Algebra and Topology

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[1] **To the students:** the material covered by these Notes goes beyond the contents of the actual course. All along the semester, the students will be informed of what is required for the exam.

Contents

Introduction

This course is a first introduction to Algebraic Topology with emphazise on Homological Algebra and Sheaf Theory. An expanded version of these Notes may be found in [27], [28].

Algebraic Topology is usually approached via the study of homology defined using chain complexes and the fundamental group, whereas, here, the accent is put on the language of categories and sheaves, with particular attention to locally constant sheaves.

Sheaves on topological spaces were invented by Jean Leray as a tool to deduce global properties from local ones. This tool turned out to be extremely powerful, and applies to many areas of Mathematics, from Algebraic Geometry to Quantum Field Theory.

The functor associating to an abelian sheaf F on a topological space X the space $F(X)$ of its global sections is left exact, but not right exact in general. The derived functors $H^j(X; F)$ encode the obstructions to pass from local to global. Given a ring k, the cohomology groups $H^j(X; k_X)$ of the sheaf k_X of k-valued locally constant functions is therefore a topological invariant of the space X. Indeed, it is a homotopy invariant, and we shall explain how to calculate $H^j(X; k_X)$ in various situations.

We also introduce the fundamental groupoid $\Pi_1(X)$ of a topological space (with suitable assumptions on the space) and prove an equivalence of categories between that of finite dimensional representations of this group and that of local systems on X. As a byproduct, we deduce the Van Kampen theorem.

The contents of these Notes is as follows.

In **Chapter 1** we expose the basic language of categories and functors. A key point is the Yoneda lemma, which asserts that a category $\mathcal{C}$ may be embedded in the category $\mathcal{C}^\wedge$ of contravariant functors on $\mathcal{C}$ with values in the category **Set** of sets. This naturally leads to the concept of representable functor. Many examples are treated, in particular in relation with the categories **Set** of sets and $\mathrm{Mod}(A)$ of A-modules, for a (non necessarily commutative) ring A.

Chapter 2 is devoted to projective and inductive limits. We start by studying special cases, namely products and coproducts, then kernels and cokernels. We give the main properties of the limits and pay special attention to inductive limits in the categories **Set** and $\mathrm{Mod}(A)$ for which we introduce the notion of filtrant inductive limits.

In **Chapter 3 and 4** we introduce additive and abelian categories and study the notions of complexes, exact sequences, left or right exact functors, injective or projective resolutions, etc. We give the construction of the derived functors of a left exact functor F of abelian categories and study their properties, with a glance to bifunctors. Finally, we study Koszul complexes and show how they naturally appear in Algebra and Analysis.

In **Chapter 5**, we study abelian sheaves on topological spaces (with a brief look at Grothendieck topologies). We construct the sheaf associated with a presheaf and the usual internal operations ($\mathcal{H}om$ and $\otimes$) and external operations (direct and inverse images). We also explain how to obtain locally constant or locally free sheaves when gluing sheaves.

In **Chapter 6** we prove that the category of abelian sheaves has enough injectives and we define the cohomology of sheaves. We construct resolutions of sheaves using closed Čech coverings and, using the fact that the cohomology of locally constant sheaves is a homotopy invariant, we show how to compute the cohomology of spaces by using cellular decomposition. We apply these techniques to deduce the cohomology of some classical manifolds. We also have a glance to soft sheaves and de Rham cohomology.

Finally, in **Chapter 7**, we define the fundamental groupoid $\Pi_1(X)$ of a locally arcwise connected space X as well as the monodromy of a locally constant sheaf. We prove that, under suitable assumptions, the monodromy functor is an equivalence from the category of locally constant sheaves to that of representations of $\Pi_1(X)$. We then deduce the Van Kampen theorem from the theorem on the gluing of sheaves. Finally, we define the notion of a covering, give some examples related with the action of groups on topological spaces and make the link with locally constant sheaves.

Thanks We thank Stéphame Guillermou for useful remarks and comments on the redaction of Chapter 7.

Some references for Chapters 1, 2, 3: [12], [24], [23], [25], [27], [31].
Some references for Chapters 5 and 6: [14], [4], [12], [21], [22], [28].
Some references for Chapters 7: [13], [26].

Chapter 1

The language of categories

In this chapter we introduce some basic notions of category theory which are of constant use in various fields of Mathematics, without spending too much time on this language. After giving the main definitions on categories and functors, we prove the Yoneda Lemma. We also introduce the notions of representable functors and adjoint functors.

We start by recalling some basic notions on sets and on modules over a (non necessarily commutative) ring.

1.1 Sets and maps

The aim of this section is to fix some notations and to recall some elementary constructions on sets.

If $f\colon X \to Y$ is a map from a set X to a set Y, we shall often say that f is a morphism from X to Y. If f is bijective we shall say that f is an isomorphism and write $f\colon X \xrightarrow{\sim} Y$. If there exists an isomorphism $f\colon X \xrightarrow{\sim} Y$, we say that X and Y are isomorphic and write $X \simeq Y$.

We shall denote by $\mathrm{Hom}_{\mathbf{Set}}(X,Y)$, or simply $\mathrm{Hom}\,(X,Y)$, the set of all maps from X to Y. If $g\colon Y \to Z$ is another map, we can define the composition $g \circ f\colon X \to Z$. Hence, we get two maps:

$$
\begin{aligned}
&g\circ\colon \mathrm{Hom}\,(X,Y) \to \mathrm{Hom}\,(X,Z),\\
&\circ f\colon \mathrm{Hom}\,(Y,Z) \to \mathrm{Hom}\,(X,Z).
\end{aligned}
$$

Notice that if $X = \{x\}$ and $Y = \{y\}$ are two sets with one element each, then there exists a unique isomorphism $X \xrightarrow{\sim} Y$. Of course, if X and Y are finite sets with the same cardinal $\pi > 1$, X and Y are still isomorphic, but the isomorphism is no more unique.

In the sequel we shall denote by $\emptyset$ the empty set and by $\{\mathrm{pt}\}$ a set with one element. Note that for any set X, there is a unique map $\emptyset \to X$ and a unique map $X \to \{\mathrm{pt}\}$.

Let $\{X_i\}_{i\in I}$ be a family of sets indexed by a set I. The product of the X_i's, denoted $\prod_{i\in I} X_i$, or simply $\prod_i X_i$, is the defined as

$$\prod_i X_i = \{\{x_i\}_{i\in I}; x_i \in X_i \text{ for all } i \in I\}. \tag{1.1}$$

If $I = \{1, 2\}$ one uses the notation $X_1 \times X_2$. If $X_i = X$ for all $i \in I$, one uses the notation X^I. Note that

$$\mathrm{Hom}\,(I, X) \simeq X^I. \tag{1.2}$$

For a set Y, there is a natural isomorphism

$$\mathrm{Hom}\,(Y, \prod_i X_i) \simeq \prod_i \mathrm{Hom}\,(Y, X_i). \tag{1.3}$$

For three sets I, X, Y, there is a natural isomorphism

$$\mathrm{Hom}\,(I \times X, Y) \ \simeq\ \mathrm{Hom}\,(I, \mathrm{Hom}\,(X, Y)). \tag{1.4}$$

If $\{X_i\}_{i\in I}$ is a family of sets indexed by a set I, one may also consider their disjoint union, also called their coproduct. The coproduct of the X_i's is denoted $\bigsqcup_{i\in I} X_i$ or simply $\bigsqcup_i X_i$. If $I = \{1, 2\}$ one uses the notation $X_1 \sqcup X_2$. If $X_i = X$ for all $i \in I$, one uses the notation $X^{(I)}$. Note that

$$X \times I \simeq X^{(I)}. \tag{1.5}$$

For a set Y, there is a natural isomorphism

$$\mathrm{Hom}\,(\bigsqcup_i X_i, Y) \simeq \prod_i \mathrm{Hom}\,(X_i, Y). \tag{1.6}$$

Consider two sets X and Y and two maps f, g from X to Y. We write for short $f, g\colon X \rightrightarrows Y$. The kernel (or equalizer) of (f, g), denoted $\mathrm{Ker}(f, g)$, is defined as

$$\mathrm{Ker}(f, g) = \{x \in X; f(x) = g(x)\}. \tag{1.7}$$

Note that for a set Z, one has

$$\mathrm{Hom}\,(Z, \mathrm{Ker}(f, g)) \simeq \mathrm{Ker}(\mathrm{Hom}\,(Z, X) \rightrightarrows \mathrm{Hom}\,(Z, Y)). \tag{1.8}$$

Let us recall a few elementary definitions.

- A relation $\mathcal{R}$ on a set X is a subset of $X \times X$. One writes $x\mathcal{R}y$ if $(x, y) \in \mathcal{R}$.
- The opposite relation $\mathcal{R}^{\mathrm{op}}$ is defined by $x\mathcal{R}^{\mathrm{op}}y$ if and only if $y\mathcal{R}x$.
- A relation $\mathcal{R}$ is reflexive if it contains the diagonal, that is, $x\mathcal{R}x$ for all $x \in X$.
- A relation $\mathcal{R}$ is symmetric if $x\mathcal{R}y$ implies $y\mathcal{R}x$.
- A relation $\mathcal{R}$ is anti-symmetric if $x\mathcal{R}y$ and $y\mathcal{R}x$ implies $x = y$.
- A relation $\mathcal{R}$ is transitive if $x\mathcal{R}y$ and $y\mathcal{R}z$ implies $x\mathcal{R}z$.
- A relation $\mathcal{R}$ is an equivalence relation if it is reflexive, symmetric and transitive.
- A relation $\mathcal{R}$ is a pre-order if it is reflexive and transitive. If moreover it is anti-symmetric, then one says that $\mathcal{R}$ is an order on X. A pre-order is often denoted $\leq$. A set endowed with a pre-order is called a poset.
- Let $(I, \leq)$ be a poset. One says that $(I, \leq)$ is filtrant (one also says "directed") if I is non empty and for any $i, j \in I$ there exists k with $i \leq k$ and $j \leq k$.
- Assume $(I, \leq)$ is a filtrant poset and let $J \subset I$ be a subset. One says that J is cofinal to I if for any $i \in I$ there exists $j \in J$ with $i \leq j$.

If $\mathcal{R}$ is a relation on a set X, there is a smaller equivalence relation which contains $\mathcal{R}$. (Take the intersection of all subsets of $X \times X$ which contain $\mathcal{R}$ and which are equivalence relations.)

Let $\mathcal{R}$ be an equivalence relation on a set X. A subset S of X is saturated if $x \in S$ and $x\mathcal{R}y$ implies $y \in S$. For $x \in X$, denote by $\widehat{x}$ the smallest saturated subset of X containing x. One then defines a new set $X/\mathcal{R}$ and a canonical map $f\colon X \to X/\mathcal{R}$ as follows: the elements of $X/\mathcal{R}$ are the sets $\widehat{x}$ and the map f associates to $x \in X$ the set $\widehat{x}$.

1.2 Modules and linear maps

All along these Notes, a ring A means an associative and unital ring, but A is not necessarily commutative and $\mathbf{k}$ denotes a commutative ring. Recall that a $\mathbf{k}$-algebra A is a ring endowed with a morphism of rings $\varphi\colon \mathbf{k} \to A$ such that the image of $\mathbf{k}$ is contained in the center of A (*i.e.,* $\varphi(x)a = a\varphi(x)$

for any $x \in \mathbf{k}$ and $a \in A$). Notice that a ring A is always a $\mathbb{Z}$-algebra. If A is commutative, then A is an A-algebra.

Since we do not assume A is commutative, we have to distinguish between left and right structures. Unless otherwise specified, a module M over A means a left A-module.

Recall that an A-module M is an additive group (whose operations and zero element are denoted $+, 0$) endowed with an external law $A \times M \to M$ (denoted $(a, m) \mapsto a \cdot m$ or simply $(a, m) \mapsto am$) satisfying:

$$\begin{cases} (ab)m = a(bm) \\ (a+b)m = am + bm \\ a(m+m') = am + am' \\ 1 \cdot m = m \end{cases}$$

where $a, b \in A$ and $m, m' \in M$.

Note that M inherits a structure of a $\mathbf{k}$-module via φ. In the sequel, if there is no risk of confusion, we shall not write φ.

We denote by A^{op} the ring A with the opposite structure. Hence the product ab in A^{op} is the product ba in A and an A^{op}-module is a right A-module.

Note that if the ring A is a field (here, a field is always commutative), then an A-module is nothing but a vector space.

Examples 1.2.1. (i) The first example of a ring is $\mathbb{Z}$, the ring of integers. Since a field is a ring, $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ are rings. If A is a commutative ring, then $A[x_1, \dots, x_n]$, the ring of polynomials in n variables with coefficients in A, is also a commutative ring. It is a sub-ring of $A[[x_1, \dots, x_n]]$, the ring of formal powers series with coefficients in A.
(ii) Let $\mathbf{k}$ be a field. Then for $n > 1$, the ring $M_n(\mathbf{k})$ of square matrices of rank n with entries in $\mathbf{k}$ is non commutative.
(iii) Let $\mathbf{k}$ be a field. The *Weyl algebra* in n variables, denoted $W_n(\mathbf{k})$, is the non commutative ring of polynomials in the variables x_i, ∂_j $(1 \leq i, j \leq n)$ with coefficients in $\mathbf{k}$ and relations :

$$[x_i, x_j] = 0,\ [\partial_i, \partial_j] = 0,\ [\partial_j, x_i] = \delta_j^i$$

where $[p, q] = pq - qp$ and δ_j^i is the Kronecker symbol.

The Weyl algebra $W_n(\mathbf{k})$ may be regarded as the ring of differential operators with coefficients in $\mathbf{k}[x_1, \dots, x_n]$, and $\mathbf{k}[x_1, \dots, x_n]$ becomes a left $W_n(\mathbf{k})$-module: x_i acts by multiplication and ∂_i is the derivation with respect to x_i.

A morphism $f\colon M \to N$ of A-modules is an A-linear map, *i.e.,* f satisfies:

$$\begin{cases} f(m+m') = f(m) + f(m') & m, m' \in M \\ f(am) = af(m) & m \in M, a \in A. \end{cases}$$

A morphism f is an isomorphism if there exists a morphism $g : N \to M$ with $f \circ g = \mathrm{id}_N, g \circ f = \mathrm{id}_M$.

If f is bijective, it is easily checked that the inverse map $f^{-1} : N \to M$ is itself A-linear. Hence f is an isomorphism if and only if f is A-linear and bijective.

A submodule N of M is a subset N of M such that $n, n' \in N$ implies $n + n' \in N$ and $n \in N, a \in A$ implies $an \in N$. A submodule of the A-module A is called an ideal of A. Note that if A is a field, it has no non trivial ideal, *i.e.,* its only ideals are $\{0\}$ and A. If $A = \mathbb{C}[x]$, then $I = \{P \in \mathbb{C}[x]; P(0) = 0\}$ is a non trivial ideal.

If N is a submodule of M, it defines an equivalence relation $m\mathcal{R}m'$ if and only if $m - m' \in N$. One easily checks that the quotient set $M/\mathcal{R}$ is naturally endowed with a structure of a left A-module. This module is called the quotient module and is denoted M/N.

Let $f\colon M \to N$ be a morphism of A-modules. One sets:

$$\begin{aligned} \operatorname{Ker} f &= \{m \in M; \quad f(m) = 0\} \\ \operatorname{Im} f &= \{n \in N; \quad \text{there exists } m \in M, \quad f(m) = n\}. \end{aligned}$$

These are submodules of M and N respectively, called the kernel and the image of f, respectively. One also introduces the cokernel and the coimage of f:

$$\operatorname{Coker} f = N/\operatorname{Im} f, \quad \operatorname{Coim} f = M/\operatorname{Ker} f.$$

Note that the natural morphism $\operatorname{Coim} f \to \operatorname{Im} f$ is an isomorphism.

Example 1.2.2. Let $W_n(\mathbf{k})$ denote as above the Weyl algebra. Consider the left $W_n(\mathbf{k})$-linear map $W_n(\mathbf{k}) \to \mathbf{k}[x_1, \ldots, x_n]$, $W_n(\mathbf{k}) \ni P \mapsto P(1) \in \mathbf{k}[x_1, \ldots, x_n]$. This map is clearly surjective and its kernel is the left ideal generated by $(\partial_1, \cdots, \partial_n)$. Hence, one has the isomorphism of left $W_n(\mathbf{k})$-modules:

$$W_n(\mathbf{k})/\sum_j W_n(\mathbf{k})\partial_j \xrightarrow{\sim} \mathbf{k}[x_1, \ldots, x_n]. \tag{1.9}$$

Products and direct sums

Let I be a set, and let $\{M_i\}_{i\in I}$ be a family of A-modules indexed by I. The set $\prod_i M_i$ is naturally endowed with a structure of a left A-module by setting

$$\begin{aligned}&\{m_i\}_i + \{m'_i\}_i = \{m_i + m'_i\}_i,\\ &a \cdot \{m_i\}_i = \{a \cdot m_i\}_i.\end{aligned}$$

The direct sum $\bigoplus_i M_i$ is the submodule of $\prod_i M_i$ whose elements are the $\{x_i\}_i$'s such that $x_i = 0$ for all but a finite number of $i \in I$. In particular, if the set I is finite, the natural injection $\bigoplus_i M_i \to \prod_i M_i$ is an isomorphism.

Linear maps

Let M and N be two A-modules. Recall that an A-linear map $f\colon M \to N$ is also called a morphism of A-modules. One denotes by $\mathrm{Hom}_A(M,N)$ the set of A-linear maps $f\colon M \to N$. This is clearly a $\mathbf{k}$-module. In fact one defines the action of $\mathbf{k}$ on $\mathrm{Hom}_A(M,N)$ by setting: $(\lambda f)(m) = \lambda(f(m))$. Hence $(\lambda f)(am) = \lambda f(am) = \lambda a f(m) = a\lambda f(m) = a(\lambda f(m))$, and $\lambda f \in \mathrm{Hom}_A(M,N)$.

There is a natural isomorphism $\mathrm{Hom}_A(A,M) \simeq M$: to $u \in \mathrm{Hom}_A(A,M)$ one associates $u(1)$ and to $m \in M$ one associates the linear map $A \to M, a \mapsto am$. More generally, if I is an ideal of A then $\mathrm{Hom}_A(A/I,M) \simeq \{m \in M; Im = 0\}$.

Note that if A is a $\mathbf{k}$-algebra and $L \in \mathrm{Mod}(\mathbf{k})$, $M \in \mathrm{Mod}(A)$, the $\mathbf{k}$-module $\mathrm{Hom}_{\mathbf{k}}(L,M)$ is naturally endowed with a structure of a left A-module. If N is a right A-module, then $\mathrm{Hom}_{\mathbf{k}}(N,L)$ becomes a left A-module.

Tensor product

Consider a right A-module N, a left A-module M and a $\mathbf{k}$-module L. Let us say that a map $f\colon N \times M \to L$ is $(A,\mathbf{k})$-bilinear if f is additive with respect to each of its arguments and satisfies $f(na,m) = f(n,am)$ and $f(n\lambda,m) = \lambda(f(n,m))$ for all $(n,m) \in N \times M$ and $a \in A, \lambda \in \mathbf{k}$.

Let us identify a set I to a subset of $\mathbf{k}^{(I)}$ as follows: to $i \in I$, we associate $\{l_j\}_{j\in I} \in \mathbf{k}^{(I)}$ given by

$$l_j = \begin{cases} 1 \text{ if } & j = i, \\ 0 \text{ if } & j \neq i. \end{cases} \tag{1.10}$$

The tensor product $N \otimes_A M$ is the $\mathbf{k}$-module defined as the quotient of $\mathbf{k}^{(N\times M)}$ by the submodule generated by the following elements (where $n, n' \in$

$N, m, m' \in M, a \in A, \lambda \in \mathbf{k}$ and $N \times M$ is identified to a subset of $\mathbf{k}^{(N\times M)}$):

$$\begin{cases} (n+n', m) - (n,m) - (n', m) \\ (n, m+m') - (n,m) - (n,m') \\ (na, m) - (n, am) \\ \lambda(n,m) - (n\lambda, m). \end{cases}$$

The image of (n,m) in $N \otimes_A M$ is denoted $n \otimes m$. Hence an element of $N \otimes_A M$ may be written (not uniquely!) as a finite sum $\sum_j n_j \otimes m_j$, $n_j \in N$, $m_j \in M$ and:

$$\begin{cases} (n+n') \otimes m = n \otimes m + n' \otimes m \\ n \otimes (m+m') = n \otimes m + n \otimes m' \\ na \otimes m = n \otimes am \\ \lambda(n \otimes m) = n\lambda \otimes m = n \otimes \lambda m. \end{cases}$$

Denote by $\beta\colon N \times M \to N \otimes_A M$ the natural map which associates $n \otimes m$ to (n,m).

Proposition 1.2.3. *The map β is $(A, \mathbf{k})$-bilinear and for any $\mathbf{k}$-module L and any $(A,\mathbf{k})$-bilinear map $f\colon N \times M \to L$, the map f factorizes uniquely through a $\mathbf{k}$-linear map $\varphi\colon N \otimes_A M \to L$.*

The proof is left to the reader.

Proposition 1.2.3 is visualized by the diagram:

$$\begin{tikzcd} N \times M \arrow[r, "\beta"] \arrow[dr, "f"'] & N \otimes_A M \arrow[d, dotted, "\varphi"] \\ & L. \end{tikzcd}$$

Consider an A-linear map $f\colon M \to L$. It defines a linear map $\mathrm{id}_N \times f\colon N \times M \to N \times L$, hence a $(A, \mathbf{k})$-bilinear map $N \times M \to N \otimes_A L$, and finally a $\mathbf{k}$-linear map

$$\mathrm{id}_N \otimes f\colon N \otimes_A M \to N \otimes_A L.$$

One constructs similarly $g \otimes \mathrm{id}_M$ associated to $g\colon N \to L$.

There is are natural isomorphisms $A \otimes_A M \simeq M$ and $N \otimes_A A \simeq N$.

Denote by $\mathrm{Bil(N \times M, L)}$ the $\mathbf{k}$-module of $(A, \mathbf{k})$-bilinear maps from $N \times M$ to L. One has the isomorphisms

$$\begin{aligned} \mathrm{Bil(N \times M, L)} &\simeq \mathrm{Hom}_{\mathbf{k}}(N \otimes_A M, L) \\ &\simeq \mathrm{Hom}_A(M, \mathrm{Hom}_{\mathbf{k}}(N, L)) \\ &\simeq \mathrm{Hom}_A(N, \mathrm{Hom}_{\mathbf{k}}(M, L)). \end{aligned} \tag{1.11}$$

For $L \in \mathrm{Mod}(\mathbf{k})$ and $M \in \mathrm{Mod}(A)$, the $\mathbf{k}$-module $L \otimes_{\mathbf{k}} M$ is naturally endowed with a structure of a left A-module. For $M, N \in \mathrm{Mod}(A)$ and $L \in \mathrm{Mod}(\mathbf{k})$, we have the isomorphisms (whose verification is left to the reader):

$$\begin{aligned} \mathrm{Hom}_A(L \otimes_{\mathbf{k}} N, M) &\simeq \mathrm{Hom}_A(N, \mathrm{Hom}_{\mathbf{k}}(L, M)) \\ &\simeq \mathrm{Hom}_{\mathbf{k}}(L, \mathrm{Hom}_A(N, M)). \end{aligned} \tag{1.12}$$

If A is commutative, there is an isomorphism: $N \otimes_A M \simeq M \otimes_A N$ given by $n \otimes m \mapsto m \otimes n$. Moreover, the tensor product is associative, that is, if L, M, N are A-modules, there are natural isomorphisms $L \otimes_A (M \otimes_A N) \simeq (L \otimes_A M) \otimes_A N$. One simply writes $L \otimes_A M \otimes_A N$.

1.3 Categories and functors

Definition 1.3.1. A category $\mathcal{C}$ consists of:

(i) a set $\mathrm{Ob}(\mathcal{C})$ whose elements are called the objects of $\mathcal{C}$,

(ii) for each $X, Y \in \mathrm{Ob}(\mathcal{C})$, a set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ whose elements are called the morphisms from X to Y,

(iii) for any $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$, a map, called the composition, $\mathrm{Hom}_{\mathcal{C}}(X, Y) \times \mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$, and denoted $(f, g) \mapsto g \circ f$,

these data satisfying:

(a) $\circ$ is associative,

(b) for each $X \in \mathrm{Ob}(\mathcal{C})$, there exists $\mathrm{id}_X \in \mathrm{Hom}\,(X, X)$ such that for all $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(Y, X)$, $f \circ \mathrm{id}_X = f$, $\mathrm{id}_X \circ g = g$.

Note that $\mathrm{id}_X \in \mathrm{Hom}\,(X, X)$ is characterized by the condition in (b).

Remark 1.3.2. There are some set-theoretical dangers, illustrated in Remark 2.5.12, and one should mention in which "universe" we are working.

We do not give in these Notes the definition of a universe, only recalling that a universe $\mathcal{U}$ is a set (a very big one) stable by many operations and containing $\mathbb{N}$. A set E is $\mathcal{U}$-small if it is isomorphic to a set which belongs to $\mathcal{U}$. Then, given a universe $\mathcal{U}$, a $\mathcal{U}$-category $\mathcal{C}$ is a category such that for any $X, Y \in \mathcal{C}$, the set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is $\mathcal{U}$-small. The category $\mathcal{C}$ is itself $\mathcal{U}$-small if moreover the set $\mathrm{Ob}(\mathcal{C})$ is $\mathcal{U}$-small.

The crucial point is Grothendieck's axiom which says that any set belong to some universe.

Notation 1.3.3. One often writes $X \in \mathcal{C}$ instead of $X \in \mathrm{Ob}(\mathcal{C})$ and $f\colon X \to Y$ (or else $f\colon Y \leftarrow X$) instead of $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$. One calls X the source and Y the target of f.

A morphism $f\colon X \to Y$ is an *isomorphism* if there exists $g\colon X \leftarrow Y$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$. In such a case, one writes $f\colon X \xrightarrow{\sim} Y$ or simply $X \simeq Y$. Of course g is unique, and one also denotes it by f^{-1}.

A morphism $f\colon X \to Y$ is a *monomorphism* (resp. an *epimorphism*) if for any morphisms g_1 and g_2, $f \circ g_1 = f \circ g_2$ (resp. $g_1 \circ f = g_2 \circ f$) implies $g_1 = g_2$. One sometimes writes $f\colon X \rightarrowtail Y$ or else $X \hookrightarrow Y$ (resp. $f\colon X \twoheadrightarrow Y$) to denote a monomorphism (resp. an epimorphism).

Two morphisms f and g are parallel if they have the same sources and targets, visualized by $f, g\colon X \rightrightarrows Y$.

One introduces the *opposite category* $\mathcal{C}^{\mathrm{op}}$:

$$\mathrm{Ob}(\mathcal{C}^{\mathrm{op}}) = \mathrm{Ob}(\mathcal{C}), \quad \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(Y,X),$$

the identity morphisms and the composition of morphisms being the obvious ones.

A category $\mathcal{C}'$ is a *subcategory* of $\mathcal{C}$, denoted $\mathcal{C}' \subset \mathcal{C}$, if: $\mathrm{Ob}(\mathcal{C}') \subset \mathrm{Ob}(\mathcal{C})$, $\mathrm{Hom}_{\mathcal{C}'}(X,Y) \subset \mathrm{Hom}_{\mathcal{C}}(X,Y)$ for any $X, Y \in \mathcal{C}'$, the composition $\circ$ in $\mathcal{C}'$ is induced by the composition in $\mathcal{C}$ and the identity morphisms in $\mathcal{C}'$ are induced by those in $\mathcal{C}$. One says that $\mathcal{C}'$ is a *full* subcategory if for all $X, Y \in \mathcal{C}'$, $\mathrm{Hom}_{\mathcal{C}'}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(X,Y)$.

A category is *discrete* if the only morphisms are the identity morphisms. Note that a set is naturally identified with a discrete category.

A category $\mathcal{C}$ is *finite* if the family of all morphisms in $\mathcal{C}$ (hence, in particular, the family of objects) is a finite set.

A category $\mathcal{C}$ is a *groupoid* if all morphisms are isomorphisms.

Examples 1.3.4. (i) **Set** is the category of sets and maps (in a given universe), $\mathbf{Set}^f$ is the full subcategory consisting of finite sets.
(ii) **Rel** is defined by: $\mathrm{Ob}(\mathbf{Rel}) = \mathrm{Ob}(\mathbf{Set})$ and $\mathrm{Hom}_{\mathbf{Rel}}(X,Y) = \mathcal{P}(X \times Y)$, the set of subsets of $X \times Y$. The composition law is defined as follows. For $f\colon X \to Y$ and $g\colon Y \to Z$, $g \circ f$ is the set

$$\{(x,z) \in X \times Z;\ \text{there exists } y \in Y \text{ with } (x,y) \in f, (y,z) \in g\}.$$

Of course, $\mathrm{id}_X = \Delta \subset X \times X$, the diagonal of $X \times X$.
(iii) Let A be a ring. The category of left A-modules and A-linear maps is denoted $\mathrm{Mod}(A)$. In particular $\mathrm{Mod}(\mathbb{Z})$ is the category of abelian groups.

We shall use the notation $\mathrm{Hom}_A(\,\bullet\,,\,\bullet\,)$ instead of $\mathrm{Hom}_{\mathrm{Mod}(A)}(\,\bullet\,,\,\bullet\,)$.

One denotes by $\mathrm{Mod}^{\mathrm{f}}(A)$ the full subcategory of $\mathrm{Mod}(A)$ consisting of finitely generated A-modules.
(iv) One associates to a pre-ordered set $(I, \leq)$ a category, still denoted by I for short, as follows. $\mathrm{Ob}(I) = I$, and the set of morphisms from i to j has a single element if $i \leq j$, and is empty otherwise. Note that I^{op} is the category associated with I endowed with the opposite order.
(v) We denote by **Top** the category of topological spaces and continuous maps.
(vi) We shall often represent by the diagram $\bullet \to \bullet$ the category which consists of two objects, say $\{a, b\}$, and one morphism $a \to b$ other than id_a and id_b. We denote this category by **Arr**.
(vii) We represent by $\bullet \rightrightarrows \bullet$ the category with two objects, say $\{a, b\}$, and two parallel morphisms $a \rightrightarrows b$ other than id_a and id_b.
(viii) Let G be a group. We may attach to it the groupoid $\mathcal{G}$ with one object, say $\{a\}$ and morphisms $\mathrm{Hom}_{\mathcal{G}}(a, a) = G$.
(ix) Let X be a topological space locally arcwise connected. We attach to it a category $\widetilde{X}$ as follows: $\mathrm{Ob}(\widetilde{X}) = X$ and for $x, y \in X$, a morphism $f \colon x \to y$ is a path form x to y.

Definition 1.3.5. (i) An object $P \in \mathcal{C}$ is called initial if for all $X \in \mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(P, X) \simeq \{\mathrm{pt}\}$. One often denotes by $\emptyset_{\mathcal{C}}$ an initial object in $\mathcal{C}$.

(ii) One says that P is terminal if P is initial in $\mathcal{C}^{\mathrm{op}}$, *i.e.,* for all $X \in \mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(X, P) \simeq \{\mathrm{pt}\}$. One often denotes by $\mathrm{pt}_{\mathcal{C}}$ a terminal object in $\mathcal{C}$.

(iii) One says that P is a zero-object if it is both initial and terminal. In such a case, one often denotes it by 0. If $\mathcal{C}$ has a zero object, for any objects $X, Y \in \mathcal{C}$, the morphism obtained as the composition $X \to 0 \to Y$ is still denoted by $0 \colon X \to Y$.

Note that initial (resp. terminal) objects are unique up to unique isomorphisms.

Examples 1.3.6. (i) In the category **Set**, $\emptyset$ is initial and $\{\mathrm{pt}\}$ is terminal.
(ii) The zero module 0 is a zero-object in $\mathrm{Mod}(A)$.
(iii) The category associated with the ordered set $(\mathbb{Z}, \leq)$ has neither initial nor terminal object.

Definition 1.3.7. Let $\mathcal{C}$ and $\mathcal{C}'$ be two categories. A functor $F \colon \mathcal{C} \to \mathcal{C}'$ consists of a map $F \colon \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{C}')$ and for all $X, Y \in \mathcal{C}$, of a map still denoted by F: $\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}'}(F(X), F(Y))$ such that

$$F(\mathrm{id}_X) = \mathrm{id}_{F(X)}, \quad F(f \circ g) = F(f) \circ F(g).$$

A contravariant functor from $\mathcal{C}$ to $\mathcal{C}'$ is a functor from $\mathcal{C}^{\mathrm{op}}$ to $\mathcal{C}'$. In other words, it satisfies $F(g \circ f) = F(f) \circ F(g)$. If one wishes to put the emphasis on the fact that a functor is not contravariant, one says it is covariant.

One denotes by $\mathrm{op} : \mathcal{C} \to \mathcal{C}^{\mathrm{op}}$ the contravariant functor, associated with $\mathrm{id}_{\mathcal{C}^{\mathrm{op}}}$.

Example 1.3.8. Let $\mathcal{C}$ be a category and let $X \in \mathcal{C}$.
(i) $\mathrm{Hom}_{\mathcal{C}}(X, \bullet)$ is a functor from $\mathcal{C}$ to **Set**. To $Y \in \mathcal{C}$, it associates the set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ and to a morphism $f\colon Y \to Z$ in $\mathcal{C}$, it associates the map

$$\mathrm{Hom}_{\mathcal{C}}(X, f)\colon \mathrm{Hom}_{\mathcal{C}}(X, Y) \xrightarrow{f\circ} \mathrm{Hom}_{\mathcal{C}}(X, Z).$$

(ii) $\mathrm{Hom}_{\mathcal{C}}(\bullet, \bullet)$ is a functor from $\mathcal{C}^{\mathrm{op}}$ to **Set**. To $Y \in \mathcal{C}$, it associates the set $\mathrm{Hom}_{\mathcal{C}}(Y, X)$ and to a morphism $f\colon Y \to Z$ in $\mathcal{C}$, it associates the map

$$\mathrm{Hom}_{\mathcal{C}}(f, X)\colon \mathrm{Hom}_{\mathcal{C}}(Z, X) \xrightarrow{\circ f} \mathrm{Hom}_{\mathcal{C}}(Y, X).$$

Example 1.3.9. Let A be a $\mathbf{k}$-algebra and let $M \in \mathrm{Mod}(A)$ Similarly as in Example 1.3.8, we have the functors

$$\mathrm{Hom}_A(M, \bullet)\colon \mathrm{Mod}(A) \to \mathrm{Mod}(\mathbf{k}),$$
$$\mathrm{Hom}_A(\bullet, M)\colon \mathrm{Mod}(A)^{\mathrm{op}} \to \mathrm{Mod}(\mathbf{k})$$

Clearly, the functor $\mathrm{Hom}_A(M, \bullet)$ commutes with products in $\mathrm{Mod}(A)$, that is,

$$\mathrm{Hom}_A(M, \prod_i N_i) \simeq \prod_i \mathrm{Hom}_A(M, N_i)$$

and the functor $\mathrm{Hom}_A(\bullet, N)$ commutes with direct sums in $\mathrm{Mod}(A)$, that is,

$$\mathrm{Hom}_A(\bigoplus_i M_i, N) \simeq \prod_i \mathrm{Hom}_A(M_i, N)$$

(ii) Let N be a right A-module. Then $N \otimes_A \bullet : \mathrm{Mod}(A) \to \mathrm{Mod}(\mathbf{k})$ is a functor. Clearly, the functor $N \otimes_A \bullet$ commutes with direct sums, that is,

$$N \otimes_A (\bigoplus_i M_i) \simeq \bigoplus_i (N \otimes_A M_i),$$

and similarly for the functor $\bullet \otimes_A M$.

Definition 1.3.10. Let $F\colon \mathcal{C} \to \mathcal{C}'$ be a functor.

(i) One says that F is faithful (resp. full, resp. fully faithful) if for $X, Y \in \mathcal{C}$ $\mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{C}'}(F(X),F(Y))$ is injective (resp. surjective, resp. bijective).

(ii) One says that F is essentially surjective if for each $Y \in \mathcal{C}'$ there exists $X \in \mathcal{C}$ and an isomorphism $F(X) \simeq Y$.

(iii) One says that F is conservative if any morphism $f\colon X \to Y$ in $\mathcal{C}$ is an isomorphism as soon as $F(f)$ is an isomorphism.

Examples 1.3.11. (i) The forgetful functor $\mathit{for}\colon \mathrm{Mod}(A) \to \mathbf{Set}$ associates to an A-module M the set M, and to a linear map f the map f. The functor for is faithful and conservative but not fully faithful.
(ii) The forgetful functor $\mathit{for}\colon \mathbf{Top} \to \mathbf{Set}$ (defined similarly as in (i)) is faithful. It is neither fully faithful nor conservative.
(iii) The forgetful functor $\mathit{for}\colon \mathbf{Set} \to \mathbf{Rel}$ is faithful and conservative.

One defines the product of two categories $\mathcal{C}$ and $\mathcal{C}'$ by :

$$\begin{aligned} \mathrm{Ob}(\mathcal{C} \times \mathcal{C}') &= \mathrm{Ob}(\mathcal{C}) \times \mathrm{Ob}(\mathcal{C}') \\ \mathrm{Hom}_{\mathcal{C}\times\mathcal{C}'}((X,X'),(Y,Y')) &= \mathrm{Hom}_{\mathcal{C}}(X,Y) \times \mathrm{Hom}_{\mathcal{C}'}(X',Y'). \end{aligned}$$

A bifunctor $F\colon \mathcal{C} \times \mathcal{C}' \to \mathcal{C}''$ is a functor on the product category. This means that for $X \in \mathcal{C}$ and $X' \in \mathcal{C}'$, $F(X, \bullet)\colon \mathcal{C}' \to \mathcal{C}''$ and $F(\bullet, X') : \mathcal{C} \to \mathcal{C}''$ are functors, and moreover for any morphisms $f : X \to Y$ in $\mathcal{C}$, $g : X' \to Y'$ in $\mathcal{C}'$, the diagram below commutes:

$$\begin{tikzcd} F(X,X') \arrow[r, "F(X,g)"] \arrow[d, "F(f,X')"'] & F(X,Y') \arrow[d, "F(f,Y')"] \\ F(Y,X') \arrow[r, "F(Y,g)"] & F(Y,Y') \end{tikzcd}$$

In fact, $(f,g) = (\mathrm{id}_Y, g) \circ (f, \mathrm{id}_{X'}) = (f, \mathrm{id}_{Y'}) \circ (\mathrm{id}_X, g)$.

Examples 1.3.12. (i) $\mathrm{Hom}_{\mathcal{C}}(\bullet, \bullet) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$ is a bifunctor.
(ii) If A is a $\mathbf{k}$-algebra, $\mathrm{Hom}_A(\bullet, \bullet)\colon \mathrm{Mod}(A)^{\mathrm{op}} \times \mathrm{Mod}(A) \to \mathrm{Mod}(\mathbf{k})$ and $\bullet \otimes_A \bullet\colon \mathrm{Mod}(A^{\mathrm{op}}) \times \mathrm{Mod}(A) \to \mathrm{Mod}(\mathbf{k})$ are bifunctors.

Definition 1.3.13. Let F_1, F_2 are two functors from $\mathcal{C}$ to $\mathcal{C}'$. A morphism of functors $\theta\colon F_1 \to F_2$ is the data for all $X \in \mathcal{C}$ of a morphism $\theta(X) : F_1(X) \to$

$F_2(X)$ such that for all $f: X \to Y$, the diagram below commutes:

$$\begin{tikzcd} F_1(X) \arrow[r, "\theta(X)"] \arrow[d, "F_1(f)"'] & F_2(X) \arrow[d, "F_2(f)"] \\ F_1(Y) \arrow[r, "\theta(Y)"] & F_2(Y) \end{tikzcd}$$

A morphism of functors is visualized by a diagram:

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F_1"] \arrow[r, bend right, "F_2"'] \arrow[r, phantom, "\Downarrow\theta"] & \mathcal{C}' \end{tikzcd}$$

Hence, by considering the family of functors from $\mathcal{C}$ to $\mathcal{C}'$ and the morphisms of such functors, we get a new category.

Notation 1.3.14. (i) We denote by $\mathrm{Fct}(\mathcal{C}, \mathcal{C}')$ the category of functors from $\mathcal{C}$ to $\mathcal{C}'$. One may also use the shorter notation $(\mathcal{C}')^{\mathcal{C}}$.

Examples 1.3.15. Let $\mathbf{k}$ be a field and consider the functor

$$\begin{aligned} {}^*\colon \mathrm{Mod}(\mathbf{k})^{\mathrm{op}} &\to \mathrm{Mod}(\mathbf{k}), \\ V &\mapsto V^* = \mathrm{Hom}_{\mathbf{k}}(V, \mathbf{k}). \end{aligned}$$

Then there is a morphism of functors $\mathrm{id} \to {}^* \circ {}^*$ in $\mathrm{Fct}(\mathrm{Mod}(\mathbf{k}), \mathrm{Mod}(\mathbf{k}))$.
(ii) We shall encounter morphisms of functors when considering pairs of adjoint functors (see (1.17)).

In particular we have the notion of an isomorphism of categories. A functor $F: \mathcal{C} \to \mathcal{C}'$ is an isomorphism of categories if there exists $G : \mathcal{C}' \to \mathcal{C}$ such that: $G \circ F = \mathrm{id}_{\mathcal{C}}$ and $F \circ G = \mathrm{id}_{\mathcal{C}'}$. In particular, for all $X \in \mathcal{C}$, $G \circ F(X) = X$. In practice, such a situation rarely occurs and is not really interesting. There is a weaker notion that we introduce below.

Definition 1.3.16. A functor $F: \mathcal{C} \to \mathcal{C}'$ is an equivalence of categories if there exists $G: \mathcal{C}' \to \mathcal{C}$ such that: $G \circ F$ is isomorphic to $\mathrm{id}_{\mathcal{C}}$ and $F \circ G$ is isomorphic to $\mathrm{id}_{\mathcal{C}'}$.

We shall not give the proof of the following important result below.

Theorem 1.3.17. *The functor $F: \mathcal{C} \to \mathcal{C}'$ is an equivalence of categories if and only if F is fully faithful and essentially surjective.*

If two categories are equivalent, all results and concepts in one of them have their counterparts in the other one. This is why this notion of equivalence of categories plays an important role in Mathematics.

Examples 1.3.18. (i) Let $\mathbf{k}$ be a field and let $\mathcal{C}$ denote the category defined by $\mathrm{Ob}(\mathcal{C}) = \mathbb{N}$ and $\mathrm{Hom}_{\mathcal{C}}(n,m) = M_{m,n}(\mathbf{k})$, the space of matrices of type (m,n) with entries in a field $\mathbf{k}$ (the composition being the usual composition of matrices). Define the functor $F\colon \mathcal{C} \to \mathrm{Mod}^f(\mathbf{k})$ as follows. To $n \in \mathbb{N}$, $F(n)$ associates $\mathbf{k}^n \in \mathrm{Mod}^f(\mathbf{k})$ and to a matrix of type (m,n), F associates the induced linear map from $\mathbf{k}^n$ to $\mathbf{k}^m$. Clearly F is fully faithful, and since any finite dimensional vector space admits a basis, it is isomorphic to $\mathbf{k}^n$ for some n, hence F is essentially surjective. In conclusion, F is an equivalence of categories.
(ii) let $\mathcal{C}$ and $\mathcal{C}'$ be two categories. There is an equivalence

$$\mathrm{Fct}(\mathcal{C},\mathcal{C}')^{\mathrm{op}} \simeq \mathrm{Fct}(\mathcal{C}^{\mathrm{op}},(\mathcal{C}')^{\mathrm{op}}). \tag{1.13}$$

(iii) Let I, J and $\mathcal{C}$ be categories. There are equivalences

$$\mathrm{Fct}(I \times J,\mathcal{C}) \simeq \mathrm{Fct}(J,\mathrm{Fct}(I,\mathcal{C})) \simeq \mathrm{Fct}(I,\mathrm{Fct}(J,\mathcal{C})). \tag{1.14}$$

1.4 The Yoneda Lemma

Definition 1.4.1. Let $\mathcal{C}$ be a category. One defines the categories

$$\mathcal{C}^\wedge = \mathrm{Fct}(\mathcal{C}^{\mathrm{op}},\mathbf{Set}), \qquad \mathcal{C}^\vee = \mathrm{Fct}(\mathcal{C}^{\mathrm{op}},\mathbf{Set}^{\mathrm{op}}),$$

and the functors

$$\begin{aligned} \mathrm{h}_{\mathcal{C}} &: \mathcal{C} \to \mathcal{C}^\wedge, \quad X \mapsto \mathrm{Hom}_{\mathcal{C}}(\,\bullet\,,X) \\ \mathrm{k}_{\mathcal{C}} &: \mathcal{C} \to \mathcal{C}^\vee, \quad X \mapsto \mathrm{Hom}_{\mathcal{C}}(X,\,\bullet\,). \end{aligned}$$

Since there is a natural equivalence of categories

$$\mathcal{C}^\vee \simeq \mathcal{C}^{\mathrm{op},\wedge,\mathrm{op}}, \tag{1.15}$$

we shall concentrate our study on $\mathcal{C}^\wedge$.

Proposition 1.4.2. (The Yoneda lemma.) *For $A \in \mathcal{C}^\wedge$ and $X \in \mathcal{C}$, there is an isomorphism $\mathrm{Hom}_{\mathcal{C}^\wedge}(\mathrm{h}_{\mathcal{C}}(X),A) \simeq A(X)$, functorial with respect to X and A.*

Proof. One constructs the morphism $\varphi\colon \mathrm{Hom}_{\mathcal{C}^\wedge}(\mathrm{h}_{\mathcal{C}}(X),A) \to A(X)$ by the chain of morphisms: $\mathrm{Hom}_{\mathcal{C}^\wedge}(\mathrm{h}_{\mathcal{C}}(X),A) \to \mathrm{Hom}_{\mathbf{Set}}(\mathrm{Hom}_{\mathcal{C}}(X,X),A(X)) \to A(X)$, where the last map is associated with id_X.

To construct $\psi\colon A(X) \to \mathrm{Hom}_{\mathcal{C}^\wedge}(\mathrm{h}_{\mathcal{C}}(X),A)$, it is enough to associate with $s \in A(X)$ and $Y \in \mathcal{C}$ a map from $\mathrm{Hom}_{\mathcal{C}}(Y,X)$ to $A(Y)$. It is defined by the chain of maps $\mathrm{Hom}_{\mathcal{C}}(Y,X) \to \mathrm{Hom}_{\mathbf{Set}}(A(X),A(Y)) \to A(Y)$ where the last map is associated with $s \in A(X)$.

One checks that φ and ψ are inverse to each other. q.e.d.

Corollary 1.4.3. *The functor* $\mathrm{h}_{\mathcal{C}}$ *is fully faithful.*

Proof. For X and Y in $\mathcal{C}$, one has $\mathrm{Hom}_{\mathcal{C}^\wedge}(\mathrm{h}_{\mathcal{C}}(X), \mathrm{h}_{\mathcal{C}}(Y)) \simeq \mathrm{h}_{\mathcal{C}}(Y)(X) = \mathrm{Hom}_{\mathcal{C}}(X, Y)$. q.e.d.

One calls $\mathrm{h}_{\mathcal{C}}$ the Yoneda embedding.

Hence, one may consider $\mathcal{C}$ as a full subcategory of $\mathcal{C}^\wedge$. In particular, for $X \in \mathcal{C}$, $\mathrm{h}_{\mathcal{C}}(X)$ determines X up to unique isomorphism, that is, an isomorphism $\mathrm{h}_{\mathcal{C}}(X) \simeq \mathrm{h}_{\mathcal{C}}(Y)$ determines a unique isomorphism $X \simeq Y$.

Corollary 1.4.4. *Let* $\mathcal{C}$ *be a category and let* $f\colon X \to Y$ *be a morphism in* $\mathcal{C}$.

(i) *Assume that for any* $Z \in \mathcal{C}$, *the map* $\mathrm{Hom}_{\mathcal{C}}(Z, X) \xrightarrow{f\circ} \mathrm{Hom}_{\mathcal{C}}(Z, Y)$ *is bijective. Then* f *is an isomorphism.*

(ii) *Assume that for any* $Z \in \mathcal{C}$, *the map* $\mathrm{Hom}_{\mathcal{C}}(Y, Z) \xrightarrow{\circ f} \mathrm{Hom}_{\mathcal{C}}(X, Z)$ *is bijective. Then* f *is an isomorphism.*

Proof. (i) By the hypothesis, $\mathrm{h}_{\mathcal{C}}(f) : \mathrm{h}_{\mathcal{C}}(X) \to \mathrm{h}_{\mathcal{C}}(Y)$ is an isomorphism in $\mathcal{C}^\wedge$. Since $\mathrm{h}_{\mathcal{C}}$ is fully faithful, this implies that f is an isomorphism. (See Exercise 1.2 (ii).)
(ii) follows by replacing $\mathcal{C}$ with $\mathcal{C}^{\mathrm{op}}$. q.e.d.

1.5 Representable functors, adjoint functors

Representable functors

Definition 1.5.1. (i) One says that a functor F from $\mathcal{C}^{\mathrm{op}}$ to **Set** is representable if there exists $X \in \mathcal{C}$ such that $F(Y) \simeq \mathrm{Hom}_{\mathcal{C}}(Y, X)$ functorially in $Y \in \mathcal{C}$. In other words, $F \simeq \mathrm{h}_{\mathcal{C}}(X)$ in $\mathcal{C}^\wedge$. Such an object X is called a representative of F.

(ii) Similarly, a functor $G\colon \mathcal{C} \to$ **Set** is representable if there exists $X \in \mathcal{C}$ such that $G(Y) \simeq \mathrm{Hom}_{\mathcal{C}}(X, Y)$ functorially in $Y \in \mathcal{C}$.

It is important to notice that the isomorphisms above determine X up to unique isomorphism. More precisely, given two isomorphisms $F \xrightarrow{\sim} \mathrm{h}_{\mathcal{C}}(X)$ and $F \xrightarrow{\sim} \mathrm{h}_{\mathcal{C}}(X')$ there exists a unique isomorphism $\theta\colon X \xrightarrow{\sim} X'$ making the diagram below commutative:

$$\begin{tikzcd} & F \arrow[dl, "\sim"'] \arrow[dr, "\sim"] & \\ \mathrm{h}_{\mathcal{C}}(X) \arrow[rr, "\mathrm{h}_{\mathcal{C}}(\theta)", "\sim"'] & & \mathrm{h}_{\mathcal{C}}(X'). \end{tikzcd}$$

Representable functors provides a categorical language to deal with universal problems. Let us illustrate this by an example.

Example 1.5.2. Let A be a $\mathbf{k}$-algebra. Let$N$ be a right A-module, M a left A-module and L a $\mathbf{k}$-module. Denote by $B(N \times M, L)$ the set of $A, \mathbf{k}$-bilinear maps from $N \times M$ to L. Then the functor $F\colon L \mapsto B(N \times M, L)$ is representable by $N \otimes_A M$ by (1.11).

Adjoint functors

Definition 1.5.3. Let $F\colon \mathcal{C} \to \mathcal{C}'$ and $G\colon \mathcal{C}' \to \mathcal{C}$ be two functors. One says that (F, G) is a pair of adjoint functors or that F is a left adjoint to G, or that G is a right adjoint to F if there exists an isomorphism of bifunctors:

$$\operatorname{Hom}_{\mathcal{C}'}(F(\,\bullet\,), \,\bullet\,) \simeq \operatorname{Hom}_{\mathcal{C}}(\,\bullet\,, G(\,\bullet\,)) \tag{1.16}$$

If G is an adjoint to F, then G is unique up to isomorphism. In fact, $G(Y)$ is a representative of the functor $X \mapsto \operatorname{Hom}_{\mathcal{C}}(F(X), Y)$.

The isomorphism (1.16) gives the isomorphisms

$$\begin{aligned}\operatorname{Hom}_{\mathcal{C}'}(F \circ G(\,\bullet\,), \,\bullet\,) &\simeq \operatorname{Hom}_{\mathcal{C}}(G(\,\bullet\,), G(\,\bullet\,)),\\ \operatorname{Hom}_{\mathcal{C}'}(F(\,\bullet\,), F(\,\bullet\,)) &\simeq \operatorname{Hom}_{\mathcal{C}}(\,\bullet\,, G \circ F(\,\bullet\,)).\end{aligned}$$

In particular, we have morphisms $X \to G \circ F(X)$, functorial in $X \in \mathcal{C}$, and morphisms $F \circ G(Y) \to Y$, functorial in $Y \in \mathcal{C}'$. In other words, we have morphisms of functors

$$F \circ G \to \mathrm{id}_{\mathcal{C}'}, \qquad \mathrm{id}_{\mathcal{C}} \to G \circ F. \tag{1.17}$$

Examples 1.5.4. (i) Let $X \in \mathbf{Set}$. Using the bijection (1.4), we get that the functor $\operatorname{Hom}_{\mathbf{Set}}(X, \,\bullet\,)\colon \mathbf{Set} \to \mathbf{Set}$ is right adjoint to the functor $\,\bullet\, \times X$.
(ii) Let A be a $\mathbf{k}$-algebra and let $L \in \mathrm{Mod}(\mathbf{k})$. Using the first isomorphism in (1.12), we get that the functor $\operatorname{Hom}_{\mathbf{k}}(L, \,\bullet\,)\colon \mathrm{Mod}(A)$ to $\mathrm{Mod}(A)$ is right adjoint to the functor $\,\bullet\, \otimes_{\mathbf{k}} L$.
(iii) Let A be a $\mathbf{k}$-algebra. Using the isomorphisms in (1.12) with $N = A$, we get that the functor *for*$\colon \mathrm{Mod}(A) \to \mathrm{Mod}(\mathbf{k})$ which, to an A-module associates the underlying $\mathbf{k}$-module, is right adjoint to the functor $A \otimes_{\mathbf{k}} \,\bullet\,\colon \mathrm{Mod}(\mathbf{k}) \to \mathrm{Mod}(A)$ (extension of scalars).

Exercises to Chapter 1

Exercise 1.1. Prove that the categories $\mathbf{Set}$ and $\mathbf{Set}^{\mathrm{op}}$ are not equivalent and similarly with the categories $\mathbf{Set}^f$ and $(\mathbf{Set}^f)^{\mathrm{op}}$.

(Hint: if $F : \mathbf{Set} \to \mathbf{Set}^{\mathrm{op}}$ were such an equivalence, then $F(\emptyset) \simeq \{\mathrm{pt}\}$ and $F(\{\mathrm{pt}\}) \simeq \emptyset$. Now compare $\mathrm{Hom}_{\mathbf{Set}}(\{\mathrm{pt}\}, X)$ and $\mathrm{Hom}_{\mathbf{Set}^{\mathrm{op}}}(F(\{\mathrm{pt}\}), F(X))$ when X is a set with two elements.)

Exercise 1.2. (i) Let $F\colon \mathcal{C} \to \mathcal{C}'$ be a faithful functor and let f be a morphism in $\mathcal{C}$. Prove that if $F(f)$ is a monomorphism (resp. an epimorphism), then f is a monomorphism (resp. an epimorphism).
(ii) Assume now that F is fully faithful. Prove that if $F(f)$ is an isomorphism, then f is an isomorphism. In other words, fully faithful functors are conservative.

Exercise 1.3. Is the natural functor $\mathbf{Set} \to \mathbf{Rel}$ full, faithful, fully faithful, conservative?

Exercise 1.4. Prove that the category $\mathcal{C}$ is equivalent to the opposite category $\mathcal{C}^{\mathrm{op}}$ in the following cases:
(i) $\mathcal{C}$ denotes the category of finite abelian groups,
(ii) $\mathcal{C}$ is the category **Rel** of relations.

Exercise 1.5. (i) Prove that in the category **Set**, a morphism f is a monomorphism (resp. an epimorphism) if and only if it is injective (resp. surjective).
(ii) Prove that in the category of rings, the morphism $\mathbb{Z} \to \mathbb{Q}$ is an epimorphism.
(iii) In the category **Top**, give an example of a morphism which is both a monomorphism and an epimorphism and which is not an isomorphism.

Exercise 1.6. Let $\mathcal{C}$ be a category. We denote by $\mathrm{id}_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}$ the identity functor of $\mathcal{C}$ and by $\mathrm{End}\,(\mathrm{id}_{\mathcal{C}})$ the set of endomorphisms of the identity functor $\mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$, that is,

$$\mathrm{End}\,(\mathrm{id}_{\mathcal{C}}) = \mathrm{Hom}_{\,\mathrm{Fct}(\mathcal{C},\mathcal{C})}(\mathrm{id}_{\mathcal{C}}, \mathrm{id}_{\mathcal{C}}).$$

Prove that the composition law on $\mathrm{End}\,(\mathrm{id}_{\mathcal{C}})$ is commutative.

Chapter 2

Limits

We construct inductive and projective limits in categories by using *projective* limits in the category **Set** and give some examples. We also analyze some related notions, in particular those of cofinal categories, filtrant categories and exact functors. Special attention will be paid to filtrant inductive limits in the categories **Set** and $\mathrm{Mod}(A)$.

2.1 Products and coproducts

Let $\mathcal{C}$ be a category and consider a family $\{X_i\}_{i\in I}$ of objects of $\mathcal{C}$ indexed by a (small) set I. Consider the two functors

$$\mathcal{C}^{\mathrm{op}} \to \mathbf{Set}, Y \mapsto \prod_i \mathrm{Hom}_{\mathcal{C}}(Y, X_i), \tag{2.1}$$

$$\mathcal{C} \to \mathbf{Set}, Y \mapsto \prod_i \mathrm{Hom}_{\mathcal{C}}(X_i, Y). \tag{2.2}$$

Definition 2.1.1. (i) Assume that the functor in (2.1) is representable. In this case one denotes by $\prod_i X_i$ a representative and calls this object the product of the X_i's. In case I has two elements, say $I = \{1, 2\}$, one simply denotes this object by $X_1 \times X_2$.

(ii) Assume that the functor in (2.2) is representable. In this case one denotes by $\coprod_i X_i$ a representative and calls this object the coproduct of the X_i's. In case I has two elements, say $I = \{1, 2\}$, one simply denotes this object by $X_1 \sqcup X_2$.

(iii) If for any family of objects $\{X_i\}_{i\in I}$, the product (resp. coproduct) exists, one says that the category $\mathcal{C}$ admits products (resp. coproducts) indexed by I.

(iv) If $X_i = X$ for all $i \in I$, one writes:

$$X^I := \prod_i X_i, \qquad X^{(I)} := \coprod_i X_i.$$

Note that the coproduct in $\mathcal{C}$ is the product in $\mathcal{C}^{\mathrm{op}}$.

By this definition, the product or the coproduct exist if and only if one has the isomorphisms, functorial with respect to $Y \in \mathcal{C}$:

$$\operatorname{Hom}_{\mathcal{C}}(Y, \prod_i X_i) \simeq \prod_i \operatorname{Hom}_{\mathcal{C}}(Y, X_i), \tag{2.3}$$

$$\operatorname{Hom}_{\mathcal{C}}(\coprod_i X_i, Y) \simeq \prod_i \operatorname{Hom}_{\mathcal{C}}(X_i, Y). \tag{2.4}$$

Assume that $\prod_i X_i$ exists. By choosing $Y = \prod_i X_i$ in (2.3), we get the morphisms

$$\pi_i \colon \prod_j X_j \to X_i.$$

Similarly, assume that $\coprod_i X_i$ exists. By choosing $Y = \coprod_i X_i$ in (2.4), we get the morphisms

$$\varepsilon_i \colon X_i \to \coprod_j X_j.$$

The isomorphism (2.3) may be translated as follows. Given an object Y and a family of morphisms $f_i \colon Y \to X_i$, this family factorizes uniquely through $\prod_i X_i$. This is visualized by the diagram

$$\begin{tikzcd}
 & & X_i \\
Y \arrow[r, dotted] \arrow[urr, "f_i"] \arrow[drr, "f_j"'] & \prod_k X_k \arrow[ur, "\pi_i"'] \arrow[dr, "\pi_j"] & \\
 & & X_j.
\end{tikzcd}$$

The isomorphism (2.4) may be translated as follows. Given an object Y and a family of morphisms $f_i \colon X_i \to Y$, this family factorizes uniquely through

$\coprod_i X_i$. This is visualized by the diagram

$$\begin{tikzcd}
X_i \arrow[dr, "\varepsilon_i"'] \arrow[drr, "f_i"] & & \\
& \coprod_k X_k \arrow[r, dotted] & Y. \\
X_j \arrow[ur, "\varepsilon_j"] \arrow[urr, "f_j"'] & &
\end{tikzcd}$$

Example 2.1.2. (i) The category **Set** admits products (that is, products indexed by small sets) and the two definitions (that given in (1.1) and that given in Definition 2.1.1) coincide.
(ii) The category **Set** admits coproducts indexed by small sets, namely, the disjoint union.
(iii) Let A be a ring. The category $\mathrm{Mod}(A)$ admits products, as defined in § 1.2. The category $\mathrm{Mod}(A)$ also admits coproducts, which are the direct sums defined in § 1.2. and are denoted $\bigoplus$.
(iv) Let X be a set and denote by $\mathfrak{X}$ the category of subsets of X. (The set $\mathfrak{X}$ is ordered by inclusion, hence defines a category.) For $S_1, S_2 \in \mathfrak{X}$, their product in the category $\mathfrak{X}$ is their intersection and their coproduct is their union.

Remark 2.1.3. The forgetful functor $\mathit{for}\colon \mathrm{Mod}(A) \to \mathbf{Set}$ commutes with products but does not commute with coproducts. That is the reason why the coproduct in the category $\mathrm{Mod}(A)$ is called and denoted differently.

2.2 Kernels and cokernels

Let $\mathcal{C}$ be a category and consider two parallel arrows $f, g : X_0 \rightrightarrows X_1$ in $\mathcal{C}$. Consider the two functors

$$\mathcal{C}^{\mathrm{op}} \to \mathbf{Set}, Y \mapsto \mathrm{Ker}(\mathrm{Hom}_{\mathcal{C}}(Y, X_0) \rightrightarrows \mathrm{Hom}_{\mathcal{C}}(Y, X_1)), \tag{2.5}$$

$$\mathcal{C} \to \mathbf{Set}, Y \mapsto \mathrm{Ker}(\mathrm{Hom}_{\mathcal{C}}(X_1, Y) \rightrightarrows \mathrm{Hom}_{\mathcal{C}}(X_0, Y)). \tag{2.6}$$

Definition 2.2.1. (i) Assume that the functor in (2.5) is representable. In this case one denotes by $\mathrm{Ker}(f, g)$ a representative and calls this object a kernel (one also says a equalizer) of (f, g).

(ii) Assume that the functor in (2.6) is representable. In this case one denotes by $\mathrm{Coker}(f, g)$ a representative and calls this object a cokernel (one also says a co-equalizer) of (f, g).

(iii) A sequence $Z \to X_0 \rightrightarrows X_1$ (resp. $X_0 \rightrightarrows X_1 \to Z$) is exact if Z is isomorphic to the kernel (resp. cokernel) of $X_0 \rightrightarrows X_1$.

(iv) Assume that the category $\mathcal{C}$ admits a zero-object 0. Let $f\colon X \to Y$ be a morphism in $\mathcal{C}$. A kernel (resp. a cokernel) of f, if it exists, is a kernel (resp. a cokernel) of $f, 0\colon X \rightrightarrows Y$. It is denoted $\mathrm{Ker}(f)$ (resp. $\mathrm{Coker}(f)$).

Note that the cokernel in $\mathcal{C}$ is the kernel in $\mathcal{C}^{\mathrm{op}}$.

By this definition, the kernel or the cokernel of $f, g\colon X_0 \rightrightarrows X_1$ exist if and only if one has the isomorphisms, functorial in $Y \in \mathcal{C}$:

$$\mathrm{Hom}_{\mathcal{C}}(Y, \mathrm{Ker}(f,g)) \simeq \mathrm{Ker}(\mathrm{Hom}_{\mathcal{C}}(Y, X_0) \rightrightarrows \mathrm{Hom}_{\mathcal{C}}(Y, X_1)), \tag{2.7}$$

$$\mathrm{Hom}_{\mathcal{C}}(\mathrm{Coker}(f,g), Y) \simeq \mathrm{Ker}(\mathrm{Hom}_{\mathcal{C}}(X_1, Y) \rightrightarrows \mathrm{Hom}_{\mathcal{C}}(X_0, Y)). \tag{2.8}$$

Assume that $\mathrm{Ker}(f,g)$ exists. By choosing $Y = \mathrm{Ker}(f,g)$ in (2.7), we get the morphism

$$h\colon \mathrm{Ker}(X_0 \rightrightarrows X_1) \to X_0.$$

Similarly, assume that $\mathrm{Coker}(f,g)$ exists. By choosing $Y = \mathrm{Coker}(f,g)$ in (2.8), we get the morphism

$$k\colon X_1 \to \mathrm{Coker}(X_0 \rightrightarrows X_1).$$

Proposition 2.2.2. *The morphism $h\colon \mathrm{Ker}(X_0 \rightrightarrows X_1) \to X_0$ is a monomorphism and the morphism $k\colon X_1 \to \mathrm{Coker}(X_0 \rightrightarrows X_1)$ is an epimorphism.*

Proof. (i) Consider a pair of parallel arrows $a, b\colon Y \rightrightarrows X$ such that $a \circ k = b \circ k = w$. Then $w \circ f = a \circ k \circ f = a \circ k \circ g = b \circ k \circ g = w \circ g$. Hence w factors uniquely through k, and this implies $a = b$.
(ii) The case of cokernels follows, by reversing the arrows. q.e.d.

The isomorphism (2.7) may be translated as follows. Given an objet Y and a morphism $u\colon Y \to X_0$ such that $f \circ u = g \circ u$, the morphism u factors uniquely through $\mathrm{Ker}(f,g)$. This is visualized by the diagram

$$\begin{tikzcd}
\mathrm{Ker}(f,g) \arrow[r, "h"] & X_0 \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & X_1. \\
& Y \arrow[ul, dotted] \arrow[u, "u"] \arrow[ur] &
\end{tikzcd}$$

The isomorphism (2.8) may be translated as follows. Given an objet Y and a morphism $v\colon X_1 \to Y$ such that $v \circ f = v \circ g$, the morphism v factors uniquely through $\mathrm{Coker}(f,g)$. This is visualized by diagram:

$$\begin{tikzcd} X_0 \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[dr] & X_1 \arrow[r, "k"] \arrow[d, "v"] & \mathrm{Coker}(f,g). \arrow[dl, dotted] \\ & Y & \end{tikzcd}$$

Example 2.2.3. (i) The category **Set** admits kernels and the two definitions (that given in (1.7) and that given in Definition 2.2.1) coincide.
(ii) The category **Set** admits cokernels. If $f, g\colon Z_0 \rightrightarrows Z_1$ are two maps, the cokernel of (f,g) is the quotient set $Z_1/\mathcal{R}$ where $\mathcal{R}$ is the equivalence relation generated by the relation $x \sim y$ if there exists $z \in Z_0$ with $f(z) = x$ and $g(z) = y$.
(iii) Let A be a ring. The category $\mathrm{Mod}(A)$ admits a zero object. Hence, the kernel or the cokernel of a morphism f means the kernel or the cokernel of $(f,0)$. As already mentioned, the kernel of a linear map $f\colon M \to N$ is the A-module $f^{-1}(0)$ and the cokernel is the quotient module $M/\operatorname{Im} f$. The kernel and cokernel are visualized by the diagrams

$$\begin{tikzcd} \mathrm{Ker}(f) \arrow[r, "h"] & X_0 \arrow[r, "f"] & X_1, \\ & Y \arrow[ul, dotted] \arrow[u, "u"] \arrow[ur, "0"'] & \end{tikzcd} \qquad \begin{tikzcd} X_0 \arrow[r, "f"] \arrow[dr, "0"'] & X_1 \arrow[r, "k"] \arrow[d, "v"] & \mathrm{Coker}(f). \arrow[dl, dotted] \\ & Y & \end{tikzcd}$$

2.3 Limits

Let us generalize and unify the preceding constructions. In the sequel, I will denote a (small) category. Let $\mathcal{C}$ be a category. A functor $\alpha\colon I \to \mathcal{C}$ (resp. $\beta\colon I^{\mathrm{op}} \to \mathcal{C}$) is sometimes called an inductive (resp. projective) system in $\mathcal{C}$ indexed by I, or else, a diagram indexed by I.

For example, if $(I, \leq)$ is a pre-ordered set, I the associated category, an inductive system indexed by I is the data of a family $(X_i)_{i\in I}$ of objects of $\mathcal{C}$ and for all $i \leq j$, a morphism $X_i \to X_j$ with the natural compatibility conditions.

Projective limits in Set

Assume first that $\mathcal{C}$ is the category **Set** and let us consider projective systems. One sets

$$\varprojlim \beta = \{\{x_i\}_i \in \prod_i \beta(i); \beta(s)(x_j) = x_i \text{ for all } s \in \operatorname{Hom}_I(i,j)\}. \tag{2.9}$$

The next result is obvious.

Lemma 2.3.1. *Let $\beta\colon I^{\mathrm{op}} \to \mathbf{Set}$ be a functor and let $X \in \mathbf{Set}$. There is a natural isomorphism*

$$\operatorname{Hom}_{\mathbf{Set}}(X, \varprojlim \beta) \xrightarrow{\sim} \varprojlim \operatorname{Hom}_{\mathbf{Set}}(X, \beta),$$

where $\operatorname{Hom}_{\mathbf{Set}}(X, \beta)$ *denotes the functor* $I^{\mathrm{op}} \to \mathbf{Set}$, $i \mapsto \operatorname{Hom}_{\mathbf{Set}}(X, \beta(i))$.

Projective and inductive limits

Consider now two functors $\beta\colon I^{\mathrm{op}} \to \mathcal{C}$ and $\alpha\colon I \to \mathcal{C}$. For $X \in \mathcal{C}$, we get functors from I^{op} to $\mathbf{Set}$:

$$\begin{aligned}
&\operatorname{Hom}_{\mathcal{C}}(X, \beta)\colon I^{\mathrm{op}} \ni i \mapsto \operatorname{Hom}_{\mathcal{C}}(X, \beta(i)) \in \mathbf{Set}, \\
&\operatorname{Hom}_{\mathcal{C}}(\alpha, X)\colon I^{\mathrm{op}} \ni i \mapsto \operatorname{Hom}_{\mathcal{C}}(\alpha, X) \in \mathbf{Set}.
\end{aligned}$$

Definition 2.3.2. (i) Assume that the functor $X \mapsto \varprojlim \operatorname{Hom}_{\mathcal{C}}(X, \beta)$ is representable. We denote by $\varprojlim \beta$ its representative and say that the functor β admits a projective limit in $\mathcal{C}$. In other words, we have the isomorphism, functorial in $X \in \mathcal{C}$:

$$\operatorname{Hom}_{\mathcal{C}}(X, \varprojlim \beta) \simeq \varprojlim \operatorname{Hom}_{\mathcal{C}}(X, \beta). \tag{2.10}$$

(ii) Assume that the functor $X \mapsto \varprojlim \operatorname{Hom}_{\mathcal{C}}(\alpha, X)$ is representable. We denote by $\varinjlim \alpha$ its representative and say that the functor α admits an inductive limit in $\mathcal{C}$. In other words, we have the isomorphism, functorial in $X \in \mathcal{C}$:

$$\operatorname{Hom}_{\mathcal{C}}(\varinjlim \alpha, X) \simeq \varprojlim \operatorname{Hom}_{\mathcal{C}}(\alpha, X), \tag{2.11}$$

Remark 2.3.3. The projective limit of the functor β is not only the object $\varprojlim \beta$ but also the isomorphism of functors given in (2.10), and similarly with inductive limits.

When $\mathcal{C} = \mathbf{Set}$ this definition of $\varprojlim \beta$ coincides with the former one, in view of Lemma 2.3.1.

Notice that both projective and inductive limits are defined using projective limits in $\mathbf{Set}$.

Assume that $\varprojlim \beta$ exists in $\mathcal{C}$. One gets:

$$\varprojlim \operatorname{Hom}_{\mathcal{C}}(\varprojlim \beta, \beta) \simeq \operatorname{Hom}_{\mathcal{C}}(\varprojlim \beta, \varprojlim \beta)$$

and the identity of $\varprojlim \beta$ defines a family of morphisms

$$\rho_i \colon \varprojlim \beta \to \beta(i).$$

Consider a family of morphisms $\{f_i \colon X \to \beta(i)\}_{i \in I}$ in $\mathcal{C}$ satisfying the compatibility conditions

$$f_j = f_i \circ f(s) \text{ for all } s \in \operatorname{Hom}_I(i,j). \tag{2.12}$$

This family of morphisms is nothing but an element of $\varprojlim\limits_i \operatorname{Hom}(X, \beta(i))$, hence by (2.10), an element of $\operatorname{Hom}(X, \varprojlim \beta, X)$. Therefore, $\varprojlim \beta$ is characterized by the "universal property":

$$\begin{cases} \text{for all } X \in \mathcal{C} \text{ and all family of morphisms } \{f_i \colon X \to \beta(i)\}_{i \in I} \\ \text{in } \mathcal{C} \text{ satisfying (2.12), all morphisms } f_i\text{'s factorize uniquely} \\ \text{through } \varprojlim \beta. \end{cases} \tag{2.13}$$

This is visualized by the diagram:

$$\begin{tikzcd}
 & & \beta(i) \\
X \arrow[urr, "f_i"] \arrow[drr, "f_j"'] \arrow[r, dotted] & \varprojlim \beta \arrow[ur, "\rho_i"] \arrow[dr, "\rho_j"'] & \\
 & & \beta(j) \arrow[uu, "\beta(s)"']
\end{tikzcd}$$

Similarly, assume that $\varinjlim \alpha$ exists in $\mathcal{C}$. One gets:

$$\varprojlim \operatorname{Hom}_{\mathcal{C}}(\alpha, \varinjlim \alpha) \simeq \operatorname{Hom}_{\mathcal{C}}(\varinjlim \alpha, \varinjlim \alpha)$$

and the identity of $\varinjlim \alpha$ defines a family of morphisms

$$\rho_i \colon \alpha(i) \to \varinjlim \alpha.$$

Consider a family of morphisms $\{f_i \colon \alpha(i) \to X\}_{i \in I}$ in $\mathcal{C}$ satisfying the compatibility conditions

$$f_i = f_j \circ f(s) \text{ for all } s \in \operatorname{Hom}_I(i,j). \tag{2.14}$$

This family of morphisms is nothing but an element of $\varprojlim_i \mathrm{Hom}\,(\alpha(i), X)$, hence by (2.11), an element of $\mathrm{Hom}\,(\varinjlim \alpha, X)$. Therefore, $\varinjlim \alpha$ is characterized by the "universal property":

$$\begin{cases} \text{for all } X \in \mathcal{C} \text{ and all family of morphisms } \{f_i \colon \alpha(i) \to X\}_{i\in I} \\ \text{in } \mathcal{C} \text{ satisfying (2.14), all morphisms } f_i\text{'s factorize uniquely} \\ \text{through } \varinjlim \alpha. \end{cases} \tag{2.15}$$

This is visualized by the diagram:

$$\begin{tikzcd}
\alpha(i) \arrow[dd, "\alpha(s)"'] \arrow[dr, "\rho_i"] \arrow[drr, "f_i"] & & \\
& \varinjlim \alpha \arrow[r, dotted] & X \\
\alpha(j) \arrow[ur, "\rho_j"] \arrow[urr, "f_j"'] & &
\end{tikzcd}$$

Example 2.3.4. Let X be a set and let $\mathfrak{X}$ be the category given in Example 2.1.2 (iv). Let $\beta \colon I^{\mathrm{op}} \to \mathfrak{X}$ and $\alpha \colon I \to \mathfrak{X}$ be two functors. Then

$$\varprojlim \beta \simeq \bigcap_i \beta(i), \qquad \varinjlim \alpha \simeq \bigcup_i \alpha(i).$$

Examples

Examples 2.3.5. (i) When the category I is discrete, projective and inductive limits indexed by I are nothing but products and coproducts indexed by I.
(ii) Consider the category I with two objects and two parallel morphisms other than identities, visualized by $\bullet \rightrightarrows \bullet$. A functor $\alpha \colon I \to \mathcal{C}$ is characterized by two parallel arrows in $\mathcal{C}$:

$$f, g \colon X_0 \rightrightarrows X_1 \tag{2.16}$$

In the sequel we shall identify such a functor with the diagram (2.16). Then, the kernel (resp. cokernel) of (f, g) is nothing but the projective (resp. inductive) limit of the functor α.
(iii) If I is the empty category and $\alpha \colon I \to \mathcal{C}$ is a functor, then $\varprojlim \alpha$ exists in $\mathcal{C}$ if and only if $\mathcal{C}$ has a terminal object $\mathrm{pt}_{\mathcal{C}}$, and in this case $\varprojlim \alpha \simeq \mathrm{pt}_{\mathcal{C}}$. Similarly, $\varinjlim \alpha$ exists in $\mathcal{C}$ if and only if $\mathcal{C}$ has an initial object $\emptyset_{\mathcal{C}}$, and in this case $\varinjlim \alpha \simeq \emptyset_{\mathcal{C}}$.

(iv) If I admits a terminal object, say i_o and if $\beta\colon I^{\mathrm{op}} \to \mathcal{C}$ and $\alpha\colon I \to \mathcal{C}$ are functors, then

$$\varprojlim \beta \simeq \beta(i_o) \qquad \varinjlim \alpha \simeq \alpha(i_o).$$

This follows immediately of (2.15) and (2.13).

If every functor from I^{op} to $\mathcal{C}$ admits a projective limit, one says that $\mathcal{C}$ admits projective limits indexed by I. If this property holds for all categories I (resp. finite categories I), one says that $\mathcal{C}$ admits projective (resp. finite projective) limits, and similarly with inductive limits.

Remark 2.3.6. Assume that $\mathcal{C}$ admits projective (resp. inductive) limits indexed by I. Then $\varprojlim\colon \mathrm{Fct}(I^{\mathrm{op}}, \mathcal{C}) \to \mathcal{C}$ (resp. $\varinjlim\colon \mathrm{Fct}(I, \mathcal{C}) \to \mathcal{C}$) is a functor.

Projective limits as kernels and products

We have seen that products and kernels (resp. coproducts and cokernels) are particular cases of projective (resp. inductive) limits. One can show that conversely, projective limits can be obtained as kernels of products and inductive limits can be obtained as cokernels of coproducts.

Recall that for a category I, $\mathrm{Mor}(I)$ denote the set of morphisms in I. There are two natural maps (source and target) from $\mathrm{Mor}(I)$ to $\mathrm{Ob}(I)$:

$$\begin{aligned} \sigma &: \quad \mathrm{Mor}(I) \to \mathrm{Ob}(I), \quad (s\colon i \to j) \mapsto i, \\ \tau &: \quad \mathrm{Mor}(I) \to \mathrm{Ob}(I), \quad (s\colon i \to j) \mapsto j. \end{aligned}$$

Let $\mathcal{C}$ be a category which admits projective limits and let $\beta\colon I^{\mathrm{op}} \to \mathcal{C}$ be a functor. For $s\colon i \to j$, we get two morphisms in $\mathcal{C}$:

$$\beta(i) \times \beta(j) \underset{\beta(s)}{\overset{\mathrm{id}_{\beta(i)}}{\rightrightarrows}} \beta(i)$$

from which we deduce two morphisms in $\mathcal{C}$: $\prod_{i \in I} \beta(i) \rightrightarrows \beta(\sigma(s))$. These morphisms define the two morphisms in $\mathcal{C}$:

$$\prod_{i \in I} \beta(i) \underset{b}{\overset{a}{\rightrightarrows}} \prod_{s \in \mathrm{Mor}(I)} \beta(\sigma(s)). \tag{2.17}$$

Similarly, assume that $\mathcal{C}$ admits inductive limits and let $\alpha\colon I \to \mathcal{C}$ be a functor. By reversing the arrows, one gets the two morphisms in $\mathcal{C}$:

$$\coprod_{s \in \mathrm{Mor}(I)} \alpha(\sigma(s)) \underset{b}{\overset{a}{\rightrightarrows}} \coprod_{i \in I} \alpha(i). \tag{2.18}$$

Proposition 2.3.7. (i) $\varprojlim \beta$ *is the kernel of* (a,b) *in* (2.17),

(ii) $\varinjlim \alpha$ *is the cokernel of* (a,b) *in* (2.18).

Sketch of proof. By the definition of projective and inductive limits we are reduced to check (i) when $\mathcal{C} = \mathbf{Set}$ and in this case this is obvious. q.e.d.

In particular, a category $\mathcal{C}$ admits finite projective limits if and only if it satisfies:

(i) $\mathcal{C}$ admits a terminal object,

(ii) for any $X, Y \in \mathrm{Ob}(\mathcal{C})$, the product $X \times Y$ exists in $\mathcal{C}$,

(iii) for any parallel arrows in $\mathcal{C}$, $f, g \colon X \rightrightarrows Y$, the kernel exists in $\mathcal{C}$.

There is a similar result for finite inductive limits, replacing a terminal object by an initial object, products by coproducts and kernels by cokernels.

Example 2.3.8. The category **Set** admits projective and inductive limits, as well as the category $\mathrm{Mod}(A)$ for a ring A. Indeed, both categories admit products, coproducts, kernels and cokernels.

2.4 Properties of limits

Double limits

For two categories I and $\mathcal{C}$, recall the notation $\mathcal{C}^I := \mathrm{Fct}(I, \mathcal{C})$ and for a third category J, recall the equivalence (1.14);

$$\mathrm{Fct}(I \times J, \mathcal{C}) \simeq \mathrm{Fct}(I, \mathrm{Fct}(J, \mathcal{C})).$$

Consider a bifunctor $\beta \colon I^{\mathrm{op}} \times J^{\mathrm{op}} \to \mathcal{C}$. It defines a functor a functor $\beta_J \colon I^{\mathrm{op}} \to \mathcal{C}^{J^{\mathrm{op}}}$ as well as a functor $\beta_I \colon J^{\mathrm{op}} \to \mathcal{C}^{I^{\mathrm{op}}}$. One easily checks that

$$\varprojlim \beta \simeq \varprojlim \varprojlim \beta_J \simeq \varprojlim \varprojlim \beta_I. \tag{2.19}$$

Similarly, if $\alpha \colon I \times J \to \mathcal{C}$ is a bifunctor, it defines a functor $\alpha_J \colon I \to \mathcal{C}^J$ as well as a functor $\alpha_I \colon J \to \mathcal{C}^I$ and one has the isomorphisms

$$\varinjlim \alpha \simeq \varinjlim (\varinjlim \alpha_J) \simeq \varinjlim (\varinjlim \alpha_I). \tag{2.20}$$

In other words:

$$\varprojlim_{i,j} \beta(i,j) \simeq \varprojlim_{j} \varprojlim_{i} (\beta(i,j)) \simeq \varprojlim_{i} \varprojlim_{j} (\beta(i,j)), \tag{2.21}$$

$$\varinjlim_{i,j} \alpha(i,j) \simeq \varinjlim_{j} (\varinjlim_{i} (\alpha(i,j)) \simeq \varinjlim_{i} \varinjlim_{j} (\alpha(i,j)). \tag{2.22}$$

Limits with values in a category of functors

Consider another category $\mathcal{A}$ and a functor $\beta\colon I^{\mathrm{op}} \to \mathrm{Fct}(\mathcal{A}, \mathcal{C})$. It defines a functor $\widetilde{\beta}\colon I^{\mathrm{op}} \times \mathcal{A} \to \mathcal{C}$, hence for each $A \in \mathcal{A}$, a functor $\widetilde{\beta}(A)\colon I^{\mathrm{op}} \to \mathcal{C}$. Assuming that $\mathcal{C}$ admits projective limits indexed by I, one checks easily that $A \mapsto \varprojlim \widetilde{\beta}(A)$ is a functor, that is, an object of $\mathrm{Fct}(\mathcal{A}, \mathcal{C})$, and is a projective limit of β. There is a similar result for inductive limits. Hence:

Proposition 2.4.1. *Let I be a category and assume that $\mathcal{C}$ admits projective limits indexed by I. Then for any category $\mathcal{A}$, the category $\mathrm{Fct}(\mathcal{A}, \mathcal{C})$ admits projective limits indexed by I. Moreover, if $\beta\colon I^{\mathrm{op}} \to \mathrm{Fct}(\mathcal{A}, \mathcal{C})$ is a functor, then $\varprojlim \beta \in \mathrm{Fct}(\mathcal{A}, \mathcal{C})$ is given by*

$$(\varprojlim \beta)(A) = \varprojlim (\beta(A)), \quad A \in \mathcal{A}.$$

Similarly, assume that $\mathcal{C}$ admits inductive limits indexed by I. Then for any category $\mathcal{A}$, the category $\mathrm{Fct}(\mathcal{A}, \mathcal{C})$ admits inductive limits indexed by I. Moreover, if $\alpha\colon I \to \mathrm{Fct}(\mathcal{A}, \mathcal{C})$ is a functor, then $\varinjlim \alpha \in \mathrm{Fct}(\mathcal{A}, \mathcal{C})$ is given by

$$(\varinjlim \alpha)(A) = \varinjlim (\alpha(A)), \quad A \in \mathcal{A}.$$

Corollary 2.4.2. *Let $\mathcal{C}$ be a category. Then the categories $\mathcal{C}^\wedge$ and $\mathcal{C}^\vee$ admit inductive and projective limits.*

Composition of limits

Let $I, \mathcal{C}$ and $\mathcal{C}'$ be categories and let $\alpha\colon I \to \mathcal{C}$, $\beta\colon I^{\mathrm{op}} \to \mathcal{C}$ and $F\colon \mathcal{C} \to \mathcal{C}'$ be functors. When $\mathcal{C}$ and $\mathcal{C}'$ admit projective or inductive limits indexed by I, there are natural morphisms

$$F(\varprojlim \beta) \to \varprojlim (F \circ \beta), \tag{2.23}$$

$$\varinjlim (F \circ \alpha) \to F(\varinjlim \alpha). \tag{2.24}$$

This follows immediately from (2.15) and (2.13).

Definition 2.4.3. Let I be a category and let $F\colon \mathcal{C} \to \mathcal{C}'$ be a functor.

(i) Assume that $\mathcal{C}$ and $\mathcal{C}'$ admit projective limits indexed by I. One says that F commutes with such limits if (2.23) is an isomorphism.

(ii) Similarly, assume that $\mathcal{C}$ and $\mathcal{C}'$ admit inductive limits indexed by I. One says that F commutes with such limits if (2.24) is an isomorphism.

Examples 2.4.4. (i) Let $\mathcal{C}$ be a category which admits projective limits indexed by I and let $X \in \mathcal{C}$. By (2.10), the functor $\mathrm{Hom}_{\mathcal{C}}(X, \bullet)\colon \mathcal{C} \to \mathbf{Set}$ commutes with projective limits indexed by I. Similarly, if $\mathcal{C}$ admits inductive limits indexed by I, then the functor $\mathrm{Hom}_{\mathcal{C}}(\bullet, X)\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ commutes with projective limits indexed by I, by (2.11).
(ii) Let I and J be two categories and assume that $\mathcal{C}$ admits projective (resp. inductive) limits indexed by $I \times J$. Then the functor $\varprojlim : \mathrm{Fct}(J^{\mathrm{op}}, \mathcal{C}) \to \mathcal{C}$ (resp. $\varinjlim : \mathrm{Fct}(J, \mathcal{C}) \to \mathcal{C}$) commutes with projective (resp. inductive) limits indexed by I. This follows from the isomorphisms (2.19) and (2.20).
(iii) Let k be a field, $\mathcal{C} = \mathcal{C}' = \mathrm{Mod}(k)$, and let $X \in \mathcal{C}$. Then the functor $\mathrm{Hom}_k(X, \bullet)$ does not commute with inductive limit if X is infinite dimensional.

Proposition 2.4.5. *Let $F\colon \mathcal{C} \to \mathcal{C}'$ be a functor and let I be a category.*

(i) *Assume that $\mathcal{C}$ and $\mathcal{C}'$ admit projective limits indexed I and F admits a left adjoint $G\colon \mathcal{C}' \to \mathcal{C}$. Then F commutes with projective limits indexed by I, that is, $F(\varprojlim_i \beta(i)) \simeq \varprojlim_i F(\beta(i))$.*

(ii) *Similarly, if $\mathcal{C}$ and $\mathcal{C}'$ admit inductive limits indexed by I and F admits a right adjoint, then F commutes with such limits.*

Proof. It is enough to prove the first assertion. To check that (2.23) is an isomorphism, we apply Corollary 1.4.4. Let $Y \in \mathcal{C}'$. One has the chain of isomorphisms

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{C}'}(Y, F(\varprojlim_i \beta(i))) &\simeq \mathrm{Hom}_{\mathcal{C}}(G(Y), \varprojlim_i \beta(i)) \\
&\simeq \varprojlim_i \mathrm{Hom}_{\mathcal{C}}(G(Y), \beta(i)) \\
&\simeq \varprojlim_i \mathrm{Hom}_{\mathcal{C}'}(Y, F(\beta(i))) \\
&\simeq \mathrm{Hom}_{\mathcal{C}'^\wedge}(Y, \varprojlim_i F(\beta(i))).
\end{aligned}$$

q.e.d.

2.5 Filtrant inductive limits

Since it admits coproducts and cokernels, the category **Set** admits inductive limits. We shall construct them more explicitely.

Let $\alpha\colon I \to \mathbf{Set}$ be a functor and consider the relation on $\bigsqcup_{i\in I}\alpha(i))$:

$$\begin{cases}\alpha(i) \ni x\mathcal{R}y \in \alpha(j) \text{ if there exists } k \in I,\ s\colon i\to k \text{ and } t\colon j \to k\\ \text{with } \alpha(s)(x) = \alpha(t)(y).\end{cases} \tag{2.25}$$

The relation $\mathcal{R}$ is reflexive and symmetric but is not transitive in general.

Proposition 2.5.1. *With the notations above, denote by $\sim$ the equivalence relation generated by $\mathcal{R}$. Then*

$$\varinjlim \alpha \ \simeq\ (\bigsqcup_{i\in I}\alpha(i))/\sim .$$

Proof. Let $S \in \mathbf{Set}$. By the definition of the projective limit in $\mathbf{Set}$ we get:

$$\begin{aligned}\varprojlim \operatorname{Hom}(\alpha, S) &\simeq \{\{u_i\}_{i\in I}; u_i\colon \alpha(i)\to S, u_j = u_i\circ\alpha(s)\\ &\qquad \text{if there exists } s\colon i\to j\},\\ &\simeq \{\{p(i,x)\}_{i\in I, x\in\alpha(i)}; p(i,x)\in S, p(i,x) = p(j,y)\\ &\qquad \text{if there exists } s\colon i \to j \text{ with } \alpha(s)(x) = y\}\\ &\simeq \operatorname{Hom}(\bigsqcup_{i\in I}\alpha(i))/\sim, S).\end{aligned}$$

q.e.d.

In the category $\mathbf{Set}$ one uses the notation $\bigsqcup$ better than $\coprod$.

For a ring A, the category $\mathrm{Mod}(A)$ admits coproducts and cokernels. Hence, the category $\mathrm{Mod}(A)$ admits inductive limits. One shall be aware that the functor $\mathit{for}\colon \mathrm{Mod}(A) \to \mathbf{Set}$ does not commute with inductive limits. For example, if I is empty and $\alpha\colon I \to \mathrm{Mod}(A)$ is a functor, then $\alpha(I) = \{0\}$ and $\mathit{for}(\{0\})$ is not an initial object in $\mathbf{Set}$.

Definition 2.5.2. A category I is called filtrant if it satisfies the conditions (i)–(iii) below.

(i) I is non empty,

(ii) for any i and j in I, there exists $k \in I$ and morphisms $i \to k, j \to k$,

(iii) for any parallel morphisms $f, g\colon i \rightrightarrows j$, there exists a morphism $h\colon j \to k$ such that $h \circ f = h \circ g$.

One says that I is cofiltrant if I^{op} is filtrant.

The conditions (ii)–(iii) of being filtrant are visualized by the diagrams:

$$
\begin{tikzcd}
i \arrow[dr, dotted] & \\
& k \\
j \arrow[ur, dotted] &
\end{tikzcd}
\qquad
\begin{tikzcd}
i \arrow[r, shift left] \arrow[r, shift right] \arrow[dr, dotted] & j \arrow[d, dotted] \\
& k
\end{tikzcd}
$$

Of course, if $(I, \leq)$ is a non-empty directed ordered set, then the associated category I is filtrant.

Proposition 2.5.3. *Let $\alpha\colon I \to \mathbf{Set}$ be a functor, with I filtrant. The relation $\mathcal{R}$ given in (2.25) on $\coprod_i \alpha(i)$ is an equivalence relation.*

Proof. Let $x_j \in \alpha(i_j)$, $j = 1, 2, 3$ with $x_1 \sim x_2$ and $x_2 \sim x_3$. There exist morphisms visualized by the diagram:

$$
\begin{tikzcd}
i_1 \arrow[r, "s_1"] & j_1 \arrow[dr, "u_1"] & & \\
i_2 \arrow[ur, "s_2"'] \arrow[dr, "t_2"] & & k_1 \arrow[r, "v"] & l \\
i_3 \arrow[r, "t_3"] & j_2 \arrow[ur, "u_2"'] & &
\end{tikzcd}
$$

such that $\alpha(s_1)x_1 = \alpha(s_2)x_2$, $\alpha(t_2)x_2 = \alpha(t_3)x_3$, and $v \circ u_1 \circ s_2 = v \circ u_2 \circ t_2$. Set $w_1 = v \circ u_1 \circ s_1$, $w_2 = v \circ u_1 \circ s_2 = v \circ u_2 \circ t_2$ and $w_3 = v \circ u_2 \circ t_3$. Then $\alpha(w_1)x_1 = \alpha(w_2)x_2 = \alpha(w_3)x_3$. Hence $x_1 \sim x_3$. q.e.d.

Corollary 2.5.4. *Let $\alpha\colon I \to \mathbf{Set}$ be a functor, with I filtrant.*

(i) *Let S be a finite subset in $\varinjlim \alpha$. Then there exists $i \in I$ such that S is contained in the image of $\alpha(i)$ by the natural map $\alpha(i) \to \varinjlim \alpha$.*

(ii) *Let $i \in I$ and let x and y be elements of $\alpha(i)$ with the same image in $\varinjlim \alpha$. Then there exists $s\colon i \to j$ such that $\alpha(s)(x) = \alpha(s)(y)$ in $\alpha(j)$.*

Proof. (i) Denote by $\alpha\colon \bigsqcup_{i \in I} \alpha(i)) \to \varinjlim \alpha$ the quotient map. Let $S = \{x_1, \ldots, x_n\}$. For $j = 1, \ldots, n$, there exists $y_j \in \alpha(i_j)$ such that $x_j = \alpha(y_j)$. Choose $k \in I$ such that there exist morphisms $s_j\colon \alpha(i_j) \to \alpha(k)$. Then $x_j = \alpha(\alpha(s_j(y_j)))$.
(ii) For $x, y \in \alpha(i)$, $x\mathcal{R}y$ if and only if there exists $s\colon i \to j$ with $\alpha(s)(x) = \alpha(s)(y)$ in $\alpha(j)$. q.e.d.

Corollary 2.5.5. *Let A be a ring and denote by for the forgetful functor $\mathrm{Mod}(A) \to \mathbf{Set}$. Then the functor for commutes with filtrant inductive limits. In other words, if I is filtrant and $\alpha\colon I \to \mathrm{Mod}(A)$ is a functor, then*

$$for \circ (\varinjlim_i \alpha(i)) = \varinjlim_i (for \circ \alpha(i)).$$

The proof is left as an exercise (see Exercise 2.8).

Inductive limits with values in **Set** indexed by filtrant categories commute with finite projective limits. More precisely:

Proposition 2.5.6. *For a filtrant category I, a finite category J and a functor $\alpha\colon I \times J^{\mathrm{op}} \to \mathbf{Set}$, one has $\varinjlim_i \varprojlim_j \alpha(i,j) \xrightarrow{\sim} \varprojlim_j \varinjlim_i \alpha(i,j)$. In other words, the functor*

$$\varinjlim : \mathrm{Fct}(I, \mathbf{Set}) \to \mathbf{Set}$$

commutes with finite projective limits.

Proof. It is enough to prove that $\varinjlim$ commutes with kernels and with finite products.

(i) $\varinjlim$ commutes with kernels. Let $\alpha, \beta\colon I \to \mathbf{Set}$ be two functors and let $f, g\colon \alpha \rightrightarrows \beta$ be two morphisms of functors. Define γ as the kernel of (f,g), that is, we have exact sequences

$$\gamma(i) \to \alpha(i) \rightrightarrows \beta(i).$$

Let Z denote the kernel of $\varinjlim_i \alpha(i) \rightrightarrows \varinjlim_i \beta(i)$. We have to prove that the natural map $\lambda\colon \varinjlim_i \gamma(i) \to Z$ is bijective.

(i) (a) The map λ is surjective. Indeed for $x \in Z$, represent x by some $x_i \in \alpha(i)$. Then $f_i(x_i)$ and $g_i(x_i)$ in $\beta(i)$ having the same image in $\varinjlim \beta$, there exists $s\colon i \to j$ such that $\beta(s)f_i(x_i) = \beta(s)g_i(x_i)$. Set $x_j = \alpha(s)x_i$. Then $f_j(x_j) = g_j(x_j)$, which means that $x_j \in \gamma(j)$. Clearly, $\lambda(x_j) = x$.

(i) (b) The map λ is injective. Indeed, let $x, y \in \varinjlim \gamma$ with $\lambda(x) = \lambda(y)$. We may represent x and y by elements x_i and y_i of $\gamma(i)$ for some $i \in I$. Since x_i and y_i have the same image in $\varinjlim \alpha$, there exists $i \to j$ such that they have the same image in $\alpha(j)$. Therefore their images in $\gamma(j)$ will be the same.

(ii) $\varinjlim$ commutes with finite products. The proof is similar to the preceding one and left to the reader. q.e.d.

Corollary 2.5.7. *Let A be a ring and let I be a filtrant category. Then the functor $\varinjlim : \mathrm{Fct}(I, \mathrm{Mod}(A)) \to \mathrm{Mod}(A)$ commutes with finite projective limits.*

Cofinal functors

Let $\varphi\colon J \to I$ be a functor. If there are no risk of confusion, we still denote by φ the associated functor $\varphi\colon J^{\mathrm{op}} \to I^{\mathrm{op}}$. For two functors $\alpha\colon I \to \mathcal{C}$ and $\beta\colon I^{\mathrm{op}} \to \mathcal{C}$, we have natural morphisms:

$$\varprojlim\,(\beta \circ \varphi) \;\leftarrow\; \varprojlim \beta, \tag{2.26}$$

$$\varinjlim\,(\alpha \circ \varphi) \;\rightarrow\; \varinjlim \alpha. \tag{2.27}$$

This follows immediately of (2.15) and (2.13).

Definition 2.5.8. Assume that φ is fully faithful and I is filtrant. One says that φ is cofinal if for any $i \in I$ there exists $j \in J$ and a morphism $s\colon i \to \varphi(j)$.

Example 2.5.9. A subset $J \subset \mathbb{N}$ defines a cofinal subcategory of $(\mathbb{N}, \leq)$ if and only if it is infinite.

Proposition 2.5.10. *Let $\varphi\colon J \to I$ be a fully faithful functor. Assume that I is filtrant and φ is cofinal. Then*

(i) *for any category $\mathcal{C}$ and any functor $\beta\colon I^{\mathrm{op}} \to \mathcal{C}$, the morphism* (2.26) *is an isomorphism,*

(ii) *for any category $\mathcal{C}$ and any functor $\alpha\colon I \to \mathcal{C}$, the morphism* (2.27) *is an isomorphism.*

Proof. Let us prove (ii), the other proof being similar. By the hypothesis, for each $i \in I$ we get a morphism $\alpha(i) \to \varinjlim\limits_{j\in J}(\alpha \circ \varphi(j))$ from which one deduce a morphism

$$\varinjlim_{i\in I} \alpha(i) \to \varinjlim_{j\in J}(\alpha \circ \varphi(j)).$$

One checks easily that this morphism is inverse to the morphism in (2.24). q.e.d.

Example 2.5.11. Let X be a topological space, $x \in X$ and denote by I_x the set of open neighborhoods of x in X. We endow I_x with the order: $U \leq V$ if $V \subset U$. Given U and V in I_x, and setting $W = U \cap V$, we have $U \leq W$ and $V \leq W$. Therefore, I_x is filtrant.

Denote by $\mathcal{C}^0(U)$ the $\mathbb{C}$-vector space of complex valued continuous functions on U. The restriction maps $\mathcal{C}^0(U) \to \mathcal{C}^0(V), V \subset U$ define an inductive system of $\mathbb{C}$-vector spaces indexed by I_x. One sets

$$\mathcal{C}^0_{X,x} = \varinjlim_{U\in I_x} \mathcal{C}^0(U). \tag{2.28}$$

An element φ of $\mathcal{C}^0_{X,x}$ is called a germ of continuous function at 0. Such a germ is an equivalence class $(U, \varphi_U)/\sim$ with U a neighborhood of x, φ_U a continuous function on U, and $(U, \varphi_U) \sim 0$ if there exists a neighborhood V of x with $V \subset U$ such that the restriction of φ_U to V is the zero function. Hence, a germ of function is zero at x if this function is identically zero in a neighborhood of x.

A set theoretical remark

Remark 2.5.12. In these notes, we have skipped problems related to questions of cardinality and universes (see Remark 1.3.2), but this is dangerous. In particular, when taking limits, we should assume that all categories ($\mathcal{C}$, $\mathcal{C}'$ etc.) belong to a given universe $\mathcal{U}$ and that all limits are indexed by $\mathcal{U}$-small categories (I, J, etc.). However some constructions force one to quit the universe $\mathcal{U}$ to a bigger one $\mathcal{V}$. We shall not develop this point here.

Let us give an example which shows that without some care, we may have troubles.

Let $\mathcal{C}$ be a category which admits products and assume there exist $X, Y \in \mathcal{C}$ such that $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ has more than one element. Set $M = \mathrm{Mor}(\mathcal{C})$, where $\mathrm{Mor}(\mathcal{C})$ denotes the "set" of all morphisms in $\mathcal{C}$, and let $\pi = \mathrm{card}(M)$, the cardinal of the set M. We have $\mathrm{Hom}_{\mathcal{C}}(X, Y^M) \simeq \mathrm{Hom}_{\mathcal{C}}(X,Y)^M$ and therefore $\mathrm{card}(\mathrm{Hom}_{\mathcal{C}}(X, Y^M) \geq 2^\pi$. On the other hand, $\mathrm{Hom}_{\mathcal{C}}(X, Y^M) \subset \mathrm{Mor}(\mathcal{C})$ which implies $\mathrm{card}(\mathrm{Hom}_{\mathcal{C}}(X, Y^M) \leq \pi$.

The "contradiction" comes from the fact that $\mathcal{C}$ does not admit products indexed by such a big set as $\mathrm{Mor}(\mathcal{C})$. (The remark was found in [10].)

Exercises to Chapter 2

Exercise 2.1. (i) Let I be a (non necessarily finite) set and $\{X_i\}_{i \in I}$ a family of sets indexed by I. Show that $\coprod_i X_i$ is the disjoint union of the sets X_i.
(ii) Construct the natural map $\coprod_i \mathrm{Hom}_{\mathbf{Set}}(Y, X_i) \to \mathrm{Hom}_{\mathbf{Set}}(Y, \coprod_i X_i)$ and prove it is injective.
(iii) Prove that the map $\coprod_i \mathrm{Hom}_{\mathbf{Set}}(X_i, Y) \to \mathrm{Hom}_{\mathbf{Set}}(\prod_i X_i, Y)$ is not injective in general.

Exercise 2.2. Let $X, Y \in \mathcal{C}$ and consider the category $\mathcal{D}$ whose objects are triplets $Z \in \mathcal{C}, f\colon Z \to X, g\colon Z \to Y$, the morphisms being the natural ones. Prove that this category admits a terminal object if and only if the product $X \times Y$ exists in $\mathcal{C}$, and that in such a case this terminal object is isomorphic to $X \times Y, X \times Y \to X, X \times Y \to Y$. Deduce that if $X \times Y$ exists, it is unique up to unique isomorphism.

Exercise 2.3. Let I and $\mathcal{C}$ be two categories and denote by Δ the functor from $\mathcal{C}$ to $\mathcal{C}^I$ which, to $X \in \mathcal{C}$, associates the constant functor $\Delta(X)\colon I \ni i \mapsto X \in \mathcal{C}$, $(i \to j) \in \mathrm{Mor}(I) \mapsto \mathrm{id}_X$. Assume that any functor from I to $\mathcal{C}$ admits an inductive limit.
(i) Prove the formula (for $\alpha\colon I \to \mathcal{C}$ and $Y \in \mathcal{C}$):

$$\mathrm{Hom}_{\mathcal{C}}(\varinjlim_i \alpha(i), Y) \simeq \mathrm{Hom}_{\,\mathrm{Fct}(I,\mathcal{C})}(\alpha, \Delta(Y)).$$

(ii) Replacing I with the opposite category, deduce the formula (assuming projective limits exist):

$$\mathrm{Hom}_{\mathcal{C}}(X, \varprojlim_i G(i)) \simeq \mathrm{Hom}_{\,\mathrm{Fct}(I^{\mathrm{op}},\mathcal{C})}(\Delta(X), G).$$

Exercise 2.4. Let $\mathcal{C}$ be a category which admits filtrant inductive limits. One says that an object X of $\mathcal{C}$ is of finite type if for any functor $\alpha\colon I \to \mathcal{C}$ with I filtrant, the natural map $\varinjlim \mathrm{Hom}_{\mathcal{C}}(X, \alpha) \to \mathrm{Hom}_{\mathcal{C}}(X, \varinjlim \alpha)$ is injective. Show that this definition coincides with the classical one when $\mathcal{C} = \mathrm{Mod}(A)$, for a ring A.
(Hint: let $X \in \mathrm{Mod}(A)$. To prove that if X is of finite type in the categorical sense then it is of finite type in the usual sense, use the fact that, denoting by S be the family of submodules of finite type of X ordered by inclusion, we have $\varinjlim_{V \in S} X/V \simeq 0$.)

Exercise 2.5. Let $\mathcal{C}$ be a category which admits filtrant inductive limits. One says that an object X of $\mathcal{C}$ is of finite presentation if for any functor $\alpha\colon I \to \mathcal{C}$ with I filtrant, the natural map $\varinjlim \mathrm{Hom}_{\mathcal{C}}(X, \alpha) \to \mathrm{Hom}_{\mathcal{C}}(X, \varinjlim \alpha)$ is bijective. Show that this definition coincides with the classical one when $\mathcal{C} = \mathrm{Mod}(A)$, for a ring A.

Exercise 2.6. Consider the category I with three objects $\{a, b, c\}$ and two morphisms other than the identities, visualized by the diagram

$$a \leftarrow c \to b.$$

Let $\mathcal{C}$ be a category. A functor $\beta\colon I^{\mathrm{op}} \to \mathcal{C}$ is nothing but the data of three objects X, Y, Z and two morphisms visualized by the diagram

$$X \xrightarrow{f} Z \xleftarrow{g} Y.$$

The fiber product $X \times_Z Y$ of X and Y over Z, if it exists, is the projective limit of β.

(i) Assume that $\mathcal{C}$ admits products (of two objects) and kernels. Prove that $X \times_Z Y$ nis isomorphic to the equalizer of $X \times Y \rightrightarrows Z$. Here, the two morphisms $X \times Y \rightrightarrows Z$ are given by f, g.
(ii) Prove that $\mathcal{C}$ admits finite projective limits if and only if it admits fiber products and a terminal object.

Exercise 2.7. Let I be a filtrant ordered set and let $A_i, i \in I$ be an inductive system of rings indexed by I.
(i) Prove that $A := \varinjlim_i A_i$ is naturally endowed with a ring structure.
(ii) Define the notion of an inductive system M_i of A_i-modules, and define the A-module $\varinjlim_i M_i$.
(iii) Let N_i (resp. M_i) be an inductive system of right (resp. left) A_i modules. Prove the isomorphism

$$\varinjlim_i (N_i \otimes_{A_i} M_i) \xrightarrow{\sim} \varinjlim_i N_i \otimes_A \varinjlim_i M_i.$$

Exercise 2.8. Let I be a filtrant ordered set and let $M_i, i \in I$ be an inductive sytem of $\mathbf{k}$-modules indexed by I. Let $M = \bigsqcup M_i / \sim$ where $\bigsqcup$ denotes the set-theoretical disjoint union and $\sim$ is the relation $M_i \ni x_i \sim y_j \in M_j$ if there exists $k \geq i, k \geq j$ such that $u_{ki}(x_i) = u_{kj}(y_j)$.

Prove that M is naturally a $\mathbf{k}$-module and is isomorphic to $\varinjlim_i M_i$.

Exercise 2.9. (i) Let $\mathcal{C}$ be a category which admits inductive limits indexed by a category I. Let $\alpha\colon I \to \mathcal{C}$ be a functor and let $X \in \mathcal{C}$. Construct the natural morhism

$$\varinjlim_i \mathrm{Hom}_{\mathcal{C}}(X, \alpha(i)) \quad\to\quad \mathrm{Hom}_{\mathcal{C}}(X, \varinjlim_i \alpha(i)). \tag{2.29}$$

(ii) Let $\mathbf{k}$ be a field and denote by $\mathbf{k}[x]^{\leq n}$ the $\mathbf{k}$-vector space consisting of polynomials of degree $\leq n$. Prove the isomorphism $\mathbf{k}[x] = \varinjlim_n \mathbf{k}[x]^{\leq n}$ and, noticing that $\mathrm{id}_{\mathbf{k}[x]} \notin \varinjlim_n \mathrm{Hom}_{\mathbf{k}}(\mathbf{k}[x], \mathbf{k}[x]^{\leq n})$, deduce that the morphism (2.29) is not an isomorphism in general.

Exercise 2.10. Let $\mathcal{C}$ be a category and recall (Proposition 2.4.1) that the category $\mathcal{C}^\wedge$ admits inductive limits. One denotes by "$\varinjlim$" the inductive limit in $\mathcal{C}^\wedge$. Let $\mathbf{k}$ be a field and let $\mathcal{C} = \mathrm{Mod}(\mathbf{k})$. Prove that the Yoneda functor $\mathrm{h}_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}^\wedge$ does not commute with inductive limits.

Exercise 2.11. Let I be a discrete set and let $\mathcal{J}$ be the set of finite subsets of I, ordered by inclusion. We consider both I and $\mathcal{J}$ as categories. Let $\mathcal{C}$ be a category and $\alpha\colon I \to \mathcal{C}$ a functor. For $J \in \mathcal{J}$ we denote by $\alpha_J\colon J \to \mathcal{C}$ the restriction of α to J.
(i) Prove that the category $\mathcal{J}$ is filtrant.
(ii) Prove the isomorphism $\varinjlim\limits_{J\in\mathcal{J}} \varinjlim\limits_{j\in J} \alpha_J \xrightarrow{\sim} \varinjlim \alpha$.

Exercise 2.12. Let $\mathcal{C}$ be a category which admits a zero-object and kernels. Prove that a morphism $f\colon X \to Y$ is a monomorphism if and only if $\operatorname{Ker} f \simeq 0$.

Chapter 3

Additive categories

Many results or constructions in the category $\mathrm{Mod}(A)$ of modules over a ring A have their counterparts in other contexts, such as finitely generated A-modules, or graded modules over a graded ring, or sheaves of A-modules, etc. Hence, it is natural to look for a common language which avoids to repeat the same arguments. This is the language of additive and abelian categories.

In this chapter, we give the main properties of additive categories.

3.1 Additive categories

Definition 3.1.1. A category $\mathcal{C}$ is additive if it satisfies conditions (i)-(v) below:

(i) for any $X, Y \in \mathcal{C}$, $\quad \mathrm{Hom}_{\mathcal{C}}(X, Y) \in \mathrm{Mod}(\mathbb{Z})$,

(ii) the composition law $\circ$ is bilinear,

(iii) there exists a zero object in $\mathcal{C}$,

(iv) the category $\mathcal{C}$ admits finite coproducts,

(v) the category $\mathcal{C}$ admits finite products.

Note that $\mathrm{Hom}_{\mathcal{C}}(X, Y) \neq \emptyset$ since it is a group and for all $X \in \mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(X, 0) = \mathrm{Hom}_{\mathcal{C}}(0, X) = 0$. (The morphism 0 should not be confused with the object 0.)

Notation 3.1.2. If X and Y are two objects of $\mathcal{C}$, one denotes by $X \oplus Y$ (instead of $X \sqcup Y$) their coproduct, and calls it their direct sum. One denotes as usual by $X \times Y$ their product. This change of notations is motivated by the fact that if A is a ring, the forgetful functor $\mathrm{Mod}(A) \to \mathbf{Set}$ does not commute with coproducts.

Lemma 3.1.3. *Let $\mathcal{C}$ be a category satisfying conditions* (i)–(iii) *in* Definition 3.1.1. *Consider the condition*

(vi) *for any two objects X and Y in $\mathcal{C}$, there exists $Z \in \mathcal{C}$ and morphisms $i_1 \colon X \to Z$, $i_2 \colon Y \to Z$, $p_1 \colon Z \to X$ and $p_2 \colon Z \to Y$ satisfying*

$$p_1 \circ i_1 = \mathrm{id}_X, \quad p_1 \circ i_2 = 0 \tag{3.1}$$
$$p_2 \circ i_2 = \mathrm{id}_Y, \quad p_2 \circ i_1 = 0, \tag{3.2}$$
$$i_1 \circ p_1 + i_2 \circ p_2 = \mathrm{id}_Z\,. \tag{3.3}$$

Then the conditions (iv), (v) *and* (vi) *are equivalent and the objects $X \oplus Y$, $X \times Y$ and Z are naturally isomorphic.*

Proof. (a) Let us assume condition (iv). The identity of X and the zero morphism $Y \to X$ define the morphism $p_1 \colon X \oplus Y \to X$ satisfying (3.1). We construct similarly the morphism $p_2 \colon X \oplus Y \to Y$ satisfying (3.2). To check (3.3), we use the fact that if $f \colon X \oplus Y \to X \oplus Y$ satisfies $f \circ i_1 = i_1$ and $f \circ i_2 = i_2$, then $f = \mathrm{id}_{X \oplus Y}$.
(b) Let us assume condition (vi). Let $W \in \mathcal{C}$ and consider morphisms $f \colon X \to W$ and $g \colon Y \to W$. Set $h := f \circ p_1 \oplus g \circ p_2$. Then $h \colon Z \to W$ satisfies $h \circ i_1 = f$ and $h \circ i_2 = g$ and such an h is unique. Hence $Z \simeq X \oplus Y$.
(c) We have proved that conditions (iv) and (vi) are equivalent and moreover that if they are satisfied, then $Z \simeq X \oplus Y$. Replacing $\mathcal{C}$ with $\mathcal{C}^{\mathrm{op}}$, we get that these conditions are equivalent to (v) and $Z \simeq X \times Y$. q.e.d.

Example 3.1.4. (i) If A is a ring, $\mathrm{Mod}(A)$ and $\mathrm{Mod}^{\mathrm{f}}(A)$ are additive categories.
(ii) **Ban**, the category of $\mathbb{C}$-Banach spaces and linear continuous maps is additive.
(iii) If $\mathcal{C}$ is additive, then $\mathcal{C}^{\mathrm{op}}$ is additive.
(iv) Let I be category. If $\mathcal{C}$ is additive, the category $\mathrm{Fct}(I, \mathcal{C})$ of functors from I to $\mathcal{C}$, is additive.
(v) If $\mathcal{C}$ and $\mathcal{C}'$ are additive, then $\mathcal{C} \times \mathcal{C}'$ is additive.

Let $F \colon \mathcal{C} \to \mathcal{C}'$ be a functor of additive categories. One says that F is additive if for $X, Y \in \mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}'}(F(X), F(Y))$ is a morphism of groups. We shall not prove here the following result.

Proposition 3.1.5. *Let $F\colon \mathcal{C} \to \mathcal{C}'$ be a functor of additive categories. Then F is additive if and only if it commutes with direct sum, that is, for X and Y in $\mathcal{C}$:*

$$\begin{aligned} F(0) &\simeq 0 \\ F(X \oplus Y) &\simeq F(X) \oplus F(Y). \end{aligned}$$

Unless otherwise specified, functors between additive categories will be assumed to be additive.

Generalization. Let k be a *commutative* ring. One defines the notion of a k-additive category by assuming that for X and Y in $\mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is a k-module and the composition is k-bilinear.

3.2 Complexes in additive categories

Let $\mathcal{C}$ denote an additive category.

A differential object $(X^{\bullet}, d_X^{\bullet})$ in $\mathcal{C}$ is a sequence of objects X^k and morphisms d^k ($k \in \mathbb{Z}$):

$$\cdots \to X^{k-1} \xrightarrow{d^{k-1}} X^k \xrightarrow{d^k} X^{k+1} \to \cdots . \tag{3.4}$$

A morphism of differential objects $f^{\bullet} : X^{\bullet} \to Y^{\bullet}$ is visualized by a commutative diagram:

$$\begin{tikzcd} \cdots \arrow[r] & X^n \arrow[r, "d_X^n"] \arrow[d, "f^n"] & X^{n+1} \arrow[r] \arrow[d, "f^{n+1}"] & \cdots \\ \cdots \arrow[r] & Y^n \arrow[r, "d_Y^n"] & X^{n+1} \arrow[r] & \cdots \end{tikzcd}$$

Hence, the category $Diff(\mathcal{C})$ of differential objects in $\mathcal{C}$ is nothing but the category $\mathrm{Fct}(\mathbb{Z}, \mathcal{C})$. In particular, it is an additive category.

Definition 3.2.1. (i) A complex is a differential object $(X^{\bullet}, d_X^{\bullet})$ such that $d^n \circ d^{n-1} = 0$ for all $n \in \mathbb{Z}$.

(ii) One denotes by $\mathrm{C}(\mathcal{C})$ the full additive subcategory of $Diff(\mathcal{C})$ consisting of complexes.

From now on, we shall concentrate our study on the category $\mathrm{C}(\mathcal{C})$.

A complex is bounded (resp. bounded below, bounded above) if $X^n = 0$ for $|n| >> 0$ (resp. $n << 0$, $n >> 0$). One denotes by $\mathrm{C}^*(\mathcal{C})(* = b, +, -)$ the full additive subcategory of $\mathrm{C}(\mathcal{C})$ consisting of bounded complexes (resp.

bounded below, bounded above). We also use the notation $\mathrm{C}^{\mathrm{ub}}(\mathcal{C}) = \mathrm{C}(\mathcal{C})$ (ub for "unbounded").

One considers $\mathcal{C}$ as a full subcategory of $\mathrm{C}^b(\mathcal{C})$ by identifying an object $X \in \mathcal{C}$ with the complex $X^\bullet$ "concentrated in degree 0":

$$X^\bullet := \cdots \to 0 \to X \to 0 \to \cdots$$

where X stands in degree 0.

Shift functor

Let $\mathcal{C}$ be an additive category, let $X \in \mathrm{C}(\mathcal{C})$ and let $p \in \mathbb{Z}$. One defines the shifted complex $X[p]$ by:

$$\begin{cases} (X[p])^n = X^{n+p} \\ d^n_{X[p]} = (-1)^p d^{n+p}_X \end{cases}$$

If $f\colon X \to Y$ is a morphism in $\mathrm{C}(\mathcal{C})$ one defines $f[p]\colon X[p] \to Y[p]$ by $(f[p])^n = f^{n+p}$.

The shift functor $[1]\colon X \mapsto X[1]$ is an automorphism (*i.e.* an invertible functor) of $\mathrm{C}(\mathcal{C})$.

Mapping cone

Definition 3.2.2. Let $f\colon X \to Y$ be a morphism in $\mathrm{C}(\mathcal{C})$. The mapping cone of f, denoted $\mathrm{Mc}(f)$, is the object of $\mathrm{C}(\mathcal{C})$ defined by:

$$\begin{aligned} \mathrm{Mc}(f)^n &= (X[1])^n \oplus Y^n \\ d^n_{\mathrm{Mc}(f)} &= \begin{pmatrix} d^n_{X[1]} & 0 \\ f^{n+1} & d^n_Y \end{pmatrix} \end{aligned}$$

Of course, before to state this definition, one should check that $d^{n+1}_{\mathrm{Mc}(f)} \circ d^n_{\mathrm{Mc}(f)} = 0$. Indeed:

$$\begin{pmatrix} -d^{n+2}_X & 0 \\ f^{n+2} & d^{n+1}_Y \end{pmatrix} \circ \begin{pmatrix} -d^{n+1}_X & 0 \\ f^{n+1} & d^n_Y \end{pmatrix} = 0$$

Notice that although $\mathrm{Mc}(f)^n = (X[1])^n \oplus Y^n$, $\mathrm{Mc}(f)$ is not isomorphic to $X[1] \oplus Y$ in $\mathrm{C}(\mathcal{C})$ unless f is the zero morphism.

There are natural morphisms of complexes

$$\alpha(f)\colon Y \to \mathrm{Mc}(f), \quad \beta(f)\colon \mathrm{Mc}(f) \to X[1]. \tag{3.5}$$

and $\beta(f) \circ \alpha(f) = 0$.

If $F\colon \mathcal{C} \to \mathcal{C}'$ is an additive functor, then $F(\mathrm{Mc}(f)) \simeq \mathrm{Mc}(F(f))$.

Homotopy

Let again $\mathcal{C}$ be an additive category.

Definition 3.2.3. (i) A morphism $f\colon X \to Y$ in $\mathrm{C}(\mathcal{C})$ is homotopic to zero if for all p there exists a morphism $s^p\colon X^p \to Y^{p-1}$ such that:

$$f^p = s^{p+1} \circ d_X^p + d_Y^{p-1} \circ s^p.$$

Two morphisms $f, g : X \to Y$ are homotopic if $f - g$ is homotopic to zero.

(ii) An object X in $\mathrm{C}(\mathcal{C})$ is homotopic to 0 if id_X is homotopic to zero.

A morphism homotopic to zero is visualized by the diagram (which is not commutative):

$$\begin{tikzcd}
X^{p-1} \arrow[r] & X^p \arrow[r, "d_X^p"] \arrow[dl, "s^p"'] \arrow[d, "f^p"] & X^{p+1} \arrow[dl, "s^{p+1}"] \\
Y^{p-1} \arrow[r, "d_Y^{p-1}"'] & Y^p \arrow[r] & Y^{p+1}
\end{tikzcd}$$

Note that an additive functor sends a morphism homotopic to zero to a morphism homotopic to zero.

Example 3.2.4. The complex $0 \to X' \to X' \oplus X'' \to X'' \to 0$ is homotopic to zero.

The homotopy category $\mathrm{K}(\mathcal{C})$

We shall construct a new category by deciding that a morphism in $\mathrm{C}(\mathcal{C})$ homotopic to zero is isomorphic to the zero morphism. Set:

$$Ht(X, Y) = \{f\colon X \to Y; f \text{ is homotopic to } 0\}.$$

If $f\colon X \to Y$ and $g\colon Y \to Z$ are two morphisms in $\mathrm{C}(\mathcal{C})$ and if f or g is homotopic to zero, then $g \circ f$ is homotopic to zero. This allows us to state:

Definition 3.2.5. The homotopy category $\mathrm{K}(\mathcal{C})$ is defined by:

$$\begin{aligned}
\mathrm{Ob}(\mathrm{K}(\mathcal{C})) &= \mathrm{Ob}(\mathrm{C}(\mathcal{C})) \\
\mathrm{Hom}_{\mathrm{K}(\mathcal{C})}(X, Y) &= \mathrm{Hom}_{\mathrm{C}(\mathcal{C})}(X, Y)/Ht(X, Y).
\end{aligned}$$

In other words, a morphism homotopic to zero in $\mathrm{C}(\mathcal{C})$ becomes the zero morphism in $\mathrm{K}(\mathcal{C})$ and a homotopy equivalence becomes an isomorphism.

One defines similarly $\mathrm{K}^*(\mathcal{C})$, $\quad (* = \mathrm{ub}, \mathrm{b}, +, -)$. They are clearly additive categories endowed with an automorphism, the shift functor $[1]\colon X \mapsto X[1]$.

3.3 Double complexes

Let $\mathcal{C}$ be as above an additive category. A double complex $(X^{\bullet,\bullet}, d_X)$ in $\mathcal{C}$ is the data of

$$\{X^{n,m}, d'^{n,m}_X, d''^{n,m}_X; (n,m) \in \mathbb{Z} \times \mathbb{Z}\}$$

where $X^{n,m} \in \mathcal{C}$ and the "differentials" $d'^{n,m}_X \colon X^{n,m} \to X^{n+1,m}$, $d''^{n,m}_X : X^{n,m} \to X^{n,m+1}$ satisfy:

$$d'^2_X = d''^2_X = 0,\ d' \circ d'' = d'' \circ d'. \tag{3.6}$$

One can represent a double complex by a commutative diagram:

(3.7)
$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & \\
{} \arrow[r] & X^{n,m} \arrow[r, "d''^{n,m}"] \arrow[d, "d'^{n,m}"'] & X^{n,m+1} \arrow[r] \arrow[d, "d'^{n,m+1}"'] & {} \\
{} \arrow[r] & X^{n+1,m} \arrow[r, "d''^{n+1,m}"'] \arrow[d] & X^{n+1,m+1} \arrow[r] \arrow[d] & {} \\
 & {} & {} &
\end{tikzcd}$$

One defines naturally the notion of a morphism of double complexes, and one obtains the additive category $\mathrm{C}^2(\mathcal{C})$ of double complexes.

There are two functors $F_I, F_{II} : \mathrm{C}^2(\mathcal{C}) \to \mathrm{C}(\mathrm{C}(\mathcal{C}))$ which associate to a double complex X the complex whose objects are the rows (resp. the columns) of X. These two functors are clearly isomorphisms of categories.

Now consider the finiteness condition:

$$\text{for all } p \in \mathbb{Z}, \quad \{(m,n) \in \mathbb{Z} \times \mathbb{Z}; X^{n,m} \neq 0, m+n=p\} \text{ is finite} \tag{3.8}$$

and denote by $C^2_f(\mathcal{C})$ the full subcategory of $\mathrm{C}^2(\mathcal{C})$ consisting of objects X satisfying (3.8). To such an X one associates its "total complex" $\mathrm{tot}(X)$ by setting:

$$\begin{aligned}
\mathrm{tot}(X)^p &= \oplus_{m+n=p} X^{n,m}, \\
d^p_{\mathrm{tot}(X)}|_{X^{n,m}} &= d'^{n,m} + (-1)^n d''^{n,m}.
\end{aligned}$$

This is visualized by the diagram:

$$\begin{tikzcd}
X^{n,m} \arrow[r, "(-)^n d''"] \arrow[d, "d'"'] & X^{n,m+1} \\
X^{n+1,m} &
\end{tikzcd}$$

[1]§ 3.3 may be skipped in a first reading.

Proposition 3.3.1. *The differential object* $\{\mathrm{tot}(X)^p, d^p_{\mathrm{tot}(X)}\}_{p\in\mathbb{Z}}$ *is a complex* (i.e., $d^{p+1}_{\mathrm{tot}(X)} \circ d^p_{\mathrm{tot}(X)} = 0$) *and* $\mathrm{tot} : C^2_f(\mathcal{C}) \to \mathrm{C}(\mathcal{C})$ *is a functor of additive categories.*

Proof. For $(n, m) \in \mathbb{Z} \times \mathbb{Z}$, one has

$$\begin{aligned} d \circ d(X^{n,m}) &= d'' \circ d''(X^{n,m}) + d' \circ d'(X^{n,m}) \\ &\quad +(-)^{n+1} d'' \circ d'(X^{n,m}) + (-)^n d' \circ d''(X^{n,m}) \\ &= 0. \end{aligned}$$

It is left to the reader to check that tot is an additive functor. q.e.d.

Example 3.3.2. Let $f^\bullet : X^\bullet \to Y^\bullet$ be a morphism in $\mathrm{C}(\mathcal{C})$. Consider the double complex $Z^{\bullet,\bullet}$ such that $Z^{-1,\bullet} = X^\bullet$, $Z^{0,\bullet} = Y^\bullet$, $Z^{i,\bullet} = 0$ for $i \neq -1, 0$, with differentials $f^j : Z^{-1,j} \to Z^{0,j}$. Then

$$\mathrm{tot}(Z^{\bullet,\bullet}) \simeq \mathrm{Mc}(f^\bullet). \tag{3.9}$$

Bifunctor

Let $\mathcal{C}, \mathcal{C}'$ and $\mathcal{C}''$ be additive categories and let $F : \mathcal{C} \times \mathcal{C}' \to \mathcal{C}''$ be an additive bifunctor (*i.e.,* $F(\,\bullet\,,\,\bullet\,)$ is additive with respect to each argument). It defines an additive bifunctor $\mathrm{C}^2(F) : \mathrm{C}(\mathcal{C}) \times \mathrm{C}(\mathcal{C}') \to \mathrm{C}^2(\mathcal{C}'')$. In other words, if $X \in \mathrm{C}(\mathcal{C})$ and $X' \in \mathrm{C}(\mathcal{C}')$ are complexes, then $\mathrm{C}^2(F)(X, X')$ is a double complex.

Example 3.3.3. Consider the bifunctor $\bullet \otimes \bullet : \mathrm{Mod}(A^{\mathrm{op}}) \times \mathrm{Mod}(A) \to \mathrm{Mod}(\mathbb{Z})$. We shall simply write $\otimes$ instead of $\mathrm{C}^2(\otimes)$. Hence, for $X \in \mathrm{C}^-(\mathrm{Mod}(A^{\mathrm{op}}))$ and $Y \in \mathrm{C}^-(\mathrm{Mod}(A))$, one has

$$\begin{aligned} &(X \otimes Y)^{n,m} = X^n \otimes Y^m, \\ &d'^{n,m} = d^n_X \otimes Y^m, d''^{n,m} = X^n \otimes d^m_Y. \end{aligned}$$

The complex $\mathrm{Hom}^\bullet$

Consider the bifunctor $\mathrm{Hom}_{\mathcal{C}} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Mod}(\mathbb{Z})$. We shall write $\mathrm{Hom}^{\bullet,\bullet}_{\mathcal{C}}$ instead of $\mathrm{C}^2(\mathrm{Hom}_{\mathcal{C}})$. If X and Y are two objects of $\mathrm{C}(\mathcal{C})$, one has

$$\begin{aligned} \mathrm{Hom}^{\bullet,\bullet}_{\mathcal{C}}(X, Y)^{n,m} &\qquad = \mathrm{Hom}_{\mathcal{C}}(X^{-m}, Y^n), \\ d'^{n,m} = \mathrm{Hom}_{\mathcal{C}}(X^{-m}, d^n_Y), &\qquad d''^{m,n} = \mathrm{Hom}_{\mathcal{C}}((-)^m d^{-m-1}_X, Y^n). \end{aligned}$$

Note that $\mathrm{Hom}^{\bullet,\bullet}_{\mathcal{C}}(X, Y)$ is a double complex in the category $\mathrm{Mod}(\mathbb{Z})$ and should not be confused with the group $\mathrm{Hom}_{\mathrm{C}(\mathcal{C})}(X, Y)$.

Let $X \in \mathrm{C}^-(\mathcal{C})$ and $Y \in \mathrm{C}^+(\mathcal{C})$. One sets

$$\mathrm{Hom}^{\bullet}_{\mathcal{C}}(X,Y) = \mathrm{tot}(\mathrm{Hom}^{\bullet,\bullet}_{\mathcal{C}}(X,Y)). \tag{3.10}$$

Hence, $\mathrm{Hom}_{\mathcal{C}}(X,Y)^n = \bigoplus_j \mathrm{Hom}_{\mathcal{C}}(X^j, Y^{n+j})$ and

$$d^n \quad : \quad \mathrm{Hom}_{\mathcal{C}}(X,Y)^n \to \mathrm{Hom}_{\mathcal{C}}(X,Y)^{n+1}$$

is defined as follows. To $f = \{f^j\}_j \in \bigoplus_{j\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(X^j, Y^{n+j})$ one associates

$$d^n f = \{g^j\}_j \in \bigoplus_{j\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(X^j, Y^{n+j+1}),$$

with

$$g^j = d'^{n+j,-j} f^j + (-)^{j+n+1} d''^{j+n+1,-j-1} f^{j+1}$$

In other words, the components of df in $\mathrm{Hom}_{\mathcal{C}}(X,Y)^{n+1}$ will be

$$(d^n f)^j = d_Y^{j+n} \circ f^j + (-)^{n+1} f^{j+1} \circ d_X^j. \tag{3.11}$$

Note that for $X, Y, Z \in \mathrm{C}(\mathcal{C})$, there is a natural composition map

$$\mathrm{Hom}^{\bullet}_{\mathcal{C}}(X,Y) \otimes \mathrm{Hom}^{\bullet}_{\mathcal{C}}(Y,Z) \xrightarrow{\circ} \mathrm{Hom}^{\bullet}_{\mathcal{C}}(X,Z). \tag{3.12}$$

Proposition 3.3.4. *Let $\mathcal{C}$ be an additive category and let $X, Y \in \mathrm{C}(\mathcal{C})$. There are isomorphisms:*

$$\begin{aligned}
Z^0(\mathrm{Hom}^{\bullet}_{\mathcal{C}}(X,Y)) = \mathrm{Ker}\, d^0 &\simeq \mathrm{Hom}_{\mathrm{C}(\mathcal{C})}(X,Y),\\
B^0(\mathrm{Hom}^{\bullet}_{\mathcal{C}}(X,Y)) = \mathrm{Im}\, d^{-1} &\simeq Ht(X,Y),\\
H^0(\mathrm{Hom}^{\bullet}_{\mathcal{C}}(X,Y)) = \mathrm{Ker}\, d^0 / \mathrm{Im}\, d^{-1} &\simeq \mathrm{Hom}_{\mathrm{K}(\mathcal{C})}(X,Y).
\end{aligned}$$

Proof. (i) Let us calculate $Z^0(\mathrm{Hom}^{\bullet}_{\mathcal{C}}(X,Y))$. By (3.11), the component of $d^0\{f^j\}_j$ in $\mathrm{Hom}_{\mathcal{C}}(X^j, Y^{j+1})$ will be zero if and only if $d_Y^j \circ f^j = f^{j+1} \circ d_X^j$, that is, if the family $\{f^j\}_j$ defines a morphism of complexes.
(ii) Let us calculate $B^0(\mathrm{Hom}^{\bullet}_{\mathcal{C}}(X,Y))$. An element $f^j \in \mathrm{Hom}_{\mathcal{C}}(X^j, Y^j)$ will be in the image of d^{-1} if it is in the sum of the image of $\mathrm{Hom}_{\mathcal{C}}(X^j, Y^{j-1})$ by d_Y^{j-1} and the image of $\mathrm{Hom}_{\mathcal{C}}(X^{j+1}, Y^j)$ by d_X^j. Hence, if it can be written as $f^j = d_Y^{j-1} \circ s^j + s^{j+1} \circ d_X^j$. q.e.d.

Remark 3.3.5. Roughly speaking, a DG-category is an additive category in which the morphisms are no more additive groups but are complexes of such groups.

The category $\mathrm{C}(\mathcal{C})$ endowed for each $X, Y \in \mathrm{C}(\mathcal{C})$ of the complex $\mathrm{Hom}^{\bullet}_{\mathcal{C}}(X,Y))$ and with the composition given by (3.12) is an example of such a DG-category.

3.4 Simplicial constructions

We shall define the simplicial category and use it to construct complexes and homotopies in additive categories.

Definition 3.4.1. (a) The simplicial category, denoted by $\mathbf{\Delta}$, is the category whose objects are the finite totally ordered sets and the morphisms are the order-preserving maps.

(b) We denote by $\mathbf{\Delta}_{inj}$ the subcategory of $\mathbf{\Delta}$ such that $\mathrm{Ob}(\mathbf{\Delta}_{inj}) = \mathrm{Ob}(\mathbf{\Delta})$, the morphisms being the injective order-preserving maps.

For integers n, m denote by $[n, m]$ the totally ordered set $\{k \in \mathbb{Z}; n \leq k \leq m\}$.

Proposition 3.4.2. (i) *the natural functor $\mathbf{\Delta} \to \mathbf{Set}^f$ is faithful,*

(ii) *the full subcategory of $\mathbf{\Delta}$ consisting of objects $\{[0, n]\}_{n \geq -1}$ is equivalent to $\mathbf{\Delta}$,*

(iii) *$\mathbf{\Delta}$ admits an initial object, namely $\emptyset$, and a terminal object, namely $\{0\}$.*

The proof is obvious.

Let us denote by

$$d_i^n \colon [0, n] \to [0, n+1] \qquad (0 \leq i \leq n+1)$$

the injective order-preserving map which does not take the value i. In other words

$$d_i^n(k) = \begin{cases} k & \text{for } k < i, \\ k+1 & \text{for } k \geq i. \end{cases}$$

One checks immediately that

$$d_j^{n+1} \circ d_i^n = d_i^{n+1} \circ d_{j-1}^n \text{ for } 0 \leq i < j \leq n+2. \tag{3.13}$$

Indeed, both morphisms are the unique injective order-preserving map which does not take the values i and j.

The category $\mathbf{\Delta}_{inj}$ is visualized by

$$\begin{tikzcd} \emptyset \arrow[r, "d_0^{-1}"] & {[0]} \arrow[r, shift left, "d_0^0"] \arrow[r, shift right, "d_1^0"'] & {[0,1]} \arrow[r, shift left=2, "d_0^1"] \arrow[r, "d_1^1"] \arrow[r, shift right=2, "d_2^1"'] & {[0,1,2]} \arrow[r, dotted, shift left=3] \arrow[r, dotted, shift left=1] \arrow[r, dotted, shift right=1] \arrow[r, dotted, shift right=3] & {} \end{tikzcd} \tag{3.14}$$

[1] § 3.4 may be skipped.

Let $\mathcal{C}$ be an additive category and $F\colon \mathbf{\Delta}_{inj} \to \mathcal{C}$ a functor. We set for $n \in \mathbb{Z}$:

$$F^n = \begin{cases} F([0,n]) & \text{for } n \geq -1, \\ 0 & \text{otherwise,} \end{cases}$$

$$d_F^n\colon F^n \to F^{n+1}, \quad d_F^n = \sum_{i=0}^{n+1} (-)^i F(d_i^n).$$

Consider the differential object

$$F^\bullet := \cdots \to 0 \to F^{-1} \xrightarrow{d_F^{-1}} F^0 \xrightarrow{d_F^0} F^1 \to \cdots \to F^n \xrightarrow{d_F^n} \cdots . \tag{3.15}$$

Theorem 3.4.3. (i) *The differential object $F^\bullet$ is a complex.*

(ii) *Assume that there exist morphisms $s_F^n\colon F^n \to F^{n-1}$ $(n \geq 0)$satisfying:*

$$\begin{cases} s_F^{n+1} \circ F(d_0^n) = \mathrm{id}_{F^n} & \textit{for } n \geq -1, \\ s_F^{n+1} \circ F(d_{i+1}^n) = F(d_i^{n-1}) \circ s_F^n & \textit{for } i > 0, n \geq 0. \end{cases}$$

Then $F^\bullet$ is homotopic to zero.

Proof. (i) By (3.13), we have

$$\begin{aligned} d_F^{n+1} \circ d_F^n &= \sum_{j=0}^{n+2} \sum_{i=0}^{n+1} (-)^{i+j} F(d_j^{n+1} \circ d_i^n) \\ &= \sum_{0 \leq j \leq i \leq n+1} (-)^{i+j} F(d_j^{n+1} \circ d_i^n) + \sum_{0 \leq i < j \leq n+2} (-)^{i+j} F(d_j^{n+1} \circ d_i^n) \\ &= \sum_{0 \leq j \leq i \leq n+1} (-)^{i+j} F(d_j^{n+1} \circ d_i^n) + \sum_{0 \leq i < j \leq n+2} (-)^{i+j} F(d_i^{n+1} \circ d_{j-1}^n) \\ &= 0\,. \end{aligned}$$

Here, we have used

$$\begin{aligned} \sum_{0 \leq i < j \leq n+2} (-)^{i+j} F(d_i^{n+1} \circ d_{j-1}^n) &= \sum_{0 \leq i < j \leq n+1} (-)^{i+j+1} F(d_i^{n+1} \circ d_j^n) \\ &= \sum_{0 \leq j \leq i \leq n+1} (-)^{i+j+1} F(d_j^{n+1} \circ d_i^n). \end{aligned}$$

(ii) We have

$$\begin{aligned}
&s_F^{n+1} \circ d_F^n + d_F^{n-1} \circ s^n \\
&\quad = \sum_{i=0}^{n+1} (-1)^i s_F^{n+1} \circ F(d_i^n) + \sum_{i=0}^{n} (-1)^i F(d_i^{n-1} \circ s_F^n) \\
&\quad = s_F^{n+1} \circ F(d_0^n) + \sum_{i=0}^{n} (-1)^{i+1} s_F^{n+1} \circ F(d_{i+1}^n) + \sum_{i=0}^{n} (-1)^i F(d_i^{n-1} \circ s_F^n) \\
&\quad = \mathrm{id}_{F^n} + \sum_{i=0}^{n} (-1)^{i+1} F(d_i^{n-1} \circ s_F^n) + \sum_{i=0}^{n} (-1)^i F(d_i^{n-1} \circ s_F^n) \\
&\quad = \mathrm{id}_{F^n}.
\end{aligned}$$

q.e.d.

Exercises to Chapter 3

Exercise 3.1. Let $\mathcal{C}$ be an additive category and let $X \in \mathrm{C}(\mathcal{C})$ with differential d_X. Define the morphism $\delta_X \colon X \to X[1]$ by setting $\delta_X^n = (-1)^n d_X^n$. Prove that δ_X is a morphism in $\mathrm{C}(\mathcal{C})$ and is homotopic to zero.

Exercise 3.2. Let $\mathcal{C}$ be an additive category, $f, g \colon X \rightrightarrows Y$ two morphisms in $\mathrm{C}(\mathcal{C})$. Prove that f and g are homotopic if and only if there exists a commutative diagram in $\mathrm{C}(\mathcal{C})$

$$\begin{tikzcd}
Y \arrow[r, "\alpha(f)"'] \arrow[d, equal] & \mathrm{Mc}(f) \arrow[r, "\beta(f)"'] \arrow[d, "u"] & X[1] \arrow[d, equal] \\
Y \arrow[r, "\alpha(g)"'] & \mathrm{Mc}(g) \arrow[r, "\beta(g)"'] & X[1].
\end{tikzcd}$$

In such a case, prove that u is an isomorphism in $\mathrm{C}(\mathcal{C})$.

Exercise 3.3. Let $\mathcal{C}$ be an additive category and let $f \colon X \to Y$ be a morphism in $\mathrm{C}(\mathcal{C})$.
Prove that the following conditions are equivalent:

(a) f is homotopic to zero,

(b) f factors through $\alpha(\mathrm{id}_X) \colon X \to \mathrm{Mc}(\mathrm{id}_X)$,

(c) f factors through $\beta(\mathrm{id}_Y)[-1] \colon \mathrm{Mc}(\mathrm{id}_Y)[-1] \to Y$,

(d) f decomposes as $X \to Z \to Y$ with Z a complex homotopic to zero.

Exercise 3.4. A category with translation $(\mathcal{A}, T)$ is a category $\mathcal{A}$ together with an equivalence $T\colon \mathcal{A} \to \mathcal{A}$. A differential object (X, d_X) in a category with translation $(\mathcal{A}, T)$ is an object $X \in \mathcal{A}$ together with a morphism $d_X\colon X \to T(X)$. A morphism $f\colon (X, d_X) \to (Y, d_Y)$ of differential objects is a commutative diagram

$$\begin{tikzcd} X \arrow[r, "d_X"] \arrow[d, "f"] & TX \arrow[d, "T(f)"] \\ Y \arrow[r, "d_Y"] & TY. \end{tikzcd}$$

One denotes by $\mathcal{A}_d$ the category consisting of differential objects and morphisms of such objects. If $\mathcal{A}$ is additive, one says that a differential object (X, d_X) in $(\mathcal{A}, T)$ is a complex if the composition $X \xrightarrow{d_X} T(X) \xrightarrow{T(d_X)} T^2(X)$ is zero. One denotes by $\mathcal{A}_c$ the full subcategory of $\mathcal{A}_d$ consisting of complexes.
(i) Let $\mathcal{C}$ be a category. Denote by $\mathbb{Z}_d$ the set $\mathbb{Z}$ considered as a discrete category and still denote by $\mathbb{Z}$ the ordered set $(\mathbb{Z}, \leq)$ considered as a category. Prove that $\mathcal{C}^{\mathbb{Z}} := \mathrm{Fct}(\mathbb{Z}_d, \mathcal{C})$ is a category with translation.
(ii) Show that the category $\mathrm{Fct}(\mathbb{Z}, \mathcal{C})$ may be identified to the category of differential objects in $\mathcal{C}^{\mathbb{Z}}$.
(iii) Let $\mathcal{C}$ be an additive category. Show that the notions of differential objects and complexes given above coincide with those in Definition 3.2.1 when choosing $\mathcal{A} = \mathrm{C}(\mathcal{C})$ and $T = [1]$.

Exercise 3.5. Consider the catgeory $\boldsymbol{\Delta}$ and for $n > 0$, denote by

$$s_i^n\colon [0, n] \to [0, n-1] \qquad (0 \leq i \leq n-1)$$

the surjective order-preserving map which takes the same value at i and $i+1$. In other words

$$s_i^n(k) \;=\; \begin{cases} k & \text{for } k \leq i, \\ k-1 & \text{for } k > i. \end{cases}$$

Checks the relations:

$$\begin{cases} s_j^n \circ s_i^{n+1} = s_{i-1}^n \circ s_j^{n+1} & \text{for } 0 \leq j < i \leq n, \\ s_j^{n+1} \circ d_i^n = d_i^{n-1} \circ s_{j-1}^n & \text{for } 0 \leq i < j \leq n, \\ s_j^{n+1} \circ d_i^n = \mathrm{id}_{[0,n]} & \text{for } 0 \leq i \leq n+1, i = j, j+1, \\ s_j^{n+1} \circ d_i^n = d_{i-1}^{n-1} \circ s_j^n & \text{for } 1 \leq j+1 < i \leq n+1. \end{cases}$$

Chapter 4

Abelian categories

Convention 4.0.4. In these Notes, when dealing with an abelian category $\mathcal{C}$ (see Definition 4.1.2 below), we shall assume that $\mathcal{C}$ is a full abelian subcategory of a category $\mathrm{Mod}(A)$ for some ring A. This makes the proofs much easier and moreover there exists a famous theorem (due to Freyd & Mitchell) that asserts that this is in fact always the case (up to equivalence of categories).

4.1 Abelian categories

Let $\mathcal{C}$ be an additive category which admits kernels and cokernels. Let $f\colon X \to Y$ be a morphism in $\mathcal{C}$. One defines:

$$\begin{aligned} \mathrm{Coim}\, f &:= \mathrm{Coker}\, h, \text{ where } h\colon \mathrm{Ker}\, f \to X \\ \mathrm{Im}\, f &:= \mathrm{Ker}\, k, \text{ where } k\colon Y \to \mathrm{Coker}\, f. \end{aligned}$$

Consider the diagram:

$$\begin{tikzcd} \mathrm{Ker}\, f \arrow[r, "h"] & X \arrow[r, "f"] \arrow[d, "s"'] & Y \arrow[r, "k"] & \mathrm{Coker}\, f \\ & \mathrm{Coim}\, f \arrow[ur, dotted, "\tilde{f}"] \arrow[r, dotted, "u"] & \mathrm{Im}\, f \arrow[u] & \end{tikzcd}$$

Since $f \circ h = 0$, f factors uniquely through $\tilde{f}$, and $k \circ f$ factors through $k \circ \tilde{f}$. Since $k \circ f = k \circ \tilde{f} \circ s = 0$ and s is an epimorphism, we get that $k \circ \tilde{f} = 0$. Hence $\tilde{f}$ factors through $\mathrm{Ker}\, k = \mathrm{Im}\, f$. We have thus constructed a canonical morphism:

$$\mathrm{Coim}\, f \xrightarrow{u} \mathrm{Im}\, f. \tag{4.1}$$

Examples 4.1.1. (i) For a ring A and a morphism f in $\mathrm{Mod}(A)$, (4.1) is an isomorphism.
(ii) The category **Ban** admits kernels and cokernels. If $f\colon X \to Y$ is a morphism of Banach spaces, define $\mathrm{Ker}\, f = f^{-1}(0)$ and $\mathrm{Coker}\, f = Y/\overline{\mathrm{Im}\, f}$ where $\overline{\mathrm{Im}\, f}$ denotes the closure of the space $\mathrm{Im}\, f$. It is well-known that there exist continuous linear maps $f\colon X \to Y$ which are injective, with dense and non closed image. For such an f, $\mathrm{Ker}\, f = \mathrm{Coker}\, f = 0$ although f is not an isomorphism. Thus $\mathrm{Coim}\, f \simeq X$ and $\mathrm{Im}\, f \simeq Y$. Hence, the morphism (4.1) is not an isomorphism.
(iii) Let A be a ring, I an ideal which is not finitely generated and let $M = A/I$. Then the natural morphism $A \to M$ in $\mathrm{Mod}^{\mathrm{f}}(A)$ has no kernel.

Definition 4.1.2. Let $\mathcal{C}$ be an additive category. One says that $\mathcal{C}$ is abelian if:

(i) any $f\colon X \to Y$ admits a kernel and a cokernel,

(ii) for any morphism f in $\mathcal{C}$, the natural morphism $\mathrm{Coim}\, f \to \mathrm{Im}\, f$ is an isomorphism.

Examples 4.1.3. (i) If A is a ring, $\mathrm{Mod}(A)$ is an abelian category. If A is noetherian, then $\mathrm{Mod}^{\mathrm{f}}(A)$ is abelian.
(ii) The category **Ban** admits kernels and cokernels but is not abelian. (See Examples 4.1.1 (ii).)
(iii) If $\mathcal{C}$ is abelian, then $\mathcal{C}^{\mathrm{op}}$ is abelian.

Proposition 4.1.4. *Let I be category and let $\mathcal{C}$ be an abelian category. Then the category $\mathrm{Fct}(I, \mathcal{C})$ of functors from I to $\mathcal{C}$ is abelian.*

Proof. (i) Let $F, G\colon I \to \mathcal{C}$ be two functors and $\varphi\colon F \to G$ a morphism of functors. Let us define a new functor H as follows. For $i \in I$, set $H(i) = \mathrm{Ker}(F(i) \to G(i))$. Let $s\colon i \to j$ be a morphism in I. In order to define the morphism $H(s)\colon H(i) \to H(j)$, consider the diagram

$$\begin{tikzcd} H(i) \arrow[r, "h_i"] \arrow[d, dotted, "H(s)"'] & F(i) \arrow[r, "\varphi(i)"] \arrow[d, "F(s)"'] & G(i) \arrow[d, "G(s)"] \\ H(j) \arrow[r, "h_j"] & F(j) \arrow[r, "\varphi(i)"] & G(j). \end{tikzcd}$$

Since $\varphi(j) \circ F(s) \circ h_i = 0$, the morphism $F(s) \circ h_i$ factorizes uniquely through $H(j)$. This gives $H(s)$. One checks immediately that for a morphism $t\colon j \to k$ in I, one has $H(t) \circ H(s) = H(t \circ s)$. Therefore H is a functor and one also easily cheks that H is a kernel of the morphism of functors φ.

(ii) One defines similarly the functor $\operatorname{Coim}\varphi$. Since, for each $i \in I$, the natural morphism $\operatorname{Coim}\varphi(i) \to \operatorname{Im}\varphi(i)$ is an isomorphism, one deduces that the natural morphism of functors $\operatorname{Coim}\varphi \to \operatorname{Im}\varphi$ is an isomorphism. q.e.d.

Corollary 4.1.5. *If $\mathcal{C}$ is abelian, then the categories of complexes* $\mathrm{C}^*(\mathcal{C})$ $(* = \mathrm{ub}, \mathrm{b}, +, -)$ *are abelian.*

Proof. It follows from Proposition 4.1.4 that the category $Diff(\mathcal{C})$ of differential objects of $\mathcal{C}$ is abelian. One checks immediately that if $f^\bullet : X^\bullet \to Y^\bullet$ is a morphism of complexes, its kernel in the category $Diff(\mathcal{C})$ is a complex and is a kernel in the category $\mathrm{C}(\mathcal{C})$, and similarly with cokernels. q.e.d.

For example, if $f \colon X \to Y$ is a morphism in $\mathrm{C}(\mathcal{C})$, the complex Z defined by $Z^n = \operatorname{Ker}(f^n \colon X^n \to Y^n)$, with differential induced by those of X, will be a kernel for f, and similarly for $\operatorname{Coker} f$.

Note the following results.

- An abelian category admits finite projective limits and finite inductive limits. (Indeed, an abelian category admits an initial object, a terminal object, finite products and finite coproducts and kernels and cokernels.)
- In an abelian category, a morphism f is a monomorphism (resp. an epimorphism) if and only if $\operatorname{Ker} f \simeq 0$ (resp. $\operatorname{Coker} f \simeq 0$) (see Exercise 2.12). Moreover, a morphism $f \colon X \to Y$ is an isomorphism as soon as $\operatorname{Ker} f \simeq 0$ and $\operatorname{Coker} f \simeq 0$. Indeed, in such a case, $X \xrightarrow{\sim} \operatorname{Coim} f$ and $\operatorname{Im} f \xrightarrow{\sim} Y$.

Unless otherwise specified, we assume until the end of this chapter that $\mathcal{C}$ is abelian.

Consider a complex $X' \xrightarrow{f} X \xrightarrow{g} X''$ (hence, $g \circ f = 0$). It defines a morphism $\operatorname{Coim} f \to \operatorname{Ker} g$, hence, $\mathcal{C}$ being abelian, a morphism $\operatorname{Im} f \to \operatorname{Ker} g$.

Definition 4.1.6. (i) One says that a complex $X' \xrightarrow{f} X \xrightarrow{g} X''$ is exact if $\operatorname{Im} f \xrightarrow{\sim} \operatorname{Ker} g$.

(ii) More generally, a sequence of morphisms $X^p \xrightarrow{d^p} \cdots \to X^n$ with $d^{i+1} \circ d^i = 0$ for all $i \in [p, n-1]$ is exact if $\operatorname{Im} d^i \xrightarrow{\sim} \operatorname{Ker} d^{i+1}$ for all $i \in [p, n-1]$.

(iii) A short exact sequence is an exact sequence $0 \to X' \to X \to X'' \to 0$

Any morphism $f \colon X \to Y$ may be decomposed into short exact sequences:

$$0 \to \operatorname{Ker} f \to X \to \operatorname{Coim} f \to 0,$$
$$0 \to \operatorname{Im} f \to Y \to \operatorname{Coker} f \to 0,$$

with $\operatorname{Coim} f \simeq \operatorname{Im} f$.

Proposition 4.1.7. *Let*

$$0 \to X' \xrightarrow{f} X \xrightarrow{g} X'' \to 0 \tag{4.2}$$

be a short exact sequence in $\mathcal{C}$. Then the conditions (a) *to* (e) *are equivalent.*

(a) *there exists $h\colon X'' \to X$ such that $g \circ h = \mathrm{id}_{X''}$.*

(b) *there exists $k\colon X \to X'$ such that $k \circ f = \mathrm{id}_{X'}$.*

(c) *there exists $\varphi = (k, g)$ and $\psi = \begin{pmatrix} f \\ h \end{pmatrix}$ such that $X \xrightarrow{\varphi} X' \oplus X''$ and $X' \oplus X'' \xrightarrow{\psi} X$ are isomorphisms inverse to each other.*

(d) *The complex* (4.2) *is homotopic to* 0*.*

(e) *The complex* (4.2) *is isomorphic to the complex $0 \to X' \to X' \oplus X'' \to X'' \to 0$.*

Proof. (a) $\Rightarrow$ (c). Since $g = g \circ h \circ g$, we get $g \circ (\mathrm{id}_X - h \circ g) = 0$, which implies that $\mathrm{id}_X - h \circ g$ factors through $\operatorname{Ker} g$, that is, through X'. Hence, there exists $k\colon X \to X'$ such that $\mathrm{id}_X - h \circ g = f \circ k$.
(b) $\Rightarrow$ (c) follows by reversing the arrows.
(c) $\Rightarrow$ (a). Since $g \circ f = 0$, we find $g = g \circ h \circ g$, that is $(g \circ h - \mathrm{id}_{X''}) \circ g = 0$. Since g is an epimorphism, this implies $g \circ h - \mathrm{id}_{X''} = 0$.
(c) $\Rightarrow$ (b) follows by reversing the arrows.
(d) By definition, the complex (4.2) is homotopic to zero if and only if there exists a diagram

$$\begin{tikzcd}
0 \arrow[r] & X' \arrow[r, "f"] \arrow[d, "\mathrm{id}"'] & X \arrow[r, "g"] \arrow[d, "\mathrm{id}"] \arrow[dl, "k"'] & X'' \arrow[r] \arrow[d, "\mathrm{id}"] \arrow[dl, "h"'] & 0 \\
0 \arrow[r] & X' \arrow[r, "f"'] & X \arrow[r, "g"'] & X'' \arrow[r] & 0
\end{tikzcd}$$

such that $\mathrm{id}_{X'} = k \circ f$, $\mathrm{id}_{X''} = g \circ h$ and $\mathrm{id}_X = h \circ g + f \circ k$.
(e) is obvious by (c). q.e.d.

Definition 4.1.8. In the above situation, one says that the exact sequence splits.

Note that an additive functor of abelian categories sends split exact sequences into split exact sequences.

If A is a field, all exact sequences split, but this is not the case in general. For example, the exact sequence of $\mathbb{Z}$-modules

$$0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$$

does not split.

4.2 Exact functors

Definition 4.2.1. Let $F\colon \mathcal{C} \to \mathcal{C}'$ be a functor of abelian categories. One says that

(i) F is left exact if it commutes with finite projective limits,

(ii) F is right exact if it commutes with finite inductive limits,

(iii) F is exact if it is both left and right exact.

Lemma 4.2.2. *Consider an additive functor $F\colon \mathcal{C} \to \mathcal{C}'$.*

(a) *The conditions below are equivalent:*

  (i) *F is left exact,*

  (ii) *F commutes with kernels, that is, for any morphism $f\colon X \to Y$, $F(\mathrm{Ker}(f)) \xrightarrow{\sim} \mathrm{Ker}(F(f))$,*

  (iii) *for any exact sequence $0 \to X' \to X \to X''$ in $\mathcal{C}$, the sequence $0 \to F(X') \to F(X) \to F(X'')$ is exact in $\mathcal{C}'$,*

  (iv) *for any exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathcal{C}$, the sequence $0 \to F(X') \to F(X) \to F(X'')$ is exact in $\mathcal{C}'$.*

(b) *The conditions below are equivalent:*

  (i) *F is exact,*

  (ii) *for any exact sequence $X' \to X \to X''$ in $\mathcal{C}$, the sequence $F(X') \to F(X) \to F(X'')$ is exact in $\mathcal{C}'$,*

  (iii) *for any exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathcal{C}$, the sequence $0 \to F(X') \to F(X) \to F(X'') \to 0$ is exact in $\mathcal{C}'$.*

There is a similar result to (a) for right exact functors.

Proof. Since F is additive, it commutes with terminal objects and products of two objects. Hence, by Proposition 2.3.7, F is left exact if and only if it commutes with kernels.
The proof of the other assertions are left as an exercise. q.e.d.

Proposition 4.2.3. (i) *The functor $\mathrm{Hom}_{\mathcal{C}}\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Mod}(\mathbb{Z})$ is left exact with respect to each of its arguments.*

(ii) *If a functor $F\colon \mathcal{C} \to \mathcal{C}'$ admits a left (resp. right) adjoint then F is left (resp. right) exact.*

(iii) *Let I be a category. The functor* $\varprojlim : \mathrm{Fct}(I^{\mathrm{op}}, \mathcal{C}) \to \mathcal{C}$ *is left exact and the functor* $\varinjlim : \mathrm{Fct}(I, \mathcal{C}) \to \mathcal{C}$ *is right exact.*

(iv) *Let A be a ring and let I be a set. The two functors* $\prod$ *and* $\bigoplus$ *from* $\mathrm{Fct}(I, \mathrm{Mod}(A))$ *to* $\mathrm{Mod}(A)$ *are exact.*

(v) *Let A be a ring and let I be a filtrant category. The functor* $\varinjlim$ *from* $\mathrm{Fct}(I, \mathrm{Mod}(A))$ *to* $\mathrm{Mod}(A)$ *is exact.*

Proof. (i) follows from (2.10) and (2.11).
(ii) Apply Proposition 2.4.5.
(iii) Apply Proposition 2.4.1.
(iv) is left as an exercise (see Exercise 4.1).
(v) follows from Corollary 2.5.7. q.e.d.

Example 4.2.4. Let A be a ring and let N be a right A-module. Since the functor $N \otimes_A \bullet$ admits a right adjoint, it is right exact. Let us show that the functors $\mathrm{Hom}_A(\bullet, \bullet)$ and $N \otimes_A \bullet$ are not exact in general. In the sequel, we choose $A = \mathbf{k}[x]$, with $\mathbf{k}$ a field, and we consider the exact sequence of A-modules:

$$0 \to A \xrightarrow{\cdot x} A \to A/Ax \to 0, \tag{4.3}$$

where $\cdot x$ means multiplication by x.
(i) Apply the functor $\mathrm{Hom}_A(\bullet, A)$ to the exact sequence (4.3). We get the sequence:

$$0 \to \mathrm{Hom}_A(A/Ax, A) \to A \xrightarrow{x\cdot} A \to 0$$

which is not exact since $x\cdot$ is not surjective. On the other hand, since $x\cdot$ is injective and $\mathrm{Hom}_A(\bullet, A)$ is left exact, we find that $\mathrm{Hom}_A(A/Ax, A) = 0$.
(ii) Apply $\mathrm{Hom}_A(A/Ax, \bullet)$ to the exact sequence (4.3). We get the sequence:

$$0 \to \mathrm{Hom}_A(A/Ax, A) \to \mathrm{Hom}_A(A/Ax, A) \to \mathrm{Hom}_A(A/Ax, A/Ax) \to 0.$$

Since $\mathrm{Hom}_A(A/Ax, A) = 0$ and $\mathrm{Hom}_A(A/Ax, A/Ax) \neq 0$, this sequence is not exact.
(iii) Apply $\bullet \otimes_A A/Ax$ to the exact sequence (4.3). We get the sequence:

$$0 \to A/Ax \xrightarrow{x\cdot} A/Ax \to A/xA \otimes_A A/Ax \to 0.$$

Multiplication by x is 0 on A/Ax. Hence this sequence is the same as:

$$0 \to A/Ax \xrightarrow{0} A/Ax \to A/Ax \otimes_A A/Ax \to 0$$

which shows that $A/Ax \otimes_A A/Ax \simeq A/Ax$ and moreover that this sequence is not exact.
(iv) Notice that the functor $\mathrm{Hom}_A(\bullet, A)$ being additive, it sends split exact sequences to split exact sequences. This shows that (4.3) does not split.

Example 4.2.5. We shall show that the functor $\varprojlim : \mathrm{Mod}(\mathbf{k})^{I^{\mathrm{op}}} \to \mathrm{Mod}(\mathbf{k})$ is not right exact in general.

Consider as above the $\mathbf{k}$-algebra $A := \mathbf{k}[x]$ over a field $\mathbf{k}$. Denote by $I = A \cdot x$ the ideal generated by x. Notice that $A/I^{n+1} \simeq \mathbf{k}[x]^{\leq n}$, where $\mathbf{k}[x]^{\leq n}$ denotes the $\mathbf{k}$-vector space consisting of polynomials of degree $\leq n$. For $p \leq n$ denote by $v_{pn} \colon A/I^n \twoheadrightarrow A/I^p$ the natural epimorphisms. They define a projective system of A-modules. One checks easily that

$$\varprojlim_n A/I^n \simeq \mathbf{k}[[x]],$$

the ring of formal series with coefficients in $\mathbf{k}$. On the other hand, for $p \leq n$ the monomorphisms $I^n \rightarrowtail I^p$ define a projective system of A-modules and one has

$$\varprojlim_n I^n \simeq 0.$$

Now consider the projective system of exact sequences of A-modules

$$0 \to I^n \to A \to A/I^n \to 0.$$

By taking the projective limit of these exact sequences one gets the sequence $0 \to 0 \to \mathbf{k}[x] \to \mathbf{k}[[x]] \to 0$ which is no more exact, neither in the category $\mathrm{Mod}(A)$ nor in the category $\mathrm{Mod}(\mathbf{k})$.

The Mittag-Leffler condition

Let us give a criterion in order that the projective limit of an exact sequence remains exact in the category $\mathrm{Mod}(A)$. This is a particular case of the so-called "Mittag-Leffler" condition (see [17]).

Proposition 4.2.6. *Let A be a ring and let $0 \to \{M'_n\} \xrightarrow{f_n} \{M_n\} \xrightarrow{g_n} \{M''_n\} \to 0$ be an exact sequence of projective systems of A-modules indexed by $\mathbb{N}$. Assume that for each n, the map $M'_{n+1} \to M'_n$ is surjective. Then the sequence*

$$0 \to \varprojlim_n M'_n \xrightarrow{f} \varprojlim_n M_n \xrightarrow{g} \varprojlim_n M''_n \to 0$$

is exact.

Proof. Let us denote for short by v_p the morphisms $M_p \to M_{p-1}$ which define the projective system $\{M_p\}$, and similarly for v'_p, v''_p. Let $\{x''_p\}_p \in \varprojlim_n M''_n$. Hence $x''_p \in M''_p$, and $v''_p(x''_p) = x''_{p-1}$.

We shall first show that $v_n\colon g_n^{-1}(x''_n) \to g_{n-1}^{-1}(x''_{n-1})$ is surjective. Let $x_{n-1} \in g_{n-1}^{-1}(x''_{n-1})$. Take $x_n \in g_n^{-1}(x''_n)$. Then $g_{n-1}(v_n(x_n) - x_{n-1})) = 0$. Hence $v_n(x_n) - x_{n-1} = f_{n-1}(x'_{n-1})$. By the hypothesis $f_{n-1}(x'_{n-1}) = f_{n-1}(v'_n(x'_n))$ for some x'_n and thus $v_n(x_n - f_n(x'_n)) = x_{n-1}$.

Then we can choose $x_n \in g_n^{-1}(x''_n)$ inductively such that $v_n(x_n) = x_{n-1}$. q.e.d.

4.3 Injective and projective objects

Definition 4.3.1. Let $\mathcal{C}$ be an abelian category.

(i) An object I of $\mathcal{C}$ is injective if the functor $\mathrm{Hom}_{\mathcal{C}}(\,\bullet\,, I)$ is exact.

(ii) One says that $\mathcal{C}$ has enough injectives if for any $X \in \mathcal{C}$ there exists a monomorphism $X \rightarrowtail I$ with I injective.

(iii) An object P is projective in $\mathcal{C}$ if it is injective in $\mathcal{C}^{\mathrm{op}}$, *i.e.,* if the functor $\mathrm{Hom}_{\mathcal{C}}(P, \bullet\,)$ is exact.

(iv) One says that $\mathcal{C}$ has enough projectives if for any $X \in \mathcal{C}$ there exists an epimorphism $P \twoheadrightarrow X$ with P projective.

Proposition 4.3.2. *The object $I \in \mathcal{C}$ is injective if and only if, for any $X, Y \in \mathcal{C}$ and any diagram in which the row is exact:*

$$\begin{tikzcd} 0 \arrow[r] & X' \arrow[r, "f"] \arrow[d, "k"'] & X \arrow[dl, dotted, "h"'] \\ & I & \end{tikzcd}$$

the dotted arrow may be completed, making the solid diagram commutative.

Proof. (i) Assume that I is injective and let X'' denote the cokernel of the morphism $X' \to X$. Applying $\mathrm{Hom}_{\mathcal{C}}(\,\bullet\,, I)$ to the sequence $0 \to X' \to X \to X''$, one gets the exact sequence:

$$\mathrm{Hom}_{\mathcal{C}}(X'', I) \to \mathrm{Hom}_{\mathcal{C}}(X, I) \xrightarrow{\circ f} \mathrm{Hom}_{\mathcal{C}}(X', I) \to 0.$$

Thus there exists $h\colon X \to I$ such that $h \circ f = k$.
(ii) Conversely, consider an exact sequence $0 \to X' \xrightarrow{f} X \xrightarrow{g} X'' \to 0$. Then the sequence $0 \to \mathrm{Hom}_{\mathcal{C}}(X'', I) \xrightarrow{\circ h} \mathrm{Hom}_{\mathcal{C}}(X, I) \xrightarrow{\circ f} \mathrm{Hom}_{\mathcal{C}}(X', I) \to 0$ is exact by the hypothesis.

To conclude, apply Lemma 4.2.2. q.e.d.

By reversing the arrows, we get that P is projective if and only if for any solid diagram in which the row is exact:

$$\begin{tikzcd} & P \arrow[dl, dotted, "h"'] \arrow[d, "k"] & \\ X \arrow[r, "f"] & X'' \arrow[r] & 0 \end{tikzcd}$$

the dotted arrow may be completed, making the diagram commutative.

Lemma 4.3.3. *Let $0 \to X' \xrightarrow{f} X \xrightarrow{g} X'' \to 0$ be an exact sequence in $\mathcal{C}$, and assume that X' is injective. Then the sequence splits.*

Proof. Applying the preceding result with $k = \mathrm{id}_{X'}$, we find $h\colon X \to X'$ such that $k \circ f = \mathrm{id}_{X'}$. Then apply Proposition 4.1.7. q.e.d.

It follows that if $F\colon \mathcal{C} \to \mathcal{C}'$ is an additive functor of abelian categories, and the hypotheses of the lemma are satisfied, then the sequence $0 \to F(X') \to F(X) \to F(X'') \to 0$ splits and in particular is exact.

Lemma 4.3.4. *Let X', X'' belong to $\mathcal{C}$. Then $X' \oplus X''$ is injective if and only if X' and X'' are injective.*

Proof. It is enough to remark that for two additive functors of abelian categories F and G, $X \mapsto F(X) \oplus G(X)$ is exact if and only if F and G are exact. q.e.d.

Applying Lemmas 4.3.3 and 4.3.4, we get:

Proposition 4.3.5. *Let $0 \to X' \to X \to X'' \to 0$ be an exact sequence in $\mathcal{C}$ and assume X' and X are injective. Then X'' is injective.*

Example 4.3.6. (i) Let A be a ring. An A-module M free if it is isomorphic to a direct sum of copies of A, that is, $M \simeq A^{(I)}$. It follows from (2.4) and Proposition 4.2.3 (iv) that free modules are projective.

Let $M \in \mathrm{Mod}(A)$. For $m \in M$, denote by A_m a copy of A and denote by $1_m \in A_m$ the unit. Define the linear map

$$\psi\colon \bigoplus_{m \in M} A_m \to M$$

by setting $\psi(1_m) = m$ and extending by linearity. This map is clearly surjective. Since the left A-module $\bigoplus_{m \in M} A_m$ is free, it is projective. This shows that the category $\mathrm{Mod}(A)$ has enough projectives.

More generally, if there exists an A-module N such that $M \oplus N$ is free then M is projective (see Exercise 4.3).

One can prove that $\mathrm{Mod}(A)$ has enough injectives (see Exercise 4.4).
(ii) If $\mathbf{k}$ is a field, then any object of $\mathrm{Mod}(\mathbf{k})$ is both injective and projective.
(iii) Let A be a $\mathbf{k}$-algebra and let $M \in \mathrm{Mod}(A^{\mathrm{op}})$. One says that M is flat if the functor $M \otimes_A \bullet : \mathrm{Mod}(A) \to \mathrm{Mod}(\mathbf{k})$ is exact. Clearly, projective modules are flat.

4.4 Complexes in abelian categories

Cohomology

Recall that the categories $\mathrm{C}^*(\mathcal{C})$ are abelian for $* = \mathrm{ub}, +, -, \mathrm{b}$.

Let $X \in \mathrm{C}(\mathcal{C})$. One defines the following objects of $\mathcal{C}$:

$$\begin{aligned} Z^n(X) &:= \operatorname{Ker} d_X^n \\ B^n(X) &:= \operatorname{Im} d_X^{n-1} \\ H^n(X) &:= Z^n(X)/B^n(X) \quad (:= \operatorname{Coker}(B^n(X) \to Z^n(X))) \end{aligned}$$

One calls $H^n(X)$ the n-th cohomology object of X. If $f\colon X \to Y$ is a morphism in $\mathrm{C}(\mathcal{C})$, then it induces morphisms $Z^n(X) \to Z^n(Y)$ and $B^n(X) \to B^n(Y)$, thus a morphism $H^n(f)\colon H^n(X) \to H^n(Y)$. Clearly, $H^n(X \oplus Y) \simeq H^n(X) \oplus H^n(Y)$. Hence we have obtained an additive functor:

$$H^n(\,\bullet\,) : \mathrm{C}(\mathcal{C}) \to \mathcal{C}.$$

Notice that $H^n(X) = H^0(X[n])$.

There are exact sequences

$$X^{n-1} \xrightarrow{d^{n-1}} \operatorname{Ker} d_X^n \to H^n(X) \to 0, \qquad 0 \to H^n(X) \to \operatorname{Coker} d_X^{n-1} \xrightarrow{d^n} X^{n+1},$$

The next result is easily checked.

Lemma 4.4.1. *The sequences below are exact:*

$$0 \to H^n(X) \to \operatorname{Coker}(d_X^{n-1}) \xrightarrow{d_X^n} \operatorname{Ker} d_X^{n+1} \to H^{n+1}(X) \to 0. \tag{4.4}$$

One defines the truncation functors:

$$\tau^{\leq n}, \widetilde{\tau}^{\leq n} \quad : \quad C(\mathcal{C}) \to C^-(\mathcal{C}) \tag{4.5}$$

$$\tau^{\geq n}, \widetilde{\tau}^{\geq n} \quad : \quad C(\mathcal{C}) \to C^+(\mathcal{C}) \tag{4.6}$$

as follows. Let $X := \cdots \to X^{n-1} \to X^n \to X^{n+1} \to \cdots$. One sets:

$$\begin{aligned}
\tau^{\leq n}(X) &:= \quad \cdots \to X^{n-2} \to X^{n-1} \to \operatorname{Ker} d_X^n \to 0 \to \cdots \\
\widetilde{\tau}^{\leq n}(X) &:= \quad \cdots \to X^{n-1} \to X^n \to \operatorname{Im} d_X^n \to 0 \to \cdots \\
\tau^{\geq n}(X) &:= \quad \cdots \to 0 \to \operatorname{Coker} d_X^{n-1} \to X^{n+1} \to X^{n+2} \to \cdots \\
\widetilde{\tau}^{\geq n}(X) &:= \quad \cdots \to 0 \to \operatorname{Im} d_X^{n-1} \to X^n \to X^{n+1} \to \cdots
\end{aligned}$$

There is a chain of morphisms in $\mathrm{C}(\mathcal{C})$:

$$\tau^{\leq n}X \to \widetilde{\tau}^{\leq n}X \to X \to \widetilde{\tau}^{\geq n}X \to \tau^{\geq n}X,$$

and there are exact sequences in $\mathrm{C}(\mathcal{C})$:

$$\begin{cases} 0 \to \widetilde{\tau}^{\leq n-1}X \to \tau^{\leq n}X \to H^n(X)[-n] \to 0, \\ 0 \to H^n(X)[-n] \to \tau^{\geq n}X \to \widetilde{\tau}^{\geq n+1}X \to 0, \\ 0 \to \tau^{\leq n}X \to X \to \widetilde{\tau}^{\geq n+1}X \to 0, \\ 0 \to \widetilde{\tau}^{\leq n-1}X \to X \to \tau^{\geq n}X \to 0. \end{cases} \tag{4.7}$$

We have the isomorphisms

$$\begin{aligned}
H^j(\tau^{\leq n}X) \xrightarrow{\sim} H^j(\widetilde{\tau}^{\leq n}X) &\simeq \begin{cases} H^j(X) & j \leq n, \\ 0 & j > n. \end{cases} \\
H^j(\widetilde{\tau}^{\geq n}X) \xrightarrow{\sim} H^j(\tau^{\geq n}X) &\simeq \begin{cases} H^j(X) & j \geq n, \\ 0 & j < n. \end{cases}
\end{aligned} \tag{4.8}$$

The verification is straightforward.

Lemma 4.4.2. *Let $\mathcal{C}$ be an abelian category and let $f \colon X \to Y$ be a morphism in $\mathrm{C}(\mathcal{C})$ homotopic to zero. Then $H^n(f) \colon H^n(X) \to H^n(Y)$ is the 0 morphism.*

Proof. Let $f^n = s^{n+1} \circ d_X^n + d_Y^{n-1} \circ s^n$. Then $d_X^n = 0$ on $\operatorname{Ker} d_X^n$ and $d_Y^{n-1} \circ s^n = 0$ on $\operatorname{Ker} d_Y^n / \operatorname{Im} d_Y^{n-1}$. Hence $H^n(f) : \operatorname{Ker} d_X^n / \operatorname{Im} d_X^{n-1} \to \operatorname{Ker} d_Y^n / \operatorname{Im} d_Y^{n-1}$ is the zero morphism. q.e.d.

In view of Lemma 4.4.2, the functor $H^0 \colon \mathrm{C}(\mathcal{C}) \to \mathcal{C}$ extends as a functor

$$H^0 \colon \mathrm{K}(\mathcal{C}) \to \mathcal{C}.$$

One shall be aware that the additive category $\mathrm{K}(\mathcal{C})$ is not abelian in general.

Definition 4.4.3. One says that a morphism $f\colon X \to Y$ in $\mathrm{C}(\mathcal{C})$ is a quasi-isomorphism (a qis, for short) if $H^k(f)$ is an isomorphism for all $k \in \mathbb{Z}$. In such a case, one says that X and Y are quasi-isomorphic. In particular, $X \in \mathrm{C}(\mathcal{C})$ is qis to 0 if and only if the complex X is exact.

Remark 4.4.4. By Lemma 4.4.2, a complex homotopic to 0 is qis to 0, but the converse is false. One shall be aware that the property for a complex of being homotopic to 0 is preserved when applying an additive functor, contrarily to the property of being qis to 0.

Remark 4.4.5. Consider a bounded complex $X^\bullet$ and denote by $Y^\bullet$ the complex given by $Y^j = H^j(X^\bullet), d_Y^j \equiv 0$. One has:

$$Y^\bullet = \oplus_i H^i(X^\bullet)[-i]. \tag{4.9}$$

The complexes $X^\bullet$ and $Y^\bullet$ have the same cohomology objects. In other words, $H^j(Y^\bullet) \simeq H^j(X^\bullet)$. However, in general these isomorphisms are neither induced by a morphism from $X^\bullet \to Y^\bullet$, nor by a morphism from $Y^\bullet \to X^\bullet$, and the two complexes $X^\bullet$ and $Y^\bullet$ are not quasi-isomorphic.

Long exact sequence

Lemma 4.4.6. (The "five lemma".) *Consider a commutative diagram:*

$$\begin{tikzcd} X^0 \arrow[r, "\alpha_0"] \arrow[d, "f^0"'] & X^1 \arrow[r, "\alpha_1"] \arrow[d, "f^1"'] & X^2 \arrow[r, "\alpha_2"] \arrow[d, "f^2"'] & X^3 \arrow[d, "f^3"'] \\ Y^0 \arrow[r, "\beta_0"'] & Y^1 \arrow[r, "\beta_1"'] & Y^2 \arrow[r, "\beta_2"'] & Y^3 \end{tikzcd}$$

and assume that the rows are exact sequences.

(i) *If f^0 is an epimorphism and f^1, f^3 are monomorphisms, then f^2 is a monomorphism.*

(ii) *If f^3 is a monomorphism and f^0, f^2 are epimorphisms, then f^1 is an epimorphism.*

According to Convention 4.0.4, we shall assume that $\mathcal{C}$ is a full abelian subcategory of $\mathrm{Mod}(A)$ for some ring A. Hence we may choose elements in the objects of $\mathcal{C}$.

Proof. (i) Let $x_2 \in X_2$ and assume that $f^2(x_2) = 0$. Then $f^3 \circ \alpha_2(x_2) = 0$ and f^3 being a monomorphism, this implies $\alpha_2(x_2) = 0$. Since the first row is exact, there exists $x_1 \in X_1$ such that $\alpha_1(x_1) = x_2$. Set $y_1 = f^1(x_1)$. Since

$\beta_1 \circ f^1(x_1) = 0$ and the second row is exact, there exists $y_0 \in Y^0$ such that $\beta_0(y_0) = f^1(x_1)$. Since f^0 is an epimorphism, there exists $x_0 \in X^0$ such that $y_0 = f^0(x_0)$. Since $f^1 \circ \alpha_0(x_0) = f^1(x_1)$ and f^1 is a monomorphism, $\alpha_0(x_0) = x_1$. Therefore, $x_2 = \alpha_1(x_1) = 0$.
(ii) is nothing but (i) in $\mathcal{C}^{\mathrm{op}}$. q.e.d.

Lemma 4.4.7. (The snake lemma.) *Consider the commutative diagram in $\mathcal{C}$ below with exact rows:*

$$\begin{tikzcd}
 & X' \arrow[r, "f"] \arrow[d, "\alpha"'] & X \arrow[r, "g"] \arrow[d, "\beta"'] & X'' \arrow[r] \arrow[d, "\gamma"'] & 0 \\
0 \arrow[r] & Y' \arrow[r, "f'"] & Y \arrow[r, "g'"] & Y'' &
\end{tikzcd}$$

Then it gives rise to an exact sequence:

$$\operatorname{Ker}\alpha \to \operatorname{Ker}\beta \to \operatorname{Ker}\gamma \xrightarrow{\delta} \operatorname{Coker}\alpha \to \operatorname{Coker}\beta \to \operatorname{Coker}\gamma.$$

The proof is similar to that of Lemma 4.4.6 and is left as an exercise.

Theorem 4.4.8. *Let $0 \to X' \xrightarrow{f} X \xrightarrow{g} X'' \to 0$ be an exact sequence in $\mathrm{C}(\mathcal{C})$.*

(i) *For each $k \in \mathbb{Z}$, the sequence $H^k(X') \to H^k(X) \to H^k(X'')$ is exact.*

(ii) *For each $k \in \mathbb{Z}$, there exists $\delta^k : H^k(X'') \to H^{k+1}(X')$ making the long sequence*

$$\cdots \to H^k(X) \to H^k(X'') \xrightarrow{\delta^k} H^{k+1}(X') \to H^{k+1}(X) \to \cdots \tag{4.10}$$

exact. Moreover, one can construct δ^k functorial with respect to short exact sequences of $\mathrm{C}(\mathcal{C})$.

Proof. Consider the commutative diagrams:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & H^k(X') \arrow[d] & H^k(X) \arrow[d] & H^k(X'') \arrow[d] & \\
 & \operatorname{Coker} d_{X'}^{k-1} \arrow[r, "f"'] \arrow[d, "d_{X'}^k"'] & \operatorname{Coker} d_X^{k-1} \arrow[r, "g"'] \arrow[d, "d_X^k"'] & \operatorname{Coker} d_{X''}^{k-1} \arrow[r] \arrow[d, "d_{X''}^k"'] & 0 \\
0 \arrow[r] & \operatorname{Ker} d_{X'}^{k+1} \arrow[r, "f"'] \arrow[d] & \operatorname{Ker} d_X^{k+1} \arrow[r, "g"'] \arrow[d] & \operatorname{Ker} d_{X''}^{k+1} \arrow[d] & \\
 & H^{k+1}(X') \arrow[d] & H^{k+1}(X) \arrow[d] & H^{k+1}(X'') \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The columns are exact by Lemma 4.4.1 and the rows are exact by the hypothesis. Hence, the result follows from Lemma 4.4.7. q.e.d.

Corollary 4.4.9. *Consider a morphism $f\colon X \to Y$ in $\mathrm{C}(\mathcal{C})$ and recall that $\mathrm{Mc}(f)$ denotes the mapping cone of f. There is a long exact sequence:*

$$\cdots \to H^{k-1}(\mathrm{Mc}(f)) \to H^k(X) \xrightarrow{f} H^k(Y) \to H^k(\mathrm{Mc}(f)) \to \cdots . \tag{4.11}$$

Proof. Using (3.5), we get a complex:

$$0 \to Y \to \mathrm{Mc}(f) \to X[1] \to 0. \tag{4.12}$$

Clearly, this complex is exact. Indeed, in degree n, it gives the split exact sequence $0 \to Y^n \to Y^n \oplus X^{n+1} \to X^{n+1} \to 0$. Applying Theorem 4.4.8, we find a long exact sequence

$$\cdots \to H^{k-1}(\mathrm{Mc}(f)) \to H^{k-1}(X[1]) \xrightarrow{\delta^{k-1}} H^k(Y) \to H^k(\mathrm{Mc}(f)) \to \cdots . \tag{4.13}$$

It remains to check that, up to a sign, the morphism $\delta^{k-1}\colon H^k(X) \to H^k(Y)$ is $H^k(f)$. We shall not give the proof here. q.e.d.

Double complexes

Consider a double complex $X^{\bullet,\bullet}$ as in (3.7).

Theorem 4.4.10. *Let $X^{\bullet,\bullet}$ be a double complex. Assume that all rows $X^{j,\bullet}$ and columns $X^{\bullet,j}$ are 0 for $j < 0$ and are exact for $j > 0$. Then $H^p(X^{0,\bullet}) \simeq H^p(X^{\bullet,0})$ for all p.*

Proof. We shall only describe the first isomorphism $H^p(X^{0,\bullet}) \simeq H^p(X^{\bullet,0})$ in the case where $\mathcal{C} = \mathrm{Mod}(A)$, by the so-called "Weil procedure". Let $x^{p,0} \in X^{p,0}$, with $d'x^{p,0} = 0$ which represents $y \in H^p(X^{\bullet,0})$. Define $x^{p,1} = d''x^{p,0}$. Then $d'x^{p,1} = 0$, and the first column being exact, there exists $x^{p-1,1} \in X^{p-1,1}$ with $d'x^{p-1,1} = x^{p,1}$. One can iterate this procedure until getting $x^{0,p} \in X^{0,p}$. Since $d'd''x^{0,p} = 0$, and d' is injective on $X^{0,p}$ for $p > 0$ by the hypothesis, we get $d''x^{0,p} = 0$. The class of $x^{0,p}$ in $H^p(X^{0,\bullet})$ will be the image of y by the Weil procedure. Of course, one has to check that this image does not depend of the various choices we have made, and that it induces an isomorphism.

This can be visualized by the diagram:

$$\begin{tikzcd}
 & & & & x^{0,p} \arrow[r, "d''"] \arrow[d, "d'"'] & 0 \\
 & & & x^{1,p-2} \arrow[r, "d''"] \arrow[d, dotted] & x^{1,p-1} & \\
 & & & {} & & \\
 & x^{p-1,1} \arrow[r, dotted] \arrow[d, "d'"'] & {} & & & \\
x^{p,0} \arrow[r, "d''"] \arrow[d, "d'"'] & x^{p,1} & & & & \\
0 & & & & &
\end{tikzcd}$$

q.e.d.

4.5 Resolutions

Solving linear equations

The aim of this subsection is to illustrate and motivate the constructions which will appear further. In this subsection, we work in the category $\mathrm{Mod}(A)$ for a $\mathbf{k}$-algebra A. Recall that the category $\mathrm{Mod}(A)$ admits enough projectives.

Suppose one is interested in studying a system of linear equations

$$\sum_{j=1}^{N_0} p_{ij}u_j = v_i, \quad (i = 1, \dots, N_1) \tag{4.14}$$

where the p_{ij}'s belong to the ring A and u_j, v_i belong to some left A-module S. Using matrix notations, one can write equations (4.14) as

$$P_0 u = v \tag{4.15}$$

where P_0 is the matrix (p_{ij}) with N_1 rows and N_0 columns, defining the A-linear map $P_0\cdot : S^{N_0} \to S^{N_1}$. Now consider the right A-linear map

$$\cdot P_0 : A^{N_1} \to A^{N_0}, \tag{4.16}$$

where $\cdot P_0$ operates on the right and the elements of A^{N_0} and A^{N_1} are written as rows. Let $(e_1, \dots, e_{N_0})$ and $(f_1, \dots, f_{N_1})$ denote the canonical basis of A^{N_0} and A^{N_1}, respectively. One gets:

$$f_i \cdot P_0 = \sum_{j=1}^{N_0} p_{ij}e_j, \quad (i = 1, \dots, N_1). \tag{4.17}$$

Hence $\mathrm{Im}\, P_0$ is generated by the elements $\sum_{j=1}^{N_0} p_{ij}e_j$ for $i = 1, \dots, N_1$. Denote by M the quotient module $A^{N_0}/A^{N_1} \cdot P_0$ and by $\psi : A^{N_0} \to M$ the natural A-linear map. Let $(u_1, \dots, u_{N_0})$ denote the images by ψ of $(e_1, \dots, e_{N_0})$. Then M is a left A-module with generators $(u_1, \dots, u_{N_0})$ and relations $\sum_{j=1}^{N_0} p_{ij}u_j = 0$ for $i = 1, \dots, N_1$. By construction, we have an exact sequence of left A-modules:

$$A^{N_1} \xrightarrow{\cdot P_0} A^{N_0} \xrightarrow{\psi} M \to 0. \tag{4.18}$$

Applying the left exact functor $\operatorname{Hom}_A(\,\bullet\,, S)$ to this sequence, we find the exact sequence of $\mathbf{k}$-modules:

$$0 \to \operatorname{Hom}_A(M, S) \to S^{N_0} \xrightarrow{P_0\cdot} S^{N_1} \tag{4.19}$$

(where $P_0\cdot$ operates on the left). Hence, the $\mathbf{k}$-module of solutions of the homogeneous equation associated to (4.14) is described by $\operatorname{Hom}_A(M, S)$.

Assume now that A is left Noetherian, that is, any submodule of a free A-module of finite rank is of finite type. In this case, arguing as in the proof of Proposition 4.5.3, we construct an exact sequence

$$\cdots \to A^{N_2} \xrightarrow{\cdot P_1} A^{N_1} \xrightarrow{\cdot P_0} A^{N_0} \xrightarrow{\psi} M \to 0.$$

In other words, we have a projective resolution $L^\bullet \to M$ of M by finite free left A-modules:

$$L^\bullet : \ \cdots \to L^n \to L^{n-1} \to \cdots \to L^0 \to 0.$$

Applying the left exact functor $\operatorname{Hom}_A(\,\bullet\,, S)$ to $L^\bullet$, we find the complex of $\mathbf{k}$-modules:

$$0 \to S^{N_0} \xrightarrow{P_0\cdot} S^{N_1} \xrightarrow{P_1\cdot} S^{N_2} \to \cdots . \tag{4.20}$$

Then

$$\begin{cases} H^0(\operatorname{Hom}_A(L^\bullet, S)) \simeq \operatorname{Ker} P_0, \\ H^1(\operatorname{Hom}_A(L^\bullet, S)) \simeq \operatorname{Ker}(P_1)/\operatorname{Im}(P_0). \end{cases}$$

Hence, a necessary condition to solve the equation $P_0 u = v$ is that $P_1 v = 0$ and this necessary condition is sufficient if $H^1(\operatorname{Hom}_A(L^\bullet, S)) \simeq 0$. As we shall see in § 4.6, the cohomology groups $H^j(\operatorname{Hom}_A(L^\bullet, S))$ do not depend, up to isomorphisms, of the choice of the projective resolution $L^\bullet$ of M.

Resolutions

Definition 4.5.1. Let $\mathcal{J}$ be a full additive subcategory of $\mathcal{C}$. We say that $\mathcal{J}$ is cogenerating if for all X in $\mathcal{C}$, there exist $Y \in \mathcal{J}$ and a monomorphism $X \rightarrowtail Y$.

If $\mathcal{J}$ is cogenerating in $\mathcal{C}^{\mathrm{op}}$, one says that $\mathcal{J}$ is generating.

Notations 4.5.2. Consider an exact sequence in $\mathcal{C}$, $0 \to X \to J^0 \to \cdots \to J^n \to \cdots$ and denote by $J^\bullet$ the complex $0 \to J^0 \to \cdots \to J^n \to \cdots$. We shall say for short that $0 \to X \to J^\bullet$ is a resolution of X. If the J^k's belong to $\mathcal{J}$, we shall say that this is a $\mathcal{J}$-resolution of X. When $\mathcal{J}$ denotes the category of injective objects one says this is an injective resolution.

Proposition 4.5.3. *Let $\mathcal{C}$ be an abelian category and let $\mathcal{J}$ be a cogenerating full additive subcategory. Then, for any $X \in \mathcal{C}$, there exists an exact sequence*

$$0 \to X \to J^0 \to \cdots \to J^n \to \cdots \tag{4.21}$$

with $J^n \in \mathcal{J}$ for all $n \geq 0$.

Proof. We proceed by induction. Assume to have constructed:

$$0 \to X \to J^0 \to \cdots \to J^n$$

For $n = 0$ this is the hypothesis. Set $B^n = \mathrm{Coker}(J^{n-1} \to J^n)$ (with $J^{-1} = X$). Then $J^{n-1} \to J^n \to B^n \to 0$ is exact. Embed B^n in an object of $\mathcal{J}$: $0 \to B^n \to J^{n+1}$. Then $J^{n-1} \to J^n \to J^{n+1}$ is exact, and the induction proceeds. q.e.d.

The sequence

$$J^\bullet := 0 \to J^0 \to \cdots \to J^n \to \cdots \tag{4.22}$$

is called a right $\mathcal{J}$-resolution of X. If $\mathcal{J}$ is the category of injective objects in $\mathcal{C}$, one says that $J^\bullet$ is an injective resolution. Note that, identifying X and $J^\bullet$ to objects of $\mathrm{C}^+(\mathcal{C})$,

$$X \to J^\bullet \text{ is a qis.} \tag{4.23}$$

Of course, there is a similar result for left resolution. If for any $X \in \mathcal{C}$ there is an exact sequence $Y \to X \to 0$ with $Y \in \mathcal{J}$, then one can construct a left $\mathcal{J}$-resolution of X, that is, a qis $Y^\bullet \to X$, where the Y^n's belong to $\mathcal{J}$. If $\mathcal{J}$ is the category of projective objects of $\mathcal{C}$, one says that $J^\bullet$ is a projective resolution.

Proposition 4.5.3 is a particular case of a result that we state without proof.

Proposition 4.5.4. *Assume $\mathcal{J}$ is cogenerating. Then for any $X^\bullet \in \mathrm{C}^+(\mathcal{C})$, there exists $Y^\bullet \in \mathrm{C}^+(\mathcal{J})$ and a quasi-isomorphism $X^\bullet \to Y^\bullet$.*

Injective resolutions

In this section, $\mathcal{C}$ denotes an abelian category and $\mathcal{I}_\mathcal{C}$ its full additive subcategory consisting of injective objects. We shall asume

$$\text{the abelian category } \mathcal{C} \text{ admits enough injectives.} \tag{4.24}$$

In other words, the category $\mathcal{I}_\mathcal{C}$ is cogenerating.

Proposition 4.5.5. (i) *Let $f^\bullet\colon X^\bullet \to I^\bullet$ be a morphism in $\mathrm{C}^+(\mathcal{C})$. Assume $I^\bullet$ belongs to $\mathcal{C}^+(\mathcal{I}_\mathcal{C})$ and $X^\bullet$ is exact. Then $f^\bullet$ is homotopic to* 0.

(ii) *Let $I^\bullet \in \mathrm{C}^+(\mathcal{I}_\mathcal{C})$ and assume $I^\bullet$ is exact. Then $I^\bullet$ is homotopic to* 0.

Proof. (i) Consider the diagram:

$$\begin{tikzcd}
X^{k-2} \arrow[r] & X^{k-1} \arrow[r] \arrow[d, "f^{k-1}"] \arrow[dl, dotted, "s^{k-1}"'] & X^k \arrow[r] \arrow[d, "f^k"] \arrow[dl, dotted, "s^k"'] & X^{k+1} \arrow[dl, dotted, "s^{k+1}"'] \\
I^{k-2} \arrow[r] & I^{k-1} \arrow[r] & I^k \arrow[r] & I^{k+1}
\end{tikzcd}$$

We shall construct by induction morphisms s^k satisfying:

$$f^k = s^{k+1} \circ d_X^k + d_I^{k-1} \circ s^k.$$

For $j << 0$, $s^j = 0$. Assume we have constructed the s^j for $j \leq k$. Define $g^k = f^k - d_I^{k-1} \circ s^k$. One has

$$\begin{aligned}
g^k \circ d_X^{k-1} &= f^k \circ d_X^{k-1} - d_I^{k-1} \circ s^k \circ d_X^{k-1} \\
&= f^k \circ d_X^{k-1} - d_I^{k-1} \circ f^{k-1} + d_I^{k-1} \circ d_I^{k-2} \circ s^{k-1} \\
&= 0.
\end{aligned}$$

Hence, g^k factorizes through $X^k / \operatorname{Im} d_X^{k-1}$. Since the complex $X^\bullet$ is exact, the sequence $0 \to X^k / \operatorname{Im} d_X^{k-1} \to X^{k+1}$ is exact. Consider

$$\begin{tikzcd}
0 \arrow[r] & X^k / \operatorname{Im} d_X^{k-1} \arrow[r] \arrow[d, "g^k"'] & X^{k+1} \arrow[dl, dotted, "s^{k+1}"'] \\
& I^k &
\end{tikzcd}$$

The dotted arrow may be completed by Proposition 4.3.2.
(ii) Apply the result of (i) with $X^\bullet = I^\bullet$ and $f = \mathrm{id}_X$. q.e.d.

Proposition 4.5.6. (i) *Let $f\colon X \to Y$ be a morphism in $\mathcal{C}$, let $0 \to X \to X^\bullet$ be a resolution of X and let $0 \to Y \to J^\bullet$ be a complex with the J^k's injective. Then there exists a morphism $f^\bullet\colon X^\bullet \to J^\bullet$ making the diagram below commutative:*

(4.25)
$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r] \arrow[d, "f"'] & X^\bullet \arrow[d, "f^\bullet"] \\
0 \arrow[r] & Y \arrow[r] & J^\bullet
\end{tikzcd}$$

(ii) *The morphism* $f^\bullet$ *in* $\mathrm{C}(\mathcal{C})$ *constructed in* (i) *is unique up to homotopy.*

Proof. (i) Let us denote by d_X (resp. d_Y) the differential of the complex $X^\bullet$ (resp. $J^\bullet$), by d_X^{-1} (resp. d_Y^{-1}) the morphism $X \to X^0$ (resp. $Y \to J^0$) and set $f^{-1} = f$.

We shall construct the f^n's by induction. Morphism f^0 is obtained by Proposition 4.3.2. Assume we have constructed $f^0, \dots, f^n$. Let $g^n = d_Y^n \circ f^n : X^n \to J^{n+1}$. The morphism g^n factorizes through $h^n : X^n / \operatorname{Im} d_X^{n-1} \to J^{n+1}$. Since $X^\bullet$ is exact, the sequence $0 \to X^n / \operatorname{Im} d_X^{n-1} \to X^{n+1}$ is exact. Since J^{n+1} is injective, h^n extends as $f^{n+1} : X^{n+1} \to J^{n+1}$.
(ii) We may assume $f = 0$ and we have to prove that in this case $f^\bullet$ is homotopic to zero. Since the sequence $0 \to X \to X^\bullet$ is exact, this follows from Proposition 4.5.5 (i), replacing the exact sequence $0 \to Y \to J^\bullet$ by the complex $0 \to 0 \to J^\bullet$. q.e.d.

4.6 Derived functors

Let $\mathcal{C}$ be an abelian category satisfying (4.24). Recall that $\mathcal{I}_\mathcal{C}$ denotes the full additive subcategory of consisting of injective objects in $\mathcal{C}$. We look at the additive category $\mathrm{K}(\mathcal{I}_\mathcal{C})$ as a full additive subcategory of the abelian category $\mathrm{K}(\mathcal{C})$.

Theorem 4.6.1. *Assuming* (4.24)*, there exists a functor* $\lambda \colon \mathcal{C} \to \mathrm{K}(\mathcal{I}_\mathcal{C})$ *and for each* $X \in \mathcal{C}$*, a qis* $X \to \lambda(X)$*, functorially in* $X \in \mathcal{C}$*.*

Proof. (i) Let $X \in \mathcal{C}$ and let $I_X^\bullet \in \mathrm{C}^+(\mathcal{I}_\mathcal{C})$ be an injective resolution of X. The image of $I_X^\bullet$ in $\mathrm{K}^+(\mathcal{C})$ is unique up to unique isomorphism, by Proposition 4.5.6.

Indeed, consider two injective resolutions $I_X^\bullet$ and $J_X^\bullet$ of X. By Proposition 4.5.6 applied to id_X, there exists a morphism $f^\bullet : I_X^\bullet \to J_X^\bullet$ making the diagram (4.25) commutative and this morphism is unique up to homotopy, hence is unique in $\mathrm{K}^+(\mathcal{C})$. Similarly, there exists a unique morphism $g^\bullet : J_X^\bullet \to I_X^\bullet$ in $\mathrm{K}^+(\mathcal{C})$. Hence, $f^\bullet$ and $g^\bullet$ are isomorphisms inverse one to each other.
(ii) Let $f \colon X \to Y$ be a morphism in $\mathcal{C}$, let $I_X^\bullet$ and $I_Y^\bullet$ be injective resolutions of X and Y respectively, and let $f^\bullet : I_X^\bullet \to I_Y^\bullet$ be a morphism of complexes such as in Proposition 4.5.6. Then the image of $f^\bullet$ in $\operatorname{Hom}_{\mathrm{K}^+(\mathcal{I}_\mathcal{C})}(I_X^\bullet, I_Y^\bullet)$ does not depend on the choice of $f^\bullet$ by Proposition 4.5.6.

In particular, we get that if $g \colon Y \to Z$ is another morphism in $\mathcal{C}$ and $I_Z^\bullet$ is an injective resolutions of Z, then $g^\bullet \circ f^\bullet = (g \circ f)^\bullet$ as morphisms in $\mathrm{K}^+(\mathcal{I}_\mathcal{C})$. q.e.d.

Let $F : \mathcal{C} \to \mathcal{C}'$ be a left exact functor of abelian categories and recall that $\mathcal{C}$ satisfies (4.24). Consider the functors

$$\mathcal{C} \xrightarrow{\lambda} \mathrm{K}^+(\mathcal{I}_\mathcal{C}) \xrightarrow{F} \mathrm{K}^+(\mathcal{C}') \xrightarrow{H^n} \mathcal{C}'.$$

Definition 4.6.2. One sets

$$R^nF = H^n \circ F \circ \lambda \tag{4.26}$$

and calls R^nF the n-th right derived functor of F.

By its definition, the receipt to construct $R^nF(X)$ is as follows:

- choose an injective resolution $I_X^\bullet$ of X, that is, construct an exact sequence $0 \to X \to I_X^\bullet$ with $I_X^\bullet \in \mathrm{C}^+(\mathcal{I}_\mathcal{C})$,
- apply F to this resolution,
- take the n-th cohomology.

In other words, $R^nF(X) \simeq H^n(F(I_X^\bullet))$. Note that

- R^nF is an additive functor from $\mathcal{C}$ to $\mathcal{C}'$,
- $R^nF(X) \simeq 0$ for $n < 0$ since $I_X^j = 0$ for $j < 0$,
- $R^0F(X) \simeq F(X)$ since F being left exact, it commutes with kernels,
- $R^nF(X) \simeq 0$ for $n \neq 0$ if F is exact,
- $R^nF(X) \simeq 0$ for $n \neq 0$ if X is injective, by the construction of $R^nF(X)$.

Definition 4.6.3. An object X of $\mathcal{C}$ such that $R^kF(X) \simeq 0$ for all $k > 0$ is called F-acyclic.

Hence, injective objects are F-acyclic for all left exact functors F.

Theorem 4.6.4. *Let $0 \to X' \xrightarrow{f} X \xrightarrow{g} X'' \to 0$ be an exact sequence in $\mathcal{C}$. Then there exists a long exact sequence:*

$$0 \to F(X') \to F(X) \to \cdots \to R^kF(X') \to R^kF(X) \to R^kF(X'') \to \cdots$$

Sketch of the proof. One constructs an exact sequence of complexes $0 \to X'^\bullet \to X^\bullet \to X''^\bullet \to 0$ whose objects are injective and this sequence is quasi-isomorphic to the sequence $0 \to X' \xrightarrow{f} X \xrightarrow{g} X'' \to 0$ in $\mathrm{C}(\mathcal{C})$. Since the objects X'^j are injective, we get a short exact sequence in $\mathrm{C}(\mathcal{C}')$:

$$0 \to F(X'^\bullet) \to F(X^\bullet) \to F(X''^\bullet) \to 0$$

Then one applies Theorem 4.4.8. q.e.d.

Definition 4.6.5. Let $\mathcal{J}$ be a full additive subcategory of $\mathcal{C}$. One says that $\mathcal{J}$ is F-injective if:

(i) $\mathcal{J}$ is cogenerating,

(ii) for any exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathcal{C}$ with $X' \in \mathcal{J}, X \in \mathcal{J}$, then $X'' \in \mathcal{J}$,

(iii) for any exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathcal{C}$ with $X' \in \mathcal{J}$, the sequence $0 \to F(X') \to F(X) \to F(X'') \to 0$ is exact.

By considering $\mathcal{C}^{\mathrm{op}}$, one obtains the notion of an F-projective subcategory, F being right exact.

Lemma 4.6.6. *Assume $\mathcal{J}$ is F-injective and let $X^\bullet \in \mathrm{C}^+(\mathcal{J})$ be a complex qis to zero* (i.e. *$X^\bullet$ is exact*). *Then $F(X^\bullet)$ is qis to zero.*

Proof. We decompose $X^\bullet$ into short exact sequences (assuming that this complex starts at step 0 for convenience):

$$\begin{array}{l} 0 \to X^0 \to X^1 \to Z^1 \to 0 \\ 0 \to Z^1 \to X^2 \to Z^2 \to 0 \\ \cdots \\ 0 \to Z^{n-1} \to X^n \to Z^n \to 0 \end{array}$$

By induction we find that all the Z^j's belong to $\mathcal{J}$, hence all the sequences:

$$0 \to F(Z^{n-1}) \to F(X^n) \to F(Z^n) \to 0$$

are exact. Hence the sequence

$$0 \to F(X^0) \to F(X^1) \to \cdots$$

is exact. q.e.d.

Theorem 4.6.7. *Assume $\mathcal{J}$ is F-injective and contains the category $\mathcal{I}_{\mathcal{C}}$ of injective objects. Let $X \in \mathcal{C}$ and let $0 \to X \to Y^\bullet$ be a resolution of X with $Y^\bullet \in \mathrm{C}^+(\mathcal{J})$. Then for each n, there is an isomorphism $R^nF(X) \simeq H^n(F(Y^\bullet))$.*

In other words, in order to calculate the derived functors $R^nF(X)$, it is enough to replace X with a right $\mathcal{J}$-resolution.

Proof. Consider a right $\mathcal{J}$-resolution $Y^\bullet$ of X and an injective resolution $I^\bullet$ of X. By the result of Proposition 4.5.6, the identity morphism $X \to X$ will extend to a morphism of complexes $f^\bullet \colon Y^\bullet \to I^\bullet$ making the diagram below commutative:

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r] \arrow[d, "\mathrm{id}"'] & Y^\bullet \arrow[d, "f^\bullet"'] \\ 0 \arrow[r] & X \arrow[r] & I^\bullet. \end{tikzcd}$$

Define the complex $K^\bullet = \mathrm{Mc}(f^\bullet)$, the mapping cone of $f^\bullet$. By the hypothesis, $K^\bullet$ belongs to $\mathrm{C}^+(\mathcal{J})$ and this complex is qis to zero by Corollary 4.4.9. By Lemma 4.6.6, $F(K^\bullet)$ is qis to zero.

On the other-hand, $F(\mathrm{Mc}(f))$ is isomorphic to $\mathrm{Mc}(F(f))$, the mapping cone of $F(f) \colon F(J^\bullet) \to F(I^\bullet)$. Applying Theorem 4.4.8 to this sequence, we find a long exact sequence

$$\cdots \to H^n(F(J^\bullet)) \to H^n(F(I^\bullet)) \to H^n(F(K^\bullet)) \to \cdots$$

Since $F(K^\bullet)$ is qis to zero, the result follows. q.e.d.

Example 4.6.8. Let $F \colon \mathcal{C} \to \mathcal{C}'$ be a left exact functor and assume that $\mathcal{C}$ admits enough injecives.
(i) The category $\mathcal{I}_\mathcal{C}$ of injective objects of $\mathcal{C}$ is F-injective.
(ii) Denote by $\mathcal{I}_F$ the full subcategory of $\mathcal{C}$ consisting of F-acyclic objects. Then $\mathcal{I}_F$ contains $\mathcal{I}_\mathcal{C}$, hence is cogenerating. It easily follows from Theorem 4.6.4 that conditions (ii) and (iii) of Definition 4.6.5 are satisfied. Hence, $\mathcal{I}_F$ is F-injective.

Theorem 4.6.9. *Let $F \colon \mathcal{C} \to \mathcal{C}'$ and $G \colon \mathcal{C}' \to \mathcal{C}''$ be left exact functors of abelian categories and assume that $\mathcal{C}$ and $\mathcal{C}'$ have enough injectives.*

(i) *Assume that G is exact. Then $R^j(G \circ F) \simeq G \circ R^j F$.*

(ii) *Assume that F is exact. There is a natural morphism $R^j(G \circ F) \to (R^j G) \circ F$.*

(iii) *Let $X \in \mathcal{C}$ and assume that $R^j F(X) \simeq 0$ for $j > 0$ and that F sends the injective objects of $\mathcal{C}$ to G-acyclic objects of $\mathcal{C}'$. Then $R^j(G \circ F)(X) \simeq (R^j G)(F(X))$.*

Proof. For $X \in \mathcal{C}$, let $0 \to X \to I_X^\bullet$ be an injective resolution of X. Then $R^j(G \circ F)(X) \simeq H^j(G \circ F(I_X^\bullet))$.
(i) If G is exact, $H^j(G \circ F(I_X^\bullet))$ is isomorphic to $G(H^j(F(I_X^\bullet))$.

(ii) Consider an injective resolution $0 \to F(X) \to J^\bullet_{F(X)}$ of $F(X)$. By the result of Proposition 4.5.6, there exists a morphism $F(I^\bullet_X) \to J^\bullet_{F(X)}$. Applying G we get a morphism of complexes: $(G \circ F)(I^\bullet_X) \to G(J^\bullet_{F(X)})$. Since $H^j((G \circ F)(I^\bullet_X)) \simeq R^j(G \circ F)(X)$ and $H^j(G(J^\bullet_{F(X)})) \simeq R^jG(F(X))$, we get the result.
(iii) Denote by $\mathcal{I}_G$ the full additive subcategory of $\mathcal{C}'$ consisting of G-acyclic objects (see Example 4.6.8). By the hypothesis, $F(I^\bullet_X)$ is qis to $F(X)$ and belongs to $\mathrm{C}^+(\mathcal{I}_G)$. Hence $R^jG(F(X)) \simeq H^j(G(F(I^\bullet_X)))$ by Theorem 4.6.7 and $H^j(G(F(I^\bullet_X))) \simeq R^j(G \circ F)(X)$. q.e.d.

Derived bifunctor

Let $F\colon \mathcal{C} \times \mathcal{C}' \to \mathcal{C}''$ be a left exact additive bifunctor of abelian categories. Assume that $\mathcal{C}$ and $\mathcal{C}'$ admit enough injectives. For $X \in \mathcal{C}$ and $Y \in \mathcal{C}'$, one can thus construct $(R^jF(X, \bullet))(Y)$ and $(R^jF(\bullet, Y))(X)$.

Theorem 4.6.10. *Assume that for each injective object $I \in \mathcal{C}$ the functor $F(I, \bullet)\colon \mathcal{C}' \to \mathcal{C}''$ is exact and for each injective object $I' \in \mathcal{C}'$ the functor $F(\bullet, I')\colon \mathcal{C} \to \mathcal{C}''$ is exact. Then, for $j \in \mathbb{Z}$, $X \in \mathcal{C}$ and $Y \in \mathcal{C}'$, there is an isomorphism, functorial in X and Y: $(R^jF(X, \bullet))(Y) \simeq (R^jF(\bullet, Y))(X)$*

Proof. Let $0 \to X \to I^\bullet_X$ and $0 \to Y \to I^\bullet_Y$ be injective resolutions of X and Y, respectively. Consider the double complex:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & F(I^0_X, Y) & \longrightarrow & F(I^1_X, Y) & \longrightarrow & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F(X, I^0_Y) & \longrightarrow & F(I^0_X, I^0_Y) & \longrightarrow & F(I^1_X, I^0_Y) & \longrightarrow & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F(X, I^1_Y) & \longrightarrow & F(I^0_X, I^1_Y) & \longrightarrow & F(I^1_X, I^1_Y) & \longrightarrow & \\
 & & \downarrow & & \downarrow & & \downarrow & &
\end{array}$$

The cohomology of the first row (resp. column) calculates $R^kF(\bullet, Y)(X)$ (resp. $R^kF(X, \bullet)(Y)$). Since the other rows and columns are exact by the hypotheses, the result follows from Theorem 4.4.10. q.e.d.

Assume that $\mathcal{C}$ has enough injectives and enough projectives. Then one can define the j-th derived functor of $\mathrm{Hom}_{\mathcal{C}}(X, \bullet)$ and the j-th derived functor of $\mathrm{Hom}_{\mathcal{C}}(\bullet, Y)$. By Theorem 4.6.10 there exists an isomorphism

$$R^j\mathrm{Hom}_{\mathcal{C}}(X, \bullet)(Y) \simeq R^j\mathrm{Hom}_{\mathcal{C}}(\bullet, Y)(X)$$

functorial with respect to X and Y. Hence, if $\mathcal{C}$ has enough injectives *or* enough projectives, we can denote by the same symbol the derived functor either of the functor $\mathrm{Hom}_{\mathcal{C}}(X, \bullet)$ or of the functor $\mathrm{Hom}_{\mathcal{C}}(\bullet, Y)$.

A similar remark applies to the bifunctor $\otimes_A\colon \mathrm{Mod}(A^{\mathrm{op}}) \times \mathrm{Mod}(A) \to \mathrm{Mod}(\mathbf{k})$.

Definition 4.6.11. (i) If $\mathcal{C}$ has enough injectives or enough projectives, one denotes by $\mathrm{Ext}^j_{\mathcal{C}}(\bullet, \bullet)$ the j-th right derived functor of $\mathrm{Hom}_{\mathcal{C}}$.

(ii) For a ring A, one denotes by $\mathrm{Tor}^A_j(\bullet, \bullet)$ the left derived functor of $\bullet \otimes_A \bullet$.

Hence, the derived functors of $\mathrm{Hom}_{\mathcal{C}}$ are calculated as follows. Let $X, Y \in \mathcal{C}$. If $\mathcal{C}$ has enough injectives, one chooses an injective resolution $I^\bullet_Y$ of Y and we get

$$\mathrm{Ext}^j_{\mathcal{C}}(X, Y) \simeq H^j(\mathrm{Hom}_{\mathcal{C}}(X, I^\bullet_Y)). \tag{4.27}$$

If $\mathcal{C}$ has enough projectives, one chooses a projective resolution $P^\bullet_X$ of X and we get

$$\mathrm{Ext}^j_{\mathcal{C}}(X, Y) \simeq H^j(\mathrm{Hom}_{\mathcal{C}}(P^\bullet_X, Y)). \tag{4.28}$$

If $\mathcal{C}$ admits both enough injectives and projectives, one can choose to use either (4.27) or (4.28). When dealing with the category $\mathrm{Mod}(A)$, projective resolutions are in general much easier to construct.

Similarly, the derived functors of $\otimes_A$ are calculated as follows. Let $N \in \mathrm{Mod}(A^{\mathrm{op}})$ and $M \in \mathrm{Mod}(A)$. One constructs a projective resolution $P^\bullet_N$ of N or a projective resolution $P^\bullet_M$ of M. Then

$$\mathrm{Tor}^A_j(N, M) \simeq H^{-j}(P^\bullet_N \otimes_A M) \simeq H^{-j}(N \otimes_A P^\bullet_M).$$

In fact, it is enough to take flat resolutions instead of projective ones.

4.7 Koszul complexes

In this section, we do not work in abstract abelian categories but in the category $\mathrm{Mod}(A)$, for a non necessarily commutative $\mathbf{k}$-algebra A.

If L is a finite free $\mathbf{k}$-module of rank n, one denotes by $\bigwedge^j L$ the $\mathbf{k}$-module consisting of j-multilinear alternate forms on the dual space L^* and calls it the j-th exterior power of L. (Recall that $L^* = \mathrm{Hom}_{\mathbf{k}}(L, \mathbf{k})$.)

Note that $\bigwedge^1 L \simeq L$ and $\bigwedge^n L \simeq \mathbf{k}$. One sets $\bigwedge^0 L = \mathbf{k}$.

If $(e_1, \dots, e_n)$ is a basis of L and $I = \{i_1 < \dots < i_j\} \subset \{1, \dots, n\}$, one sets

$$e_I = e_{i_1} \wedge \dots \wedge e_{i_j}.$$

For a subset $I \subset \{1, \dots, n\}$, one denotes by $|I|$ its cardinal. Recall that:

$$\bigwedge^{j} L \text{ is free with basis } \{e_{i_1} \wedge \dots \wedge e_{i_j}; 1 \leq i_1 < i_2 < \dots < i_j \leq n\}.$$

If $i_1, \dots, i_m$ belong to the set $(1, \dots, n)$, one defines $e_{i_1} \wedge \dots \wedge e_{i_m}$ by reducing to the case where $i_1 < \dots < i_j$, using the convention $e_i \wedge e_j = -e_j \wedge e_i$.

Let M be an A-module and let $\varphi = (\varphi_1, \dots, \varphi_n)$ be n endomorphisms of M over A which commute with one another:

$$[\varphi_i, \varphi_j] = 0, \ 1 \leq i, j \leq n.$$

(Recall the notation $[a, b] := ab - ba$.) Set $M^{(j)} = M \otimes \bigwedge^j \mathbf{k}^n$. Hence $M^{(0)} = M$ and $M^{(n)} \simeq M$. Denote by $(e_1, \dots, e_n)$ the canonical basis of $\mathbf{k}^n$. Hence, any element of $M^{(j)}$ may be written uniquely as a sum

$$m = \sum_{|I|=j} m_I \otimes e_I.$$

One defines $d \in \operatorname{Hom}_A(M^{(j)}, M^{(j+1)})$ by:

$$d(m \otimes e_I) = \sum_{i=1}^{n} \varphi_i(m) \otimes e_i \wedge e_I$$

and extending d by linearity. Using the commutativity of the φ_i's one checks easily that $d \circ d = 0$. Hence we get a complex, called a Koszul complex and denoted $K^\bullet(M, \varphi)$:

$$0 \to M^{(0)} \xrightarrow{d} \cdots \to M^{(n)} \to 0.$$

When $n = 1$, the cohomology of this complex gives the kernel and cokernel of φ_1. More generally,

$$\begin{aligned} H^0(K^\bullet(M, \varphi)) &\simeq \operatorname{Ker} \varphi_1 \cap \ldots \cap \operatorname{Ker} \varphi_n, \\ H^n(K^\bullet(M, \varphi)) &\simeq M/(\varphi_1(M) + \dots + \varphi_n(M)). \end{aligned}$$

Set $\varphi' = \{\varphi_1, \dots, \varphi_{n-1}\}$ and denote by d' the differential in $K^\bullet(M, \varphi')$. Then φ_n defines a morphism

$$\widetilde{\varphi}_n : K^\bullet(M, \varphi') \to K^\bullet(M, \varphi') \tag{4.29}$$

Lemma 4.7.1. *The complex $K^\bullet(M,\varphi)[1]$ is isomorphic to the mapping cone of $-\widetilde{\varphi}_n$.*

Proof. [1] Consider the diagram

$$\begin{tikzcd}
\mathrm{Mc}(\widetilde{\varphi}_n)^p \arrow[r, "d_M^p"'] \arrow[d, "\lambda^p"'] & \mathrm{Mc}(\widetilde{\varphi}_n)^{p+1} \arrow[d, "\lambda^{p+1}"'] \\
K^{p+1}(M,\varphi) \arrow[r, "d_K^{p+1}"'] & K^{p+2}(M,\varphi)
\end{tikzcd}$$

given explicitly by:

$$\begin{tikzcd}
(M\otimes \bigwedge^{p+1} k^{n-1})\oplus(M\otimes\bigwedge^{p} k^{n-1}) \arrow[r, "\begin{pmatrix} -d' & 0 \\ -\varphi_n & d' \end{pmatrix}"'] \arrow[d, "\mathrm{id}\oplus(\mathrm{id}\otimes e_n\wedge)"'] & (M\otimes \bigwedge^{p+2} k^{n-1})\oplus(M\otimes\bigwedge^{p+1} k^{n-1}) \arrow[d, "\mathrm{id}\oplus(\mathrm{id}\otimes e_n\wedge)"'] \\
M\otimes\bigwedge^{p+1} k^n \arrow[r, "-d"'] & M\otimes\bigwedge^{p+2} k^n
\end{tikzcd}$$

Then

$$\begin{aligned}
&d_M^p(a\otimes e_J + b\otimes e_K) = -d'(a\otimes e_J) + (d'(b\otimes e_K) - \varphi_n(a)\otimes e_J),\\
&\lambda^p(a\otimes e_J + b\otimes e_K) = a\otimes e_J + b\otimes e_n\wedge e_K.
\end{aligned}$$

(i) The vertical arrows are isomorphisms. Indeed, let us treat the first one. It is described by:

$$\sum_J a_J\otimes e_J + \sum_K b_K\otimes e_K \mapsto \sum_J a_J\otimes e_J + \sum_K b_K\otimes e_n\wedge e_K \tag{4.30}$$

with $|J| = p+1$ and $|K| = p$. Any element of $M\otimes\bigwedge^{p+1} k^n$ may uniquely be written as in the right hand side of (4.30).
(ii) The diagram commutes. Indeed,

$$\begin{aligned}
\lambda^{p+1}\circ d_M^p(a\otimes e_J + b\otimes e_K) &= -d'(a\otimes e_J) + e_n\wedge d'(b\otimes e_K) - \varphi_n(a)\otimes e_n\wedge e_J\\
&= -d'(a\otimes e_J) - d'(b\otimes e_n\wedge e_K) - \varphi_n(a)\otimes e_n\wedge e_J,\\
d_K^{p+1}\circ\lambda^p(a\otimes e_J + b\otimes e_K) &= -d(a\otimes e_J + b\otimes e_n\wedge e_K)\\
&= -d'(a\otimes e_J) - \varphi_n(a)\otimes e_n\wedge e_J - d'(b\otimes e_n\wedge e_K).
\end{aligned}$$

q.e.d.

[1] The proof may be skipped

Theorem 4.7.2. *There exists a long exact sequence*

$$(4.31)\quad \cdots \to H^j(K^\bullet(M,\varphi')) \xrightarrow{\varphi_n} H^j(K^\bullet(M,\varphi')) \to H^{j+1}(K^\bullet(M,\varphi)) \to \cdots$$

Proof. Apply Lemma 4.7.1 and the long exact sequence (4.11). q.e.d.

Definition 4.7.3. (i) If for each j, $1 \le j \le n$, φ_j is injective as an endomorphism of $M/(\varphi_1(M) + \cdots + \varphi_{j-1}(M))$, one says $(\varphi_1, \ldots, \varphi_n)$ is a regular sequence.

(ii) If for each j, $1 \le j \le n$, φ_j is surjective as an endomorphism of $\operatorname{Ker}\varphi_1 \cap \ldots \cap \operatorname{Ker}\varphi_{j-1}$, one says $(\varphi_1, \ldots, \varphi_n)$ is a coregular sequence.

Corollary 4.7.4. (i) *Assume $(\varphi_1, \ldots, \varphi_n)$ is a regular sequence. Then $H^j(K^\bullet(M,\varphi)) \simeq 0$ for $j \neq n$.*

(ii) *Assume $(\varphi_1, \ldots, \varphi_n)$ is a coregular sequence. Then $H^j(K^\bullet(M,\varphi)) \simeq 0$ for $j \neq 0$.*

Proof. Assume for example that $(\varphi_1, \ldots, \varphi_n)$ is a regular sequence, and let us argue by induction on n. The cohomology of $K^\bullet(M,\varphi')$ is thus concentrated in degree $n-1$ and is isomorphic to $M/(\varphi_1(M) + \cdots + \varphi_{n-1}(M))$. By the hypothesis, φ_n is injective on this group, and Corollary 4.7.4 follows. q.e.d.

Second proof. Let us give a direct proof of the Corollary in case $n = 2$ for coregular sequences. Hence we consider the complex:

$$0 \to M \xrightarrow{d} M \times M \xrightarrow{d} M \to 0$$

where $d(x) = (\varphi_1(x), \varphi_2(x))$, $d(y,z) = \varphi_2(y) - \varphi_1(z)$ and we assume φ_1 is surjective on M, φ_2 is surjective on $\operatorname{Ker}\varphi_1$.

Let $(y,z) \in M \times M$ with $\varphi_2(y) = \varphi_1(z)$. We look for $x \in M$ solution of $\varphi_1(x) = y, \quad \varphi_2(x) = z$. First choose $x' \in M$ with $\varphi_1(x') = y$. Then $\varphi_2 \circ \varphi_1(x') = \varphi_2(y) = \varphi_1(z) = \varphi_1 \circ \varphi_2(x')$. Thus $\varphi_1(z - \varphi_2(x')) = 0$ and there exists $t \in M$ with $\varphi_1(t) = 0, \quad \varphi_2(t) = z - \varphi_2(x')$. Hence $y = \varphi_1(t + x'), \quad z = \varphi_2(t + x')$ and $x = t + x'$ is a solution to our problem. q.e.d.

Example 4.7.5. Let $\mathbf{k}$ be a field of characteristic 0 and let $A = \mathbf{k}[x_1, \ldots, x_n]$.
(i) Denote by $x_i\cdot$ the multiplication by x_i in A. We get the complex:

$$0 \to A^{(0)} \xrightarrow{d} \cdots \to A^{(n)} \to 0$$

where:

$$d(\sum_I a_I \otimes e_I) = \sum_{j=1}^n \sum_I x_j \cdot a_I \otimes e_j \wedge e_I.$$

The sequence $(x_1\cdot, \dots, x_n\cdot)$ is a regular sequence in A, considered as an A-module. Hence the Koszul complex is exact except in degree n where its cohomology is isomorphic to $\mathbf{k}$.
(ii) Denote by ∂_i the partial derivation with respect to x_i. This is a $\mathbf{k}$-linear map on the $\mathbf{k}$-vector space A. Hence we get a Koszul complex

$$0 \to A^{(0)} \xrightarrow{d} \cdots \xrightarrow{d} A^{(n)} \to 0$$

where:

$$d(\sum_I a_I \otimes e_I) = \sum_{j=1}^{n} \sum_I \partial_j(a_I) \otimes e_j \wedge e_I.$$

The sequence $(\partial_1\cdot, \dots, \partial_n\cdot)$ is a coregular sequence, and the above complex is exact except in degree 0 where its cohomology is isomorphic to $\mathbf{k}$. Writing dx_j instead of e_j, we recognize the "de Rham complex".

Example 4.7.6. Let $\mathbf{k}$ be a field and let $A = \mathbf{k}[x,y]$, $M = \mathbf{k} \simeq A/xA + yA$ and let us calculate the $\mathbf{k}$-modules $\mathrm{Ext}^j_A(M,A)$. Since injective resolutions are not easy to calculate, it is much simpler to calculate a free (hence, projective) resolution of M. Since (x,y) is a regular sequence of endomorphisms of A (viewed as an A-module), M is quasi-isomorphic to the complex:

$$M^\bullet : 0 \to A \xrightarrow{u} A^2 \xrightarrow{v} A \to 0,$$

where $u(a) = (ya, -xa)$, $v(b,c) = xb + yc$ and the module A on the right stands in degree 0. Therefore, $\mathrm{Ext}^j_A(M,N)$ is the j-th cohomology object of the complex $\mathrm{Hom}_A(M^\bullet, N)$, that is:

$$0 \to N \xrightarrow{v'} N^2 \xrightarrow{u'} N \to 0,$$

where $v' = \mathrm{Hom}\,(v,N)$, $u' = \mathrm{Hom}\,(u,N)$ and the module N on the left stands in degree 0. Since $v'(n) = (xn, yn)$ and $u'(m,l) = ym - xl$, we find again a Koszul complex. Choosing $N = A$, its cohomology is concentrated in degree 2. Hence, $\mathrm{Ext}^j_A(M,A) \simeq 0$ for $j \neq 2$ and $\simeq \mathbf{k}$ for $j = 2$.

Example 4.7.7. Let $W = W_n(\mathbf{k})$ be the Weyl algebra introduced in Example 1.2.2, and denote by $\cdot\partial_i$ the multiplication on the right by ∂_i. Then $(\cdot\partial_1, \dots, \cdot\partial_n)$ is a regular sequence on W (considered as an W-module) and we get the Koszul complex:

$$0 \to W^{(0)} \xrightarrow{\delta} \cdots \to W^{(n)} \to 0$$

where:

$$\delta(\sum_I a_I \otimes e_I) = \sum_{j=1}^n \sum_I a_I \cdot \partial_j \otimes e_j \wedge e_I.$$

This complex is exact except in degree n where its cohomology is isomorphic to $\mathbf{k}[x]$ (see Exercise 4.9).

Remark 4.7.8. One may also encounter co-Koszul complexes. For $I = (i_1, \dots, i_k)$, introduce

$$e_j \lfloor e_I = \begin{cases} 0 & \text{if } j \notin \{i_1, \dots, i_k\} \\ (-1)^{l+1} e_{I_{\hat{l}}} := (-1)^{l+1} e_{i_1} \wedge \dots \wedge \widehat{e_{i_l}} \wedge \dots \wedge e_{i_k} & \text{if } e_{i_l} = e_j \end{cases}$$

where $e_{i_1} \wedge \dots \wedge \widehat{e_{i_l}} \wedge \dots \wedge e_{i_k}$ means that e_{i_l} should be omitted in $e_{i_1} \wedge \dots \wedge e_{i_k}$. Define δ by:

$$\delta(m \otimes e_I) = \sum_{j=1}^n \varphi_j(m) e_j \lfloor e_I.$$

Here again one checks easily that $\delta \circ \delta = 0$, and we get the complex:

$$K_\bullet(M, \varphi) : 0 \to M^{(n)} \xrightarrow{\delta} \cdots \to M^{(0)} \to 0,$$

This complex is in fact isomorphic to a Koszul complex. Consider the isomorphism

$$*\colon \bigwedge^j \mathbf{k}^n \xrightarrow{\sim} \bigwedge^{n-j} \mathbf{k}^n$$

which associates $\varepsilon_I m \otimes e_{\hat{I}}$ to $m \otimes e_I$, where $\hat{I} = (1, \dots, n) \setminus I$ and ε_I is the signature of the permutation which sends $(1, \dots, n)$ to $I \sqcup \hat{I}$ (any $i \in I$ is smaller than any $j \in \hat{I}$). Then, up to a sign, $*$ interchanges d and δ.

De Rham complexes

Let E be a real vector space of dimension n and let U be an open subset of E. Denote as usual by $\mathcal{C}^\infty(U)$ the $\mathbb{C}$-algebra of $\mathbb{C}$-valued functions on U of class C^∞. Recall that $\Omega^1(U)$ denotes the $\mathcal{C}^\infty(U)$-module of C^∞-functions on U with values in $E^* \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathrm{Hom}_{\mathbb{R}}(E, \mathbb{C})$. Hence

$$\Omega^1(U) \simeq E^* \otimes_{\mathbb{R}} \mathcal{C}^\infty(U).$$

For $p \in \mathbb{N}$, one sets

$$\begin{aligned} \Omega^p(U) &:= \bigwedge^p \Omega^1(U) \\ &\simeq (\bigwedge^p E^*) \otimes_{\mathbb{R}} \mathcal{C}^\infty(U). \end{aligned}$$

(The first exterior product is taken over the commutative ring $\mathcal{C}^\infty(U)$ and the second one over $\mathbb{R}$.) Hence, $\Omega^0(U) = \mathcal{C}^\infty(U)$, $\Omega^p(U) = 0$ for $p > n$ and $\Omega^n(U)$ is free of rank 1 over $\mathcal{C}^\infty(U)$. The differential is a $\mathbb{C}$-linear map

$$d\colon \mathcal{C}^\infty(U) \to \Omega^1(U).$$

The differential extends by multilinearity as a $\mathbb{C}$-linear map $d\colon \Omega^p(U) \to \Omega^{p+1}(U)$ satisfying

$$\begin{cases} d^2 = 0, \\ d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-)^p \omega_1 \wedge d\omega_2 \text{ for any } \omega_1 \in \Omega^p(U). \end{cases} \tag{4.32}$$

We get a complex, called the De Rham complex, that we denote by $DR(U)$:

$$\mathrm{DR}(U) := 0 \to \Omega^0(U) \xrightarrow{d} \cdots \to \Omega^n(U) \to 0. \tag{4.33}$$

Let us choose a basis $(e_1, \ldots, e_n)$ of E and denote by x_i the function which, to $x = \sum_{i=1}^n x_i \cdot e_i \in E$, associates its i-th cordinate x_i. Then $(dx_1, \ldots, dx_n)$ is the dual basis on E^* and the differential of a function φ is given by

$$d\varphi = \sum_{i=1}^n \partial_i \varphi \, dx_i.$$

where $\partial_i \varphi := \dfrac{\partial \varphi}{\partial x_i}$. By its construction, the Koszul complex of $(\partial_1, \ldots, \partial_n)$ acting on $\mathcal{C}^\infty(U)$ is nothing but the De Rham complex:

$$K^\bullet(\mathcal{C}^\infty(U), (\partial_1, \ldots, \partial_n)) = \mathrm{DR}(U).$$

Note that $H^0(\mathrm{DR}(U))$ is the space of locally constant functions on U, and therefore is isomorphic to $\mathbb{C}^{\#cc(U)}$ where $\#cc(U)$ denotes the cardinal of the set of connected components of U. Using sheaf theory, one proves that all cohomology groups $H^j(\mathrm{DR}(U))$ are topological invariants of U.

Holomorphic De Rham complexes

Replacing $\mathbb{R}^n$ with $\mathbb{C}^n$, $\mathcal{C}^\infty(U)$ with $\mathcal{O}(U)$, the space of holomorphic functions on U and the real derivation with the holomorphic derivation, one constructs similarly the holomorphic De Rham complex.

Example 4.7.9. Let $n = 1$ and let $U = \mathbb{C} \setminus \{0\}$. The holomorphic De Rham complex reduces to

$$0 \to \mathcal{O}(U) \xrightarrow{\partial_z} \mathcal{O}(U) \to 0.$$

Its cohomology is isomorphic to $\mathbb{C}$ in dimension 0 and 1.

Exercises to Chapter 4

Exercise 4.1. Prove assertion (iv) in Proposition 4.2.3, that is, prove that for a ring A and a set I, the two functors $\prod$ and $\bigoplus$ from $\mathrm{Fct}(I, \mathrm{Mod}(A))$ to $\mathrm{Mod}(A)$ are exact.

Exercise 4.2. Consider two complexes in an abelian category $\mathcal{C}$: $X_1' \to X_1 \to X_1''$ and $X_2' \to X_2 \to X_2''$. Prove that the two sequences are exact if and only if the sequence $X_1' \oplus X_2' \to X_1 \oplus X_2 \to X_1'' \oplus X_2''$ is exact.

Exercise 4.3. (i) Prove that a free module is projective.
(ii) Prove that a module P is projective if and only if it is a direct summand of a free module (*i.e.*, there exists a module K such that $P \oplus K$ is free).
(iii) An A-module M is flat if the functor $\bullet \otimes_A M$ is exact. (One defines similarly flat right A-modules.) Deduce from (ii) that projective modules are flat.

Exercise 4.4. If M is a $\mathbb{Z}$-module, set $M^\vee = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$.
(i) Prove that $\mathbb{Q}/\mathbb{Z}$ is injective in $\mathrm{Mod}(\mathbb{Z})$.
(ii) Prove that the map $\mathrm{Hom}_{\mathbb{Z}}(M, N) \to \mathrm{Hom}_{\mathbb{Z}}(N^\vee, M^\vee)$ is injective for any $M, N \in \mathrm{Mod}(\mathbb{Z})$.
(iii) Prove that if P is a right projective A-module, then $P^\vee$ is left A-injective.
(iv) Let M be an A-module. Prove that there exists an injective A-module I and a monomorphism $M \to I$.
(Hint: (iii) Use formula (1.12). (iv) Prove that $M \mapsto M^{\vee\vee}$ is an injective map using (ii), and replace M with $M^{\vee\vee}$.)

Exercise 4.5. Let $\mathcal{C}$ be an abelian category which admits inductive limits and such that filtrant inductive limits are exact. Let $\{X_i\}_{i \in I}$ be a family of objects of $\mathcal{C}$ indexed by a set I and let $i_0 \in I$. Prove that the natural morphism $X_{i_0} \to \bigoplus_{i \in I} X_i$ is a monomorphism.

Exercise 4.6. Let $\mathcal{C}$ be an abelian category.
(i) Prove that a complex $0 \to X \to Y \to Z$ is exact iff and only if for any object $W \in \mathcal{C}$ the complex of abelian groups $0 \to \mathrm{Hom}_{\mathcal{C}}(W, X) \to \mathrm{Hom}_{\mathcal{C}}(W, Y) \to \mathrm{Hom}_{\mathcal{C}}(W, Z)$ is exact.
(ii) By reversing the arrows, state and prove a similar statement for a complex $X \to Y \to Z \to 0$.

Exercise 4.7. Let $\mathcal{C}$ be an abelian category. A square is a commutative diagram:

$$\begin{tikzcd} V \arrow[r, "f'"] \arrow[d, "g'"] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z. \end{tikzcd}$$

A square is Cartesian if moreover the sequence $0 \to V \to X \times Y \to Z$ is exact, that is, if $V \simeq X \times_Z Y$ (recall that $X \times_Z Y = \mathrm{Ker}(f - g)$, where $f - g : X \oplus Y \to Z$). A square is co-Cartesian if the sequence $V \to X \oplus Y \to Z \to 0$ is exact, that is, if $Z \simeq X \oplus_V Y$ (recall that $X \oplus_Z Y = \mathrm{Coker}(f' - g')$, where $f' - g' : V \to X \times Y$).
(i) Assume the square is Cartesian and f is an epimorphism. Prove that f' is an epimorphism.
(ii) Assume the square is co-Cartesian and f' is a monomorphism. Prove that f is a monomorphism.

Exercise 4.8. Let $\mathcal{C}$ be an abelian category and consider a commutative diagram of complexes

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & X_0' \arrow[r] \arrow[d] & X_0 \arrow[r] \arrow[d] & X_0'' \arrow[d] \\ 0 \arrow[r] & X_1' \arrow[r] \arrow[d] & X_1 \arrow[r] \arrow[d] & X_1'' \arrow[d] \\ 0 \arrow[r] & X_2' \arrow[r] & X_2 \arrow[r] & X_2'' \end{tikzcd}$$

Assume that all rows are exact as well as the second and third column. Prove that all columns are exact.

Exercise 4.9. Let $\mathbf{k}$ be a field of characteristic 0, $W := W_n(\mathbf{k})$ the Weyl algebra in n variables.
(i) Denote by $x_i \cdot : W \to W$ the multiplication on the left by x_i on W (hence, the $x_i\cdot$'s are morphisms of right W-modules). Prove that $\varphi = (x_1 \cdot, \dots, x_n \cdot)$ is a regular sequence and calculate $H^j(K^\bullet(W, \varphi))$.

(ii) Denote $\cdot\partial_i$ the multiplication on the right by ∂_i on W. Prove that $\psi = (\cdot\partial_1, \dots, \cdot\partial_n)$ is a regular sequence and calculate $H^j(K^\bullet(W, \psi))$.
(iii) Now consider the left $W_n(\mathbf{k})$-module $\mathcal{O} := \mathbf{k}[x_1, \dots, x_n]$ and the $\mathbf{k}$-linear map $\partial_i : \mathcal{O} \to \mathcal{O}$ (derivation with respect to x_i). Prove that $\lambda = (\partial_1, \dots, \partial_n)$ is a coregular sequence and calculate $H^j(K^\bullet(\mathcal{O}, \lambda))$.

Exercise 4.10. Let $A = W_2(\mathbf{k})$ be the Weyl algebra in two variables. Construct the Koszul complex associated to $\varphi_1 = \cdot x_1$, $\varphi_2 = \cdot\partial_2$ and calculate its cohomology.

Exercise 4.11. Let $\mathbf{k}$ be a field, $A = \mathbf{k}[x, y]$ and consider the A-module $M = \bigoplus_{i \geq 1} \mathbf{k}[x]t^i$, where the action of $x \in A$ is the usual one and the action of $y \in A$ is defined by $y \cdot x^n t^{j+1} = x^n t^j$ for $j \geq 1$, $y \cdot x^n t = 0$. Define the endomorphisms of M, $\varphi_1(m) = x \cdot m$ and $\varphi_2(m) = y \cdot m$. Calculate the cohomology of the Kozsul complex $K^\bullet(M, \varphi)$.

Chapter 5

Abelian sheaves

In this chapter we expose basic sheaf theory in the framework of topological spaces. Although we restrict our study to topological spaces, it will be convenient to consider morphisms of sites $f\colon X \to Y$ which are not continuous maps from X to Y.
Recall that all along these Notes, k denotes a commutative unitary ring.
Some references: [14], [4], [12], [21], [22].

5.1 Presheaves

Let X be a topological space. The family of open subsets of X is ordered by inclusion and we denote by Op_X the associated category. Hence:

$$\mathrm{Hom}_{\mathrm{Op}_X}(U,V) = \begin{cases} \{\mathrm{pt}\} & \text{if } U \subset V, \\ \emptyset & \text{otherwise.} \end{cases}$$

Note that the category Op_X admits a terminal object, namely X, and finite products, namely $U \times V = U \cap V$. Indeed,

$$\mathrm{Hom}_{\mathrm{Op}_X}(W,U) \times \mathrm{Hom}_{\mathrm{Op}_X}(W,V) \simeq \mathrm{Hom}_{\mathrm{Op}_X}(W,U \cap V).$$

Definition 5.1.1. (i) One sets $\mathrm{PSh}(X) := \mathrm{Fct}((\mathrm{Op}_X)^{\mathrm{op}}, \mathbf{Set})$ calls an object of this category a presheaf of sets. In other words, a presheaf of sets in a functor from $(\mathrm{Op}_X)^{\mathrm{op}}$ to $\mathbf{Set}$.

(ii) One denotes by $\mathrm{PSh}(k_X)$ the subcategory of $\mathrm{PSh}(X)$ consisting of functors with values in $\mathrm{Mod}(k)$ and calls an object of this category a presheaf of k-modules.

Hence, a presheaf F on X associates to each open subset $U \subset X$ a set $F(U)$, and to an open inclusion $V \subset U$, a map $\rho_{VU} : F(U) \to F(V)$, such that for each open inclusions $W \subset V \subset U$, one has:

$$\rho_{UU} = \mathrm{id}_U, \qquad \rho_{WU} = \rho_{WV} \circ \rho_{VU}.$$

A morphism of presheaves $\varphi : F \to G$ is thus the data for any open set U of a map $\varphi(U) : F(U) \to G(U)$ such that for any open inclusion $V \subset U$, the diagram below commutes:

$$\begin{tikzcd} F(U) \arrow[r, "\varphi(U)"] \arrow[d] & G(U) \arrow[d] \\ F(V) \arrow[r, "\varphi(V)"] & G(V) \end{tikzcd}$$

If F is a presheaf of k-modules, then the $F(U)$'s are k-modules and the maps ρ_{VU} are k-linear.

The category $\mathrm{PSh}(k_X)$ inherits of most of the properties of the category $\mathrm{Mod}(k)$. In particular it is abelian and admits inductive and projective limits. For example, one checks easily that if F and G are two presheaves, the presheaf $U \mapsto F(U) \oplus G(U)$ is the coproduct of F and G in $\mathrm{PSh}(k_X)$. If $\varphi : F \to G$ is a morphism of presheaves, then $(\mathrm{Ker}\,\varphi)(U) \simeq \mathrm{Ker}\,\varphi(U)$ and $(\mathrm{Coker}\,\varphi)(U) \simeq \mathrm{Coker}\,\varphi(U)$ where $\varphi(U) : F(U) \to G(U)$.

More generally, if $i \mapsto F_i, i \in I$ is an inductive system of presheaves, one checks that the presheaf $U \mapsto \varinjlim_i F_i(U)$ is the inductive limit of this system in the category $\mathrm{PSh}(k_X)$ and similarly with projective limits.

Note that for $U \in \mathrm{Op}_X$, the functor $\mathrm{PSh}(k_X) \to \mathrm{Mod}(k)$, $F \mapsto F(U)$ is exact.

Notation 5.1.2. (i) One calls the morphisms ρ_{VU}, the restriction morphisms. If $s \in F(U)$, one better writes $s|_V$ instead of $\rho_{VU}s$ and calls $s|_V$ the restriction of s to V.
(ii) One denotes by $F|_U$ the presheaf on U defined by $V \mapsto F(V)$, V open in U and calls $F|_U$ the restriction of F to U.
(iii) If $s \in F(U)$, one says that s is a section of F on U, and if V is an open subset of U, one writes $s|_V$ instead of $\rho_{VU}(s)$.

Examples 5.1.3. (i) Let $M \in \mathbf{Set}$. The correspondence $U \mapsto M$ is a presheaf, called the constant presheaf on X with fiber M. For example, if $M = \mathbb{C}$, one gets the presheaf of $\mathbb{C}$-valued constant functions on X.
(ii) Let $\mathcal{C}^0(U)$ denote the $\mathbb{C}$-vector space of $\mathbb{C}$-valued continuous functions on U. Then $U \mapsto \mathcal{C}^0(U)$ (with the usual restriction morphisms) is a presheaf of $\mathbb{C}$-vector spaces, denoted $\mathcal{C}^0_X$.

Definition 5.1.4. Let $x \in X$, and let I_x denote the full subcategory of Op_X consisting of open neighborhoods of x. For a presheaf F on X, one sets:

$$F_x = \varinjlim_{U \in I_x^{\mathrm{op}}} F(U). \tag{5.1}$$

One calls F_x the stalk of F at x.

Let $x \in U$ and let $s \in F(U)$. The image $s_x \in F_x$ of s is called the germ of s at x. Note that any $s_x \in F_x$ is represented by a section $s \in F(U)$ for some open neighborhood U of x, and for $s \in F(U), t \in F(V)$, $s_x = t_x$ means that there exists an open neighborhood W of x with $W \subset U \cap V$ such that $\rho_{WU}(s) = \rho_{WV}(t)$. (See Example 2.5.11.)

Proposition 5.1.5. *The functor $F \mapsto F_x$ from $\mathrm{PSh}(k_X)$ to $\mathrm{Mod}(k)$ is exact.*

Proof. The functor $F \mapsto F_x$ is the composition

$$\mathrm{PSh}(k_X) = \mathrm{Fct}(\mathrm{Op}_X^{\mathrm{op}}, \mathrm{Mod}(k)) \to \mathrm{Fct}(I_x^{\mathrm{op}}, \mathrm{Mod}(k)) \xrightarrow{\varinjlim} \mathrm{Mod}(k).$$

The first functor associates to a presheaf F its restriction to the category I_x^{op}. It is clearly exact. Since $U, V \in I_x$ implies $U \cap V \in I_x$, the category I_x^{op} is filtrant and it follows that the functor $\varinjlim$ is exact. q.e.d.

5.2 Sheaves

Let X be a topological space and let Op_X denote the category of its open subsets. Recall that if $U, V \in \mathrm{Op}_X$, then $U \cap V$ is the product of U and V in Op_X.

We shall have to consider families $\mathcal{U} := \{U_i\}_{i \in I}$ of open subsets of U indexed by a set I. One says that $\mathcal{U}$ is an open covering of U if $\bigcup_i U_i = U$. Note that the empty family $\{U_i; i \in I\}$ with $I = \emptyset$ is an open covering of $\emptyset \in \mathrm{Op}_X$.

Let F be a presheaf on X and consider the two conditions below.

S1 For any open subset $U \subset X$, any open covering $U = \bigcup_i U_i$, any $s, t \in F(U)$ satisfying $s|_{U_i} = t|_{U_i}$ for all i, one has $s = t$.

S2 For any open subset $U \subset X$, any open covering $U = \bigcup_i U_i$, any family $\{s_i \in F(U_i), i \in I\}$ satisfying $s_i|_{U_{ij}} = s_j|_{U_{ij}}$ for all i, j, there exists $s \in F(U)$ with $s|_{U_i} = s_i$ for all i.

Definition 5.2.1. (i) One says that F is separated if it satisfies S1. One says that F is a sheaf if it satisfies S1 and S2.

(ii) One denotes by $\mathrm{Sh}(X)$ the full subcategory of $\mathrm{PSh}(X)$ whose objects are sheaves.

(iii) One denotes by $\mathrm{Mod}(k_X)$ the full k-additive subcategory of $\mathrm{PSh}(k_X)$ whose objects are sheaves and by $\iota_X : \mathrm{Mod}(k_X) \to \mathrm{PSh}(k_X)$ the forgetful functor. If there is no risk of confusion, one writes ι instead of ι_X.

(iv) One writes $\mathrm{Hom}_{k_X}(\bullet, \bullet)$ instead of $\mathrm{Hom}_{\mathrm{Mod}(k_X)}(\bullet, \bullet)$.

Let $\mathcal{U}$ be an open covering of $U \in \mathrm{Op}_X$ and let $F \in \mathrm{PSh}(k_X)$. One sets

$$F(\mathcal{U}) = \mathrm{Ker}(\prod_{V\in\mathcal{U}} F(V) \rightrightarrows \prod_{V',V''\in\mathcal{U}} F(V'\cap V'')). \tag{5.2}$$

Here the two arrows are associated with $\prod_{V\in\mathcal{U}} F(V) \to F(V') \to F(V'\cap V'')$ and $\prod_{V\in\mathcal{U}} F(V) \to F(V'') \to F(V'\cap V'')$.

In other words, a section $s \in F(\mathcal{U})$ is the data of a family of sections $\{s_V \in F(V); V \in \mathcal{U}\}$ such that for any $V', V'' \in \mathcal{U}$,

$$s_{V'}|_{V'\cap V''} = s_{V''}|_{V'\cap V''}.$$

Therefore, if F is a presheaf, there is a natural map

$$F(U) \to F(\mathcal{U}). \tag{5.3}$$

The next result is obvious.

Proposition 5.2.2. *A presheaf F is separated (resp. is a sheaf) if and only if for any $U \in \mathrm{Op}_X$ and any open covering $\mathcal{U}$ of U, the natural map $F(U) \to F(\mathcal{U})$ is injective (resp. bijective).*

One can consider $\mathcal{U}$ as a category. Assuming that $\mathcal{U}$ is stable by finite intersection, we have

$$F(\mathcal{U}) \simeq \varprojlim_{V\in\mathcal{U}} F(V). \tag{5.4}$$

Note that if F is a sheaf of sets, then $F(\emptyset) = \{\mathrm{pt}\}$. If F is a sheaf of k-modules, then $F(\emptyset) = 0$. If $\{U_i\}_{i\in I}$ is a family of disjoint open subsets, then $F(\bigsqcup_i U_i) = \prod_i F(U_i)$.

If F is a sheaf on X, then its restriction $F|_U$ to an open subset U is a sheaf.

in the sequel, we shall concentrate on sheaves of k-modules, although many results hold for sheaves of sets.

Notation 5.2.3. Let F be a sheaf of k-modules on X.
(i) One defines its support, denoted by $\operatorname{supp} F$, as the complementary of the union of all open subsets U of X such that $F|_U = 0$. Note that $F|_{X \setminus \operatorname{supp} F} = 0$.
(ii) Let $s \in F(U)$. One can define its support, denoted by $\operatorname{supp} s$, as the complementary of the union of all open subsets U of X such that $s|_U = 0$.

The next result is extremely useful. It says that to check that a morphism of sheaves is an isomorphism, it is enough to do it at each stalk.

Proposition 5.2.4. *Let $\varphi : F \to G$ be a morphism of sheaves.*

(i) *φ is a monomorphism of presheaves if and only if, for all $x \in X$, $\varphi_x : F_x \to G_x$ is injective.*

(ii) *φ is an isomorphism if and only if, for all $x \in X$, $\varphi_x : F_x \to G_x$ is an isomorphism.*

Proof. (i) The condition is necessary by Proposition 5.1.5. Assume now φ_x is injective for all $x \in X$ and let us prove that $\varphi\colon F(U) \to G(U)$ is injective. Let $s \in F(U)$ with $\varphi(s) = 0$. Then $(\varphi(s))_x = 0 = \varphi_x(s_x)$, and φ_x being injective, we find $s_x = 0$ for all $x \in U$. This implies that there exists an open covering $U = \cup_i U_i$, with $s|_{U_i} = 0$, and by S1, $s = 0$.
(ii) The condition is clearly necessary. Assume now φ_x is an isomorphism for all $x \in X$ and let us prove that $\varphi\colon F(U) \to G(U)$ is surjective. Let $t \in G(U)$. There exists an open covering $U = \cup_i U_i$ and $s_i \in F(U_i)$ such that $t|_{U_i} = \varphi(s_i)$.

Then, $\varphi(s_i)|_{U_i \cap U_j} = \varphi(s_j)|_{U_i \cap U_j}$, hence by (i), $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ and by S2, there exists $s \in F(U)$ with $s|_{U_i} = s_i$. Since $\varphi(s)|_{U_i} = t|_{U_i}$, we have $\varphi(s) = t$, by S1. q.e.d.

Examples 5.2.5. (i) The presheaf $\mathcal{C}^0_X$ is a sheaf.

(ii) Let $M \in \operatorname{Mod}(k)$. The presheaf of locally constant functions on X with values in M is a sheaf, called the constant sheaf with stalk M and denoted M_X. Note that the constant presheaf with stalk M is not a sheaf except if $M = 0$.

(iii) Let X be a topological space in (a) below, a real manifold of class C^∞ in (b)–(d), a complex analytic manifold in (e)–(h). We have the classical sheaves:
(iii)–(a) k_X: k-valued locally constant functions,
(iii)–(b) $\mathcal{C}^\infty_X$: complex valued functions of class $\mathcal{C}^\infty$,
(iii)–(c) $\mathcal{D}b_X$: complex valued distributions,
(iii)–(d) $\mathcal{C}^{\infty,(p)}_X$: p-forms of class $\mathcal{C}^\infty$, also denoted Ω^p_X,

(iii)–(e) $\mathcal{O}_X$: holomorphic functions,
(iii)–(f) Ω_X^p: holomorphic p-forms (hence, $\Omega_X^0 = \mathcal{O}_X$).

(iv) On a topological space X, the presheaf $U \mapsto \mathcal{C}_X^{0,b}(U)$ of continuous bounded functions is not a sheaf in general. To be bounded is not a local property and axiom S2 is not satisfied.

(v) Let $X = \mathbb{C}$, and denote by z the holomorphic coordinate. The holomorphic derivation $\frac{\partial}{\partial z}$ is a morphism from $\mathcal{O}_X$ to $\mathcal{O}_X$. Consider the presheaf:

$$F : U \mapsto \mathcal{O}(U)/\frac{\partial}{\partial z}\mathcal{O}(U),$$

that is, the presheaf $\mathrm{Coker}(\frac{\partial}{\partial z} : \mathcal{O}_X \to \mathcal{O}_X)$. For U an open disk, $F(U) = 0$ since the equation $\frac{\partial}{\partial z} f = g$ is always solvable. However, if $U = \mathbb{C} \setminus \{0\}$, $F(U) \neq 0$. Hence the presheaf F does not satisfy axiom S1.

(vi) If F is a sheaf on X and U is open, then $F|_U$ is sheaf on U.

5.3 Sheaf associated with a presheaf

Consider the forgetful functor

$$\iota_X : \mathrm{Mod}(k_X) \to \mathrm{PSh}(k_X) \tag{5.5}$$

which, to a sheaf F associates the underlying presheaf. In this section, we shall rapidly construct a left adjoint to this functor.

When there is no risk of confusion, we shall often omit the symbol ι_X. In other words, we shall identify a sheaf and the underlying presheaf.

Theorem 5.3.1. *The forgetful functor ι_X in (5.5) admits a left adjoint*

$$^a : \mathrm{Mod}(k_X) \to \mathrm{PSh}(k_X). \tag{5.6}$$

More precisely, one has the isomorphism, functorial with respect to $F \in \mathrm{PSh}(k_X)$ and $G \in \mathrm{Mod}(k_X)$

$$\mathrm{Hom}_{\,\mathrm{PSh}(k_X)}(F, \iota_X G) \simeq \mathrm{Hom}_{\,k_X}(F^a, G). \tag{5.7}$$

Moreover (5.7) defines a morphism of presheaves $\theta : F \to F^a$ and $\theta_x : F_x \to F_x^a$ is an isomorphism for all $x \in X$.

Note that if F is locally 0, then $F^a = 0$. If F is a sheaf, then $\theta : F \to F^a$ is an isomorphism.

If F is a presheaf on X, the sheaf F^a is called the sheaf associated with F.

Proof. Define:

$$F^a(U) = \begin{array}{l}\{s : U \to \bigsqcup_{x\in U} F_x\ ;\ s(x) \in F_x \text{ such that, for all } x \in U,\\ \text{there exists } V \ni x,\ V \text{ open in } U, \text{ and there exists } t \in\\ F(V) \text{ with } t_y = s(y) \text{ for all } y \in V\}.\end{array}$$

Define $\theta : F \to F^a$ as follows. To $s \in F(U)$, one associates the section of F^a:

$$(x \mapsto s_x) \in F^a(U).$$

One checks easily that F^a is a sheaf and any morphism of presheaves $\varphi : F \to G$ with G a sheaf will factorize uniquely through θ. In particular, any morphism of presheaves $\varphi : F \to G$ extends uniquely as a morphism of sheaves $\varphi^a : F^a \to G^a$, and $F \mapsto F^a$ is functorial. q.e.d.

Example 5.3.2. Let $M \in \mathrm{Mod}(k)$. Then the sheaf associated with the constant presheaf $U \mapsto M$ is the sheaf M_X of M-valued locally constant functions.

Theorem 5.3.3. (i) *The category* $\mathrm{Mod}(k_X)$ *admits projective limits and such limits commute with the functor* ι_X *in* (5.5). *More precisely, if* $\{F_i\}_{i\in I}$ *is a projective system of sheaves, its projective limit in* $\mathrm{PSh}(k_X)$ *is a sheaf and is a projective limit in* $\mathrm{Mod}(k_X)$.

(ii) *The functor* ^{a} *in* (5.6) *commutes with kernels. More precisely, let* $\varphi : F \to G$ *be a morphism of presheaves and let* $\varphi^a : F^a \to G^a$ *denote the associated morphism of sheaves. Then*

$$(\mathrm{Ker}\,\varphi)^a \simeq \mathrm{Ker}\,\varphi^a. \tag{5.8}$$

(iii) *The category* $\mathrm{Mod}(k_X)$ *admits inductive limits. If* $\{F_i\}_{i\in I}$ *is an inductive system of sheaves, its inductive limit is the sheaf associated with its inductive limit in* $\mathrm{PSh}(k_X)$.

(iv) *The functor* ^{a} *commutes with inductive limits* (*in particular, with cokernels*). *More precisely, if* $\{F_i\}_{i\in I}$ *is an inductive system of of presheaves, then*

$$\varinjlim_i (F_i^a) \simeq (\varinjlim_i F_i)^a, \tag{5.9}$$

where $\varinjlim$ *on the left* (*resp. right*)*is the inductive limit in the category of sheaves* (*resp. of presheaves*).

(v) *The category* $\mathrm{Mod}(k_X)$ *is abelian and the functor* ^{a} *is exact.*

(vi) *The functor* $\iota_X : \mathrm{Mod}(k_X) \to \mathrm{PSh}(k_X)$ *is fully faithful and left exact.*

(vii) *Filtrant inductive limits are exact in* $\mathrm{Mod}(k_X)$.

Let $\varphi : F \to G$ is a morphism of sheaves and let $\iota_X\varphi : \iota_X F \to \iota_X G$ denote the underlying morphism of presheaves. Then $\mathrm{Ker}\,\iota_X\varphi$ is a sheaf and coincides with $\iota_X \mathrm{Ker}\,\varphi$. On the other-hand, one shall be aware that $\mathrm{Coker}\,\iota_X\varphi$ is not necessarily a sheaf. The cokernel in the category of sheaves is the sheaf associated with this presheaf. In other words, the functor ι_X : $\mathrm{Mod}(k_X) \to \mathrm{PSh}(k_X)$ is left exact, but not right exact in general.

Proof. (i) Let $\mathcal{U}$ be an open covering of an open subset U. Since $F \mapsto F(\mathcal{U})$ commutes with projective limits, $(\varprojlim_i F_i)(U) \xrightarrow{\sim} (\varprojlim_i F_i)(\mathcal{U})$. Hence a projective limit of sheaves in the category $\mathrm{PSh}(k_X)$ is a sheaf. One has, for $G \in \mathrm{Mod}(k_X)$:

$$\begin{aligned}
\varprojlim_i \mathrm{Hom}_{k_X}(G, F_i) &\simeq \varprojlim_i \mathrm{Hom}_{\mathrm{PSh}(k_X)}(G, F_i) \\
&\simeq \mathrm{Hom}_{\mathrm{PSh}(k_X)}(G, \varprojlim_i F_i) \\
&\simeq \mathrm{Hom}_{k_X}(G, \varprojlim_i F_i).
\end{aligned}$$

It follows that $\varprojlim_i F_i$ is a projective limit in the category $\mathrm{Mod}(k_X)$.

(ii) The commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Ker}\,\varphi & \longrightarrow & F & \longrightarrow & G \\
 & & & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathrm{Ker}(\varphi^a) & \longrightarrow & F^a & \longrightarrow & G^a
\end{array}$$

defines the morphism $\mathrm{Ker}\,\varphi \to \mathrm{Ker}\,\varphi^a$, hence, the morphism $\psi : (\mathrm{Ker}\,\varphi)^a \to \mathrm{Ker}\,\varphi^a$. Since the functor $F \mapsto F_x$ commutes both with Ker and with ^{a}, ψ_x is an isomorphism for all x and it remains to apply Proposition 5.2.4.

(iii)–(iv) Let $G \in \mathrm{Mod}(k_X)$ and let $\{F_i\}_{i\in I}$ be an inductive system of of presheaves. We have the chain of isomorphisms

$$\begin{aligned}
\mathrm{Hom}_{k_X}((\varinjlim_i F_i)^a, G) &\simeq \mathrm{Hom}_{\mathrm{PSh}(k_X)}(\varinjlim_i F_i, G) \\
&\simeq \varprojlim_i \mathrm{Hom}_{\mathrm{PSh}(k_X)}(F_i, G) \\
&\simeq \varprojlim_i \mathrm{Hom}_{k_X}(F_i^a, G).
\end{aligned}$$

(v) By (i) and (iii), the category $\mathrm{Mod}(k_X)$ admits kernels and cokernels. Let $\varphi : F \to G$ be a morphism of sheaves and denote by $\iota_X\varphi$ the underlying morphism of presheaves. Using (5.9) we get that $\mathrm{Coim}\,\varphi := \mathrm{Coker}(\mathrm{Ker}\,\varphi \to F)$ is isomorphic to the sheaf associated with $\mathrm{Coim}(\iota_X\varphi)$. Using (5.8) we get that $\mathrm{Im}\,\varphi := \mathrm{Ker}(G \to (\mathrm{Coker}\,\iota_X\varphi)^a)$ is isomorphic to the sheaf associated with $\mathrm{Im}(\iota_X\varphi)$. The isomorphism of presheaves $\mathrm{Coim}(\iota_X\varphi) \xrightarrow{\sim} \mathrm{Im}(\iota_X\varphi)$ yields the isomorphism of the associated sheaves. Hence $\mathrm{Mod}(k_X)$ is abelian.

The functor $F \mapsto F^a$ is exact since it commutes with kernels by by (5.9) and with cokernels by (5.8).
(vi) The functor ι_X is fully faithful by definition. Since it admits a left adjoint, it is left exact.
(vii) Filtrant inductive limits are exact in the category $\mathrm{Mod}(k)$, whence in the category $\mathrm{PSh}(k_X)$. Then the result follows since ^{a} is exact. q.e.d.

Recall that the functor $F \mapsto F^a$ commutes with the functors of restriction $F \mapsto F|_U$, as well as with the functor $F \mapsto F_x$.

Proposition 5.3.4. (i) *Let $\varphi : F \to G$ be a morphism of sheaves and let $x \in X$. Then $(\mathrm{Ker}\,\varphi)_x \simeq \mathrm{Ker}\,\varphi_x$ and $(\mathrm{Coker}\,\varphi)_x \simeq \mathrm{Coker}\,\varphi_x$. In particular the functor $F \mapsto F_x$, from $\mathrm{Mod}(k_X)$ to $\mathrm{Mod}(k)$ is exact.*

(ii) *Let $F' \xrightarrow{\varphi} F \xrightarrow{\psi} F''$ be a complex of sheaves. Then this complex is exact if and only if for any $x \in X$, the complex $F'_x \xrightarrow{\varphi_x} F_x \xrightarrow{\psi_x} F''_x$ is exact.*

Proof. (i) The result is true in the category of presheaves. Since $\iota_X\,\mathrm{Ker}\,\varphi \simeq \mathrm{Ker}\,\iota_X\varphi$ and $\mathrm{Coker}\,\varphi \simeq (\mathrm{Coker}\,\iota_X\varphi)^a$, the result follows.
(ii) By Proposition 5.2.4, $\mathrm{Im}\,\varphi \simeq \mathrm{Ker}\,\psi$ if and only if $(\mathrm{Im}\,\varphi)_x \simeq (\mathrm{Ker}\,\psi)_x$ for all $x \in X$. Hence the result follows from (i). q.e.d.

By this statement, the complex of sheaves above is exact if and only if for each section $s \in F(U)$ defined in an open neighborhood U of x and satisfying $\psi(s) = 0$, there exists another open neighborhood V of x with $V \subset U$ and a section $t \in F'(V)$ such that $\varphi(t) = s|_V$.

On the other hand, a complex of sheaves $0 \to F' \to F \to F''$ is exact if and only if it is exact as a complex of presheaves, that is, if and only if, for any $U \in \mathrm{Op}_X$, the sequence $0 \to F'(U) \to F(U) \to F''(U)$ is exact.

Examples 5.3.5. Let X be a real analytic manifold of dimension n. The (augmented) de Rham complex is

$$0 \to \mathbb{C}_X \to \mathcal{C}_X^{\infty,(0)} \xrightarrow{d} \cdots \to \mathcal{C}_X^{\infty,(n)} \to 0 \tag{5.10}$$

where d is the differential. This complex of sheaves is exact. The same result holds with the sheaf $\mathcal{C}_X^\infty$ replaced with the sheaf $\mathcal{C}_X^\omega$ or the sheaf $\mathcal{D}b_X$.
(ii) Let X be a complex manifold of dimension n. The (augmented) holomorphic de Rham complex is

$$0 \to \mathbb{C}_X \to \Omega_X^0 \xrightarrow{d} \cdots \to \Omega_X^n \to 0 \tag{5.11}$$

where d is the holomorphic differential. This complex of sheaves is exact.

Definition 5.3.6. Let $U \in \mathrm{Op}_X$. We denote by $\Gamma(U;\bullet) : \mathrm{Mod}(k_X) \to \mathrm{Mod}(k)$ the functor $F \mapsto F(U)$.

Proposition 5.3.7. *The functor* $\Gamma(U;\bullet)$ *is left exact.*

Proof. The functor $\Gamma(U;\bullet)$ is the composition

$$\mathrm{Mod}(k_X) \xrightarrow{\iota_X} \mathrm{PSh}(k_X) \xrightarrow{\lambda_U} \mathrm{Mod}(k),$$

where λ_U is the functor $F \mapsto F(U)$. Since ι_X is left exact and λ_U is exact, the result follows. q.e.d.

The functor $\Gamma(U;\bullet)$ is not exact in general. Indeed, consider Example 5.2.5 (v). Recall that $X = \mathbb{C}$, z is a holomorphic coordinate and $U = X \setminus \{0\}$. Then the sequence of sheaves $0 \to \mathbb{C}_X \to \mathcal{O}_X \xrightarrow{\partial_z} \mathcal{O}_X \to 0$ is exact. Applying the functor $\Gamma(U;\bullet)$, the sequence one obtains is no more exact.

5.4 $\mathcal{H}om$ and $\otimes$

Definition 5.4.1. Let $F, G \in \mathrm{PSh}(k_X)$. One denotes by $\mathcal{H}om_{\,\mathrm{PSh}(k_X)}(F,G)$ or simply $\mathcal{H}om\,(F,G)$ the presheaf on X, $U \mapsto \mathrm{Hom}_{\,\mathrm{PSh}(k_U)}(F|_U, G|_U)$ and calls it the "internal hom" of F and G.

Proposition 5.4.2. *Let* $F, G \in \mathrm{Mod}(k_X)$. *Then the presheaf* $\mathcal{H}om\,(F,G)$ *is a sheaf.*

Proof. Let $U \in \mathrm{Op}_X$ and let $\mathcal{U}$ be an open covering of U. Let us show that $\mathcal{H}om\,(F,G)(U) \simeq \mathcal{H}om\,(F,G)(\mathcal{U})$. In other words, we shall prove that the sequence below is exact, (in these formulas, we write Hom instead of $\mathrm{Hom}_{\,k_W}$, W open, for short):

$$0 \to \mathrm{Hom}\,(F|_U, G|_U) \to \prod_{V\in\mathcal{U}} \mathrm{Hom}\,(F|_V, G|_V)$$
$$\rightrightarrows \prod_{V',V''\in\mathcal{U}} \mathrm{Hom}\,(F|_{V'\cap V''}, G|_{V'\cap V''}).$$

(i) Let $\varphi \in \operatorname{Hom}_{k_U}(F|_U, G|_U)$ and assume that $\varphi|_V : F|_V \to G|_V$ is zero for all $V \in \mathcal{U}$. Then for $V \in \mathcal{U}$, any $W \in \mathrm{Op}_U$ and any $s \in F(W)$, $\varphi(s)|_{W\cap V} = 0$. Since $\{W \cap V; V \in \mathcal{U}\}$ is a covering of W, $\varphi(s) \in G(W)$ is zero. This implies $\varphi = 0$.
(ii) Let $\{\varphi_V\}$ belong to $\prod_{V\in\mathcal{U}} \operatorname{Hom}(F|_V, G|_V)$. Assume that

$$\varphi_{V'}|_{V'\cap V''} = \varphi_{V''}|_{V'\cap UV''}$$

for any $V', V'' \in \mathcal{U}$. Let $W \in \mathrm{Op}_U$. Then $\{\varphi_V\}$ defines a commutative diagram

$$F(W) \xrightarrow{a_W} \prod_{V\in\mathcal{U}} G(W\cap V) \rightrightarrows \prod_{V',V''\in\mathcal{U}} G(W\cap V'\cap V'')$$

where a_W is given by $F(W) \ni s \mapsto \varphi_V(s|_{W\cap V})$. Since G is a sheaf, a_W factors uniquely as

$$F(W) \xrightarrow{\psi(W)} G(W) \to \prod_{V\in\mathcal{U}} G(W\cap V).$$

It is easy to see that $\psi : \mathrm{Op}_U \ni W \mapsto \psi(W) \in \operatorname{Hom}(F(W), G(W))$ defines an element of $\operatorname{Hom}_{k_U}(F|_U, G|_U)$. q.e.d.

The functor $\operatorname{Hom}_{k_X}(\bullet,\bullet)$ being left exact, it follows that

$$\mathcal{H}om(\bullet,\bullet) : \mathrm{Mod}(k_X)^{\mathrm{op}} \times \mathrm{Mod}(k_X) \to \mathrm{Mod}(k_X)$$

is left exact. Note that

$$\operatorname{Hom}_{k_X}(\bullet,\bullet) \simeq \Gamma(X;\bullet)\circ \mathcal{H}om(\bullet,\bullet),$$
$$\operatorname{Hom}_{k_X}(k_X,\bullet) \simeq \Gamma(X;\bullet).$$

Corollary 5.4.3. *Let $\{U_i\}_{i\in I}$ be an open covering of X, let F, G be two sheaves on X and assume to be given isomorphisms $\varphi_i \colon F|_{U_i} \xrightarrow{\sim} G|_{U_i}$, these isomorphisms satisfying $\varphi_i|_{U_i\cap U_j} = \varphi_j|_{U_i\cap U_j}$ for any $i, j \in I$. Then there exists an isomorphism $\varphi\colon F \xrightarrow{\sim} G$ such that $\varphi|_{U_i} = \varphi_i$.*

Proof. Define the isomorphism $\psi_i \in \Gamma(U_i; \mathcal{H}om(G,F))$ as $\psi_i = \varphi_i^{-1}$. Then $\psi \in \Gamma(U_i; \mathcal{H}om(G,F))$ and $\psi_i|_{U_i\cap U_j} = \psi_j|_{U_i\cap U_j}$ for any $i, j \in I$. Since $\mathcal{H}om(G,F)$ is a sheaf, there exists a morphism $\psi\colon G \to F$ such that $\psi|_{U_i} = \psi_i$. Since ψ is a local isomorphism, it is an isomorphism. q.e.d.

Since a morphism: $\varphi\colon F \to G$ defines a k-linear map $F_x \to G_x$, we get a natural morphism $(\mathcal{H}om(F,G))_x \to \operatorname{Hom}(F_x, G_x)$. In general, this map is neither injective nor surjective.

Definition 5.4.4. Let $F, G \in \mathrm{Mod}(k_X)$.

(i) One denotes by $F \overset{\mathrm{psh}}{\otimes} G$ the presheaf on X, $U \mapsto F(U) \otimes_k G(U)$.

(ii) One denotes by $F \otimes_{k_X} G$ the sheaf associated with the presheaf $F \overset{\mathrm{psh}}{\otimes} G$ and calls it the tensor product of F and G. If there is no risk of confusion, one writes $F \otimes G$ instead of $F \otimes_{k_X} G$.

Proposition 5.4.5. *Let $F, G, K \in \mathrm{Mod}(k_X)$. There is a natural isomorphism:*

$$\mathcal{H}om\,(G \otimes F, H) \simeq \mathcal{H}om\,(G, \mathcal{H}om\,(F, H)).$$

We shall skip the proof.
The functor

$$\bullet \otimes \bullet : \mathrm{Mod}(k_X) \times \mathrm{Mod}(k_X) \to \mathrm{Mod}(k_X)$$

is the composition of the right exact functor $\overset{\mathrm{psh}}{\otimes}$ and the exact functor ^{a}. This functor is thus right exact and if k is a field, it is exact. Note that for $x \in X$ and $U \in \mathrm{Op}_X$:

(i) $(F \otimes G)_x \simeq F_x \otimes G_x$,

(ii) $\mathcal{H}om\,(F, G)|_U \simeq \mathcal{H}om\,(F|_U, G|_U)$,

(iii) $\mathcal{H}om\,(k_X, F) \simeq F$,

(iv) $k_X \otimes F \simeq F$.

Example 5.4.6. Let $\mathcal{C}^\infty_X$ denote as above the sheaf of real valued $\mathcal{C}^\infty$-functions on a real manifold X. If V is a finite $\mathbb{R}$-dimensional vector space (e.g., $V = \mathbb{C}$), then the sheaf of V-valued $\mathcal{C}^\infty$-functions is nothing but $\mathcal{C}^\infty_X \otimes_{\mathbb{R}_X} V_X$.

5.5 Direct and inverse images

Let $f : X \to Y$ be a continuous map. We denote by f^t the inverse image of a set by f. Hence, we set for $V \subset Y$: $f^t(V) := f^{-1}(V)$ and $f^t : \mathrm{Op}_Y \to \mathrm{Op}_X$ is a functor. Note that f^t commutes with non empty products and coverings, that is, it satisfies

$$\begin{cases} \text{for any } U, V \in \mathrm{Op}_Y,\ f^t(U \cap V) = f^t(U) \cap f^t(V), \\ \text{for any } V \in \mathrm{Op}_Y \text{ and any covering } \mathcal{V} \text{ of } V,\ f^t(\mathcal{V}) \text{ is a covering} \\ \text{of } f^t(V). \end{cases} \tag{5.12}$$

Definition 5.5.1. Let $U \hookrightarrow X$ be an open embedding. One denotes by $j_U^t : \mathrm{Op}_U \to \mathrm{Op}_X$ the functor $\mathrm{Op}_V \ni V \mapsto V \in \mathrm{Op}_X$.

The functor j_U^t satisfies (5.12). It is a motivation to introduce the following definition which extends the notion of a continuous map.

Definition 5.5.2. A morphism of sites $f : X \to Y$ is a functor $f^t : \mathrm{Op}_Y \to \mathrm{Op}_X$ which satisfies (5.12).

One shall be aware that we do not ask that f^t commutes with finite projective limits. In particular, we do not ask $f^t(Y) = X$.

Note that if $f : X \to Y$ and $g : Y \to Z$ are morphisms of sites, then $g \circ f : X \to Z$ is also a morphism of sites.

Definition 5.5.3. Let $f : X \to Y$ be a morphism of sites and let $F \in \mathrm{PSh}(k_X)$. One defines $f_*F \in \mathrm{PSh}(k_Y)$, the direct image of F by f, by setting: $f_*F(V) = F(f^t(V))$.

Proposition 5.5.4. *Let $F \in \mathrm{Mod}(k_X)$. Then $f_*F \in \mathrm{Mod}(k_Y)$. In other words, the direct image of a sheaf is a sheaf.*

Proof. Let $V \in \mathrm{Op}_Y$ and let $\mathcal{V}$ be an open covering of V. Then $f^t(\mathcal{V})$ is an open covering of $f^t(V)$ and we get

$$f_*F(V) \simeq F(f^t(V)) \simeq F(f^t(\mathcal{V})) \simeq f_*F(\mathcal{V}).$$

q.e.d.

Definition 5.5.5. Let $f : X \to Y$ be a morphism of sites.

(i) Let $G \in \mathrm{PSh}(k_Y)$. One defines $f^\dagger G \in \mathrm{PSh}(k_X)$ by setting for $U \in \mathrm{Op}_X$:

$$f^\dagger G(U) = \varinjlim_{U \subset f^t(V)} G(V).$$

(ii) Let $G \in \mathrm{Mod}(k_Y)$. One defines $f^{-1}G \in \mathrm{Mod}(k_X)$, the inverse image of G by f, by setting

$$f^{-1}G = (f^\dagger G)^a.$$

Note that if f is a continuous map, one has for $x \in X$:

$$(f^{-1}G)_x \simeq (f^\dagger G)_x \simeq G_{f(x)}. \tag{5.13}$$

Example 5.5.6. Let $U \hookrightarrow X$ be an open embedding and let $F \in \mathrm{Mod}(k_X)$. Then $i_U^\dagger F \simeq j_{U*}F$ is already a sheaf. Hence:

$$i_U^\dagger = i_U^{-1} \simeq j_{U*}. \tag{5.14}$$

Theorem 5.5.7. *Let $f : X \to Y$ be a morphism of sites.*

(i) *The functor $f^{-1}\colon \mathrm{Mod}(k_Y) \to \mathrm{Mod}(k_X)$ is left adjoint to the functor $f_*\colon \mathrm{Mod}(k_X) \to \mathrm{Mod}(k_Y)$. In other words, we have for $F \in \mathrm{Mod}(k_X)$ and $G \in \mathrm{Mod}(k_Y)$:*

$$\mathrm{Hom}_{k_X}(f^{-1}G, F) \simeq \mathrm{Hom}_{k_Y}(G, f_*F).$$

(ii) *The functor f_* is left exact and commutes with projective limits.*

(iii) *The functor f^{-1} is exact and commutes with inductive limits.*

(iv) *There are natural morphisms of functors* $\mathrm{id} \to f_*f^{-1}$ *and* $f^{-1}f_* \to \mathrm{id}$.

Proof. (i) First we shall prove that the functor $f^\dagger\colon \mathrm{PSh}(k_Y) \to \mathrm{PSh}(k_X)$ is left adjoint to $f_*\colon \mathrm{PSh}(k_X) \to \mathrm{PSh}(k_Y)$. In other words, we have an isomorphism, functorial with respect to $F \in \mathrm{PSh}(k_X)$ and $G \in \mathrm{PSh}(k_Y)$:

$$\mathrm{Hom}_{\mathrm{PSh}(k_X)}(f^\dagger G, F) \simeq \mathrm{Hom}_{\mathrm{PSh}(k_Y)}(G, f_*F). \tag{5.15}$$

A section $\varphi \in \mathrm{Hom}_{\mathrm{PSh}(k_Y)}(G, f_*F)$ is a family of maps

$$\{\varphi_V : G(V) \to F(f^t(V))\}_{V \in \mathrm{Op}_Y}$$

compatible with the restriction morphisms. Equivalently, this is a family of maps

$$\{\varphi_U : G(V) \to F(U); U \subset f^t(V)\}_{V \in \mathrm{Op}_Y, U \in \mathrm{Op}_X}$$

compatible with the restriction morphisms. Hence it gives a family of maps

$$\{\psi_V : \varinjlim_{U \subset f^t(V)} G(V) \to F(U)\}_{U \in \mathrm{Op}_X}$$

which defines $\psi \in \mathrm{Hom}_{\mathrm{PSh}(k_X)}(f^\dagger G, F)$. Clearly, the correspondence $\varphi \mapsto \psi$ is an isomorphism.

Using the isomorphism (5.15) we get the chain of isomorphisms

$$\begin{aligned}\mathrm{Hom}_{k_Y}(G, f_*F) &\simeq \mathrm{Hom}_{\mathrm{PSh}(k_Y)}(G, f_*F) \simeq \mathrm{Hom}_{\mathrm{PSh}(k_X)}(f^\dagger G, F) \\ &\simeq \mathrm{Hom}_{k_X}((f^\dagger G)^a, F) = \mathrm{Hom}_{k_X}(f^{-1}G, F).\end{aligned}$$

(ii) With the exception of the fact that f^{-1} is left exact, the other assertions follow by the adjunction property. If f is a continuous map, f^{-1} is left exact by (5.13). Let us give a proof in the general case. Let

$$0 \to G' \to G \to G'' \tag{5.16}$$

be an exact sequence of sheaves and let $U \in \mathrm{Op}_X$, $V \in \mathrm{Op}_Y$ with $U \subset f^t(V)$. The sequence $0 \to G'(V) \to G(V) \to G''(V)$ is exact.

Consider the category $(\mathrm{Op}_Y)^U := \{V \in \mathrm{Op}_Y, U \subset f^t(V)\}$. The sequence (5.16) defines an exact sequence of functors from $((\mathrm{Op}_Y)^U)^{\mathrm{op}}$ to $\mathrm{Mod}(k)$. The category $((\mathrm{Op}_Y)^U)^{\mathrm{op}}$ is either filtrant or empty. Indeed, if $U \subset f^t(V_1)$ and $U \subset f^t(V_2)$, then $U \subset f^t(V_1 \cap V_2)$. It follows that the sequence

$$0 \to \varinjlim_{U \subset f^t(V)} G'(V) \to \varinjlim_{U \subset f^t(V)} G(V) \to \varinjlim_{U \subset f^t(V)} G''(V)$$

is exact. Hence the functor $G \mapsto f^\dagger \iota_Y G$ from $\mathrm{Mod}(k_Y)$ to $\mathrm{PSh}(k_X)$ is left exact. Since the functor ^{a} is exact, the result follows. q.e.d.

Consider morphisms of sites $f : X \to Y$, $g : Y \to Z$ and $g \circ f : X \to Z$.

Proposition 5.5.8. *One has natural isomorphisms of functors*

$$\begin{aligned} g_* \circ f_* &\simeq (g \circ f)_*, \\ f^{-1} \circ g^{-1} &\simeq (g \circ f)^{-1}. \end{aligned}$$

Proof. The functoriality of direct images is clear by its definition. The functoriality of inverse images follows by adjunction. q.e.d.

Note that inverse image commutes with the functor ^{a} (see Exercise 5.9).

Proposition 5.5.9. (i) *Let $f : X \to Y$ be a morphism of sites, let $F \in \mathrm{Mod}(k_X)$ and let $G \in \mathrm{Mod}(k_Y)$. There is a natural isomorphism in* $\mathrm{Mod}(k_Y)$

$$\mathcal{H}om_{k_Y}(G, f_*F) \xrightarrow{\sim} f_*\mathcal{H}om_{k_X}(f^{-1}G, F).$$

(ii) *Let $G_1, G_2 \in \mathrm{Mod}(k_Y)$. There is a natural isomorphism in* $\mathrm{Mod}(k_X)$

$$f^{-1}(G_1 \otimes_{k_Y} G_2) \xrightarrow{\sim} f^{-1}G_1 \otimes_{k_X} f^{-1}G_2.$$

Proof. The proof is left as an exercise. q.e.d.

Let $V \in \mathrm{Op}_Y$, set $U = f^t(V)$ and denote by f_V the restriction of f to V:

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ U \arrow[u, "i_U"] \arrow[r, "f_V"] & V. \arrow[u, "i_V"'] \end{tikzcd} \tag{5.17}$$

Proposition 5.5.10. *Let $F \in \mathrm{Mod}(k_X)$. Then $i_V^{-1} f_* F \simeq f_{V*} i_U^{-1} F$.*

Proof. Consider the morphisms of presites:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "j_U"'] & Y \arrow[d, "j_V"] \\ U \arrow[r, "f_V"] & V. \end{tikzcd} \tag{5.18}$$

By Proposition 5.5.8, one has $j_{V*} f_* F \simeq f_{V*} j_{U*} F$. q.e.d.

We denote by a_X the canonical map $a_X = X \to \{\mathrm{pt}\}$. (Recall that $\{\mathrm{pt}\}$ is the set with one element.) There is a natural equivalence of categories

$$\begin{array}{rcl} \mathrm{Mod}(k_{\mathrm{pt}}) & \xrightarrow{\sim} & \mathrm{Mod}(k), \\ F & \mapsto & \Gamma(\mathrm{pt}; F). \end{array}$$

In the sequel, we shall identify these two categories.

Examples 5.5.11. (i) Let $F \in \mathrm{Mod}(k_X)$. Then:

$$\Gamma(X; F) \simeq a_{X*} F.$$

(ii) Let $M \in \mathrm{Mod}(k)$. Recall that M_X denotes the sheaf associated with the presheaf $U \mapsto M$. Hence:

$$M_X \simeq a_X^{-1} M_{\{\mathrm{pt}\}}.$$

Let $f \colon X \to Y$ be a continuous map. Since $a_X = a_Y \circ f$, we get

$$M_X \simeq f^{-1} M_Y.$$

(iii) Let $x \in X$ and denote by $i_x : \{x\} \hookrightarrow X$ the embedding. Then

$$i_x^{-1} F \simeq F_x.$$

(iv) Let $i_U : U \hookrightarrow X$ be the inclusion of an open subset of X and let F be a sheaf on X. Then $\Gamma(V; i_U^{-1} F) \simeq \Gamma(V; F)$ for $V \in \mathrm{Op}_U$.

(v) Let $X = Y \bigsqcup Y$, the disjoint union of two copies of Y. Let $f : X \to Y$ be the natural map which induces the identity one each copy of Y. Then $f_* f^{-1} G \simeq G \oplus G$. In fact, if V is open in Y, then we have the isomorphisms

$$\begin{aligned} \Gamma(V; f_* f^{-1} G) &\simeq \Gamma(V \sqcup V; f^{-1} G) \\ &\simeq \Gamma(V; G) \oplus \Gamma(V; G). \end{aligned}$$

(vi) Let $i_S \colon S \hookrightarrow X$ be the embedding of a closed subset S. Let $G \in \mathrm{Mod}(k_S)$. Then $(i_{S*} G)_x$ is isomorphic to G_x if $x \in S$ and to 0 otherwise. It follows that the functor i_{S*} is exact.

Example 5.5.12. Let $f : X \to Y$ be a morphism of complex manifolds. To each open subset $V \subset Y$ is associated a natural "pull-back" map:

$$\Gamma(V; \mathcal{O}_Y) \to \Gamma(V; f_* \mathcal{O}_X)$$

defined by:

$$\varphi \mapsto \varphi \circ f$$

We obtain a morphism $\mathcal{O}_Y \to f_* \mathcal{O}_X$, hence a morphism:

$$f^{-1} \mathcal{O}_Y \to \mathcal{O}_X.$$

For example, if X is closed in Y and f is the injection, $f^{-1} \mathcal{O}_Y$ will be the sheaf on X of holomorphic functions on Y defined in a neighborhood of X. If f is smooth (locally on X, f is isomorphic to a projection $Y \times Z \to Y$), then $f^{-1} \mathcal{O}_Y$ will be the sub-sheaf of $\mathcal{O}_X$ consisting of functions locally constant in the fibers of f.

Examples 5.5.13. (i) Let $i_S : S \hookrightarrow X$ be the embedding of a closed subset S of X. Then the functor i_{S*} is exact.
(ii) Let $i_U : U \hookrightarrow X$ be the embedding of an open subset U of X. Let $F \in \mathrm{Mod}(k_X)$ and let $x \in X$. Then

$$(i_{U*} i_U^{-1} F)_x \simeq \varinjlim_{V \ni x} \Gamma(U \cap V; F).$$

Notation 5.5.14. On $X \times Y$, we denote by q_1 and q_2 the first and second projection, respectively. If $F \in \mathrm{Mod}(k_X)$ and $G \in \mathrm{Mod}(k_Y)$ we set:

$$F \boxtimes G = q_1^{-1} F \otimes q_2^{-1} G.$$

One can recover the functor $\otimes$ from $\boxtimes$. Denote by $\delta : X \hookrightarrow X \times X$ the diagonal embedding and let F_1 and F_2 be in $\mathrm{Mod}(k_X)$. We have:

$$\delta^{-1}(F_1 \boxtimes F_2) = \delta^{-1}(q_1^{-1} F_1 \otimes q_2^{-1} F_2) \simeq F_1 \otimes F_2.$$

5.6 Sheaves associated with a locally closed subset

Let Z be a subset of X. One denotes by

$$i_Z : Z \hookrightarrow X \tag{5.19}$$

the inclusion morphism. One endows Z with the induced topology, and for $F \in \mathrm{Mod}(k_X)$, one sets:

$$\begin{aligned} F|_Z &= i_Z^{-1}F, \\ \Gamma(Z;F) &= \Gamma(Z;F|_Z). \end{aligned}$$

If $Z := U$ is open, these definitions agree with the previous ones. The morphism $F \to i_{Z*}i_Z^{-1}F$ defines the morphism $a_{X*}F \to a_{X*}i_{Z*}i_Z^{-1}F \simeq a_{Z*}i_Z^{-1}F$, hence the morphism:

$$\Gamma(X;F) \to \Gamma(Z;F). \tag{5.20}$$

One denotes by $s|_Z$ the image of a section s of F on X by this morphism.

Replacing X with an open set U containing Z in (5.20), we get the morphism $\Gamma(U;F) \to \Gamma(Z;F)$. Denote by I_Z the category of open subsets containing Z (the morphisms are the inclusions). Then $(I_Z)^{\mathrm{op}}$ is filtrant, F defines a functor $(I_Z)^{\mathrm{op}} \to \mathrm{Mod}(k)$ and we get a morphism

$$\varinjlim_{U \supset Z} \Gamma(U;F) \to \Gamma(Z;F). \tag{5.21}$$

The morphism (5.21) is injective. Indeed, if a section $s \in \Gamma(U;F)$ is zero in $\Gamma(Z;F)$, then $s_x = 0$ for all $x \in Z$, hence $s = 0$ on an open neighborhood of Z. One shall be aware that the morphism (5.21) is not an isomorphism in general. There is a classical result which asserts that if X is paracompact (*e.g.,* if X is locally compact and countable at infinity) and Z is closed, then (5.21) is bijective.

A subset Z of a topological space X is relatively Hausdorff if two distinct points in Z admit disjoint neighborhoods in X. If $Z = X$, one says that X is Hausdorff.

Proposition 5.6.1. *Let Z be compact subset of X, relatively Hausdorff in X and let $F \in \mathrm{Mod}(k_X)$. Then the natural morphism* (5.21) *is an isomorphism.*

Proof. [1] Let $s \in \Gamma(Z;F|_Z)$. There exist a finite family of open subsets $\{U_i\}_{i=1}^n$ covering Z and sections $s_i \in \Gamma(U_i;F)$ such that $s_i|_{Z\cap U_i} = s|_{Z\cap U_i}$.

[1] The proof may be skipped

Moreover, we may find another family of open sets $\{V_i\}_{i=1}^n$ covering Z such that $Z \cap \bar{V}_i \subset U_i$. We shall glue together the sections s_i on a neighborhood of Z. For that purpose we may argue by induction on n and assume $n = 2$. Set $Z_i = Z \cap \bar{V}_i$. Then $s_1|_{Z_1 \cap Z_2} = s_2|_{Z_1 \cap Z_2}$. Let W be an open neighborhood of $Z_1 \cap Z_2$ such that $s_1|_W = s_2|_W$ and let $W_i (i = 1, 2)$ be an open subset of U_i such that $W_i \supset Z_i \setminus W$ and $W_1 \cap W_2 = \emptyset$. Such W_i's exist thanks to the hypotheses. Set $U'_i = W_i \cup W$, $(i = 1, 2)$. Then $s_1|_{U'_1 \cap U'_2} = s_2|_{U'_1 \cap U'_2}$. This defines $t \in \Gamma(U'_1 \cup U'_2; F)$ with $t|_Z = s$. q.e.d.

The case of open subsets

Let U be an open subset. Recall that we have the morphisms of sites

$$\begin{aligned} j_U \colon X \to U, &\qquad U \supset V \mapsto V \subset X, \\ i_U \colon U \to X, &\qquad X \supset V \mapsto U \cap V \subset U. \end{aligned}$$

Hence, we have the pairs of adjoint functors (i_U^{-1}, i_{U*}) and (j_U^{-1}, j_{U*}). Clearly, there is an isomorphism of functors $j_{U*} \simeq i_U^{-1} \colon \mathrm{Mod}(k_X) \to \mathrm{Mod}(k_U)$.

Definition 5.6.2. (i) One defines the functor $i_{U!} \colon \mathrm{Mod}(k_U) \to \mathrm{Mod}(k_X)$ by setting $i_{U!} := j_U^{-1}$.

(ii) For $F \in \mathrm{Mod}(k_X)$, one sets $F_U := i_{U!} i_U^{-1} F = j_U^{-1} j_{U*} F$.

(iii) For $F \in \mathrm{Mod}(k_X)$, one sets $\Gamma_U F := i_{U*} i_U^{-1} F = i_{U*} j_{U*} F$.

(iv) One sets $k_{XU} := (k_X)_U$ for short.

Hence, we have the functors

$$\begin{tikzcd} \mathrm{Mod}(k_X) \arrow[r, "i_U^{-1}" description] & \mathrm{Mod}(k_U), \arrow[l, bend right, "i_{U!}"'] \arrow[l, bend left, "i_{U*}"] \end{tikzcd}$$

and the pairs of adjoint functors

$$(i_U^{-1}, i_{U*}) \qquad (i_{U!}, i_U^{-1}) \simeq (j_U^{-1}, j_{U*}).$$

Note that $i_{U!} i_U^{-1} F \to F$ defines the morphism $F_U \to F$ and $F \to i_{U*} i_U^{-1} F$ defines the morphism $F \to \Gamma_U F$.

Moreover

$$\text{(5.22)} \qquad \begin{cases} F_U|_U & \simeq F|_U, \\ F_U|_{X \setminus U} & \simeq 0, \end{cases} \text{ in particular } \begin{cases} (F_U)_x & \simeq F_x \text{ if } x \in U, \\ (F_U)_x & \simeq 0 \text{ otherwise.} \end{cases}$$

Proposition 5.6.3. *Let $U \subset X$ be an open subset and let $F \in \mathrm{Mod}(k_X)$.*

(i) *The functor $F \mapsto F_U$ is exact.*

(ii) *Let V be another open subset. Then $(F_U)_V = F_{U\cap V}$.*

(iii) *We have $\Gamma(U;F) \simeq \mathrm{Hom}_{k_X}(k_{XU}, F)$.*

(iv) *We have $F_U \simeq F \otimes k_{XU}$.*

(v) *Let U_1 and U_2 be two open subsets of X. Then the sequence below is exact:*

$$0 \to F_{U_1\cap U_2} \xrightarrow{\alpha} F_{U_1} \oplus F_{U_2} \xrightarrow{\beta} F_{U_1\cup U_2} \to 0. \tag{5.23}$$

Here $\alpha = (\alpha_1, \alpha_2)$ and $\beta = \beta_1 - \beta_2$ are induced by the natural morphisms $\alpha_i : F_{U_1\cap U_2} \to F_{U_i}$ and $\beta_i : F_{U_i} \to F_{U_1\cup U_2}$.

Proof. (i) follows from (5.22).
(ii) By (5.22), we have the isomorphisms $F_{U\cap V} \xleftarrow{\sim} (F_{U\cap V})_V \xrightarrow{\sim} (F_U)_V$.
(iii) We have the isomorphisms

$$\begin{aligned}
\mathrm{Hom}_{k_X}(k_{XU}, F) &= \mathrm{Hom}_{k_X}(j_U^{-1} j_{U*} k_X, F)\\
&\simeq \mathrm{Hom}_{k_U}(j_{U*} k_X, j_{U*} F)\\
&\simeq \mathrm{Hom}_{k_U}(k_U, F|_U) \simeq F(U).
\end{aligned}$$

(iv)–(v) are proved similarly as (ii). q.e.d.

Note that the functor $\Gamma_U(\bullet)\colon \mathrm{Mod}(k_X) \to \mathrm{Mod}(k_X)$ is left exact and moreover $\Gamma(U;F) \simeq \Gamma(X;\Gamma_U F)$. Indeed,

$$\begin{aligned}
\Gamma(X;\Gamma_U F) &\simeq \mathrm{Hom}\,(k_X, \Gamma_U F)\\
&= \mathrm{Hom}\,(k_X, i_{U*} i_U^{-1} F)\\
&\simeq \mathrm{Hom}\,(i_U^{-1} k_X, i_U^{-1} F) \simeq \Gamma(U;F).
\end{aligned}$$

The case of closed subsets

Definition 5.6.4. Let S be a closed subset of X.

(i) For $F \in \mathrm{Mod}(k_X)$, one sets $F_S = i_{S*} i_S^{-1} F$.

(ii) One sets $k_{XS} := (k_X)_S$ for short.

Note that $F \to i_{S*}i_S^{-1}F$ defines the morphism $F \to F_S$. Moreover

$$(5.24)\quad \begin{cases} F_S|_S & \simeq F|_S, \\ F_S|_{X\setminus S} & \simeq 0, \end{cases} \text{in particular} \begin{cases} (F_S)_x & \simeq F_x \text{ if } x \in S, \\ (F_S)_x & \simeq 0 \text{ otherwise.} \end{cases}$$

Proposition 5.6.5. *Let $S \subset X$ be a closed subset and let $F \in \mathrm{Mod}(k_X)$.*

(i) *Set $U := X \setminus S$. Then the sequence $0 \to F_U \to F \to F_S \to 0$ is exact in* $\mathrm{Mod}(k_X)$.

(ii) *The functor $F \mapsto F_S$ is exact.*

(iii) *Let S' be another closed subset. Then $(F_S)_{S'} = F_{S\cap S'}$.*

(iv) *We have $F_S \simeq F \otimes k_{XS}$.*

(v) *Let S_1 and S_2 be two closed subsets of X. Then the sequence below is exact:*

$$(5.25)\qquad 0 \to F_{S_1\cup S_2} \xrightarrow{\alpha} F_{S_1} \oplus F_{S_2} \xrightarrow{\beta} F_{S_1\cap S_2} \to 0.$$

Here $\alpha = (\alpha_1, \alpha_2)$ and $\beta = \beta_1 - \beta_2$ are induced by the natural morphisms $\alpha_i : F_{S_1\cup S_2} \to F_{S_i}$ and $\beta_i : F_{S_i} \to F_{S_1\cap S_2}$.

Proof. The proof is similar to that of Proposition5.6.3. q.e.d.

The case of locally closed subsets

A subset Z of X is locally closed if there exists an open neighborhood U of Z such that Z is closed in U. Equivalently, $Z = S \cap U$ with U open and S closed in X. In this case, one sets

$$F_Z := (F_U)_S.$$

One checks easily that this definition depends only on Z, not on the choice of U and S. Moreover, (5.24) still holds with Z instead of S.

5.7 Locally constant and locally free sheaves

Locally constant sheaves

Definition 5.7.1. (i) Let M be a k-module. Recall that the sheaf M_X is the sheaf of locally constant M-valued functions on X. It is also the sheaf associated with the constant presheaf $U \mapsto M$.

(ii) A sheaf F on X is constant if it is isomorphic to a sheaf M_X, for some $M \in \mathrm{Mod}(k)$.

(iii) A sheaf F on X is locally constant if there exists an open covering $X = \bigcup_i U_i$ such that $F|_{U_i}$ is a constant sheaf of U_i.

Recall that a morphism of sheaves which is locally an isomorphism is an isomorphism of sheaves. However, given two sheaves F and G, it may exist an open covering $\{U_i\}_{i\in I}$ of X and isomorphisms $F|_{U_i} \xrightarrow{\sim} G|_{U_i}$ for all $i \in I$, although these isomorphisms are not induced by a globally defined isomorphism $F \to G$.

Example 5.7.2. Consider $X = \mathbb{R}$ and consider the $\mathbb{C}$-valued function $t \mapsto \exp(t)$, that we simply denote by $\exp(t)$. Consider the sheaf $\mathbb{C}_X \cdot \exp(t)$ consisting of functions which are locally a constant multiple of $\exp(t)$. Clearly $\mathbb{C}_X \cdot \exp(t)$ is isomorphic to the constant sheaf $\mathbb{C}_X$, hence, is a constant sheaf. Note that this sheaf may also be defined by the exact sequence

$$0 \to \mathbb{C}_X \cdot \exp(t) \to C^\infty_X \xrightarrow{P} C^\infty_X \to 0$$

where P is the differential operator $\dfrac{\partial}{\partial t} - 1$.

Examples 5.7.3. (i) If X is not connected it is easy to construct locally constant sheaves which are not constant. Indeed, let $X = U_1 \sqcup U_2$ be a covering by two non-empty open subsets, with $U_1 \cap U_2 = \emptyset$. Let $M \in \mathrm{Mod}(k)$ with $M \neq 0$. Then the sheaf which is 0 on U_1 and M_{U_2} on U_2 is locally constant and not constant.
(ii) Let $X = Y = \mathbb{C} \setminus \{0\}$, and let $f : X \to Y$ be the map $z \mapsto z^2$, where z denotes a holomorphic coordinate on $\mathbb{C}$. If D is an open disk in Y, $f^{-1}D$ is isomorphic to the disjoint union of two copies of D. Hence, the sheaf $f_*k_X|_D$ is isomorphic to k_D^2, the constant sheaf of rank two on D. However, $\Gamma(Y; f_*k_X) = \Gamma(X; k_X) = k$, which shows that the sheaf f_*k_X is locally constant but not constant.
(iii) Let $X = \mathbb{C} \setminus \{0\}$ with holomorphic coordinate z and consider the differential operator $P = z\frac{\partial}{\partial z} - \alpha$, where $\alpha \in \mathbb{C} \setminus \mathbb{Z}$. Let us denote by K_α the kernel of P acting on $\mathcal{O}_X$.

Let U be an open disk in X centered at z_0, and let $A(z)$ denote a primitive of α/z in U. We have a commutative diagram of sheaves on U:

$$\begin{tikzcd}
\mathcal{O}_X \arrow[r, "z\frac{\partial}{\partial z}-\alpha"] \arrow[d, "\exp(-A(z))"'] & \mathcal{O}_X \arrow[d, "\frac{1}{z}\exp(-A(z))"] \\
\mathcal{O}_X \arrow[r, "\frac{\partial}{\partial z}"] & \mathcal{O}_X
\end{tikzcd}$$

Therefore, one gets an isomorphism of sheaves $K_\alpha|_U \xrightarrow{\sim} \mathbb{C}_X|_U$, which shows that K_α is locally constant, of rank one.

On the other hand, $f \in \mathcal{O}(X)$ and $Pf = 0$ implies $f = 0$. Hence $\Gamma(X; K_\alpha) = 0$, and K_α is a locally constant sheaf of rank one on $\mathbb{C} \setminus \{0\}$ which is not constant.

Let us show that all locally constant sheaf on the interval $[0,1]$ are constant.

Recall that for $M \in \mathrm{Mod}(k)$, M_X is the constant sheaf with stalk M.

Lemma 5.7.4. *Let $M, N \in \mathrm{Mod}(k)$. Then*

(i) $(M \otimes N)_X \simeq M_X \otimes N_X$,

(ii) $(\mathrm{Hom}\,(M, N))_X \simeq \mathcal{H}om_{k_X}(M_X, N_X)$.

The proof is left as an exercise.

Lemma 5.7.5. *let $X = U_1 \cup U_2$ be a covering of X by two open sets. Let F be a sheaf on X and assume that:*

(i) *$U_{12} = U_1 \cap U_2$ is connected and non empty,*

(ii) *$F|_{U_i}$ $(i = 1, 2)$ is a constant sheaf.*

Then F is a constant sheaf.

Proof. By the hypothesis, there is $M_i \in \mathrm{Mod}(k)$ and isomorphisms $\theta_i : F|_{U_i} \xrightarrow{\sim} (M_i)_X|_{U_i}$ $(i = 1, 2)$. Since $U_1 \cap U_2$ is non empty and connected, $M_1 \simeq M_2$ and we may assume $M_1 = M_2 = M$. define the isomorphism $\theta_{12} = \theta_1 \circ \theta_2^{-1} \colon M_X|_{U_1 \cap U_2} \xrightarrow{\sim} M_X|_{U_1 \cap U_2}$. Since $U_1 \cap U_2$ is connected and non empty, $\Gamma(U_1 \cap U_2; \mathcal{H}om\,(M_X, M_X)) \simeq \mathrm{Hom}\,(M, M)$ by Lemma 5.7.4. Hence, θ_{12} defines an invertible element of $\mathrm{Hom}\,(M, M)$. Using the map $\mathrm{Hom}\,(M, M) \to \Gamma(X; \mathcal{H}om\,(M_X, M_X))$, we find that θ_{12} extends as an isomorphism $\theta : M_X \simeq M_X$ all over X. Now define the isomorphisms: $\alpha_i : F|_{U_i} \xrightarrow{\sim} (M_X)|_{U_i}$ by $\alpha_1 = \theta_1$ and $\alpha_2 = \theta|_{U_2} \circ \theta_2$. By Corollary 5.4.3, α_1 and α_2 will glue together to define an isomorphism $F \xrightarrow{\sim} M_X$. q.e.d.

Proposition 5.7.6. *Let I denote the interval* $[0, 1]$.

(i) *Let F be a locally constant sheaf on I. Then F is a constant sheaf.*

(ii) *In particular, if $t \in I$, the morphism $\Gamma(I; F) \to F_t$ is an isomorphism.*

(iii) *Moreover, if* $F = M_I$ *for a* k*-module* M*, then the composition*

$$M \simeq F_0 \xleftarrow{\sim} \Gamma(I; M_I) \xrightarrow{\sim} F_1 \simeq M$$

is the identity of M*.*

Proof. (i) We may find a finite open covering $U_i, (i = 1, \dots, n)$ such that F is constant on U_i, $U_i \cap U_{i+1}$ $(1 \le i < n)$ is non empty and connected and $U_i \cap U_j = \emptyset$ for $|i - j| > 1$. By induction, we may assume that $n = 2$. Then the result follows from Lemma 5.7.5.
(ii)–(iii) are obvious. q.e.d.

Locally free sheaves

A sheaf of k-algebras (or, equivalently, a k_X-algebra) $\mathcal{A}$ on X is a sheaf of k-modules such that for each $U \subset X$, $\mathcal{A}(U)$ is endowed with a structure of a k-algebra, and the operations (addition, multiplication) commute to the restriction morphisms. A sheaf of $\mathbb{Z}$-algebras is simply called a sheaf of rings. If $\mathcal{A}$ is a sheaf of rings, one defines in an obvious way the notion of a sheaf F of (left) $\mathcal{A}$-modules (or simply, an $\mathcal{A}$-module) as follows: for each open set $U \subset X$, $F(U)$ is an $\mathcal{A}(U)$-module and the action of $\mathcal{A}(U)$ on $F(U)$ commutes to the restriction morphisms. One also naturally defines the notion of an $\mathcal{A}$-linear morphism of $\mathcal{A}$-modules. Hence we have defined the category $\mathrm{Mod}(\mathcal{A})$ of $\mathcal{A}$-modules.

Examples 5.7.7. (i) Let A be a k-algebra. The constant sheaf A_X is a sheaf of k-algebras.
(ii) On a topological space, the sheaf $\mathcal{C}^0_X$ is a $\mathbb{C}_X$-algebra. If X is open in $\mathbb{R}^n$, the sheaf $\mathcal{C}^\infty_X$ is a $\mathbb{C}_X$-algebra. The sheaf $\mathcal{D}b_X$ is a $\mathcal{C}^\infty_X$-module.
(iii) If X is open in $\mathbb{C}^n$, the sheaf $\mathcal{O}_X$ is a $\mathbb{C}_X$-algebra.

The category $\mathrm{Mod}(\mathcal{A})$ is clearly an additive subcategory of $\mathrm{Mod}(k_X)$. Moreover, if $\varphi : F \to G$ is a morphism of $\mathcal{A}$-modules, then $\operatorname{Ker}\varphi$ and $\operatorname{Coker}\varphi$ will be $\mathcal{A}$-modules. One checks easily that the category $\mathrm{Mod}(\mathcal{A})$ is abelian, and the natural functor $\mathrm{Mod}(\mathcal{A}) \to \mathrm{Mod}(k_X)$ is exact and faithful (but not fully faithful). Moreover, the category $\mathrm{Mod}(\mathcal{A})$ admits inductive and projective limits and filtrant inductive limits are exact. Now consider a sheaf of rings $\mathcal{A}$.

Definition 5.7.8. (i) A sheaf $\mathcal{L}$ of $\mathcal{A}$-modules is locally free of rank k (resp. of finite rank) if there exists an open covering $X = \cup_i U_i$ such that $\mathcal{L}|_{U_i}$ is isomorphic to a direct sum of k copies (resp. to a finite direct sum) of $\mathcal{A}|_{U_i}$.

(ii) A locally free sheaf of rank one is called an invertible sheaf.

We shall construct locally constant and locally free sheaves by gluing sheaves in the §5.8.

5.8 Gluing sheaves

Let X be a topological space, and let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open covering of X. One sets $U_{ij} = U_i \cap U_j, U_{ijk} = U_{ij} \cap U_k$. First, consider a sheaf F on X, set $F_i = F|_{U_i}$, $\theta_i \colon F|_{U_i} \xrightarrow{\sim} F_i$, $\theta_{ji} = \theta_j \circ \theta_i^{-1}$. Then clearly:

$$\left.\begin{aligned} \theta_{ii} &= \text{id on } U_i, \\ \theta_{ij} \circ \theta_{jk} &= \theta_{ik} \text{ on } U_{ijk}. \end{aligned}\right\} \tag{5.26}$$

The family of isomorphisms $\{\theta_{ij}\}$ satisfying conditions (5.26) is called a 1-cocycle. Let us show that one can reconstruct F from the data of a 1-cocycle.

Theorem 5.8.1. *Let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open covering of X and let F_i be a sheaf on U_i. Assume to be given for each pair (i, j) an isomorphism of sheaves $\theta_{ji} : F_i|_{U_{ij}} \xrightarrow{\sim} F_j|_{U_{ij}}$, these isomorphisms satisfying the conditions (5.26).*

Then there exists a sheaf F on X and for each i isomorphisms $\theta_i : F|_{U_i} \xrightarrow{\sim} F_i$ such that $\theta_j = \theta_{ji} \circ \theta_i$. Moreover, $(F, \{\theta_i\}_{i\in I})$ is unique up to unique isomorphism.

Clearly, if the F_i's are locally constant, then F is locally constant.

Sketch of proof. (i) Existence. For each open subset V of X, define $F(V)$ as the submodule of $\prod_{i\in I} F_i(V \cap U_i)$ consisting of families $\{s_i\}_i$ such that for any $(i, j) \in I \times I$, $\theta_{ji}(s_i|_{V\cap U_{ji}}) = s_j|_{V\cap U_{ji}}$. One checks that the presheaf so obtained is a sheaf, and the isomorphisms θ_i's are induced by the projections $\prod_{k\in I} F_k(V \cap U_k) \to F_i(V \cap U_i)$.
(ii) Unicity. Let $\theta_i : F|_{U_i} \xrightarrow{\sim} F_i$ and $\lambda_i : G|_{U_i} \xrightarrow{\sim} F_i$. Then the isomorphisms $\lambda_i^{-1} \circ \theta_i : F|_{U_i} \to G|_{U_i}$ will glue as an isomorphism $G \xrightarrow{\sim} F$ on X, by Proposition 5.4.2. q.e.d.

Remark 5.8.2. Let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open covering of X and let $\{F_i, G_i\}_{i\in I}$ be families of sheaves on the U_i's as in Theorem 5.8.1, both satisfying conditions (5.26). Let F and G be the associated sheaves on X given by the theorem. Then on easily checks that

$$\operatorname{Hom}_{k_X}(F, G) \simeq \varprojlim_i \operatorname{Hom}_{k_{U_i}}(F_i, G_i). \tag{5.27}$$

Example 5.8.3. Assume k is a field, and recall that $k^\times$ denote the multiplicative group $k \setminus \{0\}$. Let $X = \mathbb{S}^1$ be the 1-sphere, and consider a covering of X by two open connected intervals U_1 and U_2. Let $U_{12}^\pm$ denote the two connected components of $U_1 \cap U_2$. Let $\alpha \in k^\times$. One defines a locally constant sheaf L_α on X of rank one over k by gluing k_{U_1} and k_{U_2} as follows. Let $\theta_\varepsilon : k_{U_1}|_{U_{12}^\varepsilon} \to k_{U_2}|_{U_{12}^\varepsilon}$ ($\varepsilon = \pm$) be defined by $\theta_+ = 1$, $\theta_- = \alpha$.

Assume that $k = \mathbb{C}$. One can give a more intuitive description of the sheaf L_α as follows. Let us identify $\mathbb{S}^1$ with $[0, 2\pi]/\sim$, where $\sim$ is the relation which identifies 0 and 2π. Choose $\beta \in \mathbb{C}$ with $\exp(i\beta) = \alpha$. If $\beta \notin \mathbb{Z}$, the function $\theta \mapsto \exp(i\beta\theta)$ is not well defined on $\mathbb{S}^1$ since it does not take the same value at 0 and at 2π. However, the sheaf $\mathbb{C}_X \cdot \exp(i\beta\theta)$ of functions which are a constant multiple of the function $\exp(i\beta\theta)$ is well-defined on each of the intervals U_1 and U_2, hence is well defined on $\mathbb{S}^1$, although it does not have any global section.

Example 5.8.4. Consider an n-dimensional real manifold X of class $\mathcal{C}^\infty$, and let $\{X_i, f_i\}$ be an atlas, that is, the X_i are open subsets of X and $f_i : X_i \xrightarrow{\sim} U_i$ is a $\mathcal{C}^\infty$-isomorphism with an open subset U_i of $\mathbb{R}^n$. Let $U_{ij}^i = f_i(X_{ij})$ and denote by f_{ji} the map

$$f_{ji} = f_j|_{X_{ij}} \circ f_i^{-1}|_{U_{ij}^i} : U_{ij}^i \to U_{ij}^j. \tag{5.28}$$

The maps f_{ji} are called the transition functions. They are isomorphisms of class $\mathcal{C}^\infty$. Denote by J_f the Jacobian matrix of a map $f : \mathbb{R}^n \supset U \to V \subset \mathbb{R}^n$. Using the formula $J_{g \circ f}(x) = J_g(f(x)) \circ J_f(x)$, one gets that the locally constant function on X_{ij} defined as the sign of the Jacobian determinant $\det J_{f_{ji}}$ of the f_{ji}'s is a 1-cocycle. It defines a sheaf locally isomorphic to $\mathbb{Z}_X$ called the orientation sheaf on X and denoted by or_X.

Remark 5.8.5. In the situation of Theorem 5.8.1, if $\mathcal{A}$ is a sheaf of k-algebras on X and if all F_i's are sheaves of $\mathcal{A}|_{U_i}$ modules and the isomorphisms θ_{ji} are $\mathcal{A}|_{U_{ij}}$-linear, the sheaf F constructed in Theorem 5.8.1 will be naturally endowed with a structure of a sheaf of $\mathcal{A}$-modules.

Example 5.8.6. (i) Let $X = \mathbb{P}^1(\mathbb{C})$, the Riemann sphere. Then $\Omega_X := \Omega_X^1$ is locally free of rank one over $\mathcal{O}_X$. Since $\Gamma(X; \Omega_X) = 0$, this sheaf is not globally free.
(ii) Consider the covering of X by the two open sets $U_1 = \mathbb{C}$, $U_2 = X \setminus \{0\}$. One can glue $\mathcal{O}_X|_{U_1}$ and $\mathcal{O}_X|_{U_2}$ on $U_1 \cap U_2$ by using the isomorphism $f \mapsto z^p f$ ($p \in \mathbb{Z}$). One gets a locally free sheaf of rank one. For $p \neq 0$ this sheaf is not free.

Exercises to Chapter 5

Exercise 5.1. Let S (resp. U) be a closed (resp. an open) subset of X and let $F \in \mathrm{Mod}(k_X)$.
(i) Prove the isomorphism $\Gamma(X; F_S) \simeq \Gamma(S; F|_S)$.
(ii) Construct the morphism $\Gamma(X; F_U) \to \Gamma(U; F)$ and prove that it is not an isomorphism in general.

Exercise 5.2. Assume that $X = \mathbb{R}$, let S be a non-empty closed interval and let $U = X \setminus S$.
(i) Prove that the natural map $\Gamma(X; k_X) \to \Gamma(X; k_{XS})$ is surjective and deduce that $\Gamma(X; k_{XU}) \simeq 0$.
(ii) Let $x \in \mathbb{R}$.Prove that the morphism $k_X \to k_{X\{x\}}$ does not split.

Exercise 5.3. Let $F \in \mathrm{Mod}(k_X)$. Define $\widetilde{F} \in \mathrm{Mod}(k_X)$ by $\widetilde{F} = \bigoplus_{x \in X} F_{\{x\}}$. (Here, $F_{\{x\}} \in \mathrm{Mod}(k_X)$ and the direct sum is calculated in $\mathrm{Mod}(k_X)$, not in $\mathrm{PSh}(k_X)$.) Prove that F_x and $\widetilde{F}_x$ are isomorphic for all $x \in X$, although F and $\widetilde{F}$ are not isomorphic in general.

Exercise 5.4. Let $Z = Z_1 \sqcup Z_2$ be the disjoint union of two sets Z_1 and Z_2 in X.
(i) Assume that Z_1 and Z_2 are both open (resp. closed) in X. Prove that $k_{XZ} \simeq k_{XZ_1} \oplus k_{XZ_2}$.
(ii) Give an example which shows that (i) is no more true if one only assume that Z_1 and Z_2 are both locally closed.

Exercise 5.5. Let $X = \mathbb{R}^2$, $Y = \mathbb{R}$, $S = \{(x, y) \in X; xy \geq 1\}$, and let $f : X \to Y$ be the map $(x, y) \mapsto y$. calculate $f_* k_{XS}$.

Exercise 5.6. Let $f : X \to Y$ be a continuous map, and let Z be a closed subset of Y. Construct the natural isomorphism $f^{-1} k_{YZ} \xrightarrow{\sim} k_{X(f^{-1}Z)}$.

Exercise 5.7. Assume that X is a compact space and let $\{F_i\}_{i \in I}$ be a filtrant inductive system of sheaves on X. Prove the isomorphism $\varinjlim\limits_i \Gamma(X; F_i) \xrightarrow{\sim} \Gamma(X; \varinjlim\limits_i F_i)$.

Exercise 5.8. Let S be a set endowed with the discrete topology, let $p : X \times S \to X$ denote the projection and let $F \in \mathrm{Mod}(k_{X \times S})$ and set $F_s = F|_{X \times \{s\}}$. Prove that $p_* F \simeq \prod_{s \in S} F_s$.

Exercise 5.9. Let $f : X \to Y$ be a continuous map and let $G \in \mathrm{PSh}(k_Y)$. Prove the isomorphism $(f^\dagger G)^a \xrightarrow{\sim} f^{-1}(G^a)$.

Exercise 5.10. Let $X = \bigcup_i U_i$ be an open covering of X and let $F \in \mathrm{PSh}(k_X)$. Assume that $F|_{U_i}$ is a sheaf for all $i \in I$. Prove that F is a sheaf. (Hint: compare F with the sheaf constructed in Theorem 5.8.1.)

Exercise 5.11. Let M be a k-module and let X be an open subset of $\mathbb{R}^n$. Let F be a presheaf such that for any non empty convex open subsets $U \subset X$, there exists an isomorphism $F(U) \simeq M$ and this isomorphism is compatible to the restriction morphisms for $V \subset U$. Prove that the associated sheaf is locally constant.

Exercise 5.12. Prove Lemma 5.7.4.

Exercise 5.13. Assume k is a field, and let L be a locally constant sheaf of rank one over k (hence, L is locally isomorphic to the sheaf k_X). Set $L^* = \mathcal{H}om\,(L, k_X)$.
(i) Prove the isomorphisms $L^* \otimes L \xrightarrow{\sim} k_X$ and $k_X \xrightarrow{\sim} \mathcal{H}om\,(L, L)$.
(ii) Assume that k is a field, X is connected and $\Gamma(X; L) \neq 0$. Prove that $L \simeq k_X$. (Hint: $\Gamma(X; L) \simeq \Gamma(X; \mathcal{H}om\,(k_X, L))$.)

Chapter 6

Cohomology of sheaves

We first show that the category of abelian sheaves has enough injective objects. This allows us to derive all left exact functors we have constructed.

Next we prove an important theorem which asserts that the cohomology of constant sheaves is a homotopy invariant. Using Čech's resolutions by closed contractible coverings, this allows us to calculate the cohomology of some classical manifolds. We also have a glance to soft sheaves and De Rham cohomology.

Some references: [14], [4], [12], [21], [22].

6.1 Cohomology of sheaves

A sheaf F of k-modules is injective if it is an injective object in the category $\mathrm{Mod}(k_X)$.

Lemma 6.1.1. (i) *Let X and Y be two topological spaces and let $f : X \to Y$ be a morphism of sites. Assume that $F \in \mathrm{Mod}(k_X)$ is injective. Then f_*F is injective in $\mathrm{Mod}(k_Y)$.*

(ii) *Let $i_U : U \hookrightarrow X$ be an open embedding and let $F \in \mathrm{Mod}(k_X)$ be injective. Then $i_U^{-1}F$ is injective in $\mathrm{Mod}(k_U)$.*

Proof. (i) follows immediately from the adjunction formula:

$$\mathrm{Hom}_{k_X}(f^{-1}(\cdot), F) \simeq \mathrm{Hom}_{k_Y}(\cdot, f_*F)$$

and the fact that the functor f^{-1} is exact.
(ii) follows from (i) since $i_U^{-1} \simeq j_{U*}$. q.e.d.

Theorem 6.1.2. *The category $\mathrm{Mod}(k_X)$ admits enough injectives.*

Proof. (i) When $X = \{\mathrm{pt}\}$, the result follows from $\mathrm{Mod}(k_X) \simeq \mathrm{Mod}(k)$.
(ii) Assume X is discrete. Then for $F, G \in \mathrm{Mod}(k_X)$, the natural morphism

$$\mathrm{Hom}_{k_X}(G, F) \to \prod_{x \in X} \mathrm{Hom}_k(G_x, F_x)$$

is an isomorphism. Since products are exact in $\mathrm{Mod}(k)$, it follows that a sheaf F is injective as soon as each F_x is injective. For each $x \in X$, choose an injective module I_x together with a monomorphism $F_x \rightarrowtail I_x$ and define the sheaf F^0 on X by setting $\Gamma(U, F^0) = \prod_{x \in U} I_x$. Since the topology on X is discrete, $(F^0)_x = I_x$. Therefore the sequence $0 \to F \to F^0$ is exact and F^0 is injective.
(iii) Let $\hat{X}$ denote the set X endowed with the discrete topology, and let $f : \hat{X} \to X$ be the identity map. Let $F \in \mathrm{Mod}(k_X)$. There exists an injective sheaf G^0 on $\hat{X}$ and a monomorphism $0 \to f^{-1}F \to G^0$. Then f_*G^0 is injective in $\mathrm{Mod}(k_X)$ and the sequence $0 \to f_*f^{-1}F \to f_*G^0$ is exact. To conclude, notice that the morphism $F \to f_*f^{-1}F$ is a monomorphism, since on an open subset U of X it is defined by $F(U) \to \prod_{x \in U} F_x$. q.e.d.

It is now possible to derive all left exact functors defined on the category of sheaves, as well as the bifunctors Hom_{k_X} and $\mathcal{H}om_{k_X}$. The derived functors of these two bifunctors are respectively denoted by $\mathrm{Ext}^j_{k_X}$ and $\mathcal{E}xt^j_{k_X}$. Recall that, for $G, F \in \mathrm{Mod}(k_X)$, the k-module $\mathrm{Ext}^j_{k_X}(G, F)$ is calculated as follows. Choose an injective resolution $F^\bullet$ of F. Then

$$\mathrm{Ext}^j_{k_X}(G, F) \simeq H^j(\mathrm{Hom}_{k_X}(G, F^\bullet)). \tag{6.1}$$

Let $F \in \mathrm{Mod}(k_X)$ and let U (resp. S, resp. Z) be an open (resp. a closed, resp. a locally closed) subset of X.

As usual, one denotes by $i_Z : Z \hookrightarrow X$ the embedding of Z in X and by a_Z the map $Z \to \{\mathrm{pt}\}$. Note that $a_Z = a_X \circ i_Z$. Recall that for a sheaf F on X, we have set: $\Gamma(Z; F) = \Gamma(Z; F|_Z)$. One sets

$$H^j(U; F) \;=\; R^j\Gamma(U; \cdot)(F). \tag{6.2}$$

By (6.1), we have

$$H^j(U; F) \simeq \mathrm{Ext}^j_{k_X}(k_{XU}, F). \tag{6.3}$$

Proposition 6.1.3. (i) *If U is open in X, then $H^j(U; F) \simeq H^j(U; F|_U)$.*

(ii) *If S is closed in X, then $H^j(X; F_S) \simeq H^j(S; F|_S)$.*

(iii) *If K is compact and relatively Hausdorff in X, then the natural morphism* $\varinjlim_{U \supset K} H^j(U;F) \to H^j(K;F)$ *is an isomorphism.*

Proof. (i) We have the chain of isomorphisms:

$$\begin{aligned} H^j(U;F) &\simeq R^j(a_{U*}i_U^{-1})F \simeq (R^j a_{U*})i_U^{-1}F \\ &\simeq H^j(U;F|_U). \end{aligned}$$

The second isomorphism follows from the fact that i_U^{-1} is exact and sends injective sheaves to injective sheaves (Lemma 4.6.9 (iv)).
(ii) We have the chain of isomorphisms:

$$\begin{aligned} H^j(X;F_S) &\simeq (R^j a_{X*})(i_{S*}i_S^{-1})F \simeq R^j(a_X \circ i_{S*})i_S^{-1}F \\ &\simeq H^j(S;F|_S). \end{aligned}$$

The second isomorphism follows from the fact that i_{S*} is exact and sends injective sheaves to injective sheaves (Lemma 4.6.9 (iv)).
(iii) The result is true for $j=0$ by Proposition 5.6.1. Consider an injective resolution $F \to F^\bullet$ of F. Then

$$\begin{aligned} H^j(K;F) &\simeq H^j(\Gamma(K;F^\bullet)) \simeq H^j(\varinjlim_{U \supset K} \Gamma(U;F^\bullet)) \\ &\simeq \varinjlim_{U \supset K} H^j(\Gamma(U;F^\bullet)) \simeq \varinjlim_{U \supset K} H^j(U;F). \end{aligned}$$

q.e.d.

Let $0 \to F' \to F \to F'' \to 0$ be an exact sequence of sheaves. Applying a left exact functor Ψ to it, we obtain a long exact sequence

$$0 \to \Psi(F') \to \Psi(F) \to \cdots \to R^j\Psi(F) \to R^j\Psi(F'') \to R^{j+1}\Psi(F') \to \cdots$$

For example, applying the functor $\Gamma(X;\cdot)$, we get a long exact sequence

$$H^{k-1}(X;F'') \to H^k(X;F') \to H^k(X;F) \to H^k(X;F'') \to \cdots \tag{6.4}$$

There are similar results, replacing $\Gamma(X;\cdot)$ with other functors, such as f_*.

Proposition 6.1.4. *Let S_1 and S_2 be two closed subsets of X and set $S_{12} = S_1 \cap S_2$. Let $F \in \mathrm{Mod}(k_X)$. There is a long exact sequence*

$$\begin{aligned} \cdots \to H^j(X;F_{S_1\cup S_2}) \to H^j(X;F_{S_1}) \oplus H^j(X;F_{S_2}) \to H^j(X;F_{S_{12}}) \\ \to H^{j+1}(X;F_{S_1\cup S_2}) \to \cdots \end{aligned}$$

Proof. Apply the functor $\Gamma(X;\cdot)$ to the exact sequence of sheaves $0 \to F_{S_1\cup S_2} \to F_{S_1} \oplus F_{S_2} \to F_{S_{12}} \to 0$. q.e.d.

Proposition 6.1.5. *Let U_1 and U_2 be two open subsets of X and set $U_{12} = U_1 \cap U_2$. Let $F \in \mathrm{Mod}(k_X)$. There is a long exact sequence*

$$\begin{aligned} \cdots \to H^j(U_1 \cup U_2; F) \to H^j(U_1; F) \oplus H^j(U_2; F) \to H^j(U_{12}; F) \\ \to H^{j+1}(U_1 \cup U_2; F) \to \cdots \end{aligned}$$

Proof. Apply the functor $\mathrm{Hom}_{k_X}(\,\bullet\,, F)$ to the exact sequence of sheaves $0 \to k_{XU_{12}} \to k_{XU_1} \oplus k_{XU_2} \to k_{U_1\cup U_2} \to 0$ and use (6.3). q.e.d.

6.2 Čech complexes for closed coverings

Let I be a finite totally ordered set. In this section, we shall follow the same notations as for Koszul complexes. For $J \subset I$, we denote by $|J|$ its cardinal and for $J = \{i_0 < \cdots < i_p\}$, we set

$$e_J := e_{i_0} \wedge \cdots \wedge e_{i_p} \in \bigwedge^{p+1} \mathbb{Z}^{|I|}.$$

Let $\mathcal{S} = \{S_i\}_{i\in I}$ be a family indexed by I of closed subsets of X and let $F \in \mathrm{Mod}(k_X)$. For $J \subset I$ we set

$$\begin{aligned} &S_J := \cap_{j\in J} S_j, \quad S = \bigcup_{i\in I} S_i \\ &F_{\mathcal{S}}^p := \bigoplus_{|J|=p+1} F_{S_J} \otimes e_J, \quad F_{\mathcal{S}}^{-1} = F_S. \end{aligned}$$

For $J \subset I_{\mathrm{ord}}$ and $a \in I, a \notin J$, we denote by $J_a = J \cup \{a\}$ the ordered subset of the ordered set I and we denote by $r_{J,a} \colon F_{S_J} \to F_{S_{Ja}}$ the natural restriction morphism.

We set

$$\delta_{J,a} := r_{J,a}(\,\bullet\,) \otimes e_a \wedge \bullet \colon F_{S_J} \otimes e_J \to F_{S_{Ja}} \otimes e_{Ja}. \tag{6.5}$$

The morphisms $\delta_{J,a}$ ($J \subset I$, $a \notin J$) in (6.5) define the morphisms

$$d^p \colon F_{\mathcal{S}}^p \to F_{\mathcal{S}}^{p+1}.$$

Clearly, $d^{p+1} \circ d^p = 0$ and we obtain a complex

$$F_{\mathcal{S}}^\bullet := 0 \to F_S \xrightarrow{d^{-1}} F_{\mathcal{S}}^0 \xrightarrow{d^0} F_{\mathcal{S}}^1 \xrightarrow{d^1} \cdots. \tag{6.6}$$

Proposition 6.2.1. *Consider a family $\mathcal{S} = \{S_i\}_{i\in I}$ of closed subsets of X indexed by a finite totally ordered set I. Then the complex* (6.6) *is exact.*

Proof. It is enough to check that the stalk of the complex (6.6) at each $x \in X$ is exact. Hence, we may assume that x belongs to all S_i's. Set $M = F_x$. Then the complex (6.6) is a Koszul complex $K^\bullet(M, \varphi)$ where $\varphi = \{\varphi_i\}_{i\in I}$ and all φ_i are id_M. The sequence φ being both regular and coregular, this complex is exact. q.e.d.

Example 6.2.2. Assume that $X = S_0 \cup S_1 \cup S_2$, where the S_i's are closed subsets. We get the exact complex of sheaves

$$0 \to F \xrightarrow{d^{-1}} F_{S_0} \oplus F_{S_1} \oplus F_{S_2} \xrightarrow{d^0} F_{S_{12}} \oplus F_{S_{02}} \oplus F_{S_{01}} \xrightarrow{d^1} F_{S_{012}} \to 0.$$

Let us denote by

$$s_i \colon F \to F_{S_i}, \quad s_{ij}^a \colon F_{S_a} \to F_{S_{ij}}, \quad s_{\hat{k}} \colon F_{S_{ij}} \to F_{S_{012}}\, (a, i, j, k) \in \{0, 1, 2\}),$$

the natural morphisms. Then

$$d^{-1} = \begin{pmatrix} s_0, \\ s_1, \\ s_2 \end{pmatrix}, \quad d^0 = \begin{pmatrix} 0 & -s_{12}^1 & s_{12}^2 \\ s_{02}^0 & 0 & -s_{02}^2 \\ -s_{01}^0 & s_{01}^1 & 0 \end{pmatrix} \quad d^1 = (s_{\hat{2}}, -s_{\hat{0}}, s_{\hat{1}}).$$

6.3 Invariance by homotopy

In this section, we shall prove that the cohomology of locally constant sheaves is an homotopy invariant. First, we define what it means.

In the sequel, we denote by I the closed interval $I = [0, 1]$.

Definition 6.3.1. Let X and Y be two topological spaces.

(i) Let f_0 and f_1 be two continuous maps from X to Y. One says that f_0 and f_1 are homotopic if there exists a continuous map $h : I \times X \to Y$ such that $h(0, \cdot) = f_0$ and $h(1, \cdot) = f_1$.

(ii) Let $f : X \to Y$ be a continuous map. One says that f is a homotopy equivalence if there exists $g : Y \to X$ such that $f \circ g$ is homotopic to id_Y and $g \circ f$ is homotopic to id_X. In such a case one says that X and Y are homotopic.

(iii) One says that a topological space X is contractible if X is homotopic to a point $\{x_0\}$.

If $f_0, f_1 : X \rightrightarrows Y$ are homotopic, one gets the diagram

$$\begin{tikzcd} X \simeq \{t\} \times X \arrow[r, hook, "i_t"] & I \times X \arrow[r, "h"] \arrow[d, "p"] & Y \\ & X & \end{tikzcd} \tag{6.7}$$

where $t \in I$, $i_t : X \simeq \{t\} \times X \to I \times X$ is the embedding, p is the projection and $f_t = h \circ i_t$, $t = 0, 1$.

One checks easily that the relation "f_0 is homotopic to f_1" is an equivalence relation.

A topological space is contractible if and only if there exist $g : \{x_0\} \to X$ and $f : X \to \{x_0\}$ such that $f \circ g$ is homotopic to id_X. Replacing x_0 with $g(x_0)$, this means that there exists $h : I \times X \to X$ such that $h(0, x) = \mathrm{id}_X$ and $h(1, x)$ is the map $x \mapsto x_0$. Note that contractible implies non empty.

Examples 6.3.2. (i) Let V be a real vector space. A non empty convex set in V as well as a closed non empty cone are contractible sets.
(ii) Let $X = \mathbb{S}^{n-1}$ be the unit sphere of the Euclidian space $\mathbb{R}^n$ and let $Y = \mathbb{R}^n \setminus \{0\}$. The embedding $f : \mathbb{S}^{n-1} \hookrightarrow \mathbb{R}^n \setminus \{0\}$ is a homotopy equivalence. Indeed, denote by $g : \mathbb{R}^n \setminus \{0\} \to \mathbb{S}^{n-1}$ the map $x \mapsto x/||x||$. Then $g \circ f = \mathrm{id}_X$ and $f \circ g$ is homotopic to id_Y. The homotopy is given by the map $h(x, t) = (t/||x|| + 1 - t)x$.

Statement of the main theorem

Let $f : X \to Y$ be a continuous map and let $G \in \mathrm{Mod}(k_Y)$. Remark that $a_X \simeq a_Y \circ f$. The morphism of functors $\mathrm{id} \to f_* \circ f^{-1}$ defines the morphism

$$\begin{aligned} R^j a_{Y*} &\to R^j(a_{Y*} \circ f_* \circ f^{-1}) \\ &\simeq R^j(a_{X*} \circ f^{-1}). \end{aligned}$$

Using Theorem 4.6.9, we get the morphism of functors

$$R^j(a_{X*} \circ f^{-1}) \to (R^j a_{X*}) \circ f^{-1}.$$

Applying this morphism to G we get

$$f^{\sharp j} : H^j(Y; G) \to H^j(X; f^{-1}G). \tag{6.8}$$

Lemma 6.3.3. *Let $f : X \to Y$ and $g : Y \to Z$ be continuous maps. Then*

$$g^{\sharp j} \circ f^{\sharp j} = (f \circ g)^{\sharp j}$$

Proof. We have a commutative diagram of morphisms of functors

$$\begin{tikzcd}
a_{Z*} \arrow[r, "g^\sharp"] \arrow[dr, "(g\circ f)^\sharp"'] & a_{Z*}\circ g_*\circ g^{-1} \arrow[d, "f^\sharp"] \\
 & a_{Z*}\circ (g\circ f)_*\circ (g\circ f)^{-1} \simeq a_{Z*}\circ g_*\circ f_*\circ f^{-1}\circ g^{-1}
\end{tikzcd}$$

Applying the functor $R^j(\,\bullet\,)$ we find the commutative diagram:

$$\begin{tikzcd}
R^j a_{Z*} \arrow[r] \arrow[dr] & R^j(a_{Z*}\circ g_*\circ g^{-1}) \arrow[r] \arrow[d] & R^j(a_{Z*}\circ g_*)\circ g^{-1} \arrow[d] \\
 & R^j(a_{Z*}\circ g_*\circ f_*\circ f^{-1}\circ g^{-1}) \arrow[r] \arrow[dr] & R^j(a_{Z*}\circ g_*\circ f_*\circ f^{-1})\circ g^{-1} \arrow[d] \\
 & & R^j(a_{Z*}\circ g_*\circ f_*)\circ f^{-1}\circ g^{-1}
\end{tikzcd}$$

The composition of the arrows on the top gives the morphism $g^{\sharp j}$, the composition of the vertical arrows gives the morphism $f^{\sharp j}$ and the composition of the diagonal arrows gives the morphism $(g\circ f)^{\sharp j}$. q.e.d.

The aim of this section is to prove:

Theorem 6.3.4 (Invariance by homotopy Theorem)**.** [1] *Let $f_0, f_1 : X \rightrightarrows Y$ be two homotopic maps, and let G be a locally constant sheaf on Y. Consider the two morphisms $f_t^{\sharp j} : H^j(Y;G) \to H^j(X; f_t^{-1}G)$, for $t = 0, 1$. Then there exists an isomorphism $\theta^j : H^j(X; f_0^{-1}G) \to H^j(X; f_1^{-1}G)$ such that $\theta^j \circ f_0^{\sharp j} = f_1^{\sharp j}$.*

If $G = M_Y$ for some $M \in \mathrm{Mod}(k)$, then, identifying $f_t^{-1}M_Y$ with M_X $(t = 0, 1)$, we have $f_1^{\sharp j} = f_0^{\sharp j}$.

Proof of the main theorem

In order to prove Theorem 6.3.4, we need several preliminary results.

For $F \in \mathrm{Mod}(k_I)$ and a closed interval $[0,t] \subset I$, we write $H^j([0,t];F)$ instead of $H^j([0,t];F|_{[0,t]})$ for short.

Lemma 6.3.5. *Let $F \in \mathrm{Mod}(k_I)$. Then:*

(i) *For $j > 1$, one has $H^j(I;F) = 0$.*

(ii) *If $F(I) \to F_t$ is an epimorphism for all $t \in I$, then $H^1(I;F) = 0$.*

[1] The proof of this theorem may be skipped

Proof. Let $j \geq 1$ and let $s \in H^j(I;F)$. For $0 \leq t_1 \leq t_2 \leq 1$, consider the morphism:

$$f_{t_1,t_2} : H^j(I;F) \to H^j([t_1,t_2];F)$$

and let

$$J = \{t \in [0,1];\ f_{0,t}(s) = 0\}.$$

Since $H^j(\{0\};F) = 0$ for $j \geq 1$, we have $0 \in J$. Since $f_{0,t}(s) = 0$ implies $f_{0,t'}(s) = 0$ for $0 \leq t' \leq t$, J is an interval. Since $H^j([0,t_0];F) = \varinjlim_{t>t_0} H^j([0,t];F)$, this interval is open. It remains to prove that J is closed. For $0 \leq t \leq t_0$, consider the Mayer-Vietoris sequence (Proposition 6.1.4):

$$\cdots \to H^j([0,t_0];F) \to H^j([0,t];F) \oplus H^j([t,t_0];F) \to H^j(\{t\};F) \to \cdots$$

For $j > 1$, or else for $j = 1$ assuming $H^0(I;F) \to H^0(\{t\};F)$ is surjective, we obtain:

$$H^j([0,t_0];F) \simeq H^j([0,t];F) \oplus H^j([t,t_0];F). \tag{6.9}$$

Let $t_0 = \sup\ \{t; t \in J\}$. Then $f_{0,t}(s) = 0$, for all $t < t_0$. On the other hand,

$$\varinjlim_{t<t_0} H^j([t,t_0];F) = 0.$$

Hence, there exists $t < t_0$ with $f_{t,t_0}(s) = 0$. By (6.9), this implies $f_{0,t_0}(s) = 0$. Hence $t_0 \in J$. q.e.d.

Recall that the maps $p\colon I \times X \to X$ and $i_t\colon X \to I \times X$ are defined in (6.7). We also introduce the notation $I_x := I \times \{x\}$.

Lemma 6.3.6. *Let $G \in \mathrm{Mod}(k_{I\times X})$. Then $(R^jp_*G)_x \simeq H^j(I_x; G|_{I_x})$.*

Proof. Let $G^\bullet$ be an injective resolution of G. We have the isomorphisms

$$\begin{aligned}
(R^jp_*G)_x &\simeq (H^j(p_*G^\bullet))_x \simeq H^j((p_*G^\bullet)_x) \\
&\simeq H^j(\varinjlim_{x\in U} \Gamma(I\times U;G^\bullet)) \\
&\simeq \varinjlim_{x\in U} H^j(I\times U;G) \simeq H^j(I_x;G|_{I_x}).
\end{aligned}$$

where the last isomorphism follows from Proposition 6.1.3 (iii). q.e.d.

Lemma 6.3.7. *Let $G \in \mathrm{Mod}(k_{I\times X})$ be a locally constant sheaf. Then the natural morphism $p^{-1}p_*G \to G$ is an isomorphism.*

Proof. One has

$$\begin{aligned}(p^{-1}p_*G)_{(t,x)} &\simeq (p_*G)_x\\ &\simeq \Gamma(I_x; G|_{I_x}) \simeq G_{(t,x)}.\end{aligned}$$

Here the last isomorphism follows from Proposition 5.7.6. q.e.d.

Lemma 6.3.8. *Let* $F \in \mathrm{Mod}(k_X)$. *Then* $F \xrightarrow{\sim} p_*p^{-1}F$ *and* $(R^jp_*)p^{-1}F = 0$ *for* $j \geq 1$.

Proof. Let $x \in X$ and let $t \in I$. Using Lemma 6.3.6 one gets the isomorphism $((R^jp_*)p^{-1}F)_x \simeq H^j(I_x; p^{-1}F|_{I_x})$. Then this group is 0 for $j > 0$ by Lemma 6.3.5 and is isomorphic to $(p^{-1}F)_{t,x} \simeq F_x$ for $j = 0$. q.e.d.

Lemma 6.3.9. *Let* $F \in \mathrm{Mod}(k_X)$.

(i) *The morphisms* $p^{\sharp j} : H^j(X; F) \to H^j(I \times X; p^{-1}F)$ *are isomorphisms.*

(ii) *The morphisms* $i_t^{\sharp j} : H^j(I \times X; p^{-1}F) \to H^j(X; F)$ *are isomorphisms and do not depend on* $t \in I$.

Proof. (i) We know that $R^jp_*(p^{-1}F) = 0$ for $j \geq 1$. Applying Proposition 4.6.9 (iii) to the functors a_{X*}, p_* and the object $p^{-1}F$, we get that $R^j(a_{X*}p_*)(p^{-1}F) \simeq R^ja_{X*}(p_*p^{-1}F) \simeq R^ja_{X*}F$. Hence $p^{\sharp j}$ is an isomorphism.
(ii) By Lemma 6.3.3, $i_t^{\sharp j} \circ p^{\sharp j}$ is the identity, and $p^{\sharp j}$ is an isomorphism by (i). Hence, $i_t^{\sharp j}$ which is the inverse of $p^{\sharp j}$ does not depend on t. q.e.d.

End of the proof of Theorem 6.3.4. (i) Consider the diagram

$$\begin{tikzcd}
H^j(I \times X; p^{-1}p_*h^{-1}G) \arrow[r, "i_t^{\sharp j}"] \arrow[d] & H^j(X; i_t^{-1}p^{-1}p_*h^{-1}G) \arrow[d] \\
H^j(I \times X; h^{-1}G) \arrow[r, "i_t^{\sharp j}"] & H^j(X; i_t^{-1}h^{-1}G).
\end{tikzcd}$$

Since $h^{-1}G$ is locally constant, the morphism $p^{-1}p_*h^{-1}G \to h^{-1}G$ is an isomorphism by Lemma 6.3.7 and the two vertical arrows are isomorphisms.

By Lemma 6.3.9 (ii), the row on the top is an isomorphism. Therefore, the row on the bottom is an isomorphism.

By Lemma 6.3.3, the diagram below commutes:

$$\begin{tikzcd}
H^j(Y; G) \arrow[r, "h^{\sharp j}"] \arrow[dr, "f_t^{\sharp j}"'] & H^j(I \times X; h^{-1}G) \arrow[d, "i_t^{\sharp j}"] \\
& H^j(X; f_t^{-1}G)
\end{tikzcd}$$

Set $\theta = i_0^{\sharp j} {i_1^{\sharp j}}^{-1}$. Then

$$f_0^{\sharp j} = i_0^{\sharp j} \circ h^{\sharp j} = i_0^{\sharp j} {i_1^{\sharp j}}^{-1} i_1^{\sharp j} \circ h^{\sharp j} = \theta \circ f_0^{\sharp j}.$$

(ii) If $G = M_Y$, then $h^{-1}G \simeq M_{I\times X} = p^{-1}M_X$ and $i_t^{\sharp j}$ does not depend on t by Lemma 6.3.9. q.e.d.

Applications of Theorem 6.3.4

Corollary 6.3.10. *Assume $f : X \to Y$ is a homotopy equivalence and let G be a locally constant sheaf on Y. Then $H^j(X, f^{-1}G) \simeq H^j(Y; G)$.*

In other words, the cohomology of locally constant sheaves on topological spaces is a homotopy invariant.

Proof. Let $g : Y \to X$ be a map such that $f\circ g$ and $g\circ f$ are homotopic to the identity of Y and X, respectively. Consider $f^{\sharp j} : H^j(Y;G) \to H^j(X; f^{-1}G)$ and $g^{\sharp j} : H^j(X; f^{-1}G) \to H^j(Y;G)$. Then: $(f \circ g)^{\sharp j} = g^{\sharp j} \circ f^{\sharp j} \simeq \mathrm{id}_X^{\sharp j} = \mathrm{id}$ and $(g \circ f)^{\sharp j} = f^{\sharp j} \circ g^{\sharp j} \simeq \mathrm{id}_Y^{\sharp j} = \mathrm{id}\,.$ q.e.d.

Corollary 6.3.11. *If X is contractible and $M \in \mathrm{Mod}(k)$, then $\Gamma(X; M_X) \simeq M$ and $H^j(X; M_X) \simeq 0$ for $j > 0$.*

We shall apply this result together with the technique of Mayer-Vietoris sequences to calculate the cohomology of various spaces. We shall follow the notations of Section 5.6.

Theorem 6.3.12. *Let $X = \bigcup_{i\in I} Z_i$ be a finite covering of X by closed subsets satisfying the condition*

(6.10) *for each non empty subset $J \subset I$, Z_J is contractible or empty.*

Let F be a locally constant sheaf on X. Then $H^j(X; F)$ is isomorphic to the j-th cohomology object of the complex

$$\Gamma(X; F^\bullet_{\mathcal{Z}}) := 0 \to \Gamma(X; F^0_{\mathcal{Z}}) \xrightarrow{d} \Gamma(X; F^1_{\mathcal{Z}}) \to \cdots$$

Proof. Recall (Proposition 6.1.3) that if Z is closed in X, then $\Gamma(X; F_Z) \simeq \Gamma(Z; F|_Z)$. Therefore the sheaves $F^p_{\mathcal{Z}}$ $(p \geq 0)$ are acyclic with respect to the functor $\Gamma(X; \cdot)$, by Corollary 6.3.11. Applying Proposition 6.2.1, the result follows from Theorem 4.6.7. q.e.d.

Corollary 6.3.13. (A particular case of the universal coefficients formula.) *In the situation of Theorem 6.3.12, let M be a flat k-module. Then for all j there are natural isomorphisms $H^j(X; M_X) \simeq H^j(X; k_X) \otimes M$.*

Proof. The k-module $H^j(X; M_X)$ is the j-th cohomology object of the complex $\Gamma(X; M^\bullet_{\mathcal{Z}})$. Clearly,

$$\Gamma(X; M^\bullet_{\mathcal{Z}}) \simeq \Gamma(X; k^\bullet_{\mathcal{Z}}) \otimes M$$

Since M is flat we have for any bounded complex of modules $N^\bullet$:

$$H^j(N^\bullet \otimes M) \simeq H^j(N^\bullet) \otimes M.$$

Hence,

$$H^j(\Gamma(X; M^\bullet_{\mathcal{Z}})) \simeq H^j(\Gamma(X; k^\bullet_{\mathcal{Z}})) \otimes M$$

To conclude, apply Theorem 6.3.12 to both sides. q.e.d.

Proposition 6.3.14. (A particular case of the Künneth formula.) *Let X and Y be two topological spaces which both admit finite closed coverings satisfying condition (6.10). Let F (resp. G) be a locally constant sheaf on X with fiber M, (resp. on Y with fiber N). Assume that k is a field. Then there are natural isomorphisms:*

$$H^p(X \times Y; F \boxtimes G) \simeq \bigoplus_{i+j=p} H^i(X; F) \otimes H^j(Y; G).$$

Sketch of proof. First, notice that $F \boxtimes G$ is locally constant on $X \times Y$. Next, denote by $\mathcal{S} = \{S_i\}_{i \in I}$ (resp. $\mathcal{Z} = \{Z_j\}_{j \in J}$) a finite covering of X (resp. Y) satisfying condition (6.10). Since the product of two contractible sets is clearly contractible, the covering $\mathcal{S} \times \mathcal{Z} = \{S_i \times Z_j\}_{(i,j) \in I \times J}$ is a finite covering of $X \times Y$ satisfying condition (6.10). Then $H^p(X \times Y; F \boxtimes G)$ is the p-th cohomology object of the complex

$$\Gamma(X \times Y; (F \boxtimes G)^\bullet_{\mathcal{S} \times \mathcal{Z}}).$$

One checks that this complex is the simple complex associated with the double complex

$$\Gamma(X \times Y; F^\bullet_{\mathcal{S}} \boxtimes G^\bullet_{\mathcal{Z}})$$

and this double complex is isomorphic to

$$\Gamma(X; F^\bullet_{\mathcal{S}}) \otimes \Gamma(Y; G^\bullet_{\mathcal{Z}}).$$

q.e.d.

It may be convenient to reformulate the Künneth formula by saying that $H^p(X \times Y; F \boxtimes G)$ is isomorphic to the p-th cohomology object of the complex

$$(\bigoplus_i H^i(X; F)[-i]) \otimes (\bigoplus_j H^j(Y; G)[-j]).$$

6.4 Cohomology of some classical manifolds

Here, k denotes as usual a commutative unitary ring and M denotes a k-module.

Example 6.4.1. Let X be the circle $\mathbb{S}^1$ and let Z_j's be a closed covering by intervals such that the Z_{ij}'s are single points and $Z_{012} = \emptyset$. Applying Theorem 6.3.12, we find that if F is a locally constant sheaf on X, the cohomology groups $H^j(X; F)$ are the cohomology objects of the complex:

$$0 \to \Gamma(X; F_{Z_0}) \oplus \Gamma(X; F_{Z_1}) \oplus \Gamma(X; F_{Z_2}) \xrightarrow{d} \Gamma(X; F_{Z_{12}}) \oplus \Gamma(X; F_{Z_{20}}) \oplus \Gamma(X; F_{Z_{01}}) \to 0.$$

Recall Example 5.8.3: $\mathbb{S}^1 = U_1 \cup U_2$, $U_1 \cap U_2$ has two connected components U_{12}^+ and U_{12}^-, k is a field, $\alpha \in k^\times$ and L_α denotes the locally constant sheaf of rank one over k obtained by glueing k_{U_1} and k_{U_2} by the identity on U_{12}^+ and by multiplication by $\alpha \in k^\times$ on U_{12}^-.

Then for $j = 0$ (resp. for $j = 1$), $H^j(\mathbb{S}^1; L_\alpha)$ is the kernel (resp. the cokernel) of the matrix $\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -\alpha \\ -1 & 1 & 0 \end{pmatrix}$ acting on k^3. (See Example 6.2.2.) Note that these kernel and cokernel are zero except in case of $\alpha = 1$ which corresponds to the constant sheaf k_X.

It follows that if M is a k-module, then $H^j(\mathbb{S}^1; M_{\mathbb{S}^1}) \simeq M$ for $j = 0, 1$ and 0 otherwise.

Example 6.4.2. Consider the topological n-sphere $\mathbb{S}^n$. Recall that it can be defined as follows. Let $\mathbb{E}$ be an $\mathbb{R}$-vector space of dimension $n+1$ and denote by $\dot{\mathbb{E}}$ the set $\mathbb{E} \setminus \{0\}$. Then

$$\mathbb{S}^n \simeq \dot{\mathbb{E}}/\mathbb{R}^+,$$

where $\mathbb{R}^+$ denotes the multiplicative group of positive real numbers and $\mathbb{S}^n$ is endowed with the quotient topology. (See Definition 7.3.4 below.) In other words, $\mathbb{S}^n$ is the set of all half-lines in $\mathbb{E}$. If one chooses an Euclidian norm on $\mathbb{E}$, then one may identify $\mathbb{S}^n$ with the unit sphere in $\mathbb{E}$.

We have $\mathbb{S}^n = \bar{D}^+ \cup \bar{D}^-$, where $\bar{D}^+$ and $\bar{D}^-$ denote the closed hemispheres, and $\bar{D}^+ \cap \bar{D}^- \simeq \mathbb{S}^{n-1}$. Let us prove that for $n \geq 1$:

$$H^j(\mathbb{S}^n; M_{\mathbb{S}^n}) = \begin{cases} M & j = 0 \text{ or } j = n, \\ 0 & \text{otherwise.} \end{cases} \tag{6.11}$$

Consider the Mayer-Vietoris long exact sequence

$$
\begin{aligned}
(6.12) \qquad \to H^j(\bar{D}^+; M_{\bar{D}^+}) \oplus H^j(\bar{D}^-; M_{\bar{D}^-}) &\to H^j(\mathbb{S}^{n-1}; M_{\mathbb{S}^{n-1}}) \\
&\to H^{j+1}(\mathbb{S}^n; M_{\mathbb{S}^n}) \to \cdots
\end{aligned}
$$

Then the result follows by induction on n since the closed hemispheres being contractible, their cohomology is concentrated in degree 0.

Let $\mathbb{E}$ be a real vector space of dimension $n+1$, and let $X = \mathbb{E} \setminus \{0\}$. Assume $\mathbb{E}$ is endowed with a norm $|\cdot|$. The map $x \mapsto x((1-t)+t/|x|)$ defines an homotopy of X with the sphere $\mathbb{S}^n$. Hence the cohomology of a constant sheaf with stalk M on $V \setminus \{0\}$ is the same as the cohomology of the sheaf $M_{\mathbb{S}^n}$

As an application, one obtains that the dimension of a finite dimensional vector space is a topological invariant. In other words, if V and W are two real finite dimensional vector spaces and are topologically isomorphic, they have the same dimension. In fact, if V has dimension n, then $V \setminus \{0\}$ is homotopic to $\mathbb{S}^{n-1}$.

Notice that $\mathbb{S}^n$ is not contractible, although one can prove that any locally constant sheaf on $\mathbb{S}^n$ for $n \geq 2$ is constant.

Example 6.4.3. Denote by a the antipodal map on $\mathbb{S}^n$ (the map deduced from $x \mapsto -x$) and denote by $a^{\sharp n}$ the action of a on $H^n(\mathbb{S}^n; M_{\mathbb{S}^n})$. Using (6.12), one deduces the commutative diagram:

$$
(6.13) \qquad \begin{array}{ccc}
H^{n-1}(\mathbb{S}^{n-1}; M_{\mathbb{S}^{n-1}}) & \xrightarrow{u} & H^n(\mathbb{S}^n; M_{\mathbb{S}^n}) \\
{\scriptstyle a^{\sharp n-1}}\downarrow & & \downarrow{\scriptstyle a^{\sharp n}} \\
H^{n-1}(\mathbb{S}^{n-1}; M_{\mathbb{S}^{n-1}}) & \xrightarrow{-u} & H^n(\mathbb{S}^n; M_{\mathbb{S}^n})
\end{array}
$$

For $n = 1$, the map a is homotopic to the identity (in fact, it is the same as a rotation of angle π). By (6.13), we deduce:

$$
(6.14) \qquad a^{\sharp n} \text{ acting on } H^n(\mathbb{S}^n; M_{\mathbb{S}^n}) \text{ is } (-)^{n+1}.
$$

Example 6.4.4. Let $\mathbb{T}^n$ denote the n-dimensional torus, $\mathbb{T}^n \simeq (\mathbb{S}^1)^n$. Using the Künneth formula, one gets that (if k is a field) $\bigoplus_j H^j(\mathbb{T}^n; k_{\mathbb{T}^n}) \simeq (k \oplus k[-1])^{\otimes n}$. For example, $H^j(\mathbb{T}^2; k_{\mathbb{T}^2})$ is k for $j = 0, 2$, is k^2 for $j = 1$ and is 0 otherwise.

Let us recover this result (when $n = 2$) by using Mayer-Vietoris sequences. One may represent $\mathbb{T}^2$ as follows. Consider the two cylinders $Z_0 = \mathbb{S}^1 \times I_0$, $Z_1 = \mathbb{S}^1 \times I_1$ where $I_0 = I_1 = [0, 1]$. Then

$$
\mathbb{T}^2 \simeq (Z_0 \sqcup Z_1)/\sim
$$

where $\sim$ is the relation which identifies $\mathbb{S}^1 \times \{0\} \subset Z_0$ with $\mathbb{S}^1 \times \{0\} \subset Z_1$ and $\mathbb{S}^1 \times \{1\} \subset Z_0$ with $\mathbb{S}^1 \times \{1\} \subset Z_1$. Then

$$Z_{01} := Z_0 \cap Z_1 \simeq \mathbb{S}^1 \sqcup \mathbb{S}^1.$$

We have a short exact sequence of sheaves

$$0 \to k_{\mathbb{T}^2} \xrightarrow{\alpha} k_{Z_0} \oplus k_{Z_1} \xrightarrow{\beta} k_{Z_{01}} \to 0 \tag{6.15}$$

Write $H^j(X)$ instead of $H^j(X; k_X)$ for short. Applying the functor $\Gamma(\mathbb{T}^2; \cdot)$, we get the long exact sequence of Theorem 6.1.4:

$$\begin{aligned} 0 \to H^0(\mathbb{T}^2) \xrightarrow{\alpha_0} H^0(Z_0) \oplus H^0(Z_1) \xrightarrow{\beta_0} H^0(Z_{01}) \to H^1(\mathbb{T}^2) \\ \xrightarrow{\alpha_1} H^1(Z_0) \oplus H^1(Z_1) \xrightarrow{\beta_1} H^1(Z_{01}) \to H^2(\mathbb{T}^2) \to 0. \end{aligned}$$

Although we know the groups $H^j(Z_0), H^j(Z_1)$ and $H^j(Z_{01})$ (since Z_0 and Z_1 are homotopic to $\mathbb{S}^1$), this sequence does not allow us to conclude, unless we know the morphism β. Since Z_0 and Z_1 are homotopic to $\mathbb{S}^1$, we may identify the sequence (6.15) with the sequence

$$0 \to k_{\mathbb{T}^2} \xrightarrow{\alpha} k_S \oplus k_S \xrightarrow{\beta} k_S \oplus k_S \to 0 \tag{6.16}$$

where β is given by the matrix $\begin{pmatrix} \mathrm{id}_{\mathbb{S}^1} & -\mathrm{id}_{\mathbb{S}^1} \\ \mathrm{id}_{\mathbb{S}^1} & -\mathrm{id}_{\mathbb{S}^1} \end{pmatrix}$. Identifying $H^j(\mathbb{S}^1; k_{\mathbb{S}^1})$ with k for $j = 0, 1$, β_0 and β_1 will is given by the matrix $\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$ and one gets that $H^j(\mathbb{T}^2; k_{\mathbb{T}^2})$ is k for $j = 0, 2$ and k^2 for $j = 1$.

6.5 De Rham cohomology

The aim of this section is to explain why the de Rham complex allows us to calculate the cohomology of the constant sheaf $\mathbb{C}_X$ on a real smooth manifold X.

In this section, we assume that X is locally compact. Recall that for a sheaf F on X and a compact subset K of X, one has $\Gamma(K; F) \simeq \varinjlim_U \Gamma(U; F)$, where U ranges over the family of open neighborhoods of K.

Definition 6.5.1. A sheaf F on X is c-soft if for any compact subset K of X, the map $\Gamma(X; F) \to \Gamma(K; F)$ is onto.

We shall admit the following result.

Theorem 6.5.2. *Assume that X is locally compact and countable at infinity. Then the category of c-soft sheaves is injective with respect to the functor $\Gamma(X;\bullet)$.*

Let X be a real $\mathcal{C}^\infty$-manifold of dimension n (this implies in particular that X is locally compact and countable at infinity). If $n > 0$, the sheaf $\mathbb{C}_X$ is not acyclic for the functor $\Gamma(X;\cdot)$ in general. In fact consider two connected open subsets U_1 and U_2 such that $U_1 \cap U_2$ has two connected components, V_1 and V_2. The sequence:

$$0 \to \Gamma(U_1 \cup U_2;\mathbb{C}_X) \to \Gamma(U_1;\mathbb{C}_X) \oplus \Gamma(U_2;\mathbb{C}_X) \to \Gamma(U_1 \cap U_2;\mathbb{C}_X) \to 0$$

is not exact since the locally constant function $\varphi = 1$ on V_1, $\varphi = 2$ on V_2 may not be decomposed as $\varphi = \varphi_1 - \varphi_2$, with φ_j constant on U_j. By the Mayer-Vietoris long exact sequence, this implies:

$$H^1(U_1 \cup U_2;\mathbb{C}_X) \neq 0.$$

On the other hand, for K a compact subset of X and U an open neighborhood of K in X, there exists a real $\mathcal{C}^\infty$-function φ with compact support contained in U and which is identically 1 in a neighborhood of K (existence of "partition of unity"). This implies that the sheaf $\mathcal{C}^\infty_X$ is c-soft, as well as any sheaf of $\mathcal{C}^\infty_X$-modules.

Denote by $\mathcal{C}^{\infty,(p)}_X$ or else, Ω^p_X, the sheaf on X differential forms of degree p with $\mathcal{C}^\infty_X$ coefficients. These sheaves are c-soft and in particular $\Gamma(X;\cdot)$ acyclic.

Consider the complex of sheaves on X:

$$\mathrm{DR}_X \ := \ 0 \to \Omega^0_X \xrightarrow{d} \cdots \to \Omega^n_X \to 0.$$

We call it the De Rham complex on X with $\mathcal{C}^\infty$ coefficients.

Lemma 6.5.3. (The Poincaré lemma.) *Let $I = (]0,1[)^n$ be the unit open cube in $\mathbb{R}^n$. The complex below is exact.*

$$0 \to \mathbb{C} \to \mathcal{C}^{\infty,(0)}(I) \xrightarrow{d} \cdots \to \mathcal{C}^{\infty,(n)}(I) \to 0.$$

Proof. Consider the Koszul complex $K^\bullet(M,\varphi)$ over the ring $\mathbb{C}$, where $M = \mathcal{C}^\infty(I)$ and $\varphi = (\partial_1,\ldots,\partial_n)$ (with $\partial_j = \frac{\partial}{\partial x_j}$). This complex is nothing but the complex:

$$0 \to \mathcal{C}^{\infty,(0)}(I) \xrightarrow{d} \cdots \to \mathcal{C}^{\infty,(n)}(I) \to 0.$$

Clearly $H^0(K^\bullet(M,\varphi)) \simeq \mathbb{C}$, and it is enough to prove that the sequence $(\partial_1,\ldots,\partial_n)$ is coregular. Let $M_{j+1} = \mathrm{Ker}(\partial_1) \cap \cdots \cap \mathrm{Ker}(\partial_j)$. This is the space of $\mathcal{C}^\infty$-functions on I constant with respect to the variables $x_1,\ldots,x_j$. Clearly, ∂_{j+1} is surjective on this space. q.e.d.

Lemma 6.5.3 implies:

Lemma 6.5.4. *Let X be a $\mathcal{C}^\infty$-manifold of dimension n. Then the natural morphisms $\mathbb{C}_X \to \mathrm{DR}_X$ is a quasi-isomorphism.*

Corollary 6.5.5. (The de Rham theorem.) *Let X be a $\mathcal{C}^\infty$-manifold of dimension n. Then $H^j(X;\mathbb{C}_X)$ is isomorphic to $H^j(\Gamma(X;\mathrm{DR}_X))$.*

Note that this result in particular implies that $H^j(\Gamma(X;\mathrm{DR}_X))$ is a topological invariant of X.

Exercises to Chapter 6

Exercise 6.1. Let $\{F_i\}_{i\in I}$ be a family of sheaves on X. Assume that this family is locally finite, that is, each $x \in X$ has an open neighborhood U such that all but a finite number of the $F_i|_U$'s are zero.
(i)Prove that $(\prod_i F_i)_x \simeq \prod_i (F_i)_x$.
(ii) Prove that if each F_i is injective, then $\prod_i F_i$ is injective.
(iii) Let $F_i^\bullet$ be an injective resolution of F_i. Prove that $\prod_i F_i^\bullet$ is an injective resolution of $\prod_i F_i$.
(iv) Prove that $H^j(X;\prod_i F_i) \simeq \prod_i H^j(X;F_i)$.

Exercise 6.2. In this exercise, we shall admit the following theorem: for any open subset U of the complex line $\mathbb{C}$, one has $H^j(U;\mathcal{O}_\mathbb{C}) \simeq 0$ for $j > 0$.

Let ω be an open subset of $\mathbb{R}$, and let $U_1 \subset U_2$ be two open subsets of $\mathbb{C}$ containing ω as a closed subset.
(i) Prove that the natural map $\mathcal{O}(U_2 \setminus \omega)/\mathcal{O}(U_2) \to \mathcal{O}(U_1 \setminus \omega)/\mathcal{O}(U_1)$ is an isomorphism. One denote by $\mathcal{B}(\omega)$ this quotient.
(ii) Construct the restriction morphism to get the presheaf $\omega \to \mathcal{B}(\omega)$, and prove that this presheaf is a sheaf (the sheaf $\mathcal{B}_\mathbb{R}$ of Sato's hyperfunctions on $\mathbb{R}$).
(iii) Prove that the restriction morphisms $\mathcal{B}(\mathbb{R}) \to \mathcal{B}(\omega)$ are surjective.
(iv) Let Ω an open subset of $\mathbb{C}$ and let $P = \sum_{j=1}^m a_j(z)\frac{\partial}{\partial z}^j$ be a holomorphic differential operator (the coefficients are holomorphic in Ω). Recall the Cauchy theorem which asserts that if Ω is simply connected and if $a_m(z)$ does not vanish on Ω, then P acting on $\mathcal{O}(\Omega)$ is surjective. Prove that if ω is an open subset of $\mathbb{R}$ and if P is a holomorphic differential operator defined in a open neighborhood of ω, then P acting on $\mathcal{B}(\omega)$ is surjective

Exercise 6.3. Let X and Y be two topological spaces, S and S' two closed subsets of X and Y respectively, $f : S \simeq S'$ a topological isomorphism.

Define $X \sqcup_S Y$ as the quotient space $X \sqcup Y/\sim$ where $\sim$ is the relation which identifies $x \in X$ and $y \in Y$ if $x \in S$, $y \in S'$ and $f(x) = y$. One still denotes by X, Y, S the images of X, Y, $S \sqcup S'$ in $X \sqcup_S Y$.
(i) Let F be a sheaf on $X \sqcup_S Y$. Write the long exact Mayer-Vietoris sequence associated with X, Y, S.
(ii) Application (a). Let $\mathbb{S}^n$ denote the unit sphere of the Euclidian space $\mathbb{R}^{n+1}$, B the intersection of $\mathbb{S}^n$ with an open ball of radius ε $(0 < \varepsilon << 1)$ centered in some point of $\mathbb{S}^n$, Σ its boundary in $\mathbb{S}^n$. Set $X = \mathbb{S}^n \setminus B$, $S = \Sigma$ and let Y and S' be a copy of X and S, respectively. Calculate $H^j(X \sqcup_S Y; k_{X \sqcup_S Y})$.
(iii) Application (b). Same question by replacing the sphere $\mathbb{S}^n$ by the torus $\mathbb{T}^2$ embedded into $\mathbb{R}^3$.

Exercise 6.4. Let $X = \mathbb{R}^4$ and consider the locally closed subset $Z = \{(x, y, z, t) \in \mathbb{R}^4; t^4 = x^2 + y^2 + z^2; t > 0\}$. Denote by $f : Z \hookrightarrow X$ the natural injection. Calculate $(R^j f_* k_Z)_0$ for $j \geq 0$.

Exercise 6.5. Let p, q, n be integers ≥ 1 with $n = p + q$ and let $X = \mathbb{R}^n$ endowed with the coordinates $x = (x_1, \ldots, x_n)$. Set $S_0 = \{x \in X; \sum_{i=1}^n x_i^2 = 1\}$, $S_1 = \{x \in X; \sum_{i=1}^p x_i^2 + 2\sum_{i=p+1}^n x_i^2 = 1\}$, $S = S_0 \cup S_1$. Calculate $H^j(S; k_S)$ for all j.

Exercise 6.6. Let $\gamma = \{(x, y, z, t) \in X = \mathbb{R}^4; x^2 + y^2 + z^2 = t^2\}$ and let $U = X \setminus \gamma$.
(i) Show that γ is contractible.
(ii) Calculate $H^j(X; k_{XU})$ for all j. (Recall that there exists an exact sequence $0 \to k_{XU} \to k_X \to k_{X\gamma} \to 0$.)

Exercise 6.7. A closed subset Z of a space X is called a retract of X if there exists a continuous map $f\colon X \to Z$ which induces the identity on Z. Show that $\mathbb{S}^1$ is not a retract of the closed disk $\bar{D}$ in $\mathbb{R}^2$.
(Hint: denote by $\iota\colon \mathbb{S}^1 \hookrightarrow \bar{D}$ the embedding and assume that there exists a continuous map $f\colon \bar{D} \to \mathbb{S}^1$ such that the composition $f \circ \iota$ is the identity. We get that the composition

$$H^1(\mathbb{S}^1; \mathbb{Z}_{\mathbb{S}^1}) \xrightarrow{f^{\sharp 1}} H^1(\bar{D}; \mathbb{Z}_{\bar{D}}) \xrightarrow{\iota^{\sharp 1}} H^1(\mathbb{S}^1; \mathbb{Z}_{\mathbb{S}^1})$$

is the identity.)

Exercise 6.8. Consider the unit ball $B_{n+1} = \{x \in \mathbb{E}; |x| \leq 1\}$ and consider a map $f : B_{n+1} \to B_{n+1}$. Prove that f has at least one fixed point. (Hint: otherwise, construct a map $g : B_{n+1} \to \mathbb{S}^n$ which induces the identity on $\mathbb{S}^n$ and use the same argument as in Exercise 6.7.)
(Remark: the result of this exercise is known as the Brouwer's Theorem.)

Chapter 7

Homotopy and fundamental groupoid

In this chapter we study locally constant sheaves of sets and sheaves of k-modules and introduce the fundamental group of locally connected topological spaces. We define the monodromy of a locally constant sheaf and prove the equivalence between the category of representations of the fundamental group and that of locally constant sheaves. [1]
Some references: [13], [8], [15], [4], [26]. In this chapter, we shall admit some results treated with all details in [13].

7.1 Fundamental groupoid

Let us recall some classical notions of topology. We denote as usual by I the closed interval $[0,1]$ and by $\mathbb{S}^1$ the circle. Note that topologically $\mathbb{S}^1 \simeq I/\sim$ where $\sim$ is the equivalence relation on I which identifies the two points 0 and 1. We shall also consider the space

$$D := I \times I/\sim$$

where $\sim$ is the equivalence relation which identifies $I \times \{0\}$ to a single point (denoted a_0) and $I \times \{1\}$ to a single point (denoted a_1). Note that topologically, D is isomorphic to the closed unit disk, or else, to $I \times I$.

Let X denote a topological space.

Definition 7.1.1. (i) A path from x_0 to x_1 in X is a continuous map $\sigma : I \to X$, with $\sigma(0) = x_0$ and $\sigma(1) = x_1$. The two points x_0 and x_1 are called the ends of the path.

[1] The contents of this chapter will not be part of the exam 2007/2008

(ii) Two paths σ_0 and σ_1 are called homotopic if there exists a continuous function $\varphi\colon I \times I \to X$ such that $\varphi(i,t) = \sigma_i(t)$ for $i = 0, 1$. (See Definition 6.3.1.)

(iii) If the two paths have the same ends, x_0 and x_1, one says they are homotopic with fixed ends if moreover $\varphi(s, 0) = x_0, \varphi(s, 1) = x_1$ for all s. This is equivalent to saying that there exists a continuous function $\psi : D \to X$ such that $\psi(i,t) = \sigma_i(t)$ for $i = 0, 1$.

(iv) A loop in X is a continuous map $\gamma : \mathbb{S}^1 \to X$. One can also consider a loop as a path γ such that $\gamma(0) = \gamma(1)$. A trivial loop is a constant map $\gamma : \mathbb{S}^1 \to \{x_0\}$. Two loops are homotopic if they are homotopic as paths with fixed ends.

It is left to the reader to check that "homotopy" (of paths or paths with fixed ends) is an equivalence relation. We denote by $[\sigma]$ the homotopy class of a path σ.

Product. If σ is a path from x_0 to x_1 and τ a path from x_1 to x_2 one can define a new path $\tau\sigma$ (in this order) from x_0 to x_2 by setting

$$\tau\sigma(t) = \begin{cases} \sigma(2t) & \text{for } 0 \le t \le 1/2, \\ \tau(2t-1) & \text{for } 1/2 \le t \le 1. \end{cases}$$

If σ is a path from x_0 to x_1, one can define the path σ^{-1} from x_1 to x_0 by setting $\sigma^{-1}(t) = \sigma(1-t)$.

It is easily checked that the homotopy class of $\tau\sigma$ only depends on the homotopy classes of σ and τ. Hence, we can define $[\tau][\sigma]$ as $[\tau\sigma]$. The product of paths isn't associative but the following lemma says that it induces an associative product on paths modulo homotopy (the proof is left as an exercise).

Lemma 7.1.2. *For four points x_i, $i = 1, \ldots, 4$, in X and three paths σ_i from x_i to x_{i+1} the paths $\sigma_1(\sigma_2\sigma_3)$ and $(\sigma_1\sigma_2)\sigma_3$ are homotopic. The path $\sigma_1\sigma_1^{-1}$ is homotopic to the trivial loop at x_2.*

When all x_i are the same fixed point x_0 this lemma implies that the set of homotopy classes of loops at x_0 is a group.

Fundamental group

Definition 7.1.3. The set of homotopy classes of loops at x_0 endowed with the above product is called the fundamental group of X at x_0 and denoted $\pi_1(X; x_0)$.

If σ is a path from x_0 to x_1 in X, then the map $[\gamma] \mapsto [\sigma\gamma\sigma^{-1}] = [\sigma][\gamma][\sigma^{-1}]$ defines an isomorphism

$$u_{[\sigma]}\colon \pi_1(X;x_0) \xrightarrow{\sim} \pi_1(X;x_1),$$

with inverse $u_{[\sigma^{-1}]}$.

Definition 7.1.4. Let X be a topological space.

(i) X is arcwise connected (or "path connected") if given x_0 and x_1 in X, there exists a path with ends x_0 and x_1,

(ii) X is simply connected if any loop in X is homotopic to a trivial loop,

(iii) X is locally connected (resp. locally arcwise connected, resp. locally simply connected) if each $x \in X$ has a neighborhood system consisting of connected (resp. arcwise connected, resp. simply connected) open subsets.

Hence, if X is arcwise connected, all groups $\pi_1(X;x)$ are isomorphic for $x \in X$ and one often denotes by $\pi_1(X)$ one of the groups $\pi_1(X;x)$. Clearly, if X is arcwise connected, it is connected. If X is locally arcwise connected and connected, it is arcwise connected.

Example 7.1.5. In $\mathbb{R}^2$ set

$$\begin{aligned} X = \{(x,y) \in \mathbb{R}^2; y = \sin(1/x),\ x > 0\} \\ \cup\{(x,y) \in \mathbb{R}^2; x = 0, -1 \le y \le 1\} \cup \{(x,y) \in \mathbb{R}^2; y = 0, x \ge 0\}. \end{aligned}$$

Then X is arcwise connected but not locally arcwise connected.

The next result is left as an exercise.

Lemma 7.1.6. *If X is simply connected, two paths γ and τ with the same ends are homotopic.*

In the sequel, we shall make the hypothesis

$$X \text{ is locally arcwise connected.} \tag{7.1}$$

Fundamental group of the spheres

Lemma 7.1.7. *Let X be a topological space, U, V open subsets such that $X = U \cup V$. Let $\sigma\colon I \to X$ be a path with $\sigma(0), \sigma(1) \in U \cap V$. Then there exists a finite sequence $0 = t_0 \leq t_1 \leq \cdots \leq t_{2n} = 1$ such that $\forall i = 0, \dots, n-1$, $\sigma([t_{2i}, t_{2i+1}]) \subset U$, $\sigma([t_{2i+1}, t_{2i+2}]) \subset V$.*

Proof. We set $K = \sigma^{-1}(X \setminus U)$. This is a compact subset of I. For any $s \in K$ we have $\sigma(s) \in V$; hence there exists an open interval $I_s \subset I$ containing s such that $\sigma(\overline{I_s}) \subset V$.

Since K is compact we may choose finitely many s_i, $i = 1, \dots, k$, such that $J = \bigcup_{i=1,\dots,k} I_{s_i}$ contains K. Joining overlapping intervals we may write $J =]t_1, t_2[\sqcup]t_3, t_4[\sqcup \ldots \sqcup]t_{2l-1}, t_{2l}[$, with $0 \leq t_1 < t_2 < \cdots < t_{2l} \leq 1$. q.e.d.

Proposition 7.1.8. *Assume* (7.1) *and let U, V be two connected open subsets of X such that $X = U \cup V$ and $U \cap V$ is non empty and connected. Then, for $x \in U \cap V$, $\pi_1(X; x)$ is generated by the images of $\pi_1(U; x)$ and $\pi_1(V; x)$.*

Remark 7.1.9. The Van Kampen Theorem below (Theorem 7.2.16) gives a more general and more precise result.

Proof. We let $\sigma\colon I \to X$ be a loop at x and choose a sequence $0 = t_0 \leq t_1 \leq \cdots \leq t_{2n} = 1$ as in Lemma 7.1.7. We set $x_k = \sigma(t_k)$ and we let σ_k be the path from x_k to x_{k+1} given by $\sigma|_{[t_k, t_{k+1}]}$ (that is $\sigma_k(t) = \sigma(t_k + t(t_{k+1} - t_k))$). Since $x_k \in U \cap V$ and $U \cap V$ is connected we may choose a path τ_k in $U \cap V$ from x to x_k. We define $\sigma'_k = \tau_{k+1}^{-1}(\sigma_k \tau_k)$ which is a path contained either in U or in V. Now the result follows from the equality $[\sigma] = [\sigma'_{n-1}][\sigma'_{n-2}] \cdots [\sigma'_0]$. q.e.d.

Corollary 7.1.10. *One has $\pi_1(\mathbb{S}^n) = 0$ for $n \geq 2$.*

Proof. We choose two distinct points $N, S \in \mathbb{S}^n$ and we apply Proposition 7.1.8 with the open subsets $U = \mathbb{S}^n \setminus \{N\}$, $V = \mathbb{S}^n \setminus \{S\}$.

We have $U \simeq V \simeq \mathbb{R}^n$ and it is easy to see that $\pi_1(\mathbb{R}^n; 0) \simeq 0$ using the homotopy $h\colon I \times \mathbb{R}^n \to \mathbb{R}^n$, $(s, x) \mapsto (1-s)x$ contracting the whole $\mathbb{R}^n$ to 0 (see also Corollary 7.1.19). q.e.d.

Theorem 7.1.11. *One has $\pi_1(\mathbb{S}^1) \simeq \mathbb{Z}$.*

In order to prove this result, we need some lemmas. In the sequel, we regard $\mathbb{S}^1$ as embedded in $\mathbb{C}$, we denote by p the map:

$$p\colon \mathbb{R} \to \mathbb{S}^1, \quad t \mapsto \exp(2i\pi t)$$

and we set $e = p(0)$.

Lemma 7.1.12. *For any continuous map $f\colon I \to \mathbb{S}^1$ satisfying $f(0) = e$, there exists a unique continuous map $\widetilde{f}\colon I \to \mathbb{R}$ with $\widetilde{f}(0) = 0$ such that $f = p \circ \widetilde{f}$.*

Proof. (i) Unicity. We first note that if $h_1, h_2\colon I \to \mathbb{R}$ are two continuous maps such that $ph_1 = ph_2$ and $h_1(t) = h_2(t)$ for some $t \in I$ then there exists an open interval $J \subset I$ containing t such that $h_1|_J = h_2|_J$. Indeed we take J such that $h_1(J)$ and $h_2(J)$ are contained in an open interval K of length less than 1 and we remark that $p|_K$ is injective.

Now we consider $\widetilde{g}\colon I \to \mathbb{R}$ with $\widetilde{g}(0) = e$ such that $f = p \circ \widetilde{g}$ and define $I' = \{t \in ;\ \widetilde{f}(t) = \widetilde{g}(t)\}$. By the above remark I' is open in I. Since I' is obviously closed and nonempty we have $I' = I$ so that $\widetilde{f} = \widetilde{g}$.

(ii) Existence. Let q be the right inverse of p on $]-\frac{1}{2}, \frac{1}{2}[$, that is $q\colon \mathbb{S}^1 \setminus \{-e\} \xrightarrow{\sim}]-\frac{1}{2}, \frac{1}{2}[$ satisfies $p \circ q = \mathrm{id}$. Let n be a positive integer such that $|f(t) - f(t')| \leq 1$ for $|t - t'| \leq \frac{1}{n}$. Then $f(t)\overline{f(t')} \neq -e$ for $|t - t'| \leq \frac{1}{n}$. One defines the map $\widetilde{f}\colon I \to \mathbb{R}$ by setting:

$$\widetilde{f}(t) = q(f(t)\overline{f(j/n)}) + \sum_{i=1}^{j} q(f(i/n)\overline{f((i-1)/n)}) \text{ for } j/n \leq t < (j+1)/n.$$

The map $\widetilde{f}$ is continuous and satisfies $\widetilde{f}(0) = 0$, $p \circ \widetilde{f} = f$. q.e.d.

Lemma 7.1.13. *For any continuous map $h\colon I \times I \to \mathbb{S}^1$ satisfying $h(s, 0) = e$, there exists a unique continuous map $\widetilde{h}\colon I \times I \to \mathbb{R}$ with $\widetilde{h}(s, 0) = e$ such that $h = p \circ \widetilde{h}$.*

The proof is similar to that of Lemma 7.1.12.

Definition 7.1.14. Let $f\colon I \to \mathbb{S}^1$ satisfying $f(0) = f(1) = e$ and let $\widetilde{f}$ be as in Lemma 7.1.12. Then $\widetilde{f}(1) \in \mathbb{Z}$ is called the degree of f and denoted $\deg(f)$.

For $n \in \mathbb{Z}$ let γ_n be the loop in $\mathbb{S}^1$, $[0, 1] \ni t \mapsto \exp(2i\pi nt)$. Then $\deg(\gamma_n) = n$.

Lemma 7.1.15. *Two loops $f, g\colon I \rightrightarrows \mathbb{S}^1$ satisfying $f(0) = f(1) = e$, $g(0) = g(1) = e$ are homotopic if and only if they have the same degree.*

Proof. (i) If f and g are homotopic then they have the same degree by Lemma 7.1.13.

(ii) Conversely let us shown that if $\deg(f) = n$ then f is homotopic to the loop γ_n defined above. We consider the map $\widetilde{f}\colon I \to \mathbb{R}$ given by Lemma 7.1.12 and define $h\colon I \times I \to \mathbb{R}$, $(s, t) \mapsto snt + (1 - s)\widetilde{f}(t)$. Then ph is a homotopy between f and γ_n. q.e.d.

By Lemma 7.1.15, if $[\gamma]$ is a loop in $\mathbb{S}^1$, its degree $\deg([\gamma])$ is well defined and one checks immediately that for two loops γ and τ in $\mathbb{S}^1$,

$$\deg([\tau]\cdot[\gamma]) = \deg([\tau]) + \deg([\gamma]).$$

Hence, we have a morphism of groups

$$\deg\colon \pi_1(\mathbb{S}^1, e) \to \mathbb{Z}.$$

Proof of Theorem 7.1.11. Denote by γ the loop $[0,1] \ni t \mapsto \exp(2i\pi t)$. The morphism of groups

$$\theta\colon \mathbb{Z} \to \pi_1(\mathbb{S}^1, e), \quad n \mapsto [\gamma]^n.$$

is inverse of the morphism deg. q.e.d.

Fundamental groupoid

Assume (7.1). Consider the category $\Pi_1(X)$ given by

$$\begin{cases} \mathrm{Ob}(\Pi_1(X)) = X, \\ \mathrm{Hom}_{\Pi_1(X)}(x_0, x_1) = \{\text{the set of homotopy classes of paths from} \\ x_0 \text{ to } x_1\}. \end{cases}$$

Clearly, any morphism in $\Pi_1(X)$ is an isomorphism, that is, $\Pi_1(X)$ is a groupoid.

Definition 7.1.16. The category $\Pi_1(X)$ defined above is called the fundamental groupoid of X.

Note that for $x \in X$, $\mathrm{Hom}_{\Pi_1(X)}(x, x) = \pi_1(X, x)$.

Remark 7.1.17. (i) Assume (7.1) and moreover X is non empty and connected. Let $x_0 \in X$ and consider the category $\Pi_1(X; x_0)$ with a single object $\{x_0\}$ and $\mathrm{Hom}_{\Pi_1(X;x_0)}(x_0, x_0) = \pi_1(X; x_0)$. Then the natural functor $\Pi_1(X; x_0) \to \Pi_1(X)$ is an equivalence.
(ii) Remark that X being arcwise connected, it is simply connected if and only if $\Pi_1(X) \simeq \{1\}$.

Consider a continuous map $f\colon X \to Y$. If γ is a path in X, then $f \circ \gamma$ is a path in Y, and if two paths γ_0 and γ_1 are homotopic in X, then $f \circ \gamma_0$ and $f \circ \gamma_1$ are homotopic in Y. Hence, we get a functor:

$$f_* : \Pi_1(X) \to \Pi_1(Y). \tag{7.2}$$

In particular, if $i_U\colon U \hookrightarrow X$ denotes the embedding of an open subset U of X, we get the functor

$$i_{U*}\colon \Pi_1(U) \to \Pi_1(X). \tag{7.3}$$

Proposition 7.1.18. *Let $f_0, f_1\colon X \to Y$ be two continuous maps and assume f_0 and f_1 are homotopic. Then the two functors f_{0*} and f_{1*} are isomorphic.*

In particular, if $f : X \to Y$ is a homotopy equivalence, then the two groupoids $\Pi_1(Y)$ and $\Pi_1(X)$ are equivalent.

Proof. Let $h\colon I \times X \to Y$ be a continuous map such that $h(i,\cdot) = f_i(\cdot)$, $i = 0,1$. We define a morphism of functors $\theta_h\colon f_{0*} \to f_{1*}$ as follows. For $x \in X = \mathrm{Ob}(\Pi_1(X))$ we define $\theta_h(x) \in \mathrm{Hom}_{\Pi_1(Y)}(f_0(x), f_1(x))$ as the class of the path $t \mapsto h(t,x)$. It is easy to check that if σ is a path in X then $[\theta_h(y)\, f_{0*}(\sigma)] = [f_{1*}(\sigma)\,\theta_h(x)]$, which shows that θ is a morphism of functors.

Using $\bar{h}$ defined by $\bar{h}(t,x) = h(1-t,x)$ we define in the same way the morphism of functors $\theta_{\bar{h}}\colon f_{1*} \to f_{0*}$. It is clearly an inverse to θ_h because, for any $x \in X$, $\theta_h(x) = \theta_{\bar{h}}(x)^{-1}$. q.e.d.

Corollary 7.1.19. *A contractible space is simply connected.*

Let X and Y be two topological spaces satisfying (7.1). Denote by p_i the projection from $X \times Y$ to X and Y respectively. These projections define functors $p_{1*} : \Pi_1(X \times Y) \to \Pi_1(X)$ and $p_{2*} : \Pi_1(X \times Y) \to \Pi_1(Y)$, hence a functor

$$(p_{1*} \times p_{2*})\colon \Pi_1(X \times Y) \to \Pi_1(X) \times \Pi_1(Y). \tag{7.4}$$

Proposition 7.1.20. *The functor in* (7.4) *is an equivalence.*

Proof. We built an inverse, say q, to $p_{1*} \times p_{2*}$ as follows. For objects $x \in \mathrm{Ob}(\Pi_1(X))$, $y \in \mathrm{Ob}(\Pi_1(Y))$ we set $q(x,y) = (x,y) \in \mathrm{Ob}(\Pi_1(X \times Y))$.

If σ is a path from x to x' in X and τ a path from y to y' in Y then $\sigma \times \tau$ is a path from (x,y) to (x',y'). This construction is compatible with the homotopy relation and we may define $q([\sigma] \times [\tau]) = [\sigma \times \tau]$.

Now it is easy to check that q is a functor and that it is inverse to $p_{1*} \times p_{2*}$. q.e.d.

7.2 Monodromy of locally constant sheaves

Locally constant sheaves

Remark 7.2.1. We shall work here with sheaves of k-modules, but many results remain true without any change for sheaves of sets.

Let $M \in \mathrm{Mod}(k)$. Recall that a constant sheaf F with stalk M on X is a sheaf isomorphic to the sheaf M_X of locally constant functions with values in M. We shall denote by $\mathrm{LCSH}(k_X)$ (resp. $\mathrm{CSH}(k_X)$) the full additive subcategory of $\mathrm{Mod}(k_X)$ consisting of locally constant (resp. of constant) sheaves.

Denote as usual by a_X the map $X \to \mathrm{pt}$.

Proposition 7.2.2. (i) *Assume X is non empty and connected. Then the two functors*

$$\mathrm{CSH}(\mathrm{k_X}) \underset{a_X^{-1}}{\overset{a_{X*}}{\rightleftarrows}} \mathrm{Mod}(k)$$

are equivalences of categories, inverse one to each other. In particular, the category $\mathrm{CSH}(\mathrm{k_X})$ is abelian.

(ii) *Assume X is locally connected. Then the category $\mathrm{LCSH}(k_X)$ is abelian. If M and N are two k-modules then $\mathcal{H}om_{k_X}(M_X, N_X) \simeq (\mathrm{Hom}_k(M, N))_X$.*

Proof. (i) If F is a constant sheaf on X, and if one sets $M = \Gamma(X; F)$, then $F \simeq M_X$. Therefore, if $M \in \mathrm{Mod}(k)$, then $a_{X*}a_X^{-1}M \simeq M$ and $a_X^{-1}a_{X*}M_X \simeq M_X$.
(ii) Let $\varphi : F \to G$ be a morphism of locally constant sheaves, and let $x \in X$. If U is a sufficiently small connected open neighborhood of x, the restriction to U of $\mathrm{Ker}\,\varphi$ and $\mathrm{Coker}\,\varphi$ will be constant sheaves, by (i). Hence, $\mathrm{LCSH}(k_X)$ is a full additive subcategory of an abelian category (namely $\mathrm{Mod}(k_X)$) stable by kernels and cokernels. This implies it is abelian. q.e.d.

Lemma 7.2.3. *Let F be a locally constant sheaf on $X = I \times I$. Then F is a constant sheaf.*

Proof. Since $I \times I$ is compact, there exists finite coverings of I by intervals $\{U_i\}_{1 \le i \le N_0}$ and $\{V_j\}_{1 \le j \le N_1}$ such that $F|_{U_i \times V_j}$ is a constant sheaf. Since $(U_i \times V_j) \cap (U_{i+1} \times V_j)$ is connected, the argument of the proof of Proposition 5.7.6 shows that $F|_{I \times V_j}$ is a constant sheaf for all j. Since $(I \times V_j) \cap (I \times V_{j+1})$ is connected, the argument of the proof of Proposition 5.7.6 shows that F is constant. q.e.d.

Representations

For a group G and a ring k one defines the category $\mathrm{Rep}(G, \mathrm{Mod}(k))$ of representations of G in $\mathrm{Mod}(k)$ as follows. An object is a pair (M, μ_M) with $M \in \mathrm{Mod}(k)$ and $\mu_M \in \mathrm{Hom}\,(G, \mathbb{G}l(M))$. A morphism $\mu_f : (M, \mu_M) \to (N, \mu_N)$ is a k-linear map $f : M \to N$ which satisfies $\mu_N \circ f = f \circ \mu_M$ (i.e. $\mu_N(g) \circ f = f \circ \mu_M(g)$ for any $g \in G$). Note that $\mathrm{Rep}(G, \mathrm{Mod}(k))$ contains the full abelian subcategory $\mathrm{Mod}(k)$, identified to the trivial representations of G.

We identify G with a category $\mathcal{G}$ with one object c, the morphisms being given by $\mathrm{Hom}_{\mathcal{G}}(c,c)= G$. One gets that

$$\mathrm{Rep}(G, \mathrm{Mod}(k)) \simeq \mathrm{Fct}(\mathcal{G}, \mathrm{Mod}(k)). \tag{7.5}$$

The next lemma will be useful. Its proof is left as an exercise (see Exercise 7.3).

Lemma 7.2.4. *Let $u\colon G_1 \to G_2$ be a morphism of groups. Assume that u induces an isomorphism*

$$\widetilde{u}\colon \mathrm{Rep}(G_2, \mathrm{Mod}(k)) \xrightarrow{\sim} \mathrm{Rep}(G_1, \mathrm{Mod}(k)).$$

Then u is an isomorphism.

Now we shall consider the category $\mathrm{Fct}(\Pi_1(X), \mathrm{Mod}(k))$, a generalization of the category of representations $\mathrm{Rep}(\pi_1(X), \mathrm{Mod}(k))$. In fact, if X is connected, non empty and satisfies (7.1), then all $x \in \Pi_1(X)$ are isomorphic, and the groupoids $\Pi_1(X)$ is equivalent to the group $\pi_1(X, x_0)$ identified to the category with one object x_0 and morphisms $\pi_1(X, x_0)$. In this case, the two categories $\mathrm{Rep}(\pi_1(X), \mathrm{Mod}(k))$ and $\mathrm{Fct}(\Pi_1(X), \mathrm{Mod}(k))$ are equivalent.

Monodromy

In this section, we make the hypothesis

(7.6) X is locally arcwise connected and locally simply connected.

Definition 7.2.5. (i) One calls an object of $\mathrm{Fct}(\Pi_1(X), \mathrm{Mod}(k))$ a representation of the groupoid $\Pi_1(X)$ in $\mathrm{Mod}(k)$.

(ii) Let $\theta \in \mathrm{Fct}(\Pi_1(X), \mathrm{Mod}(k))$. One says that θ is a trivial representation if θ is isomorphic to a constant functor Δ_M which associates the module M to any $x \in X$, and id_M to any $[\gamma] \in \mathrm{Hom}_{\Pi_1(X)}(x, y)$.

Let $\mathrm{Fct}_0(\Pi_1(X), \mathrm{Mod}(k))$ be the full subcategory of $\mathrm{Fct}(\Pi_1(X), \mathrm{Mod}(k))$ consisting of trivial representations. Then the functor $M \mapsto \Delta_M$ from $\mathrm{Mod}(k)$ to $\mathrm{Fct}_0(\Pi_1(X), \mathrm{Mod}(k))$ is an equivalence of categories.

Let F be a locally constant sheaf of k-modules on X. Let γ be a path from x_0 to x_1. We shall construct an isomorphism

$$\mu(F)(\gamma) : F_{x_0} \simeq F_{x_1}.$$

Since $\gamma^{-1}F$ is a locally constant sheaf on $[0,1]$, it is a constant sheaf. We get the isomorphisms, which define $\mu(F)(\gamma)$:

$$F_{x_0} = (\gamma^{-1}F)_0 \xleftarrow{\sim} \Gamma(I;\gamma^{-1}F) \xrightarrow{\sim} (\gamma^{-1}F)_1 = F_{x_1}. \tag{7.7}$$

Example 7.2.6. (i) Let $X = I$, denote by t a coordinate on I, and consider the constant sheaf $F = \mathbb{C}_X \exp(\alpha t)$. Then $\mu(F)(I)\colon F_0 \xrightarrow{\sim} F_1$ is the multiplication by $\exp(\alpha)$.
(ii) Let $X = \mathbb{S}^1 \simeq [0,2\pi]/\sim$ (where $\sim$ identifies 0 and 2π) and denote by θ a coordinate on $\mathbb{S}^1$. Consider the locally constant sheaf $F = \mathbb{C}_X \exp(i\beta\theta)$. (See Example 5.8.3.) Let γ be the identity loop. Then $\mu(F)(\gamma) : F_{x_0} \xrightarrow{\sim} F_{x_0}$ is the multiplication by $\exp(2i\pi\beta)$.

Lemma 7.2.7. *The isomorphism $\mu(F)(\gamma)$ depends only on the homotopy class of γ in X.*

Proof. Let φ be a continuous function $D \to X$ such that $\varphi(i,t) = \gamma_i(t)$, $i = 0,1$. The sheaf $\varphi^{-1}F$ is constant by Lemma 7.2.3. The isomorphisms $\mu(F)(\gamma_i)$ $(i = 0,1)$ are described by the commutative diagram:

$$\begin{tikzcd}
F_{x_0} = (\varphi^{-1}F)_{a_0} \arrow[d, "="'] & \Gamma(D;\varphi^{-1}F) \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[d, "\sim"'] & (\varphi^{-1}F)_{a_1} = F_{x_1} \arrow[d, "="'] \\
F_{x_0} = (\gamma_i^{-1}F)_0 & \Gamma(I;\gamma_i^{-1}F) \arrow[l, "\sim"'] \arrow[r, "\sim"] & (\gamma_i^{-1}F)_1 = F_{x_1}
\end{tikzcd} \tag{7.8}$$

This shows that $\mu(F)(\gamma_0) = \mu(F)(\gamma_1)$. q.e.d.

If τ is another path from x_1 to x_2, then:

$$\mu(F)(\gamma\tau) = \mu(F)(\gamma) \circ \mu(F)(\tau).$$

Hence we have constructed a functor $\mu(F)\colon \Pi_1(X) \to \mathrm{Mod}(k)$ given by $\mu(F)(x) = F_x$, $\mu(F)([\gamma]) = \mu(F)(\gamma)$ where γ is a representative of $[\gamma]$.

Lemma 7.2.8. *The correspondence $F \mapsto \mu(F)$ defines a functor*

$$\mu\colon \mathrm{LCSH(k_X)} \to \mathrm{Fct}(\Pi_1(X), \mathrm{Mod(k)}). \tag{7.9}$$

Proof. Let $u\colon F \to G$ be a morphism of locally constant sheaves and let γ be a path from x_0 to x_1. One has to check that the diagram below commutes:

$$\begin{tikzcd}
F_{x_0} \arrow[r, "\mu(F)(\gamma)"] \arrow[d, "u"'] & F_{x_1} \arrow[d, "u"'] \\
G_{x_0} \arrow[r, "\mu(G)(\gamma)"'] & G_{x_1}.
\end{tikzcd}$$

By applying the functor γ^{-1} we are reduced to check the commutativity of the diagram below, which is obvious.

$$\begin{tikzcd}
(\gamma^{-1}F)_0 \arrow[d, "u"'] & \Gamma(I;\gamma^{-1}F) \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[d, "u"] & (\gamma^{-1}F)_1 \arrow[d, "u"'] \\
(\gamma^{-1}G)_0 & \Gamma(I;\gamma^{-1}G) \arrow[l, "\sim"'] \arrow[r, "\sim"] & (\gamma^{-1}G)_1 .
\end{tikzcd}$$

q.e.d.

Definition 7.2.9. The functor μ in (7.9) is called the monodromy functor.

The functor μ is also functorial with respect to the space X. More precisely, let $f\colon X \to Y$ be a continuous map, and assume that both X and Y are locally arcwise connected. We have the commutative (up to isomorphism) diagram of categories and functors:

$$\begin{tikzcd}
\mathrm{LCSH}(k_Y) \arrow[r, "f^{-1}"] \arrow[d, "\mu"] & \mathrm{LCSH}(k_X) \arrow[d, "\mu"] \\
\mathrm{Fct}(\Pi_1(Y), \mathrm{Mod}(k)) \arrow[r, "\widetilde{f}_*"] & \mathrm{Fct}(\Pi_1(X), \mathrm{Mod}(k))
\end{tikzcd} \tag{7.10}$$

Proposition 7.2.10. *Assume* (7.1) *and X is non empty. Then the monodromy functor μ induces an equivalence*

$$\mu_0\colon \mathrm{CSH(k_X)} \xrightarrow{\sim} \mathrm{Fct_0(\Pi_1(X), Mod(k))}. \tag{7.11}$$

Proof. Since the functor $\mathrm{Mod}(k) \to \mathrm{CSH(X)}$, $M \mapsto M_X$, is an equivalence, it is enough to check that the functor μ_0 induces an equivalence $\mu_0\colon \mathrm{Mod}(k) \xrightarrow{\sim} \mathrm{Fct}_0(\Pi_1(X), \mathrm{Mod}(k))$, which is obvious. q.e.d.

Theorem 7.2.11. *Assume* (7.1). *The functor μ in* (7.9) *is fully faithful.*

Proof. (i) μ is faithful. Let $\varphi, \psi\colon F \to G$ be morphisms of locally constant sheaves and assume that $\mu(\varphi) = \mu(\psi)$. This implies that $\varphi_{x_0} = \psi_{x_0} : F_{x_0} \to G_{x_0}$ for any $x_0 \in X$. Hence $\varphi = \psi$.

(ii) μ is full. Consider a morphism $u : \mu(F) \to \mu(G)$. It defines a morphism $\varphi_x\colon F_x \to G_x$ for any $x \in X$.

We choose a neighborhood U_x of x such that $F|_{U_x}$ and $G|_{U_x}$ are constant. Then φ_x extends uniquely to a morphism $\widetilde{\varphi_x}\colon F|_{U_x} \to G|_{U_x}$. By (7.1) we may assume that U_x is arcwise connected. For $y \in U_x$ we choose a path σ in U_x from x to y. Then $(\widetilde{\varphi_x})_y = \mu(G)(\sigma) \circ (\widetilde{\varphi_x})_x \circ \mu(F)(\sigma^{-1})$.

But we also have $(\widetilde{\varphi_x})_x = \varphi_x$ and $\varphi_y = \mu(G)(\sigma) \circ \varphi_x \circ \mu(F)(\sigma^{-1})$ because u is a morphism of functors. Hence $(\widetilde{\varphi_x})_y = \varphi_y$.

In particular for any two points x, x' we obtain $\forall y \in U_x \cap U_{x'}$, $(\widetilde{\varphi_x})_y = (\widetilde{\varphi_{x'}})_y$. This proves that the $\widetilde{\varphi_x}$ glue together in a morphism of sheaves φ. We clearly have $\mu(\varphi) = u$ as desired. q.e.d.

Corollary 7.2.12. *Assume that X is connected, locally arcwise connected and simply connected. Then any locally constant sheaf F on X is a constant sheaf.*

Proof. By the hypothesis, $\mathrm{Fct}(\Pi_1(X), \mathrm{Mod}(k)) \simeq \mathrm{Mod}(k)$. Let F be a locally constant sheaf. Then $\mu(F) \in \mathrm{Mod}(k)$ and there exists $G \in \mathrm{CSH}(\mathrm{k_X})$ such that $\mu(F) \simeq \mu(G)$. Since μ is fully faithful, this implies $F \simeq G$. q.e.d.

Theorem 7.2.13. *Assume*

$$\left\{\begin{array}{l}\text{(i) } X \text{ is locally arcwise connected,}\\ \text{(ii) there exists an open covering stable by finite inter-}\\ \text{sections by connected and simply connected subsets.}\end{array}\right. \tag{7.12}$$

Then the functor μ in (7.9) *is an equivalence of categories.*

Proof. (i) By Theorem 7.2.11, it remains to show that μ is essentially surjective.
(ii) By Corollary 7.2.12, the theorem is true if X is connected and simply connected.
(iii) Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open covering of X as in (7.12). The functors $i_{U_i *} \colon \Pi_1(U_i) \to \Pi_1(X)$ define functors λ_i from $\mathrm{Fct}(\Pi_1(X), \mathrm{Mod}(k))$ to $\mathrm{Fct}(\Pi_1(U_i), \mathrm{Mod}(k))$. Let $A \in \mathrm{Fct}(\Pi_1(X), \mathrm{Mod}(k))$ and set $A_i = \lambda_i(A)$. Using the result in (ii) for U_i, we find sheaves F_i such that $\mu(F_i) = A_i$.

For i, j such that $U_j \subset U_i$ the functor $i_{U_j *}$ factorizes through $\Pi_1(U_i)$ and the result in (ii) for U_j implies that we have an isomorphism $\theta_{ji} : F_i|_{U_j} \xrightarrow{\sim} F_j$. For $U_k \subset U_j \subset U_i$ we have $\theta_{ki} = \theta_{kj} \circ \theta_{ji}|_{U_k}$. Since $\mathcal{U}$ is stable by finite intersections Theorem 5.8.1 gives the existence of a sheaf F on X which is locally isomorphic to the F_i's, hence locally constant.

(iv) It remains to show that $\mu(F) \simeq A$.

We have isomorphisms $\theta(x) \colon F_x \simeq A(x)$ for any $x \in X$ because $x \in U_i$ for some i.

Let $[\gamma] \in \mathrm{Hom}_{\Pi_1(X)}(x, y)$ and let γ be a path which represents $[\gamma]$. We have to show that $A([\gamma]) \circ \theta(x) = \theta(y) \circ \mu(F)([\gamma])$.

We first assume γ is contained in some U_i, and denote by $[\gamma']$ the corresponding element in $\mathrm{Hom}_{\Pi_1(U_i)}(x, y)$. Then $A([\gamma]) = A_i([\gamma'])$ and $\mu(F)([\gamma]) = \mu(F_i)([\gamma'])$ and the result follows by construction.

In general, iterating Lemma 7.1.7, we may decompose γ as $\gamma = \gamma_1 \cdots \gamma_n$, each γ_j ($1 \leq j \leq n$) being contained in some U_{i_j}. Since the result holds for each γ_j it holds for γ by composition. q.e.d.

Corollary 7.2.14. *Assume* (7.12) *and X is connected. Then X is simply connected if and only if any locally constant sheaf on X is constant.*

Proof. We may assume X is non empty and choose $x \in X$. Since X is connected, the category $\Pi_1(X)$ is equivalent to the category associated with the group $\pi_1(X;x)$. Then the natural functor $\mathrm{Mod}(k) \to \mathrm{Fct}(\Pi_1(X), \mathrm{Mod}(k))$ is an equivalence if and only if $\pi_1(X;x)$ is the trivial group with one element by Lemma 7.2.4. On the other hand, it follows from Theorem 7.2.13 that $\mathrm{Mod}(k) \to \mathrm{Fct}(\Pi_1(X), \mathrm{Mod}(k))$ is an equivalence if and only if any locally constant sheaf on X is constant. q.e.d.

Example 7.2.15. A locally constant sheaf of $\mathbb{C}$-vector spaces of finite rank on X is called a local system. Hence, it is now possible to classify all local systems on the space $X = \mathbb{R}^2 \setminus \{0\}$. In fact, $\pi_1(X) \simeq \mathbb{Z}$, hence, $\mathrm{Hom}\,(\pi_1(X), \mathbb{G}l(\mathbb{C}^n)) = \mathbb{G}l(\mathbb{C}^n)$. A local system F of rank n is determined, up to isomorphism, by its monodromy $\mu(F) \in \mathbb{G}l(\mathbb{C}^n)$. The classification of such sheaves is thus equivalent to that of invertible $n \times n$ matrices over $\mathbb{C}$ up to conjugation, a well known theory (Jordan-Hölder decomposition). In particular, when $n = 1$, $\mathbb{G}l(\mathbb{C}) = \mathbb{C}^\times$.

Hence, a local system of rank one is determined, up to isomorphism, by its monodromy $\alpha \in \mathbb{C}^\times$.

The Van Kampen Theorem

We shall deduce a particular case of the Van Kampen theorem.

Theorem 7.2.16. *Assume that X satisfies the hypothesis* (7.12) *and let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open covering of X stable by finite intersection. Assume*

(a) *the U_i's are connected and simply connected,*

(b) *there exists x which belongs to all U_i's.*

Then the natural morphism $\varinjlim_{i \in I} \pi_1(U_i; x) \to \pi_1(X; x)$ is an isomorphism.

Proof. By Theorem 7.2.13, $\mathrm{Fct}(\Pi_1(X), \mathrm{Mod}(k)) \simeq \mathrm{LCSH}(\mathrm{k_X})$ and X being connected, the category $\mathrm{Fct}(\Pi_1(X), \mathrm{Mod}(k))$ is equivalent to the set

$\mathrm{Rep}(\pi_1(\mathrm{X},\mathrm{x}),\mathrm{Mod}(\mathrm{k}))$. The same results hold with X replaced by the U_i's. Then Theorem 5.8.1 gives the isomorphism

$$\begin{aligned}\mathrm{Hom}\,(\pi_1(X),\mathbb{G}l(M)) &\simeq \varprojlim_i \mathrm{Hom}\,(\pi_1(U_i),\mathbb{G}l(M)) \\ &\simeq \mathrm{Hom}\,(\varinjlim_i \pi_1(U_i),\mathbb{G}l(M)) \text{ for all } M\in\mathrm{Mod}(\mathbb{Z}).\end{aligned}$$

Therefore,

$$\mathrm{Rep}(\pi_1(\mathrm{X},\mathrm{x}),\mathrm{Mod}(\mathrm{k}))\simeq \mathrm{Rep}(\varinjlim_{\mathrm{i}} \pi_1(\mathrm{U_i}),\mathrm{Mod}(\mathrm{k})),$$

and the result follows from Lemma 7.2.4. q.e.d.

7.3 Coverings

Let S be a set. We endow it with the discrete topology. Then $X\times S\simeq\bigsqcup_{s\in S}X_s$ where $X_s=X\times\{s\}$ is a copy of X, and the coproduct is taken in the category of topological spaces. In particular each X_s is open.

Definition 7.3.1. (i) A continuous map $f: Z\to X$ is a trivial covering if there exists a non empty set S, a topological isomorphism $h: Z \xrightarrow{\sim} X\times S$ where S is endowed with the discrete topology, and $f=p\circ h$ where $p: X\times S\to X$ is the projection.

(ii) A continuous map $f: Z\to X$ is a covering [2] if f is surjective and any $x\in X$ has an open neighborhood U such that $f|_{f^{-1}(U)}: f^{-1}(U)\to U$ is a trivial covering.

(iii) If $f: Z\to X$ is a covering, a section u of f is a continuous map $u: X\to Z$ such that $f\circ u=\mathrm{id}_X$. A local section is a section defined on an open subset U of X.

(iv) A morphism of coverings $f: Z\to X$ to $f': Z'\to X$ is a continuous map $h: Z\to Z'$ such that $f=f'\circ h$.

Hence, we have defined the category $\mathrm{Cov}(X)$ of coverings above X, and the full subcategory of trivial coverings. Roughly speaking, a covering is locally isomorphic to a trivial covering.

Notation 7.3.2. Let $f: Z\to X$ be a covering. One denotes by $\mathrm{Aut}\,(f)$ the group of automorphisms of this covering, that is, the group of isomorphisms of the object $(f: Z\to X)\in\mathrm{Cov}(X)$.

[2] "revêtement" in French, not to be confused with "recouvrement".

The definition of a covering is visualized as follows.

$$\begin{tikzcd}
\bigsqcup_{s\in S} U_s \arrow[dr, "p"'] & f^{-1}U \arrow[l, "\sim", "h"'] \arrow[r] \arrow[d, "f"] & Z \arrow[d, "f"] \\
& U \arrow[r] & X
\end{tikzcd}$$

If X is connected and S is finite for some x, S will be finite for all x, with the same cardinal, say n. In this case one says that f is a finite (or an n-)covering.

Example 7.3.3. Let $Z = \mathbb{C} \setminus \{0, \pm i, \pm i\sqrt{2}\}, X = \mathbb{C} \setminus \{0, 1\}$ and let $f : Z \to X$ be the map $z \mapsto (z^2 + 1)^2$. Then f is a 4-covering.

Many coverings appear naturally as the quotient of a topological space by a discrete group.

Definition 7.3.4. Let X be a locally compact topological space and let G be a group, that we endow with the discrete topology. We denote by e the unit in G.

(i) An action μ of G on X is a map $\mu : G \times X \to X$ such that:

  (a) for each $g \in G$, $\mu(g) : X \to X$ is continuous,

  (b) $\mu(e) = \mathrm{id}_X$,

  (c) $\mu(g_1 \circ g_2) = \mu(g_1) \circ \mu(g_2)$.

  In the sequel, we shall often write $g \cdot x$ instead of $\mu(g)(x)$, for $g \in G$ and $x \in X$.

(ii) Let $x \in X$. The orbit of x in X is the subset $G \cdot x$ of X.

(iii) One says that G acts transitively on X if for any $x \in X$, $X = G \cdot x$.

(iv) One says that G acts properly on X if for any compact subset K of X, the set $G_K = \{g \in G; g \cdot K \cap K = \emptyset\}$ is finite.

(v) One says that G acts freely if for any $x \in X$, the group $G_x = \{g \in G; g \cdot x = x\}$ is trivial, that is, is reduced to $\{e\}$.

If a group G acts on X, it defines an equivalence relation on X, namely, $x \sim y$ if and only if there exists $g \in G$ with $x = g \cdot y$. One denotes by X/G the quotient space, endowed with the quotient topology.

Theorem 7.3.5. *Assume that a discrete group G acts properly and freely on a locally compact space X. Then $p : X \to X/G$ is a covering. Moreover, if X is connected, then* $\mathrm{Aut}\,(p) = G$.

For the proof, we refer to [13].

Examples 7.3.6. (i) The map $p : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ (where $\mathbb{Z}$ acts on $\mathbb{R}$ by translation) is a covering, and the map $t \mapsto \exp(2i\pi t) : \mathbb{R} \to \mathbb{S}^1$ induces an isomorphism $h : \mathbb{R}/\mathbb{Z} \xrightarrow{\sim} \mathbb{S}^1$ such that $\exp(2i\pi t) = h \circ p$. Therefore, $t \mapsto \exp(2i\pi t) : \mathbb{R} \to \mathbb{S}^1$ is a covering.

Similarly, the map $p : \mathbb{C} \to \mathbb{C}/2i\pi\mathbb{Z}$ is a covering and the map $z \mapsto \exp(z) : \mathbb{C} \to \mathbb{C} \setminus \{0\}$ induces an isomorphism $h : \mathbb{C}/2i\pi\mathbb{Z} \xrightarrow{\sim} \mathbb{C} \setminus \{0\}$ such that $\exp(z) = h \circ p$. Therefore, $z \mapsto \exp(z) : \mathbb{C} \to \mathbb{C} \setminus \{0\}$ is a covering.
(ii) Consider the group H_n of n-roots of unity in $\mathbb{C}$, that is, the subgroup of $\mathbb{C}^\times$ generated by $\exp(2i\pi/n)$. Then $p : \mathbb{S}^1 \to \mathbb{S}^1/H_n$ is a covering and the map $z \mapsto z^n : \mathbb{S}^1 \to \mathbb{S}^1$ induces an isomorphism $h : \mathbb{S}^1/H_n \xrightarrow{\sim} \mathbb{S}^1$ such that $z^n = h \circ p$. Therefore, $z \mapsto z^n : \mathbb{S}^1 \to \mathbb{S}^1$ is an n-covering.
(iii) The projection $\mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$ is a covering, and there is are isomorphisms $\mathbb{R}^n/\mathbb{Z}^n \simeq (\mathbb{R}/\mathbb{Z})^n \simeq (\mathbb{S}^1)^n$.

Example 7.3.7. The projective space of dimension n, denoted $\mathbb{P}^n(\mathbb{R})$, is constructed as follows. Let $\mathbb{E}$ be an $n+1$-dimensional $\mathbb{R}$-vector space. Then $\mathbb{P}^n(\mathbb{R})$ is the set of lines in $\mathbb{E}$, in other words,

$$\mathbb{P}^n(\mathbb{R}) = \dot{\mathbb{E}}/\mathbb{R}^\times, \tag{7.13}$$

where $\dot{\mathbb{E}} = \mathbb{E} \setminus \{0\}$, and $\mathbb{R}^\times$ is the multiplicative group of non-zero elements of $\mathbb{R}$.

Identifying $\mathbb{E}$ with $\mathbb{R}^{n+1}$, a point $x \in \mathbb{R}^{n+1}$ is written $x = (x_0, x_1, \cdots, x_n)$ and $x \in \mathbb{P}^n(\mathbb{R})$ may be written as $x = [x_0, x_1, \cdots, x_n]$, with the relation $[x_0, x_1, \cdots, x_n] = [\lambda x_0, \lambda x_1, \cdots, \lambda x_n]$ for any $\lambda \in \mathbb{R}^\times$. One says that $[x_0, x_1, \cdots, x_n]$ are homogeneous coordinates.

The map $\mathbb{R}^n \to \mathbb{P}^n(\mathbb{R})$ given by $(y_1, \cdots, y_n) \mapsto [1, y_1, \cdots, y_n]$ allows us to identify $\mathbb{R}^n$ to the open subset of $\mathbb{P}^n(\mathbb{R})$ consisting of the set $\{x = [x_0, x_1, \cdots, x_n]; x_0 \neq 0\}$.

In the sequel, we shall often write for short $\mathbb{P}^n$ instead of $\mathbb{P}^n(\mathbb{R})$. Since $\mathbb{S}^n = \dot{\mathbb{E}}/\mathbb{R}^+$, we get $\mathbb{P}^n \simeq \mathbb{S}^n/a$ where a is the antipodal relation on $\mathbb{S}^n$ which identifies x and $-x$. The map a defines an action of the group $\mathbb{Z}/2\mathbb{Z}$ on $\mathbb{S}^n$, and this action is clearly proper and free. Denote by

$$\gamma : \mathbb{S}^n \to \mathbb{P}^n \tag{7.14}$$

the natural map. This is a 2-covering and $\mathbb{P}^n \simeq \mathbb{S}^n/(\mathbb{Z}/2\mathbb{Z})$.

Coverings and locally constant sheaves

Let $f\colon Z \to X$ be a covering. We associates a sheaf of sets F_f on X as follows. For U open in X, $F_f(U)$ is the set of sections of $f|_U\colon f^{-1}(U) \to U$. Since, locally, f is isomorphic to the projection $U \times S \to U$, the sheaf F is locally isomorphic the the constant sheaf with values in S. We have thus constructed a functor

$$\Phi\colon \operatorname{Cov}(X) \to \mathrm{LCSH(X)}. \tag{7.15}$$

One checks immediately that the functor Φ induces an equivalence between trivial coverings and constant sheaves. Form this result, one deduces:

Proposition 7.3.8. *The functor* Φ *in* (7.15) *is an equivalence.*

Now assume (7.6) and (7.12). In this case, Theorem 7.2.13 is also true when replacing sheaves of k-modules with sheaves of sets. We get the equivalences

$$\operatorname{Cov}(X) \simeq \mathrm{LCSH(X)} \simeq \mathrm{Fct}(\Pi_1(\mathrm{X}), \mathbf{Set}).$$

Exercises to Chapter 7

Exercise 7.1. Classify all locally constant sheaves of $\mathbb{C}$-vector spaces on the space $X = \mathbb{S}^1 \times \mathbb{S}^1$.

Exercise 7.2. Let $\gamma = \{(x, y, z, t) \in X = \mathbb{R}^4; x^2 + y^2 + z^2 = t^2\}$ $\dot{\gamma} = \gamma \setminus \{0\}$. Classify all locally constant sheaves of rank one of $\mathbb{C}$-vector spaces on $\dot{\gamma}$. (Hint: one can use the fact that $\dot{\gamma}$ is homotopic to its intersection with the unit sphere $\mathbb{S}^3$ of $\mathbb{R}^4$.)

Exercise 7.3. (i) Let G be a group and assume that all representations of G in $\mathrm{Mod}(k)$ are trivial. Prove that $G = \{1\}$.
(ii) Let $u\colon G_1 \to G_2$ is a morphism of groups which induces an isomorphism $\operatorname{Hom}(G_2, \mathbb{G}l(M)) \xrightarrow{\sim} \operatorname{Hom}(G_1, \mathbb{G}l(M))$ for all $M \in \mathrm{Mod}(\mathbb{Z})$. Prove that u is an isomorphism.
(Hint: one may use the free k-module $k[G]$ generated over k by the elements $g \in G$.)

Exercise 7.4. Assume X satisfies (7.1), $X = U_1 \cup U_2$, U_1 and U_2 are connected and simply connected and $U_1 \cap U_2$ is connected.
(i) Prove that X is simply connected.
(ii) Deduce that for $n > 1$, the sphere $\mathbb{S}^n$ as well as $\mathbb{R}^{n+1} \setminus \{0\}$ are simply connected.

Exercise 7.5. Let X be the subset of $\mathbb{C}$ defined by

$$X = \{z \in \mathbb{C}; (|z-1|-1)\cdot(|z+1|-1) = 0\}.$$

Using the Van Kampen theorem, calculate $\pi_1(X;0)$ and show that this group is not commutative.

Exercise 7.6. Consider three spheres S_1, S_2, S_3 in $\mathbb{R}^3$ of radius > 0 and tangent two by two. Assume that each S_i is contained in the exterior of the open ball delimited by S_j. Using the Van Kampen theorem, calculate $\pi_1(X)$ for $X = S_1 \cup S_2 \cup S_3$.

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