Commun. Math. Phys. 273, 395–414 (2007)
Digital Object Identifier (DOI) 10.1007/s00220-007-0250-2

Communications in
**Mathematical
Physics**

An Algebra of Deformation Quantization for Star-Exponentials on Complex Symplectic Manifolds

Giuseppe Dito[1], **Pierre Schapira**[2]

[1] Institut de Mathématiques de Bourgogne, Université de Bourgogne, B.P. 47870, 21078 Dijon Cedex, France.
E-mail: giuseppe.dito@u-bourgogne.fr
[2] Institut de Mathématiques, Université Pierre et Marie Curie, 175, rue du Chevaleret, 75013 Paris, France.
E-mail: schapira@math.jussieu.fr

Received: 10 July 2006 / Accepted: 4 January 2007

Abstract: The cotangent bundle T^*X to a complex manifold X is classically endowed with the sheaf of $\mathbf{k}$-algebras $\mathcal{W}_{T^*X}$ of deformation quantization, where $\mathbf{k} := \mathcal{W}_{\{\mathrm{pt}\}}$ is a subfield of $\mathbb{C}[[\hbar, \hbar^{-1}]$. Here, we construct a new sheaf of $\mathbf{k}$-algebras $\mathcal{W}^t_{T^*X}$ which contains $\mathcal{W}_{T^*X}$ as a subalgebra and an extra central parameter t. We give the symbol calculus for this algebra and prove that quantized symplectic transformations operate on it. If P is any section of order zero of $\mathcal{W}_{T^*X}$, we show that $\exp(t\hbar^{-1}P)$ is well defined in $\mathcal{W}^t_{T^*X}$.

Introduction

The cotangent bundle T^*X to a complex manifold X is endowed with the sheaf of filtered $\mathbb{C}$-algebras $\mathcal{E}_{T^*X}$ constructed functorially by Sato-Kashiwara-Kawai in [9] and called the sheaf of microdifferential operators. This sheaf is conic and is associated with the homogeneous symplectic structure of T^*X. Another *no more conic* sheaf of filtered algebras on T^*X, denoted here by $\widehat{\mathcal{W}}_{T^*X}$ and defined over $\mathbb{C}[[\hbar, \hbar^{-1}]$, has been constructed in the framework of formal associative deformations by many authors after [1]. (This construction has been extended to Poisson manifolds in [7].) Its analytic counterpart $\mathcal{W}_{T^*X}$ is constructed in [8]. The sheaf $\mathcal{W}_{T^*X}$ is similar to the sheaf $\mathcal{E}_{T^*X}$ of microdifferential operators of [9], but with an extra central parameter $\hbar$, a substitute to the lack of homogeneity [1]. Here $\hbar$ belongs to the field $\mathbf{k} := \mathcal{W}_{\{\mathrm{pt}\}}$, a subfield of $\mathbb{C}[[\hbar, \hbar^{-1}]$. (Note that the notation $\tau = \hbar^{-1}$ is used in [8].) When X is affine and one denotes by $(x; u)$ a point of T^*X, a section P of this sheaf on an open subset $U \subset T^*X$ is represented by its total symbol $\sigma_{\mathrm{tot}}(P) = \sum_{-\infty<j\leq m} p_j(x; u)\hbar^{-j}$, with $m \in \mathbb{Z}$, $p_j \in \mathcal{O}_{T^*X}(U)$, the p_j's satisfying suitable inequalities and the product, denoted here by $\star$, being given by the Leibniz formula.

[1] In this paper, we write $\mathcal{E}_{T^*X}$ and $\mathcal{W}_{T^*X}$ instead of the classical notations $\mathcal{E}_X$ and $\mathcal{W}_X$.

A fundamental tool for spectral analysis in deformation quantization is the star-exponential of the Hamiltonian H (see [1]):

$$\exp_{\star}(t\hbar^{-1}H) = \sum_{n\geq 0} \frac{(t\hbar^{-1}H)^{\star n}}{n!}.$$

However, at the formal level, the star-exponential does not make sense as a formal series in $\hbar$ and $\hbar^{-1}$. The goal of this article is to construct a new sheaf of algebras on the cotangent bundle T^*X to a complex manifold X in which the star-exponential has a meaning and such that quantized symplectic transformations operate on such algebras. More precisely, we construct a new sheaf of $\mathbf{k}$-algebras $\mathcal{W}^t_{T^*X}$, with an extra central holomorphic parameter t defined in a neighborhood of $t = 0$, with the property that complex symplectic transformations may be locally quantized as isomorphisms of algebras and there are natural morphisms of $\mathbf{k}$-algebras $\mathcal{W}_{T^*X} \xrightarrow{\iota} \mathcal{W}^t_{T^*X} \xrightarrow{\mathrm{res}} \mathcal{W}_{T^*X}$ whose composition is the identity on $\mathcal{W}_{T^*X}$. We give the symbol calculus on $\mathcal{W}^t_{T^*X}$, which extends naturally that of $\mathcal{W}_{T^*X}$ (however, now we get series in $\hbar^j$ with $-\infty < j < \infty$), and finally we show that, if P is a section of $\mathcal{W}_{T^*X}$ of order 0, then $\exp(t\hbar^{-1}P)$ is well defined in $\mathcal{W}^t_{T^*X}$. We also briefly discuss the case where T^*X is replaced with a general symplectic manifold.

Our construction is as follows. First, we add a central holomorphic parameter $s \in \mathbb{C}$ and consider the sheaf $\mathcal{W}_{\mathbb{C}\times T^*X}$, the subsheaf of $\mathcal{W}_{T^*(\mathbb{C}\times X)}$ consisting of sections not depending on ∂_s. Denoting by $a\colon \mathbb{C} \times T^*X \to T^*X$ the projection, we first define an algebra $\mathcal{W}^s_{T^*X} := R^1a_!\mathcal{W}_{\mathbb{C}\times T^*X}$. The algebra structure with respect to the s-variable is given by convolution, as in the case of the space $H^1_c(\mathbb{C}; \mathcal{O}_{\mathbb{C}})$. In order to replace this convolution product by an usual product, we define the sheaf $\mathcal{W}^t_{T^*X}$ as the "formal" Laplace transform with respect to the variables $s\hbar^{-1}$ of the algebra $\mathcal{W}^s_{T^*X}$.

In a deformation quantization context, the existence of $\exp(t\hbar^{-1}P)$ in $\mathcal{W}^t_{T^*X}$ gives a precise meaning to the star-exponential [1] of P which is heuristically related to the Feynman Path Integral of P.

1. Symbols

The fields $\widehat{\mathbf{k}}$ *and* $\mathbf{k}$. We set $\widehat{\mathbf{k}} := \mathbb{C}[[\hbar, \hbar^{-1}]$. Hence, an element $a \in \widehat{\mathbf{k}}$ is a series

$$a = \sum_{-\infty<j\leq m} a_j\hbar^{-j}, \quad a_j \in \mathbb{C}, \quad m \in \mathbb{Z}.$$

Consider the following condition on a:

$$\begin{cases} \text{there exist positive constants } C, \varepsilon \text{ such that } |a_j| \leq C\varepsilon^{-j}(-j)! \\ \text{for all } j < 0. \end{cases} \tag{1.1}$$

We denote by $\mathbf{k}$ the subfield of $\widehat{\mathbf{k}}$ consisting of series satisfying (1.1).

Convention. We endow $\widehat{\mathbf{k}}$, hence $\mathbf{k}$, with the filtration associated to

$$\mathrm{ord}(\hbar) = -1. \tag{1.2}$$

The fields $\widehat{\mathbf{k}}$ and $\mathbf{k}$ are $\mathbb{Z}$-filtered[2] and contain the subrings $\widehat{\mathbf{k}}(0)$ and $\mathbf{k}(0)$, respectively. Note that $\widehat{\mathbf{k}}(0) = \mathbb{C}[[\hbar]]$ and $\mathbf{k}(0) = \mathbf{k} \cap \widehat{\mathbf{k}}(0)$.

Remark 1.1. $\widehat{\mathbf{k}}$ is flat over $\widehat{\mathbf{k}}(0)$ and $\mathbf{k}$ is flat over $\mathbf{k}(0)$.

[2] In the sequel, we shall say "filtered" instead of "$\mathbb{Z}$-filtered".

The sheaves $\widehat{\mathcal{O}}_X^\hbar$ and $\mathcal{O}_X^\hbar$. Let $(X, \mathcal{O}_X)$ be a complex manifold.

Definition 1.2. (i) *We denote by $\widehat{\mathcal{O}}_X^\hbar$ the sheaf $\mathcal{O}_X[[\hbar, \hbar^{-1}]$. In other words, $\widehat{\mathcal{O}}_X^\hbar$ is the filtered $\widehat{\mathbf{k}}$-algebra defined as follows: A section $f(x,\hbar)$ of $\widehat{\mathcal{O}}_X^\hbar$ of order $\le m$ ($m \in \mathbb{Z}$) on an open set U of X is a series*

$$f(x,\hbar) = \sum_{-\infty<j\le m} f_j(x)\hbar^{-j}, \tag{1.3}$$

with $f_j \in \mathcal{O}_X(U)$.

(ii) *We denote by $\mathcal{O}_X^\hbar$ the filtered $\mathbf{k}$-subalgebra of $\widehat{\mathcal{O}}_X^\hbar$ consisting of sections $f(x,\hbar)$ as above satisfying:*

$$\begin{cases}\text{for any compact subset } K \text{ of } U \text{ there exist positive constants } C, \varepsilon \text{ such that}\\ \sup_K |f_j| \le C\varepsilon^{-j}(-j)! \text{ for all } j<0.\end{cases} \tag{1.4}$$

Note that

$$\widehat{\mathcal{O}}_X^\hbar \simeq \widehat{\mathcal{O}}_X^\hbar(0) \otimes_{\widehat{\mathbf{k}}(0)} \widehat{\mathbf{k}}, \;\; \mathcal{O}_X^\hbar \simeq \mathcal{O}_X^\hbar(0) \otimes_{\mathbf{k}(0)} \mathbf{k}. \tag{1.5}$$

(To be correct, we should have written $\mathbf{k}_X$, the constant sheaf with values in $\mathbf{k}$, instead of $\mathbf{k}$ in these formulas, and similarly for $\mathbf{k}(0)$, $\widehat{\mathbf{k}}(0)$ and $\widehat{\mathbf{k}}$.)

Also note that there exist isomorphisms of sheaves (not of algebras)

$$\widehat{\mathcal{O}}_X^\hbar(0) \simeq \mathcal{O}_{X\times\mathbb{C}}\hat{|}_{X\times\{0\}}, \tag{1.6}$$

$$\mathcal{O}_X^\hbar(0) \simeq \mathcal{O}_{X\times\mathbb{C}}|_{X\times\{0\}}, \tag{1.7}$$

where $\mathcal{O}_{X\times\mathbb{C}}\hat{|}_{X\times\{0\}}$ is the formal completion of $\mathcal{O}_{X\times\mathbb{C}}$ along the hypersurface $X\times\{0\}$ of $X\times\mathbb{C}$ and $\mathcal{O}_{X\times\mathbb{C}}|_{X\times\{0\}}$ is the restriction of $\mathcal{O}_{X\times\mathbb{C}}$ to $X\times\{0\}$.

Denoting by t the coordinate on $\mathbb{C}$, the isomorphism (1.7) is given by the map

$$\mathcal{O}_X^\hbar(0) \ni \sum_{j\le 0} f_j\hbar^{-j} \mapsto \sum_{j\ge 0} f_{-j}\frac{t^j}{j!} \in \mathcal{O}_{X\times\mathbb{C}}|_{X\times\{0\}}.$$

The convolution algebra $H_c^1(\mathbb{C};\mathcal{O}_\mathbb{C})$. The results of this subsection are well known and elementary. We recall them for the reader's convenience.

We consider the complex line $\mathbb{C}$ endowed with a holomorphic coordinate s. Using this coordinate, we identify the sheaf $\mathcal{O}_\mathbb{C}$ of holomorphic functions on $\mathbb{C}$ and the sheaf $\Omega_\mathbb{C}$ of holomorphic forms on $\mathbb{C}$.

The space $H_c^1(\mathbb{C};\mathcal{O}_\mathbb{C})$ is endowed with a structure of an algebra by

$$\begin{aligned} H_c^1(\mathbb{C};\mathcal{O}_\mathbb{C}) \times H_c^1(\mathbb{C};\mathcal{O}_\mathbb{C}) &\to H_c^2(\mathbb{C}^2;\mathcal{O}_{\mathbb{C}^2}) \\ &\to H_c^1(\mathbb{C};\mathcal{O}_\mathbb{C}), \end{aligned}$$

where the first arrow is the cup product and the second arrow is the integration along the fibers of the map $\mathbb{C}^2 \to \mathbb{C}$, $(s,s') \mapsto s+s'$.

When representing the cohomology classes by holomorphic functions, the convolution product is described as follows.

For a compact subset K of $\mathbb{C}$, we identify the vector space $H^1_K(\mathbb{C};\mathcal{O}_\mathbb{C})$ with the quotient space $\Gamma(\mathbb{C}\setminus K;\mathcal{O}_\mathbb{C})/\Gamma(\mathbb{C};\mathcal{O}_\mathbb{C})$ and, if $f\in\Gamma(\mathbb{C}\setminus K;\mathcal{O}_\mathbb{C})$, we still denote by f its image in $H^1_K(\mathbb{C};\mathcal{O}_\mathbb{C})$ or in $H^1_c(\mathbb{C};\mathcal{O}_\mathbb{C})$. Let K and L be compact subsets of $\mathbb{C}$, let $f\in\Gamma(\mathbb{C}\setminus K;\mathcal{O}_\mathbb{C})$ and $g\in\Gamma(\mathbb{C}\setminus L;\mathcal{O}_\mathbb{C})$. The convolution product $f*g$ is given by

$$f*g(z)=\frac{1}{2i\pi}\int_\gamma f(z-w)g(w)dw, \tag{1.8}$$

where γ is a counter-clockwise oriented circle which contains L and $|z|$ is chosen big enough so that $z+K$ is outside of the disc bounded by γ. It is an easy exercise to show that this definition does not depend on the representatives f and g, and that to interchange the role of f and g in the formula (1.8) modifies the result by a function defined all over $\mathbb{C}$, hence gives the same result in $H^1_c(\mathbb{C};\mathcal{O}_\mathbb{C})$. Therefore, we obtain a commutative algebra structure on $H^1_c(\mathbb{C};\mathcal{O}_\mathbb{C})$.

Example 1.3.

$$\frac{1}{z^{n+1}}*\frac{1}{z^{m+1}}=\frac{(n+m)!}{n!m!}\frac{1}{z^{n+m+1}}.$$

The sheaf $\mathcal{O}^{s,\hbar}_X$. From now on, we shall concentrate our study on $\mathcal{O}^\hbar_X$.

Notataion 1.4. We shall often denote by $\mathbb{C}_s$ the complex line $\mathbb{C}$ endowed with the coordinate s.

Lemma 1.5. *Let Y be a complex manifold and Z a Stein submanifold of Y. Then $H^j(Z;\mathcal{O}^\hbar_Y(0)|_Z)$ vanishes for $j\neq 0$.*

Proof. Using the isomorphism (1.7), we may replace the sheaf $\mathcal{O}^\hbar_Y(0)$ with the sheaf $\mathcal{O}_{Y\times\mathbb{C}_t}|_{t=0}$. By a theorem of Siu [11], $Z\times\{0\}$ admits a fundamental system of open Stein neighborhoods in $Y\times\mathbb{C}_t$ and the result follows. □

Let X be a complex manifold. The manifold $\mathbb{C}_s\times X$ is thus endowed with the **k**-filtered sheaf $\mathcal{O}^\hbar_{\mathbb{C}_s\times X}$. Let $a\colon\mathbb{C}_s\times X\to X$ denote the projection.

Lemma 1.6. (i) *One has the isomorphism*

$$R^ja_!\mathcal{O}^\hbar_{\mathbb{C}_s\times X}\simeq R^ja_!\mathcal{O}^\hbar_{\mathbb{C}_s\times X}(0)\otimes_{\mathbf{k}(0)}\mathbf{k}.$$

(ii) $R^ja_!\mathcal{O}^\hbar_{\mathbb{C}_s\times X}(0)\simeq 0$ *for* $j\neq 1$.

(iii) *Let $U\subset\subset V\subset\subset W$ be three open subsets of X and assume that W is Stein. Then the natural morphism $\Gamma(W;R^1a_!\mathcal{O}^\hbar_{\mathbb{C}_s\times X})\to\Gamma(U;R^1a_!\mathcal{O}^\hbar_{\mathbb{C}_s\times X})$ factorizes through*

$$\varinjlim_{K\subset\mathbb{C}_s}\Gamma((\mathbb{C}_s\setminus K)\times V;\mathcal{O}^\hbar_{\mathbb{C}_s\times X})/\Gamma(\mathbb{C}_s\times V;\mathcal{O}^\hbar_{\mathbb{C}_s\times X}),$$

where K ranges over the family of compact subsets of $\mathbb{C}$.

Proof. (i) follows from the projection formula for sheaves (*i.e.,* $Ra_!(F\overset{\mathrm{L}}{\otimes}a^{-1}G)\simeq Ra_!F\overset{\mathrm{L}}{\otimes}G$) and (1.5), since $\mathbf{k}$ is flat over $\mathbf{k}(0)$.

(ii) For $x \in X$, we have

$$H^j(Ra_!\mathcal{O}^{\hbar}_{\mathbb{C}_s\times X}(0))_x \simeq \varinjlim_K H^j_K(\mathbb{C}_s \times \{x\}; \mathcal{O}^{\hbar}_{\mathbb{C}_s\times X}(0)|_{\mathbb{C}_s\times\{x\}}).$$

Applying the distinguished triangle of functors

$$\mathrm{R}\Gamma_K(\mathbb{C}_s \times \{x\}; \bullet) \to \mathrm{R}\Gamma(\mathbb{C}_s \times \{x\}; \bullet) \to \mathrm{R}\Gamma((\mathbb{C}_s \setminus K) \times \{x\}; \bullet) \xrightarrow{+1}$$

to the sheaf $\mathcal{O}^{\hbar}_{\mathbb{C}_s\times X}(0)|_{\mathbb{C}_s\times\{x\}}$ we get the result by Lemma 1.5 for $j > 1$ and the case $j = 0$ follows from the principle of analytic continuation.

(iii) Recall first that if W is a Stein manifold and if $W_1 \subset\subset W$ is open, there exists a Stein open subset W_2 of W with $W_1 \subset\subset W_2 \subset\subset W$.
For a compact subset L of X, $\Gamma(L; R^1a_!\mathcal{O}^{\hbar}_{\mathbb{C}_s\times X}) \simeq \Gamma(L; R^1a_!\mathcal{O}^{\hbar}_{\mathbb{C}_s\times X}(0)) \otimes_{\mathbf{k}(0)} \mathbf{k}$.
Hence, it is enough to prove the result for $\mathcal{O}^{\hbar}_{\mathbb{C}_s\times X}(0)$.

By Lemma 1.5, $H^j(D \times U; \mathcal{O}^{\hbar}_{\mathbb{C}_s\times X}(0))$ vanishes for D open in $\mathbb{C}_s$, U Stein open in X and $j \neq 0$. Therefore, $H^j_{K\times U}(\mathbb{C}_s \times U; \mathcal{O}^{\hbar}_{\mathbb{C}_s\times X}(0))$ vanishes for $j \neq 1$ and we get the exact sequence:

$$\begin{aligned}0 \to \Gamma(\mathbb{C}_s \times U; \mathcal{O}^{\hbar}_{\mathbb{C}_s\times X}(0)) &\to \Gamma((\mathbb{C}_s \setminus K) \times U; \mathcal{O}^{\hbar}_{\mathbb{C}_s\times X}(0)) \\ &\to H^1_{K\times U}(\mathbb{C}_s \times U; \mathcal{O}^{\hbar}_{\mathbb{C}_s\times X}(0)) \to 0.\end{aligned}$$

□

Definition 1.7. *We set* $\mathcal{O}^{s,\hbar}_X := R^1a_!\mathcal{O}^{\hbar}_{\mathbb{C}_s\times X}$.

Clearly, $\mathcal{O}^{s,\hbar}_X$ is a sheaf of filtered $\mathbf{k}$-modules. By Lemma 1.6, a section $f(s, x, \hbar)$ of order m of the sheaf $\mathcal{O}^{s,\hbar}_X$ on a Stein open subset W of X may be written on any relatively compact open subset U of W as a series

$$f(s, x, \hbar) = \sum_{-\infty<j\le m} f_j(s, x)\hbar^{-j},$$

where $f_j(s, x)$ is a holomorphic function on $(\mathbb{C}_s \setminus K_0) \times U$ for a compact set K_0 not depending on j and the f_j's satisfy an estimate (1.4) on each compact subset K of $(\mathbb{C}_s \setminus K_0) \times U$.

We shall extend the product (1.8) to $\mathcal{O}^{s,\hbar}_X$ as follows. For two sections $f(s, x, \hbar) = \sum_{-\infty<j\le m} f_j(s, x)\hbar^{-j}$ and $g(s, x, \hbar) = \sum_{-\infty<j\le m'} g_j(s, x)\hbar^{-j}$ of $\mathcal{O}^{s,\hbar}_X$, we set:

$$\begin{cases} f(s, x, \hbar) * g(s, x, \hbar) = \sum_{-\infty<j\le m+m'} h_j(s, x)\hbar^{-j}, \\ h_k(s, x) = \sum_{i+j=k} \frac{1}{2i\pi} \int_\gamma f_i(s - w, x)g_j(w, x)dw. \end{cases} \tag{1.9}$$

Proposition 1.8. *The sheaf* $\mathcal{O}^{s,\hbar}_X$ *has a structure of a filtered commutative* $\mathbf{k}$- *algebra.*

Proof. It is easily checked that multiplication by $\hbar^{-1}$ induces an isomorphism of sheaves of $\mathbf{k}$-modules $\mathcal{O}_X^{s,\hbar}(m) \xrightarrow{\sim} \mathcal{O}_X^{s,\hbar}(m+1)$. Hence we just need to check that the product of two sections of order 0 is a section of order 0. Let $f(s,x,\hbar) = \sum_{-\infty<i\le 0} f_i(s,x)\hbar^{-i}$ and $g(s,x,\hbar) = \sum_{-\infty<j\le 0} g_j(s,x)\hbar^{-j}$ be in $\mathcal{O}_X^{s,\hbar}(0)$ and let K be a compact subset of $(\mathbb{C}_s \setminus K_0)\times U$. Let γ be a counter-clockwise oriented circle which contains K_0 and $s > R$ big enough so that $s+K_0$ does not meet γ. Then for $w \in \gamma$ and $x \in K \cap (\mathbb{C}_s \setminus K_0)\times U$, we have:

$$
\begin{aligned}
|\sum_{i+j=k, i,j\le 0} f_i(s-w,x) g_j(w,x)| &\le C^2(-k)! \\
\times \sum_{i+j=k,i,j\le 0} \varepsilon^{-i-j}\frac{(-i)!(-j)!}{(-k)!} &\le 3C^2\varepsilon^{-k}(-k)!.
\end{aligned}
$$

Hence $h(s,x,\hbar) = \sum_{-\infty<j\le 0} h_k(s,x)\hbar^{-k}$ defined by (1.9) is in $\mathcal{O}_X^{s,\hbar}(0)$. □

The Laplace transform and the algebra $\mathcal{O}_X^{t,\hbar}$. In order to replace the convolution product in the s-variable with the ordinary product, we shall apply a kind of Laplace transform to $\mathcal{O}_X^{s,\hbar}$.

Definition 1.9. *On a complex manifold X, we denote by* $\mathcal{O}_X^{t,\hbar}$ *the filtered sheaf of* $\mathbf{k}$*-modules defined as follows. A section* $f(t,x,\hbar)$ *of* $\mathcal{O}_X^{t,\hbar}(m)$ (i.e., *a section of order m) on an open set U of X is a series*

$$
f(t,x,\hbar) = \sum_{-\infty<j<\infty} f_j(t,x)\hbar^{-j}, \quad f_j \in \Gamma(U; \mathcal{O}_{\mathbb{C}\times X|_{t=0}}), \tag{1.10}
$$

with the condition that for any compact subset K of U there exists $\eta > 0$ *such that* $f_j(t,x)$ *is holomorphic in a neighborhood of* $\{|t| \le \eta\} \times K$ *and satisfies*

$$
\begin{cases} \text{there exist positive constants } C, \varepsilon \text{ such that} \\ \sup_{x\in K, |t|\le\eta} |f_j(t,x)| \le C\cdot\varepsilon^{-j}(-j)! \text{ for all } j<0, \end{cases} \tag{1.11}
$$

$$
\begin{cases} \text{there exist positive constants } M \text{ and } R \text{ such that} \\ \sup_{x\in K} |f_j(t,x)| \le M\dfrac{R^{j-m}}{(j-m)!}|t|^{j-m} \text{ for } |t|\le\eta \text{ and all } j\ge m. \end{cases} \tag{1.12}
$$

Let $f(t,x,\hbar) = \sum_{-\infty<j<\infty} f_j(t,x)\hbar^{-j}$ and $g(t,x,\hbar) = \sum_{-\infty<j<\infty} g_j(t,x)\hbar^{-j}$ be two sections of $\mathcal{O}_X^{t,\hbar}$ of order m and m' respectively. Define formally

$$
h(t,x,\hbar) = \sum_{-\infty<j<\infty} h_j(t,x)\hbar^{-j}, \quad h_k(t,x) = \sum_{i+j=k} f_i(t,x)g_j(t,x). \tag{1.13}
$$

Lemma 1.10. (i) *Multiplication by* $\hbar^{-1}$ *induces an isomorphism of sheaves of* $\mathbf{k}(0)$-*modules* $\mathcal{O}_X^{t,\hbar}(m) \xrightarrow{\sim} \mathcal{O}_X^{t,\hbar}(m+1)$.

(ii) *The product* (1.13) *of a section* $f(t,x,\hbar) \in \mathcal{O}_X^{t,\hbar}(m)$ *and a section* $g(t,x,\hbar) \in \mathcal{O}_X^{t,\hbar}(m')$ *is well defined and belongs to* $\mathcal{O}_X^{t,\hbar}(m+m')$.

Proof. (i) (a) Let $f(t,x,\hbar)=\sum_{-\infty<j<\infty} f_j(t,x)\hbar^{-j}\in\mathcal{O}_X^{t,\hbar}(m)$. Then $\hbar^{-1}f(t,x,\hbar)=\sum_{-\infty<j<\infty}\tilde{f}_j(t,x)\hbar^{-j}$, with $\tilde{f}_j=f_{j-1}$. For any integer $j<0$, we have:

$$\sup_{x\in K,|t|\le\eta}|\tilde{f}_j(t,x)|=\sup_{x\in K,|t|\le\eta}|f_{j-1}(t,x)|\le C\varepsilon^{-j+1}(-j+1)!\le(C\varepsilon)(\varepsilon e)^{-j}(-j)!.$$

Hence Condition (1.11) is satisfied.

For $j\ge m+1$, we have:

$$\sup_{x\in K}|\tilde{f}_j(t,x)|=\sup_{x\in K}|f_{j-1}(t,x)|\le M\frac{R^{j-m-1}}{(j-m-1)!}|t|^{j-m-1},$$

which is simply Condition (1.12) for $m+1$ and $\hbar^{-1}f(t,x,\hbar)\in\mathcal{O}_X^{t,\hbar}(m+1)$.
(b) Let $\hbar f(t,x,\hbar)=\sum_{-\infty<j<\infty}\tilde{f}_j(t,x)\hbar^{-j}$, with $\tilde{f}_j=f_{j+1}$. For any integer $j<-1$, we have:

$$\sup_{x\in K,|t|\le\eta}|\tilde{f}_j(t,x)|=\sup_{x\in K,|t|\le\eta}|f_{j+1}(t,x)|\le C\varepsilon^{-j-1}(-j-1)!\le\frac{C}{\varepsilon}\varepsilon^{-j}(-j)!.$$

For $j=-1$, we have:

$$\sup_{x\in K,|t|\le\eta}|\tilde{f}_{-1}(t,x)|=\sup_{x\in K,|t|\le\eta}|f_0(t,x)|=A\ge 0,$$

since $f_0(t,x)$ is holomorphic in a neighborhood of $|t|\le\eta\times K$. Set $C'=\max\{\frac{A}{\varepsilon},\frac{C}{\varepsilon}\}$, then for all integers $j<0$, we have:

$$\sup_{x\in K,|t|\le\eta}|\tilde{f}_j(t,x)|\le C'\varepsilon^{-j}(-j)!,$$

and Condition (1.11) is satisfied.

For $j\ge m-1$, we have:

$$\sup_{x\in K}|\tilde{f}_j(t,x)|=\sup_{x\in K}|f_{j+1}(t,x)|\le M\frac{R^{j-m+1}}{(j-m+1)!}|t|^{j-m+1},$$

which is Condition (1.12) for $m-1$ and $\hbar f(t,x,\hbar)\in\mathcal{O}_X^{t,\hbar}(m-1)$. Therefore, multiplication by $\hbar^{-1}$ induces an isomorphism $\mathcal{O}_X^{t,\hbar}(m)\xrightarrow{\sim}\mathcal{O}_X^{t,\hbar}(m+1)$.
(ii) By (i), we may assume $m=m'=0$. Let $f=\sum_{-\infty<i<\infty}f_i(t,x)\hbar^{-i}$ and $g=\sum_{-\infty<j<\infty}g_j(t,x)\hbar^{-j}$ be in $\mathcal{O}_X^{t,\hbar}(0)$. Let K be a compact set. There exists $\eta>0$ such that $f_i(t,x)$ and $g_j(t,x)$ are holomorphic in a neighborhood of $\{|t|\le\eta\}\times K$. Conditions (1.11) and (1.12) guarantee the existence of the positive constants C_1, ε_1, M_1 and R_1 for the f_i's, and C_2, ε_2, M_2 and R_2 for the g_j's. We set $C=\max\{C_1,C_2\}$, $\varepsilon=\max\{\varepsilon_1,\varepsilon_2\}$, $M=\max\{M_1,M_2\}$ and $R=\max\{R_1,R_2\}$.

We shall show that the product (1.13) is well defined. Let $h_k(t,x)=\sum_{i+j=k}f_i(t,x)\,g_j(t,x)$.
(a) Consider the case $k<0$. The sum defining h_k can be divided into three parts:

$$h_k=\sum_{k<i<0}f_ig_{k-i}+\sum_{i\ge 0}f_ig_{k-i}+\sum_{j\ge 0}f_{k-j}g_j.\tag{1.14}$$

The first sum is finite and defines a holomorphic function in a neighborhood of $\{|t| \leq \eta\} \times K$.

In the second sum, $k - i$ is strictly negative and, for each term in this sum, Conditions (1.11) and (1.12) give the following estimates when $x \in K$ and $|t| \leq \eta$:

$$
\begin{aligned}
|f_i(t,x)g_{k-i}(t,x)| &\leq M\frac{(R\eta)^i}{i!}C\varepsilon^{i-k}(i-k)! \\
&\leq CM\varepsilon^{-k}(-k)!(R\eta\varepsilon)^i\binom{i-k}{i}.
\end{aligned}
$$

Recall that $\sum_{i\geq 0}\alpha^i\binom{n+i}{i} = \frac{1}{(1-\alpha)^{n+1}}$ for $|\alpha| < 1$. When $R\eta\varepsilon < 1$, $(R\eta\varepsilon)^i\binom{i-k}{i}$ is the general term of an absolutely convergent series. Let $\tilde{\eta} = \min\{\eta, \frac{1}{2R\varepsilon}\}$. Then the second sum in (1.14) converges uniformly on $\{|t| \leq \tilde{\eta}\} \times K$.

The third sum is handled in a similar way and one gets the estimate:

$$
|f_{k-j}(t,x)g_j(t,x)| \leq CM\varepsilon^{-k}(-k)!(R\eta\varepsilon)^j\binom{j-k}{j}.
$$

It follows that, for $k < 0$, h_k is a holomorphic function in a neighborhood of $\{|t| \leq \tilde{\eta}\}\times K$.

Let us show that h_k satisfies Condition (1.11). For $x \in K$ and $|t| \leq \tilde{\eta}$, the first sum in (1.14) is bounded by:

$$
|\sum_{k<i<0} f_i(t,x)g_{k-i}(t,x)| \leq C^2\sum_{k<i<0}\varepsilon^{-i}\varepsilon^{i-k}(-i)!(i-k)! \leq C^2\varepsilon^{-k}(-k)!.
$$

For the second and third sums we have:

$$
\begin{aligned}
|\sum_{i\geq 0} f_i(t,x)g_{k-i}(t,x)| &\leq CM\varepsilon^{-k}(-k)!\frac{1}{(1-R\tilde{\eta}\varepsilon)^{-k+1}}, \\
|\sum_{j\geq 0} f_{k-j}(t,x)g_j(t,x)| &\leq CM\varepsilon^{-k}(-k)!\frac{1}{(1-R\tilde{\eta}\varepsilon)^{-k+1}}.
\end{aligned}
$$

Let $\tilde{\varepsilon} = \max\{\varepsilon, \frac{\varepsilon}{(1-R\tilde{\eta}\varepsilon)}\}$. For $x \in K$ and $|t| \leq \tilde{\eta}$, we find that:

$$
|h_k(t,x)| \leq (C^2 + \frac{2CM}{(1-R\tilde{\eta}\varepsilon)})\tilde{\varepsilon}^{-k}(-k)!.
$$

Hence h_k satisfies Condition (1.11).

(b) The case $k \geq 0$. We again split the sum defining h_k into three parts:

$$
h_k = \sum_{0\leq i\leq k} f_i g_{k-i} + \sum_{i<0} f_i g_{k-i} + \sum_{j<0} f_{k-j}g_j. \tag{1.15}
$$

The first sum is a holomorphic function in a neighborhood of $\{|t| \leq \eta\} \times K$.

For each term in the second sum, we have the following estimates when $x \in K$ and $|t| \leq \eta$:

$$
|f_i(t,x)g_{k-i}(t,x)| \leq C\varepsilon^{-i}(-i)!M\frac{R^{k-i}}{(k-i)!}|t|^{k-i} \leq CM(\varepsilon R\eta)^{-i}\frac{R^k}{k!}|t|^k.
$$

Since $(\varepsilon R\eta)^{-i}$ is the general term of the geometric series, the second sum in (1.15) defines a holomorphic function in a neighborhood of $\{|t| \le \tilde{\eta}\} \times K$, where $\tilde{\eta} = \min\{\eta, \frac{1}{2R\varepsilon}\}$.

Similarly, for the third sum we have:

$$|f_{k-j}(t,x)g_j(t,x)| \le CM(\varepsilon R\eta)^{-j}\frac{R^k}{k!}|t|^k.$$

Therefore, for $k \ge 0$, h_k is a holomorphic function in a neighborhood of $\{|t| \le \tilde{\eta}\} \times K$.

Let us show that h_k satisfies Condition (1.12) with $m = 0$. For $x \in K$ and $|t| \le \tilde{\eta}$, the first sum in (1.15) is bounded by:

$$|\sum_{0\le i\le k} f_i(t,x)g_{k-i}(t,x)| \le M^2\frac{R^k}{k!}|t|^k \sum_{0\le i\le k}\binom{k}{i} \le M^2\frac{(2R)^k}{k!}|t|^k.$$

For the second and third sums we find:

$$|\sum_{i<0} f_i(t,x)g_{k-i}(t,x)| \le CM\frac{R\tilde{\eta}\varepsilon}{1-R\tilde{\eta}\varepsilon}\frac{R^k}{k!}|t|^k,$$

$$|\sum_{j<0} f_{k-j}(t,x)g_j(t,x)| \le CM\frac{R\tilde{\eta}\varepsilon}{1-R\tilde{\eta}\varepsilon}\frac{R^k}{k!}|t|^k.$$

For $x \in K$ and $|t| \le \tilde{\eta}$, we have:

$$|h_k(t,x)| \le (M^2 + 2CM\frac{R\tilde{\eta}\varepsilon}{1-R\tilde{\eta}\varepsilon})\frac{(2R)^k}{k!}|t|^k.$$

Hence h_k satisfies Condition (1.12) with $m = 0$.

The product of $f \in \mathcal{O}_X^{t,\hbar}(0)$ and $g \in \mathcal{O}_X^{t,\hbar}(0)$ is well defined and $fg \in \mathcal{O}_X^{t,\hbar}(0)$. □

Therefore:

Proposition 1.11. *The sheaf $\mathcal{O}_X^{t,\hbar}$ is naturally endowed with a structure of a commutative filtered* **k**-*algebra.*

Let U be an open subset of X and let $f(s,x,\hbar) \in \Gamma((\mathbb{C}_s \setminus K)\times U; \mathcal{O}^{\hbar}_{\mathbb{C}_s\times X})$. We formally define the Laplace transform $\mathcal{L}(f)$ of f by

$$\mathcal{L}(f)(t,x,\hbar) = \frac{1}{2i\pi}\int_\gamma f(s,x,\hbar)\exp(st\hbar^{-1})\,ds,$$

where γ is a counter-clockwise oriented circle centered at 0 with radius $R \gg 0$.

Example 1.12.

$$\mathcal{L}(s^{-n-1}) = \hbar^{-n}t^n/n!,\;\; \mathcal{L}(\frac{1}{s-1}) = \exp(t\hbar^{-1}).$$

Lemma 1.13. *The Laplace transform induces a* **k**-*linear monomorphism*

$$s^{-1}\cdot \mathcal{O}_X[[s^{-1}]][[\hbar,\hbar^{-1}] \hookrightarrow \mathcal{O}_X[[t]][[\hbar,\hbar^{-1}]].$$

Proof. One notices that the Laplace transform is given by:

$$\sum_{-\infty<j\le m}\sum_{n\ge 0} a_{n,j}s^{-n-1}\hbar^{-j} \mapsto \sum_{j\le m}\sum_{n\ge 0}\frac{a_{n,j}}{n!}t^n\hbar^{-n-j},$$

and the result follows. $\square$

Theorem 1.14. *The Laplace transform induces a* **k***-linear isomorphism of filtered* **k***-algebras*

$$\mathcal{L}\colon \mathcal{O}_X^{s,\hbar} \xrightarrow{\sim} \mathcal{O}_X^{t,\hbar}. \tag{1.16}$$

Proof. (i) By Lemma 1.10, it is enough to check that $\mathcal{L}$ induces an isomorphism $\mathcal{O}_X^{s,\hbar}(0) \xrightarrow{\sim} \mathcal{O}_X^{t,\hbar}(0)$.
(ii) Let W be a Stein open subset of X and let U be a relatively compact open subset of W. Let us develop a section $f(s,x,\hbar)$ of $\Gamma(W; R^1a_!\mathcal{O}^{\hbar}_{\mathbb{C}_s\times X}(0))$ with respect to s^{-1} for $s > R$. We get

$$\begin{aligned}\tilde{f}(s,x,\hbar) &= \sum_{-\infty<j\le 0}\tilde{f}_j(s,x)\hbar^{-j}\\ &= \sum_{-\infty<j\le 0}\sum_{n\ge 0} f_{j,n}(x)s^{-n-1}\hbar^{-j}\end{aligned}$$

with the following Cauchy estimates:

$$\begin{cases}\text{for any compact subset } K \text{ of } U \text{ there exist positive constants}\\ C,\,\varepsilon,\,R \text{ such that } \sup\limits_{x\in K}|f_{j,n}(x)| \le C\varepsilon^{-j}(-j)!R^n.\end{cases}$$

Applying the Laplace transform to $\tilde{f}(s,x,\hbar)$ means to replace s^{-n-1} with $\dfrac{t^n}{n!}\hbar^{-n}$. Hence, we find

$$\mathcal{L}(\tilde{f})(t,x,\hbar) = \sum_{-\infty<j<\infty} f_j(t,x)\hbar^{-j} = \sum_{-\infty<j\le 0}\sum_{n\ge 0} f_{j,n}(x)\frac{t^n}{n!}\hbar^{-j-n},$$

where

$$f_j(t,x) = \sum_{j\le n,0\le n} f_{j-n,n}(x)\frac{t^n}{n!}$$

satisfies

$$|f_j(t,x)| \le C\sum_{j\le n,0\le n}\varepsilon^{n-j}\frac{(n-j)!}{n!}(|t|R)^n.$$

Let $\eta < (\varepsilon R)^{-1}$. It follows that $f_j(t,x)$ is holomorphic in a neighborhood of $\{|t| \le \eta\}\times K$.

Assume $j < 0$, $|t| \le \eta$ and $x \in K$. We get

$$|f_j(t,x)| \le C\varepsilon^{-j}(-j)!\sum_{0\le n}\frac{(n-j)!}{(-j)!n!}(\eta\varepsilon R)^n \le \frac{C}{1-\eta\varepsilon R}\left(\frac{\varepsilon}{1-\eta\varepsilon R}\right)^{-j}(-j)!.$$

Hence Condition (1.11) is satisfied.

Assume $j \geq 0$. We get for $|t| \leq \eta$ and $x \in K$,

$$|f_j(t,x)| \leq C\frac{\varepsilon^{-j}}{j!}\sum_{j\leq n}\frac{j!(n-j)!}{n!}(|t|R\varepsilon)^n \leq \frac{C}{1-\eta\varepsilon R}\frac{R^j}{j!}|t|^j.$$

Hence Condition (1.12) for $m = 0$ is satisfied and $\mathcal{L}(f)(t,x,\hbar)$ is in $\mathcal{O}_X^{t,\hbar}(0)$.
(iii) Conversely, let $f(t,x,\hbar)$ be a section of $\mathcal{O}_X^{t,\hbar}(0)$. We develop f as

$$f(t,x,\hbar) = \sum_{-\infty<j<\infty} f_j(t,x)\hbar^{-j} = \sum_{-\infty<j<\infty}\sum_{n\geq 0} n! f_{j,n}(x)\frac{t^n}{n!}\hbar^{-n}\hbar^{-j+n}. \quad (1.17)$$

For any compact set K, there exists $\eta > 0$ such that $f_j(t,x)$ is holomorphic in a neighborhood of $\{|t| \leq \eta\} \times K$. Conditions (1.11) and (1.12) give the Cauchy estimates

$$|f_{j,n}(x)| \leq C\varepsilon^{-j}(-j)!\eta^{-n} \text{ for } j < 0,$$
$$|f_{j,n}(x)| \leq M\frac{R^j}{j!}\eta^{j-n} \text{ for } j \geq 0.$$

Notice that Condition (1.12) for $j > 0$ implies that

$$f_j(0,x) = \frac{\partial f_j}{\partial t}(0,x) = \cdots = \frac{\partial^{j-1} f_j}{\partial t^{j-1}}(0,x) = 0, \quad (1.18)$$

or $f_{j,n}(x) = 0$ for $0 \leq n \leq j-1$.

The inverse Laplace transform consists formally in replacing $\frac{t^n}{n!}\hbar^{-n}$ by s^{-n-1} in (1.17). We then get

$$\mathcal{L}^{-1}(f)(s,x,\hbar) = \tilde{f}(s,x,\hbar) = \sum_{-\infty<j<\infty}\sum_{n\geq 0} n! f_{j+n,n}(x)s^{-n-1}\hbar^{-j}.$$

Writing $\tilde{f}(s,x,\hbar) = \sum_{-\infty<j<\infty}\tilde{f}_j(s,x)\hbar^{-j}$, (1.18) implies that $\tilde{f}_j(s,x) = 0$ for $j \geq 1$.

Let $R_1 > R$ be large enough so that $(\eta R_1)^{-1} \leq 1$. We shall check that the sum $\tilde{f}_j(s,x) = \sum_{n\geq 0} n! f_{j+n,n}(x)s^{-n-1}$ defines a holomorphic function in a neighborhood of $\{|s| \geq R_1\} \times K$ for any $j \leq 0$.

For $j \leq 0$, let us split the sum $\tilde{f}_j(s,x)$ as

$$\tilde{f}_j(s,x) = \sum_{n\geq -j} n! f_{j+n,n}(x)s^{-n-1} + \sum_{0\leq n<-j} n! f_{j+n,n}(x)s^{-n-1}. \quad (1.19)$$

In the first sum we have $n+j \leq 0$ and for $|s| \geq R_1$ and $x \in K$, we get from the Cauchy estimates

$$|n! f_{j+n,n}(x)s^{-n-1}| \leq n! M\frac{R^{n+j}}{(n+j)!}\eta^j|s|^{-n-1} \leq \frac{M}{R_1}(\eta R)^j(-j)!\binom{n}{-j}(\frac{R}{R_1})^n. \quad (1.20)$$

The right-hand side is the general term of a convergent series since $R_1 > R$ and we get the result by noticing that the second sum in (1.19) is finite.

Finally we shall show that $\tilde{f}_j(s,x)$ satisfies the required estimates. From (1.20), the first sum in (1.19) is bounded by

$$
\begin{aligned}
|\sum_{n\geq -j} n! f_{j+n,n}(x)s^{-n-1}| &\leq \frac{M}{R_1}(\eta R)^j(-j)! \sum_{n\geq -j}\binom{n}{-j}(\frac{R}{R_1})^n \\
&\leq \frac{M}{R_1-R}\Big(\frac{1}{\eta(R_1-R)}\Big)^{-j}(-j)!.
\end{aligned}
$$

Similarly, for the second sum we have

$$
|\sum_{0\leq n<-j} n! f_{j+n,n}(x)s^{-n-1}| \leq \frac{C}{R_1}\varepsilon^{-j}(-j)! \sum_{0\leq n<-j}\frac{1}{\binom{-j}{n}}(\frac{1}{\eta R_1})^n \leq \frac{2C}{R_1}\varepsilon^{-j}(-j)!,
$$

where the last inequality follows from $(\eta R_1)^{-1}\leq 1$ and $\sum_{0\leq n<-j}\frac{1}{\binom{-j}{n}}\leq 2$.

Combining these estimates we get for $j\leq 0$,

$$
|\tilde{f}_j(s,x)| \leq \tilde{C}\tilde{\varepsilon}^{-j}(-j)!,
$$

with $\tilde{C}=\max\{\frac{M}{R_1-R},\frac{2C}{R_1}\}$ and $\tilde{\varepsilon}=\max\{\varepsilon,\frac{1}{\eta(R_1-R)}\}$.

Therefore $\tilde{f}(s,x,\hbar)=\sum_{j\leq 0}\tilde{f}_j(s,x)\hbar^{-j}$ is a section of $\mathcal{O}_X^{s,\hbar}(0)$ and $\mathcal{L}(\tilde{f})(t,x,\hbar)=f(t,x,\hbar)$.

(iv) The fact that $\mathcal{L}$ is a morphism of algebras follows easily from Example 1.3. □

The ring $\mathrm{gr}\,\mathcal{O}_X^{t,\hbar}$. If $\mathcal{A}$ is a filtered sheaf of rings, we denote as usual by $\mathrm{gr}\,\mathcal{A}$ the associated graded ring.

Let $\mathbb{C}_u$ be the complex line endowed with the coordinate u and denote by $b\colon X\times\mathbb{C}_u\to X$ the projection.

Definition 1.15. (i) *One denotes by* $\mathcal{O}_X^{\exp u}$ *the subsheaf of* $\mathbb{C}$*-algebras on* X *of the sheaf* $b_*\mathcal{O}_{X\times\mathbb{C}_u}$ *whose sections on an open set* $U\subset X$ *are the holomorphic functions* $f(x,u)$ *on* $U\times\mathbb{C}_u$ *satisfying:*

$$
\begin{cases}
\text{for any compact subset } K \text{ of } U \text{ there exist positive constants}\\
C,\ R \text{ such that } \sup_{x\in K}|f(x,u)|\leq C\exp(R|u|).
\end{cases}
$$

(ii) One sets $\mathcal{O}_X^{\exp t\hbar^{-1}}[\hbar,\hbar^{-1}]=\mathcal{O}_X^{\exp t\hbar^{-1}}\otimes_{\mathbb{C}}\mathbb{C}[\hbar,\hbar^{-1}]$.

Proposition 1.16. *There is a natural isomorphism of graded sheaves of rings*

$$
\mathrm{gr}\,\mathcal{O}_X^{t,\hbar}\simeq\mathcal{O}_X^{\exp t\hbar^{-1}}[\hbar,\hbar^{-1}].
$$

Proof. First note the isomorphism

$$
\mathcal{O}_X^{s,\hbar}(0)/\mathcal{O}_X^{s,\hbar}(-1)\simeq R^1a_!\mathcal{O}_{\mathbb{C}_s\times X},
$$

from which we deduce the isomorphism

$$
\mathrm{gr}\,\mathcal{O}_X^{s,\hbar}\simeq R^1a_!\mathcal{O}_{\mathbb{C}_s\times X}\otimes_{\mathbb{C}}\mathbb{C}[\hbar,\hbar^{-1}].
$$

The classical Paley-Wiener theorem says that the Laplace transform induces an isomorphism between $H^1_c(\mathbb{C}; \mathcal{O}_{\mathbb{C}})$ and the space of entire functions of exponential type. An extension of this result with holomorphic parameters provides an isomorphism

$$\mathcal{L}\colon R^1 a_! \mathcal{O}_{\mathbb{C}_s \times X} \xrightarrow{\sim} \mathcal{O}_X^{\exp t\hbar^{-1}},$$

and the result follows. □

The formal case. It is possible to replace $\mathcal{O}_X^\hbar$ with $\widehat{\mathcal{O}}_X^\hbar$ in the preceding constructions and to set

$$\widehat{\mathcal{O}}_X^{s,\hbar} := R^1 a_! \widehat{\mathcal{O}}^\hbar_{\mathbb{C}_s \times X}. \tag{1.21}$$

However the Laplace transform of $\widehat{\mathcal{O}}_X^{s,\hbar}$ does not seem to have an easy description. Indeed, its sections are no longer germs of holomorphic functions with respect to t as shown in the next example.

Example 1.17. Consider a sequence $\{c_j\}_{j\le 0}$ of complex numbers and the section f of $\widehat{\mathcal{O}}_X^{s,\hbar}$ given by

$$f(s,\hbar) = \sum_{j\le 0} \frac{c_j}{(s-1)} \hbar^{-j}.$$

Then, formally, the Laplace transform of f is given by

$$\mathcal{L}(f)(t,\hbar) = \sum_{j\le 0}\sum_{n\ge 0} c_j \frac{t^n}{n!} \hbar^{-n-j},$$

and the coefficient of $\hbar^0$ is $\sum_{n\ge 0} c_{-n}\frac{t^n}{n!}$, which does not belong to $\mathcal{O}_{\mathbb{C}_t}|_{t=0}$ in general.

2. The Algebra $\mathcal{W}_{T^*X}$

Let $(X, \mathcal{O}_X)$ be a complex manifold. The cotangent bundle T^*X is a homogeneous symplectic manifold endowed with the $\mathbb{C}^\times$-conic sheaf of rings $\mathcal{E}_{T^*X}$ of finite-order microdifferential operators. This ring is filtered and contains in particular the subring $\mathcal{E}_{T^*X}(0)$ of operators of order ≤ 0. This ring is constructed in [9] and we assume that the reader is familiar with this theory, referring to [5] or [10] for an exposition.

On the symplectic manifold T^*X there exists another (no more conic) useful sheaf of rings constructed as follows (see [8]). Let $\mathbb{C}$ be the complex line endowed with the coordinate t and $(t;\tau)$ the associated coordinates on $T^*\mathbb{C}$. Set $T^*_{\{\tau\neq 0\}}(X\times\mathbb{C}) = \{(x,t;\xi,\tau);\ \tau\neq 0\}$ and consider the map

$$\rho\colon T^*_{\{\tau\neq 0\}}(X\times\mathbb{C}) \to T^*X, \quad (x,t;\xi,\tau)\mapsto (x;\xi/\tau). \tag{2.1}$$

Set

$$\mathcal{E}_{T^*(X\times\mathbb{C}),\hat{t}} = \{P \in \mathcal{E}_{T^*(X\times\mathbb{C})};\ [P, \partial/\partial_t] = 0\}. \tag{2.2}$$

The ring $\mathcal{W}_{T^*X}$ on T^*X is given by

$$\mathcal{W}_{T^*X} := \rho_*(\mathcal{E}_{T^*(X\times\mathbb{C}),\hat{t}}).$$

In the sequel we set

$$\hbar := \tau^{-1}. \tag{2.3}$$

The ring $\mathcal{W}_{T^*X}$ is filtered and we denote by $\mathcal{W}_{T^*X}(j)$ the subsheaf of $\mathcal{W}_{T^*X}$ consisting of sections of order less than or equal to j. The following result was obtained in [8].

Theorem 2.1. (i) *The sheaf* $\mathcal{W}_{T^*X}$ *is naturally endowed with a structure of a filtered* $\mathbf{k}$*-algebra and* $\operatorname{gr}\mathcal{W}_{T^*X} \simeq \mathcal{O}_{T^*X}[\hbar, \hbar^{-1}]$.
(ii) *Consider two complex manifolds* X *and* Y*, two open subsets* $U_X \subset T^*X$ *and* $U_Y \subset T^*Y$ *and a symplectic isomorphism* $\psi : U_X \xrightarrow{\sim} U_Y$*. Then, locally,* ψ *may be quantized as an isomorphism of filtered* $\mathbf{k}$*-algebras* $\Psi\colon \mathcal{W}_{T^*X} \xrightarrow{\sim} \mathcal{W}_{T^*Y}$ *such that the isomorphism induced on the graded algebras coincides with the isomorphism* $\mathcal{O}_{T^*X}[\hbar, \hbar^{-1}] \xrightarrow{\sim} \mathcal{O}_{T^*Y}[\hbar, \hbar^{-1}]$ *induced by* ψ.

Total symbols. Assume that X is affine of dimension n, that is, X is open in some $\mathbb{C}$-vector space V of dimension n.

Theorem 2.2. *Assume* X *is affine. There is an isomorphism of filtered sheaves of* $\mathbf{k}$*-modules* (*not of algebras*)*, called the "total symbol" morphism:*

$$\sigma_{\mathrm{tot}}\colon \mathcal{W}_{T^*X} \xrightarrow{\sim} \mathcal{O}^{\hbar}_{T^*X}. \tag{2.4}$$

The total symbol of a product is given by the Leibniz formula. Denote by (x) *a local coordinate system on* X *and denote by* (x, u) *the associated local symplectic coordinate system on* T^*X*. If* Q *is an operator of total symbol* $\sigma_{\mathrm{tot}}(Q)$*, then*

$$\sigma_{\mathrm{tot}}(P \circ Q) = \sum_{\alpha \in \mathbb{N}^n} \frac{\hbar^{|\alpha|}}{\alpha!} \partial_u^{\alpha} \sigma_{\mathrm{tot}}(P) \cdot \partial_x^{\alpha} \sigma_{\mathrm{tot}}(Q). \tag{2.5}$$

The total symbol of a section $P \in \mathcal{W}_{T^*X}(U)$ is thus written as a formal series:

$$\sigma_{\mathrm{tot}}(P) = \sum_{-\infty \le j \le m} p_j(x; u)\hbar^{-j}, \quad m \in \mathbb{Z}, \quad p_j \in \mathcal{O}_{T^*X}(U), \tag{2.6}$$

with the condition (1.4).

Note that (2.5) does not depend of the choice of a local coordinate system on X but only on the affine structure of V. Indeed, (2.5) may be rewritten as

$$\sigma_{\mathrm{tot}}(P \circ Q) = (\exp(\hbar\langle d_u, d_y\rangle)\sigma_{\mathrm{tot}}(P)(x, u)\sigma_{\mathrm{tot}}(Q)(y, v))|_{x=y,u=v},$$

where $\langle d_u, d_y\rangle = \sum_{i=1}^{n} \partial_{u_i}\partial_{y_i}$ does not depend on the affine coordinate system.

Remark 2.3. Let us identify X with the zero section of T^*X. Then the sheaf $\mathcal{O}^{\hbar}_X$ (see Def. 1.2) is isomorphic to the left coherent $\mathcal{W}_{T^*X}$-module obtained as the quotient of $\mathcal{W}_{T^*X}$ by the left ideal generated by the vector fields on X.

3. The Algebra $\mathcal{W}^s_{T^*X}$.

Operations on $\mathcal{W}$. Let S be a complex manifold of complex dimension d_S. One defines the sheaf $\mathcal{W}_{S\times T^*X}$ on $S \times T^*X$ as the subsheaf of $\mathcal{W}_{T^*(S\times X)}$ consisting of sections which commute with the holomorphic functions on S. Heuristically, $\mathcal{W}_{S\times T^*X}$ is the sheaf $\mathcal{W}_{T^*X}$ with holomorphic parameters on S. For a morphism of complex manifolds $f\colon S \to Z$ we shall still denote by f the map $S \times X \to Z \times X$, as well as the map $S \times T^*X \to Z \times T^*X$. One denotes as usual by Ω_S the sheaf of holomorphic forms of maximal degree and one sets for short:

$$\mathcal{W}^{(d_S)}_{S\times T^*X} = \mathcal{W}_{S\times T^*X} \otimes_{\mathcal{O}_S} \Omega_S. \tag{3.1}$$

Let us recall well-known operations of the theory of microdifferential operators. Although these results do not seem to be explicitly written in the literature, their proofs are straightforward and will not be given here.

Let $f\colon S \to Z$ be a morphism of complex manifolds. The usual operations of inverse image $f^*\colon f^{-1}\mathcal{O}_Z \to \mathcal{O}_S$ and of direct image $\int_f\colon Rf_!\Omega_S[d_S] \to \Omega_Z[d_Z]$ extend to $\mathcal{W}_{S\times T^*X}$. More precisely, there exist morphisms of sheaves of $\mathbf{k}$-modules (the second morphism holds in the derived category $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{Z\times T^*X})$):

$$f^*\colon f^{-1}\mathcal{W}_{Z\times T^*X} \to \mathcal{W}_{S\times T^*X}, \tag{3.2}$$

$$\int_f\colon Rf_!(\mathcal{W}^{(d_S)}_{S\times T^*X}[d_S]) \to \mathcal{W}^{(d_Z)}_{Z\times T^*X}[d_Z], \tag{3.3}$$

these morphisms having the following properties:

- they are functorial with respect to f, that is, for a morphism of complex manifolds $g\colon Z \to W$, one has $(g\circ f)^* \simeq f^*\circ g^*$ and $\int_{g\circ f} = \int_g \circ \int_f$, and moreover the inverse (resp. direct) image of the identity morphism is the identity,
- when X is affine, f^* and $\int_f$ commute with the total symbol morphism (2.4).

As a convention, we choose the morphism in (3.3) so that the integral of $\dfrac{ds}{s} \in H^1_c(\mathbb{C}_s; \Omega_{\mathbb{C}_s})$ is 1. In other words,

$$\int_a \frac{1}{s} = \frac{1}{2i\pi}\int_\gamma \frac{ds}{s},$$

where γ is a counter-clockwise oriented circle around the origin.

The algebra $\mathcal{W}^s_{T^*X}$. Denote by

$$a\colon \mathbb{C}_s \times T^*X \to T^*X \tag{3.4}$$

the projection. Then, after identifying the sheaves $\mathcal{O}_{\mathbb{C}_s}$ and $\Omega_{\mathbb{C}_s}$ by $f(s) \mapsto f(s)ds$, the sheaf $R^1a_!\mathcal{W}_{\mathbb{C}_s\times T^*X}$ is endowed with a structure of a filtered $\mathbf{k}$-algebra by

$$\begin{aligned}
H^1_c(\mathbb{C}_s \times T^*X; \mathcal{W}_{\mathbb{C}_s\times T^*X}) \times H^1_c(\mathbb{C}_{s'} \times T^*X; \mathcal{W}_{\mathbb{C}_{s'}\times T^*X})&\\
\to H^2_c(\mathbb{C}^2_{s,s'} \times T^*X; \mathcal{W}_{\mathbb{C}^2_{s,s'}\times T^*X})&\\
\to H^1_c(\mathbb{C}_s; \mathcal{W}_{\mathbb{C}_s\times T^*X}),&
\end{aligned}$$

where the first arrow is the cup product and the second arrow is the integration along the fibers of the map $\mathbb{C}^2 \to \mathbb{C}$, $(s, s') \mapsto s + s'$.

Definition 3.1. *The sheaf* $\mathcal{W}^s_{T^*X}$ *of* $\mathbf{k}$*-modules on* T^*X *is given by*

$$\mathcal{W}^s_{T^*X} = R^1 a_!(\mathcal{W}_{\mathbb{C}_s\times T^*X}). \tag{3.5}$$

After identifying the holomorphic function $\frac{1}{s}$ with the cohomology class it defines in $H^1_c(\mathbb{C}_s;\mathcal{O}_{\mathbb{C}_s})$, we define the morphism of sheaves

$$\iota\colon \mathcal{W}_{T^*X} \to \mathcal{W}^s_{T^*X},\quad P\mapsto \frac{1}{s}P. \tag{3.6}$$

Clearly, the morphism (3.6) is a monomorphism of sheaves of $\mathbf{k}$-algebras.

We define the morphism of sheaves

$$\mathrm{res}\colon \mathcal{W}^s_{T^*X} \to \mathcal{W}_{T^*X} \tag{3.7}$$

by the integration morphism (3.3) associated to the map (3.4). Clearly, the morphism (3.7) is a morphism of sheaves of $\mathbf{k}$-algebras. Hence:

Theorem 3.2. (i) *The sheaf* $\mathcal{W}^s_{T^*X}$ *is naturally endowed with a structure of a filtered* $\mathbf{k}$*-algebra and* $\mathrm{gr}\,\mathcal{W}^s_{T^*X}\simeq R^1a_!\mathcal{O}_{\mathbb{C}_s\times T^*X}[\hbar,\hbar^{-1}]$.

(ii) *The monomorphism* ι *in* (3.6) *is a morphism of filtered* $\mathbf{k}$*-algebras, the integration morphism* res *in* (3.7) *is a morphism of filtered* $\mathbf{k}$*-algebras and the composition* $\mathrm{res}\circ\iota\colon \mathcal{W}_{T^*X}\to\mathcal{W}^s_{T^*X}\to\mathcal{W}_{T^*X}$ *is the identity.*

(iii) *Consider two complex manifolds* X *and* Y*, two open subsets* $U_X\subset T^*X$ *and* $U_Y\subset T^*Y$ *and a symplectic isomorphism* $\psi : U_X \xrightarrow{\sim} U_Y$*. Then, locally,* ψ *may be quantized as an isomorphism of filtered* $\mathbf{k}$*-algebras* $\Psi\colon \mathcal{W}^s_{T^*X}\xrightarrow{\sim}\mathcal{W}^s_{T^*Y}$ *such that the isomorphism induced on the graded algebras coincides with the isomorphism* $R^1a_!\mathcal{O}_{\mathbb{C}_s\times T^*X}[\hbar,\hbar^{-1}]\xrightarrow{\sim}R^1a_!\mathcal{O}_{\mathbb{C}_s\times T^*Y}[\hbar,\hbar^{-1}]$ *induced by* ψ.

(iv) *Assume* X *is affine. There is an isomorphism of filtered sheaves of* $\mathbf{k}$*-modules* (*not of algebras*)*, called the "total symbol" morphism:*

$$\sigma_{\mathrm{tot}}\colon \mathcal{W}^s_{T^*X}\xrightarrow{\sim}\mathcal{O}^{s,\hbar}_{T^*X}. \tag{3.8}$$

The total symbol of a product is given by the Leibniz formula with a convolution product in the s *variable* (*see* (3.10)).

Proof. These results follow immediately from Theorem 2.1. □

Assume that X is affine. For each Stein open subset W of T^*X and each relatively compact open subset $U\subset\subset W$, a section P of $\mathcal{W}^s_{T^*X}$ on W admits a total symbol

$$\sigma_{\mathrm{tot}}(P)(s,x,u) = \sum_{-\infty<j\le m} p_j(s,x;u)\hbar^{-j},\quad m\in\mathbb{Z}, \tag{3.9}$$

where p_j belongs to $\Gamma((\mathbb{C}_s\setminus K_0)\times U;\mathcal{O}_{\mathbb{C}_s\times T^*X})$, for a compact subset K_0 of $\mathbb{C}_s$ which depends only on P and U, and the p_j's satisfy an estimate as in (1.4) on each compact subset K of $(\mathbb{C}_s\setminus K_0)\times U$.

Consider now two sections P and Q of $\mathcal{W}^s_{T^*X}$ on a Stein open set W with total symbols as in (3.9) (replacing p_j with q_j and m with m' for Q). Then the total symbol of $P\circ Q$ is given by the Leibniz formula:

$$\sigma_{\mathrm{tot}}(P\circ Q) = \sum_{\alpha\in\mathbb{N}^n}\frac{\hbar^{|\alpha|}}{\alpha!}\partial_u^\alpha\sigma_{\mathrm{tot}}(P) * \partial_x^\alpha\sigma_{\mathrm{tot}}(Q), \tag{3.10}$$

where, setting $f(s,x,u)=\partial_u^\alpha\sigma_{\mathrm{tot}}(P)(s,x;u)$ and $g(s,x,u)=\partial_x^\alpha\sigma_{\mathrm{tot}}(Q)(s,x;u)$, the product $f*g$ is given by (1.9).

4. The Laplace Transform and the Algebra $\mathcal{W}^t_{T^*X}$

The filtered $\mathbf{k}$-algebra $\mathcal{W}^t_{T^*X}$ on T^*X is the algebra $\mathcal{W}^s_{T^*X}$, but with a different symbol calculus.

Definition 4.1. *We set $\mathcal{W}^t_{T^*X} := \mathcal{W}^s_{T^*X}$. For X affine, the total symbol morphism of $\mathbf{k}$-modules (not of algebras)*

$$\sigma_{\mathrm{tot}} \colon \mathcal{W}^t_{T^*X} \xrightarrow{\sim} \mathcal{O}^{t,\hbar}_{T^*X} \tag{4.1}$$

is the composition $\mathcal{W}^s_{T^*X} \xrightarrow[\sigma_{\mathrm{tot}}]{\sim} \mathcal{O}^{s,\hbar}_{T^*X} \xrightarrow[\mathcal{L}]{\sim} \mathcal{O}^{t,\hbar}_{T^*X}$.

For P a section of $\mathcal{W}^t_{T^*X}$ on a Stein open subset V of T^*X and an open subset $U \subset\subset V$, $\sigma_{\mathrm{tot}}(P)$ is written as a series

$$\sigma_{\mathrm{tot}}(P)(t,x,u,\hbar) = \sum_{-\infty<j<\infty} p_j(t,x,u)\hbar^{-j}, \quad p_j \in \mathcal{O}_{\mathbb{C}\times T^*X|_{t=0}}(U)$$

satisfying (1.11) and (1.12).

Applying Theorem 3.2, we get:

Theorem 4.2. (i) *$\mathcal{W}^t_{T^*X}$ is a filtered $\mathbf{k}$-algebra and $\mathrm{gr}\,\mathcal{W}^t_{T^*X} \simeq \mathcal{O}^{\exp t\hbar^{-1}}_{T^*X}[\hbar,\hbar^{-1}]$ (see Definition 1.15).*

(ii) *The morphism ι in (3.6) induces a monomorphism of filtered $\mathbf{k}$-algebras $\iota \colon \mathcal{W}_{T^*X} \hookrightarrow \mathcal{W}^t_{T^*X}$, the morphism res in (3.7) induces a morphism of filtered $\mathbf{k}$-algebras $\mathrm{res}\colon \mathcal{W}^t_{T^*X} \to \mathcal{W}_{T^*X}$ and the composition $\mathcal{W}_{T^*X} \to \mathcal{W}^t_{T^*X} \to \mathcal{W}_{T^*X}$ is the identity.*

(iii) *Consider two complex manifolds X and Y, two open subsets $U_X \subset T^*X$ and $U_Y \subset T^*Y$ and a symplectic isomorphism $\psi : U_X \xrightarrow{\sim} U_Y$. Then, locally, ψ may be quantized as an isomorphism of filtered $\mathbf{k}$-algebras $\Psi\colon \mathcal{W}^t_{T^*X} \xrightarrow{\sim} \mathcal{W}^t_{T^*Y}$ such that the isomorphism induced on the graded algebras coincides with the isomorphism $\mathcal{O}^{\exp t\hbar^{-1}}_{T^*X}[\hbar,\hbar^{-1}] \xrightarrow{\sim} \mathcal{O}^{\exp t\hbar^{-1}}_{T^*Y}[\hbar,\hbar^{-1}]$ induced by ψ.*

(iv) *Assume X is affine. There is an isomorphism of filtered sheaves of $\mathbf{k}$-modules (not of algebras), called the "total symbol" morphism:*

$$\sigma_{\mathrm{tot}} \colon \mathcal{W}^t_{T^*X} \xrightarrow{\sim} \mathcal{O}^{t,\hbar}_{T^*X}. \tag{4.2}$$

The total symbol of a product is given by the Leibniz formula.

For P and Q two sections of $\mathcal{W}^t_{T^*X}$ on an open subset U of T^*X, with X affine, the total symbol of $P \circ Q$ is thus given by the formula:

$$\sigma_{\mathrm{tot}}(P\circ Q) = \sum_{\alpha\in\mathbb{N}^n} \frac{\hbar^{|\alpha|}}{\alpha!}\partial_u^\alpha \sigma_{\mathrm{tot}}(P)\cdot \partial_x^\alpha \sigma_{\mathrm{tot}}(Q), \tag{4.3}$$

where the product $\partial_u^\alpha\sigma_{\mathrm{tot}}(P)\cdot\partial_x^\alpha\sigma_{\mathrm{tot}}(Q)$ is given by the usual commutative algebra structure of $\mathcal{O}^{t,\hbar}_{T^*X}$ of Lemma 1.10.

Remark 4.3. In Theorem 4.2, the monomorphism $\mathcal{W}_{T^*X} \to \mathcal{W}^t_{T^*X}$ is given on symbols by $\sigma_{\mathrm{tot}}(P) \mapsto \sigma_{\mathrm{tot}}(P)$ and the morphism $\mathcal{W}^t_{T^*X} \to \mathcal{W}_{T^*X}$ is given on symbols by $\sigma_{\mathrm{tot}}(P)(t,x;u,\hbar) \mapsto \sigma_{\mathrm{tot}}(P)(0,x;u,\hbar)$.

The formal case. The above constructions also work when replacing the sheaf $\mathcal{W}_{T^*X}$ with its formal counterpart, the sheaf $\widehat{\mathcal{W}}_{T^*X}$. Let us briefly explain it.

Let X be a complex manifold, as above. Replacing the sheaf of rings $\mathcal{E}_{T^*X}$ on T^*X with the sheaf of rings $\widehat{\mathcal{E}}_{T^*X}$ of formal microdifferential operators and proceeding as for $\mathcal{W}_{T^*X}$, we get the sheaf of rings $\widehat{\mathcal{W}}_{T^*X}$ of finite-order formal WKB-operators on T^*X. It is defined by

$$\widehat{\mathcal{W}}_{T^*X} := \rho_*(\widehat{\mathcal{E}}_{T^*(X\times\mathbb{C}),\hat{\tau}}).$$

When X is affine of dimension n, the total symbol morphism induces an isomorphism of $\widehat{\mathbf{k}}$-modules

$$\sigma_{\mathrm{tot}}\colon \widehat{\mathcal{W}}_{T^*X} \xrightarrow{\sim} \widehat{\mathcal{O}}^{\hbar}_{T^*X},$$

and the symbol $\sigma_{\mathrm{tot}}(P\circ Q)$ is given by the Leibniz formula (2.5). Then by a similar construction as for $\mathcal{W}^s_{T^*X}$ we construct the filtered sheaf of $\widehat{\mathbf{k}}$-algebras $\widehat{\mathcal{W}}^s_{T^*X}$. Namely, we set

$$\widehat{\mathcal{W}}^s_{T^*X} := R^1a_!\widehat{\mathcal{W}}_{\mathbb{C}\times T^*X}.$$

If X is affine, the total symbol morphism induces an isomorphism of $\widehat{\mathbf{k}}$-modules $\widehat{\mathcal{W}}^s_{T^*X} \xrightarrow{\sim} \widehat{\mathcal{O}}^{s,\hbar}_{T^*X}$ and the product is again given by the Leibniz formula (3.10).

However, as already noticed, the Laplace transform does not seem to behave as well for the formal case as for the analytic case, and we shall not construct the Laplace transform of $\widehat{\mathcal{O}}^{s,\hbar}_{T^*X}$.

5. Remark: The Algebra $\mathcal{W}^s_{\mathfrak{X}}$ on a Symplectic Manifold $\mathfrak{X}$

The complex case. Consider a complex symplectic manifold $\mathfrak{X}$. There exists an open covering $\mathfrak{X} = \bigcup_i U_i$ and complex symplectic isomorphisms $\varphi_i\colon U_i \xrightarrow{\sim} V_i$ where the V_i's are open in some cotangent bundles T^*X_i of complex manifolds X_i. Set $\mathcal{W}_{U_i} := \varphi_i^{-1}\mathcal{W}_{T^*X_i|V_i}$. In general, the $\mathcal{W}_{U_i}$'s do not glue in order to give a globally defined sheaf of algebras $\mathcal{W}_{\mathfrak{X}}$ on $\mathfrak{X}$. However the prestack $\mathfrak{S}$ on $\mathfrak{X}$ (roughly speaking, a prestack is a sheaf of categories) $U_i \mapsto \mathrm{Mod}(\mathcal{W}_{U_i})$ is a stack and the category $\mathrm{Mod}(\mathcal{W}_{\mathfrak{X}}) := \mathfrak{S}(\mathfrak{X})$ is well defined. Moreover, one can give a precise meaning to $\mathcal{W}_{\mathfrak{X}}$ by replacing the notion of a sheaf of algebras with that of an algebroid. We refer to [4] for the construction of (an analogue of) this stack in the contact complex case and to [6] in the symplectic complex case for $\widehat{\mathcal{W}}_{\mathfrak{X}}$ and for the definition of an algebroid. See also [8] for a construction of $\mathcal{W}_{\mathfrak{X}}$ (by a different method). By adapting the construction of [8], one easily constructs the algebroid $\mathcal{W}^s_{\mathfrak{X}}$ associated with the locally defined sheaves of algebras $\mathcal{W}^s_{U_i}$. Details are left to the reader.

The real case. Let M be a real analytic manifold, X a complexification of M and denote by ω_X the canonical 2-form on T^*X. The conormal bundle T^*_MX is Lagrangian for $\mathrm{Re}\,\omega_X$ and symplectic for $\mathrm{Im}\,\omega_X$. In particular, the real manifold T^*_MX is symplectic. For an open subset U of T^*_MX, we set $\mathcal{W}_U := \mathcal{W}_{T^*X}|_U$.

Now, consider a real analytic symplectic manifold $\mathfrak{M}$. It is well known that it is possible to construct a globally defined sheaf of algebras $\mathcal{W}_{\mathfrak{M}}$ on $\mathfrak{M}$ such that:

- there exists an open covering $\mathfrak{M} = \bigcup_{i \in I} U_i$ and real symplectic isomorphisms $\varphi_i : U_i \xrightarrow{\sim} V_i$ where the V_i's are open in the conormal bundles $T^*_{M_i} X_i$ for some real manifolds M_i with complexification X_i,
- $\mathcal{W}_{\mathfrak{M}}|_{V_i} \simeq \varphi_i^{-1} \mathcal{W}_{V_i}$ for all $i \in I$.

Replacing $\mathfrak{M}$ with $\mathbb{C}_s \times \mathfrak{M}$, one easily constructs the sheaf of algebras $\mathcal{W}_{\mathbb{C}_s \times \mathfrak{M}}$ of sections with holomorphic parameter $s \in \mathbb{C}_s$. Setting

$$\mathcal{W}^s_{\mathfrak{M}} := R^1 a_! \mathcal{W}_{\mathbb{C}_s \times \mathfrak{M}}$$

we get a filtered **k**-algebra similar to the algebra $\mathcal{W}^s_{T^*X}$ of Definition 3.1. Then, if P belongs to $\mathcal{W}_{\mathfrak{M}}$ and has order 0, the section $\dfrac{1}{s-P}$ is well defined in $\mathcal{W}^s_{\mathfrak{M}}$.

6. Applications

As an application, let us construct the exponential of sections of order 0 of $\mathcal{W}_{T^*X}$.

Consider a section P of $\mathcal{W}_{T^*X}(0)$ on an open subset U of T^*X. For each compact subset K of U, there exists $R > 0$ such that the section $s - P$ of $\mathcal{W}^s_{T^*X}$ defined on $\mathbb{C}_s \times U$ is invertible on $(\mathbb{C}_s \setminus D(0, R)) \times K$, where $D(0, R)$ denotes the closed disc centered at 0 with radius R. Therefore $\dfrac{1}{s-P}$ defines an element of $H^1_c(\mathbb{C}_s \times U; \mathcal{W}_{\mathbb{C}_s \times T^*X})$, hence, an element of $\Gamma(U; \mathcal{W}^s_{T^*X})$. We still denote this section of $\mathcal{W}^s_{T^*X}$ on U by $\dfrac{1}{s-P}$.

By developing $\dfrac{1}{s-P}$ as $\sum_{n \geq 0} \dfrac{P^n}{s^{n+1}}$ and applying the Laplace transform, we get formally: $\mathcal{L}(\frac{1}{s-P}) = \exp(t\hbar^{-1}P)$.

Notataion 6.1. We denote by $\exp(t\hbar^{-1}P)$ the image in $\mathcal{W}^t_{T^*X}$ of the section $\dfrac{1}{s-P}$ of $\mathcal{W}^s_{T^*X}$.

Proposition 6.2. *For $P \in \mathcal{W}_{T^*X}(0)$, there is a section $\exp(t\hbar^{-1}P) \in \mathcal{W}^t_{T^*X}$ such that, when X is affine:*

$$\sigma_{\mathrm{tot}}(\exp(t\hbar^{-1}P)) = \sum_{n \geq 0} \frac{(t\hbar^{-1}\sigma_{\mathrm{tot}}(P))^{\star n}}{n!},$$

where the star-product $f^{\star n}$ means the product given by the Leibniz formula (2.5).

Remark 6.3. The Leibniz formula (2.5) is nothing but the standard or normal or Wick star-product and Proposition 6.2 tells us that the star-exponential [1] of P makes sense in $\mathcal{W}^t_{T^*X}$.

In a holomorphic deformation quantization context, the star-exponential of P is heuristically related to the Feynman Path Integral $\mathcal{FPI}(P)$ of P. Indeed, the Feynman Path Integral of a Hamiltonian H is the symbol of the evolution operator associated to H, the precise relation being given (see [2]) by

$$\exp(-xu\hbar^{-1})\mathcal{FPI}(P) = \sigma_{\mathrm{tot}}(\exp(t\hbar^{-1}P)).$$

Example 6.4. As a simple example, take $X = \mathbb{C}$ and $P \in \mathcal{W}_{T^*X}(0)$ with $\sigma_{\mathrm{tot}}(P) = p_0(t, x; u) = \theta x u$, $\theta \in \mathbb{C}$. Up to a change of holomorphic symplectic coordinates, $\sigma_{\mathrm{tot}}(P)$ represents the Hamiltonian of the harmonic oscillator in the holomorphic representation. Clearly P is in $\mathcal{W}_{\mathbb{C}}(0)$, and the total symbol of $\exp(t\hbar^{-1}P)$ is easily computed:

$$\begin{aligned}\frac{\partial}{\partial t}\sigma_{\mathrm{tot}}(\exp(t\hbar^{-1}P)) &= \sigma_{\mathrm{tot}}(\hbar^{-1}P \circ \exp(t\hbar^{-1}P))\\ &= \hbar^{-1}\Big(\sigma_{\mathrm{tot}}(P)\sigma_{\mathrm{tot}}(\exp(t\hbar^{-1}P))\\ &\quad + \hbar\frac{\partial}{\partial u}\sigma_{\mathrm{tot}}(P)\frac{\partial}{\partial x}\sigma_{\mathrm{tot}}(\exp(t\hbar^{-1}P))\Big)\\ &= \hbar^{-1}\theta u x \sigma_{\mathrm{tot}}(\exp(t\hbar^{-1}P)) + \theta x\frac{\partial}{\partial x}\sigma_{\mathrm{tot}}(\exp(t\hbar^{-1}P)).\end{aligned}$$

Since $\sigma_{\mathrm{tot}}(\exp(t\hbar^{-1}P))|_{t=0} = 1$, the solution to the preceding equation is:

$$\sigma_{\mathrm{tot}}(\exp(t\hbar^{-1}P)) = \exp\big((\exp(\theta t) - 1)xu\hbar^{-1}\big).$$

The Feynman Path Integral for the harmonic oscillator is well known in the Physics literature and is given by $\exp\big(\exp(\theta t)xu\hbar^{-1}\big)$ [3].

Acknowledgement. We would like to thank Masaki Kashiwara for extremely useful conversations and helpful insights. The first named author thanks Yoshiaki Maeda for warm hospitality at Keio University where this work was finalized, and the JSPS for financial support.

References

1. Bayen, F., Flato, M., Fronsdal, C., Lichnerowicz, A., Sternheimer, D.: Deformation theory and quantization I,II, Ann. Phys. **111**:61–110, 111–151 (1978)
2. Dito, J.: Star product approach to quantum field theory: The free scalar field. Lett. Math. Phys. **20**, 125–134 (1990)
3. Faddeev, L.D., Slanov, A.A.: *Gauge Fields. Introduction to Quantum Theory*. Reading MA: Benjamin Cummings Publishing, 1980
4. Kashiwara, M.: Quantization of contact manifolds. Publ. RIMS, Kyoto Univ. **32**, 1–5 (1996)
5. Kashiwara, M.: D-modules and Microlocal Calculus, Translations of Mathematical Monographs **217**, Providence, RI: American Math. Soc., 2003
6. Kontsevich, M.: Deformation quantization of algebraic varieties, In: *EuroConférence Moshé Flato, Part III (Dijon, 2000)*. Lett. Math. Phys. **56**, 271–294 (2001)
7. Kontsevich, M.: Deformation quantization of Poisson manifolds. Lett. Math. Phys. **66**, 157–216 (2003)
8. Polesello, P., Schapira, P.: Stacks of quantization-deformation modules over complex symplectic manifolds. Int. Math. Res. Notices **49**, 2637–2664 (2004)
9. Sato, M., Kawai, T., Kashiwara, M.: Microfunctions and pseudo-differential equations. In: Komatsu, H. (ed.), *Hyperfunctions and pseudo-differential equations*. Proceedings, Katata 1971. Lecture Notes in Math. **287**. New-York: Springer-Verlag, 1973, pp. 265–529
10. Schapira, P.: *Microdifferential Systems in the Complex Domain*. Grundlehren der Math. Wiss. **269**. Berlin: Springer-Verlag, 1985
11. Siu, Y.T.: Every Stein subvariety admits a Stein neighborhood. Invent. Math. **38**, 89–100 (1976/77)

Communicated by L. Takhtajan