paper, we mainly follow the notations of [16].
(i) We denote by $\mathbf{k}$ a field and by $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$ the bounded derived category of sheaves of $\mathbf{k}$-vector spaces on a topological space X. More generally, if $\mathcal{A}$ is a sheaf of rings on X, we denote by $\mathrm{Mod}(\mathcal{A})$ the category of left $\mathcal{A}$-modules and by $\mathrm{D}^{\mathrm{b}}(\mathcal{A})$ its bounded derived category. If there is no risk of confusion, we write $R\mathcal{H}om$ instead of $R\mathcal{H}om_{\mathbf{k}_X}$ and similarly for RHom and $\otimes$.
(ii) If Z is a locally closed subset of the topological space X, we denote by $\mathbf{k}_{XZ}$ the sheaf which is the constant sheaf with stalk $\mathbf{k}$ on Z and which is 0 on $X \setminus Z$. If there is no risk of confusion, we write $\mathbf{k}_Z$ instead of $\mathbf{k}_{XZ}$.
(iii) For a real manifold X, we denote by $\dim_X$ its dimension and by by or_X the orientation sheaf. For a morphism of manifolds $f\colon X \to Y$, we set $\mathrm{or}_{X/Y} = \mathrm{or}_X \otimes f^{-1}\mathrm{or}_Y$.

(iv) We denote by $\mathrm{D}'_X(\,\bullet\,) = R\mathcal{H}om_{\mathbf{k}_X}(\,\bullet\,, \mathbf{k}_X)$ the duality functor in $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$ and by ω_X the dualizing complex. Recall that $\omega_X \simeq \mathrm{or}_X\,[\dim_X]$.
(v) For a (real or complex) manifold X, we denote by $\tau\colon TX \to X$ and $\pi\colon T^*X \to X$ its tangent bundle and cotangent bundle, respectively. Let $f\colon X \to Y$ be a morphism of (real or complex) manifolds. To f are associated the maps

$$\begin{tikzcd} TX \arrow[r, "f'"] \arrow[dr, "\tau"'] & X \times_Y TY \arrow[r, "f_\tau"] \arrow[d, "\tau"] & TY, \arrow[d, "\tau"] \\ & X \arrow[r, "f"] & Y \end{tikzcd} \qquad \begin{tikzcd} T^*X \arrow[dr, "\pi"'] & X \times_Y T^*Y \arrow[l, "f_d"'] \arrow[r, "f_\pi"] \arrow[d, "\pi"] & T^*Y \arrow[d, "\pi"] \\ & X \arrow[r, "f"] & Y. \end{tikzcd}$$

We denote by $T^*_X X$ the zero-section of T^*X and by $T^*_X Y = f_d^{-1} T^*_X X$ the conormal bundle to f. If f is an embedding, we identify $T^*_X Y$ to a sub-bundle of T^*Y and call it the conormal bundle to X.
(vi) For a complex manifold X, we denote by $\mathcal{O}_X$ the sheaf of holomorphic functions and by Ω_X the sheaf of holomorphic forms of maximal degree. We denote by d_X the complex dimension of X.

1 Generalized functions

In the sixties, people used to work in various spaces of generalized functions on a real manifold. The situation drastically changed with Sato's definition of hyperfunctions [28].

Consider first the case where M is an open subset of the real line $\mathbb{R}$ and let X an open neighborhood of M in the complex line $\mathbb{C}$ satisfying $X \cap \mathbb{R} = M$. The space $\mathcal{B}(M)$ of hyperfunctions on M is given by

$$\mathcal{B}(M) = \mathcal{O}(X \setminus M)/\mathcal{O}(X).$$

It is easily proved that this space depends only on M, not on the choice of X, and that the correspondence $U \mapsto \mathcal{B}(U)$ (U open in M) defines a flabby sheaf $\mathcal{B}_M$ on M.

Classically, the "boundary value" of a holomorphic function $\varphi(z)$ defined in the open set $X \cap \{\mathrm{Im}\, z > 0\}$ of the complex line, if it exists, is the limit (for a suitable topology) of the function $\varphi(x + iy)$ as $y \xrightarrow{>} 0$. With Sato's definition, the boundary value always exists and is no more a limit. Indeed, it is the class of the holomorphic function $\psi(z) \in \mathcal{O}(X \setminus M)$ given by $\psi(z) = \varphi(z)$ for $\mathrm{Im}\, z > 0$ and $\psi(z) = 0$ for $\mathrm{Im}\, z < 0$.

On a manifold M of dimension n, the sheaf $\mathcal{B}_M$ was originally defined as

$$\mathcal{B}_M = H^n_M(\mathcal{O}_X) \otimes \mathrm{or}_M$$

where X is a complexification of M. Since X is oriented, Poincaré's duality gives the isomorphism $\mathrm{D}'_X(\mathbb{C}_M) \simeq \mathrm{or}_M\,[-n]$. An equivalent definition of hyperfunctions is thus given by

$$\mathcal{B}_M = R\mathcal{H}om_{\mathbb{C}_X}(\mathrm{D}'_X(\mathbb{C}_M), \mathcal{O}_X). \tag{1.1}$$

The importance of Sato's definition is twofold: first, it is purely algebraic (starting with the analytic object $\mathcal{O}_X$), and second it highlights the link between real and complex geometry.

Let us define the notion of "boundary value" in this settings. Consider a subanalytic open subset Ω of X and denote by $\overline{\Omega}$ its closure. Assume that:

$$\begin{cases} \mathrm{D}'_X(\mathbb{C}_\Omega) \simeq \mathbb{C}_{\overline{\Omega}}, \\ M \subset \overline{\Omega}. \end{cases}$$

The morphism $\mathbb{C}_{\overline{\Omega}} \to \mathbb{C}_M$ defines by duality the morphism $\mathrm{D}'_X(\mathbb{C}_M) \to \mathrm{D}'_X(\mathbb{C}_{\overline{\Omega}}) \simeq \mathbb{C}_\Omega$. Applying the functor $\mathrm{RHom}\,(\bullet\,, \mathcal{O}_X)$, we get the boundary value morphism

$$\mathrm{b}\colon \mathcal{O}(\Omega) \to \mathcal{B}(M). \tag{1.2}$$

Sato's sheaf of hyperfunctions is an example of a sheaf of generalized holomorphic functions. Another example of such a sheaf is as follows.

Consider a closed complex hypersurface Z of the complex manifold X and denote by U its complementary. Let $j\colon U \hookrightarrow X$ denote the embedding. Then $j_*j^{-1}\mathcal{O}_X$ represents the sheaf on X of functions holomorphic on U with possible (essential) singularities on Z. One has

$$j_*j^{-1}\mathcal{O}_X \simeq R\mathcal{H}om_{\mathbb{C}_X}(\mathbb{C}_U, \mathcal{O}_X). \tag{1.3}$$

Both examples (1.1) and (1.3) are described by a sheaf of the type $R\mathcal{H}om\,(G, \mathcal{O}_X)$, with G a *constructible* sheaf (see [16] for an exposition). Recall that a sheaf G on a real analytic manifold X is $\mathbb{R}$-constructible if there exists a subanalytic stratification of X on which G is locally constant of finite rank (over the field $\mathbf{k}$). One denotes by $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(\mathbf{k}_X)$ the full triangulated subcategory of $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$ consisting of objects with $\mathbb{R}$-constructible cohomology. On a complex manifold, replacing the subanalytic stratifications by complex analytic stratifications, one also gets the category $\mathrm{D}^b_{\mathbb{C}-\mathrm{c}}(\mathbf{k}_X)$ of $\mathbb{C}$-constructible sheaves.

The advantage of considering the category $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(\mathbf{k}_X)$ is that the properties of being constructible (in the derived sense) is stable by the six Grothendieck operations (with suitable properness hypotheses).

To summarize, the classical functional spaces are now replaced by the "functorial spaces" $R\mathcal{H}om_{\mathbb{C}_X}(G, \mathcal{O}_X)$, where $G \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(\mathbb{C}_X)$.

2 $\mathcal{D}$-modules

References for the theory of $\mathcal{D}$-modules are made to [8, 9].

The theory of $\mathcal{D}$-modules appeared in the 70's with Kashiwara's thesis [8] and Bernstein's paper [2]. However, already in the 60's, Sato had the main ideas of the theory in mind and gave talks at Tokyo University on these topics. Unfortunately, Sato did not write anything and it seems that his ideas were not understood at this time. (See [1, 30].)

Let X be a complex manifold. One denotes by $\mathcal{D}_X$ the sheaf of rings of holomorphic (finite order) differential operators. A system of linear differential equations on X is a left coherent $\mathcal{D}_X$-module. The link with the intuitive notion of a system of linear differential equations is as follows. Locally on X, $\mathcal{M}$ may be represented as the cokernel of a matrix $\cdot P_0$ of differential operators acting on the right. By classical arguments of analytic geometry (Hilbert's syzygies theorem), one shows that, locally, $\mathcal{M}$ admits a bounded resolution by free modules of finite type, that is, $\mathcal{M}$ is locally isomorphic to the cohomology of a bounded complex

$$\mathcal{M}^\bullet := 0 \to \mathcal{D}_X^{N_r} \to \cdots \to \mathcal{D}_X^{N_1} \xrightarrow{\cdot P_0} \mathcal{D}_X^{N_0} \to 0. \tag{2.1}$$

Let us introduce the notation:

$$\mathrm{Sol}(\,\bullet\,) := R\mathcal{H}om_{\mathcal{D}_X}(\,\bullet\,, \mathcal{O}_X). \tag{2.2}$$

The complex $\mathrm{Sol}(\mathcal{M})$ of holomorphic solutions of $\mathcal{M}$ may be locally calculated by applying the functor $\mathcal{H}om_{\mathcal{D}_X}(\,\bullet\,, \mathcal{O}_X)$ to $\mathcal{M}^\bullet$. Hence

$$\mathrm{Sol}(\mathcal{M}) \simeq 0 \to \mathcal{O}_X^{N_0} \xrightarrow{P_0\cdot} \mathcal{O}_X^{N_1} \to \cdots \to \mathcal{O}_X^{N_r} \to 0, \tag{2.3}$$

where now $P_0\cdot$ operates on the left.

One defines naturally the characteristic variety $\mathrm{char}(\mathcal{M})$ of a coherent $\mathcal{D}_X$-module $\mathcal{M}$. This is a closed complex analytic subset of T^*X, conic with respect to the action of $\mathbb{C}^\times$ on T^*X. For example, if $\mathcal{M}$ has a single generator u with relation $\mathcal{I}u = 0$, where $\mathcal{I}$ is a locally finitely generated ideal of $\mathcal{D}_X$, then

$$\mathrm{char}(\mathcal{M}) = \{(x;\xi) \in T^*X; \sigma(P)(x;\xi) = 0 \text{ for all } P \in \mathcal{I}\}$$

where $\sigma(P)$ denotes the principal symbol of P. A fundamental result of [29] asserts that for a coherent $\mathcal{D}$-module $\mathcal{M}$, $\mathrm{char}(\mathcal{M})$ is an involutive (*i.e.,* coisotropic) subset of T^*X. The proof uses infinite order microdifferential operators and quantized contact transformations. Of course, the involutivity theorem has a longer history, including the previous work of Guillemin-Quillen-Sternberg [6], and culminating with the purely algebraic proof of Gabber [5].

We denote by $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X)$ the full triangulated subcategory of $\mathrm{D}^{\mathrm{b}}(\mathcal{D}_X)$ consisting of objects with coherent cohomologies. However, we need a refined notion. A $\mathcal{D}_X$-module $\mathcal{M}$ is "good" if on each relatively compact open subset U of X, $\mathcal{M}$ is generated by a coherent $\mathcal{O}_X|_U$-modules. We denote by $\mathrm{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_X)$ the full triangulated subcategory of $\mathrm{D}^{\mathrm{b}}(\mathcal{D}_X)$ consisting of objects with good cohomologies.

Operations on $\mathcal{D}$-modules

Let $f\colon X \to Y$ be a morphism of complex manifolds. The sheaf

$$\mathcal{D}_{X\to Y} := \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y \tag{2.4}$$

is naturally endowed with a structure of a $(\mathcal{D}_X, f^{-1}\mathcal{D}_Y)$-bimodule, but one shall be aware that the left action of $\mathcal{D}_X$ is not simply its action on $\mathcal{O}_X$ (see [9] for details).

The inverse image functor $\mathrm{D}f^{-1}$ for $\mathcal{D}$-modules is given by

$$\mathrm{D}f^{-1}\mathcal{N} = \mathcal{D}_{X\to Y} \overset{\mathrm{L}}{\otimes}_{f^{-1}\mathcal{D}_Y} f^{-1}\mathcal{N}, \quad \mathcal{N} \in \mathrm{D}^{\mathrm{b}}(\mathcal{D}_Y).$$

One says that f is non characteristic for $\mathcal{N} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_Y)$ if (see Notations 0.1)

$$f_\pi^{-1}\,\mathrm{char}(\mathcal{N}) \cap (T^*_XY) \subset X \times_Y T^*_YY.$$

In this case, $\mathrm{D}f^{-1}\mathcal{N} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X)$ and the Cauchy-Kowalevsky-Kashiwara theorem asserts that there is a natural isomorphism

$$f^{-1}\mathrm{Sol}(\mathcal{N}) \xrightarrow{\sim} \mathrm{Sol}(\mathrm{D}f^{-1}\mathcal{N}). \tag{2.5}$$

The proper direct image functor $\mathrm{D}f_!$ for (right) $\mathcal{D}$-modules is given by

$$\mathrm{D}f_!\mathcal{M} := Rf_!(\mathcal{M}\overset{\mathrm{L}}{\otimes}_{\mathcal{D}_X}\mathcal{D}_{X\to Y}), \quad \mathcal{M}\in \mathrm{D}^{\mathrm{b}}(\mathcal{D}_X^{\mathrm{op}}).$$

One defines the direct image functor $\mathrm{D}f_!$ for left $\mathcal{D}$-modules by using the line bundles Ω_X and Ω_Y (or their inverse) which interwine the left and right structures. If $\mathcal{M}\in \mathrm{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_X)$ and f is proper on the support of $\mathcal{M}$, one deduces from Grauert's direct images theorem that $\mathrm{D}f_!\mathcal{M}$ belongs to $\mathrm{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_Y)$. Moreover, there is a natural isomorphism

$$Rf_!\mathrm{Sol}(\mathcal{M})\,[d_X] \simeq \mathrm{Sol}(\mathrm{D}f_!\mathcal{M})\,[d_Y]. \tag{2.6}$$

(Recall that d_X (resp. d_Y) is the complex dimension of X (resp. Y).)

The product $\mathcal{M}\overset{\mathrm{L}}{\otimes}_{\mathcal{O}_X}\mathcal{L}$ of two left $\mathcal{D}_X$-modules is naturally endowed with a structure of a left $\mathcal{D}_X$-module. We denote it by $\mathcal{M}\overset{\mathrm{D}}{\otimes}\mathcal{L}$.

Holonomic systems

Since the characteristic variety of a coherent $\mathcal{D}_X$-module $\mathcal{M}$ is involutive, it is natural to study with a particular attention the extreme case where this characteristic variety has minimal dimension.

Definition 2.1. A holonomic system on X is a coherent $\mathcal{D}_X$-module whose characteristic variety is Lagrangian in T^*X.

Denote by $\mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)$ the full triangulated subcategory $\mathrm{D}^{\mathrm{b}}(\mathcal{D}_X)$ consisting of objects with holonomic cohomology and recall that we set $\mathrm{Sol}(\,\bullet\,) = R\mathcal{H}om_{\mathcal{D}}(\,\bullet\,,\mathcal{O}_X)$. The first fundamental result of the theory of holonomic $\mathcal{D}$-modules is Kashiwara's constructibility theorem:

Theorem 2.2. [10] *The functor* Sol *induces a functor*

$$\mathrm{Sol}\colon (\mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X))^{\mathrm{op}} \to \mathrm{D}^{\mathrm{b}}_{\mathbb{C}-\mathrm{c}}(\mathbb{C}_X).$$

Simple examples in dimension one show that this functor is not fully faithful, but Kashiwara-Oshima [15] and Kashiwara-Kawai [14] gave the definition of a "regular holonomic" $\mathcal{D}$-module and Kashiwara [11] proved the Riemann-Hilbert correspondence, that is, the equivalence of categories

$$(\mathrm{D}^{\mathrm{b}}_{\mathrm{reg-hol}}(\mathcal{D}_X))^{\mathrm{op}} \xrightarrow{\sim} \mathrm{D}^{b}_{\mathbb{C}-c}(\mathbb{C}_X)$$

where now $\mathrm{D}^{\mathrm{b}}_{\mathrm{reg-hol}}(\mathcal{D}_X)$ denotes the full triangulated subcategory of $\mathrm{D}^{\mathrm{b}}(\mathcal{D}_X)$ consisting of objects with regular holonomic cohomology.

Integral transforms

In this subsection, we shall study the action of integral transforms on $\mathcal{D}$-modules and their sheaves of generalized solutions.

Consider two complex manifolds X and Y, of complex dimension d_X and d_Y respectively, and the correspondence:

$$\begin{tikzcd} & X \times Y \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd}$$

For $G \in \mathrm{D}^{\mathrm{b}}(\mathbb{C}_Y)$ and $L \in \mathrm{D}^{\mathrm{b}}(\mathbb{C}_{X\times Y})$, we set

$$L \circ G \;=\; Rf_!(L \otimes g^{-1}G). \tag{2.7}$$

For $\mathcal{M} \in \mathrm{D}^{\mathrm{b}}(\mathcal{D}_X)$ and $\mathcal{L} \in \mathrm{D}^{\mathrm{b}}(\mathcal{D}_{X\times Y})$, we set

$$\mathcal{M} \overset{\mathrm{D}}{\circ} \mathcal{L} \;=\; \mathrm{D}g_!(\mathrm{D}f^{-1}\mathcal{M} \overset{\mathrm{D}}{\otimes} \mathcal{L}). \tag{2.8}$$

Now we consider

$$\mathcal{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_X),\ \mathcal{L} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{reg-hol}}(\mathcal{D}_{X\times Y}),\ G \in \mathrm{D}^{\mathrm{b}}(\mathbb{C}_Y),$$

and we make the hypotheses:

$$\begin{cases} f^{-1}\operatorname{supp}(\mathcal{M}) \cap \operatorname{supp}(\mathcal{L}) \text{ is proper over } Y, \\ (\operatorname{char}(\mathcal{M}) \times T^*_Y Y) \cap \operatorname{char}(\mathcal{L}) \subset T^*_{X\times Y}(X \times Y). \end{cases} \tag{2.9}$$

Theorem 2.3. (See [4].) *Set* $L = R\mathcal{H}om_{\mathcal{D}_{X\times Y}}(\mathcal{O}_{X\times Y}, \mathcal{L})$ *and assume* (2.9). *One has the isomorphism*

$$\mathrm{RHom}_{\mathbb{C}_Y}(G, \mathrm{Sol}(\mathcal{M} \overset{\mathrm{D}}{\circ} \mathcal{L})) \simeq \mathrm{RHom}_{\mathbb{C}_X}(L \circ G, \mathrm{Sol}(\mathcal{M}))\,[d_X - 2d_Y].$$

This result was first stated (in a slightly less general formulation) in [4]. Its proof makes use of the isomorphisms (2.5) and (2.6).

This result admits several variants: one may replace $\mathcal{O}_X$ with its temperate version or formal version (see § 5 below and [17, Ch. 7]), there are twisted versions and also G-equivariant versions (see [13]).

Many applications of this theorem are exposed in [3], in particular to projective duality, Penrose transform and, more generally, Grasmanniann correspondences.

Remark 2.4. The kernel $\mathcal{L}$ appearing in Theorem 2.3 is regular holonomic. However, irregular kernels may naturally appear which makes the situation much more intricate. The Laplace transform is such an example of irregular kernel (see [20]).

3 Microsupport

References for this section are made to [16].

Let X denote a *real* manifold of class C^∞ and let $F \in \mathrm{D^b}(\mathbf{k}_X)$. The microsupport $\mathrm{SS}(F)$ of F is the closed conic subset of T^*X defined as follows.

Definition 3.1. Let U be an open subset of T^*X. Then $U \cap \mathrm{SS}(F) = \emptyset$ if and only if for any $x_0 \in X$ and any real C^∞-function $\varphi\colon X \to \mathbb{R}$ such that $\varphi(x_0) = 0, d\varphi(x_0) \in U$, one has:

$$(\mathrm{R}\Gamma_{\varphi \geq 0}(F))_{x_0} = 0.$$

In other words, F has no cohomology supported by the closed half spaces whose conormals do not belong to its microsupport. One again proves that the microsupport is a closed conic *involutive* subset of T^*X.

Assume now that X is a complex manifold, that we identify with its real underlying manifold. The link between the microsupport of sheaves and the characteristic variety of coherent $\mathcal{D}$-modules is given by:

Theorem 3.2. *Let $\mathcal{M}$ be a coherent $\mathcal{D}$-module. Then* $\mathrm{SS}(\mathrm{Sol}(\mathcal{M})) = \mathrm{char}(\mathcal{M})$.

The inclusion $\mathrm{SS}(\mathrm{Sol}(\mathcal{M})) \subset \mathrm{char}(\mathcal{M})$ is the most useful in practice. Its proof only makes use of the Cauchy-Kowalevsky theorem in its precise form given by Leray (see [23] or [7, § 9.4]) and of purely algebraic arguments.

Now consider a space of generalized functions $R\mathcal{H}om\,(G, \mathcal{O}_X)$ associated with an $\mathbb{R}$-constructible sheaf G, and a system of linear partial differential equations, that is, a coherent $\mathcal{D}_X$-module $\mathcal{M}$. The complex of generalized functions solution of this system is given by the complex

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, R\mathcal{H}om\,(G, \mathcal{O}_X)) \simeq R\mathcal{H}om\,(G, \mathrm{Sol}(\mathcal{M})).$$

Setting $F = \mathrm{Sol}(\mathcal{M})$, we are reduced to study $R\mathcal{H}om\,(G, F)$. Our only information is now purely geometrical, this is the microsupport of G and that of F (this last one being the characteristic variety of $\mathcal{M}$). Now, we can forget that we are working on a complex manifold and that we are dealing with partial differential equations. We are reduced to the microlocal study of sheaves on a real manifold [16].

Application: ellipticity

Let us show how the classical Petrowsky regularity theorem may be obtained with the only use of the Cauchy-Kowalevsky-Leray Theorem, and some sheaf theory.

The regularity theorem for sheaves is as follows.

Theorem 3.3. *Let X be a real analytic manifold and let $F, G \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$. Assume that G is $\mathbb{R}$-constructible and $\mathrm{SS}(G) \cap \mathrm{SS}(F) \subset T^*_X X$. Then the natural morphism*

$$R\mathcal{H}om\,(G, \mathbf{k}_X) \otimes F \to R\mathcal{H}om\,(G, F) \tag{3.1}$$

is an isomorphism.

Let us come back to the situation where X is a complexification of a real manifold M and choose $\mathbf{k} = \mathbb{C}$. Set $G = \mathrm{D}'_X(\mathbb{C}_M)$ and $F = R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$. A differential operator P on X is elliptic (with respect to M) if its principal symbol $\sigma(P)$ does not vanish on the conormal bundle $T^*_M X$, outside of the zero-section. More generally a coherent $\mathcal{D}_X$-module $\mathcal{M}$ is elliptic with respect to M if

$$\mathrm{char}(\mathcal{M}) \cap T^*_M X \subset T^*_X X.$$

By Theorem 3.2, $\mathrm{SS}(F) \cap T^*_M X \subset T^*_X X$. Let $\mathcal{A}_M = \mathbb{C}_M \otimes \mathcal{O}_X$ denotes the sheaf of analytic functions on M. The regularity theorem for sheaves gives the isomorphism

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{A}_X) \xrightarrow{\sim} R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_X).$$

In other words, the two complexes of real analytic and hyperfunction solutions of an elliptic system are quasi-isomorphic. This is the Petrowsky's theorem for $\mathcal{D}$-modules.

Of course, this result extends to other sheaves of generalized holomorphic functions, replacing the constant sheaf $\mathbb{C}_M$ with an $\mathbb{R}$-constructible sheaf G. For further developments, see [31].

4 Microlocal analysis

For a detailed exposition, see [16].

Fourier-Sato transform

Let $\tau \colon E \to M$ be a finite dimensional real vector bundle over a real manifold M with fiber dimension n, $\pi \colon E^* \to M$ the dual vector bundle. Denote by p_1 and p_2 the first and second projection defined on $E \times_M E^*$, and define:

$$P = \{(x, y) \in E \times_M E^*; \langle x, y \rangle \geq 0\}$$
$$P' = \{(x, y) \in E \times_M E^*; \langle x, y \rangle \leq 0\}$$

Consider the diagram:

$$\begin{tikzcd}[column sep=small]
 & E\times_M E^* \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
E \arrow[dr, "\tau"'] & & E^* \arrow[dl, "\pi"] \\
 & M &
\end{tikzcd}$$

Denote by $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathbf{k}_E)$ the full triangulated subcategory of $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_E)$ consisting of conic objects, that is, objects F such that for all j, $H^j(F)$ is locally constant on the orbits of the action of $\mathbb{R}^+$ on E. One defines the two functors

$$\mathrm{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathbf{k}_E) \underset{(\bullet)^\vee}{\overset{(\bullet)^\wedge}{\rightleftarrows}} \mathrm{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathbf{k}_{E^*})$$

by setting for $F \in \mathrm{D}_{\mathbb{R}^+}(\mathbf{k}_E)$ and $G \in \mathrm{D}_{\mathbb{R}^+}(\mathbf{k}_{E^*})$

$$F^\wedge = Rp_{2!}(p_1^{-1}F)_{P'}, \quad G^\vee = Rp_{1!}(p_2^!G)_P.$$

The main result of the theory is the following.

Theorem 4.1. *The two functors $(\bullet)^\wedge$ and $(\bullet)^\vee$ are equivalences of categories inverse to each other. In particular, for F_1 and F_2 in $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathbf{k}_E)$, there is a natural isomorphism:*

$$\mathrm{RHom}\,(F_1, F_2) \simeq \mathrm{RHom}\,(F_1^\wedge, F_2^\wedge). \tag{4.1}$$

Example 4.2. (i) For a convex cone γ in E, denote $\mathrm{Int}\gamma$ its interior, by $\gamma^a = -\gamma$ its image by the antipodal map and by γ° its polar cone in E^*. Then, for a closed convex proper cone γ with $M \subset \gamma$,

$$(\mathbf{k}_\gamma)^\wedge \simeq \mathbf{k}_{\mathrm{Int}\gamma^\circ}.$$

For an open convex cone λ in E:

$$(\mathbf{k}_\lambda)^\wedge \simeq \mathbf{k}_{\lambda^{\circ a}} \otimes \mathrm{or}_{E^*/M}\,[-n].$$

(ii) Let E denote the Euclidian space $\mathbb{R}^n$ and let $x = (x_1, \dots, x_n)$ be the coordinates. Let $n = p + q$ with $p, q \geq 1$ and set $x = (x', x'')$ with $x' = (x_1, \dots, x_p)$, $x'' = (x_{p+1}, \dots, x_n)$. We denote by $u = (u', u'')$ the dual coordinates on the dual space E^*.

Let γ denote the closed solid cone in E,

$$\gamma = \{x; {x'}^2 - {x''}^2 \geq 0\}$$

and let λ denote the closed solid cone in E^*:

$$\lambda = \{u; u'^2 - u''^2 \leq 0\}.$$

We have (see [20]):

$$\mathbf{k}_\gamma^\wedge \simeq \mathbf{k}_\lambda\,[-p].$$

Specialization and microlocalization

Let X be a *real* manifold (say of class $\mathcal{C}^\infty$), M a closed submanifold. Denote by $\tau\colon T_MX \to M$ and $\pi\colon T^*_MX \to M$ the normal bundle and the conormal bundle to M in X, respectively. Let $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$. The specialization of F along M, denoted $\nu_M(F)$, is an object of $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathbf{k}_{T_MX})$. Its cohomology objects are described as follows. If V is an open cone in T_MX, then

$$H^j(V;\nu_M(F)) \simeq \varinjlim_U H^j(U;F)$$

where U ranges over the family of open subsets of X which are "tangent" to V, that is, open tuboids in X with wedge M whose "profiles" is V. (For a precise definition, refer to [16, § 4.2].)

The Sato's microlocalization of F along M, denoted $\mu_M(F)$, is the Fourier-Sato transform of $\nu_M(F)$, an object of $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathbf{k}_{T^*_MX})$. It satisfies:

$$\begin{aligned}
&R\pi_*\mu_M(F) \simeq \mathrm{R}\Gamma_M(F),\\
&H^j(\mu_M(F))_{(x_0;\xi_0)} \simeq \varinjlim_{U,Z} H^j_{U\cap Z}(U;F),
\end{aligned}$$

where, in the last formula, $(x_0;\xi_0) \in T^*_MX$, U ranges over the family of open neighborhoods of x_0 in X and Z ranges over the family of closed tuboids in X with wedge M whose profiles λ in T_MX satisfy $(x_0;\xi_0) \in \mathrm{Int}\lambda^{\circ a}$.

Using the diagonal Δ of $X \times X$, one defines the bifunctor μhom by setting:

$$\mu hom\,(G,F) = \mu_\Delta R\mathcal{H}om\,(q_2^{-1}G, q_1^!F)$$

where q_i $(i=1,2)$ denotes the i-th projection on $X \times X$. Note that

$$\begin{aligned}
&R\pi_*\mu hom\,(G,F) \simeq R\mathcal{H}om\,(G,F),\\
&\mu hom\,(\mathbf{k}_M,F) \simeq \mu_M(F),\\
&\mathrm{supp}\,\mu hom\,(G,F) \subset \mathrm{SS}(G) \cap \mathrm{SS}(F).
\end{aligned}$$

Now assume that M is a real analytic manifold and X is a complexification of M. The sheaf of Sato's microfunctions on $T^*_M X$ is defined by:

$$\mathcal{C}_M = \mu hom\,(\mathrm{D}'_X \mathbb{C}_M, \mathcal{O}_X).$$

(It is proved that this complex is concentrated in degree 0.) Hence, a hyperfunction is nothing but a microfunction globally defined on $T^*_M X$. Denote by spec the natural isomorphism:

$$\mathrm{spec}\colon \mathcal{B}_M \xrightarrow{\sim} \pi_* \mathcal{C}_M.$$

If u is an hyperfunction, Sato defines its analytic wave front set as:

$$\mathrm{WF}(u) = \mathrm{supp}(\mathrm{spec}(u)),$$

a closed conic subset of $T^*_M X$. As an application, consider the situation of the construction of the boundary value morphism in (1.2). Let $\varphi \in \mathcal{O}(\Omega)$ and assume that φ is solution of a system of differential equations, that is, $\varphi \in \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$ for a coherent $\mathcal{D}_X$-module $\mathcal{M}$. Set $F_0 = \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$. Since the boundary value morphism factorizes through $\mu hom\,(\mathbb{C}_\Omega, F_0)$ and $SS(F_0) \subset \mathrm{char}(\mathcal{M})$, we get

$$\mathrm{WF}(\mathrm{b}(\varphi)) \subset T^*_M X \cap \mathrm{SS}(\mathbb{C}_\Omega) \cap \mathrm{char}(\mathcal{M}). \tag{4.2}$$

Remark 4.3. A new microlocalization functor μ has been constructed in [21], taking its values in the category of ind-sheaves on T^*X. It is related to the functor μhom by the formula

$$\mu hom\,(G, F) \simeq R\mathcal{H}om\,(\pi^{-1}G, \mu(F)).$$

Microdifferential operators

The sheaf of microfunctions allows us to analyze hyperfunctions microlocally, that is, in the cotangent bundle. A similar localization may be performed with respect to the sheaf of differential operators. Let X be a complex manifold of dimension d_X and denote by Δ the diagonal of $X \times X$. The sheaf of microlocal operators is defined in [29] by

$$\mathcal{E}^{\mathbb{R}}_X := \mu_\Delta(\mathcal{O}^{(0,d_X)}_{X\times X})\,[d_X].$$

Here, $\mathcal{O}^{(0,d_X)}_{X\times X} = \mathcal{O}_{X\times X} \otimes_{q_2^{-1}\mathcal{O}_X} q_2^{-1}\Omega_X$. One proves that $\mathcal{E}^{\mathbb{R}}_X$ is concentrated in degree 0 and is naturally endowed with a structure of a sheaf of

$\mathbb{C}$-algebras. Moreover, for $G \in \mathrm{D}^{\mathrm{b}}(\mathbb{C}_X)$, the object $\mu hom(G, \mathcal{O}_X)$ is well defined in $\mathrm{D}^{\mathrm{b}}(\mathcal{E}^{\mathbb{R}}_X)$ (see [21]). This applies in particular to the sheaf $\mathcal{C}_M$ of microfunctions.

The algebra $\mathcal{E}^{\mathbb{R}}_X$ is extremely difficult to manipulate, but it contains the $\mathbb{C}$-algebra $\mathcal{E}_X$ of microdifferential operators which is filtered and admits a symbol calculus. When X is affine and U is open in T^*X, a section $P \in \mathcal{E}_{T^*X}(U)$ is described by its "total symbol"

$$\sigma_{\mathrm{tot}}(P)(x;\xi) = \sum_{-\infty<j\leq m} p_j(x;\xi),\ m \in \mathbb{Z}, \quad p_j \in \Gamma(U; \mathcal{O}_{T^*X}(j)),$$

with the condition:

$$\begin{cases} \text{for any compact subset } K \text{ of } U \text{ there exists a positive constant} \\ C_K \text{ such that } \sup\limits_K |p_j| \leq C_K^{-j}(-j)! \text{ for all } j<0. \end{cases}$$

(Here $\mathcal{O}_{T^*X}(j)$ denotes the subsheaf of $\mathcal{O}_{T^*X}$ of functions homogeneous of degree j in the fiber variable.) The total symbol of the product is given by the Leibniz rule. If Q is an operator of total symbol $\sigma_{\mathrm{tot}}(Q)$, then

$$\sigma_{\mathrm{tot}}(P \circ Q) \quad = \quad \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!} \partial^\alpha_\xi \sigma_{\mathrm{tot}}(P) \partial^\alpha_x \sigma_{\mathrm{tot}}(Q).$$

If $\varphi\colon T^*X \supset U \xrightarrow{\sim} V \subset T^*Y$ is a homogeneous complex symplectic isomorphism, it can be locally "quantized" (see [29]), that is, extended as an isomorphism of algebras

$$\Phi\colon \varphi_*\mathcal{E}_{T^*X} \xrightarrow{\sim} \mathcal{E}_{T^*Y}.$$

However, this isomorphism Φ is only locally defined and not unique. That is why, in general, it is not possible to define such sheaves $\mathcal{E}_{\mathfrak{X}}$ of microdifferential operators on a homogeneous complex symplectic manifold $\mathfrak{X}$. However, Kashiwara [12] has shown that such a construction was possible when weakening the notion of a sheaf of algebras by that of an algebroid stack. (Refer to [19] for an introduction to stacks.)

Further developments and open problem

A complex cotangent bundle T^*X is endowed with the sheaf $\mathcal{E}_{T^*X}$. This sheaf is conic for the action of $\mathbb{C}^\times$ on T^*X and one can eliminate this homogeneity by adding an extra central variable $\hbar$. One gets the algebra $\mathcal{W}_{T^*X}$ over a field $\mathbf{k}$, a subfield of $\widehat{\mathbf{k}} = \mathbb{C}((\hbar))$. The algebra $\mathcal{W}_{T^*X}$ admits a formal version $\widehat{\mathcal{W}}_{T^*X}$ over $\widehat{\mathbf{k}}$ which is extremely popular and known as a deformation-quantization algebra. If X is affine, a section $P \in \widehat{\mathcal{W}}_{T^*X}(U)$ on an open subset $U \subset T^*X$ is a series, called its total symbol:

$$\sigma_{\mathrm{tot}}(P)(x;u,\hbar) = \sum_{m \leq j < \infty} p_j(x;u)\hbar^j,\ m \in \mathbb{Z}, \quad p_j \in \mathcal{O}_{T^*X}(U), \tag{4.3}$$

and the total symbol of the product $P \circ Q$ is given by the Leibniz rule:

$$\sigma_{\mathrm{tot}}(P \circ Q) \quad = \quad \sum_{\alpha \in \mathbb{N}^n} \frac{\hbar^{|\alpha|}}{\alpha!} \partial_u^\alpha(\sigma_{\mathrm{tot}}P)\partial_x^\alpha(\sigma_{\mathrm{tot}}Q).$$

Note that $\widehat{\mathcal{W}}_{T^*X}$ has a natural filtration for which $\hbar$ has order -1 and $\widehat{\mathcal{W}}_{T^*X}(0)$ may naturally be considered as a deformation of the Poisson algebra $\mathcal{O}_{T^*X}$.

By replacing again the notion of a sheaf of algebras by that of an algebroid stack, one can define such objects on complex symplectic manifolds (see [22, 27]).

A natural question would be to perform an analogous construction for sheaves on real manifolds, that is, to construct a non conic microlocal theory of sheaves on real symplectic manifolds. This problem is closely related to Mirror Symmetry (see [25]).

5 The use of Grothendieck topologies

References for this section are made to [17, Ch. 7].

Let X be a *real* analytic manifold. The usual topology on X does not allow one to treat usual spaces of analysis with the tools of sheaf theory. For example, the property of being temperate is not local, and there is no sheaf of temperate distributions. One way to overcome this difficulty is to introduce a Grothendieck topology on X. Recall that a Grothendieck topology is not a topology, and in fact is not defined on a space but on a category. The objects of the category playing the role of the open subsets of the space, it is an axiomatization of the notion of a covering. A site is a category endowed with a Grothendieck topology.

We denote by Op_X the category whose objects are the open subsets of X and the morphisms are the inclusions of open subsets. One defines a Grothendieck topology on Op_X by deciding that a family $\{U_i\}_{i\in I}$ of subobjects of $U \in \mathrm{Op}_X$ is a covering of U if it is a covering in the usual sense.

Definition 5.1. Denote by $\mathrm{Op}_{X_{\mathrm{sa}}}$ the full subcategory of Op_X consisting of subanalytic and relatively compact open subsets. The site X_{sa} is obtained by deciding that a family $\{U_i\}_{i\in I}$ of subobjects of $U \in \mathrm{Op}_{X_{\mathrm{sa}}}$ is a covering of U if there exists a finite subset $J \subset I$ such that $\bigcup_{j\in J} U_j = U$.

Let us denote by

$$\rho\colon X \to X_{\mathrm{sa}} \tag{5.1}$$

the natural morphism of sites. As usual, we have a pair of adjoint functors (ρ^{-1}, ρ_*):

$$\mathrm{Mod}(\mathbf{k}_X) \underset{\rho^{-1}}{\overset{\rho_*}{\rightleftarrows}} \mathrm{Mod}(\mathbf{k}_{X_{\mathrm{sa}}}).$$

The functor ρ^{-1} also admits a right adjoint $\rho_!$. For $F \in \mathrm{Mod}(\mathbf{k}_X)$, $\rho_! F$ is the sheaf associated to the presheaf $U \mapsto F(\overline{U})$, $U \in \mathrm{Op}_{X_{\mathrm{sa}}}$.

Let us denote by $\mathcal{C}^\infty_X$ the sheaf of complex valued $\mathcal{C}^\infty$-functions on X.

Definition 5.2. Let $f \in \mathcal{C}^\infty_X(U)$. One says that f has *polynomial growth* at $p \in X$ if it satisfies the following condition. For a local coordinate system $(x_1, \dots, x_n)$ around p, there exist a sufficiently small compact neighborhood K of p and a positive integer N such that

$$\sup_{x\in K\cap U} \big(\mathrm{dist}(x, K\setminus U)\big)^N |f(x)| < \infty\,. \tag{5.2}$$

It is obvious that f has polynomial growth at any point of U. We say that f is temperate at p if all its derivatives have polynomial growth at p. We say that f is temperate if it is temperate at any point.

For an open subanalytic subset U of X, denote by $\mathcal{C}^{\infty,t}_X(U)$ the subspace of $\mathcal{C}^\infty_X(U)$ consisting of temperate functions.

Using Lojasiewicz's inequalities [24], one easily proves that the presheaf $\mathcal{C}^{\infty,t}_{X_{\mathrm{sa}}}$, given by $U \mapsto \mathcal{C}^{\infty,t}_X(U)$, is a sheaf on X_{sa}. One calls it the sheaf of temperate $\mathcal{C}^\infty$-functions on X_{sa}. Note that the sheaf $\rho_*\mathcal{D}_X$ does not operate on $\mathcal{C}^{\infty,t}_{X_{\mathrm{sa}}}$ but $\rho_!\mathcal{D}_X$ does.

Now let X be a *complex* manifold. We still denote by X the real underlying manifold and we denote by $\overline{X}$ the complex manifold conjugate to X. One defines the sheaf of temperate holomorphic functions $\mathcal{O}^t_{X_{\mathrm{sa}}}$ as the Dolbeault complex with coefficients in $\mathcal{C}^{\infty,t}_{X_{\mathrm{sa}}}$. More precisely

$$\mathcal{O}^t_{X_{\mathrm{sa}}} = R\mathcal{H}om_{\rho_!\mathcal{D}_{\overline{X}}}(\rho_!\mathcal{O}_{\overline{X}}, \mathcal{C}^{\infty,t}_{X_{\mathrm{sa}}}). \tag{5.3}$$

Note that the object $\mathcal{O}^t_{X_{\mathrm{sa}}} \in \mathrm{D}^{\mathrm{b}}(\rho_! \mathcal{D}_X)$ is not concentrated in degree zero in dimension > 1. Nevertheless, it should have many important applications. Let us mention two of them:

(a) One proves that Sato's construction of hyperfunctions, when applied to $\mathcal{O}^t_{X_{\mathrm{sa}}}$, gives the sheaf of Schwartz's distributions.

(b) Very little is known on irregular holonomic $\mathcal{D}$-modules (see [26]) in dimension higher than one, and even in dimension one, there is no (to our opinion) totally satisfactory results. In [18], one calculates the temperate holomorphic solutions of the $\mathcal{D}_X$-module $\mathcal{M} := \mathcal{D}_X \exp(1/z) = \mathcal{D}_X/\mathcal{D}_X(z^2\partial_z + 1)$, where $X = \mathbb{C}$. The result obtained shows that the sheaf of temperate holomorphic solutions gives more informations than the classical sheaf of holomorphic solutions.

The sheaf $\mathcal{O}^t_{X_{\mathrm{sa}}}$ is simply an example which shows that the methods of Algebraic Analysis may be applied to treat generalized functions with growth conditions.

References

[1] E. Andronikof, *Interview with Mikio Sato,* Notices of the AMS, **54** Vol 2, (2007).

[2] I. Bernstein *Modules over a ring of differential operators,* Funct. Analysis and Appl. **5** pp 89–101 (1971).

[3] A. D'Agnolo, *Sheaves and D-modules in integral geometry,* in Contemp. Math., **251** Amer. Math. Soc., pp 141 (2000).

[4] A. D'Agnolo and P. Schapira, *Leray's quantization of projective duality,* Duke Math. Journ. **84** pp 453–496 (1996).

[5] O. Gabber, *The integrability of the characteristic variety,* Amer. Journ. Math. **103** pp 445–468 (1981).

[6] V. Guillemin, D. Quillen and S. Sternberg, *The integrability of characteristics,* Comm. Pure and Appl. Math. **23** 39–77 (1970)

[7] L. Hörmander, *The analysis of linear partial differential operators,* Grundlehren der Math. Wiss. **256** Springer-Verlag (1983).

[8] M. Kashiwara, *Algebraic study of systems of partial diffential equations,* Thesis, Tokyo Univ. (1970), translated by A. D'Agnolo and J-P. Schneiders, Mémoires Soc. Math. France **63** (1995).

[9] ———, *D-modules and Microlocal Calculus,* Translations of Mathematical Monographs, **217** American Math. Soc. (2003).

[10] ———, *On the maximally overdetermined systems of linear differential equations I,* Publ. RIMS, Kyoto Univ. **10** pp 563–579 (1975).

[11] ———, *The Riemann-Hilbert problem for holonomic systems,* Publ. RIMS, Kyoto Univ. **20** pp 319–365 (1984).

[12] ———, *Quantization of contact manifolds,* Publ. RIMS, Kyoto Univ. **32** p. 1–5 (1996).

[13] ———, *Equivariant derived category and representation of real semisimple Lie groups.* Representation theory and complex analysis, Lecture Notes in Math. Springer-Verlag, to appear

[14] M. Kashiwara and T. Kawai, *On the holonomic systems of microdifferential equations III,* Publ. RIMS, Kyoto Univ. **17** pp 813–979 (1981).

[15] M. Kashiwara and T. Oshima, *Systems of differential equations with regular singularities and their boundary value problems*, Ann. of Math. **106** (1977), p. 145–200.

[16] M. Kashiwara and P. Schapira, *Sheaves on Manifolds,* Grundlehren der Math. Wiss. **292** Springer-Verlag (1990).

[17] ———, *Ind-sheaves,* Astérisque **271** Soc. Mathématique de France (2001).

[18] ———, *Microsupport and regularity for ind-sheaves I,* Astérique **284** pp 143–164 (2003). ArXiv: math.AG/0108065

[19] ———, *Categories and Sheaves,* Grundlehren der Math. Wiss. Springer-Verlag (2005).

[20] ———, *Integral transforms with exponential kernels and Laplace transform,* Journ. of the A.M.S. **10** pp 939–972 (1997).

[21] M. Kashiwara, P. Schapira, F. Ivorra and I. Waschkies, *Microlocalization of indsheaves.* in Studies in Lie Theory, Progress in Math. **243** pp 171–221 (2006).

[22] M. Kontsevich, *Deformation quantization of algebraic varieties,* in: EuroConférence Moshé Flato, Part III (Dijon, 2000) Lett. Math. Phys. **56** (3) p. 271–294 (2001).

[23] J. Leray, *Scientific work,* Springer-Verlag & Soc. Mathématique de France (1997).

[24] S. Lojaciewicz, *Sur le problème de la division,* Studia Math. **8** pp. 87–156, (1961).

[25] D. Nadler and E. Zaslow, *Constructible Sheaves and the Fukaya Category,* `arXiv:math/0604379`

[26] C. Sabbah, *Equations différentielles à points singuliers irréguliers et phénomène de Stokes en dimension* 2*,* Astérisque **263** Soc. Mathématique de France (2000).

[27] P. Polesello and P. Schapira, *Stacks of quantization-deformation modules over complex symplectic manifolds,* Int. Math. Res. Notices **49** p. 2637–2664 (2004).

[28] M. Sato, *Theory of hyperfunctions,* I & II Journ. Fac. Sci. Univ. Tokyo, **8** 139–193 487–436 (1959–1960).

[29] M. Sato, T. Kawai, and M. Kashiwara, *Microfunctions and pseudo-differential equations,* in Komatsu (ed.), *Hyperfunctions and pseudo-differential equations,* Proceedings Katata 1971, Lecture Notes in Math. Springer-Verlag **287** pp. 265–529 (1973).

[30] P. Schapira, *Mikio Sato, a visionary of mathematics,* Notices of the AMS, **54** Vol 2, (2007).

[31] P. Schapira and J-P. Schneiders, *Index theorem for elliptic pairs,* Astérisque **224** Soc. Mathématique de France (1994).

Pierre Schapira
Université Pierre et Marie Curie
Institut de Mathématiques
175, rue du Chevaleret, 75013 Paris France
schapira@math.jussieu.fr
http://www.math.jussieu.fr/˜schapira/