Constructibility and duality for simple holonomic modules on complex symplectic manifolds

Masaki Kashiwara and Pierre Schapira

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Abstract

Consider a complex symplectic manifold $\mathfrak{X}$ and the algebroid stack $\mathcal{W}_{\mathfrak{X}}$ of deformation-quantization. For two regular holonomic $\mathcal{W}_{\mathfrak{X}}$-modules $\mathcal{L}_i$ $(i = 0, 1)$ supported by smooth Lagrangian submanifolds, we prove that the complex $R\mathcal{H}om_{\mathcal{W}_{\mathfrak{X}}}(\mathcal{L}_1, \mathcal{L}_0)$ is perverse over the field $\mathcal{W}_{\mathrm{pt}}$ and dual to the complex $R\mathcal{H}om_{\mathcal{W}_{\mathfrak{X}}}(\mathcal{L}_0, \mathcal{L}_1)$.

Introduction

Consider a complex symplectic manifold $\mathfrak{X}$. A local model for $\mathfrak{X}$ is an open subset of the cotangent bundle T^*X to a complex manifold X, and T^*X is endowed with the filtered sheaf of rings $\mathcal{W}_{T^*X}$ of deformation-quantization. This sheaf of rings is similar to the sheaf $\mathcal{E}_{T^*X}$ of microdifferential operators of Sato-Kawai-Kashiwara [18], but with an extra central parameter τ, a substitute to the lack of homogeneity (see § 1). This is an algebra over the field $\mathbf{k} = \mathcal{W}_{\mathrm{pt}}$, a subfield of the field of formal Laurent series $\mathbb{C}[[\tau^{-1}, \tau]$.

It would be tempting to glue the locally defined sheaves of algebras $\mathcal{W}_{T^*X}$ to give rise to a sheaf on $\mathfrak{X}$, but the procedure fails, and one is lead to replace the notion of a sheaf of algebras by that of an "algebroid stack". A canonical algebroid stack $\mathcal{W}_{\mathfrak{X}}$ on $\mathfrak{X}$, locally equivalent to $\mathcal{W}_{T^*X}$, has been constructed by Polesello-Schapira [16] after Kontsevich [15] had treated the formal case (in the general setting of Poisson manifolds) by a different method.

In this paper, we study regular holonomic $\mathcal{W}_{\mathfrak{X}}$-modules supported by smooth Lagrangian submanifolds of $\mathfrak{X}$. For example, a regular holonomic module along the zero-section T^*_XX of T^*X is locally isomorphic to a finite sum of copies of the sheaf $\mathcal{O}^\tau_X$ whose sections are series $\sum_{-\infty<j\leq m} f_j\tau^j$

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($m \in \mathbb{Z}$), where the f_j's are sections of $\mathcal{O}_X$ and satisfy certain growth conditions.

Denote by $\mathrm{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathcal{W}_{\mathfrak{X}})$ the full subcategory of the bounded derived category of $\mathcal{W}_{\mathfrak{X}}$-modules on $\mathfrak{X}$ consisting of objects with regular holonomic cohomologies. The main theorem of this paper asserts that if $\mathcal{L}_i$ ($i = 0, 1$) are objects of $\mathrm{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathcal{W}_{\mathfrak{X}})$ supported by smooth Lagrangian manifolds Λ_i and if one sets $F := R\mathcal{H}om_{\mathcal{W}_{\mathfrak{X}}}(\mathcal{L}_1, \mathcal{L}_0)$, then F is $\mathbb{C}$-constructible over the field $\mathbf{k}$, its microsupport is contained in the normal cone $C(\Lambda_0, \Lambda_1)$ and F is dual over $\mathbf{k}$ to the object $R\mathcal{H}om_{\mathcal{W}_{\mathfrak{X}}}(\mathcal{L}_0, \mathcal{L}_1)$. If $\mathcal{L}_0$ and $\mathcal{L}_1$ are concentrated in degree 0, then F is perverse. We make the conjecture that the hypothesis on the smoothness of the Λ_i's may be removed.

The strategy of our proof is as follows.

First, assuming only that $\mathcal{L}_0$ and $\mathcal{L}_1$ are coherent, we construct the canonical morphism

$$R\mathcal{H}om_{\mathcal{W}_{\mathfrak{X}}}(\mathcal{L}_1, \mathcal{L}_0) \to \mathrm{D}_{\mathfrak{X}}\big(R\mathcal{H}om_{\mathcal{W}_{\mathfrak{X}}}(\mathcal{L}_0, \mathcal{L}_1\,[\dim_{\mathbb{C}} \mathfrak{X}])\big), \tag{0.1}$$

where $\mathrm{D}_{\mathfrak{X}}$ is the duality functor for sheaves of $\mathbf{k}_{\mathfrak{X}}$-modules. By using a kind of Serre's duality for the sheaf $\mathcal{O}^{\tau}_X$, we prove that (0.1) is an isomorphism as soon as $R\mathcal{H}om_{\mathcal{W}_{\mathfrak{X}}}(\mathcal{L}_0, \mathcal{L}_1)$ is constructible. For that purpose, we need to develop some functional analysis over nuclear algebras, in the line of Houzel [6].

In order to prove the constructibility result for $F = R\mathcal{H}om_{\mathcal{W}_{\mathfrak{X}}}(\mathcal{L}_1, \mathcal{L}_0)$, we may assume that $\mathfrak{X} = T^*X$, $\Lambda_0 = T^*_X X$, $\mathcal{L}_0 = \mathcal{O}^{\tau}_X$ and Λ_1 is the graph of the differential of a holomorphic function φ defined on X. Consider the sheaf of rings $\mathcal{D}_X[\tau^{-1}] = \mathcal{D}_X \otimes \mathbb{C}[\tau^{-1}]$. We construct a coherent $\mathcal{D}_X[\tau^{-1}]$-module $\mathcal{M}$ which generates $\mathcal{L}_1$ and, setting $F_0 := R\mathcal{H}om_{\mathcal{D}_X[\tau^{-1}]}(\mathcal{M}, \mathcal{O}^{\tau}_X(0))$, we prove that the microsupport of F_0 is a closed complex analytic Lagrangian subset of T^*X contained in $C(\Lambda_0, \Lambda_1)$. Using a deformation argument as in [7] and some functional analysis (extracted from [6]) over the base ring $\mathbf{k}_0 = \mathcal{W}_{\mathrm{pt}}(0)$, we deduce that the fibers of the cohomology of F_0 are finitely generated, and the result follows from the isomorphism $F \simeq F_0 \otimes_{\mathbf{k}_0} \mathbf{k}$.

In this paper, we also consider a complex compact contact manifold $\mathfrak{Y}$ and the algebroid stack $\mathcal{E}_{\mathfrak{Y}}$ of microdifferential operators on it. Denote by $\mathrm{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathcal{E}_{\mathfrak{Y}})$ the $\mathbb{C}$-triangulated category consisting of $\mathcal{E}_{\mathfrak{Y}}$-modules with regular holonomic cohomology. Using results of Kashiwara-Kawai [10], we prove that this category has finite Ext, admits a Serre functor in the sense of Bondal-Kapranov [2] and this functor is nothing but a shift by $\dim_{\mathbb{C}} \mathfrak{Y} + 1$. In other words, $\mathrm{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathcal{E}_{\mathfrak{Y}})$ is a Calabi-Yau category. Similar results hold over $\mathbf{k}$ for a complex compact symplectic manifold $\mathfrak{X}$.

1 $\mathcal{W}$-modules on T^*X

Let X be a complex manifold, $\pi\colon T^*X \to X$ its cotangent bundle. The homogeneous symplectic manifold T^*X is endowed with the $\mathbb{C}^\times$–conic $\mathbb{Z}$-filtered sheaf of rings $\mathcal{E}_{T^*X}$ of finite–order microdifferential operators, and its subring $\mathcal{E}_{T^*X}(0)$ of operators of order ≤ 0 constructed in [18] (with other notations). We assume that the reader is familiar with the theory of $\mathcal{E}$-modules, referring to [9] or [19] for an exposition.

On the symplectic manifold T^*X there exists another (no more conic) sheaf of rings $\mathcal{W}_{T^*X}$, called ring of deformation-quantization by many authors (see Remark 1.2 below). The study of the relations between the ring $\mathcal{E}_{T^*X}$ on a complex homogeneous symplectic manifold and the sheaf $\mathcal{W}_{T^*X}$ on a complex symplectic manifold is systematically performed in [16], where it is shown in particular how quantized symplectic transformations act on $\mathcal{W}_{T^*X}$-modules. We follow here their presentation.

Let $t \in \mathbb{C}$ be the coordinate and set

$$\mathcal{E}_{T^*(X\times\mathbb{C}),\hat{t}} = \left\{P \in \mathcal{E}_{T^*(X\times\mathbb{C})}\,;\,[P,\partial_t] = 0\right\}.$$

Set $T^*_{\tau\neq 0}(X\times\mathbb{C}) = \{(x,t;\xi,\tau) \in T^*(X\times\mathbb{C})\,;\,\tau \neq 0\}$, and consider the map

$$\rho\colon T^*_{\tau\neq 0}(X\times\mathbb{C}) \to T^*X \tag{1.1}$$

given in local coordinates by $\rho(x,t;\xi,\tau) = (x;\xi/\tau)$. The ring $\mathcal{W}_{T^*X}$ on T^*X is given by

$$\mathcal{W}_{T^*X} := \rho_*(\mathcal{E}_{X\times\mathbb{C},\hat{t}}|_{T^*_{\tau\neq 0}(X\times\mathbb{C})}).$$

In a local symplectic coordinate system $(x;u)$ on T^*X, a section P of $\mathcal{W}_{T^*X}$ on an open subset U is written as a formal series, called its total symbol:

$$\sigma_{\mathrm{tot}}(P) = \sum_{-\infty<j\leq m} p_j(x;u)\tau^j,\ m \in \mathbb{Z} \quad p_j \in \mathcal{O}_{T^*X}(U), \tag{1.2}$$

with the condition

$$\begin{cases}\text{for any compact subset } K \text{ of } U \text{ there exists a positive constant}\\ C_K \text{ such that } \sup\limits_K |p_j| \leq C_K^{-j}(-j)! \text{ for all } j<0.\end{cases} \tag{1.3}$$

The product is given by the Leibniz rule. If Q is an operator of total symbol $\sigma_{\mathrm{tot}}(Q)$, then

$$\sigma_{\mathrm{tot}}(P\circ Q) = \sum_{\alpha\in\mathbb{N}^n} \frac{\tau^{-|\alpha|}}{\alpha!}\partial_u^\alpha\sigma_{\mathrm{tot}}(P)\partial_x^\alpha\sigma_{\mathrm{tot}}(Q).$$

Denote by $\mathcal{W}_{T^*X}(m)$ the subsheaf of $\mathcal{W}_{T^*X}$ consisting of sections P whose total symbol $\sigma_{\mathrm{tot}}(P)$ satisfies: p_j is zero for $j > m$. The ring $\mathcal{W}_{T^*X}$ is $\mathbb{Z}$-filtered by the $\mathcal{W}_{T^*X}(m)$'s, and $\mathcal{W}_{T^*X}(0)$ is a subring. If $P \in \mathcal{W}_{T^*X}(m)$ and $P \notin \mathcal{W}_{T^*X}(m-1)$, P is said of order m. Hence 0 has order $-\infty$. Then there is a well-defined principal symbol morphism, which does not depend on the local coordinate system on X:

$$\sigma_m \colon \mathcal{W}_{T^*X}(m) \to \mathcal{O}_{T^*X} \cdot \tau^m. \tag{1.4}$$

If $P \in \mathcal{W}_{T^*X}$ has order m on a connected open subset of T^*X, $\sigma_m(P)$ is called the principal symbol of P and is denoted $\sigma(P)$. Note that a section P in $\mathcal{W}_{T^*X}$ is invertible on an open subset U of T^*X if and only if its principal symbol is nowhere vanishing on U.

The principal symbol map induces an isomorphism of graded algebras

$$\mathrm{gr}\,\mathcal{W}_{T^*X} \xrightarrow{\sim} \mathcal{O}_{T^*X}[\tau, \tau^{-1}].$$

We set

$$\mathbf{k} := \mathcal{W}_{\{\mathrm{pt}\}}, \quad \mathbf{k}(j) := \mathcal{W}_{\{\mathrm{pt}\}}(j)\,(j \in \mathbb{Z}), \quad \mathbf{k}_0 := \mathbf{k}(0).$$

Hence, $\mathbf{k}$ is a field and an element $a \in \mathbf{k}$ is written as a formal series

$$a = \sum_{j \le m} a_j \tau^j, \quad a_j \in \mathbb{C}, \quad m \in \mathbb{Z} \tag{1.5}$$

satisfying (1.3), that is,

$$\begin{cases} \text{there exists a positive constant } C \text{ such that } |a_j| \le C^{-j}(-j)! \text{ for} \\ \text{all } j < 0. \end{cases} \tag{1.6}$$

Note that $\mathbf{k}_0$ is a discrete valuation ring.

Convention 1.1. *In the sequel, we shall identify ∂_t and its total symbol τ. Hence, we consider τ as a section of $\mathcal{W}_{T^*X}$.*

Note that

- $\mathcal{W}_{T^*X}$ is flat over $\mathcal{W}_{T^*X}(0)$ and in particular $\mathbf{k}$ is flat over $\mathbf{k}_0$,
- $\mathbf{k}_0$ is faithfully flat over $\mathbb{C}[\tau^{-1}]$,
- the sheaves of rings $\mathcal{W}_{T^*X}$ and $\mathcal{W}_{T^*X}(0)$ are right and left Noetherian (see [9, Appendix]) and in particular coherent,

- if $\mathcal{M}$ is a coherent $\mathcal{W}_{T^*X}$-module, its support is a closed complex analytic involutive (by Gabber's theorem) subset of T^*X,
- $\mathcal{W}_{T^*X}$ is a $\mathbf{k}$-algebra and $\mathcal{W}_{T^*X}(0)$ is a $\mathbf{k}_0$-algebra,
- there are natural monomorphisms of sheaves of $\mathbb{C}$-algebras

$$\pi^{-1}\mathcal{D}_X \hookrightarrow \mathcal{E}_{T^*X} \hookrightarrow \mathcal{W}_{T^*X}. \tag{1.7}$$

On an affine chart, morphism (1.7) is described on symbols as follows. To a section of $\mathcal{E}_{T^*X}$ of total symbol $\sum_{-\infty<j\leq m} p_j(x;\xi)$ (the p_j's are homogeneous in ξ of degree j) one associates the section of $\mathcal{W}_{T^*X}$ of total symbol $\sum_{-\infty<j\leq m} p_j(x;u)\tau^j$, with $u=\tau^{-1}\xi$.

Remark 1.2. (i) Many authors consider the filtered ring of formal operators, defined by

$$\widehat{\mathcal{W}}_{T^*X}(m) = \varprojlim_{j\leq m} \mathcal{W}_{T^*X}(m)/\mathcal{W}_{T^*X}(j) \quad m\in\mathbb{Z}, \qquad \widehat{\mathcal{W}}_{T^*X} = \bigcup_m \widehat{\mathcal{W}}_{T^*X}(m).$$

Then

$$\widehat{\mathbf{k}} := \widehat{\mathcal{W}}_{\{\mathrm{pt}\}} \simeq \mathbb{C}[[\tau^{-1},\tau], \quad \widehat{\mathbf{k}}_0 := \widehat{\mathcal{W}}_{\{\mathrm{pt}\}}(0) \simeq \mathbb{C}[[\tau^{-1}]].$$

Note that $\widehat{\mathcal{W}}_{T^*X}$ is faithfully flat over $\mathcal{W}_{T^*X}$ and $\widehat{\mathcal{W}}_{T^*X}$ is flat over $\widehat{\mathcal{W}}_{T^*X}(0)$.
(ii) Many authors also prefer to use the symbol $\hbar=\tau^{-1}$ instead of τ.

Notation 1.3. Let $\mathcal{I}_X$ be the left ideal of $\mathcal{W}_{T^*X}$ generated by the vector fields on X. We set

$$\mathcal{O}_X^\tau = \mathcal{W}_{T^*X}/\mathcal{I}_X, \quad \mathcal{O}_X^\tau(m) = \mathcal{W}_{T^*X}(m)/(\mathcal{I}_X\cap\mathcal{W}_{T^*X}(m))\,(m\in\mathbb{Z}).$$

Note that $\mathcal{O}_X^\tau$ is a coherent $\mathcal{W}_{T^*X}$-module supported by the zero-section T^*_XX. A section $f(x,\tau)$ of this module may be written as a series:

$$f(x,\tau) = \sum_{-\infty<j\leq m} f_j(x)\tau^j, \quad m\in\mathbb{Z}, \tag{1.8}$$

the f_j's satisfying the condition (1.3).

Also note that $\mathcal{O}_X$ is a direct summand of $\mathcal{O}_X^\tau$ as a sheaf.

Lemma 1.4. *After identifying X and $X\times\{0\}\subset X\times\mathbb{C}$, there is an isomorphism of sheaves of $\mathbb{C}$-vector spaces* (*not of algebras*)

$$\mathcal{O}_X^\tau(0) \simeq \mathcal{O}_{X\times\mathbb{C}}|_{X\times\{0\}}. \tag{1.9}$$

Proof. By its construction, $\mathcal{O}^\tau_X(0)$ is isomorphic to the sheaf of holomorphic microfunctions $\mathcal{C}_{X\times\{0\}|X\times\mathbb{C}}(0)$ of [18], and this last sheaf is isomorphic to $\mathcal{O}_{X\times\mathbb{C}}|_{X\times\{0\}}$ by loc. cit. q.e.d.

Note that isomorphism (1.9) corresponds to the map

$$\mathcal{O}^\tau_X(0) \ni \sum_{j\le 0} f_j\tau^j \mapsto \sum_{j\ge 0} f_{-j}\frac{t^j}{j!} \in \mathcal{O}_{X\times\mathbb{C}}|_{X\times\{0\}}.$$

We shall use the following sheaves of rings. We set

$$\mathcal{D}_X[\tau^{-1}] := \mathcal{D}_X \otimes \mathbb{C}[\tau^{-1}], \qquad \mathcal{D}_X[\tau^{-1},\tau] := \mathcal{D}_X \otimes \mathbb{C}[\tau,\tau^{-1}]. \tag{1.10}$$

Note that $\mathcal{O}^\tau_X(0)$ is a left $\mathcal{D}_X[\tau^{-1}]$-module and $\mathcal{O}^\tau_X$ a left $\mathcal{D}_X[\tau^{-1},\tau]$-module.

Lemma 1.5. (i) *$\mathcal{D}_X[\tau^{-1}]$ is a right and left Noetherian sheaf of $\mathbb{C}$-algebras,*

(ii) *if $\mathcal{N} \subset \mathcal{M}$ are two coherent $\mathcal{D}_X[\tau^{-1}]$-modules and $\mathcal{M}_0$ a coherent $\mathcal{D}_X$-submodule of $\mathcal{M}$, then $\mathcal{N}\cap\mathcal{M}_0$ is $\mathcal{D}_X$-coherent,*

(iii) *if $\mathcal{M}$ is a coherent $\mathcal{D}_X[\tau^{-1}]$-module and $\mathcal{M}_0$ a $\mathcal{D}_X$-submodule of $\mathcal{M}$ of finite type, then $\mathcal{M}_0$ is $\mathcal{D}_X$-coherent.*

Proof. (i) follows from [9, Th. A. 3].
(ii) Set $\mathcal{A} := \mathcal{D}_X[\tau^{-1}]$ and let $\mathcal{A}_n \subset \mathcal{A}$ be the $\mathcal{D}_X$-submodule consisting of sections of order $\le n$ with respect to τ^{-1}. We may reduce to the case where $\mathcal{M} = \mathcal{A}^N$. By [9, Th. A.29, Lem. A21], $\mathcal{N}\cap\mathcal{A}_n$ is $\mathcal{D}_X$-coherent. Since $\{\mathcal{N}\cap\mathcal{A}_n\cap\mathcal{M}_0\}_n$ is an increasing sequence of coherent $\mathcal{D}_X$-submodules of $\mathcal{M}_0$ and this last module is finitely generated, the sequence is stationary.
(iii) Let us keep the notation in (ii). Here again, we may reduce to the case where $\mathcal{M} = \mathcal{A}^N$. Then $\mathcal{M}_0\cap\mathcal{A}_n$ is $\mathcal{D}_X$-coherent, and the proof goes as in (ii). q.e.d.

Lemma 1.6. (i) *$\mathcal{D}_X[\tau^{-1},\tau]$ is flat over $\mathcal{D}_X[\tau^{-1}]$,*

(ii) *if $\mathcal{I}$ is a finitely generated left ideal of $\mathcal{D}_X[\tau^{-1},\tau]$, then $\mathcal{I}\cap\mathcal{D}_X[\tau^{-1}]$ is a locally finitely generated left ideal of $\mathcal{D}_X[\tau^{-1}]$.*

Proof. (i) $\mathbb{C}[\tau,\tau^{-1}]$ is flat over $\mathbb{C}[\tau^{-1}]$.
(ii) Let $\mathcal{I}_0 \subset \mathcal{I}$ be a coherent $\mathcal{D}_X[\tau^{-1}]$-module which generates $\mathcal{I}$. Set $\mathcal{J}_n := \tau^n\mathcal{I}_0 \cap \mathcal{D}_X[\tau^{-1}]$. Then $\mathcal{I}\cap\mathcal{D}_X[\tau^{-1}] = \bigcup_n \mathcal{J}_n$ and this increasing sequence of coherent ideals of $\mathcal{D}_X[\tau^{-1}]$ is locally stationary. q.e.d.

Lemma 1.7. *$\mathcal{W}_{T^*X}$ is flat over $\pi^{-1}\mathcal{D}_X[\tau^{-1},\tau]$.*

Proof. One proves that $\mathcal{W}_{T^*X}$ is flat over $\pi^{-1}\mathcal{D}_X[\tau^{-1}]$ exactly as one proves that the ring of microdifferential operators $\mathcal{E}_{T^*X}$ is flat over $\pi^{-1}\mathcal{D}_X$. Since we have $\mathcal{D}_X[\tau^{-1},\tau] \otimes_{\mathcal{D}_X[\tau^{-1}]} \mathcal{M} \simeq \mathcal{M}$ for any $\mathcal{D}_X[\tau^{-1},\tau]$-module $\mathcal{M}$, the result follows. q.e.d.

2 Regular holonomic $\mathcal{W}$-modules

The following definition adapts a classical definition of [11] to $\mathcal{W}_{T^*X}$-modules (see also [4]).

Definition 2.1. Let Λ be a smooth Lagrangian submanifold of T^*X.

(a) Let $\mathcal{L}(0)$ be a coherent $\mathcal{W}_{T^*X}(0)$-module supported by Λ. One says that $\mathcal{L}(0)$ is regular (resp. simple) along Λ if $\mathcal{L}(0)/\mathcal{L}(-1)$ is a coherent $\mathcal{O}_\Lambda$-module (resp. an invertible $\mathcal{O}_\Lambda$-module). Here, $\mathcal{L}(-1) = \mathcal{W}_{T^*X}(-1)\mathcal{L}(0)$.

(b) Let $\mathcal{L}$ be a coherent $\mathcal{W}_{T^*X}$-module supported by Λ. One says that $\mathcal{L}$ is regular (resp. simple) along Λ if there exists locally a coherent $\mathcal{W}_{T^*X}(0)$-submodule $\mathcal{L}(0)$ of $\mathcal{L}$ such that $\mathcal{L}(0)$ generates $\mathcal{L}$ over $\mathcal{W}_{T^*X}$ and is regular (resp. simple) along Λ.

Note that in (b), $\mathcal{L}(0)/\mathcal{L}(-1)$ is necessarily a locally free $\mathcal{O}_\Lambda$-module.

Example 2.2. The sheaf $\mathcal{O}_X^\tau$ is a simple $\mathcal{W}_{T^*X}$-module along the zero-section T^*_XX.

The following result is easily proved (see [4]).

Proposition 2.3. *Let Λ be a smooth Lagrangian submanifold of T^*X and let $\mathcal{M}$ be a coherent $\mathcal{W}_{T^*X}$-module.*

(i) *If $\mathcal{M}$ is regular, then it is locally a finite direct sum of simple modules.*

(ii) *Any two $\mathcal{W}_{T^*X}$-modules simple along Λ are locally isomorphic. In particular, any simple module along T^*_XX is locally isomorphic to $\mathcal{O}_X^\tau$.*

(iii) *If $\mathcal{M}$, $\mathcal{N}$ are simple $\mathcal{W}_{T^*X}$-modules along Λ, then $R\mathcal{H}om_{\mathcal{W}_{T^*X}}(\mathcal{M},\mathcal{N})$ is concentrated in degree 0 and is a $\mathbf{k}$-local system of rank one on Λ.*

Definition 2.4. Let $\mathcal{M}$ be a coherent $\mathcal{W}_{T^*X}$-module and let Λ denote its support (a closed $\mathbb{C}$-analytic subset of T^*X).

(i) We say that $\mathcal{M}$ is holonomic if Λ is Lagrangian.

(ii) Assume $\mathcal{M}$ is holonomic. We say that $\mathcal{M}$ is regular holonomic if there is an open subset $U \subset \mathfrak{X}$ such that $U \cap \Lambda$ is a dense subset of the regular locus Λ_{reg} of Λ and $\mathcal{M}|_U$ is regular along $U \cap \Lambda$.

In other words, a holonomic module $\mathcal{M}$ is regular if it is so at the generic points of its support. Note that when Λ is smooth, Definition 2.4 is compatible with Definition 2.1. This follows from Gabber's theorem. Indeed, we have the following theorem, analogous of [9, Th. 7.34]:

Theorem 2.5. *Let U be an open subset of T^*X, $\mathcal{M}$ a coherent $\mathcal{W}_{T^*X}|_U$-module and $\mathcal{N} \subset \mathcal{M}$ a sub-$\mathcal{W}_{T^*X}(0)$-module. Assume that $\mathcal{N}$ is a small filtrant inductive limit of coherent $\mathcal{W}_{T^*X}(0)$-modules. Let V be the set of $p \in U$ in a neighborhood of which $\mathcal{N}$ is $\mathcal{W}_{T^*X}(0)$-coherent. Then $U \setminus V$ is an analytic involutive subset of U.*

The fact that regularity of holonomic $\mathcal{W}$-modules is a generic property follows as in loc. cit. Prop. 8.28.

3 $\mathcal{W}$-modules on a complex symplectic manifold

We refer to [15] for the definition of an algebroid stack and to [3] for a more systematic study.

Let $\mathbb{K}$ be a commutative unital ring and let X be a topological space. We denote by $\mathrm{Mod}(\mathbb{K}_X)$ the abelian category of sheaves of $\mathbb{K}$-modules and by $\mathrm{D}^{\mathrm{b}}(\mathbb{K}_X)$ its bounded derived category.

If A is a $\mathbb{K}$-algebra, we denote by A^+ the category with one object and having A as endomorphisms of this object. If $\mathcal{A}$ is a sheaf of $\mathbb{K}$-algebras on X, we denote by $\mathcal{A}^+$ the $\mathbb{K}$-linear stack associated with the prestack $U \mapsto \mathcal{A}(U)^+$ (U open in X) and call it the $\mathbb{K}$-algebroid stack associated with $\mathcal{A}$. It is equivalent to the stack of right $\mathcal{A}$-modules locally isomorphic to $\mathcal{A}$, and $\mathcal{A}$-linear homomorphisms.

The projective cotangent bundle P^*Y to a complex manifold Y is endowed with the sheaf of rings $\mathcal{E}_{P^*Y}$ of microdifferential operators. (This sheaf is the direct image of the sheaf $\mathcal{E}_{T^*Y}$ of § 1 by the map $T^*Y \setminus T^*_YY \to P^*Y$.) A complex contact manifold $\mathfrak{Y}$ is locally isomorphic to an open subset of a projective cotangent bundle P^*Y and on such a contact manifold, a canonical algebroid stack $\mathcal{E}_{\mathfrak{Y}}$ locally equivalent to the stack associated with the sheaf of rings $\mathcal{E}_{P^*Y}$ has been constructed in [8].

This construction has been adapted to the symplectic case by [16]. A complex symplectic manifold $\mathfrak{X}$ is locally isomorphic to the cotangent bundle T^*X to a complex manifold X and a canonical algebroid stack $\mathcal{W}_{\mathfrak{X}}$ locally equivalent to the stack associated with the sheaf of rings $\mathcal{W}_{T^*X}$ of § 1 is constructed in loc. cit., after Kontsevich [15] had treated the general case of complex Poisson manifolds in the formal setting by a different approach.

Denote by $\mathfrak{X}^a$ the complex manifold $\mathfrak{X}$ endowed with the symplectic form $-\omega$, where ω is the symplectic form on $\mathfrak{X}$. There is a natural equivalence of algebroid stacks $\mathcal{W}_{\mathfrak{X}^a} \simeq (\mathcal{W}_{\mathfrak{X}})^{\mathrm{op}}$.

Let $\mathfrak{X}$ and $\mathfrak{Y}$ be two complex symplectic manifolds. There exist a natural $\mathbf{k}$-algebroid stack $\mathcal{W}_{\mathfrak{X}} \boxtimes \mathcal{W}_{\mathfrak{Y}}$ on $\mathfrak{X} \times \mathfrak{Y}$ and a natural functor of $\mathbf{k}$-algebroid stacks $\mathcal{W}_{\mathfrak{X}} \boxtimes \mathcal{W}_{\mathfrak{Y}} \to \mathcal{W}_{\mathfrak{X}\times\mathfrak{Y}}$ which locally corresponds to the morphism of sheaves of rings $\mathcal{W}_{T^*X} \boxtimes \mathcal{W}_{T^*Y} \to \mathcal{W}_{T^*(X\times Y)}$.

One sets

$$\mathrm{Mod}(\mathcal{W}_{\mathfrak{X}}) = \mathrm{Fct}_{\mathbf{k}}(\mathcal{W}_{\mathfrak{X}}, \mathfrak{Mod}(\mathbf{k}_{\mathfrak{X}})),$$

where $\mathrm{Fct}_{\mathbf{k}}(\,\bullet\,,\,\bullet\,)$ denotes the $\mathbf{k}$-linear category of $\mathbf{k}$-linear functors of stacks and $\mathfrak{Mod}(\mathbf{k}_{\mathfrak{X}})$ is the stack of sheaves of $\mathbf{k}$-modules on $\mathfrak{X}$. We denote by $\mathfrak{Mod}(\mathcal{W}_{\mathfrak{X}})$ the stack on $\mathfrak{X}$ given by $U \mapsto \mathrm{Mod}(\mathcal{W}_{\mathfrak{X}}|_U)$.

Then $\mathrm{Mod}(\mathcal{W}_{\mathfrak{X}})$ is a Grothendieck abelian category. We denote by $\mathrm{D^b}(\mathcal{W}_{\mathfrak{X}})$ its bounded derived category and call an object of this derived category a $\mathcal{W}_{\mathfrak{X}}$-module on $\mathfrak{X}$, for short. Objects of $\mathrm{Mod}(\mathcal{W}_{\mathfrak{X}})$ are described with some details in [4].

We denote by $\mathcal{H}om_{\mathcal{W}_{\mathfrak{X}}}$ the hom-functor of the stack $\mathfrak{Mod}(\mathcal{W}_{\mathfrak{X}})$, a functor from $\mathrm{Mod}(\mathcal{W}_{\mathfrak{X}})^{\mathrm{op}} \times \mathrm{Mod}(\mathcal{W}_{\mathfrak{X}})$ to $\mathrm{Mod}(\mathbf{k}_{\mathfrak{X}})$. The object $R\mathcal{H}om_{\mathcal{W}_{\mathfrak{X}}}(\mathcal{M},\mathcal{N})$ is thus well defined in $\mathrm{D^b}(\mathbf{k}_{\mathfrak{X}})$ for two objects $\mathcal{M}$ and $\mathcal{N}$ of $\mathrm{D^b}(\mathcal{W}_{\mathfrak{X}})$.

The natural functor $\mathcal{W}_{\mathfrak{X}} \boxtimes \mathcal{W}_{\mathfrak{Y}} \to \mathcal{W}_{\mathfrak{X}\times\mathfrak{Y}}$ defines a functor of stacks

$$\mathit{for}\colon\ \mathfrak{Mod}(\mathcal{W}_{\mathfrak{X}\times\mathfrak{Y}}) \to \mathfrak{Mod}(\mathcal{W}_{\mathfrak{X}} \boxtimes \mathcal{W}_{\mathfrak{Y}}) \tag{3.1}$$

and this last functor admits an adjoint (since it does locally). For $\mathcal{M} \in \mathrm{D^b}(\mathcal{W}_{\mathfrak{X}})$ and $\mathcal{N} \in \mathrm{D^b}(\mathcal{W}_{\mathfrak{Y}})$, it thus exists a canonically defined object $\mathcal{M}\underline{\boxtimes}\mathcal{N} \in \mathrm{D^b}(\mathcal{W}_{\mathfrak{X}\times\mathfrak{Y}})$ such that, locally, $\mathcal{M}\underline{\boxtimes}\mathcal{N} \simeq \mathcal{W}_{T^*(X\times Y)} \boxtimes_{(\mathcal{W}_{T^*X}\boxtimes\mathcal{W}_{T^*Y})} (\mathcal{M} \boxtimes \mathcal{N})$ in $\mathrm{D^b}(\mathcal{W}_{T^*(X\times Y)})$.

Being local, the notions of coherent or holonomic, regular holonomic or simple object of $\mathrm{Mod}(\mathcal{W}_{\mathfrak{X}})$ make sense. We denote by

- $\mathrm{D^b_{coh}}(\mathcal{W}_{\mathfrak{X}})$ the full triangulated subcategory of $\mathrm{D^b}(\mathcal{W}_{\mathfrak{X}})$ consisting of objects with coherent cohomologies,
- $\mathrm{D^b_{hol}}(\mathcal{W}_{\mathfrak{X}})$ the full triangulated subcategory of $\mathrm{D^b_{coh}}(\mathcal{W}_{\mathfrak{X}})$ consisting of objects with holonomic cohomologies, or equivalently, of objects with Lagrangian supports in $\mathfrak{X}$,

- $\mathrm{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathcal{W}_{\mathfrak{X}})$ the full triangulated subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{W}_{\mathfrak{X}})$ consisting of objects with regular holonomic cohomologies.

The support of an object $\mathcal{M}$ of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{W}_{\mathfrak{X}})$ is denoted by $\mathrm{supp}(\mathcal{M})$ and is also called its characteristic variety. This is a closed complex analytic involutive subset of $\mathfrak{X}$.

In the sequel, we shall denote by $\Delta_{\mathfrak{X}}$ the diagonal of $\mathfrak{X}\times\mathfrak{X}^a$ and identify it with $\mathfrak{X}$ by the first projection.

The next lemma follows from general considerations on stacks and its verification is left to the reader.

Lemma 3.1. *There exists a canonical simple $\mathcal{W}_{\mathfrak{X}\times\mathfrak{X}^a}$-module $\mathcal{C}_{\Delta_{\mathfrak{X}}}$ supported by the diagonal $\Delta_{\mathfrak{X}}$ such that if U is open in $\mathfrak{X}$ and isomorphic to an open subset V of a cotangent bundle T^*X, then $\mathcal{C}_{\Delta_{\mathfrak{X}}}|_U$ is isomorphic to $\mathcal{W}_{T^*X}|_V$ as a $\mathcal{W}_{T^*X}\otimes(\mathcal{W}_{T^*X})^{\mathrm{op}}$-module.*

Definition 3.2. Let $\mathcal{M}\in\mathrm{D}^{\mathrm{b}}(\mathcal{W}_{\mathfrak{X}})$. Its dual $\mathrm{D}'_{\mathrm{w}}\mathcal{M}\in\mathrm{D}^{\mathrm{b}}(\mathcal{W}_{\mathfrak{X}^a})$ is given by

$$\mathrm{D}'_{\mathrm{w}}\mathcal{M} := R\mathcal{H}om_{\mathcal{W}_{\mathfrak{X}}}(\mathcal{M},\mathcal{C}_{\Delta_{\mathfrak{X}}}). \tag{3.2}$$

Let Λ be a smooth Lagrangian submanifold of $\mathfrak{X}$. Consider the cohomology class $\mathbf{c}_{1/2}\in H^2(X;\mathbb{C}^\times_\Lambda)$ defined as $\mathbf{c}_{1/2}=\beta(\frac{1}{2}\alpha([\Omega_\Lambda]))$ where $[\Omega_\Lambda]\in H^1(X;\mathcal{O}^\times_\Lambda)$ is the class of the line bundle Ω_Λ and α,β are the morphisms of the exact sequence

$$H^1(X;\mathcal{O}^\times_\Lambda)\xrightarrow{\alpha}H^1(X;d\mathcal{O}_\Lambda)\xrightarrow{\beta}H^2(X;\mathbb{C}^\times_\Lambda).$$

The main theorem of [4] asserts that simple $\mathcal{W}_{\mathfrak{X}}$-modules along Λ are in one-to-one correspondence with twisted local systems of rank one on Λ with twist $\mathbf{c}_{1/2}$.

Proposition 3.3. *Let $\mathcal{M},\mathcal{N}$ be two objects of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{W}_{\mathfrak{X}})$. There is a natural isomorphism in $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathfrak{X}})$:*

$$R\mathcal{H}om_{\mathcal{W}_{\mathfrak{X}}}(\mathcal{M},\mathcal{N})\simeq R\mathcal{H}om_{\mathcal{W}_{\mathfrak{X}\times\mathfrak{X}^a}}(\mathcal{M}\underline{\boxtimes}\mathrm{D}'_{\mathrm{w}}\mathcal{N},\mathcal{C}_{\Delta_{\mathfrak{X}}}).$$

Proof. We have the isomorphism in $\mathrm{D}^{\mathrm{b}}(\mathcal{W}_{\mathfrak{X}})$:

$$\mathcal{N}\simeq R\mathcal{H}om_{\mathcal{W}_{\mathfrak{X}^a}}(\mathrm{D}'_{\mathrm{w}}\mathcal{N},\mathcal{C}_{\Delta_{\mathfrak{X}}}),$$

from which we deduce

$$\begin{aligned} R\mathcal{H}om_{\mathcal{W}_{\mathfrak{X}\times\mathfrak{X}^a}}(\mathcal{M}\underline{\boxtimes}\mathrm{D}'_{\mathrm{w}}\mathcal{N},\mathcal{C}_{\Delta_{\mathfrak{X}}}) &\simeq R\mathcal{H}om_{\mathcal{W}_{\mathfrak{X}}}(\mathcal{M},R\mathcal{H}om_{\mathcal{W}_{\mathfrak{X}^a}}(\mathrm{D}'_{\mathrm{w}}\mathcal{N},\mathcal{C}_{\Delta_{\mathfrak{X}}}))\\ &\simeq R\mathcal{H}om_{\mathcal{W}_{\mathfrak{X}}}(\mathcal{M},\mathcal{N}). \end{aligned}$$

q.e.d.

4 Functional analysis I

In this section, we will use techniques elaborated by Houzel [6] and will follow his terminology. (See also Kiel-Verdier [14] for related results.)

We call a bornological convex $\mathbb{C}$-vector space (resp. $\mathbb{C}$-algebra), a *bc*-space (resp. *bc*-algebra) and we denote by $\mathrm{Mod}^{\mathrm{bc}}(\mathbb{C})$ the category of *bc*-spaces and bounded linear maps. This additive category admits small inductive and projective limits, but is not abelian.

Let A be a *bc*-algebra. We denote by $\mathrm{Mod}^{\mathrm{bc}}(A)$ the additive category of bornological A-modules and bounded A-linear maps. For $E, F \in \mathrm{Mod}^{\mathrm{bc}}(A)$, we set:

$$\begin{aligned}
&\mathrm{Bhom}_A(E, F) = \mathrm{Hom}_{\mathrm{Mod}^{\mathrm{bc}}(A)}(E, F),\\
&E^\vee = \mathrm{Bhom}_A(E, A).
\end{aligned}$$

Let $E \in \mathrm{Mod}^{\mathrm{bc}}(A)$ and let $B \subset E$ be a convex circled bounded subset of E. For $x \in E$, one sets

$$||x||_B = \inf_{x \in cB, c \in \mathbb{C}} |c|.$$

For $u \in \mathrm{Bhom}_A(E, F)$, B bounded in E and B' convex circled bounded in F, one sets

$$||u||_{BB'} = \sup_{x \in B} ||u(x)||_{B'}.$$

One says that a sequence $\{u_n\}_n$ in $\mathrm{Bhom}_A(E, F)$ is bounded if for any bounded subset $B \subset E$ there exists a convex circled bounded subset $B' \subset F$ such that $\sup_n ||u_n||_{BB'} < \infty$.

One says that $u \in \mathrm{Bhom}_A(E, F)$ is A-nuclear if there exist a bounded sequence $\{y_n\}_n$ in F, a bounded sequence $\{u_n\}_n$ in $\mathrm{Bhom}_A(E, A)$ and a summable sequence $\{\lambda_n\}_n$ in $\mathbb{R}_{\geq 0}$ such that

$$u(x) = \sum_n \lambda_n \, u_n(x) \, y_n \text{ for all } x \in E.$$

For $E, F \in \mathrm{Mod}^{\mathrm{bc}}(A)$, there is a natural structure of *bc*-space on $E \otimes_A F$ and one denotes by $E \widehat{\otimes}_A F$ the completion of $E \otimes_A F$. Assuming F is complete, there is a natural linear map

$$E^\vee \widehat{\otimes}_A F \to \mathrm{Bhom}_A(E, F).$$

An element $u \in \operatorname{Bhom}_A(E,F)$ is A-nuclear if and only if it is in the image of $E^\vee \widehat{\otimes}_A F$.

Note that, if $u\colon E \to F$ is a nuclear $\mathbb{C}$-linear map, $u \widehat{\otimes} 1\colon E \widehat{\otimes} A \to F \widehat{\otimes} A$ is A-nuclear.

Recall that a $\mathbb{C}$-vector space E is called a DFN-space if it is an inductive limit $\{(E_n, u_n)\}_{n\in\mathbb{N}}$ of Banach spaces such that the maps $u_n\colon E_n \to E_{n+1}$ are $\mathbb{C}$-nuclear and injective. Note that any bounded subset of E is contained in E_n for some n.

In the sequel, we will consider a DFN-algebra A and the full subcategory $\mathrm{Mod}^{\mathrm{dfn}}(A)$ of $\mathrm{Mod}^{\mathrm{bc}}(A)$ consisting of DFN-spaces. Note that any epimorphism $u\colon E \to F$ in $\mathrm{Mod}^{\mathrm{bc}}(A)$ is semi-strict, that is, any bounded sequence in F is the image by u of a bounded sequence in E. Also note that for E and F in $\mathrm{Mod}^{\mathrm{dfn}}(A)$, $\operatorname{Bhom}_A(E,F)$ is the subspace of $\operatorname{Hom}_A(E,F)$ consisting of continuous maps. Since the category $\mathrm{Mod}^{\mathrm{dfn}}(A)$ is not abelian, we introduce the following definition, referring to [20] for a more systematic treatment of homological algebra in terms of quasi-abelian categories.

Definition 4.1. (i) Let A be a DFN-algebra. A complex $0 \to E' \to E \to E'' \to 0$ in $\mathrm{Mod}^{\mathrm{dfn}}(A)$ is called a short exact sequence if it is an exact sequence in $\mathrm{Mod}(A)$.

(ii) Let A and B be two DFN-algebra. An additive functor from $\mathrm{Mod}^{\mathrm{dfn}}(A)$ to $\mathrm{Mod}^{\mathrm{dfn}}(B)$ is called exact if it sends short exact sequences to short exact sequences.

The following result is well-known and follows from [5].

Proposition 4.2. *Let A be a DFN-algebra. The functor $\bullet \widehat{\otimes} A\colon \mathrm{Mod}^{\mathrm{dfn}}(\mathbb{C}) \to \mathrm{Mod}^{\mathrm{dfn}}(A)$ is exact.*

Recall that a *bc*-algebra A is multiplicatively convex if for any bounded set $B \subset A$, there exist a constant $c > 0$ and a convex circled bounded set B' such that $B \subset c \cdot B'$ and $B' \cdot B' \subset B'$.

As usual, for an additive category $\mathcal{C}$, we denote by $\mathrm{C}^{\mathrm{b}}(\mathcal{C})$ the category of bounded complexes in $\mathcal{C}$ and, for $a \le b$ in $\mathbb{Z}$, by $\mathrm{C}^{[a,b]}(\mathcal{C})$ the full subcategory consisting of complexes concentrated in degrees $j \in [a,b]$. We denote by $\mathrm{K}^{\mathrm{b}}(\mathcal{C})$ the homotopy category associated with $\mathrm{C}^{\mathrm{b}}(\mathcal{C})$. Finally, we denote by $\mathrm{Ind}(\mathcal{C})$ and $\mathrm{Pro}(\mathcal{C})$ the categories of ind-objects and pro-objects of $\mathcal{C}$, respectively.

Theorem 4.3. *Let A be a multiplicatively convex DFN-algebra and assume that A is a Noetherian ring* (*when forgetting the topology*)*. Consider an inductive system $\{(E_n^\bullet, u_n^\bullet)\}_{n\in\mathbb{N}}$ in $\mathrm{C}^{[a,b]}(\mathrm{Mod}^{\mathrm{dfn}}(A))$ for $a \le b \in \mathbb{Z}$. Assume:*

(i) $u_n^\bullet\colon E_n^\bullet \to E_{n+1}^\bullet$ *is a quasi-isomorphism for all* $n \geq 0$,

(ii) $u_n^j\colon E_n^j \to E_{n+1}^j$ *is* A*-nuclear for all* $j \in \mathbb{Z}$ *and all* $n \geq 0$.

Then $H^j(E_n^\bullet)$ *is finitely generated over* A *for all* $j \in \mathbb{Z}$ *and all* $n \geq 0$.

This is a particular case of [6, § 6 Th. 1, Prop. A.1].

Theorem 4.4. *Let* A *be a DFN-algebra, and consider an inductive system* $\{E_n^\bullet, u_n^\bullet\}_{n\in\mathbb{N}}$ *in* $\mathrm{C}^{[a,b]}(\mathrm{Mod}^{\mathrm{dfn}}(A))$ *for* $a \leq b$ *in* $\mathbb{Z}$*. Assume*

(i) *for each* $i \in \mathbb{Z}$ *and* $n \in \mathbb{N}$*, the map* $u_n^i\colon E_n^i \to E_{n+1}^i$ *is* A*-nuclear,*

(ii) *for each* $i \in \mathbb{Z}$*,* $\text{“}\varinjlim_n\text{”}\, H^i(E_n^\bullet) \simeq 0$ *in* $\mathrm{Ind}(\mathrm{Mod}(\mathbb{C}))$.

Then $\text{“}\varinjlim_n\text{”}\, E_n^\bullet \simeq 0$ *in* $\mathrm{Ind}(\mathrm{K}^{[a,b]}(\mathrm{Mod}^{\mathrm{dfn}}(A)))$.

First, we need a lemma.

Lemma 4.5. *Consider the solid diagram in* $\mathrm{Mod}^{\mathrm{dfn}}(A)$*:*

$$\begin{tikzcd} E \arrow[r, "u"] \arrow[d, dashed, "v"'] & F \arrow[d, "v'"] \\ E' \arrow[r, "u'"'] & F'. \end{tikzcd}$$

Assume that u *is* A*-nuclear and* $\mathrm{Im}\, v' \subset \mathrm{Im}\, u'$*. Then there exists a morphism* $v\colon E \to E'$ *making the whole diagram commutative.*

Proof. The morphism $w\colon E' \times_{F'} F \to F$ is well defined in the category $\mathrm{Mod}^{\mathrm{dfn}}(A)$ and is surjective by the hypothesis. We get a diagram

$$\begin{tikzcd} & E \arrow[d, "u"] \\ E' \times_{F'} F' \arrow[r, two heads, "w"'] & F, \end{tikzcd}$$

In this situation, the nuclear map u factors through $E' \times_{F'} F'$ by [6, §4 Cor. 2]. q.e.d.

Proof of Theorem 4.4. We may assume that $H^i(E_n^\bullet) \to H^i(E_{n+1}^\bullet)$ is the zero morphism for all $i \in \mathbb{Z}$ and all $n \in \mathbb{N}$. Consider the solid diagram

$$\begin{tikzcd} E_{n-1}^b \arrow[r, "u_{n-1}^b"] \arrow[d, dotted, "k_n^b"'] & E_n^b \arrow[d, "u_n^b"] \\ E_{n+1}^{b-1} \arrow[r, "d_{n+1}^{b-1}"'] & F_{n+1}^b. \end{tikzcd}$$

Since $H^b(E_p^\bullet) \simeq \operatorname{Coker} d_p^{b-1}$ for all p, $\operatorname{Im} u_n^b \subset \operatorname{Im} d_{n+1}^{b-1}$. Moreover, u_{n-1}^b is A-nuclear by the hypothesis. Therefore we may apply Lemma 4.5 and we obtain a map $k_n^b \colon E_{n-1}^b \to E_{n+1}^{b-1}$ making the whole diagram commutative. Set $v_n^i = u_n^i \circ u_{n-1}^i$ and $h_n^b = d_{n+1}^{b-1} \circ k_n^b$. Consider the diagram

$$\begin{tikzcd} \cdots \arrow[r] & E_{n-1}^{b-2} \arrow[r, "d_{n-1}^{b-2}"] \arrow[d, "v_n^{b-2}"] & E_{n-1}^{b-1} \arrow[r, "d_{n-1}^{b-1}"] \arrow[d, "v_n^{b-1}"] & E_{n-1}^b \arrow[r] \arrow[d, "v_n^b"] \arrow[dl, "h_n^b"'] & 0 \\ \cdots \arrow[r] & E_{n+1}^{b-2} \arrow[r, "d_{n+1}^{b-2}"] & E_{n+1}^{b-1} \arrow[r, "d_{n+1}^{b-1}"] & E_{n+1}^b \arrow[r] & 0. \end{tikzcd}$$

The morphisms v_n^i's define a morphism of complex $v_n \colon E_{n-1}^\bullet \to E_{n+1}^\bullet$. We define $h_n^i \colon E_{n-1}^i \to E_{n+1}^i$ by setting $h_n^i = 0$ for $i \neq b$. Now denote by $\sigma^{\leq b-1} E_n^\bullet$ the stupid truncated complex obtained by replacing E_n^b with 0. The morphism

$$v_n - h_n \circ d_{n-1} - d_{n+1} \circ h_n \colon E_{n-1}^\bullet \to E_{n+1}^\bullet$$

factorizes through $\sigma^{\leq b-1} E_n^\bullet$. Hence, we get an isomorphism

$$\text{“}\varinjlim_n\text{”}\, E_n^\bullet \xrightarrow{\sim} \text{“}\varinjlim_n\text{”}\, \sigma^{\leq b-1} E_n^\bullet$$

in $\mathrm{Ind}(\mathrm{K}^{[a,b]}(\mathrm{Mod}^{\mathrm{dfn}}(A)))$. By repeating this argument, we find the isomorphism $\text{“}\varinjlim_n\text{”}\, E_n^\bullet \simeq \text{“}\varinjlim_n\text{”}\, \sigma^{\leq b-p} E_n^\bullet$ for any $p \in \mathbb{N}$. This completes the proof.

q.e.d.

Theorem 4.6. *Let A be a DFN-field, let $a \leq b$ in $\mathbb{Z}$, consider an inductive system $\{E_n^\bullet, u_n^\bullet\}_{n\in\mathbb{N}}$ in $\mathrm{C}^{[a,b]}(\mathrm{Mod}^{\mathrm{dfn}}(A))$ and set $F_n^\bullet = (E_n^\bullet)^\vee = \mathrm{Bhom}_A(E_n^\bullet, A)$. Assume*

(i) *for each $i \in \mathbb{Z}$ and $n \in \mathbb{N}$, the map $u_n^i \colon E_n^i \to E_{n+1}^i$ is A-nuclear,*

(ii) *for each* $i \in \mathbb{Z}$, $\text{“}\varinjlim_n\text{”}\, H^i(E_n^\bullet)$ *belongs to* $\mathrm{Mod}^{\mathrm{f}}(A)$ (*the category of finite-dimensional* A*-vector spaces*).

Then we have the isomorphism

$$\text{“}\varprojlim_n\text{”}\, F_n^\bullet \simeq \text{“}\varprojlim_n\text{”}\, \mathrm{Hom}_A(E_n^\bullet, A) \text{ in } \mathrm{Pro}(\mathrm{K}^{\mathrm{b}}(\mathrm{Mod}(A))). \tag{4.1}$$

In particular, for each $i \in \mathbb{Z}$, $\text{“}\varprojlim_n\text{”}\, H^{-i}(F_n^\bullet)$ *belongs to* $\mathrm{Mod}^{\mathrm{f}}(A)$ *and is dual of* $\text{“}\varinjlim_n\text{”}\, H^i(E_n^\bullet)$.

Proof. Recall (see [13, Exe 15.1] that for an abelian category $\mathcal{C}$ and an inductive system $\{X_j\}_{j\in J}$ in $\mathrm{D}^{[a,b]}(\mathcal{C})$ indexed by a small filtrant category J, if the object $\text{“}\varinjlim_j\text{”}\, H^i(X_j)$ of $\mathrm{Ind}(\mathcal{C})$ is representable for all $i \in \mathbb{Z}$, then $\text{“}\varinjlim_j\text{”}\, X_j \in \mathrm{Ind}(\mathrm{D}^{\mathrm{b}}(\mathcal{C}))$ is representable.

Applying this result to our situation, we find that the object $\text{“}\varinjlim_n\text{”}\, E_n^\bullet$ of $\mathrm{Ind}(\mathrm{D}^{\mathrm{b}}(\mathrm{Mod}(A)))$ is representable in $\mathrm{D}^{\mathrm{b}}(\mathrm{Mod}(A))$.

Denote by $L^\bullet$ the complex given by $L^i = \text{“}\varinjlim_n\text{”}\, H^i(E_n^\bullet)$ and zero differentials. Since A is a field, there exists an isomorphism $L^\bullet \xrightarrow{\sim} \text{“}\varinjlim_n\text{”}\, E_n^\bullet$ in $\mathrm{D}^{\mathrm{b}}(\mathrm{Mod}(A))$, hence a quasi-isomorphism $u\colon L^\bullet \to \text{“}\varinjlim_n\text{”}\, E_n^\bullet$ in $\mathrm{C}^{\mathrm{b}}(\mathrm{Mod}(A))$. There exists $n \in \mathbb{N}$ such that u factorizes through $L^\bullet \to E_n^\bullet$ for some n and we may assume $n = 0$. Since $L^\bullet$ belongs to $\mathrm{C}^{\mathrm{b}}(\mathrm{Mod}^{\mathrm{f}}(A))$, $L^\bullet \to E_0^\bullet$ is well defined in $\mathrm{C}^{\mathrm{b}}(\mathrm{Mod}^{\mathrm{dfn}}(A))$. For any n, let $u_n\colon L^\bullet \to E_n^\bullet$ be the induced morphism. Let $G_n^\bullet$ be the mapping cone of u_n. The morphism u_n is embedded in a distinguished triangle in $\mathrm{K}^{\mathrm{b}}(\mathrm{Mod}^{\mathrm{dfn}}(A))$

$$L^\bullet \xrightarrow{u_n} E_n^\bullet \to G_n^\bullet \xrightarrow{+1} .$$

By Theorem 4.4, $\text{“}\varinjlim_n\text{”}\, H^i(G_n^\bullet) \simeq 0$ implies $\text{“}\varinjlim_n\text{”}\, G_n^\bullet \simeq 0$ in $\mathrm{Ind}(\mathrm{K}^{\mathrm{b}}(\mathrm{Mod}^{\mathrm{dfn}}(A)))$. Hence, for any $K \in \mathrm{K}^{\mathrm{b}}(\mathrm{Mod}^{\mathrm{dfn}}(A))$, the morphism

$$\mathrm{Hom}_{\mathrm{K}^{\mathrm{b}}(\mathrm{Mod}^{\mathrm{dfn}}(A))}(K, L^\bullet) \to \varinjlim_n \mathrm{Hom}_{\mathrm{K}^{\mathrm{b}}(\mathrm{Mod}^{\mathrm{dfn}}(A))}(K, E_n^\bullet)$$

is an isomorphism. By the Yoneda lemma, we have thus obtained the isomorphism

$$L^\bullet \xrightarrow{\sim} \text{“}\varinjlim_n\text{”}\, E_n^\bullet \text{ in } \mathrm{Ind}(\mathrm{K}^{\mathrm{b}}(\mathrm{Mod}^{\mathrm{dfn}}(A))).$$

Since $L^\bullet$ belongs to $\mathrm{K^b}(\mathrm{Mod^f}(A))$, $(L^\bullet)^\vee \simeq \mathrm{Hom}_A(L^\bullet, A)$ and we obtain

$$\text{“}\varprojlim_n\text{”}\, F_n^\bullet \simeq (L^\bullet)^\vee \simeq \mathrm{Hom}_A(L^\bullet, A) \simeq \text{“}\varprojlim_n\text{”}\, \mathrm{Hom}_A(E_n^\bullet, A)$$

in $\mathrm{Pro}(\mathrm{K^b}(\mathrm{Mod}(A)))$. The isomorphisms for the cohomologies follow since “$\varprojlim$” commutes with $H^{-i}(\,\bullet\,)$ and $\mathrm{Hom}_A(\,\bullet\,, A)$. q.e.d.

A similar result to Theorem 4.6 holds for projective system.

Theorem 4.7. *Let A be a DFN-field, let $a \le b$ in $\mathbb{Z}$, consider a projective system $\{F_n^\bullet, v_n^\bullet\}_{n\in\mathbb{N}}$ in $\mathrm{C}^{[a,b]}(\mathrm{Mod^{dfn}}(A))$ and set $E_n^\bullet = \mathrm{Bhom}_A(F_n^\bullet, A)$. Assume*

(i) *for each $i \in \mathbb{Z}$ and $n \in \mathbb{N}$, the map $v_n^i\colon F_{n+1}^i \to F_n^i$ is A-nuclear,*

(ii) *for each $i \in \mathbb{Z}$, “$\varprojlim_n$” $H^i(F_n^\bullet)$ belongs to $\mathrm{Mod^f}(A)$ (the category of finite-dimensional A-vector spaces).*

Then we have the isomorphism

$$\text{“}\varinjlim_n\text{”}\, E_n^\bullet \simeq \text{“}\varinjlim_n\text{”}\, \mathrm{Hom}_A(F_n^\bullet, A) \text{ in } \mathrm{Ind}(\mathrm{K^b}(\mathrm{Mod}(A))). \tag{4.2}$$

In particular, for each $i \in \mathbb{Z}$, “$\varinjlim_n$” $H^{-i}(E_n^\bullet)$ belongs to $\mathrm{Mod^f}(A)$ and is dual of “$\varprojlim_n$” $H^i(F_n^\bullet)$.

The proof being similar to the one of Theorem 4.6, we shall not repeat it.

In the course of § 5 below, we shall also need the next lemma.

Lemma 4.8. *Let A be a DFN-algebra and let $u\colon E_0 \to E_1$ be a $\mathbb{C}$-nuclear map of DFN-spaces. Recall that $(\,\bullet\,)^\vee = \mathrm{Bhom}_A(\,\bullet\,, A)$ and set $(\,\bullet\,)^\star = \mathrm{Bhom}_{\mathbb{C}}(\,\bullet\,, \mathbb{C})$. Then the solid commutative diagram below may be completed with the dotted arrow as a commutative diagram:*

$$\begin{tikzcd}
E_1^\star \widehat{\otimes} A \arrow[r, "u^\star \widehat{\otimes} A"] \arrow[d] & E_0^\star \widehat{\otimes} A \arrow[d] \\
(E_1 \widehat{\otimes} A)^\vee \arrow[r, "(u\widehat{\otimes} A)^\vee"'] \arrow[ur, dotted] & (E_0 \widehat{\otimes} A)^\vee.
\end{tikzcd}$$

Proof. Consider the commutative diagram

$$\begin{tikzcd}
E_0^\star \widehat{\otimes} E_1 \arrow[r] \arrow[d] & \mathrm{Bhom}_{\mathbb{C}}(E_0, E_1) \arrow[d] \\
\mathrm{Bhom}_A((E_1\widehat{\otimes} A)^\vee, E_0^\star\widehat{\otimes} A) \arrow[r] & \mathrm{Bhom}_A((E_1\widehat{\otimes} A)^\vee, (E_0\widehat{\otimes} A)^\vee).
\end{tikzcd}$$

Since u is nuclear, it is the image of an element of $E_0^\star\widehat{\otimes} E_1$. q.e.d.

5 Functional analysis II

This section will provide the framework for apply Theorems 4.3 and 4.6.

Here, X will denote a complex manifold. For a locally free $\mathcal{O}_X$-module $\mathcal{F}$ of finite rank, we set

$$\mathcal{F}^\tau := \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X^\tau \quad \mathcal{F}^\tau(0) := \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X^\tau(0).$$

Lemma 5.1. (i) *The $\mathbb{C}$-algebra $\mathbf{k}_0$ is a multiplicatively convex DFN-algebra,*

(ii) *the $\mathbb{C}$-algebra $\mathbf{k}$ is a DFN-algebra,*

(iii) *the functor $\bullet \widehat{\otimes}\, \mathbf{k}_0 \colon \mathrm{Mod}^{\mathrm{dfn}}(\mathbb{C}) \to \mathrm{Mod}^{\mathrm{dfn}}(\mathbf{k}_0)$ is well-defined and exact,*

(iv) *the functor $\bullet \widehat{\otimes}\, \mathbf{k} \colon \mathrm{Mod}^{\mathrm{dfn}}(\mathbb{C}) \to \mathrm{Mod}^{\mathrm{dfn}}(\mathbf{k})$ is well-defined and exact, and is isomorphic to the functor $(\bullet \widehat{\otimes}\, \mathbf{k}_0) \otimes_{\mathbf{k}_0} \mathbf{k}$.*

Proof. (i) Define the subalgebra $\mathbf{k}_0(r)$ of $\mathbf{k}_0$ by

$$(5.1) \qquad \begin{cases} u = \sum_{j\le 0} a_j \tau^j \text{ belongs to } \mathbf{k}_0(r) \text{ if and only if} \\ |u|_r := \sum_{j\le 0} \dfrac{r^{-j}}{(-j)!}\, |a_j| < \infty. \end{cases}$$

Then, for u,v in $\mathbf{k}_0(r)$, we have

$$|u \cdot v|_r \le |u|_r \cdot |v|_r.$$

Hence, $(\mathbf{k}_0(r), |\bullet|_r)$ is a Banach algebra and $\mathbf{k}_0$ is multiplicatively convex since it is the inductive limit of the $\mathbf{k}_0(r)$'s. Moreover, $\mathbf{k}_0$ is a DFN-space because the linear maps $\mathbf{k}_0(r) \to \mathbf{k}_0(r')$ are nuclear for $0 < r' < r$.
(ii)–(iv) are clear. q.e.d.

Note that $\mathbf{k}$ is not multiplicatively convex.

Let M be a real analytic manifold, X a complexification of M. We denote as usual by $\mathcal{A}_M$ the sheaf on M of real analytic functions, that is, $\mathcal{A}_M = \mathcal{O}_X|_M$. Recall that, for K compact in M, $\Gamma(K;\mathcal{A}_M)$ is a DFN-space. We set

$$\mathcal{A}_M^\tau = \mathcal{O}_X^\tau|_M. \tag{5.2}$$

Lemma 5.2. *Let M be a real analytic manifold and K a compact subset of M. Then*

(i) *the sheaf $\mathcal{A}_M^\tau$ is $\Gamma(K;\bullet)$-acyclic,*

(ii) $\Gamma(K;\mathcal{A}_M^\tau) \simeq \Gamma(K;\mathcal{A}_M)\widehat{\otimes}\,\mathbf{k}$,

(iii) *the same result holds with $\mathcal{A}_M^\tau$ and $\mathbf{k}$ replaced with $\mathcal{A}_M^\tau(0)$ and $\mathbf{k}_0$, respectively.*

Proof. Applying Lemma 1.4, we have isomorphisms for each holomorphically convex compact subset K of X:

$$\begin{aligned}
\Gamma(K;\mathcal{O}_X^\tau(0)) &\simeq \Gamma(K\times\{0\};\mathcal{O}_{X\times\mathbb{C}}) \\
&\simeq \Gamma(K;\mathcal{O}_X)\widehat{\otimes}\,\mathcal{O}_{\mathbb{C},0} \\
&\simeq \Gamma(K;\mathcal{O}_X)\widehat{\otimes}\,\mathbf{k}_0.
\end{aligned}$$

This proves (i)–(ii) for $\mathcal{A}_M^\tau(0)$ and $\mathbf{k}_0$. The other case follows since $\mathcal{A}_M^\tau \simeq \mathcal{A}_M^\tau(0)\otimes_{\mathbf{k}_0}\mathbf{k}$. q.e.d.

Let us denote, as usual, by $\mathcal{D}b_M$ the sheaf of Schwartz's distribution on M. Recall that $\Gamma_c(M;\mathcal{D}b_M)$ is a DFN-space.

Lemma 5.3. *Let M be a real analytic manifold. There is a unique* (*up to unique isomorphism*) *sheaf of $\mathbf{k}$-modules $\mathcal{D}b_M^\tau$ on M which is soft and satisfies*

$$\Gamma_c(U;\mathcal{D}b_M^\tau) \simeq \Gamma_c(U;\mathcal{D}b_M)\widehat{\otimes}\mathbf{k}$$

for each open subset U of M. The same result holds with $\mathbf{k}$ replaced with $\mathbf{k}_0$. In this case, we denote by $\mathcal{D}b_M^\tau(0)$ the sheaf of $\mathbf{k}_0$-modules so obtained.

Proof. For two open subsets U_0 and U_1, the sequence

$$\begin{aligned}
0 \to \Gamma_c(U_0\cap U_1;\mathcal{D}b_M)\widehat{\otimes}\,\mathbf{k} \to (\Gamma_c(U_0;\mathcal{D}b_M)\widehat{\otimes}\,\mathbf{k}) \oplus (\Gamma_c(U_1;\mathcal{D}b_M)\widehat{\otimes}\,\mathbf{k}) \\
\to \Gamma_c(U_0\cup U_1;\mathcal{D}b_M)\widehat{\otimes}\,\mathbf{k} \to 0
\end{aligned}$$

is exact, and similarly with $\mathbf{k}$ replaced with $\mathbf{k}_0$. The results then easily follow. q.e.d.

Denote by $\overline{X}$ the complex conjugate manifold to X and by $X_{\mathbb{R}}$ the real underlying manifold, identified with the diagonal of $X \times \overline{X}$. We shall write for short $\mathcal{A}_X^\tau$ and $\mathcal{D}b_X^\tau$ instead of $\mathcal{A}_{X_{\mathbb{R}}}^\tau$ and $\mathcal{D}b_{X_{\mathbb{R}}}^\tau$, respectively. We set $\mathcal{A}_X^{(p,q),\tau} = \mathcal{A}_X^{(p,q)} \otimes_{\mathcal{A}_X} \mathcal{A}_X^\tau$, and similarly with $\mathcal{D}b_X$ instead of $\mathcal{A}_X$.

Consider the Dolbeault-Grothendieck complexes of sheaves of $\mathbf{k}$-modules

$$0 \to \mathcal{A}_X^{(0,0),\tau} \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \mathcal{A}_X^{(0,d),\tau} \to 0, \tag{5.3}$$

$$0 \to \mathcal{D}b_X^{(0,0),\tau} \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \mathcal{D}b_X^{(0,d),\tau} \to 0. \tag{5.4}$$

Lemma 5.4. *Both complexes* (5.3) *and* (5.4) *are qis to* $\mathcal{O}_X^\tau$. *The same result holds when replacing* $\mathcal{A}_X^\tau$, $\mathcal{D}b_X^\tau$ *and* $\mathcal{O}_X^\tau$ *with* $\mathcal{A}_X^\tau(0)$, $\mathcal{D}b_X^\tau(0)$ *and* $\mathcal{O}_X^\tau(0)$, *respectively.*

The easy proof is left to the reader.

Lemma 5.5. *Let X be a complex Stein manifold, K a holomorphically convex compact subset and $\mathcal{F}$ a locally free $\mathcal{O}_X$-module of finite rank. Then*

(i) *one has an isomorphism* $\Gamma(K;\mathcal{F}^\tau) \simeq \Gamma(K;\mathcal{F}) \widehat{\otimes}\, \mathbf{k}$,

(ii) *the* $\mathbb{C}$*-vector space* $\Gamma(K;\mathcal{F}^\tau)$ *is naturally endowed with a topology of DFN-space,*

(iii) $\mathrm{R}\Gamma(K;\mathcal{F}^\tau)$ *is concentrated in degree* 0,

(iv) *for K_0 and K_1 two compact subsets of X such that K_0 is contained in the interior of K_1, the morphism $\Gamma(K_1;\mathcal{F}^\tau) \to \Gamma(K_0;\mathcal{F}^\tau)$ is* $\mathbf{k}$*-nuclear,*

(v) *the same results hold with $\mathcal{F}^\tau$ and $\mathbf{k}$ replaced with $\mathcal{F}^\tau(0)$ and $\mathbf{k}_0$, respectively.*

Proof. (i)–(iii) By the hypothesis, the sequence

$$0 \to \Gamma(K;\mathcal{O}_X) \to \Gamma(K;\mathcal{A}_X^{(0,0)}) \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \Gamma(K;\mathcal{A}_X^{(0,d)}) \to 0 \tag{5.5}$$

is exact. It remains exact after applying the functor $\bullet \widehat{\otimes}\, \mathbf{k}$. The result will follow when comparing the sequence so obtained with the complex

$$0 \to \Gamma(K;\mathcal{O}_X^\tau) \to \Gamma(K;\mathcal{A}_X^{(0,0),\tau}) \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \Gamma(K;\mathcal{A}_X^{(0,d),\tau}) \to 0, \tag{5.6}$$

The case of $\mathcal{F}$ is treated similarly, replacing $\mathcal{A}_X$ with $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{A}_X$.
(iv) follows from (i) and the corresponding result for $\mathcal{O}_X$.
(iv) The proof for $\mathcal{F}^\tau(0)$ and $\mathbf{k}_0$ is similar. q.e.d.

Lemma 5.6. *Let X be a complex manifold of complex dimension d and let $\mathcal{F}$ be a locally free $\mathcal{O}_X$-module of finite rank. Assume that X is Stein. Then*

(i) *one has the isomorphism $H^d_c(X;\mathcal{F}^\tau) \simeq H^d_c(X;\mathcal{F})\widehat{\otimes}\mathbf{k}$,*

(ii) *the $\mathbb{C}$-vector space $H^d_c(X;\mathcal{F}^\tau)$ is naturally endowed with a topology of DFN-space,*

(iii) *$\mathrm{R}\Gamma_c(X;\mathcal{F}^\tau_X)$ is concentrated in degree d,*

(iv) *for U_0 and U_1 two Stein open subset of X with $U_0 \subset\subset U_1$, the map $H^d_c(U_0;\mathcal{F}^\tau_X) \to H^d_c(U_1;\mathcal{F}^\tau_X)$ is $\mathbf{k}$-nuclear,*

(v) *the same results hold with $\mathcal{F}^\tau$ and $\mathbf{k}$ replaced with $\mathcal{F}^\tau(0)$ and $\mathbf{k}_0$, respectively.*

Proof. The proof is similar to that of Lemma 5.5. By the hypothesis, the sequence

$$0 \to \Gamma_c(X;\mathcal{D}b_X^{(0,0)}) \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \Gamma_c(X;\mathcal{D}b_X^{(0,d)}) \to H^d_c(X;\mathcal{O}_X) \to 0 \tag{5.7}$$

is exact and $H^d_c(X;\mathcal{O}_X)$ is a DFN-space. This sequence will remain exact after applying the functor $\bullet\widehat{\otimes}\mathbf{k}$. The result will follow when comparing the obtained sequence with

$$0 \to \Gamma_c(X;\mathcal{D}b_X^{(0,0),\tau}) \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \Gamma_c(X;\mathcal{D}b_X^{(0,d),\tau}) \to H^d_c(X;\mathcal{O}^\tau_X) \to 0. \tag{5.8}$$

The case of $\mathcal{F}$ is treated similarly, replacing $\mathcal{D}b_X$ with $\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{D}b_X$. q.e.d.

Proposition 5.7. *Let $f\colon X \to Y$ be a morphism of complex manifolds of complex dimension d_X and d_Y, respectively. There is a $\mathbf{k}$-linear morphism*

$$\int_f \colon Rf_!\Omega^\tau_X\,[d_X] \to \Omega^\tau_Y\,[d_Y] \tag{5.9}$$

functorial with respect to f, which sends $Rf_!\Omega^\tau_X(0)\,[d_X]$ to $\Omega^\tau_Y(0)\,[d_Y]$ and which induces the classical integration morphism $\int_f\colon Rf_!\Omega_X\,[d_X] \to \Omega_Y\,[d_Y]$ when identifying Ω_X and Ω_Y with a direct summand of Ω^τ_X and Ω^τ_Y, respectively.

In particular, there is a $\mathbf{k}$-linear morphism

$$\int_X \colon H^{d_X}_c(X;\Omega^\tau_X) \to \mathbf{k} \tag{5.10}$$

which induces the classical residues morphism $H^{d_X}_c(X;\Omega_X) \to \mathbb{C}$ when identifying $H^{d_X}_c(X;\Omega_X)$ (resp. $\mathbb{C}$) with a direct summand of $H^{d_X}_c(X;\Omega^\tau_X)$ (resp. $\mathbf{k}$).

In the particular case where X is Stein and $Y = \mathrm{pt}$, this follows easily from Lemma 5.6. Since we shall not use the general case, we leave the proof to the reader.

Proposition 5.8. *Let X be a Stein complex manifold of complex dimension d and let K be a holomorphically convex compact subset of X. Then the pairing for a Stein open subset U of X*

$$H^d_c(U; \Omega^\tau_X) \times \Gamma(U; \mathcal{O}^\tau_X) \to \mathbf{k}, \qquad (f dx, g) \mapsto \int_U g f dx, \tag{5.11}$$

defines the isomorphisms $\text{“}\varprojlim_{U \supset K}\text{”}\, \mathrm{Bhom}_{\mathbf{k}}(\Gamma(\overline{U}; \mathcal{O}^\tau_X), \mathbf{k}) \simeq \text{“}\varprojlim_{U \supset K}\text{”}\, H^d_c(U; \Omega^\tau_X)$ *and* $\text{“}\varinjlim_{U \supset K}\text{”}\, \mathrm{Bhom}_{\mathbf{k}}(H^d_c(U; \Omega^\tau_X), \mathbf{k}) \simeq \text{“}\varinjlim_{U \supset K}\text{”}\, \Gamma(\overline{U}; \mathcal{O}^\tau_X)$*. Here, U ranges over the family of Stein open neighborhoods of K.*

Proof. We shall prove the first isomorphism, the other case being similar.

By Lemmas 5.5, 5.6 and 4.8, we have:

$$\begin{aligned}
\text{“}\varprojlim_{U \supset K}\text{”}\, \mathrm{Bhom}_{\mathbf{k}}(\Gamma(\overline{U}; \mathcal{O}^\tau_X), \mathbf{k}) &\simeq \text{“}\varprojlim_{U \supset K}\text{”}\, (\mathcal{O}_X(\overline{U}) \widehat{\otimes}\, \mathbf{k})^\vee \\
&\simeq \text{“}\varprojlim_{U \supset K}\text{”}\, (\mathcal{O}_X(\overline{U}))^\star \widehat{\otimes}\, \mathbf{k} \\
&\simeq \text{“}\varprojlim_{U \supset K}\text{”}\, (\mathcal{O}_X(U))^\star \widehat{\otimes}\, \mathbf{k} \\
&\simeq \text{“}\varprojlim_{U \supset K}\text{”}\, (H^d_c(U; \Omega_X)) \widehat{\otimes}\, \mathbf{k} \\
&\simeq \text{“}\varprojlim_{U \supset K}\text{”}\, H^d_c(U; \Omega^\tau_X).
\end{aligned}$$

q.e.d.

In the proof of Corollary 7.6, we shall need the following result.

Lemma 5.9. *Let Y be a smooth closed submanifold of codimension l of X. Then $H^j(\mathrm{R}\Gamma_Y(\mathcal{O}^\tau_X))$ and $H^j(\mathrm{R}\Gamma_Y(\mathcal{O}^\tau_X(0)))$ vanish for $j < l$.*

Proof. Since the problem is local, we may assume that $X = Y \times \mathbb{C}^l$ where $Y \subset X$ is identified with $Y \times \{0\}$.

Let U_n be the open ball of $\mathbb{C}^l$ centered at 0 with radius $1/n$. Using the Mittag-Leffler theorem (see [12, Prop. 2.7.1]), it is enough to prove that for any holomorphically convex compact subset K of Y

$$H^j_c(K \times U_n; \mathcal{O}^\tau_X) = 0 \text{ for } j < l,$$

and similarly with $\mathcal{O}^\tau_X(0)$. It is enough to prove the result for the sheaf $\mathcal{O}^\tau_X(0)$. Then we may replace $\mathcal{O}^\tau_X(0)$ with the sheaf $\mathcal{O}_{X\times\mathbb{C}}|_{X\times\{0\}}$ and we are reduced to the well-known result

$$H^j_c(K\times U_n\times\{0\};\mathcal{O}_{X\times\mathbb{C}})=0 \text{ for } j<l.$$

q.e.d.

Remark 5.10. Related results have been obtained, in a slightly different framework, in [17]

6 Duality for $\mathcal{W}$-modules

We mainly follow the notations of [12]. Let X be a real manifold and $\mathbb{K}$ a field. For $F\in \mathrm{D}^{\mathrm{b}}(\mathbb{K}_X)$, we denote by $SS(F)$ its microsupport, a closed $\mathbb{R}^+$-conic subset of T^*X. Recall that this set is *involutive* (see loc. cit. Def. 6.5.1).

We denote by D_X the duality functor:

$$\mathrm{D}_X\colon (\mathrm{D}^{\mathrm{b}}(\mathbb{K}_X))^{\mathrm{op}}\to \mathrm{D}^{\mathrm{b}}(\mathbb{K}_X),\quad F\mapsto R\mathcal{H}om_{\mathbb{K}_X}(F,\omega_X),$$

where ω_X is the dualizing complex.

Assume now that X is a complex manifold. We denote by $\dim_{\mathbb{C}}X$ its complex dimension. We identify the orientation sheaf on X with the constant sheaf, and the dualizing complex ω_X with $\mathbb{K}_X\,[2\dim_{\mathbb{C}}X]$.

Recall that an object $F\in \mathrm{D}^{\mathrm{b}}(\mathbb{K}_X)$ is weakly-$\mathbb{C}$-constructible if there exists a complex analytic stratification $X=\bigsqcup_{\alpha\in A}X_\alpha$ such that $H^j(F)|_{X_\alpha}$ is locally constant for all $j\in\mathbb{Z}$ and all $\alpha\in A$. The object F is $\mathbb{C}$-constructible if moreover $H^j(F)_x$ is finite-dimensional for all $x\in X$ and all $j\in\mathbb{Z}$. We denote by $\mathrm{D}^{\mathrm{b}}_{\mathrm{w-\mathbb{C}-c}}(\mathbb{K}_X)$ the full subcategory of $\mathrm{D}^{\mathrm{b}}(\mathbb{K}_X)$ consisting of weakly-$\mathbb{C}$-constructible objects and by $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\mathrm{-c}}(\mathbb{K}_X)$ the full subcategory consisting of $\mathbb{C}$-constructible objects.

Recall ([12]) that $F\in \mathrm{D}^{\mathrm{b}}(\mathbb{K}_X)$ is weakly-$\mathbb{C}$-constructible if and only if its microsupport is a closed $\mathbb{C}$-conic complex analytic Lagrangian subset of T^*X or, equivalently, if it is contained in a closed $\mathbb{C}$-conic complex analytic isotropic subset of T^*X.

From now on, our base field is $\mathbf{k}$.

Theorem 6.1. *Let $\mathfrak{X}$ be a complex symplectic manifold and let $\mathcal{L}_0$ and $\mathcal{L}_1$ be two objects of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{W}_{\mathfrak{X}})$.*

(i) *There is a natural morphism*

$$\text{(6.1)}\quad R\mathcal{H}om_{\mathcal{W}_{\mathfrak{X}}}(\mathcal{L}_1,\mathcal{L}_0)\to \mathrm{D}_{\mathfrak{X}}\big(R\mathcal{H}om_{\mathcal{W}_{\mathfrak{X}}}(\mathcal{L}_0,\mathcal{L}_1\,[\dim_{\mathbb{C}}\mathfrak{X}])\big).$$

(ii) *Assume that* $R\mathcal{H}om_{\mathcal{W}_{\mathfrak{X}}}(\mathcal{L}_0,\mathcal{L}_1)$ *belongs to* $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}-\mathrm{c}}(\mathbf{k}_{\mathfrak{X}})$. *Then the morphism* (6.1) *is an isomorphism. In particular,* $R\mathcal{H}om_{\mathcal{W}_{\mathfrak{X}}}(\mathcal{L}_1,\mathcal{L}_0)$ *belongs to* $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}-\mathrm{c}}(\mathbf{k}_{\mathfrak{X}})$.

Proof. By Proposition 3.3, we may assume that $\mathcal{L}_0$ is a simple module along a smooth Lagrangian manifold Λ_0. In this case, $\mathbf{k}_{\Lambda_0}\to R\mathcal{H}om_{\mathcal{W}_{\mathfrak{X}}}(\mathcal{L}_0,\mathcal{L}_0)$ is an isomorphism.

(i) The natural $\mathbf{k}$-linear morphism

$$R\mathcal{H}om_{\mathcal{W}_{\mathfrak{X}}}(\mathcal{L}_1,\mathcal{L}_0)\otimes_{\mathbf{k}} R\mathcal{H}om_{\mathcal{W}_{\mathfrak{X}}}(\mathcal{L}_0,\mathcal{L}_1)\to R\mathcal{H}om_{\mathcal{W}_{\mathfrak{X}}}(\mathcal{L}_0,\mathcal{L}_0)\simeq\mathbf{k}_{\Lambda_0}$$

defines the morphism

$$\text{(6.2)}\qquad R\mathcal{H}om_{\mathcal{W}_{\mathfrak{X}}}(\mathcal{L}_0,\mathcal{L}_1)\to R\mathcal{H}om_{\mathbf{k}_{\Lambda_0}}(R\mathcal{H}om_{\mathcal{W}_{\mathfrak{X}}}(\mathcal{L}_1,\mathcal{L}_0),\mathbf{k}_{\Lambda_0}).$$

To conclude, remark that for an object $F\in\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathfrak{X}})$ supported by Λ_0, we have

$$\mathrm{D}_{\mathfrak{X}}F\simeq R\mathcal{H}om_{\mathbf{k}_{\Lambda_0}}(F,\mathbf{k}_{\Lambda_0})\,[\dim_{\mathbb{C}}\mathfrak{X}].$$

(ii) Let us prove that (6.2) is an isomorphism by adapting the proof of [7, Prop. 5.1].

Since this is a local problem, we may assume that $\mathfrak{X}=T^*X$, X is an open subset of $\mathbb{C}^d$, $\Lambda_0=T^*_XX$ and $\mathcal{L}_0=\mathcal{O}^\tau_X$. Since $R\mathcal{H}om_{\mathcal{W}_{\mathfrak{X}}}(\mathcal{O}^\tau_X,\mathcal{L}_1)$ is constructible, we are reduced to prove the isomorphisms for each $x\in X$

$$\text{(6.3)}\ H^j(R\mathcal{H}om_{\mathcal{W}_{\mathfrak{X}}}(\mathcal{L}_1,\mathcal{O}^\tau_X))_x\simeq\varinjlim_{U\ni x}(H^j\mathrm{R}\Gamma_c(U;R\mathcal{H}om_{\mathcal{W}_{\mathfrak{X}}}(\mathcal{O}^\tau_X,\mathcal{L}_1))\,[2d])^\star$$

where U ranges over the family of Stein open neighborhoods of x and * denotes the duality functor in the category of $\mathbf{k}$-vector spaces.

We choose a finite free resolution of $\mathcal{L}_1$ on a neighborhood of x:

$$0\to\mathcal{W}^{N_r}_{T^*X}\xrightarrow{\cdot P_{r-1}}\cdots\xrightarrow{\cdot P_0}\mathcal{W}^{N_0}_{T^*X}\to\mathcal{L}_1\to 0\text{ for some }r\geq 0.$$

For a sufficiently small holomorphically convex compact neighborhood K of x, the object $\mathrm{R}\Gamma(K;R\mathcal{H}om_{\mathcal{W}_{T^*X}}(\mathcal{L}_1,\mathcal{O}^\tau_X))$ is represented by the complex

$$E^\bullet(K):=\qquad 0\to(\Gamma(K;\mathcal{O}^\tau_X))^{N_0}\xrightarrow{P_0\cdot}\cdots\xrightarrow{P_{r-1}\cdot}(\Gamma(K;\mathcal{O}^\tau_X))^{N_r}\to 0,$$

where $(\Gamma(K;\mathcal{O}_X^\tau))^{N_0}$ stands in degree 0. Since

$$R\mathcal{H}om_{\mathcal{W}_{T^*X}}(\mathcal{O}_X^\tau, \mathcal{W}_{T^*X}\,[d]) \simeq \Omega_X^\tau,$$

the object $R\mathcal{H}om_{\mathcal{W}}(\mathcal{O}_X^\tau, \mathcal{L}_1\,[d])$ is represented by the complex

$$0 \to (\Omega_X^\tau)^{N_r} \xrightarrow{\cdot P_{r-1}} \cdots \xrightarrow{\cdot P_0} (\Omega_X^\tau)^{N_0} \to 0,$$

where $(\Omega_X^\tau)^{N_0}$ stands in degree 0. Hence, for a sufficiently small Stein open neighborhood U of x, $\mathrm{R}\Gamma_c(U; R\mathcal{H}om_{\mathcal{W}}(\mathcal{O}_X^\tau, \mathcal{L}_1\,[2d]))$ is represented by the complex

$$F_c^\bullet(U) := \qquad 0 \to (H_c^d(U;\Omega_X^\tau))^{N_r} \xrightarrow{\cdot P_{r-1}} \cdots \xrightarrow{\cdot P_0} (H_c^d(U;\Omega_X^\tau))^{N_0} \to 0$$

where $(H_c^d(U;\Omega_X^\tau))^{N_0}$ stands in degree 0.

Let U_n be the open ball of X centered at x with radius $1/n$. By the hypothesis, all morphisms

$$F_c^\bullet(U_p) \to F_c^\bullet(U_n)$$

are quasi-isomorphisms for $p \geq n \gg 0$, and the cohomologies are finite-dimensional over $\mathbf{k}$. Therefore, the hypotheses of Theorem 4.7 are satisfied by Lemma 5.6 and we get

$$\text{“}\varinjlim_n\text{”}\, \mathrm{Hom}_{\mathbf{k}}(F_c^\bullet(U_n), \mathbf{k}) \simeq \text{“}\varinjlim_n\text{”}\, (F_c^\bullet(U_n))^\vee \text{ in } \mathrm{Ind}(\mathrm{K}^{\mathrm{b}}(\mathrm{Mod}^{\mathrm{dfn}}(\mathbf{k}))).$$

Applying Proposition 5.8, we obtain

$$\begin{aligned} H^j(R\mathcal{H}om_{\mathcal{W}}(\mathcal{L}_1, \mathcal{O}_X^\tau))_x &\simeq \varinjlim_n H^j(E^\bullet(\overline{U}_n)) \\ &\simeq \varinjlim_n H^j\mathrm{Hom}_{\mathbf{k}}(F_c^\bullet(U_n), \mathbf{k}) \\ &\simeq \varinjlim_n (H^j \mathrm{R}\Gamma_c(U; R\mathcal{H}om_{\mathcal{W}_{\mathfrak{X}}}(\mathcal{O}_X^\tau, \mathcal{L}_1))\,[2d])^\star. \end{aligned}$$

q.e.d.

Corollary 6.2. *In the situation of* Theorem 6.1 (ii), *assume moreover that $\mathfrak{X}$ is compact of complex dimension $2n$. Then the* $\mathbf{k}$*-vector spaces* $\mathrm{Ext}^j_{\mathcal{W}_{\mathfrak{X}}}(\mathcal{L}_1, \mathcal{L}_0)$ *and* $\mathrm{Ext}^{2n-j}_{\mathcal{W}_{\mathfrak{X}}}(\mathcal{L}_0, \mathcal{L}_1)$ *are finite-dimensional and dual to each other.*

7 Statement of the main theorem

For two subsets V and S of the real manifold X, the normal cone $C(S,V)$ is well defined in TX. If V is a smooth and closed submanifold of X, one denotes by $C_V(S)$ the image of $C(S,V)$ in the normal bundle T_VX.

Consider now a complex symplectic manifold $(\mathfrak{X},\omega)$. The 2-form ω gives the Hamiltonian isomorphism H from the cotangent to the tangent bundle to $\mathfrak{X}$:

$$H\colon T^*\mathfrak{X} \xrightarrow{\sim} T\mathfrak{X}, \quad \langle\theta, v\rangle = \omega(v, H(\theta)), \quad v\in T\mathfrak{X},\ \theta\in T^*\mathfrak{X}. \tag{7.1}$$

For a smooth Lagrangian submanifold Λ of $\mathfrak{X}$, the isomorphism (7.1) induces an isomorphism between the normal bundle to Λ in $\mathfrak{X}$ and its cotangent bundle:

$$T_\Lambda\mathfrak{X} \simeq T^*\Lambda. \tag{7.2}$$

Let us recall a few notations and conventions (see [12]). For a complex manifold X and a complex analytic subvariety Z of X, one denotes by Z_{reg} the smooth locus of Z, a complex submanifold of X. For a holomorphic p-form θ on X, one says that θ vanishes on Z and one writes $\theta|_Z = 0$ if $\theta|_{Z_{\mathrm{reg}}} = 0$.

Proposition 7.1. *Let $\mathfrak{X}$ be a complex symplectic manifold and let Λ_0 and Λ_1 be two closed complex analytic isotropic subvarieties of $\mathfrak{X}$. Then, after identifying $T\mathfrak{X}$ and $T^*\mathfrak{X}$ by* (7.1)*, the normal cone $C(\Lambda_0,\Lambda_1)$ is a complex analytic $\mathbb{C}^\times$-conic isotropic subvariety of $T^*\mathfrak{X}$.*

Note that the same result holds for real analytic symplectic manifolds, replacing "complex analytic variety" with "subanalytic subset" and "$\mathbb{C}^\times$-conic" with "$\mathbb{R}^+$-conic".

First we need two lemmas.

Lemma 7.2. *Let X be a complex manifold and θ a p-form on X. Let $Z\subset Y$ be closed subvarieties of X. If $\theta|_Y = 0$, then $\theta|_Z = 0$.*

Proof. By Whitney's theorem, we can find an open dense subset Z' of Z_{reg} such that

$$\begin{cases}\text{for any sequence } \{y_n\}_n \text{ in } Y_{\mathrm{reg}} \text{ such that it converges to a point } z\in Z' \\ \text{and } \{T_{y_n}Y\}_n \text{ converges to a linear subspace } \tau\subset T_zX,\ \tau \text{ contains } T_zZ'.\end{cases}$$

Since θ vanishes on $T_{y_n}Y$, it vanishes also on τ and hence on T_zZ'. q.e.d.

Lemma 7.3. *Let X be a complex manifold, Y a closed complex subvariety of X and $f\colon X \to \mathbb{C}$ a holomorphic function. Set $Z{:=}f^{-1}(0)$, $Y'{:=}\overline{(Y \setminus Z)} \cap Z$. Consider a p-form η, a $(p-1)$-form θ on X and set*

$$\omega = df \wedge \theta + f\eta.$$

Assume that $\omega|_Y = 0$. Then $\theta|_{Y'} = 0$ and $\eta|_{Y'} = 0$.

Proof. We may assume that $Y = \overline{(Y \setminus Z)}$.

Using Hironaka's desingularization theorem, we may find a smooth manifold $\widetilde{Y}$ and a proper morphism $p\colon \widetilde{Y} \to Y$ such that, in a neighborhood of each point of $\widetilde{Y}$, p^*f may be written in a local coordinate system $(y_1, \dots, y_n)$ as a product $\prod_{i=1}^n y_i^{a_i}$, where the a_i's are non-negative integers. Let $\widetilde{Z} = p^{-1}(Z)$. Then $\overline{(\widetilde{Y} \setminus \widetilde{Z})} \cap \widetilde{Z} \to Y'$ is proper and surjective.

Hence, we may assume from the beginning that Y is smooth and then $Y = X$. Moreover, it is enough to prove the result at generic points of Y'. Hence, we may assume, setting $(y_1, \dots, y_n) = (t, x)$ $(x = (y_2, \dots, y_n))$, that $f(t, x) = t^a$ for some $a > 0$. Write

$$\theta = t\theta_0 + dt \wedge \theta_1 + \theta_2,$$

where θ_1 and θ_2 depend only on x and dx.

By the hypothesis,

$$0 = df \wedge \theta + f\eta = at^a dt \wedge \theta_0 + at^{a-1} dt \wedge \theta_2 + t^a \eta.$$

It follows that θ_2 is identically zero. Hence:

$$\theta = t\theta_0 + dt \wedge \theta_1, \quad \eta = -a dt \wedge \theta_0.$$

Therefore, $\theta|_{t=0} = \eta|_{t=0} = 0$. q.e.d.

Proof of Proposition 7.1. Recall that $\mathfrak{X}^a$ denotes the complex manifold $\mathfrak{X}$ endowed with the symplectic form $-\omega$ and that Δ denotes the diagonal of $\mathfrak{X} \times \mathfrak{X}^a$. Using the isomorphisms

$$\begin{aligned} &T\mathfrak{X} \simeq T_\Delta(\mathfrak{X} \times \mathfrak{X}^a), \\ &C(\Lambda_0, \Lambda_1) \simeq C(\Delta, \Lambda_0 \times \Lambda_1^a), \end{aligned}$$

(the first isomorphism is associated with the first projection on $\mathfrak{X} \times \mathfrak{X}^a$) we are reduced to prove the result when Λ_0 is smooth.

Let (x,u) be a local symplectic coordinate system on X such that

$$\Lambda_0 = \{(x;u)\,;\, u=0\}\,, \qquad \omega = \sum_{i=1}^n du_i \wedge dx_i.$$

Consider the normal deformation $\widetilde{\mathfrak{X}}_{\Lambda_0}$ of $\mathfrak{X}$ along Λ_0. Recall that we have a diagram

$$\begin{tikzcd}
\mathfrak{X} & \widetilde{\mathfrak{X}}_{\Lambda_0} \arrow[l, "p"'] \arrow[r, "t"] & \mathbb{C} \\
\Lambda_1 \arrow[u, hook] & p^{-1}\Lambda_1 \arrow[l, "p"'] \arrow[u, hook] &
\end{tikzcd}$$

such that, denoting by $(x;\xi,t)$ the coordinates on $\widetilde{\mathfrak{X}}_{\Lambda_0}$,

$$\begin{aligned}
&p(x;\xi,t) = (x;t\xi),\\
&T_{\Lambda_0}\mathfrak{X} \simeq \Big\{(x;\xi,t)\in \widetilde{\mathfrak{X}}_{\Lambda_0}\,;\, t=0\Big\}\,,\\
&C_{\Lambda_0}(\Lambda_1) \simeq \overline{p^{-1}(\Lambda_1)\setminus t^{-1}(0)} \cap t^{-1}(0).
\end{aligned}$$

Clearly, $C_{\Lambda_0}(\Lambda_1)$ is a complex analytic variety. Moreover,

$$p^*\omega = \sum_{i=1}^n d(t\xi_i)\wedge dx_i = dt \wedge \big(\sum_{i=1}^n \xi_i \wedge dx_i\big) + t\sum_{i=1}^n d\xi_i\wedge dx_i.$$

Since $p^*\omega$ vanishes on $(p^{-1}\Lambda_1)_{\mathrm{reg}}$, $\sum_{i=1}^n d\xi_i \wedge dx_i$ vanishes on $C_{\Lambda_0}(\Lambda_1)$ by Lemma 7.3. q.e.d.

Theorem 7.4. *Let $\mathfrak{X}$ be a complex symplectic manifold and let $\mathcal{L}_i$ $(i=0,1)$ be two objects of $\mathrm{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathcal{W}_{\mathfrak{X}})$ supported by smooth Lagrangian submanifolds Λ_i. Then*

(i) *the object $R\mathcal{H}om_{\mathcal{W}_{\mathfrak{X}}}(\mathcal{L}_1,\mathcal{L}_0)$ belongs to $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}-\mathrm{c}}(\mathbf{k}_{\mathfrak{X}})$ and its microsupport is contained in the normal cone $C(\Lambda_0,\Lambda_1)$,*

(ii) *the natural morphism*

$$R\mathcal{H}om_{\mathcal{W}_{\mathfrak{X}}}(\mathcal{L}_1,\mathcal{L}_0) \to \mathrm{D}_{\mathfrak{X}}\big(R\mathcal{H}om_{\mathcal{W}_{\mathfrak{X}}}(\mathcal{L}_0,\mathcal{L}_1\,[\dim_{\mathbb{C}}\mathfrak{X}])\big)$$

is an isomorphism.

The proof of (i) will be given in § 8 and (ii) is a particular case of Theorem 6.1.

Conjecture 7.5. Theorem 7.4 remains true without assuming that the Λ_i's are smooth.

Remark that the analogous of Conjecture 7.5 for complex contact manifolds is true over the field $\mathbb{C}$, as we shall see in § 9.

Corollary 7.6. *Let $\mathcal{L}_0$ and $\mathcal{L}_1$ be two regular holonomic $\mathcal{W}_{\mathfrak{X}}$-modules supported by smooth Lagrangian submanifolds. Then the object $R\mathcal{H}om_{\mathcal{W}_{\mathfrak{X}}}(\mathcal{L}_1, \mathcal{L}_0)$ of $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}-\mathrm{c}}(\mathbf{k}_{\mathfrak{X}})$ is perverse.*

Proof. Since the problem is local, we may assume that $\mathfrak{X} = T^*X$, $\Lambda_0 = T^*_X X$ and $\mathcal{L}_0 = \mathcal{O}^\tau_X$.

By Theorem 7.4 (ii), it is enough to check that if Y is a locally closed smooth submanifold of X of codimension l, then $H^j(\mathrm{R}\Gamma_Y(R\mathcal{H}om_{\mathcal{W}_{T^*X}}(\mathcal{L}_1, \mathcal{L}_0))|_Y$ vanishes for $j < l$. This follows from Lemma 5.9. q.e.d.

Remark 7.7. It would be interesting to compare $R\mathcal{H}om_{\mathcal{W}_{\mathfrak{X}}}(\mathcal{L}_1, \mathcal{L}_0)$ with the complexes obtained in [1].

8 Proof of Theorem 7.4

In this section X denotes a complex manifold. As usual, $\mathcal{O}_X$ is the structure sheaf and $\mathcal{D}_X$ is the sheaf of rings of (finite–order) differential operators. For a coherent $\mathcal{D}_X$-module $\mathcal{M}$, we denote by $\mathrm{char}(\mathcal{M})$ its characteristic variety. This notion extends to the case where $\mathcal{M}$ is a countable union of coherent sub-$\mathcal{D}_X$-modules. In this case, one sets

$$\mathrm{char}(\mathcal{M}) = \overline{\bigcup_{\mathcal{N} \subset \mathcal{M}} \mathrm{char}(\mathcal{N})}$$

where $\mathcal{N}$ ranges over the family of coherent $\mathcal{D}_X$-submodules of $\mathcal{M}$, and, for a subset S of T^*X, $\overline{S}$ means the closure of S.

Lemma 8.1. *Let $\mathcal{M}_0$ be a coherent $\mathcal{D}_X$-module. Then*

$$SS(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_0, \mathcal{O}^\tau_X(0))) \subset \mathrm{char}(\mathcal{M}_0). \tag{8.1}$$

Proof. Apply [19, Ch.3 Th. 3.2.1]. q.e.d.

Lemma 8.2. *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X[\tau^{-1}]$-module. Then*

$$SS(R\mathcal{H}om_{\mathcal{D}_X[\tau^{-1}]}(\mathcal{M}, \mathcal{O}^\tau_X(0))) \subset \mathrm{char}(\mathcal{M}). \tag{8.2}$$

Proof. Let $\mathcal{M}_0 \subset \mathcal{M}$ be a coherent $\mathcal{D}_X$-submodule which generates $\mathcal{M}$. Then

$$\mathrm{char}(\mathcal{D}_X[\tau^{-1}] \otimes_{\mathcal{D}_X} \mathcal{M}_0) = \mathrm{char}(\mathcal{M}_0) \subset \mathrm{char}(\mathcal{M}).$$

Consider the exact sequence of coherent $\mathcal{D}_X[\tau^{-1}]$-modules

$$0 \to \mathcal{N} \to \mathcal{D}_X[\tau^{-1}] \otimes_{\mathcal{D}_X} \mathcal{M}_0 \to \mathcal{M} \to 0. \tag{8.3}$$

Applying the functor $R\mathcal{H}om_{\mathcal{D}_X[\tau^{-1}]}(\bullet, \mathcal{O}_X^\tau)$ to the exact sequence (8.3), we get a distinguished triangle $G' \to G \to G'' \xrightarrow{+1}$. Note that $G \simeq R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_0, \mathcal{O}_X^\tau)$, since $\mathcal{D}_X[\tau^{-1}]$ is flat over $\mathcal{D}_X$.

Let $\theta = (x_0; p_0) \in T^*X$ with $\theta \notin \mathrm{char}(\mathcal{M})$ and let ψ be a real function on X such that $\psi(x_0) = 0$ and $d\psi(x_0) = p_0$. Consider the distinguished triangle

$$(\mathrm{R}\Gamma_{\psi \geq 0}(G'))_{x_0} \to (\mathrm{R}\Gamma_{\psi \geq 0}(G))_{x_0} \to (\mathrm{R}\Gamma_{\psi \geq 0}(G''))_{x_0} \xrightarrow{+1} . \tag{8.4}$$

By Lemma 8.1, we have:

$$H^j((\mathrm{R}\Gamma_{\psi \geq 0}(G))_{x_0}) \simeq 0 \text{ for all } j \in \mathbb{Z}.$$

The objects of the distinguished triangle (8.4) are concentrated in degree ≥ 0. Therefore, $H^j((\mathrm{R}\Gamma_{\psi \geq 0}(G'))_{x_0}) \simeq 0$ for $j \leq 0$.

Since $\mathrm{char}(\mathcal{N}) \subset \mathrm{char}(\mathcal{D}_X[\tau^{-1}] \otimes_{\mathcal{D}_X} \mathcal{M}_0)$, we get $H^j((\mathrm{R}\Gamma_{\psi \geq 0}(G''))_{x_0}) \simeq 0$ for $j \leq 0$. By repeating this argument, we deduce that $H^j((\mathrm{R}\Gamma_{\psi \geq 0}(G'))_{x_0}) \simeq H^{j-1}((\mathrm{R}\Gamma_{\psi \geq 0}(G''))_{x_0}) \simeq 0$ for all $j \in \mathbb{Z}$. q.e.d.

Note that the statement of Theorem 7.4 is local and invariant by quantized symplectic transformation. From now on, we denote by $(x; u)$ a local symplectic coordinate system on $\mathfrak{X}$ such that

$$X := \Lambda_0 = \{(x; u) \in \mathfrak{X} \,;\, u = 0\}.$$

We denote by $(x; \xi)$ the associated homogeneous symplectic coordinates on T^*X. The differential operator ∂_{x_i} on X has order 1 and principal symbol ξ_i. The monomorphism (1.7) extends as a monomorphism of rings

$$\mathcal{D}_X[\tau^{-1}, \tau] \hookrightarrow \mathcal{W}_{T^*X}.$$

Note that the total symbol of the operator ∂_{x_i} of $\mathcal{W}_{T^*X}$ is $u_i \cdot \tau$.

We may assume that there exists a holomorphic function $\varphi\colon X \to \mathbb{C}$ such that

$$\Lambda_1 = \{(x;u) \in \mathfrak{X}\,;\, u = \operatorname{grad}\varphi(x)\}\,. \tag{8.5}$$

Here, $\operatorname{grad}\varphi = (\varphi_1', \dots, \varphi_n')$ and $\varphi_i' = \frac{\partial\varphi}{\partial x_i}$.

If $\Lambda_0 = \Lambda_1$, Theorem 7.4 is immediate. We shall assume that $\Lambda_0 \neq \Lambda_1$ and thus that φ is not a constant function. We may assume that

$$\begin{cases} \mathcal{L}_0 = \mathcal{O}_X^\tau, \\ \mathcal{L}_1 = \mathcal{W}_{T^*X}/\mathcal{I}_1,\ \mathcal{I}_1 \text{ being the ideal generated by } \{\partial_{x_i} - \varphi_i'\tau\}_{i=1,\dots,n}. \end{cases} \tag{8.6}$$

To $\varphi\colon X \to \mathbb{C}$ are associated the maps

$$T^*X \xleftarrow{\varphi_d} X \times_{\mathbb{C}} T^*\mathbb{C} \xrightarrow{\varphi_\pi} T^*\mathbb{C},$$

and the $(\mathcal{D}_X, \mathcal{D}_{\mathbb{C}})$-bimodule $\mathcal{D}_{X \xrightarrow{\varphi} \mathbb{C}}$. Let t be the coordinate on $\mathbb{C}$. By identifying ∂_t and τ, we regard $\mathcal{D}_{X \xrightarrow{\varphi} \mathbb{C}}$ as a $\mathcal{D}_X[\tau]$-module. We set

$$V = \overline{\operatorname{Im}\varphi_d}, \text{ the closure of } \varphi_d(T^*\mathbb{C} \times_{\mathbb{C}} X).$$

Lemma 8.3. *Regarding $\mathcal{D}_{X \xrightarrow{\varphi} \mathbb{C}}$ as a $\mathcal{D}_X$-module, one has* $\operatorname{char}(\mathcal{D}_{X \xrightarrow{\varphi} \mathbb{C}}) \subset V$.

Note that $\mathcal{D}_{X \xrightarrow{\varphi} \mathbb{C}}$ is not a coherent $\mathcal{D}_X$-module in general.

Proof. Let Γ_φ be the graph of φ in $X \times \mathbb{C}$ and let $\mathcal{B}_\varphi$ be the coherent left $\mathcal{D}_{X\times\mathbb{C}}$-module associated to this submanifold. (Note that $\operatorname{char}(\mathcal{B}_\varphi) = \Lambda_\varphi$, the conormal bundle to Γ_φ in $T^*(X \times \mathbb{C})$.) We shall identify Γ_φ with X by the first projection on $X \times \mathbb{C}$ and the left $\mathcal{D}_X$ modules $\mathcal{B}_\varphi$ with $\mathcal{D}_{X \xrightarrow{\varphi} \mathbb{C}}$.

Denote by $\delta(t-\varphi)$ the canonical generator of $\mathcal{B}_\varphi$. Then

$$\mathcal{D}_{X \xrightarrow{\varphi} \mathbb{C}} = \mathcal{D}_{X\times\mathbb{C}} \cdot \delta(t-\varphi) = \sum_{n\in\mathbb{N}} \mathcal{D}_X \partial_t^n \cdot \delta(t-\varphi).$$

Define

$$\mathcal{N}_\varphi := \sum_{n\in\mathbb{N}} \mathcal{D}_X (t\partial_t)^n \cdot \delta(t-\varphi).$$

By [9, Th. 6.8], the $\mathcal{D}_X$-module $\mathcal{N}_\varphi$ is coherent and its characteristic variety is contained in V. All $\mathcal{D}_X$-submodules $\partial_t^n\mathcal{N}_\varphi$ of $\mathcal{D}_{X \xrightarrow{\varphi} \mathbb{C}}$ are isomorphic since ∂_t is injective on $\mathcal{B}_\varphi$. Then the result follows from

$$\mathcal{D}_{X \xrightarrow{\varphi} \mathbb{C}} = \sum_{n\in\mathbb{N}} \partial_t^n \mathcal{N}_\varphi.$$

q.e.d.

We consider the left ideal and the modules:

$$
(8.7)\quad \begin{cases} \mathcal{I}{:=} \text{ the left ideal of } \mathcal{D}_X[\tau^{-1},\tau] \text{ generated by } \{\partial_{x_i} - \varphi_i'\tau\}_{i=1,\dots,n}, \\ \mathcal{M} = \mathcal{D}_X[\tau^{-1}]/(\mathcal{I}\cap\mathcal{D}_X[\tau^{-1}]), \\ \mathcal{M}_0 = \mathcal{D}_X/(\mathcal{I}\cap\mathcal{D}_X). \end{cases}
$$

Lemma 8.4. (i) *$\mathcal{M}$ is $\mathcal{D}_X[\tau^{-1}]$-coherent,*

(ii) *$\mathcal{M}_0$ is $\mathcal{D}_X$-coherent,*

(iii) *we have an isomorphism of $\mathcal{D}_X$-modules $\mathcal{D}_X\cdot\delta(t-\varphi)\simeq\mathcal{M}_0$, where $\mathcal{D}_X\cdot\delta(t-\varphi)$ is the $\mathcal{D}_X$-submodule of $\mathcal{D}_{X\xrightarrow{\varphi}\mathbb{C}}$ generated by $\delta(t-\varphi)$,*

(iv) $\mathrm{char}(\mathcal{M})=\mathrm{char}(\mathcal{M}_0)\subset V$.

Proof. (i) follows from Lemma 1.6 (ii).
(ii) follows from Lemma 1.5 (iii).
(iii) Clearly, $\mathcal{D}_X\cdot\delta(t-\varphi)\simeq\mathcal{D}_X/\mathcal{D}_X\cap\mathcal{I}$.
(iv) (a) By (iii) and Lemma 8.3 we get the inclusion $\mathrm{char}(\mathcal{M}_0)\subset V$.
(iv) (b) Since $\mathcal{M}_0\subset\mathcal{M}$, the inclusion $\mathrm{char}(\mathcal{M}_0)\subset\mathrm{char}(\mathcal{M})$ is obvious.
(iv) (c) Denote by u the canonical generator of $\mathcal{M}$. Then $\mathcal{M}=\bigcup_{n\le 0}\mathcal{D}_X\tau^n u$. Since there are epimorphisms $\mathcal{M}_0\twoheadrightarrow\mathcal{D}_X\tau^n u$, the inclusion $\mathrm{char}(\mathcal{M})\subset\mathrm{char}(\mathcal{M}_0)$ follows. q.e.d.

Set

$$
F_0 = R\mathcal{H}om_{\mathcal{D}_X[\tau^{-1}]}(\mathcal{M},\mathcal{O}^\tau_X(0)), \tag{8.8}
$$

where the module $\mathcal{M}$ is defined in (8.7).

Lemma 8.5. *$SS(F_0)$ is a closed $\mathbb{C}^\times$-conic complex analytic Lagrangian subset of T^*X contained in $C(\Lambda_0,\Lambda_1)$.*

Proof. By Lemmas 8.2 and 8.4, $SS(F_0)\subset V\cap(\pi^{-1}(\Lambda_0\cap\Lambda_1))$, and one immediately checks that

$$
V\cap\pi^{-1}(\Lambda_0\cap\Lambda_1)=C(\Lambda_0,\Lambda_1). \tag{8.9}
$$

Since $SS(F_0)$ is involutive by [12] and is contained in a $\mathbb{C}^\times$-conic analytic isotropic subset by Proposition 7.1, it is a closed $\mathbb{C}^\times$-conic complex analytic Lagrangian subset of $T^*\Lambda_0$ by [12, Prop. 8.3.13]. q.e.d.

Lemma 8.6. *Let F_0 be as in (8.8). Then for each $x\in X$ and each $j\in\mathbb{Z}$, the $\mathbf{k}_0$-module $H^j(F_0)_x$ is finitely generated.*

Proof. Let $x_0 \in X$ and choose a local coordinate system around x_0. Denote by $B(x_0;\varepsilon)$ the closed ball of center x_0 and radius ε. By a result of [7] (see also [12, Prop. 8.3.12]), we deduce from Lemma 8.5 that the natural morphisms

$$\mathrm{R}\Gamma(B(x_0;\varepsilon_1);F_0) \to \mathrm{R}\Gamma(B(x_0;\varepsilon_0);F_0)$$

are isomorphisms for $0 < \varepsilon_0 \leq \varepsilon_1 \ll 1$.

We represent F_0 by a complex:

$$0 \to (\mathcal{O}^\tau(0))^{N_0} \xrightarrow{d_0} \cdots \to (\mathcal{O}^\tau(0))^{N_n} \to 0, \tag{8.10}$$

where the differentials are $\mathbf{k}_0$-linear. It follows from Lemma 5.5 and Theorem 4.3 that the cohomology objects $H^j(F_0)_{x_0}$ are finitely generated. q.e.d.

End of the proof of Theorem 7.4. As already mentioned, part (ii) is a particular case of Theorem 6.1.

Let us prove part (i). By "dévissage" we may assume that $\mathcal{L}_0$ and $\mathcal{L}_1$ are concentrated in degree 0. We may also assume that $\mathcal{L}_0$ and $\mathcal{L}_1$ are as in (8.6).

Let F_0 be as in (8.8), and set $F = R\mathcal{H}om_{\mathcal{W}_{T^*X}}(\mathcal{L}_1, \mathcal{O}^\tau_X)$. Since $\mathcal{L}_1 \simeq \mathcal{W}_{T^*X} \otimes_{\mathcal{D}_X[\tau^{-1}]} \mathcal{M}$, we obtain

$$F \simeq F_0 \otimes_{\mathbf{k}_0} \mathbf{k}$$

by Lemmas 1.6 and 1.7.

Hence we have $SS(F) \subset SS(F_0)$, and the weak constructibility as well as the estimate of $SS(F)$ follows by Lemma 8.2 and Lemma 8.5. The finiteness result follows from Lemma 8.6. q.e.d.

9 Serre functors on contact and symplectic manifolds

In the definition below, $\mathbb{K}$ is a field and * denotes the duality functor for $\mathbb{K}$-vector spaces.

Definition 9.1. Consider a $\mathbb{K}$-triangulated category $\mathcal{T}$.

(i) The category $\mathcal{T}$ is Ext-finite if for any $L_0, L_1 \in \mathcal{T}$, $\mathrm{Ext}^j_{\mathcal{T}}(L_1, L_0)$ is finite-dimensional over $\mathbb{K}$ for all $j \in \mathbb{Z}$ and is zero for $|j| \gg 0$.

(ii) Assume that $\mathcal{T}$ is Ext-finite. A Serre functor (see [2]) S on $\mathcal{T}$ is an equivalence of $\mathbb{K}$-triangulated categories $S\colon \mathcal{T} \to \mathcal{T}$ satisfying

$$\bigl(\mathrm{Hom}_{\mathcal{T}}(L_1, L_0)\bigr)^{\star} \simeq \mathrm{Hom}_{\mathcal{T}}(L_0, S(L_1))$$

functorially in $L_0, L_1 \in \mathcal{T}$.

(iii) If moreover there exists an integer d such that S is isomorphic to the shift by d, then one says that $\mathcal{T}$ is a $\mathbb{K}$-triangulated Calabi-Yau category of dimension d.

Let $\mathfrak{Y}$ be a complex contact manifold. The algebroid stack $\mathcal{E}_{\mathfrak{Y}}$ of microdifferential operators on $\mathfrak{Y}$ has been constructed in [8] and the triangulated categories $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{E}_{\mathfrak{Y}})$, $\mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{E}_{\mathfrak{Y}})$ and $\mathrm{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathcal{E}_{\mathfrak{Y}})$ are naturally defined.

Theorem 9.2. *For a complex contact manifold $\mathfrak{Y}$, we have*

(i) *for $\mathcal{M}$ and $\mathcal{N}$ in $\mathrm{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathcal{E}_{\mathfrak{Y}})$, the object $F = R\mathcal{H}om_{\mathcal{E}_{\mathfrak{Y}}}(\mathcal{M}, \mathcal{N})$ belongs to $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}-\mathrm{c}}(\mathbb{C}_{\mathfrak{Y}})$,*

(ii) *if $\mathfrak{Y}$ is compact, then $\mathrm{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathcal{E}_{\mathfrak{Y}})$ is a Calabi-Yau $\mathbb{C}$-triangulated category of dimension $\dim_{\mathbb{C}} \mathfrak{Y} - 1$.*

Sketch of proof. (i) is well-known and follows from [10] (see [21] for further developments). The idea of the proof is as follows. The assertion being local and invariant by quantized contact transformations, we may assume that $\mathfrak{Y}$ is an open subset of the projective cotangent bundle P^*Y to a complex manifold. Then, using the diagonal procedure, we reduce to the case $F = R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}, \mathcal{C}_{Z|Y})$, where $\mathcal{M}$ is a regular holonomic $\mathcal{D}_Y$-module and $\mathcal{C}_{Z|Y}$ is the $\mathcal{E}_{P^*Y}$-module associated to a complex hypersurface Z of Y.
(ii) follows from (i) as in the proof of Theorem 6.1. q.e.d.

Remark 9.3. (i) If Conjecture 7.5 is true, that is, if Theorem 7.4 holds for any Lagrangian varieties, then, for any compact complex symplectic manifold $\mathfrak{X}$, the $\mathbf{k}$-triangulated category $\mathrm{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathcal{W}_{\mathfrak{X}})$ is a Calabi-Yau $\mathbf{k}$-triangulated category of dimension $\dim_{\mathbb{C}} \mathfrak{X}$. Note that this result is true when replacing the notion of regular holonomic module by the notion of good modules, i.e., coherent modules admitting globally defined $\mathcal{W}_{\mathfrak{X}}(0)$-submodules which generate them. This follows from a theorem of Schapira-Schneiders to appear.
(ii) Note that Proposition 1.4.8 of [10] is not true, but this proposition is not used in loc. cit. Indeed, for a compact complex manifold X, if $\mathcal{T} := \mathrm{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathcal{D}_X)$ denotes the full triangulated subcategory of $\mathrm{D}^{\mathrm{b}}(\mathcal{D}_X)$ consisting

of objects with regular holonomic cohomologies, it is well known that the duality functor is not a Serre functor on $\mathcal{T}$.
(iii) It may be interesting to notice that the duality functor is not a Serre functor in the setting of regular holonomic $\mathcal{D}$-modules, but is a Serre functor in the microlocal setting, that is, for regular holonomic $\mathcal{E}$-modules. This may be compared to [12, Prop. 8.4.14].

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Masaki Kashiwara
Research Institute for Mathematical Sciences
Kyoto University
Kyoto, 606–8502, Japan
e-mail: masaki@kurims.kyoto-u.ac.jp

Pierre Schapira
Institut de Mathématiques
Université Pierre et Marie Curie
175, rue du Chevaleret, 75013 Paris, France
e-mail: schapira@math.jussieu.fr
http://www.math.jussieu.fr/∼schapira/