From $\mathcal{D}$-modules to deformation quantization modules

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Contents

Introduction

The aim of these Notes is first to introduce the reader to the theory of $\mathcal{D}$-modules in the analytical setting and also to make a link with the theory of deformation quantization (DQ for short) in the complex setting.

As we shall see, the advantage of the DQ-framework is that it unifies, in some sense, the theory of $\mathcal{O}$-modules (when the star product is commutative) and the theory of $\mathcal{D}$-modules (when the star product is symplectic).

This text is a short introduction, not a systematic study. In particular many proofs are skipped and the reader is encouraged to consult the literature. To our opinion, the best reference to $\mathcal{D}$-modules is [22], and, in fact, Chapters 1 and 2 are extracted from this book.

References for $\mathcal{D}$-modules. Some classical titles are [20], [4] and, in the algebraic setting, [3], [5]. A nice introduction may also be found in [10], [36]. Although the ring of microdifferential operators will be introduced, its detailed study will not be treated here. References for microdifferential operators are made to [21], [37] and [31].

References for DQ-algebroids and DQ-modules. We refer to [25] and the references therein.

References for categories, homological algebra and sheaves. The reader is assumed to be familiar with sheaf theory as well as homological algebra, including derived categories. An exhaustive treatment may be found in [24] and a pedagogical treatment is provided in [32], [33]. Among numerous other references, see [15], [23, Ch. 1, 2], [38].

History. An outline of $\mathcal{D}$-module theory, including holonomic systems, was sketched by Mikio Sato in the early 60's in a series of lectures at Tokyo University (see [34]). However, it seems that Sato's vision has not been understood until his student, Masaki Kashiwara, wrote his thesis in 1970 (see [20]). Independently and at the same time, J. Bernstein, a student of I. Gelfand at Moscow's University, developed a very similar theory in the algebraic setting (see [2]).

We shall not give any reference to the vast literature on deformation quantization, referring to [25]. Indeed, most of the authors are interested in constructing or classifying DQ-algebras (or DQ-algebroids), but the study of modules on such algebras seems to be essentially performed in loc. cit.

Chapter 1

The ring $\mathcal{D}_X$

In all these Notes, we denote by $\mathbb{K}$ a commutative unital ring. If R is a ring, an R-module means a left R-module and we denote by $\mathrm{Mod}(R)$ the abelian category of such modules. We denote by R^{op} the opposite ring. Hence, $\mathrm{Mod}(R^{\mathrm{op}})$ denotes the category of right R-modules. If a, b belong to R, their bracket $[a, b]$ is given by $[a, b] = ab - ba$. We use similar conventions and notations for a sheaf of rings $\mathcal{R}$ on a topological space X. In particular, $\mathrm{Mod}(\mathcal{R})$ denotes the category of sheaves of left $\mathcal{R}$-modules on X.

1.1 Construction of $\mathcal{D}_X$

Coherent and Noetherian sheaves

Let $\mathcal{R}$ be a sheaf of rings on a topological space X.

- An $\mathcal{R}$-module $\mathcal{M}$ is locally finitely generated if there locally exists an exact sequence $\mathcal{R}^{N_0} \to \mathcal{M} \to 0$.

- An $\mathcal{R}$-module $\mathcal{M}$ is locally of finite presentation if there locally exists an exact sequence $\mathcal{R}^{N_1} \xrightarrow{\cdot P_0} \mathcal{R}^{N_0} \to \mathcal{M} \to 0$, where $\cdot P_0$ means that P_0 acts on the right.

- An $\mathcal{R}$-module $\mathcal{F}$ is pseudo-coherent if for any open subset U of X, any $\mathcal{R}|_U$-submodule of $\mathcal{F}|_U$ locally finitely generated is locally of finite presentation.

- A pseudo-coherent $\mathcal{R}$-module which is locally of finite presentation is called coherent.

- If $\mathcal{R}$ is coherent, then the category $\mathrm{Mod}_{\mathrm{coh}}(\mathcal{R})$ of coherent $\mathcal{R}$-modules is a thick abelian subcategory of the abelian category $\mathrm{Mod}(\mathcal{R})$ and a coherent $\mathcal{R}$-module $\mathcal{M}$ is nothing but an $\mathcal{R}$-module locally of finite 1-presentation.

- The sheaf of rings $\mathcal{R}$ is Noetherian if it is coherent, if for any $x \in X$, the stalk $\mathcal{R}_x$ is a Noetherian ring and if for any open subset U of X and any family $\{\mathcal{J}_i\}_{i\in I}$ of coherent ideals of $\mathcal{R}|_U$, the ideal $\sum_i \mathcal{J}_i$ of $\mathcal{R}|_U$ is $\mathcal{R}|_U$-coherent.

$\mathcal{O}$-modules

Let X denote a complex manifold, $\mathcal{O}_X$ its structural sheaf, that is, the sheaf of holomorphic functions on X. Unless otherwise specified, we denote by d_X the complex dimension of X. We denote by Ω_X^p the sheaf of holomorphic p-forms and one sets $\Omega_X = \Omega_X^{d_X}$. One also sets

$$\Omega^\bullet = \bigoplus_p \Omega_X^p. \tag{1.1}$$

We denote by $\mathrm{Mod}(\mathbb{C}_X)$ the abelian category of sheaves of $\mathbb{C}$-vector spaces on X, and we denote by $\mathcal{H}om$ and $\otimes$ the internal Hom and tensor product in this category. For $F \in \mathrm{Mod}(\mathbb{C}_X)$, we set $\mathcal{E}nd(F) = \mathcal{H}om\,(F, F)$.

Similarly, we denote by $\mathrm{Mod}(\mathcal{O}_X)$ the abelian category of sheaves of $\mathcal{O}_X$-modules, and we denote by $\mathcal{H}om_{\mathcal{O}}$ and $\otimes_{\mathcal{O}}$ the internal Hom and tensor product in this category. We denote by $\mathrm{Mod}_{\mathrm{coh}}(\mathcal{O}_X)$ the full abelian subcategory consisting of coherent sheaves.

The sheaf $\mathcal{O}_X$ is Noetherian.

One denotes by Θ_X the sheaf of Lie algebras of holomorphic vector fields. Hence, $\Theta_X = \mathcal{H}om_{\mathcal{O}}(\Omega_X^1, \mathcal{O}_X)$.

The sheaf Θ_X has two actions on $\Omega^\bullet$, that we recall. Let $v \in \Theta_X$. The interior derivative $i_v \in \mathcal{E}nd(\Omega_X^\bullet)$ is characterized by the conditions

$$\begin{cases} i_v(a) = 0,\ a \in \mathcal{O}_X \\ i_v(\omega) = \langle v, \omega\rangle,\ \omega \in \Omega^1, \\ i_v(\omega_1 \wedge \omega_2) = (i_v\omega_1) \wedge \omega_2 + (-)^p \omega_1 \wedge (i_v\omega_2),\ \omega_1 \in \Omega_X^p. \end{cases} \tag{1.2}$$

Note that $i_v : \Omega_X^p \to \Omega_X^{p-1}$ is of degree -1.

On the other-hand, the Lie derivative $L_v \in \mathcal{E}nd(\Omega_X^\bullet)$ is characterized by the conditions

$$\begin{cases} L_v(a) = v(a) = \langle v, da\rangle, a \in \mathcal{O}_X, \\ d \circ L_v = L_v \circ d, \\ L_v(\omega_1 \wedge \omega_2) = (L_v\omega_1) \wedge \omega_2 + \omega_1 \wedge (L_v\omega_2), \end{cases} \tag{1.3}$$

The Lie derivative is of degree 0 and satisfies

$$[L_u, L_v] = L_{[u,v]},\ u, v \in \Theta_X. \tag{1.4}$$

One has the relations

$$L_v = d \circ i_v + i_v \circ d. \tag{1.5}$$

Using $v \mapsto L_v$, one may regard Θ_X as a subsheaf of $\mathcal{E}nd(\mathcal{O}_X)$.

The ring $\mathcal{D}_X$

Definition 1.1.1. One denotes by $\mathcal{D}_X$ the subalgebra of $\mathcal{E}nd(\mathcal{O}_X)$ generated by $\mathcal{O}_X$ and Θ_X.

If $(x_1, \dots, x_n)$ is a local coordinate system on a local chart U of X, then a section P of $\mathcal{D}_X$ on U may be uniquely written as a polynomial

$$P = \sum_{|\alpha| \le m} a_\alpha \partial^\alpha \tag{1.6}$$

where $a_\alpha \in \mathcal{O}_X$, $\partial_i = \partial_{x_i} = \frac{\partial}{\partial x_i}$ and we use the classical notations for multi-indices:

$$\begin{cases} \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n, \\ |\alpha| = \alpha_1 + \cdots + \alpha_n, \\ \text{ if } X = (X_1, \dots, X_n), \text{ then } X^\alpha = X^{\alpha_1} \dots X^{\alpha_n}. \end{cases}$$

Proposition 1.1.2. *Let $\mathcal{R}$ be a sheaf of $\mathbb{C}_X$-algebras and let $\iota : \mathcal{O}_X \to \mathcal{R}$ and $\varphi : \Theta_X \to \mathcal{R}$ be $\mathbb{C}_X$-linear morphisms satisfying:*

(i) *$\iota : \mathcal{O}_X \to \mathcal{R}$ is a ring morphism,*

(ii) *$\varphi : \Theta_X \to \mathcal{R}$ is left $\mathcal{O}_X$-linear (the $\mathcal{O}_X$-module structure on $\mathcal{R}$ is given by* (i)*),*

(iii) *$\varphi : \Theta_X \to \mathcal{R}$ is a morphism of Lie algebras (the Lie bracket on $\mathcal{R}$ being given by the commutator)*

(iv) *$[\varphi(v), \iota(a)] = \iota(v(a))$ for any $v \in \Theta_X$ and $a \in \mathcal{O}_X$.*

Then there exists a unique morphism of $\mathbb{C}_X$-algebras $\Psi : \mathcal{D}_X \to \mathcal{R}$ such that the composition $\mathcal{O}_X \to \mathcal{D}_X \to \mathcal{R}$ coincides with ι and the composition $\Theta_X \to \mathcal{D}_X \to \mathcal{R}$ coincides with φ.

Corollary 1.1.3. *Let $\mathcal{M}$ be an $\mathcal{O}_X$-module and let $\mu\colon \mathcal{O}_X \to \mathcal{E}nd(\mathcal{M})$ be the action of $\mathcal{O}_X$ on $\mathcal{M}$. Let $\psi : \Theta_X \to \mathcal{E}nd(\mathcal{M})$ be a $\mathbb{C}_X$-linear morphism satisfying:*

(i) $\mu(a) \circ \psi(v) = \psi(av)$ *(resp.* $\psi(v) \circ \mu(a) = \psi(av)$*).*

(ii) $[\psi(v), \psi(w)] = \psi([v, w])$ *(resp.* $[\psi(v), \psi(w)] = -\psi([v, w])$*),*

(iii) $[\psi(v), \mu(a)] = \mu(v(a))$, *(resp.* $[\psi(v), \mu(a)] = -\mu(v(a))$*).*

Then there exists one and only one structure of a left (resp. right) $\mathcal{D}_X$-module on $\mathcal{M}$ which extends the action of Θ_X.

Proof. For the structure of a left module, apply Proposition 1.1.2 to $\mathcal{R} = \mathcal{E}nd(\mathcal{M})$. The case of right modules follows since the bracket $[a, b]$ in $\mathcal{D}_X^{\mathrm{op}}$ is $-[a, b]$, where now $[a, b]$ is the bracket in $\mathcal{D}_X$. q.e.d.

Examples 1.1.4. (i) The sheaf $\mathcal{O}_X$ is naturally endowed with a structure of a left $\mathcal{D}_X$-module and $1 \in \mathcal{O}_X$ is a generator. Since the anihilator of 1 is the left ideal generated by Θ_X, we find an exact sequence of left $\mathcal{D}_X$-modules

$$\mathcal{D}_X \cdot \Theta_X \to \mathcal{D}_X \to \mathcal{O}_X \to 0.$$

Note that if X is connected and f is a section of $\mathcal{O}_X$, $f \neq 0$ (*i.e.*, f is not identically zero), then f is also a generator of $\mathcal{O}_X$ over $\mathcal{D}_X$. This follows from the Weierstrass Preparation Lemma. Indeed, choosing a local coordinate system $(x_1, \ldots, x_n)$, one may write $f = \sum_{j=0}^{m} a_j(x')x_1^j$, with $a_m \equiv 1$. Then $\partial_1^m(f) = m!$.
(ii) The sheaf Ω_X is naturally endowed with a structure of a right $\mathcal{D}_X$-module, by

$$v(\omega) = -L_v(\omega), \quad v \in \Theta_X, \omega \in \Omega_X.$$

(iii) Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then $\mathcal{D}_X \otimes_{\mathcal{O}} \mathcal{F}$ is a left $\mathcal{D}_X$-module.
(iv) Let Z be a closed complex submanifold of X of codimension d. Then $H_Z^d(\mathcal{O}_X)$ is a left $\mathcal{D}_X$-module.
(v) Let X be a complex manifold and let P be a differential operator on X. The differential equation $Pu = v$ may be studied via the left $\mathcal{D}_X$-module $\mathcal{D}_X / \mathcal{D}_X \cdot P$. (See below.)
(vi) Let $X = \mathbb{C}^n$ and consider the differential operators $P = \sum_{j=1}^{n} \partial_j^2$, $Q_{ij} = x_i\partial_j - x_j\partial_i$. Consider the left ideal $\mathcal{J}$ of $\mathcal{D}_X$ generated by P and the family $\{Q_{ij}\}_{i<j}$. The left $\mathcal{D}_X$-module $\mathcal{D}_X / \mathcal{J}$ is naturally associated to the operator P.

Internal hom and tens

The sheaf $\mathcal{D}_X$ is a sheaf of non commutative rings and $\mathbb{C}_X$ is contained (in fact, is equal, but we have not proved it here) in its center. It follows that we have functors:

$$\begin{aligned}&\mathcal{H}om_{\mathcal{D}} : (\mathrm{Mod}(\mathcal{D}_X))^{\mathrm{op}} \times \mathrm{Mod}(\mathcal{D}_X) \to \mathrm{Mod}(\mathbb{C}_X),\\&\otimes_{\mathcal{D}} : \mathrm{Mod}(\mathcal{D}_X^{\mathrm{op}}) \times \mathrm{Mod}(\mathcal{D}_X) \to \mathrm{Mod}(\mathbb{C}_X).\end{aligned}$$

We shall now study hom and tens over $\mathcal{O}_X$.
Let $\mathcal{M}, \mathcal{N}$ and $\mathcal{P}$ be left $\mathcal{D}_X$-modules and let $\mathcal{M}'$ and $\mathcal{N}'$ be two right $\mathcal{D}_X$-modules.

(a) One endows $\mathcal{M} \otimes_{\mathcal{O}} \mathcal{N}$ with a structure of a left $\mathcal{D}_X$-module by setting

$$v(m \otimes n) = v(m) \otimes n + m \otimes v(n), \quad m \in \mathcal{M}, n \in \mathcal{N}, v \in \Theta_X.$$

(b) One endows $\mathcal{H}om_{\mathcal{O}}(\mathcal{M}, \mathcal{N})$ with a structure of a left $\mathcal{D}_X$-module by setting

$$v(f)(m) = v(f(m)) - f(v(m)), \quad m \in \mathcal{M}, f \in \mathcal{H}om_{\mathcal{O}}(\mathcal{M}, \mathcal{N}), v \in \Theta_X.$$

(c) One endows $\mathcal{N}' \otimes_{\mathcal{O}} \mathcal{M}$ with a structure of a right $\mathcal{D}_X$-module by setting

$$(n \otimes m)v = nv \otimes m - n \otimes vm, \quad m \in \mathcal{M}, n \in \mathcal{N}', v \in \Theta_X.$$

(d) One endows and $\mathcal{H}om_{\mathcal{O}}(\mathcal{M}', \mathcal{N}')$ with a structure of a left $\mathcal{D}_X$-module by setting

$$v(f)(m) = f(mv) - f(m)v \ m \in \mathcal{M}', f \in \mathcal{H}om_{\mathcal{O}}(\mathcal{M}', \mathcal{N}'), v \in \Theta_X.$$

(e) One endows and $\mathcal{H}om_{\mathcal{O}}(\mathcal{M}, \mathcal{N}')$ with a structure of a right $\mathcal{D}_X$-module by setting

$$(fv)(m) = f(m)v + f(vm) \ m \in \mathcal{M}, f \in \mathcal{H}om_{\mathcal{O}}(\mathcal{M}, \mathcal{N}'), v \in \Theta_X.$$

There are isomorphisms of $\mathbb{C}_X$-modules;

$$\begin{aligned}&\mathcal{H}om_{\mathcal{D}}(\mathcal{M} \otimes_{\mathcal{O}} \mathcal{N}, \mathcal{P}) \simeq \mathcal{H}om_{\mathcal{D}}(\mathcal{M}, \mathcal{H}om_{\mathcal{O}}(\mathcal{N}, \mathcal{P})),\\&\mathcal{H}om_{\mathcal{D}}(\mathcal{M}' \otimes_{\mathcal{O}} \mathcal{M}, \mathcal{N}) \simeq \mathcal{H}om_{\mathcal{D}}(\mathcal{M}, \mathcal{H}om_{\mathcal{O}}(\mathcal{M}', \mathcal{N})),\\&(\mathcal{M}' \otimes_{\mathcal{O}} \mathcal{M}) \otimes_{\mathcal{D}} \mathcal{N} \simeq \mathcal{M}' \otimes_{\mathcal{D}} (\mathcal{M} \otimes_{\mathcal{O}} \mathcal{N}).\end{aligned}$$

To summarize, we have functors

$$\begin{aligned}
&\otimes_{\mathcal{O}}\colon \mathrm{Mod}(\mathcal{D}_X)\times\mathrm{Mod}(\mathcal{D}_X)\to\mathrm{Mod}(\mathcal{D}_X),\\
&\otimes_{\mathcal{O}}\colon \mathrm{Mod}(\mathcal{D}_X^{\mathrm{op}})\times\mathrm{Mod}(\mathcal{D}_X)\to\mathrm{Mod}(\mathcal{D}_X^{\mathrm{op}}),\\
&\mathcal{H}om_{\mathcal{O}}\colon \mathrm{Mod}(\mathcal{D}_X)^{\mathrm{op}}\times\mathrm{Mod}(\mathcal{D}_X)\to\mathrm{Mod}(\mathcal{D}_X),\\
&\mathcal{H}om_{\mathcal{O}}\colon \mathrm{Mod}(\mathcal{D}_X^{\mathrm{op}})^{\mathrm{op}}\times\mathrm{Mod}(\mathcal{D}_X^{\mathrm{op}})\to\mathrm{Mod}(\mathcal{D}_X),\\
&\mathcal{H}om_{\mathcal{O}}\colon \mathrm{Mod}(\mathcal{D}_X)^{\mathrm{op}}\times\mathrm{Mod}(\mathcal{D}_X^{\mathrm{op}})\to\mathrm{Mod}(\mathcal{D}_X^{\mathrm{op}}).
\end{aligned}$$

Remark 1.1.5. Following [17] who call it the Oda's rule, one way to memorize the left an right actions is to use the correspondence left $=0$, right $=1$, $a\otimes b=a+b$ and $\mathcal{H}om\,(a,b)=-a+b$.

Applying these results, we get:

Proposition 1.1.6. *The functor $\mathcal{M}\mapsto\Omega\otimes_{\mathcal{O}}\mathcal{M}$ induces an equivalence of categories $\mathrm{Mod}(\mathcal{D}_X)\xrightarrow{\sim}\mathrm{Mod}(\mathcal{D}_X^{\mathrm{op}})$. A quasi-inverse is given by $\mathcal{N}\mapsto\mathcal{H}om_{\mathcal{O}}(\Omega_X,\mathcal{N})\simeq\Omega_X^{\otimes-1}\otimes_{\mathcal{O}}\mathcal{N}$.*

Remark 1.1.7. Suppose to be given a volume form dv on X. Then $f\mapsto f dv$ gives an isomorphism $\mathcal{O}_X\xrightarrow{\sim}\Omega_X$ and we get an isomorphism $\mathcal{D}_X\simeq\mathcal{D}_X^{\mathrm{op}}$. The image of a section $P\in\mathcal{D}_X$ by this isomorphism is called its adjoint with respect to dv and is denoted by P^*. Hence, for a left $\mathcal{D}_X$-module $\mathcal{M}$ and a section u of $\mathcal{M}$, we have

$$P\cdot u=(u\cdot dv)\cdot P^*.$$

Clearly $(Q\circ P)^*=P^*\circ Q^*$. If $(x_1,\dots,x_n)$ is a local coordinate system on X and $dv=dx_1\wedge\cdots\wedge dx_n$, one checks that $x_i^*=x_i$ and $\partial_{x_i}^*=-\partial_{x_i}$.

1.2 Filtration on $\mathcal{D}_X$

Filtered rings and modules

Let X be a topological space.

- A filtered sheaf $\mathrm{Fl}\,\mathcal{M}$ of $\mathbb{K}_X$-modules is a sheaf $\mathcal{M}$ of $\mathbb{K}_X$-modules together with a family $\{\mathrm{Fl}_j\mathcal{M}\}_{j\in\mathbb{Z}}$ of subsheaves satisfying : $\mathrm{Fl}_j\mathcal{M}\subset\mathrm{Fl}_{j+1}\mathcal{M}$, $\varinjlim_j\mathrm{Fl}_j\mathcal{M}=\mathcal{M}$.

- The shifted filtration $\mathrm{Fl}^{[p]}\mathcal{M}$ is given by $\mathrm{Fl}^{[p]}_j\mathcal{M}=\mathrm{Fl}_{p+j}\mathcal{M}$.

- A morphism of filtered sheaves $\mathrm{Fl}\, f : \mathrm{Fl}\, \mathcal{M} \to \mathrm{Fl}\mathcal{N}$ is a morphism of sheaves $f : \mathcal{M} \to \mathcal{N}$ such that $f(\mathrm{Fl}_j\mathcal{M}) \subset \mathrm{Fl}_j\mathcal{N}$ for all j.

- The graded sheaf $\mathrm{gr}\, \mathcal{M}$ associated to $\mathrm{Fl}\, \mathcal{M}$ is the sheaf $\bigoplus_j \mathrm{gr}_j\mathcal{M}$, where $\mathrm{gr}_j\mathcal{M} = \mathrm{Fl}_j\mathcal{M}/\mathrm{Fl}_{j-1}\mathcal{M}$. If $\mathrm{Fl}\, f : \mathrm{Fl}\, \mathcal{M} \to \mathrm{Fl}\mathcal{N}$ is a filtered morphism, one denotes by $\mathrm{gr}\, f : \mathrm{gr}\, \mathcal{M} \to \mathrm{gr}\, \mathcal{N}$ the associated morphism of graded sheaves.

- One denote by $\sigma_j : \mathrm{Fl}_j\mathcal{M} \to \mathrm{gr}_j\mathcal{M}$ the canonical morphism and calls it the "symbol of order j" morphism. One denotes by $\sigma : \mathrm{Fl}\, \mathcal{M} \to \mathrm{gr}\, \mathcal{M}$ the morphism deduced from the σ_j's and calls it the"principal symbol" morphism. (One shall be aware that σ_j is an additive morphism, contrarily to σ.)

- Consider an exact sequence of sheaves $0 \to \mathcal{M}' \xrightarrow{f} \mathcal{M} \xrightarrow{g} \mathcal{M}'' \to 0$ and assume that $\mathcal{M}$ is endowed with a filtration $\mathrm{Fl}\, \mathcal{M}$. The induced filtration on $\mathcal{M}'$ is given by $\mathrm{Fl}_j\mathcal{M}' = f^{-1}(\mathrm{Fl}_j\mathcal{M})$. The image filtration on $\mathcal{M}''$ is given by $\mathrm{Fl}_j\mathcal{M}'' = g(\mathrm{Fl}_j\mathcal{M})$. Note that in this case, the associated sequence $0 \to \mathrm{gr}\, \mathcal{M}' \xrightarrow{f} \mathrm{gr}\, \mathcal{M} \xrightarrow{g} \mathrm{gr}\, \mathcal{M}'' \to 0$ is exact.

- A filtered ring $\mathrm{Fl}\, \mathcal{R}$ on X is a filtered sheaf of rings satisfying: $1 \in \mathrm{Fl}_0\mathcal{R}$ and $\mathrm{Fl}_i\mathcal{R} \cdot \mathrm{Fl}_j\mathcal{R} \subset \mathrm{Fl}_{i+j}\mathcal{R}$ for all i, j.

- A filtered $\mathcal{R}$-module $\mathrm{Fl}\, \mathcal{M}$, or equivalently an $\mathrm{Fl}\, \mathcal{R}$-module, is an $\mathcal{R}$-module endowed with a filtration satisfying: $\mathrm{Fl}_i\mathcal{R} \cdot \mathrm{Fl}_j\mathcal{M} \subset \mathrm{Fl}_{i+j}\mathcal{M}$.

- A filtration $\mathrm{Fl}\, \mathcal{M}$ on $\mathcal{M}$ is locally finite free if it is locally isomorphic to a finite direct sum of $\mathrm{Fl}^{[i]}\mathcal{R}$.

- A coherent filtration (one also says "a good filtration") on $\mathcal{M}$ is a locally finitely generated filtration, that is, $\mathrm{Fl}\, \mathcal{M}$ is locally the image of a finite free filtration. Note that the image of a good filtration is good and that any finitely generated $\mathcal{R}$-module $\mathcal{M}$ may be endowed with a good filtration. Namely, if $\mathcal{R}^m \twoheadrightarrow \mathcal{M}$ is an epimorphism, one endows $\mathcal{M}$ with the image filtration.

Remark 1.2.1. Let us denote by $\mathrm{Mod}^{\mathrm{fi}}(\mathbb{K}_X)$ the category of filtered sheaves. Clearly, the category $\mathrm{Mod}^{\mathrm{fi}}(\mathbb{K}_X)$ is additive and admits kernels and cokernels. One shall be aware that the category $\mathrm{Mod}^{\mathrm{fi}}(\mathbb{K}_X)$ is not abelian, even when $X = \mathrm{pt}$. Indeed, consider a filtered k-module $\mathrm{Fl}\, M$ and the identity morphism $u : \mathrm{Fl}\, M \to \mathrm{Fl}^{[1]}M$. Its kernel and cokernel are zero, although this morphism is not an isomorphism in general.

Total symbol of differential operators

Assume X is affine, that is, X is open in a finite dimensional complex vector space E. Let P be a section of $\mathcal{D}_X$. One defines its total symbol

$$\sigma_{\mathrm{tot}}(P)(x;\xi) := \exp\langle -x,\xi\rangle P(\exp\langle x,\xi\rangle) \;=\; \sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha. \tag{1.7}$$

Using (1.6), one gets that $\sigma_{\mathrm{tot}}(P)$ is a function on $X \times E^*$, polynomial with respect to $\xi \in E^*$. This function highly depends on the affine structure, but its order (a locally constant function on X) does not. It is called the order of P and denoted $\mathrm{ord}(P)$.

If Q is another differential operator with total symbol $\sigma_{\mathrm{tot}}(Q)$, it follows easily from the Leibniz formula that the total symbol $\sigma_{\mathrm{tot}}(R)$ of $R = P\cdot Q$ is given by:

$$\sigma_{\mathrm{tot}}(R) = \sum_{\alpha\in\mathbb{N}^n} \frac{1}{\alpha!}\partial_\xi^\alpha(\sigma_{\mathrm{tot}}(P))\partial_x^\alpha(\sigma_{\mathrm{tot}}(Q)). \tag{1.8}$$

By this formula, one gets that

$$\begin{aligned}&\mathrm{ord}(P\cdot Q) = \mathrm{ord}(P) + \mathrm{ord}(Q),\\ &\mathrm{ord}([P,Q]) \le \mathrm{ord}(P) + \mathrm{ord}(Q) - 1.\end{aligned}$$

The ring $\mathcal{D}_X$ is now endowed with the filtration " by the order",

$$\mathrm{Fl}_m(\mathcal{D}_X) = \{P\in\mathcal{D}_X; \mathrm{ord}(P)\le m\}.$$

One can give a more intrinsic definition of the filtration.

Filtration on $\mathcal{D}_X$

Definition 1.2.2. The filtration $\mathrm{Fl}\,\mathcal{D}_X$ on $\mathcal{D}_X$ is given by

$$\mathrm{Fl}_{-1}\mathcal{D}_X = \{0\}, \quad \mathrm{Fl}_m\mathcal{D}_X = \{P\in\mathcal{D}_X; [P,\mathcal{O}_X]\in\mathrm{Fl}_{m-1}\mathcal{D}_X\}.$$

Note that

$$\begin{cases}\mathrm{Fl}_0\mathcal{D}_X = \mathcal{O}_X, \quad \mathrm{Fl}_1\mathcal{D}_X = \mathcal{O}_X\oplus\Theta_X,\\ \mathrm{Fl}_m\mathcal{D}_X\cdot\mathrm{Fl}_l\mathcal{D}_X \subset \mathrm{Fl}_{m+l}\mathcal{D}_X, \quad [\mathrm{Fl}_m\mathcal{D}_X,\mathrm{Fl}_l\mathcal{D}_X]\subset\mathrm{Fl}_{m+l-1}\mathcal{D}_X.\end{cases} \tag{1.9}$$

One denotes by $\mathrm{gr}\,\mathcal{D}_X$ the associated graded ring, by $\sigma : \mathrm{Fl}\,\mathcal{D}_X \to \mathrm{gr}\,\mathcal{D}_X$ the "principal symbol map" and by $\sigma_m : \mathrm{Fl}_m\mathcal{D}_X \to \mathrm{gr}_m\mathcal{D}_X$ the map "symbol of order m".

One shall not confuse the total symbol, which is defined on affine charts, and the principal symbol, which is well defined on manifolds.

It follows from (1.8) that $\sigma(P)\sigma(Q) = \sigma(Q)\sigma(P) = \sigma(P \cdot Q)$. Hence, $\mathrm{gr}\,(\mathcal{D}_X)$ is a *commutative* graded ring. Moreover, $\mathrm{gr}_0(\mathcal{D}_X) \simeq \mathcal{O}_X$ and $\mathrm{gr}_1(\mathcal{D}_X) \simeq \Theta_X$.

Denote by $S_{\mathcal{O}}(\Theta_X)$ the symmetric $\mathcal{O}_X$-algebra associated with the locally free $\mathcal{O}_X$-module Θ_X. By the universal property of symmetric algebras, the morphism $\Theta_X \to \mathrm{gr}\,(\mathcal{D}_X)$ may be extended to a morphism of symmetric algebra

$$S_{\mathcal{O}}(\Theta_X) \to \mathrm{gr}\,\mathcal{D}_X. \tag{1.10}$$

Proposition 1.2.3. *The morphism* (1.10) *is an isomorphism.*

Proof. Choose a local coordinate system $(x_1, \ldots, x_n)$ on X. Then $\Theta_X \simeq \bigoplus_{i=1}^n \mathcal{O}_X \partial_i$ and the correspondence $\partial_i \mapsto \xi_i$ gives the isomorphism

$$S_{\mathcal{O}}(\Theta_X) \simeq \bigoplus_{\alpha} \mathcal{O}_X \partial^\alpha \simeq \mathcal{O}_X[\xi_1, \ldots, \xi_n] \simeq \mathrm{gr}\,\mathcal{D}_X.$$

q.e.d.

Denote by $\pi : T^*X \to X$ the projection. There is a natural monomorphism

$$\Theta_X \hookrightarrow \pi_* \mathcal{O}_{T^*X}.$$

Indeed, a vector field on X is a section of the tangent bundle TX, hence defines a linear function on T^*X.

By the universal property of symmetric algebra, we get a monomorphism $S_{\mathcal{O}}(\Theta_X) \hookrightarrow \pi_* \mathcal{O}_{T^*X}$. Applying Proposition 1.2.3, we get an embedding of $\mathbb{C}_X$-algebras:

$$\mathrm{gr}\,\mathcal{D}_X \hookrightarrow \pi_* \mathcal{O}_{T^*X}.$$

In the sequel, we shall still denote by

$$\sigma\colon \mathcal{D}_X \to \pi_* \mathcal{O}_{T^*X} \text{ and } \sigma_m\colon \mathrm{Fl}_m \mathcal{D}_X \to \pi_* \mathcal{O}_{T^*X},$$

the maps obtained by applying the inverse of the isomorphism (1.10) to σ and σ_m.

Theorem 1.2.4. *The sheaf of rings $\mathcal{D}_X$ is right and left Noetherian.*

Proof. This follows from Proposition 1.2.3 and general results on filtered ring with associated commutative graded ring. (See [22, Th. A.20].) q.e.d.

1.3 Characteristic variety

Gabber's theorem

Recall that if B is a commutative ring and I an ideal, the radical $\sqrt{I}$ of I is given by

$$x \in \sqrt{I} \Leftrightarrow \text{ there exists } k \geq 0 \text{ with } x^k \in I.$$

If N is a B-module, the annihilator I_N of N is the ideal given by

$$x \in I_N \Leftrightarrow xu = 0 \text{ for all } u \in N.$$

If $0 \to M' \to M \to M'' \to 0$ is an exact sequence in $\mathrm{Mod}(B)$, then clearly

$$\sqrt{I_M} = \sqrt{I_{M'}} \cap \sqrt{I_{M''}} = \sqrt{I_{M'} \cdot I_{M''}}. \tag{1.11}$$

In other words, the map $M \mapsto \sqrt{I_M}$ is additive. If $\mathrm{gr}\, M$ is a graded $\mathrm{gr}\, A$-module, then $\sqrt{I_{\mathrm{gr}\, M}}$ is a graded ideal.

Consider now a filtered ring $\mathrm{Fl}\, A$ with $\mathrm{gr}\, A$ commutative. The ring $\mathrm{gr}\, A$ is endowed with a Poisson bracket as follows. Let $\overline{a}$ and $\overline{b}$ be two elements of $\mathrm{gr}\, A$, homogeneous say of degree p and q, respectively. Choose a and b in A such that $\sigma_p(a) = \overline{a}$ and $\sigma_q(b) = \overline{b}$. One sets

$$\{\overline{a}, \overline{b}\} = \sigma_{p+q-1}([a, b]). \tag{1.12}$$

One checks easily that this bracket satisfies the Jacobi identities

$$\begin{cases} \{\overline{a}, \overline{b}\} = -\{\overline{b}, \overline{a}\}, \\ \{\overline{a}, \overline{b}\overline{c}\} = \overline{c}\{\overline{a}, \overline{b}\} + \overline{b}\{\overline{a}, \overline{c}\}, \\ \{\{\overline{a}, \overline{b}\}, \overline{c}\} + \{\{\overline{b}, \overline{c}\}, \overline{a}\} + \{\{\overline{c}, \overline{a}\}, \overline{b}\} = 0. \end{cases} \tag{1.13}$$

Let M be a finitely generated A-module. Recall that a good filtration on M is the image of a finite free filtration.

Lemma 1.3.1. *Let M be a finitely generated A-module, let $\mathrm{Fl}\, M$ be a good filtration on M and let $\mathrm{gr}\, M$ be the associated graded module. Then*

$$\mathrm{Icar}(M) := \sqrt{I_{\mathrm{gr}\, M}}, \tag{1.14}$$

does not depend on the choice of the filtration.

Proof. Let $\bar{a} \in \sqrt{I_{\operatorname{gr} M}}$ of order p. There exists q such that $\bar{a}^p \in I_{\operatorname{gr} M}$ and there exists $a \in \mathrm{Fl}_p M$ such that $\sigma(a) = \bar{a}$. Then

$$\begin{aligned} a^q \mathrm{Fl}_k M &\subset \mathrm{Fl}_{k+pq-1} \text{ for all } k, \\ a^{lq} \mathrm{Fl}_k M &\subset \mathrm{Fl}_{k+lpq-l} \text{ for all } k. \end{aligned}$$

If $\mathrm{Fl}' M$ is another filtration, there exists r such that $\mathrm{Fl}'_{k-r} M \subset \mathrm{Fl}_k M \subset \mathrm{Fl}'_{k+r} M$. Hence, $a^{lq} \mathrm{Fl}'_k \subset \mathrm{Fl}'_{k+lpq-1}$ for $l >> 0$. q.e.d.

Theorem 1.3.2. (Gabber's Theorem.) *Assume that* $\operatorname{gr} A$ *is a commutative Noetherian* $\mathbb{Q}$*-algebra. Let* M *be a finitely generated* A*-module. Then* $\operatorname{Icar}(M)$ *is involutive, that is, is stable by Poisson bracket.*

Note that if $\bar{a}$ and $\bar{b}$ belong to $I_{\operatorname{gr} M}$, then $\{\bar{a}, \bar{b}\}$ obviously belongs to $I_{\operatorname{gr} M}$. The difficulty is that one assumes that $\bar{a}$ and $\bar{b}$ belong to the radical of $I_{\operatorname{gr} M}$.

Poisson's structures

The graded ring $\operatorname{gr}(\mathcal{D}_X)$ is endowed with a natural Poisson bracket induced by the commutator in $\mathcal{D}_X$.

On the other hand, the sheaf $\mathcal{O}_{T^*X}$ (hence, the sheaf $\pi_* \mathcal{O}_{T^*X}$) is endowed with the Poisson bracket induced by the symplectic structure of T^*X. Recall that if $(x_1, \dots, x_n; \xi_1, \dots, \xi_n)$ is a local symplectic coordinate system on T^*X, this Poisson bracket is given by

$$\{f, g\} = \sum_{i=1}^{n} \partial_{\xi_i} f \partial_{x_i} g - \partial_{\xi_i} g \partial_{x_i} f.$$

Proposition 1.3.3. *The Poisson bracket on* $\pi_* \mathcal{O}_{T^*X}$ *induces the Poisson bracket on* $\operatorname{gr}(\mathcal{D}_X)$.

Proof. Let $P \in \mathrm{Fl}_m(\mathcal{D}_X)$ and $Q \in \mathrm{Fl}_l(\mathcal{D}_X)$. Then $[P, Q] \in \mathrm{Fl}_{m+l-1}(\mathcal{D}_X)$ and it follows from (1.8) that

$$\sigma_{m+l-1}([P, Q]) = \sum_{i=1}^{n} \big(\partial_{\xi_i} \sigma_m(P) \partial_{x_i} \sigma_l(Q) - \partial_{\xi_i} \sigma_l(Q) \partial_{x_i} \sigma_m(P)\big). \tag{1.15}$$

Hence, $\sigma_{m+l-1}([P, Q]) = \{\sigma_m(P), \sigma_l(Q)\}$. q.e.d.

Good filtration

Recall that a good filtration on a coherent $\mathcal{D}_X$-module $\mathcal{M}$ is a filtration which is locally the image of a finite free filtration. Hence, a filtration $\mathrm{Fl}\,\mathcal{M}$ on $\mathcal{M}$ is good if and only if,

$$\begin{cases} \text{locally on } X,\ \mathrm{Fl}_j\mathcal{M} = 0 \text{ for } j \ll 0, \\ \mathrm{Fl}_j\mathcal{M} \text{ is } \mathcal{O}_X\text{-coherent}, \\ \text{locally on } X,\ (\mathrm{Fl}_k\mathcal{D}_X)\cdot(\mathrm{Fl}_j\mathcal{M}) = \mathrm{Fl}_{k+j}\mathcal{M} \text{ for } j \gg 0 \text{ and} \\ \text{all } k \geq 0. \end{cases} \tag{1.16}$$

Lemma 1.3.4. *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module, $\mathcal{N} \subset \mathcal{M}$ a coherent submodule. Assume that $\mathcal{M}$ is endowed with a good filtration $\mathrm{Fl}\,\mathcal{M}$. Then the induced filtration on $\mathcal{N}$ defined by $\mathrm{Fl}_j\mathcal{N} = \mathcal{N} \cap \mathrm{Fl}_j\mathcal{M}$ is good.*

Proof. We do not give the proof here. q.e.d.

Denote by $\mathrm{Mod}^{\mathrm{gr}}_{\mathrm{coh}}(\mathrm{gr}\,\mathcal{D}_X)$ the abelian category of coherent graded $\mathrm{gr}\,\mathcal{D}_X$-modules and consider the functor

$$\begin{aligned} \widetilde{\ }: \mathrm{Mod}^{\mathrm{gr}}_{\mathrm{coh}}(\mathrm{gr}\,\mathcal{D}_X) &\to \mathrm{Mod}_{\mathrm{coh}}(\pi_*\mathcal{O}_{T^*X}), \\ \mathrm{gr}\,\mathcal{M} &\mapsto \pi_*\mathcal{O}_{T^*X} \otimes_{\mathrm{gr}\,\mathcal{D}_X} \mathrm{gr}\,\mathcal{M}. \end{aligned}$$

This functor is exact and faithful. If $\mathcal{M}$ is a coherent $\mathcal{D}_X$-module endowed with a good filtration, the $\pi_*\mathcal{O}_{T^*X}$-module

$$\widetilde{\mathrm{gr}\,\mathcal{M}} = \pi_*\mathcal{O}_{T^*X} \otimes_{\mathrm{gr}\,\mathcal{D}_X} \mathrm{gr}\,\mathcal{M}$$

is thus coherent and its support satisfies:

$$\mathrm{supp}(\widetilde{\mathrm{gr}\,\mathcal{M}}) = \{p \in T^*X; \sigma(P)(p) = 0 \text{ for any } P \in \mathrm{Icar}(\mathcal{M})\}.$$

In the sequel, we shall often confuse $\mathrm{gr}\,\mathcal{M}$ and $\widetilde{\mathrm{gr}\,\mathcal{M}}$.

Definition 1.3.5. The characteristic variety of $\mathcal{M}$, denoted $\mathrm{char}(\mathcal{M})$, is the closed subset of T^*X characterized as follows: for any open subset U of X such that $\mathcal{M}|_U$ is endowed with a good filtration, $\mathrm{char}(\mathcal{M})|_{T^*U}$ is the support of $\widetilde{\mathrm{gr}\,\mathcal{M}|_U}$.

Theorem 1.3.6. (i) $\mathrm{char}(\mathcal{M})$ *is a closed $\mathbb{C}^\times$-conic analytic subset of T^*X.*

(ii) $\mathrm{char}(\mathcal{M})$ *is involutive for the Poisson structure of T^*X, and in particular,* $\mathrm{codim}(\mathrm{char}(\mathcal{M})) \leq d_X$.

(iii) *If* $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$ *is an exact sequence of coherent* $\mathcal{D}_X$*-modules, then*

$$\mathrm{char}(\mathcal{M}) = \mathrm{char}(\mathcal{M}') \cup \mathrm{char}(\mathcal{M}'').$$

Proof. (i) is obvious, (ii) follows from Gabber's theorem and (iii) follows from Lemma 1.3.4. q.e.d.

Note that the involutivity theorem has first been proved by Sato, Kashiwara and Kawai [37] using analytical tools, before Gabber gave is purely algebraic proof.

Suppose that a coherent $\mathcal{D}_X$-module $\mathcal{M}$ is generated by a single section u. Then $\mathcal{M} \simeq \mathcal{D}_X/\mathcal{I}$, where $\mathcal{I}$ is the anihilator of u. There is a natural filtration on $\mathcal{M}$, the image of $\mathrm{Fl}\,\mathcal{D}_X$. Put $\mathrm{Fl}_j\mathcal{I} = \mathcal{I} \cap \mathrm{Fl}_j\mathcal{D}_X$. It follows from the fact that $\mathrm{gr}\,\mathcal{D}_X$ is Noetherian that the graded ideal $\mathrm{gr}\,\mathcal{I}$ is coherent. Moreover, since $\mathrm{gr}\,\mathcal{M} = \mathrm{gr}\,\mathcal{D}_X/\mathrm{gr}\,\mathcal{I}$, we get

$$\mathrm{char}(\mathcal{M}) = \{p \in T^*X; \sigma_j(P)(p) = 0 \text{ for all } P \in \mathrm{Fl}_j(\mathcal{I})\}. \tag{1.17}$$

If $\{P_0, \ldots, P_N\}$ generates $\mathcal{I}$ it follows that

$$\mathrm{char}(\mathcal{M}) \subset \bigcap_j \sigma(P_j)^{-1}(0).$$

In general the equality does not hold, since the family of the P_j's may generate $\mathcal{I}$ although the family of the $\sigma_{m_j}(P_j)$'s does not generate $\mathrm{gr}\,\mathcal{I}$.

Example 1.3.7. If $X = \mathbb{A}^1(\mathbb{C})$, the affine line, the ideal generated by ∂ and x is $\mathcal{D}_X$, but the ideal generated by their principal symbols is not $\mathcal{O}_{T^*X}$.

Corollary 1.3.8. *Let* $\mathcal{M}$ *be a coherent* $\mathcal{D}_X$*-module, let* $p \in T^*X$ *and assume that* $p \notin \mathrm{char}(\mathcal{M})$. *Let* $u \in \mathcal{M}$. *Then there exists a section* $P \in \mathcal{D}_X$ *defined in a neighborhood of* $\pi(p)$ *with* $Pu = 0$ *and* $\sigma(P)(p) \neq 0$.

Proof. Consider the sub-$\mathcal{D}_X$-module $\mathcal{D}_X u$ generated by u. It is coherent and its characteristic variety is contained in that of $\mathcal{M}$. Let $\mathcal{I}$ denotes the anihilator ideal of u in $\mathcal{D}_X$ and let $P_1, \ldots, P_N$ denotes sections of this ideal such that $\sigma(P_1), \ldots, \sigma(P_N)$ generate the graded ideal $\mathrm{gr}\,\mathcal{I}$. Such a finite family exists since $\mathrm{gr}\,\mathcal{I}$ is coherent. Since $p \notin \mathrm{char}(\mathcal{D}_X u)$, there exists j with $\sigma(P_j)(p) \neq 0$. q.e.d.

Remark 1.3.9. Using the notion of characteristic cycles for coherent $\mathcal{O}_X$-modules, one defines similarly the notion of characteristic cycles for coherent $\mathcal{D}_X$-modules, see [20].

Example 1.3.10. (i) $\mathrm{char}(\mathcal{O}_X) = T^*_X X$, the zero-section of T^*X.
(ii) $\mathrm{char}(\mathcal{D}_X/\mathcal{D}_X \cdot P) = \{p \in T^*X; \sigma(P)(p) = 0\}$.

1.4 De Rham and Spencer complexes

If A is a ring, M is an A-module, and $\varphi := (\varphi_1, \dots, \varphi_n)$ are n-commuting endomorphisms of M, one can define the Koszul complex $K^\bullet(M;\varphi)$ and the co-Koszul complex $K_\bullet(M;\varphi)$. We refer to [31] for an exposition.

Also recall the De Rham complex

$$\mathrm{DR_X}(\mathcal{O}_\mathrm{X}) := 0 \to \Omega^0_\mathrm{X} \xrightarrow{\mathrm{d}} \Omega^1_\mathrm{X} \xrightarrow{\mathrm{d}} \cdots \to \Omega^{\mathrm{d_X}}_\mathrm{X} \to 0, \tag{1.18}$$

where d is the differential.

Let $\mathcal{M}$ be a left $\mathcal{D}_X$-module. One defines the differential $d\colon \mathcal{M} \to \Omega^1_X \otimes_\mathcal{O} \mathcal{M}$ as follows. In a local coordinate system $(x_1, \dots, x_{d_X})$ on X, the differential d is given by

$$\mathcal{M} \to \Omega^1_X \otimes_\mathcal{O} \mathcal{M}, \quad m \mapsto \sum_i dx_i \otimes \partial_i m$$

and one checks easily that this does not depend on the choice of the local coordinate system.

One defines the De Rham complex of $\mathcal{M}$, denoted $\mathrm{DR_X}(\mathcal{M})$, as the complex

$$\mathrm{DR_X}(\mathcal{M}) := \quad 0 \to \Omega^0_\mathrm{X} \otimes_\mathcal{O} \mathcal{M} \xrightarrow{\mathrm{d}} \cdots \to \Omega^{\mathrm{d_X}}_\mathrm{X} \otimes_\mathcal{O} \mathcal{M} \to 0, \tag{1.19}$$

where $\Omega^0_X \otimes_\mathcal{O} \mathcal{M}$ is in degree 0 and the differential d is characterized by:

$$d(\omega \otimes m) = d\omega \otimes m + (-)^p \omega \wedge dm, \quad \omega \in \Omega^p_X, m \in \mathcal{M}.$$

Note that $\mathrm{DR_X}(\mathcal{D}_\mathrm{X}) \in \mathrm{C^b}(\mathrm{Mod}(\mathcal{D}^{\mathrm{op}}_\mathrm{X}))$, the category of bounded complexes of right $\mathcal{D}_X$-modules, and

$$\mathrm{DR_X}(\mathcal{M}) \simeq \mathrm{DR_X}(\mathcal{D}_\mathrm{X}) \otimes_\mathcal{D} \mathcal{M}. \tag{1.20}$$

Recall that there is a natural right $\mathcal{D}$-linear morphism $\Omega_X \otimes_\mathcal{O} \mathcal{D}_X \to \Omega_X$. Moreover, one checks easily that the composition

$$\Omega^{d_X - 1}_X \otimes_\mathcal{O} \mathcal{D}_X \to \Omega^{d_X}_X \otimes_\mathcal{O} \mathcal{D}_X \to \Omega_X$$

is zero. Hence, we get a morphism in the derived category $\mathrm{D}^b(\mathcal{D}^{\mathrm{op}}_X)$

$$\mathrm{DR_X}(\mathcal{D}_\mathrm{X}) \to \Omega_\mathrm{X}[-\mathrm{d_X}]. \tag{1.21}$$

Proposition 1.4.1. *The morphism* (1.21) *induces an isomorphism in* $\mathrm{D}^b(\mathcal{D}^{\mathrm{op}}_X)$.

Proof. Since the morphism is well defined on X, we may argue locally and choose a local coordinate system. In this case, there is an isomorphism of complexes

$$\mathrm{DR_X}(\mathcal{D}_\mathrm{X}) \simeq \mathrm{K}^\bullet(\mathcal{D}_\mathrm{X}; \partial_1\cdot, \ldots, \partial_{\mathrm{d_X}}\cdot) \tag{1.22}$$

where the right hand side is the Koszul complex of the the sequence $\partial_1\cdot, \ldots, \partial_n\cdot$ acting on the left on $\mathcal{D}_X$. Since this sequence is clearly regular, the result follows. q.e.d.

Applying Proposition 1.4.1 and isomorphism (1.20), we get:

Corollary 1.4.2. *Let $\mathcal{M}$ be a left $\mathcal{D}_X$-module. Then*

$$\mathrm{DR_X}(\mathcal{M}) \simeq \Omega_\mathrm{X} \overset{\mathrm{L}}{\otimes}_{\mathcal{D}} \mathcal{M}\,[-\mathrm{d_X}].$$

Let us apply the contravariant functor $\mathcal{H}om_{\mathcal{D}^{\mathrm{op}}}(\,\bullet\,, \mathcal{D}_X)$ to the complex $\mathrm{DR_X}(\mathcal{D}_\mathrm{X})$. One sets

$$\mathrm{SP_X}(\mathcal{D}_\mathrm{X}) \ := \ \mathcal{H}om_{\mathcal{D}}(\mathrm{DR_X}(\mathcal{D}_\mathrm{X}), \mathcal{D}_\mathrm{X}), \tag{1.23}$$

and calls $\mathrm{SP_X}(\mathcal{D}_\mathrm{X})$ the Spencer complex.

$$\mathrm{SP_X}(\mathcal{D}_\mathrm{X}) := \quad 0 \to \mathcal{D}_\mathrm{X} \otimes_{\mathcal{O}} \bigwedge^{\mathrm{d_x}} \Theta_\mathrm{X} \xrightarrow{\mathrm{d}} \cdots \to \mathcal{D}_\mathrm{X} \otimes_{\mathcal{O}} \Theta_\mathrm{X} \to \mathcal{D}_\mathrm{X} \to 0, \tag{1.24}$$

One deduces from (1.22) the isomorphism of complexes

$$\mathrm{SP_X}(\mathcal{D}_\mathrm{X}) \simeq \mathrm{K}_\bullet(\mathcal{D}_\mathrm{X}; \cdot\partial_1, \ldots, \cdot\partial_{\mathrm{d_X}}) \tag{1.25}$$

where the right hand side is the co-Koszul complex of the sequence $\cdot\partial_1, \ldots, \cdot\partial_{d_X}$ acting on the right on $\mathcal{D}_X$. Since this sequence is clearly regular, we obtain:

Proposition 1.4.3. *The left $\mathcal{D}$-linear morphism $\mathcal{D}_X \to \mathcal{O}_X$ induces an isomorphism $\mathrm{SP_X}(\mathcal{D}_\mathrm{X}) \xrightarrow{\sim} \mathcal{O}_\mathrm{X}$ in $\mathrm{D^b}(\mathcal{D}_X)$.*

Corollary 1.4.4. *Let $\mathcal{M}$ be a left $\mathcal{D}_X$-module. There is an isomorphism in $\mathrm{D}^b(\mathbb{C}_X)$*

$$R\mathcal{H}om_{\mathcal{D}}(\mathcal{O}_X, \mathcal{M}) \simeq \mathrm{DR_X}(\mathcal{M}).$$

Proof. Since $\mathrm{SP_X}(\mathcal{D}_\mathrm{X})$ is a complex of locally free $\mathcal{D}_X$-modules of finite rank, one has

$$\begin{aligned} R\mathcal{H}om_{\mathcal{D}}(\mathcal{O}_X, \mathcal{M}) &\simeq \mathcal{H}om_{\mathcal{D}}(\mathrm{SP_X}(\mathcal{D}_\mathrm{X}), \mathcal{M}) \\ &\simeq \mathcal{H}om_{\mathcal{D}}(\mathrm{SP_X}(\mathcal{D}_\mathrm{X}), \mathcal{D}_\mathrm{X}) \otimes_{\mathcal{D}} \mathcal{M} \\ &\simeq \mathrm{DR_X}(\mathcal{D}_\mathrm{X}) \otimes_{\mathcal{D}} \mathcal{M} \\ &\simeq \mathrm{DR_X}(\mathcal{M}). \end{aligned}$$

q.e.d.

Proposition 1.4.5. *One has the isomorphism*

$$R\mathcal{H}om_{\mathcal{D}}(\mathcal{O}_X, \mathcal{D}_X)[d_X] \simeq \Omega_X$$
$$R\mathcal{H}om_{\mathcal{D}^{\mathrm{op}}}(\Omega_X, \mathcal{D}_X)[d_X] \simeq \mathcal{O}_X$$
$$R\mathcal{H}om_{\mathcal{D}}(\mathcal{O}_X, \mathcal{O}_X) \simeq \mathbb{C}_X.$$

Proof. (i) One has the chain of isomorphisms

$$\begin{aligned} R\mathcal{H}om_{\mathcal{D}}(\mathcal{O}_X, \mathcal{D}_X)[d_X] &\simeq R\mathcal{H}om_{\mathcal{D}}(\mathrm{SP_X}(\mathcal{D}_\mathrm{X}), \mathcal{D}_\mathrm{X})[-\mathrm{d_X}] \\ &\simeq \mathcal{H}om_{\mathcal{D}}(\mathrm{SP_X}(\mathcal{D}_\mathrm{X}), \mathcal{D}_\mathrm{X})[-\mathrm{d_X}] \\ &\simeq \mathrm{DR}(\mathcal{D}_\mathrm{X})[-\mathrm{d_X}] \simeq \Omega_\mathrm{X}. \end{aligned}$$

(ii) The proof is similar.
(iii) The canonical morphism $\mathbb{C}_X \to \mathcal{H}om_{\mathcal{D}}(\mathcal{O}_X, \mathcal{O}_X)$ induces the morphism

$$\begin{aligned} \mathbb{C}_X &\to R\mathcal{H}om_{\mathcal{D}}(\mathcal{O}_X, \mathcal{O}_X) \\ &\simeq \mathcal{H}om_{\mathcal{D}}(\mathrm{SP_X}(\mathcal{D}_\mathrm{X}), \mathcal{O}_\mathrm{X}) \\ &\simeq \Omega^\bullet_X. \end{aligned}$$

The isomorphism $\mathbb{C}_X \xrightarrow{\sim} \Omega^\bullet_X$ is the classical Poincaré lemma. q.e.d.

1.5 $\mathcal{D}$-module associated with a submanifold

Let Z be a hypersurface of X. One denotes by $\mathcal{O}_X(*Z)$ the sheaf of meromorphic functions on X with poles in Z. Hence, if $\{f = 0\}$ is a local equation of Z, a section u of $\mathcal{O}_X(*Z)$ is locally written as a quotient $u = g/f^m$, for some $m \in \mathbb{N}$ and g a section of $\mathcal{O}_X$. Clearly, $\mathcal{O}_X(*Z)$ is a left $\mathcal{D}_X$-module.

One also introduces the left $\mathcal{D}_X$-module $\mathcal{B}_{Z|X}$ by the exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(*Z) \to \mathcal{B}_{Z|X} \to 0.$$

If $\{f = 0\}$ is a local equation of Z, then

$$\mathcal{O}_X(*Z) \simeq (\mathcal{O}_X[1/f])/\mathcal{O}_X.$$

Also note that

$$\mathcal{B}_{Z|X} \simeq H^1_{[Z]}(\mathcal{O}_X),$$

where $H^p_{[Z]}(\mathcal{O}_X)$ denotes the p-th algebraic cohomology with support in Z of $\mathcal{O}_X$.

Let $Z = \{f_j = 0; j = 1, \dots, d\}$ be a complete intersection. One sets

$$\mathcal{B}_{Z|X} \simeq \mathcal{O}_X[1/f_1 \dots f_d]/\sum_i \mathcal{O}_X[1/f_1 \dots \widehat{f_i} \dots f_d]. \tag{1.26}$$

One checks that this does not depend on the choice of the $f_j's$.

Proposition 1.5.1. *Let Z be a closed smooth submanifold of X. Then $\mathcal{B}_{Z|X}$ is a coherent $\mathcal{D}_X$-module and its characteristic variety is $T^*_Z X$, the conormal bundle to Z in X.*

This result extends to the non smooth case, but the proof is then much more difficult, using the whole theory of holonomic systems.

Proof. The problem is local and we may assume to be given a local coordinate system $x = (x', x'')$ on X, with $x' = (x_1, \ldots, x_d)$ and $Z = \{x' = 0\}$. Then

$$\mathcal{B}_{Z|X} \simeq \mathcal{D}_X/\mathcal{J} \qquad \text{where } \mathcal{J} \text{ is the left ideal generated by } (x', \partial_{x''}).$$

It follows that $\mathrm{char}(\mathcal{B}_{Z|X}) \subset T^*_Z X$. The converse inclusion follows from the fact that $\mathrm{char}(\mathcal{B}_{Z|X})$ is non empty and involutive. q.e.d.

Proposition 1.5.2. *Let Z be a complete intersection of codimension d and assume $\mathcal{I}_Z = \mathcal{O}_X f_1 + \cdots + \mathcal{O}_X f_d$. Then the section*

$$\delta(f_1) \otimes \cdots \otimes \delta(f_d) \otimes df_1 \wedge \cdots \wedge df_d \in \mathcal{B}_{Z|X} \otimes_{\mathcal{O}} \Omega^d_X$$

does not depend on the choice of the sequence $(f_1, \ldots, f_d)$.

Proof. Let $(f'_1, \ldots, f'_d)$ be another sequence defining the ideal $\mathcal{I}_Z$. There exists a section $A \in Gl(\mathcal{O}_X, d)$ which interchanges these two sequences. The group $Gl(\mathcal{O}_X, d)$ is generated by the transformations

(i) $(f_1, \ldots, f_d) \mapsto (af_1, \ldots, f_d)$, with $a \in \mathcal{O}_X^\times$,

(ii) $(f_1, \ldots, f_i, f_{i+1}, \ldots f_d) \mapsto (f_1, \ldots, f_{i+1}, f_i, \ldots, f_d)$

(iii) $(f_1, \ldots, f_d) \mapsto (f_1, f_2 + bf_1, \ldots, f_d)$

Then, it is enough to notice that

$$\begin{aligned}
&1/af_1 \cdot 1/f_2 d(af_1) \wedge df_2 = 1/f_1 \cdot 1/f_2 df_1 \wedge df_2,\\
&1/f_2 \cdot 1/f_1 df_2 \wedge df_1 = 1/f_1 \cdot 1/f_2 df_1 \wedge df_2\\
&1/f_1 \cdot 1/(f_2 + bf_1) df_1 \wedge d(f_2 + bf_1) = 1/f_1 \cdot 1/f_2 df_1 \wedge df_2.
\end{aligned}$$

q.e.d.

Definition 1.5.3. Assume that Z is smooth of codimension d. We shall denote by δ_Z the canonical section of $\mathcal{B}_{Z|X} \otimes_{\mathcal{O}} \Omega^d_X$ constructed in Proposition 1.5.2. One calls it the fundamental class of Z in X.

Note that δ_Z belongs to $\bigwedge^d \mathcal{L}_Z$ where $\mathcal{L}_Z$ denotes the subsheaf of Ω^1_X consisting of sections with values in the conormal bundle $T^*_Z X$.

Denote by Δ the diagonal in $X \times X$ and by q_1 and q_2 the first and second projections $X \times X \to X$. The projection q_2 allows us to identify $T^*_\Delta(X \times X)$ with T^*X.

Proposition 1.5.4. *There is a natural $\mathcal{D}_X \otimes \mathcal{D}_X^{\mathrm{op}}$-linear isomorphism*

$$\mathcal{D}_X \xrightarrow{\sim} \mathcal{B}_{\Delta|X\times X} \otimes_{q_2^{-1}\mathcal{O}} q_2^{-1}\Omega_X, \tag{1.27}$$

which associates δ_Δ to $1 \in \mathcal{D}_X$.

Proof. Consider the left ideal $\mathcal{J}$ of $\mathcal{D}_X \otimes \mathcal{D}_X^{\mathrm{op}}$ generated by the sections $P \otimes 1 - 1 \otimes P$ ($P \in \mathcal{D}_X$). Then $\mathcal{D}_X$ is isomorphic as a $\mathcal{D}_X \otimes \mathcal{D}_X^{\mathrm{op}}$-module to $(\mathcal{D}_X \otimes \mathcal{D}_X^{\mathrm{op}})/\mathcal{J}$.

Let us show that the section δ_Δ of $\mathcal{B}_{\Delta|X\times X} \otimes_{q_2^{-1}\mathcal{O}} q_2^{-1}\Omega_X$ satifies $\mathcal{J} \cdot \delta_\Delta = 0$, which will prove that the morphism (1.27) is well defined and let us show at the same time that this morphism is an isomorphism.

Both questions are local and we may choose a local coordinate system (x) on X. Denote by (y) a copy of this system and by dx the section $dx_1 \wedge \cdots \wedge dx_n$. Identifying $\mathcal{D}_X^{\mathrm{op}}$ to $\mathcal{D}_X$ by this section, the ideal $\mathcal{J}$ is identified to the left ideal of $\mathcal{D}_X \otimes \mathcal{D}_X$ generated by the sections $P \otimes 1 - 1 \otimes P^*$ where P^* denotes the adjoint of P with respect to dx. On the other hand, $\mathcal{B}_{\Delta|X\times X} \otimes_{q_2^{-1}\mathcal{O}} q_2^{-1}\Omega_X$ is identified to $\mathcal{B}_{\Delta|X\times X}$ which is isomorphic to $(\mathcal{D}_X \otimes \mathcal{D}_X)/\mathcal{J}'$ where $\mathcal{J}'$ is the left ideal generated by $(x - y, \partial_x + \partial_y)$. It remains to notice that $\mathcal{J} = \mathcal{J}'$ by Remark (1.1.7). q.e.d.

1.6 Homological properties of $\mathcal{D}_X$

Vanishing theorems and dimension

Recall the classical theorem for $\mathcal{O}$-modules:

Theorem 1.6.1. *Let X be a smooth manifold and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then*

(i) *$\mathcal{E}xt^k_{\mathcal{O}}(\mathcal{F}, \mathcal{O}_X)$ is coherent for all k and is 0 for $k < \operatorname{codim} \operatorname{supp}(\mathcal{F})$,*

(ii) $\operatorname{codim}(\operatorname{supp}(\mathcal{E}xt^k_{\mathcal{O}}(\mathcal{F}, \mathcal{O}_X))) \geq k$.

There is a corresponding theorem for $\mathcal{D}$-modules.

Theorem 1.6.2. *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. Then*

(i) $\mathcal{E}xt^k_{\mathcal{D}}(\mathcal{M}, \mathcal{D}_X)$ *is coherent for all* k *and is* 0 *for* $k < \operatorname{codim}(\operatorname{char}(\mathcal{M}))$,

(ii) $\operatorname{codim}(\operatorname{char}(\mathcal{E}xt^k_{\mathcal{D}}(\mathcal{M}, \mathcal{D}_X))) \geq k$,

(iii) $\operatorname{char}(\mathcal{E}xt^k_{\mathcal{D}}(\mathcal{M}, \mathcal{D}_X)) \subset \operatorname{char}(\mathcal{M})$,

(iv) $\mathcal{E}xt^k_{\mathcal{D}}(\mathcal{M}, \mathcal{D}_X) = 0$ *for* $k > d_X$.

Corollary 1.6.3. *Let* $\mathcal{M}$ *be a coherent* $\mathcal{D}_X$*-module. Then the support of* $\mathcal{E}xt^{d_X}_{\mathcal{D}}(\mathcal{M}, \mathcal{D}_X)$ *has pure dimension* d_X*.*

Proof. First we construct by induction a finite free filtered resolution of $\mathrm{Fl}\,\mathcal{M}$, that is, a filtered exact sequence of $\mathrm{Fl}\,\mathcal{D}_X$-modules

$$\cdots \to \mathrm{Fl}\,\mathcal{L}_1 \to \mathrm{Fl}\,\mathcal{L}_0 \to \mathrm{Fl}\,\mathcal{M} \to 0$$

where the $\mathrm{Fl}\,\mathcal{L}_j$'s are filtered finite free. We denote by d^j the differential. Set:

$$\begin{aligned} \mathrm{Fl}\,\mathcal{L}_\bullet &:= \cdots \to \mathrm{Fl}\,\mathcal{L}_1 \to \mathrm{Fl}\,\mathcal{L}_0 \to 0, \\ \operatorname{gr}\mathcal{L}_\bullet &:= \cdots \to \operatorname{gr}\mathcal{L}_1 \to \operatorname{gr}\mathcal{L}_0 \to 0. \end{aligned}$$

Then

$$\cdots \to \operatorname{gr}\mathcal{L}_1 \to \operatorname{gr}\mathcal{L}_0 \to \operatorname{gr}\mathcal{M} \to 0$$

is exact. Put

$$\begin{aligned} \mathcal{L}^*_j &= \mathcal{H}om_{\mathcal{D}}(\mathcal{L}_j, \mathcal{D}_X), \\ \mathcal{L}^*_\bullet &= \mathcal{H}om_{\mathcal{D}}(\mathcal{L}_\bullet, \mathcal{D}_X) = 0 \to \mathcal{L}^*_0 \to \mathcal{L}^*_1 \to \cdots \end{aligned}$$

One defines a filtration $\mathrm{Fl}\,\mathcal{L}^*_j$ on $\mathcal{L}^*_j$ by setting

$$\mathrm{Fl}_m\mathcal{L}^*_j = \{\varphi \in \mathcal{H}om_{\mathcal{D}}(\mathcal{L}_j, \mathcal{D}_X); \varphi(\mathrm{Fl}_k\mathcal{L}_j) \subset \mathrm{Fl}_{k+m}\mathcal{D}_X \text{ for all } k\}.$$

Clearly, this filtration on $\mathcal{L}^*_j$ is good and moreover $\mathcal{H}om_{\operatorname{gr}\mathcal{D}}(\operatorname{gr}\mathcal{L}_j, \operatorname{gr}\mathcal{D}) \simeq \operatorname{gr}\mathcal{L}^*_j$. In other words,

$$\mathcal{H}om_{\operatorname{gr}\mathcal{D}}(\operatorname{gr}\mathcal{L}_\bullet, \operatorname{gr}\mathcal{D}) \simeq \operatorname{gr}\mathcal{L}^*_\bullet.$$

Put

$$\mathcal{Z}^k = \operatorname{Ker}(\mathcal{L}_k \xrightarrow{d^k} \mathcal{L}_{k+1}), \quad \mathcal{I}^k = \operatorname{Im}(\mathcal{L}_{k-1} \to \mathcal{L}_k) \quad H^k(\mathcal{L}^*_\bullet) = \mathcal{Z}^k/\mathcal{I}^k.$$

We endow $\mathcal{Z}^k$ with the induced filtration and $H^k(\mathcal{L}^*_\bullet)$ with the filtration image of $\mathrm{Fl}\,\mathcal{Z}^k$. Since $\mathcal{E}xt^k_{\mathcal{D}}(\mathcal{M}, \mathcal{D}_X) \simeq H^k(\mathcal{L}^*_\bullet)$, we get a filtration $\mathrm{Fl}\,\mathcal{E}xt^k_{\mathcal{D}}(\mathcal{M}, \mathcal{D}_X)$ on this module. Moreover $\mathcal{E}xt^k_{\operatorname{gr}\mathcal{D}}(\operatorname{gr}\mathcal{M}, \operatorname{gr}\mathcal{D}_X)) \simeq H^k(\operatorname{gr}\mathcal{L}^*_\bullet)$.

In order to complete the proof, we need a lemma. q.e.d.

Lemma 1.6.4. $\mathrm{gr}\, H^k(\mathcal{L}^*_\bullet)$ *is a subquotient of* $H^k(\mathrm{gr}\,\mathcal{L}^*_\bullet)$.

Proof of Lemma 1.6.4.

$$\begin{aligned} H^k(\mathrm{gr}_m \mathcal{L}^*_\bullet) &= \frac{\mathrm{Fl}_m(\mathcal{L}^*_k) \cap (d^k)^{-1} \mathrm{Fl}_{m-1}\mathcal{L}^*_{k+1}}{\mathrm{Fl}_{m-1}(\mathcal{L}^*_k) + d^{k-1}\mathrm{Fl}_m \mathcal{L}^*_{k-1}} \\ &\supset \frac{\mathrm{Fl}_m(\mathcal{Z}^k)}{\mathrm{Fl}_{m-1}(\mathcal{Z}_k) + d^{k-1}\mathrm{Fl}_m \mathcal{L}^*_{k-1}}. \end{aligned}$$

On the other-hand,

$$\mathrm{gr}_m H^k(\mathcal{L}^*_\bullet) = \frac{\mathrm{Fl}_m(\mathcal{Z}^k)}{\mathrm{Fl}_{m-1}(\mathcal{Z}_k) + \mathcal{I}^k \cap \mathrm{Fl}_m(\mathcal{Z}_k)}.$$

The result then follows from

$$\mathrm{Fl}_{m-1}(\mathcal{Z}_k) + d^{k-1}\mathrm{Fl}_m \mathcal{L}^*_{k-1} \subset \mathrm{Fl}_{m-1}(\mathcal{Z}_k) + \mathcal{I}^k \cap \mathrm{Fl}_m(\mathcal{Z}_k).$$

q.e.d.

End of proof of Theorem 1.6.2. It follows that

$$\mathrm{char}(\mathcal{E}xt^k_{\mathcal{D}}(\mathcal{M}, \mathcal{D}_X)) \subset \mathrm{supp}(\mathcal{E}xt^k_{\mathrm{gr}\,\mathcal{D}}(\mathrm{gr}\,\mathcal{M}, \mathrm{gr}\,\mathcal{D}_X))). \tag{1.28}$$

(i) By Theorem 1.6.1, $\mathcal{E}xt^k_{\mathcal{O}}(\widetilde{\mathrm{gr}\,\mathcal{M}}, \mathcal{O}_{T^*X})) = 0$ for $k < \mathrm{codim}(\mathrm{char}(\mathcal{M}))$. By (1.28), we get that $\mathcal{E}xt^k_{\mathcal{D}}(\mathcal{M}, \mathcal{D}_X) = 0$ for $k < \mathrm{codim}(\mathrm{char}(\mathcal{M}))$.

(ii) By Theorem 1.6.1, $\mathrm{codim}(\mathrm{supp}(\mathcal{E}xt^k_{\mathrm{gr}\,\mathcal{D}}(\mathrm{gr}\,\mathcal{M}, \mathrm{gr}\,\mathcal{D}_X))) \geq k$. By (1.28), we get that $\mathrm{codim}(\mathrm{char}(\mathcal{E}xt^k_{\mathcal{D}}(\mathcal{M}, \mathcal{D}_X))) \geq k$.

(iii) follows from the inclusion

$$\mathrm{supp}(\mathcal{E}xt^k_{\mathrm{gr}\,\mathcal{D}}(\mathrm{gr}\,\mathcal{M}, \mathrm{gr}\,\mathcal{D}_X)) \subset \mathrm{supp}(\mathrm{gr}\,\mathcal{M}).$$

(iv) follows from (ii) and the involutivity of the characteristic variety of $\mathcal{E}xt^k_{\mathcal{D}}(\mathcal{M}, \mathcal{D}_X)$. q.e.d.

Example 1.6.5. Let $d_X = 1$. Then any coherent ideal $\mathcal{I}$ of $\mathcal{D}_X$ is projective since $\mathcal{E}xt^j_{\mathcal{D}}(\mathcal{D}_X/\mathcal{I}, \mathcal{D}_X) = 0$ for $j > 1$.

Let t denote a local holomorphic coordinate. The left ideal of $\mathcal{D}_X$ generated by t^2 and $t\partial_t - 1$ is projective. By Theorem 1.6.2, its characteristic is T^*X. Since it is contained in $\mathcal{D}_X$, its multiplicity on T^*X is 1. This module does not admits a single generator, and it follows that it is not free.

Free resolutions

Theorem 1.6.6. *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. Then, locally on X, $\mathcal{M}$ admits a finite free resolution of length $\leq d_X$. In other words, there locally exists an exact sequence*

$$0 \to \mathcal{L}^{d_X} \to \cdots \to \mathcal{L}^0 \to \mathcal{M} \to 0,$$

where the $\mathcal{L}^i$'s are free of finite rank over $\mathcal{D}_X$ and $n \leq d_X$.

Proof. Set $n = d_X$. Since we argue locally, we may endow $\mathcal{M}$ with a good filtration $\mathrm{Fl}\,\mathcal{M}$. We may locally find a finite free filtered resolution

$$\cdots \to \mathrm{Fl}\,\mathcal{L}^n \to \cdots \to \mathrm{Fl}\,\mathcal{L}^0 \to \mathrm{Fl}\,\mathcal{M} \to 0.$$

On the other-hand, we know that $\mathcal{E}xt^j_{\mathrm{gr}\,\mathcal{D}}(\mathrm{gr}\,\mathcal{M}, \mathrm{gr}\,\mathcal{D}_X) = 0$ for $j > n$. Set $\mathcal{K}^n = \mathrm{Ker}(\mathcal{L}^{n-1} \to \mathcal{L}^{n-2})$ and let us endow $\mathcal{K}_n$ with the induced filtration. Then the sequence

$$0 \to \mathrm{gr}\,\mathcal{K}^n \to \mathrm{gr}\,\mathcal{L}^{n-1} \to \cdots \to \mathrm{gr}\,\mathcal{L}^0 \to \mathrm{gr}\,\mathcal{M} \to 0$$

is exact and it follows that $\mathrm{gr}\,\mathcal{K}^n$ is projective. Since projective modules over $\mathrm{gr}\,\mathcal{D}_X$ are stably free, there exists a finite free $\mathcal{D}_X$ module $\mathcal{L}$ such that $\mathrm{gr}\,\mathcal{K}^n \oplus \mathrm{gr}\,\mathcal{L}$ is free and this implies that $\mathcal{K}^n \oplus \mathcal{L}$ is a free $\mathcal{D}_X$-module. The sequence

$$0 \to \mathcal{K}^n \oplus \mathcal{L} \to \mathcal{L}^{n-1} \oplus \mathcal{L} \to \cdots \to \mathcal{L}^0 \to \mathcal{M} \to 0$$

is a finite free resolution of $\mathcal{M}$. q.e.d.

Exercises to Chapter 1

Exercise 1.1. Let A be a filtered ring and assume that $\mathrm{gr}\,A$ is commutative and has no zero divisors. Let $a \neq 0, b \neq 0$ in A.
(i) Prove that the sequence below in $\mathrm{Mod}(A)$ of is exact.

$$0 \to A/(A \cdot a \cap A \cdot b) \to A/A \cdot a \oplus A/A \cdot b \to A/(A \cdot a + A \cdot b) \to 0$$

(ii) Deduce that $A \cdot a \cap A \cdot b \neq \{0\}$.

Exercise 1.2. Let Z_1 and Z_2 be two smooth submanifolds of X and assume they are transversal. Calculate $R\mathcal{H}om_{\mathcal{D}}(\mathcal{B}_{Z_1|X}, \mathcal{B}_{Z_2|X})$.

Exercise 1.3. Calculate $R\mathcal{H}om_{\mathcal{D}}(\mathcal{B}_{Z|X}, \mathcal{D}_X)$ for a smooth submanifold Z of X.

Exercise 1.4. (i) Prove that the $\mathcal{O}_X$-module $\mathcal{D}_X$ is flat.
(ii) Prove that if a $\mathcal{D}_X$-module $\mathcal{I}$ is injective in the category $\mathrm{Mod}(\mathcal{D}_X)$, then it is injective in the category $\mathrm{Mod}(\mathcal{O}_X)$.

Exercise 1.5. Let $\mathcal{M}, \mathcal{N} \in \mathrm{Mod}(\mathcal{D}_X)$. Prove that

$$R\mathcal{H}om_{\mathcal{D}}(\mathcal{M}, \mathcal{N}) \simeq R\mathcal{H}om_{\mathcal{D}}(\mathcal{O}_X, R\mathcal{H}om_{\mathcal{O}}(\mathcal{M}, \mathcal{N})).$$

Exercise 1.6. Prove that $\mathrm{Ext}^{d_X}_{\mathcal{D}_{X,x}}(\mathcal{O}_{X,x}, \mathcal{D}_{X,x}) \neq 0$.

Exercise 1.7. Recall that the weak global dimension $\mathrm{wgld}(R)$ of a ring R is the smallest integer d such that $\mathrm{Tor}^R_j(N, M) = 0$ for $j > d$ and any left R-module M and right module N.

Prove that $\mathrm{wgld}(\mathcal{D}_{X,x}) = d_X$. (Hint: use [38, Ch. 4].)

Exercise 1.8. Recall that the global dimension $\mathrm{gld}(\mathcal{R})$ of a sheaf of rings $\mathcal{R}$ is the smallest integer d such that $\mathrm{Ext}^j_{\mathcal{D}}(\mathcal{M}, \mathcal{N}) = 0$ for $j > d$ and any left $\mathcal{R}$-modules $\mathcal{M}$ and $\mathcal{N}$.

Prove that $\mathrm{gld}(\mathcal{D}_X) = 2d_X + 1$. (Hint: see [22].)

Chapter 2

Operations on $\mathcal{D}$-modules

2.1 Operations on $\mathcal{O}$-modules

For a complex manifold X, one denotes by $\mathrm{Mod}_{\mathrm{coh}}(\mathcal{O}_X)$ the thick abelian subcategory of $\mathrm{Mod}(\mathcal{O}_X)$ consisting of coherent modules. One denotes by $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{O}_X)$ the full triangulated category of the bounded derived category $\mathrm{D}^{\mathrm{b}}(\mathcal{O}_X)$ consisting of objects with coherent cohomology.

We shall also encounter the duality functors for $\mathcal{O}$-modules:

$$\mathbb{D}'_{\mathscr{O}}\mathcal{F} := R\mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{O}_X),$$
$$\mathbb{D}_{\mathscr{O}}\mathcal{F} := R\mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \Omega_X\,[d_X]).$$

Recall that d_X is the complex dimension of X and $\Omega_X = \Omega_X^{d_X}$.

Let X and Y be two manifolds. For an $\mathcal{O}_X$-module $\mathcal{F}$ and an $\mathcal{O}_Y$-module $\mathcal{G}$, we define their external product, denoted $\mathcal{F}\underline{\boxtimes}\mathcal{G}$, by

$$\mathcal{F}\underline{\boxtimes}\mathcal{G} = \mathcal{O}_{X\times Y} \otimes_{\mathcal{O}_X\boxtimes\mathcal{O}_Y} (\mathcal{F}\boxtimes\mathcal{G}).$$

Note that the functor $\mathcal{F} \mapsto \mathcal{F}\underline{\boxtimes}\mathcal{G}$ is exact. Clearly, if $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{O}_X)$ and $\mathcal{G} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{O}_Y)$, then $\mathcal{F}\underline{\boxtimes}\mathcal{G} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{O}_{X\times Y})$.

Let $f\colon X \to Y$ be a morphism of complex manifolds. There is a natural morphism of rings $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$. Using this morphism, the direct images $f_*\mathcal{F}$ and $f_*\mathcal{F}$ of an $\mathcal{O}_X$-module are well defined as $\mathcal{O}_Y$-modules. One denotes as usual by Rf_* and $Rf_!$ their derived functors. The inverse image of an $\mathcal{O}_Y$-module $\mathcal{G}$ is defined by $f^* := \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{G}$. Its right derived functor is denoted Lf^*. The following result is left as an exercise.

Proposition 2.1.1. *Let $\mathcal{G} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{O}_Y)$. Then $Lf^*\mathcal{G} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{O}_X)$ and there is a natural isomorphism*

$$Lf^*\mathbb{D}'_{\mathscr{O}}\mathcal{G} \simeq \mathbb{D}'_{\mathscr{O}}Lf^*\mathcal{G}.$$

There is a similar result for direct images:

Theorem 2.1.2. Grauert's theorem. *Let* $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{O}_X)$ *and assume that* f *is proper on* $\mathrm{supp}(\mathcal{F})$. *Then* $Rf_!\mathcal{F} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{O}_Y)$ *and there is a natural isomorphism*

$$Rf_!\mathbb{D}_{\mathscr{O}}\mathcal{F} \simeq \mathbb{D}_{\mathscr{O}}Rf_!\mathcal{F}.$$

Note that Grauert's theorem is a relative version of the Cartan-Serre's finiteness theorem and the Serre's duality theorem.

2.2 Real and complex microlocal geometry

In these Notes, unless otherwise specified, a real manifold means a real analytic manifold.

For a complex manifold X we denote by $X_{\mathbb{R}}$ the real underlying submanifold to X. When there is no risk of confusion, we simply write X instead of $X_{\mathbb{R}}$.

We denote by $\overline{X}$ the complex conjugate manifold to X. (Recall that $\overline{X} = X$ as a topological space, but the sheaf of holomorphic functions on $\overline{X}$ is the sheaf of anti-holomorphic functions on X.) Then, identifying X with the diagonal of $X \times \overline{X}$, the complex manifold $X \times \overline{X}$ is a complexification of $X_{\mathbb{R}}$.

Denote by $d\alpha_X$ the symplectic form on T^*X and by $d\alpha_{X_{\mathbb{R}}}$ the symplectic form on $T^*X_{\mathbb{R}}$. Then

$$d\alpha_{X_{\mathbb{R}}} = 2\Re\, d\alpha_X.$$

Let $f\colon X \to Y$ be a morphism of real manifolds. To f are associated the tangent morphisms

$$\begin{tikzcd} TX \arrow[r, "f'"] \arrow[d] & X \times_Y TY \arrow[r, "f_\tau"] \arrow[d] & TY \arrow[d] \\ X \arrow[r, equal] & X \arrow[r, "f"] & Y. \end{tikzcd} \tag{2.1}$$

By duality, we deduce the diagram:

$$\begin{tikzcd} T^*X & X \times_Y T^*Y \arrow[l, "f_d"'] \arrow[r, "f_\pi"] \arrow[d, "\pi"] & T^*Y \arrow[d, "\pi"] \\ X \arrow[u, leftarrow, "\pi"'] \arrow[r, equal] & X \arrow[r, "f"] & Y. \end{tikzcd} \tag{2.2}$$

Definition 2.2.1. Consider a morphism $f: X \to Y$ of real manifolds.

(i) The conormal bundle to X in Y is the sub-vector bundle of $X \times_Y T^*Y$ given by $f_d^{-1}(T^*_X X)$.

(ii) Let $\Lambda \subset T^*Y$ be a closed $\mathbb{R}^+$-conic subset. One says that f is non-characteristic for Λ (or else, Λ is non-characteristic for f, or f and Λ are transversal) if,

$$f_\pi^{-1}(\Lambda) \cap T^*_X Y \subset X \times_Y T^*_Y Y.$$

of course, the same notions hold for complex manifolds.

Lemma 2.2.2. *Let Λ be a closed $\mathbb{R}^+$-conic subset of T^*Y. Then a morphism $f: X \to Y$ is non characteristic for Λ if and only if $f_d: X \times_Y T^*Y \to T^*X$ is proper on $f_\pi^{-1}(\Lambda)$. In this case:*

(i) *$f_d f_\pi^{-1}(\Lambda)$ is closed and $\mathbb{R}^+$-conic in T^*X.*

(ii) *If moreover f is a morphism of complex manifolds and Λ is a complex analytic $\mathbb{C}^\times$-conic subset, then f_d is finite on $f_\pi^{-1}(\Lambda)$ and $f_d f_\pi^{-1}(\Lambda)$ is a complex analytic $\mathbb{C}^\times$-conic subset of T^*X.*

Example 2.2.3. Let Z be a closed and smooth submanifold of Y. Then f is non-characteristic for $T^*_Z Y$ if and only if f is transversal to Z.

The next lemma will allows us to decompse a morphism into a smooth morphism and a closed embedding when studying operations on $\mathcal{D}$-modules.

Lemma 2.2.4. *Consider morphisms of real manifolds $X \xrightarrow{f} Y \xrightarrow{g} Z$ and set $h = g \circ f$. Let Λ be a closed $\mathbb{R}^+$-conic subset of T^*Z.*

(i) *Assume that g is non characteristic for Λ and f is non characteristic for $g_d g_\pi^{-1}(\Lambda)$. Then h is non characteristic for Λ.*

(ii) *Assume that h is non characteristic for Λ. Then g is non characteristic for Λ on a neighborhood of $f(X)$ and f is non characteristic for $g_d g_\pi^{-1}\Lambda$.*

2.3 Internal operations, duality and external product

For a complex manifold X, one denotes by $\mathrm{Mod}_{\mathrm{coh}}(\mathcal{D}_X)$ the thick abelian subcategory of $\mathrm{Mod}(\mathcal{D}_X)$ consisting of coherent modules. One denotes by

$\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X)$ the full triangulated category of the bounded derived category $\mathrm{D}^{\mathrm{b}}(\mathcal{D}_X)$ consisting of objects with coherent cohomology.

If $\mathcal{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X)$, we set

$$\operatorname{char}(\mathcal{M}) = \bigcup_j \operatorname{char}(H^j(\mathcal{M})). \tag{2.3}$$

Internal operations

We denote by $R\mathcal{H}om_{\mathcal{O}}$ the right derived functor of $\mathcal{H}om_{\mathcal{O}}$ and by $\overset{\mathrm{D}}{\otimes}$ the left derived functor of $\otimes_{\mathcal{O}}$ acting on $\mathcal{D}$-modules. Hence, we get the functors

$$\begin{aligned}
\bullet\overset{\mathrm{D}}{\otimes}\bullet \quad &: \quad \mathrm{D}^{\mathrm{b}}(\mathcal{D}_X) \times \mathrm{D}^{\mathrm{b}}(\mathcal{D}_X) \to \mathrm{D}^{\mathrm{b}}(\mathcal{D}_X),\\
\bullet\overset{\mathrm{D}}{\otimes}\bullet \quad &: \quad \mathrm{D}^{\mathrm{b}}(\mathcal{D}_X^{\mathrm{op}}) \times \mathrm{D}^{\mathrm{b}}(\mathcal{D}_X) \to \mathrm{D}^{\mathrm{b}}(\mathcal{D}_X^{\mathrm{op}}),\\
R\mathcal{H}om_{\mathcal{O}}(\,\bullet\,,\bullet\,) \quad &: \quad \mathrm{D}^{\mathrm{b}}(\mathcal{D}_X)^{\mathrm{op}} \times \mathrm{D}^{\mathrm{b}}(\mathcal{D}_X) \to \mathrm{D}^{\mathrm{b}}(\mathcal{D}_X),\\
R\mathcal{H}om_{\mathcal{O}}(\,\bullet\,,\bullet\,) \quad &: \quad \mathrm{D}^{\mathrm{b}}(\mathcal{D}_X^{\mathrm{op}})^{\mathrm{op}} \times \mathrm{D}^{\mathrm{b}}(\mathcal{D}_X^{\mathrm{op}}) \to \mathrm{D}^{\mathrm{b}}(\mathcal{D}_X).
\end{aligned}$$

The tensor product is commutative and associative, that is, for $\mathcal{L}, \mathcal{M}, \mathcal{N}$ in $\mathrm{D}^{\mathrm{b}}(\mathcal{D}_X)$ there are natural isomorphisms $\mathcal{M}\overset{\mathrm{D}}{\otimes}\mathcal{N} \simeq \mathcal{N}\overset{\mathrm{D}}{\otimes}\mathcal{M}$ and $(\mathcal{M}\overset{\mathrm{D}}{\otimes}\mathcal{N})\overset{\mathrm{D}}{\otimes}\mathcal{L} \simeq \mathcal{M}\overset{\mathrm{D}}{\otimes}(\mathcal{N}\overset{\mathrm{D}}{\otimes}\mathcal{L})$. Moreover $\mathcal{O}_X\overset{\mathrm{D}}{\otimes}\mathcal{M} \simeq \mathcal{M}$.

There are also natural functors

$$\begin{aligned}
R\mathcal{H}om_{\mathcal{D}}(\,\bullet\,,\bullet\,) \quad &: \quad \mathrm{D}^{\mathrm{b}}(\mathcal{D}_X)^{\mathrm{op}} \times \mathrm{D}^{\mathrm{b}}(\mathcal{D}_X) \to \mathrm{D}^{\mathrm{b}}(\mathbb{C}_X),\\
\bullet\overset{\mathrm{L}}{\otimes}_{\mathcal{D}}\bullet \quad &: \quad \mathrm{D}^{\mathrm{b}}(\mathcal{D}_X^{\mathrm{op}}) \times \mathrm{D}^{\mathrm{b}}(\mathcal{D}_X) \to \mathrm{D}^{\mathrm{b}}(\mathbb{C}_X).
\end{aligned}$$

These functors are related by the formulas (2.4) and (2.5) below.

Proposition 2.3.1. *For $\mathcal{L}, \mathcal{M}, \mathcal{N}$ in $\mathrm{D}^{\mathrm{b}}(\mathcal{D}_X)$ and $\mathcal{K}$ in $\mathrm{D}^{\mathrm{b}}(\mathcal{D}_X^{\mathrm{op}})$ there are natural isomorphisms*

$$\mathcal{K}\overset{\mathrm{L}}{\otimes}_{\mathcal{D}}(\mathcal{M}\overset{\mathrm{D}}{\otimes}\mathcal{N}) \simeq (\mathcal{K}\overset{\mathrm{D}}{\otimes}\mathcal{M})\overset{\mathrm{L}}{\otimes}_{\mathcal{D}}\mathcal{N}, \tag{2.4}$$

$$R\mathcal{H}om_{\mathcal{D}}(\mathcal{L}, R\mathcal{H}om_{\mathcal{O}}(\mathcal{M},\mathcal{N})) \simeq R\mathcal{H}om_{\mathcal{D}}(\mathcal{L}\overset{\mathrm{D}}{\otimes}\mathcal{M},\mathcal{N}). \tag{2.5}$$

Duality

We define the duality functors on $\mathrm{D}^{\mathrm{b}}(\mathcal{D}_X)$ or $\mathrm{D}^{\mathrm{b}}(\mathcal{D}_X^{\mathrm{op}})$, all denoted by $\mathbb{D}'_{\mathcal{D}}$ and $\mathbb{D}_{\mathcal{D}}$, by setting

$$\mathbb{D}'_{\mathcal{D}}(\mathcal{M}) := R\mathcal{H}om_{\mathcal{D}}(\mathcal{M}, \mathcal{D}_X) \ (\mathcal{M} \in \mathrm{D}^{\mathrm{b}}(\mathcal{D}_X) \text{ or } \mathcal{M} \in \mathrm{D}^{\mathrm{b}}(\mathcal{D}_X^{\mathrm{op}})), \tag{2.6}$$

$$\mathbb{D}_{\mathcal{D}}(\mathcal{M}) := R\mathcal{H}om_{\mathcal{D}}(\mathcal{M}, \mathcal{D}_X \otimes_{\mathcal{O}} \Omega_X^{\otimes -1}[d_X]) \ (\mathcal{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X)), \tag{2.7}$$

$$\mathbb{D}_{\mathcal{D}}(\mathcal{M}) := R\mathcal{H}om_{\mathcal{D}}(\mathcal{M}, \Omega_X[d_X] \otimes_{\mathcal{O}} \mathcal{D}_X) \ (\mathcal{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X^{\mathrm{op}})). \tag{2.8}$$

Proposition 2.3.2. *For $\mathcal{M}, \mathcal{N}$ in $\mathrm{D}^{\mathrm{b}}(\mathcal{D}_X)$, we have*

$$R\mathcal{H}om_{\mathcal{D}}(\mathcal{M}, \mathcal{N}) \simeq R\mathcal{H}om_{\mathcal{D}}(\mathcal{O}_X, \mathbb{D}_{\mathcal{D}}\mathcal{M} \overset{\mathrm{D}}{\otimes} \mathcal{N}). \tag{2.9}$$

Proof. We have the isomorphism

$$\begin{aligned} R\mathcal{H}om_{\mathcal{D}}(\mathcal{O}_X, \mathbb{D}_{\mathcal{D}}\mathcal{M} \overset{\mathrm{D}}{\otimes} \mathcal{N}) &\simeq R\mathcal{H}om_{\mathcal{D}}(\mathcal{O}_X, \mathcal{D}_X) \overset{\mathrm{L}}{\otimes}_{\mathcal{D}} (\mathbb{D}_{\mathcal{D}}\mathcal{M} \overset{\mathrm{D}}{\otimes} \mathcal{N}) \\ &\simeq \Omega_X \overset{\mathrm{L}}{\otimes}_{\mathcal{D}} (\mathbb{D}_{\mathcal{D}}\mathcal{M} \overset{\mathrm{D}}{\otimes} \mathcal{N})\,[-d_X] \\ &\simeq (\Omega_X \overset{\mathrm{D}}{\otimes} \mathbb{D}_{\mathcal{D}}\mathcal{M}) \overset{\mathrm{L}}{\otimes}_{\mathcal{D}} \mathcal{N}\,[-d_X] \\ &\simeq \mathbb{D}'_{\mathcal{D}}\mathcal{M} \overset{\mathrm{L}}{\otimes}_{\mathcal{D}} \mathcal{N} \simeq R\mathcal{H}om_{\mathcal{D}}(\mathcal{M}, \mathcal{N}). \end{aligned}$$

q.e.d.

Proposition 2.3.3. (i) *The functor $\mathbb{D}'_{\mathcal{D}} \colon \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X)^{\mathrm{op}} \to \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X^{\mathrm{op}})$ is well-defined and satisfies $\mathbb{D}'_{\mathcal{D}} \circ \mathbb{D}'_{\mathcal{D}} \simeq \mathrm{id}$ and similarly with $\mathbb{D}_{\mathcal{D}}$.*

(ii) *If $\mathcal{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X)$, then $\mathrm{char}(\mathbb{D}'_{\mathcal{D}}(\mathcal{M})) = \mathrm{char}(\mathcal{M})$.*

Proof. (i) There is a natural morphism $\mathrm{id} \to \mathbb{D}'_{\mathcal{D}} \circ \mathbb{D}'_{\mathcal{D}}$. To prove it is an isomorphism, we argue by induction on the amplitude of $\mathcal{M}$ and reduce to the case where $\mathcal{M}$ is a coherent $\mathcal{D}_X$-module. More precisely, assume $H^j(\mathcal{M}) = 0$ for $j \notin [j_0, j_1]$ and the result has been proved for modules with amplitude $j_1 - j_0 - 1$. Consider the distinguished triangle (d.t. for short)

$$H^{j_0}(\mathcal{M})[-j_0] \to \mathcal{M} \to \tau^{>j_0}(\mathcal{M}) \xrightarrow{+1} \tag{2.10}$$

and apply the functor $\mathbb{D}'_{\mathcal{D}} \circ \mathbb{D}'_{\mathcal{D}}$. We get a new d.t. with two objects isomorphic to two objects of the d.t. (2.10). hence the third objects of these d.t. will be isomorphic.

Hence, we are reduced to treat the case of $\mathcal{M} \in \mathrm{Mod}_{\mathrm{coh}}(\mathcal{D}_X)$. We may argue locally and replace $\mathcal{M}$ with a bounded complex of finite free $\mathcal{D}_X$-modules. It reduces to the case where $\mathcal{M} = \mathcal{D}_X$.

(ii) It is enough to prove the inclusion $\mathrm{char}(\mathbb{D}'_{\mathcal{D}}(\mathcal{M})) \subset \mathrm{char}(\mathcal{M})$. We argue by induction on the amplitude of $\mathcal{M}$. Assume $H^j(\mathcal{M}) = 0$ for $j \notin [j_0, j_1]$. Consider the distinguished triangle (2.10) Applying the functor $\mathbb{D}'_{\mathcal{D}}$ we find the d.t.

$$\mathbb{D}(\tau^{>j_0}\mathcal{M}) \to \mathbb{D}'_{\mathcal{D}}\mathcal{M} \to \mathbb{D}'_{\mathcal{D}}(H^{j_0}(\mathcal{M}))[j_0] \xrightarrow{+1}$$

Since $\mathrm{char}(\mathcal{M}) = \mathrm{char}(H^{j_0}(\mathcal{M})) \cup \mathrm{char}(\tau^{>j_0}(\mathcal{M}))$, the induction proceeds, and we are reduced to the case where $\mathcal{M}$ is a coherent $\mathcal{D}_X$-module. Then the result follows from Theorem 1.6.2 (iii). q.e.d.

External product

Let X and Y be two manifolds. For a $\mathcal{D}_X$-module $\mathcal{M}$ and a $\mathcal{D}_Y$-module $\mathcal{N}$, we define their external product, denoted $\mathcal{M}\underline{\boxtimes}\mathcal{N}$, by

$$\mathcal{M}\underline{\boxtimes}\mathcal{N} := \mathcal{D}_{X\times Y} \otimes_{\mathcal{D}_X\boxtimes\mathcal{D}_Y} (\mathcal{M}\boxtimes\mathcal{N}).$$

Note that the functor $\mathcal{M} \mapsto \mathcal{M}\underline{\boxtimes}\mathcal{N}$ is exact.

Theorem 2.3.4. *Let $\mathcal{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X)$ and $\mathcal{N} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_Y)$. Then $\mathcal{M}\underline{\boxtimes}\mathcal{N} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_{X\times Y})$ and $\mathrm{char}(\mathcal{M}\underline{\boxtimes}\mathcal{N}) = \mathrm{char}(\mathcal{M}) \times \mathrm{char}(\mathcal{N})$.*

Proof. (o) By dévissage, one reduces to the case where $\mathcal{M} \in \mathrm{Mod}_{\mathrm{coh}}(\mathcal{D}_X)$ and $\mathcal{N} \in \mathrm{Mod}_{\mathrm{coh}}(\mathcal{D}_Y)$.

(i) Let us show that $\mathcal{M}\underline{\boxtimes}\mathcal{N}$ is coherent. Consider finite free resolutions of $\mathcal{M}$ and $\mathcal{N}$:

$$(\mathcal{D}_X)^{N_1} \xrightarrow{\cdot P} (\mathcal{D}_X)^{N_0} \to \mathcal{M} \to 0, \qquad (\mathcal{D}_Y)^{M_1} \xrightarrow{\cdot Q} (\mathcal{D}_Y)^{M_0} \to \mathcal{N} \to 0.$$

Then

$$(\mathcal{D}_X\boxtimes\mathcal{D}_Y)^{N_1+M_1} \xrightarrow{\left(\begin{smallmatrix} P & 0 \\ 0 & Q \end{smallmatrix}\right)} (\mathcal{D}_X\boxtimes\mathcal{D}_Y)^{N_0+M_0} \to \mathcal{M}\boxtimes\mathcal{N} \to 0$$

is a finite free resolution of $\mathcal{M}\boxtimes\mathcal{N}$ over $\mathcal{D}_X\boxtimes\mathcal{D}_Y$. To conclude, apply the exact functor $\mathcal{D}_{X\times Y}\otimes_{\mathcal{D}_X\boxtimes\mathcal{D}_Y} \bullet$ to this sequence.

(ii) Let us endow $\mathcal{M}$ and $\mathcal{N}$ with good filtrations $\mathrm{Fl}\,\mathcal{M}$ and $\mathrm{Fl}\,\mathcal{N}$. Set

$$\mathrm{Fl}_k(\mathcal{M}\underline{\boxtimes}\mathcal{N}) = \sum_{i+j=k} \mathrm{Fl}_i(\mathcal{M})\underline{\boxtimes}\mathrm{Fl}_j(\mathcal{N}).$$

Then $\{\mathrm{Fl}_k(\mathcal{M}\underline{\boxtimes}\mathcal{N})\}_k$ is a good filtration on $\mathcal{M}\underline{\boxtimes}\mathcal{N}$ and the result follows from

$$\mathrm{gr}\,(\mathcal{M}\underline{\boxtimes}\mathcal{N}) \simeq \mathrm{gr}\,(\mathcal{M}) \overset{\mathrm{gr}\,\mathcal{D}}{\boxtimes} \mathrm{gr}\,(\mathcal{N}).$$

q.e.d.

Example 2.3.5. Let Z_i be a smooth submanifold of X_i $(i = 1, 2)$. Then

$$\mathcal{B}_{Z_1\times Z_2|X_1\times X_2} \simeq \mathcal{B}_{Z_1|X_1}\underline{\boxtimes}\mathcal{B}_{Z_2|X_2}.$$

2.4 Transfert bimodule

Let $f\colon X \to Y$ be a morphism of complex manifolds. We shall construct a $(\mathcal{D}_X, f^{-1}\mathcal{D}_Y)$-bimodule denoted $\mathcal{D}_{X\to Y}$ which shall allow one to pass from left $\mathcal{D}_Y$-modules to left $\mathcal{D}_X$-modules and from right $\mathcal{D}_X$-modules to right $\mathcal{D}_Y$-modules.

Set

$$\mathcal{D}_{X\to Y} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y.$$

This sheaf on X is naturally endowed with a structure of an $(\mathcal{O}_X, f^{-1}\mathcal{D}_Y)$-bimodule. We shall endow it of a structure of a left $\mathcal{D}_X$-module by defining the action Θ_X and verifying that this action satisfies the hypothesis of Lemma **??**. Let $v \in \Theta_X$. Then $f'_*v \in \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\Theta_Y$. Hence

$$f'_*v = \sum_j a_j \otimes w_j,$$

with $a_j \in \mathcal{O}_X$ and $w_j \in f^{-1}\Theta_Y$. Define the action of v on $a\otimes P \in \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y$ by setting

$$v(a\otimes P) = v(a)\otimes P + \sum_j aa_j \otimes w_j \circ P. \tag{2.11}$$

If one chooses a local coordinate system $(y_1, \ldots, y_m)$ on Y and writes $f = (f_1, \ldots, f_m)$, then

$$v(f^*\varphi) = \sum_{j=1}^m v(f_j)\frac{\partial\varphi}{\partial y_j},$$

which implies

$$f'_*v = \sum_{j=1}^m v(f_j)\otimes \partial_{y_j}.$$

A section P of $\mathcal{D}_{X\to Y}$ may formally be written as $P = \sum_\alpha a_\alpha(x)\partial_y^\alpha$.

By composing the monomorphism $\mathcal{D}_Y \hookrightarrow \mathcal{H}om_{\mathbb{C}_Y}(\mathcal{O}_Y, \mathcal{O}_Y)$ with $\mathcal{D}_{X\to Y} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y$ we get the monomorphisms

$$\begin{aligned}\mathcal{D}_{X\to Y} &\hookrightarrow \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{H}om_{\mathbb{C}_Y}(\mathcal{O}_Y, \mathcal{O}_Y)\\ &\hookrightarrow \mathcal{H}om_{\mathbb{C}_X}(f^{-1}\mathcal{O}_Y, \mathcal{O}_X)\end{aligned}$$

and the section $1_{X\to Y} := 1 \otimes 1 \in \mathcal{D}_{X\to Y}$ corresponds to the canonical morphism

$$\begin{array}{rcl} f^{-1}\mathcal{O}_Y & \to & \mathcal{O}_X \\ \varphi & \mapsto & \varphi \circ f. \end{array}$$

Note that $\mathcal{D}_Y$ being flat over $\mathcal{O}_Y$,

$$\mathcal{D}_{X\to Y} \simeq \mathcal{O}_X \overset{\mathrm{L}}{\otimes}_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y.$$

One also introduces the $(f^{-1}\mathcal{D}_Y, \mathcal{D}_X)$-bimodule $\mathcal{D}_{Y\leftarrow X}$ by setting

$$\mathcal{D}_{Y\leftarrow X} = \Omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_{X\to Y} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\Omega_Y^{\otimes -1}.$$

Proposition 2.4.1. *Let $f\colon X \to Y$, $g\colon Y \to Z$ be morphisms of manifolds and set $h = g \circ f\colon X \to Z$. Then there is an isomorphism of $(\mathcal{D}_X, h^{-1}\mathcal{D}_Z)$-bimodules*

$$\mathcal{D}_{X\to Y} \overset{\mathrm{L}}{\otimes}_{f^{-1}\mathcal{D}_Y} f^{-1}\mathcal{D}_{Y\to Z} \simeq \mathcal{D}_{X\to Z}. \tag{2.12}$$

In particular, the left hand side is concentrated in degree zero.

Proof. One has the isomorphisms of $(\mathcal{O}_X, h^{-1}\mathcal{D}_Z)$-bimodules:

$$\begin{aligned} \mathcal{D}_{X\to Y} \overset{\mathrm{L}}{\otimes}_{f^{-1}\mathcal{D}_Y} f^{-1}\mathcal{D}_{Y\to Z} &= (\mathcal{O}_X \overset{\mathrm{L}}{\otimes}_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y) \overset{\mathrm{L}}{\otimes}_{f^{-1}\mathcal{D}_Y} f^{-1}(\mathcal{O}_Y \overset{\mathrm{L}}{\otimes}_{g^{-1}\mathcal{O}_Z} g^{-1}\mathcal{D}_Z) \\ &\simeq \mathcal{O}_X \overset{\mathrm{L}}{\otimes}_{f^{-1}\mathcal{O}_Y} (f^{-1}\mathcal{D}_Y \overset{\mathrm{L}}{\otimes}_{f^{-1}\mathcal{D}_Y} f^{-1}\mathcal{O}_Y \overset{\mathrm{L}}{\otimes}_{h^{-1}\mathcal{O}_Z} h^{-1}\mathcal{D}_Z) \\ &\simeq \mathcal{O}_X \overset{\mathrm{L}}{\otimes}_{h^{-1}\mathcal{O}_Z} h^{-1}\mathcal{D}_Z \simeq \mathcal{O}_X \otimes_{h^{-1}\mathcal{O}_Z} h^{-1}\mathcal{D}_Z. \end{aligned}$$

(Recall that $\mathcal{D}_Z$ is flat over $\mathcal{O}_Z$.) Then, one checks that these isomorphisms extend as isomorphisms of $(\mathcal{D}_X, h^{-1}\mathcal{D}_Z)$-bimodules. q.e.d.

Proposition 2.4.2. (i) *Assume f is smooth. Then $\mathcal{D}_{X\to Y}$ is $\mathcal{D}_X$-coherent and $f^{-1}\mathcal{D}_Y$-flat.*

(ii) *Assume f is a closed embedding. Then $\mathcal{D}_{X\to Y}$ is $\mathcal{D}_Y$-coherent and $\mathcal{D}_X$-flat.*

Proof. (i) Since the problem is local on X, we may assume that $X = Z \times Y$ and f is the second projection. In this case, $\mathcal{D}_{X\to Y} \simeq \mathcal{O}_Z \boxtimes \mathcal{D}_Y$. Note that if $x = (t, y)$ is a local coordinate system on $Z \times Y$ with $t = (t_1, \ldots, t_m)$, then

$$\mathcal{D}_{X\to Y} \simeq \mathcal{D}_X / \mathcal{D}_X \cdot \partial_t$$

where $\mathcal{D}_X \cdot \partial_t$ denotes the left ideal generated by $(\partial_{t_1}, \dots, \partial_{t_m})$.
(ii) For a local coordinate system $y = (t, x)$ on Y such that $X = \{t = 0\}$, we have

$$\mathcal{D}_{X\to Y} \simeq \mathcal{D}_Y / t \cdot \mathcal{D}_Y$$

where $t \cdot \mathcal{D}_Y$ denotes the right ideal generated by $(t_1, \dots, t_m)$. q.e.d.

If f is smooth, one has

$$\mathcal{D}_{X\to Y} \simeq \mathcal{D}_X / \mathcal{D}_X \cdot \Theta_f$$

where $\mathcal{D}_X \cdot \Theta_f$ denotes the left ideal generated by the vector fields tangent to the leaves of f.

If f is a closed embedding, one has

$$\mathcal{D}_{X\to Y} \simeq \mathcal{D}_Y / \mathcal{I}_X \cdot \mathcal{D}_Y$$

where $\mathcal{I}_X \cdot \mathcal{D}_Y$ denotes the right ideal generated sections of $\mathcal{O}_Y$ vanishing on X.

Notice that any morphism $f \colon X \to Y$ may be decomposed as

$$f \colon X \hookrightarrow X \times Y \to Y$$

where the first map is the graph (closed) embedding and the second map is the (smooth) projection.

Example 2.4.3. One has $\mathcal{D}_{X\to \mathrm{pt}} \simeq \mathcal{O}_X$ and $\mathcal{D}_{\mathrm{pt}\leftarrow X} \simeq \Omega_X$.

Proposition 2.4.4. *There is a natural isomorphism of* $(\mathcal{D}_X, f^{-1}\mathcal{D}_Y)$*-modules*

$$\mathcal{D}_{X\to Y} \xrightarrow{\sim} \mathcal{B}_{\Gamma_f|X\times Y} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\Omega_Y$$

which associates δ_{Γ_f} *to* $\mathbf{1}_{X\to Y} \in \mathcal{D}_{X\to Y}$.

Proof. Applying Proposition 1.5.4, we get

$$\mathcal{D}_{X\to Y} \xrightarrow{\sim} \mathcal{D}_{X\to Y} \overset{\mathrm{L}}{\otimes}_{\mathcal{D}_Y} \mathcal{B}_{\Delta|Y\times Y} \otimes_{q_2^{-1}\mathcal{O}} q_2^{-1}\Omega_Y.$$

Hence, it is enough to prove the isomorphism

$$\mathcal{D}_{X\to Y} \overset{\mathrm{L}}{\otimes}_{f^{-1}\mathcal{D}_Y} f^{-1}\mathcal{B}_{\Delta|Y\times Y} \simeq \mathcal{B}_{\Gamma_f|X\times Y}.$$

This result follows from Exercise 2.1. q.e.d.

Inverse and direct images of $\mathcal{D}$-modules

Definition 2.4.5. Let $f\colon X \to Y$ be a morphism of complex manifolds.

(i) One defines the inverse image functor $f_{\mathcal{D}}^{-1}\colon \mathrm{D}^{\mathrm{b}}(\mathcal{D}_Y) \to \mathrm{D}^{\mathrm{b}}(\mathcal{D}_X)$ by setting for $\mathcal{N} \in \mathrm{D}^{\mathrm{b}}(\mathcal{D}_Y)$:

$$f_{\mathcal{D}}^{-1}\mathcal{N} := \mathcal{D}_{X\to Y} \otimes^{L}_{f^{-1}\mathcal{D}_Y} f^{-1}\mathcal{N}.$$

(ii) One defines the direct image functors $f_*^{\mathcal{D}}, f_!^{\mathcal{D}}\colon \mathrm{D}^{\mathrm{b}}(\mathcal{D}_X) \to \mathrm{D}^{\mathrm{b}}(\mathcal{D}_Y)$ by setting for $\mathcal{M} \in \mathrm{D}^{\mathrm{b}}(\mathcal{D}_X^{\mathrm{op}})$:

$$f_*^{\mathcal{D}}\mathcal{M} := Rf_*(\mathcal{M} \otimes^{L}_{\mathcal{D}} \mathcal{D}_{X\to Y}), \qquad f_!^{\mathcal{D}}\mathcal{M} := Rf_!(\mathcal{M} \otimes^{L}_{\mathcal{D}} \mathcal{D}_{X\to Y}).$$

Using the bimodule $\mathcal{D}_{Y\leftarrow X}$, one defines similarly the inverse image of a right $\mathcal{D}_Y$-module or the direct images of a left $\mathcal{D}_X$-module. Note that, if $g\colon Y \to Z$ is another morphism of complex manifolds, we have

$$(g\circ f)_{\mathcal{D}}^{-1} \simeq f_{\mathcal{D}}^{-1} \circ g_{\mathcal{D}}^{-1}, \tag{2.13}$$

$$(g\circ f)_*^{\mathcal{D}} \simeq g_*^{\mathcal{D}} \circ f_*^{\mathcal{D}}, \tag{2.14}$$

$$(g\circ f)_!^{\mathcal{D}} \simeq g_!^{\mathcal{D}} \circ f_!^{\mathcal{D}}. \tag{2.15}$$

2.5 Non characteristic inverse images

Definition 2.5.1. Let $\mathcal{N}$ be a coherent $\mathcal{D}_Y$-module. One says that f is non characteristic for $\mathcal{N}$ (or $\mathcal{N}$ is non characteristic for f) if f is non characteristic for $\mathrm{char}(\mathcal{N})$. (See Definition 2.2.1.)

Example 2.5.2. (i) Since $\mathrm{char}(\mathcal{O}_Y) = T^*_Y Y$, the $\mathcal{D}_Y$-module $\mathcal{O}_Y$ is non characteristic for any morphism $f\colon X \to Y$. Note that $f_{\mathcal{D}}^{-1}\mathcal{O}_Y \simeq \mathcal{O}_X$.
(ii) See Exercise 2.1.

Example 2.5.3. Assume to be given a coordinate system $(y) = (x_1, \dots, x_n, t) = (x, t)$ on Y such that $X = \{t = 0\}$. Let P be a differential operator of order m. Then X is non-characteristic with respect to P (*i.e.*, for the $\mathcal{D}_Y$-module $\mathcal{D}_Y/\mathcal{D}_Y \cdot P$) in a neighborhood of $(x_0, 0) \in X$ if and only if P is written as

$$P(x, t; \partial_x, \partial_t) = \sum_{0\leq j\leq m} a_j(x, t, \partial_x)\partial_t^j \tag{2.16}$$

where $a_j(x, t, \partial_x)$ is a differential operator not depending on ∂_t of order $\leq m - j$ and $a_m(x, t)$ (which is a holomorphic function on Y) satisfies: $a_m(x_0, 0) \neq 0$.

Lemma 2.5.4. *Let X, Y and P be as in Example 2.5.3. Let $\mathcal{N} = \mathcal{D}_Y/\mathcal{D}_Y \cdot P$. Then $\mathcal{D}_{X\to Y} \otimes_{\mathcal{D}_Y} \mathcal{N} \simeq \mathcal{D}_X^m$.*

Proof. Notice that

$$\mathcal{D}_{X\to Y} \otimes_{\mathcal{D}} \mathcal{N} \simeq \mathcal{D}_Y/(t \cdot \mathcal{D}_Y + \mathcal{D}_Y \cdot P).$$

By the Weierstrass preparation theorem, any $Q(x, t, \partial_x, \partial_t) \in \mathcal{D}_Y$ may be written uniquely as

$$Q(x, t, \partial_x, \partial_t) = S(x, t, \partial_x, \partial_t) \cdot P(x, t, \partial_x, \partial_t) + \sum_{j=0}^{m-1} R_j(x, t, \partial_x)\partial_t^j.$$

Hence, $Q(x, t, \partial_x, \partial_t) \in \mathcal{D}_Y$ may be written uniquely as

$$\begin{aligned} Q(x, t, \partial_x, \partial_t) &= \\ &S(x, t, \partial_x, \partial_t) \cdot P(x, t, \partial_x, \partial_t) + t \cdot T(x, t, \partial_x) + \sum_{j=0}^{m-1} P_j(x, \partial_x)\partial_t^j. \end{aligned}$$

q.e.d.

Theorem 2.5.5. *Let $\mathcal{N} \in \mathrm{Mod}_{\mathrm{coh}}(\mathcal{D}_Y)$ and assume that f is non characteristic for $\mathcal{N}$. Then*

(a) *$f_{\mathcal{D}}^{-1}\mathcal{N}$ is concentrated in degree 0,*

(b) *$f_{\mathcal{D}}^{-1}\mathcal{N}$ is $\mathcal{D}_X$-coherent,*

(c) $\mathrm{char}(f_{\mathcal{D}}^{-1}\mathcal{N}) \subset f_d f_\pi^{-1} \mathrm{char}(\mathcal{N})$.

Remark 2.5.6. In fact, there is a better result, namely $\mathrm{char}(f_{\mathcal{D}}^{-1}\mathcal{N}) = f_d f_\pi^{-1} \mathrm{char}(\mathcal{N})$ and the characteristic cycle of $f_{\mathcal{D}}^{-1}\mathcal{N}$ is the image by $f_d f_\pi^{-1}$ of the characteristic cycle of $\mathcal{N}$ (see [20]).

Proof. The map $f\colon X \to Y$ decomposes as

$$X \xrightarrow{h} X \times Y \xrightarrow{p} Y$$

where h is the graph embedding and p is the projection. Using (2.13) and Lemma 2.2.4, it is enough to prove the result for p and for h. Hence, we shall treat separately the case where f is smooth and the case where f is a closed embedding.

(i) Assume $f\colon X \to Y$ is smooth. The problem is local on X. Hence, we may assume $X = Y \times Z$ and f is the projection. In this case, $f_{\mathcal{D}}^{-1}(\,\bullet\,) \simeq \mathcal{O}_X \boxtimes \bullet$. Hence, this functor is exact and the result follows from Theorem 2.3.4.

(ii) Assume $f\colon X \to Y$ is a closed embedding. Let d denote the codimension of X in Y. Since our problem is local, we may assume that there are submanifolds $X = X_0 \subset X_1 \subset \cdots \subset X_d = Y$. Using (2.13) and Lemma 2.2.4 again, we are reduced to treat the case $d = 1$. Since the problem is local we may assume to be given a local coordinate system in a neighborhood of $x_0 \in X$, $(y) = (x_1, \dots, x_n, t) = (x, t)$ on Y such that $X = \{t = 0\}$. Let $(x, t; \xi, \tau)$ denote the associated coordinate system on T^*Y. Set $\Lambda = \mathrm{char}(\mathcal{N})$. By the hypothesis, $(x_0, 0; 0, 1) \notin \Lambda$. By Corollary 1.3.8, for each section u of $\mathcal{N}$ defined in a neighborhood of $(x_0, 0)$, there exists a differential operator P, say of order m, such that

$$Pu = 0, \qquad \sigma_m(P)(x_0, 0; 0, 1) \neq 0. \tag{2.17}$$

(iii) Let us prove that $f_{\mathcal{D}}^{-1}\mathcal{N}$ is concentrated in degree 0. Since $\mathcal{D}_{X\to Y} \simeq \mathcal{D}_Y / t \cdot \mathcal{D}_Y$, $f_{\mathcal{D}}^{-1}\mathcal{N}$ is isomorphic to the complex $\mathcal{N} \xrightarrow{t\cdot} \mathcal{N}$. Hence, we have to show that $t\cdot$ acting on $\mathcal{N}$ is injective. Let $u \in \mathcal{N}$ with $tu = 0$. Let P satisfying (2.17). Set $\mathrm{Ad}(P) = [P, \bullet]$. We obtain

$$\mathrm{Ad}^m(P)(t)u = m!u = 0.$$

Hence, $u = 0$.

(iv) Let us prove that $f_{\mathcal{D}}^{-1}\mathcal{N}$ is $\mathcal{D}_X$-coherent. Let $(u_1, \dots, u_N)$ be a system of generators of $\mathcal{N}$ in a neighborhood of $(x_0, 0)$. For each $j, 1 \le j \le N$, there exists a differential operator P_j of order m_j, such that $P_j u_j = 0$ and $\sigma_{m_j}(P_j)(x_0, 0; 0, 1) \neq 0$. Set

$$\mathcal{M} = \oplus_{j=1}^{N} \mathcal{D}_Y / \mathcal{D}_Y \cdot P_j.$$

It follows from (iii) and Lemma 2.5.4 that $f_{\mathcal{D}}^{-1}\mathcal{M}$ is concentrated in degree 0 and is $\mathcal{D}_X$-coherent.

Denote by v_j the canonical generator of $\mathcal{D}_Y / \mathcal{D}_Y \cdot P_j$, the image of $1 \in \mathcal{D}_Y$. There is a well-defined $\mathcal{D}_Y$-linear epimorphism $\psi\colon \mathcal{M} \twoheadrightarrow \mathcal{N}$ which associates u_j to v_j. The functor $f_{\mathcal{D}}^{-1}$ being right exact, the epimorphism ψ defines the epimorphism $f_{\mathcal{D}}^{-1}\mathcal{M} \twoheadrightarrow f_{\mathcal{D}}^{-1}\mathcal{N}$. Therefore, $f_{\mathcal{D}}^{-1}\mathcal{N}$ is locally finitely generated.

Define the coherent $\mathcal{D}_Y$-module $\mathcal{L}$ by the exact sequence

$$0 \to \mathcal{L} \to \mathcal{M} \to \mathcal{N} \to 0. \tag{2.18}$$

It follows from (iii) that the sequence

$$0 \to f_{\mathcal{D}}^{-1}\mathcal{L} \to f_{\mathcal{D}}^{-1}\mathcal{M} \to f_{\mathcal{D}}^{-1}\mathcal{N} \to 0 \tag{2.19}$$

is exact. Since X is non-characteristic for $\mathcal{M}$, it is non-characteristic for its submodule $\mathcal{L}$. Therefore, $f_{\mathcal{D}}^{-1}\mathcal{L}$ is locally finitely generated and $f_{\mathcal{D}}^{-1}\mathcal{M}$ being coherent, this implies that $f_{\mathcal{D}}^{-1}\mathcal{N}$ is coherent.

(v) Let us prove (c).
(v)–(a) Let us choose a local coordinate system (x, t) on Y such that $X = \{(x, t); t = 0\}$. Then $f_{\mathcal{D}}^{-1}\mathcal{N} \simeq \mathcal{N}/t \cdot \mathcal{N}$. Set

$$\mathcal{M} := f_{\mathcal{D}}^{-1}\mathcal{N}.$$

Let $\mathrm{Fl}\,\mathcal{N} = \{\mathcal{N}_j\}_{j\in\mathbb{Z}}$ be a good filtration on $\mathcal{N}$. We define a filtration on $\mathrm{Fl}\,\mathcal{M} = \{\mathcal{M}_j\}_{j\in\mathbb{Z}}$ by setting

$$\mathcal{M}_j \;=\; \mathcal{N}_j/(t \cdot \mathcal{N} \cap \mathcal{N}_j). \tag{2.20}$$

(v)–(b) Let us show that $\mathrm{Fl}\,\mathcal{M}$ is a good filtration. It is enough to check that the $\mathcal{M}_j$'s are $\mathcal{O}_X$-coherent. Since

$$t \cdot \mathcal{N} \cap \mathcal{N}_j = \bigcup_k (t \cdot \mathcal{N}_k \cap \mathcal{N}_j),$$

and $\mathcal{N}_j$ is $\mathcal{O}_Y$-coherent, this sequence is locally stationary. It follows that $\mathcal{M}_j$ is $\mathcal{O}_Y$-coherent. Being supported by X, $\mathcal{M}_j$ is $\mathcal{O}_X$-coherent.

(v)–(c) The exact sequence $0 \to \mathcal{N}_{j-1} \to \mathcal{N}_j \to \mathrm{gr}_j\mathcal{N} \to 0$ gives rise to the exact sequence

$$\mathcal{N}_{j-1}/t \cdot \mathcal{N}_{j-1} \to \mathcal{N}_j/t \cdot \mathcal{N}_j \to \mathrm{gr}_j\mathcal{N}/t \cdot \mathrm{gr}_j\mathcal{N} \to 0. \tag{2.21}$$

We deduce from (2.20) and (2.21) an epimorphism $\mathrm{gr}_j\mathcal{N}/t \cdot \mathrm{gr}_j\mathcal{N} \twoheadrightarrow \mathrm{gr}_j\mathcal{M}$, hence, an epimorphism

$$\mathrm{gr}\,\mathcal{N}/t \cdot \mathrm{gr}\,\mathcal{N} \twoheadrightarrow \mathrm{gr}\,\mathcal{M}. \tag{2.22}$$

It follows that the support of $\mathrm{gr}\,\mathcal{M}$ in $X \times_Y T^*Y$ (*i.e.*, as an $\mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathrm{gr}\,\mathcal{D}_Y$-module) is contained in $f_\pi^{-1}\mathrm{char}(\mathcal{N})$. Since f_d is finite over $f_\pi^{-1}\mathrm{char}(\mathcal{N})$, the support of $\mathrm{gr}\,\mathcal{M}$ as a $\mathrm{gr}\,\mathcal{D}_Y$-module is contained in $f_d f_\pi^{-1}\mathrm{char}(\mathcal{N})$. q.e.d.

We shall deduce some corollaries of Theorem 2.5.5.

Proposition 2.5.7. . *For $\mathcal{M}, \mathcal{N} \in \mathrm{D^b}(\mathcal{D}_X)$, one has*

$$\mathcal{M} \overset{\mathrm{D}}{\otimes} \mathcal{N} \simeq \delta_{\mathcal{D}}^{-1}(\mathcal{M} \boxtimes \mathcal{N}),$$

where $\delta\colon X \to X \times X$ is the diagonal embedding.

Proof. Let us identify X with Δ, the diagonal of $X \times X$. One has the chain of isomorphisms

$$\begin{aligned}\delta_{\mathcal{D}}^{-1}(\mathcal{M}\underline{\boxtimes}\mathcal{N}) &\simeq \mathcal{O}_\Delta \overset{\mathrm{L}}{\otimes}_{\mathcal{O}} \mathcal{D}_{X\times X} \overset{\mathrm{L}}{\otimes}_{\mathcal{D}} (\mathcal{M}\underline{\boxtimes}\mathcal{N}) \\ &\simeq \mathcal{O}_\Delta \overset{\mathrm{L}}{\otimes}_{\mathcal{O}} (\mathcal{M}\underline{\boxtimes}\mathcal{N}) \simeq \mathcal{M}\overset{\mathrm{D}}{\otimes}\mathcal{N}.\end{aligned}$$

q.e.d.

Corollary 2.5.8. *Let $f\colon X \to Y$ be a morphism of complex manifolds. For $\mathcal{N}_1, \mathcal{N}_2 \in \mathrm{D^b}(\mathcal{D}_Y)$, one has*

$$f_{\mathcal{D}}^{-1}(\mathcal{N}_1 \overset{\mathrm{D}}{\otimes} \mathcal{N}_2) \simeq f_{\mathcal{D}}^{-1}\mathcal{N}_1 \overset{\mathrm{D}}{\otimes} f_{\mathcal{D}}^{-1}\mathcal{N}_2.$$

Proof. Denote by δ_X the diagonal embedding $X \to X \times X$ and similarly with δ_Y, and denote by $\widetilde{f}\colon X \times X \to Y \times Y$ the map associated with f. One has the chain of isomorphisms

$$\begin{aligned}f_{\mathcal{D}}^{-1}(\mathcal{N}_1 \overset{\mathrm{D}}{\otimes} \mathcal{N}_2) &\simeq f_{\mathcal{D}}^{-1}\delta_{Y\mathcal{D}}^{-1}(\mathcal{N}_1\underline{\boxtimes}\mathcal{N}_2) \simeq \delta_{X\mathcal{D}}^{-1}\widetilde{f}_{\mathcal{D}}^{-1}(\mathcal{N}_1\underline{\boxtimes}\mathcal{N}_2) \\ &\simeq \delta_{X\mathcal{D}}^{-1}(f_{\mathcal{D}}^{-1}\mathcal{N}_1\underline{\boxtimes} f_{\mathcal{D}}^{-1}\mathcal{N}_2) \simeq f_{\mathcal{D}}^{-1}\mathcal{N}_1 \overset{\mathrm{D}}{\otimes} f_{\mathcal{D}}^{-1}\mathcal{N}_2.\end{aligned}$$

q.e.d.

Corollary 2.5.9. *Let $\mathcal{M}, \mathcal{N} \in \mathrm{Mod_{coh}}(\mathcal{D}_X)$ and assume that* $\mathrm{char}(\mathcal{M}) \cap \mathrm{char}(\mathcal{N}) \subset T^*_X X$. *Then $\mathcal{M}\overset{\mathrm{D}}{\otimes}\mathcal{N}$ is $\mathcal{D}_X$-coherent and*

$$\mathrm{char}(\mathcal{M}\overset{\mathrm{D}}{\otimes}\mathcal{N}) \subset \mathrm{char}(\mathcal{M}) \underset{X}{+} \mathrm{char}(\mathcal{N}).$$

Recall that for two conic subsets Λ_1 and Λ_2 of T^*X ,

$$\Lambda_1 \underset{X}{+} \Lambda_2 := \{(x;\xi_1+\xi_2); (x;\xi_j \in \Lambda_j, j = 1,2\}.$$

Proof. Apply Proposition 2.5.7 and Theorem 2.5.5. q.e.d.

Duality and inverse images

Let $\mathcal{N} \in \mathrm{D^b}(\mathcal{D}_Y)$. Recall that its dual, $\mathbb{D}_{\mathcal{D}}\mathcal{N} \in \mathrm{D^b}(\mathcal{D}_Y)$ has been constructed in (2.8)

Theorem 2.5.10. *Let $f\colon X \to Y$ be a morphism of complex manifolds and let $\mathcal{N} \in \mathrm{D^b_{coh}}(\mathcal{D}_Y)$. Assume that f is non characteristic for $\mathcal{N}$. Then there exists a natural isomorphism :*

$$\psi\colon \mathbb{D}_{\mathcal{D}} f_{\mathcal{D}}^{-1}\mathcal{N} \xrightarrow{\sim} f_{\mathcal{D}}^{-1}\mathbb{D}_{\mathcal{D}}\mathcal{N}.$$

Proof. First, we shall construct the morphism ψ. By Proposition 2.3.2, we have an isomorphism

$$\mathrm{Hom}_{\mathrm{D^b}(\mathcal{D}_Y)}(\mathcal{N},\mathcal{N}) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{D^b}(\mathcal{D}_Y)}(\mathcal{O}_Y, \mathbb{D}_{\mathcal{D}}\mathcal{N} \overset{\mathrm{D}}{\otimes} \mathcal{N}).$$

It defines the morphism $\mathcal{O}_Y \to \mathbb{D}_{\mathcal{D}}\mathcal{N} \overset{\mathrm{D}}{\otimes} \mathcal{N}$. Applying the functor $f_{\mathcal{D}}^{-1}$ we get the morphisms

$$\begin{aligned} f_{\mathcal{D}}^{-1}\mathcal{O}_Y \simeq \mathcal{O}_X &\to f_{\mathcal{D}}^{-1}\mathbb{D}_{\mathcal{D}}\mathcal{N} \overset{\mathrm{D}}{\otimes} f_{\mathcal{D}}^{-1}\mathcal{N} \\ &\to f_{\mathcal{D}}^{-1}\mathbb{D}_{\mathcal{D}}\mathcal{N} \overset{\mathrm{D}}{\otimes} \mathbb{D}_{\mathcal{D}}\mathbb{D}_{\mathcal{D}} f_{\mathcal{D}}^{-1}\mathcal{N}. \end{aligned}$$

Hence, we have obtained a morphism

$$\begin{aligned} \psi &\in \mathrm{Hom}_{\mathrm{D^b}(\mathcal{D}_X)}(\mathcal{O}_X, f_{\mathcal{D}}^{-1}\mathbb{D}_{\mathcal{D}}\mathcal{N} \overset{\mathrm{D}}{\otimes} \mathbb{D}_{\mathcal{D}}\mathbb{D}_{\mathcal{D}} f_{\mathcal{D}}^{-1}\mathcal{N}) \\ &\simeq \mathrm{Hom}_{\mathrm{D^b}(\mathcal{D}_X)}(\mathbb{D}_{\mathcal{D}} f_{\mathcal{D}}^{-1}\mathcal{N}, f_{\mathcal{D}}^{-1}\mathbb{D}_{\mathcal{D}}\mathcal{N}). \end{aligned}$$

To prove that ψ is an isomorphism, we proceed as in the proof of Theorem 2.5.5 and reduce to the case where X is a closed hypersurface of Y and $\mathcal{N} = \mathcal{D}_Y/\mathcal{D}_Y \cdot P$ for a differential operator P of order m. In this case, $f_{\mathcal{D}}^{-1}\mathcal{N} \simeq \mathcal{D}_X^m$ and $\mathbb{D}_{\mathcal{D}} f_{\mathcal{D}}^{-1}\mathcal{N} \simeq \mathcal{D}_X^m\,[d_X]$. On the other hand, $\mathcal{N}$ is represented by the complex $0 \to \mathcal{D}_Y \xrightarrow{\cdot P} \mathcal{D}_Y \to 0$ and it follows that

$$\mathbb{D}_{\mathcal{D}}\mathcal{N} \simeq \mathcal{N}\,[d_Y - 1].$$

Therefore, $f_{\mathcal{D}}^{-1}\mathbb{D}_{\mathcal{D}}\mathcal{N} \simeq \mathcal{D}_X^m\,[d_Y - 1]$. q.e.d.

Holomorphic solutions of inverse images

Let $f\colon X \to Y$ be a morphism of complex manifolds and let $\mathcal{N}_1, \mathcal{N}_2 \in \mathrm{Mod}(\mathcal{D}_Y)$. There is a natural morphism

$$f^{-1}R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{N}_1,\mathcal{N}_2) \to R\mathcal{H}om_{\mathcal{D}_X}(f_{\mathcal{D}}^{-1}\mathcal{N}_1, f_{\mathcal{D}}^{-1}\mathcal{N}_2). \tag{2.23}$$

obtained as the composition

$$\begin{aligned} f^{-1}R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{N}_1,\mathcal{N}_2) &\to R\mathcal{H}om_{f^{-1}\mathcal{D}_Y}(f^{-1}\mathcal{N}_1, f^{-1}\mathcal{N}_2) \\ &\to R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_{X\to Y} \overset{\mathrm{L}}{\otimes}_{f^{-1}\mathcal{D}} f^{-1}\mathcal{N}_1, \mathcal{D}_{X\to Y} \overset{\mathrm{L}}{\otimes}_{f^{-1}\mathcal{D}} f^{-1}\mathcal{N}_2). \end{aligned}$$

Also recall the natural isomorphism

$$f_{\mathcal{D}}^{-1}\mathcal{O}_Y \simeq \mathcal{O}_X. \tag{2.24}$$

Theorem 2.5.11. (Cauchy-Kowalevski-Kashiwara) *Let $f\colon X \to Y$ be a morphism of complex manifolds and let $\mathcal{N} \in \mathrm{Mod}(\mathcal{D}_Y)$. Assume that f is non characteristic for $\mathcal{N}$. Then there exists a natural isomorphism :*

$$f^{-1}R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{N},\mathcal{O}_Y) \xrightarrow{\sim} R\mathcal{H}om_{\mathcal{D}_X}(f_{\mathcal{D}}^{-1}\mathcal{N},\mathcal{O}_X). \tag{2.25}$$

Proof. As in the proof of Theorem 2.5.5, we may check separately the case of a projection and a closed embedding.

(a) If f is smooth, the morphism (2.23) is an isomorphism. Indeed, we may reduce to the case where $\mathcal{N}_1 = \mathcal{N}_2 = \mathcal{D}_Y$. In such a case, the isomorphism reduces to:

$$f^{-1}\mathcal{D}_Y \;\simeq\; R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_{X\to Y},\mathcal{D}_{X\to Y}).$$

We may assume f is the projection $X = Y \times Z \to Y$, and the result is a relative version of the De Rham isomorphism $\mathbb{C}_Z \simeq R\mathcal{H}om_{\mathcal{D}_Z}(\mathcal{O}_Z,\mathcal{O}_Z)$.

(b) Now assume f is a closed embedding. Again, we reduce to the case where X is a hypersurface. First we treat the case where $\mathcal{N} = \mathcal{D}_Y/\mathcal{D}_Y \cdot P$. We may assume that we have a local coordinate system (x,t) such that $X = \{(x,t); t = 0\}$ and P is a differential operator of order m as in Lemma 2.5.3. The complex $R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{N},\mathcal{O}_Y)$ is represented by the complex $0 \to \mathcal{O}_Y|_X \xrightarrow{P\cdot} \mathcal{O}_Y|_X \to 0$, where $\mathcal{O}_Y|_X$ on the left is in degree 0. Since $\mathcal{N}_{\mathcal{D}}^{-1} \simeq \mathcal{D}_X^m$, the complex $R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{N}_{\mathcal{D}}^{-1},\mathcal{O}_X)$ is represented by the complex $\mathcal{O}_X^m$ in degree 0. The morphism (2.25) reduces to the morphism

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_Y|_X \arrow[r, "P"] \arrow[d, "\gamma"] & \mathcal{O}_Y|_X \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_X^m \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

Here, the vertical arrow γ is the morphism which, to $f \in \mathcal{O}_Y|_X$ associates the first m traces of f

$$\gamma(f) = f|_X, \partial_t f|_X, \ldots, \partial_t^{m-1} f|_X.$$

Then the theorem asserts that P acting on $\mathcal{O}_Y|_X$ is an epimorphism and $\operatorname{Ker} P$ acting on this sheaf is isomorphic by γ to $\mathcal{O}_X^m$. This is the Cauchy-Kovalevski theorem.

(c) As in the proof of Theorem 2.5.5, we construct an exact sequence (2.18) $0 \to \mathcal{L} \to \mathcal{M} \to \mathcal{N} \to 0$ where $\mathcal{M}$ is a finite direct sum of modules of the type $\mathcal{D}_Y/\mathcal{D}_Y \cdot P$. let us apply the functor $R\mathcal{H}om_{\mathcal{D}_Y}(\bullet,\mathcal{O}_Y)$ to the sequence (2.18) and the functor $R\mathcal{H}om_{\mathcal{D}_X}(\bullet,\mathcal{O}_X)$ to the image by $(\overset{-1}{\mathcal{D}}\bullet)$ of the sequence (2.18). Let us set for short

$$\mathcal{S}\mathrm{ol}_Y(\bullet) := R\mathcal{H}om_{\mathcal{D}_Y}(\bullet,\mathcal{O}_Y)$$

and similarly with $\mathcal{S}\mathrm{ol}_X(\bullet)$. We find the morphism of distinguished triangles

$$\begin{tikzcd}
f^{-1}\mathcal{S}\mathrm{ol}_Y(\mathcal{N}) \arrow[r] \arrow[d] & f^{-1}\mathcal{S}\mathrm{ol}_Y(\mathcal{M}) \arrow[r] \arrow[d] & f^{-1}\mathcal{S}\mathrm{ol}_Y(\mathcal{L}) \arrow[r, "+1"] \arrow[d] & {} \\
\mathcal{S}\mathrm{ol}_Y(f_{\mathcal{D}}^{-1}\mathcal{N}) \arrow[r] & \mathcal{S}\mathrm{ol}_Y(f_{\mathcal{D}}^{-1}\mathcal{M}) \arrow[r] & \mathcal{S}\mathrm{ol}_Y(f_{\mathcal{D}}^{-1}\mathcal{L}) \arrow[r, "+1"] & {}
\end{tikzcd}$$

Let us apply the cohomology functor H^0 to this morphism of distinguished triangles. We find a morphism of long exact sequences

$$\begin{tikzcd}
0 \arrow[r] & H^0(A_1) \arrow[r] \arrow[d, "u_1^0"] & H^0(A_2) \arrow[r] \arrow[d, "u_2^0"] & H^0(A_3) \arrow[r] \arrow[d, "u_3^0"] & H^1(A_1) \arrow[r] \arrow[d, "u_1^1"] & \cdots \\
0 \arrow[r] & H^0(B_1) \arrow[r] & H^0(B_2) \arrow[r] & H^0(B_3) \arrow[r] & H^1(B_1) \arrow[r] & \cdots.
\end{tikzcd}$$

By (b), all morphisms $u_2^n, n \geq 0$ are isomorphisms. It follows that u_1^0 is a monomorphism, and the module $\mathcal{M}$ satisfying the non characteristicity hypothesis, the morphism u_3^0 is also a monomorphism. Therefore, u_1^0 is an isomorphism, hence u_3^0 is also an isomorphism. By induction, we get that all u_1^n are isomorphism. q.e.d.

2.6 Direct image

Good $\mathcal{D}$-modules

Definition 2.6.1. (i) Let $\mathcal{F} \in \mathrm{Mod}(\mathcal{O}_X)$. One says that $\mathcal{F}$ is good if for any relatively compact open subset $U \subset\subset X$, there exists a small and filtrant category I, an inductive system $\{\mathcal{F}_i\}_{i\in I}$ of coherent $\mathcal{O}_U$-modules and an isomorphism $\varinjlim_i \mathcal{F}_i \xrightarrow{\sim} \mathcal{F}|_U$.

(ii) One denotes by $\mathrm{Mod}_{\mathrm{gd}}(\mathcal{O}_X)$ the full subcategory of $\mathrm{Mod}(\mathcal{O}_X)$ consisting of good $\mathcal{O}_X$-modules.

(iii) A coherent $\mathcal{D}_X$-module $\mathcal{M}$ is good if it is good as an $\mathcal{O}_X$-module.

(iv) One denotes by $\mathrm{Mod}_{\mathrm{gd}}(\mathcal{D}_X)$ the full subcategory of $\mathrm{Mod}_{\mathrm{coh}}(\mathcal{D}_X)$ consisting of good $\mathcal{O}_X$-modules.

Note that $\mathcal{D}_X$ is good. For generally, if a coherent $\mathcal{D}_X$-module may be endowed with a good filtration, then it is good. However, there exist coherent $\mathcal{D}_X$-modules which are not good.

Lemma 2.6.2. *The category* $\mathrm{Mod}_{\mathrm{gd}}(\mathcal{O}_X)$ *is a thick abelian subcategory of the category* $\mathrm{Mod}_{\mathrm{coh}}(\mathcal{D}_X)$. *In particular, the full subcategory* $\mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathcal{D}_X)$ *of* $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X)$ *consisting of objects* $\mathcal{M}$ *such that* $H^j(\mathcal{M})$ *is good for all* j *is triangulated.*

Proof. For the proof, we refer to [22]. q.e.d.

Lemma 2.6.3. *Let* $\mathcal{M} \in \mathrm{Mod}_{\mathrm{coh}}(\mathcal{D}_X)$. *Then* $\mathcal{M}$ *is good if and only if, for any relatively compact open subset* $U \subset\subset X$, *there exists* $\mathcal{F} \subset \mathcal{M}|_U$ *with* $\mathcal{F} \in \mathrm{Mod}_{\mathrm{coh}}(\mathcal{O}_U)$ *and an epimorphism of* $\mathcal{D}_U$*-modules* $\mathcal{F} \otimes_{\mathcal{O}_U} \mathcal{D}_U \twoheadrightarrow \mathcal{M}|_U$.

Proof. After replacing X with a relatively compact open subset of X containing the closure of U, we may assume that $\mathcal{M} = \varinjlim_i \mathcal{F}_i$ where I is small and filtrant and $\mathcal{F}_i$ is $\mathcal{O}_X$-coherent. Set

$$\mathcal{L}_i = \mathrm{Im}(\mathcal{F}_i \otimes_{\mathcal{O}_X} \mathcal{D}_X \to \mathcal{M}).$$

Since $\mathcal{M}$ is $\mathcal{D}_X$-coherent, the family $\{\mathcal{L}_i\}_{i\in I}$ of coherent $\mathcal{D}_X$-modules is locally stationary hence is stationary on the closure of U. q.e.d.

Coherency

Theorem 2.6.4. *Let* $f\colon X \to Y$ *be a morphism of complex manifolds and let* $\mathcal{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathcal{D}_X^{\mathrm{op}})$. *Assume that* f *is proper on* $\mathrm{supp}(\mathcal{M})$. *Then*

(i) $f_*^{\mathcal{D}}\mathcal{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathcal{D}_Y^{\mathrm{op}})$,

(ii) $\mathrm{char}(f_*^{\mathcal{D}}\mathcal{M}) \subset f_\pi f_d(\mathrm{char}(\mathcal{M}))$.

(iii) *Moreover, if* f *is finite on* $\mathrm{supp}(\mathcal{M})$, *the above inclusion is an equality.*

Proof. (i)–(a) By "dévissage", we reduce to the case where $\mathcal{M}$ is a good $\mathcal{D}_X$-module. More precisely, assume $H^j(\mathcal{M}) = 0$ for $j \notin [j_0, j_1]$ and the result has been proved for modules with amplitude $j_1 - j_0 - 1$. Consider the distinguished triangle

$$H^{j_0}(\mathcal{M})[-j_0] \to \mathcal{M} \to \tau^{>j_0}(\mathcal{M}) \xrightarrow{+1}$$

Applying the functor $f_!^{\mathcal{D}}$ to this d.t., we get the d.t.:

$$f_!^{\mathcal{D}}(H^{j_0}(\mathcal{M}))[-j_0] \to f_!^{\mathcal{D}}\mathcal{M} \to f_!^{\mathcal{D}}(\tau^{>j_0}(\mathcal{M})) \xrightarrow{+1}$$

It follows from the induction hypothesis and Lemma 2.6.2 that $f_!^{\mathcal{D}}\mathcal{M}$ belongs to $\mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathcal{D}_Y^{\mathrm{op}})$.

(i)–(b) First, assume that $\mathcal{M} \simeq \mathcal{F} \otimes_{\mathcal{O}} \mathcal{D}_X$ for a coherent $\mathcal{O}_X$-module $\mathcal{F}$ and f is proper on $\mathrm{supp}(\mathcal{F})$. Then

$$\begin{aligned} f_*^{\mathcal{D}}\mathcal{M} &\simeq Rf_!(\mathcal{F} \otimes_{\mathcal{O}} \mathcal{D}_X \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y}) \\ &\simeq Rf_!(\mathcal{F} \otimes_{\mathcal{O}} \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y) \\ &\simeq Rf_!\mathcal{F} \otimes_{\mathcal{O}} \mathcal{D}_Y. \end{aligned}$$

The coherence of $Rf_!\mathcal{F}$ follows from Grauert's theorem.

(i)–(c) Since the problem is local on Y and f is proper on $\mathrm{supp}(\mathcal{M})$, we may assume by Lemma 2.6.3 that there exists an exact sequence in $\mathrm{Mod}(\mathcal{D}_X^{\mathrm{op}})$:

$$0 \to \mathcal{M}' \to \mathcal{F} \otimes_{\mathcal{O}} \mathcal{D}_X \to \mathcal{M} \to 0$$

and f is proper on $\mathrm{supp}(\mathcal{F})$. We apply the functor $f_!^{\mathcal{D}}$ to this sequence and take the cohomology. Setting $\mathcal{L} = \mathcal{F} \otimes_{\mathcal{O}} \mathcal{D}_X$ we find a long exact sequence

$$\cdots \to H^j(f_!^{\mathcal{D}}\mathcal{M}') \to H^j(f_!^{\mathcal{D}}\mathcal{L}) \to H^j(f_!^{\mathcal{D}}\mathcal{M}) \to H^{j+1}(f_!^{\mathcal{D}}\mathcal{M}') \to \cdots .$$

Assume $H^j(f_!^{\mathcal{D}}\mathcal{M})$ is good for all $\mathcal{M}$ and all $j > j_0$. Set

$$\mathcal{K}^j := \mathrm{Ker}(H^{j_0+1}(f_!^{\mathcal{D}}\mathcal{M}') \to H^{j_0+1}(f_!^{\mathcal{D}}\mathcal{L})).$$

Then $\mathcal{K}^j$ is good. Moreover, we have an exact sequence

$$H^j(f_!^{\mathcal{D}}\mathcal{L}) \to H^j(f_!^{\mathcal{D}}\mathcal{M}) \to \mathcal{K}^j \to 0$$

from which we deduce that $H^j(f_!^{\mathcal{D}}\mathcal{M})$ is locally finitely generated over $\mathcal{D}_Y^{\mathrm{op}}$. Set

$$\mathcal{R}^j := \mathrm{Coker}\, H^j(f_!^{\mathcal{D}}\mathcal{M}') \to H^j(f_!^{\mathcal{D}}\mathcal{L}).$$

Being a quotient of a good $\mathcal{D}_Y^{\mathrm{op}}$-module by a finitely generated module, it is a good $\mathcal{D}_Y^{\mathrm{op}}$-module. By the exact sequence

$$0 \to \mathcal{R}^j \to H^j(f_!^{\mathcal{D}}\mathcal{M}) \to \mathcal{K}^j \to 0$$

we conclude that $H^{j_0}(f_!^{\mathcal{D}}\mathcal{M})$ is a good $\mathcal{D}_X$-module and the induction proceeds.

(ii)–(iii) We shall not give the proofs here. q.e.d.

Example 2.6.5. (i) Assume X is compact and let $\mathcal{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathcal{D}_X^{\mathrm{op}})$. Denote by a_X the projection $X \to \{\mathrm{pt}\}$. Then ${a_X}_*^{\mathcal{D}}\mathcal{M} \simeq \mathrm{R}\Gamma(X; \mathcal{M}\overset{\mathrm{L}}{\otimes}_{\mathcal{D}}\mathcal{O}_X)$ and for all $j \in \mathbb{Z}$, $H^j(\mathrm{R}\Gamma(X; \mathcal{M}\overset{\mathrm{L}}{\otimes}_{\mathcal{D}}\mathcal{O}_X)$ is a finite-dimensional $\mathbb{C}$-vector space.
(ii) Let $f\colon X \to Y$ be a proper map and assume that Y is a curve (*i.e.,* $d_Y = 1$). The object $f_!^{\mathcal{D}}\mathcal{O}_X$ is called the Gauss-Manin connection on Y associated with f. It is of particular importance when f is finite (hence, X is again a curve). Note that the characteristic variety of the Gauss-Manin connection satisfies

$$\begin{aligned}
\mathrm{char}(f_!^{\mathcal{D}}\mathcal{O}_X) &\subset f_\pi f_d^{-1}(T^*_X X) \\
&= \{(y;\eta) \in T^*Y;\ \text{there exist } x \in X \text{ with } f_d(x)\eta = 0\}.
\end{aligned}$$

In other words, this characteristic variety is contained in the union of the zero-section of T^*Y and the conormal bundles to the points $y \in Y$ which are critical values of f.

We state without proof an important result due to Kashiwara.

Theorem 2.6.6. *Let $j\colon Z \hookrightarrow X$ be a closed embedding of a smooth manifold. Then the functor $j_*^{\mathcal{D}}$ induces an equivalence of categories* $\mathrm{Mod}(\mathcal{D}_Z) \xrightarrow{\sim} \mathrm{Mod}_Z(\mathcal{D}_X)$*, where* $\mathrm{Mod}_Z(\mathcal{D}_X)$ *denotes the full abelian subcategory of* $\mathrm{Mod}(\mathcal{D}_X)$ *consisting of objects with support contained in Z. Moreover, this equivalence induces an equivalence of the subcategories consisting of coherent modules.*

A quasi-inverse functor to $j_*^{\mathcal{D}}$ is given by $j^{-1}\mathcal{H}om_{\mathcal{D}}(\mathcal{D}_{X\leftarrow Z}, \bullet)$.
Although we do not give its proof here, the next result will be used in the sequel.

Theorem 2.6.7. Projection formula for $\mathcal{D}$-modules *Let $f\colon X \to Y$ be a morphism of complex manifolds. Let $\mathcal{M} \in \mathrm{D}^{\mathrm{b}}(\mathcal{D}_X^{\mathrm{op}})$ and let $\mathcal{N} \in \mathrm{D}^{\mathrm{b}}(\mathcal{D}_Y)$. There is a natural isomorphism in* $\mathrm{D}^{\mathrm{b}}(\mathcal{D}_Y)$

$$f_!^{\mathcal{D}}(\mathcal{M}\overset{\mathrm{D}}{\otimes} f_{\mathcal{D}}^{-1}\mathcal{N}) \simeq f_!^{\mathcal{D}}\mathcal{M}\overset{\mathrm{D}}{\otimes}\mathcal{N}. \tag{2.26}$$

Trace morphism

Theorem 2.6.8. *For each morphism of complex manifoldsf$\colon X \to Y$ there exists a "trace morphism" in* $\mathrm{D}^{\mathrm{b}}(\mathcal{D}_Y^{\mathrm{op}})$

$$\mathrm{tr}_f\colon f_!^{\mathcal{D}}\Omega_X\,[d_X] \to \Omega_Y\,[d_Y] \tag{2.27}$$

with the following properties:

(i) tr_f *is functorial in* f, *that is,* $\mathrm{tr}_{\mathrm{id}_X} = \mathrm{id}$ *and* $\mathrm{tr}_{g\circ f} = \mathrm{tr}_g \circ \mathrm{tr}_f$ *for morphisms* $X \xrightarrow{f} Y \xrightarrow{g} Z$,

(ii) *when* X *is a curve and* $Y = \{\mathrm{pt}\}$, tr_f *induces the residues morphism on* $H^1_c(X;\Omega_X)$.

Using the direct images functor for left $\mathcal{D}$-modules, (2.27) gives the functorial morphism

$$\mathrm{tr}_f\colon f_!^{\mathcal{D}}\mathcal{O}_X\,[d_X] \to \mathcal{O}_Y\,[d_Y]. \tag{2.28}$$

Proof. Recall that $\Omega_X[-d_X]$ is quasi-isomorphic in $\mathrm{D^b}(\mathcal{D}_X^{\mathrm{op}})$ to the De Rham complex $\mathrm{DR_X}(\mathcal{D}_\mathrm{X})$ (see (1.21)):

$$\mathrm{DR_X}(\mathcal{D}_\mathrm{X}) := \quad 0 \to \Omega^0_\mathrm{X} \otimes_{\mathcal{O}} \mathcal{D}_\mathrm{X} \xrightarrow{\mathrm{d}} \cdots \to \Omega_\mathrm{X} \otimes_{\mathcal{O}} \mathcal{D}_\mathrm{X} \to 0,$$

where the differential d is characterized by:

$$d(\omega\otimes m) = d\omega\otimes P + (-)^p\omega\wedge dP, \quad \omega\in\Omega^p_X, P\in\mathcal{D}_X$$

and $dP = \sum_i dx_i\otimes\partial_i\circ P$ in a local coordinate system.

Let us identify $X_\mathbb{R}$, the real analytic manifold underlying the complex manifold X with the diagonal of $X\times\overline{X}$. Hence, the real tangent bundle $TX_\mathbb{R}$ is isomorphic to $TX\times_{X_\mathbb{R}} T\overline{X}$ and the differential $d_{X_\mathbb{R}}$ splits as

$$d_{X_\mathbb{R}} = \partial\oplus\overline{\partial}.$$

Denote by $\mathcal{D}b_{X_\mathbb{R}}$ the sheaf of distributions on the real analytic manifold $X_\mathbb{R}$. The sheaf Ω^p_X is quasi-isomorphic to the Dolbeault complex

$$0\to\mathcal{D}b^{(p,0)}_{X_\mathbb{R}} \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \mathcal{D}b^{(p,d_X)}_{X_\mathbb{R}} \to 0,$$

where $\mathcal{D}b^{(p,q)}_{X_\mathbb{R}}$ is the sheaf of forms of type (p,q) with coefficients in $\mathcal{D}b_{X_\mathbb{R}}$. It follows that there is a qis

$$\Omega_X[-d_X] \to \mathcal{D}b^{\bullet,\bullet}_{X_\mathbb{R}} \otimes_{\mathcal{O}} \mathcal{D}_X, (\partial,\overline{\partial}) \tag{2.29}$$

where the bidifferential $(\partial,\overline{\partial})$ satisfies

$$\begin{aligned}
\partial(u\otimes P) &= \partial u\otimes P + (-)^p u\wedge dP, && (2.30)\\
\overline{\partial}(u\otimes P) &= \overline{\partial}u\otimes P. && (2.31)
\end{aligned}$$

Denote by $\mathcal{C}^{\infty(p,q)}_{X_\mathbb{R}}$ the sheaf of forms of type (p,q) with coefficients in the sheaf $\mathcal{C}^{\infty}_{X_\mathbb{R}}$ of complex valued $\mathcal{C}^\infty$-functions on $X_\mathbb{R}$. There is a natural morphism

$$f^*\colon f^{-1}\mathcal{C}^{\infty(p,q)}_{Y_\mathbb{R}} \to \mathcal{C}^{\infty(p,q)}_{X_\mathbb{R}}. \tag{2.32}$$

Since $\Gamma_c(X;\mathcal{D}b^{(p+d_X,q+d_X)}_{X_\mathbb{R}})$ is the dual of the space $\Gamma_c(X;\mathcal{C}^{\infty(p,q)}_{X_\mathbb{R}})$, the morphism (2.32) defines the morphism

$$\int_f\colon f_!\mathcal{D}b^{(p+d_X,q+d_X)}_{X_\mathbb{R}} \to \mathcal{D}b^{(p+d_Y,q+d_Y)}_{Y_\mathbb{R}}. \tag{2.33}$$

Moreover, $\int_f$ commutes with $\overline{\partial}$ and ∂.

The object $\Omega_X[d_X]\overset{\mathrm{L}}{\otimes}_{\mathcal{D}}\mathcal{D}_{X\to Y}$ of $\mathrm{D^b}(\mathcal{D}_X^{\mathrm{op}})$ is isomorphic to the complex $\mathcal{D}b^{\bullet,\bullet}_{X_\mathbb{R}}\otimes_{\mathcal{O}} f^{-1}\mathcal{D}_Y\,[2d_X]$ where $\overline{\partial}(u\otimes P)=\overline{\partial}u\otimes P$ and the action of ∂ is given by (2.30) and (2.11). Noticing that the sheaves $\mathcal{D}b^{(p,q)}_{X_\mathbb{R}}$ are soft, we get the chain of morphisms and isomorphisms

$$\begin{aligned}
f^{\mathcal{D}}_!\Omega_X\,[d_X] &\simeq f_!(\mathcal{D}b^{\bullet,\bullet}_{X_\mathbb{R}}\otimes_{\mathcal{O}}\mathcal{D}_X\otimes_{\mathcal{D}}\mathcal{O}_X\otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y)[2d_X]\\
&\simeq f_!(\mathcal{D}b^{\bullet,\bullet}_{X_\mathbb{R}}\otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y)[2d_X]\\
&\xrightarrow{\int_f} \mathcal{D}b^{\bullet,\bullet}_{Y_\mathbb{R}}\otimes_{\mathcal{O}}\mathcal{D}_Y[2d_Y]\\
&\simeq \Omega_Y\,[d_Y].
\end{aligned}$$

The properties (i) and (ii) of the morphism tr_f are easily checked. q.e.d.

Corollary 2.6.9. *Let $\mathcal{N}\in\mathrm{D^b}(\mathcal{D}_Y)$. There exists a canonical morphism in $\mathrm{D^b}(\mathcal{D}_Y)$:*

$$f^{\mathcal{D}}_!(f^{-1}_{\mathcal{D}}\mathcal{N}\otimes_{\mathcal{O}}\Omega_X\,[d_X])\to\mathcal{N}\otimes_{\mathcal{O}}\Omega_Y\,[d_Y]. \tag{2.34}$$

Proof. By Theorem 2.6.7, we have an isomorphism

$$\begin{aligned}
f^{\mathcal{D}}_!(f^{-1}_{\mathcal{D}}\mathcal{N}\otimes_{\mathcal{O}}\Omega_X\,[d_X]) &= f^{\mathcal{D}}_!(f^{-1}_{\mathcal{D}}\mathcal{N}\overset{\mathrm{D}}{\otimes}\Omega_X\,[d_X])\\
&\simeq \mathcal{N}\overset{\mathrm{D}}{\otimes}f^{\mathcal{D}}_!\Omega_X\,[d_X].
\end{aligned}$$

To conclude, apply the trace morphism $f^{\mathcal{D}}_!\Omega_X\,[d_X]\to\Omega_Y\,[d_Y]$. q.e.d.

Corollary 2.6.10. *Let $\mathcal{M}\in\mathrm{D^b}(\mathcal{D}_X)$ and let $\mathcal{N}\in\mathrm{D^b}(\mathcal{D}_Y)$. There is a canonical morphism*

$$Rf_*R\mathcal{H}om_{\mathcal{D}}(\mathcal{M},f^{-1}_{\mathcal{D}}\mathcal{N})\,[d_X]\to R\mathcal{H}om_{\mathcal{D}}(f^{\mathcal{D}}_!\mathcal{M},\mathcal{N})\,[d_Y]. \tag{2.35}$$

Proof. Consider the chain of morphisms

$$\begin{aligned}
&Rf_* R\mathcal{H}om_{\mathcal{D}}(\mathcal{M}, f_{\mathcal{D}}^{-1}\mathcal{N})\,[d_X] \\
&\qquad \to Rf_* R\mathcal{H}om_{\mathcal{D}}(\mathcal{D}_{Y\leftarrow X} \otimes_{\mathcal{D}} \mathcal{M}, \mathcal{D}_{Y\leftarrow X} \otimes_{\mathcal{D}} f_{\mathcal{D}}^{-1}\mathcal{N})\,[d_X] \\
&\qquad \to R\mathcal{H}om_{\mathcal{D}}(Rf_!(\mathcal{D}_{Y\leftarrow X} \otimes_{\mathcal{D}} \mathcal{M}), Rf_*(\mathcal{D}_{Y\leftarrow X} \otimes_{\mathcal{D}} f_{\mathcal{D}}^{-1}\mathcal{N}))\,[d_X] \\
&\qquad \simeq R\mathcal{H}om_{\mathcal{D}}(f_!^{\mathcal{D}}\mathcal{M}, f_!^{\mathcal{D}} f_{\mathcal{D}}^{-1}\mathcal{N})\,[d_X] \\
&\qquad \to R\mathcal{H}om_{\mathcal{D}}(f_!^{\mathcal{D}}\mathcal{M}, \mathcal{N})\,[d_Y]
\end{aligned}$$

where the last morphism follows from (2.34). q.e.d.

Duality and direct images

Let again $f\colon X \to Y$ be a morphism of complex manifolds.

Lemma 2.6.11. *Let $\mathcal{M} \in \mathrm{D}^{\mathrm{b}}(\mathcal{D}_X^{\mathrm{op}})$. There is a canonical morphism in $\mathrm{D}^{\mathrm{b}}(\mathcal{D}_Y^{\mathrm{op}})$:*

$$f_!^{\mathcal{D}}\mathbb{D}_{\mathcal{D}}\mathcal{M} \to \mathbb{D}_{\mathcal{D}} f_!^{\mathcal{D}}\mathcal{M}. \tag{2.36}$$

Proof. By choosing $\mathcal{N} = \mathcal{D}_Y$ in Corollary 2.6.10, we get the chain of morphisms

$$\begin{aligned}
f_!^{\mathcal{D}}\mathbb{D}_{\mathcal{D}}\mathcal{M} &= Rf_!(R\mathcal{H}om_{\mathcal{D}}(\mathcal{M}, \mathcal{D}_X \otimes_{\mathcal{O}} \Omega_X\,[d_X]) \otimes_{\mathcal{D}} \mathcal{D}_{X\to Y}) \\
&\to Rf_!(R\mathcal{H}om_{\mathcal{D}}(\mathcal{M}, \Omega_X \otimes_{\mathcal{O}} \mathcal{D}_{X\to Y})\,[d_X] \\
&\simeq Rf_!(R\mathcal{H}om_{\mathcal{D}}(\mathcal{M}, f_{\mathcal{D}}^{-1}\Omega_Y)\,[d_X]) \\
&\to R\mathcal{H}om_{\mathcal{D}}(f_!^{\mathcal{D}}\mathcal{M}, \mathcal{D}_Y \otimes_{\mathcal{O}} \Omega_Y)\,[d_Y] \\
&= \mathbb{D}_{\mathcal{D}} f_!^{\mathcal{D}}\mathcal{M}.
\end{aligned}$$

q.e.d.

Theorem 2.6.12. *Let $\mathcal{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathcal{D}_X^{\mathrm{op}})$ and assume that f is proper on* $\mathrm{supp}(\mathcal{M})$. *Then the morphism* (2.36) *is an isomorphism.*

Proof. We may reduce to the case where $\mathcal{M} \in \mathrm{Mod}_{\mathrm{gd}}(\mathcal{D}_X^{\mathrm{op}})$ and, as in the proof of Theorem 2.6.4, that $\mathcal{M} = \mathcal{F} \otimes_{\mathcal{O}} \mathcal{D}_X$ for a coherent $\mathcal{O}_X$-module $\mathcal{F}$. In this case,

$$\begin{aligned}
f_!^{\mathcal{D}}\mathbb{D}_{\mathcal{D}}\mathcal{M} &\simeq Rf_!(R\mathcal{H}om_{\mathcal{D}}(\mathcal{F} \otimes_{\mathcal{O}} \mathcal{D}_X, \mathcal{D}_{Y\leftarrow X} \otimes_{\mathcal{O}} f^{-1}\Omega_Y)\,[d_X] \\
&\simeq Rf_! R\mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{O}_X) \otimes_{\mathcal{O}} \mathcal{D}_Y\,[d_X] \otimes_{\mathcal{O}} \Omega_Y \\
&\simeq R\mathcal{H}om_{\mathcal{O}}(Rf_!\mathcal{F}, \mathcal{O}_Y) \otimes_{\mathcal{O}} \mathcal{D}_Y \otimes_{\mathcal{O}} \Omega_Y\,[d_Y] \\
&\simeq R\mathcal{H}om_{\mathcal{D}}(Rf_!\mathcal{F} \otimes_{\mathcal{O}} \mathcal{D}_Y, \mathcal{D}_Y) \otimes_{\mathcal{O}} \Omega_Y\,[d_Y] \\
&\simeq \mathbb{D}_{\mathcal{D}} f_!^{\mathcal{D}}\mathcal{M}.
\end{aligned}$$

Here, we have used the fact that proper direct images commute with duality for $\mathcal{O}$-modules (Theorem 2.1.2). q.e.d.

Theorem 2.6.13. *Let* $\mathcal{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathcal{D}_X^{\mathrm{op}})$ *and assume that* f *is proper on* $\mathrm{supp}(\mathcal{M})$. *Then the morphism* (2.34) *is an isomorphism.*

Proof. Since $\mathcal{M}$ and $f_!^{\mathcal{D}}\mathcal{M}$ have coherent cohomologies, we have the isomorphisms

$$\begin{aligned} R\mathcal{H}om_{\mathcal{D}}(\mathcal{M}, f_{\mathcal{D}}^{-1}\mathcal{N}) &\simeq R\mathcal{H}om_{\mathcal{D}}(\mathcal{M}, \mathcal{D}_{X\leftarrow Y})\overset{\mathrm{L}}{\otimes}_{f^{-1}\mathcal{D}_Y} f^{-1}\mathcal{N}, \\ R\mathcal{H}om_{\mathcal{D}}(f_!^{\mathcal{D}}\mathcal{M}, \mathcal{N}) &\simeq R\mathcal{H}om_{\mathcal{D}}(f_!^{\mathcal{D}}\mathcal{M}, \mathcal{D}_Y)\overset{\mathrm{L}}{\otimes}_{\mathcal{D}}\mathcal{N}. \end{aligned}$$

Hence, we are reduced to prove the result when $\mathcal{N} = \mathcal{D}_Y$, and it follows immediately from Theorem 2.6.12. q.e.d.

Corollary 2.6.14. *Let* $\mathcal{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathcal{D}_X)$ *and assume* f *is proper on* $\mathrm{supp}(\mathcal{M})$. *There is a canonical isomorphism*

$$Rf_* R\mathcal{H}om_{\mathcal{D}}(\mathcal{M}, \mathcal{O}_X)\,[d_X] \xrightarrow{\sim} R\mathcal{H}om_{\mathcal{D}}(f_!^{\mathcal{D}}\mathcal{M}, \mathcal{O}_Y)\,[d_Y].$$

Exercises to Chapter 2

Exercise 2.1. Let $f\colon X \to Y$ be a morphism of complex manifolds and let Z be a smooth closed submanifold of Y. Assume that f is transversal to Z, that is, f is non characteristic for $T^*_Z Y$, or, equivalently, for $\mathcal{B}_{Z|Y}$. Prove that $S := f^{-1}Z$ is a smooth closed submanifold of X and that $f_{\mathcal{D}}^{-1}\mathcal{B}_{Z|Y} \simeq \mathcal{B}_{S|X}$.

Exercise 2.2. Let Z_1 and Z_2 be smooth submanifolds of X and assume they are transversal. Calculate $\mathcal{B}_{Z_1|X}\overset{\mathrm{D}}{\otimes}\mathcal{B}_{Z_2|X}$.

Exercise 2.3. Denote by $j\colon Z \hookrightarrow X$ the closed embedding of a smooth submanifold Z of X. Prove that $\mathcal{B}_{Z|X} \simeq j_!^{\mathcal{D}}\mathcal{O}_Z$.

Exercise 2.4. Let $\mathcal{M} \in \mathrm{Mod}_{\mathrm{coh}}(\mathcal{D}_X)$ and assume that $\mathrm{char}(\mathcal{M}) \subset T^*_X X$. Prove that locally on X, there is an isomorphism of $\mathcal{D}_X$-modules $\mathcal{M} \simeq \mathcal{O}_X^N$ for some integer N. (Hint: see [22, Prop. 4.43]).

Exercise 2.5. Let $f\colon X \to Y$ be a morphism of complex manifolds. Let $\mathcal{M} \in \mathrm{D}^{\mathrm{b}}(\mathcal{D}_X^{\mathrm{op}})$ and let $\mathcal{N} \in \mathrm{D}^{\mathrm{b}}(\mathcal{D}_Y)$. Prove that there is a natural isomorphism in $\mathrm{D}^{\mathrm{b}}(\mathbb{C}_Y)$

$$Rf_!(\mathcal{M}\overset{\mathrm{L}}{\otimes}_{\mathcal{D}} f_{\mathcal{D}}^{-1}\mathcal{N}) \simeq f_!^{\mathcal{D}}\mathcal{M}\overset{\mathrm{L}}{\otimes}_{\mathcal{D}}\mathcal{N}.$$

Exercise 2.6. Let X and Y be two complex manifolds and denote by q_i the i-th projection defined on $X \times Y$ and by p_i the i-th projection defined on $T^*X \times T^*Y$ $(i = 1, 2)$. Let $\mathcal{M} \in \mathrm{D}^{\mathrm{b}}(\mathcal{D}_X)$ and $\mathcal{L} \in \mathrm{D}^{\mathrm{b}}(\mathcal{D}^{\mathrm{op}}_{X\times Y})$.
(i) Prove the isomorphism

$$\mathcal{L} \circ \mathcal{M} := {q_{2!}}^{\mathcal{D}}(\mathcal{L} \overset{\mathrm{D}}{\otimes} {q_1}_{\mathcal{D}}^{-1}\mathcal{M}) \simeq Rq_{2!}(\mathcal{L} \overset{\mathrm{L}}{\otimes}_{\mathcal{D}} q_1^{-1}\mathcal{M}).$$

(ii) Assume now that $\mathcal{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathcal{D}_X)$, $\mathcal{L} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathcal{D}^{\mathrm{op}}_{X\times Y})$ and that p_2 is proper on $p_1^{-1}\mathrm{char}(\mathcal{M}) \cap \mathrm{char}(\mathcal{L})$. Prove that $p_1^{-1}\mathrm{char}(\mathcal{M}) \cap \mathrm{char}(\mathcal{L}) \subset T^*_{X\times Y}X \times Y$ and that $\mathcal{L} \circ \mathcal{M}\mathcal{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathcal{D}^{\mathrm{op}}_Y)$.
(iii) Show that the construction of the inverse or direct image of a $\mathcal{D}$-module can be obtained by this procedure.

Chapter 3

$\mathbb{DQ}$-algebras and $\mathbb{DQ}$-modules

The results of Section 3.1 concerning microdifferential operators are due to [37]. The results of sections 3.2 and 3.3 are extracted from [25].

3.1 Construction of $\widehat{\mathscr{E}}_{T^*X}$ and $\widehat{\mathscr{W}}_{T^*X}$

Assume X is open in a finite dimensional complex vector space E. In this case, the total symbol of a differential operator P is a section of $\pi_*\mathscr{O}_{T^*X}$, polynomial in the fiber variable.

If order to invert $\sigma_{\mathrm{tot}}(P)(x;\xi)$ on an open set on which its principal symbol does not vanish, one is lead to consider the new ring

$$\widehat{\mathscr{S}}_{T^*X} = \varinjlim_{m\in\mathbb{Z}} \prod_{-\infty<j\leq m} \mathscr{O}_{T^*X}(j)$$

where $\mathscr{O}_{T^*X}(j)$ denotes the subsheaf of $\mathscr{O}_{T^*X}$ consisting of section homogeneous of degree j in the fiber variable. Hence a section f of $\widehat{\mathscr{S}}_{T^*X}$ on an open subset U of T^*X is a formal series

$$f(x;\xi) = \sum_{-\infty<j\leq m} f_j(x;\xi)$$

where $f_j \in \Gamma(U;\mathscr{O}_{T^*X})$ is homogeneous of degree j in ξ.

If f_m is not identically zero, we say f is of order m and call f_m the principal symbol of f. We set

$$\sigma(f) = f_m.$$

One extends the product (1.8) to this new sheaf. Hence, we define the algebra $(\widehat{\mathscr{S}}_{T^*X}, \star)$ by setting for for $f, g \in \widehat{\mathscr{S}}_{T^*X}$

$$(f\star g)(x,\xi) = \sum_{\alpha\in\mathbb{N}^n} \frac{1}{\alpha!}\partial_\xi^\alpha(f(x;\xi))\partial_x^\alpha(g(x;\xi)). \tag{3.1}$$

We call this product $\star$, the Leibniz product. The next result is obvious.

Proposition 3.1.1. *The sheaf* $\widehat{\mathscr{S}_{T^*X}}$ *endowed with the Leibniz product given by* (3.1) *is a sheaf of filtered unitary* $\mathbb{C}$*-algebras.*

One denotes by $\widehat{\mathscr{S}_{T^*X}}(m)$ the subring of $\widehat{\mathscr{S}_{T^*X}}$ consisting of sections of order $\leq m$.

Proposition 3.1.2. *Let* U *be an open subset of* T^*X *and let* $f \in \Gamma(U; \widehat{\mathscr{S}_{T^*X}})$. *Assume that* $\sigma(f)(x;\xi) \neq 0$ *for any* $(x;\xi) \in U$. *Then* f *is invertible in* $\Gamma(U; \widehat{\mathscr{S}_{T^*X}})$.

Proof. We may assume U is connected and f is of order $m \in \mathbb{Z}$. The operator g with total symbol f_m^{-1} is well defined. By Proposition 3.1.1, the operator $g \circ f$ has order 0 and principal symbol 1. Hence

$$g \circ f = 1 - r$$

where $r \in \widehat{\mathscr{S}_{T^*X}}(-1)$. Therefore the series $\sum_{j=0}^{\infty} r^j$ is well defined in $\widehat{\mathscr{S}_{T^*X}}(0)$ and is a right and left inverse of $g \circ f$, which shows that f admits a left inverse. One proves similarly that f admits a right inverse. q.e.d.

Theorem 3.1.3. (Sato-Kashiwara-Kawai) *For each complex manifold* X, *there exists a filtered sheaf* $\widehat{\mathscr{E}}_{T^*X}$ *of unitary* $\mathbb{C}$*-algebras together with*

(i) *a natural isomorphism* $\operatorname{gr} \widehat{\mathscr{E}}_{T^*X} \simeq \bigoplus_{m \in \mathbb{Z}} \mathscr{O}_{T^*X}(m)$,

(ii) *a monomorphism of algebras* $\pi^{-1}\mathcal{D}_X \hookrightarrow \widehat{\mathscr{E}}_{T^*X}$,

(iii) *and, when* X *is affine, a canonical isomorphism of filtered rings*

$$\sigma_{\mathrm{tot}} \colon \widehat{\mathscr{E}}_{T^*X} \xrightarrow{\sim} (\widehat{\mathscr{S}_{T^*X}}, \star),$$

such that (i) *and* (ii) *are induced by* (iii).

In practice, when X is affine, one identifies these two sheaves.

Proposition 3.1.4. *Let* X *be a complex manifold.*

(a) *The sheaf of rings* $\widehat{\mathscr{E}}_{T^*X}$ *is right and left Noetherian.*

(b) *The ring* $\widehat{\mathscr{E}}_{T^*X}$ *is flat over* $\pi^{-1}\mathcal{D}_X$ *and over* $\widehat{\mathscr{E}}_{T^*X}(0)$.

Proof. These properties are local and we may assume that X is affine. Then they follows from the corresponding properties on $\operatorname{gr} \widehat{\mathscr{E}}_{T^*X}$ and general results on filtered and graded rings. q.e.d.

Characteristic variety of $\widehat{\mathscr{E}}_{T^*X}$-modules

One defines the notion of good filtrations on coherent $\widehat{\mathscr{E}}_{T^*X}$-modules similarly as for $\mathcal{D}_X$-modules. For a coherent $\widehat{\mathscr{E}}_{T^*X}$-module endowed with a good filtration, we set

$$\widetilde{\mathrm{gr}\,(\mathcal{M})} := \mathscr{O}_{T^*X} \otimes_{\mathrm{gr}\,\widehat{\mathscr{E}}_{T^*X}} \mathrm{gr}\,(\mathcal{M}).$$

Outside of the zero section T^*_XX, to give a good filtration on a coherent $\widehat{\mathscr{E}}_{T^*X}$-module $\mathcal{M}$ is equivalent to give a coherent $\widehat{\mathscr{E}}_{T^*X}(0)$-module $\mathcal{M}_0 \subset \mathcal{M}$ which generates $\mathcal{M}$.

Theorem 3.1.5. *Let $\mathcal{M}$ be a coherent $\widehat{\mathscr{E}}_{T^*X}$-module defined on an open subset U of T^*X. Assume $\mathcal{M}$ is endowed with a good filtration.*

(i) *One has* $\mathrm{supp}(\mathcal{M}) = \mathrm{supp}(\widetilde{\mathrm{gr}\,(\mathcal{M})})$.

(ii) $\mathrm{supp}(\mathcal{M})$ *is a closed $\mathbb{C}^\times$-conic complex analytic subset of $U \subset T^*X$.*

(iii) $\mathrm{supp}(\mathcal{M})$ *is an involutive subset of $U \subset T^*X$.*

Proof. The inclusion $\mathrm{supp}(\mathcal{M}) \supset \mathrm{supp}(\widetilde{\mathrm{gr}\,(\mathcal{M})})$ is obvious. The converse inclusion comes from the fact that the filtration of $\mathcal{M}$ is separated. q.e.d.

Corollary 3.1.6. *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. Then*

$$\mathrm{char}(\mathcal{M}) = \mathrm{supp}(\widehat{\mathscr{E}}_{T^*X} \otimes_{\pi^{-1}\mathcal{D}_X} \pi^{-1}\mathcal{M}).$$

Construction of $\widehat{\mathscr{W}}_{T^*X}$

Notation 3.1.7. We denote by $\mathbf{k}_0$ the ring $\mathbb{C}[[\hbar]]$ of formal power series in an indeterminate $\hbar$ and by $\mathbf{k}$ the field $\mathbb{C}((\hbar))$ of Laurent series in $\hbar$. Then $\mathbf{k}$ is the fraction field of $\mathbf{k}_0$.

We set

$$\begin{aligned}
\mathcal{O}_X[[\hbar]] &:= \varprojlim_n \mathcal{O}_X \otimes (\mathbf{k}_0/\hbar^n\mathbf{k}_0) \simeq \prod_{n\geq 0} \mathcal{O}_X\hbar^n,\\
\mathcal{O}_X((\hbar)) &:= \mathbf{k} \otimes_{\mathbf{k}_0} \mathcal{O}_X[[\hbar]].
\end{aligned}$$

Recall that a section f of $\widehat{\mathscr{S}}_{T^*X}$ on an open subset U of T^*X is a formal series

$$f(x;\xi) = \sum_{-\infty<j\leq m} f_j(x;\xi)$$

where $f_j \in \Gamma(U; \mathscr{O}_{T^*X})$ is homogeneous of degree j in ξ. Let us introduce a formal parameter $\hbar$ and set

$$u_i = \hbar\xi_i, (i = 1, \dots, n).$$

By this change of variable, we may embed the sheaf $\widehat{\mathscr{S}}_{T^*X}$ into the sheaf $\mathscr{O}_{T^*X}((\hbar))$. The Leibniz product on $\widehat{\mathscr{S}}_{T^*X}$ defines the star product on $\mathcal{O}_{T^*X}((\hbar))$:

$$f \star g \;=\; \sum_{\alpha\in\mathbb{N}^n} \frac{\hbar^{|\alpha|}}{\alpha!}(\partial_u^\alpha f)(\partial_x^\alpha g). \tag{3.2}$$

Theorem 3.1.8. *For each complex manifold X, there exists canonically a filtered sheaf $\widehat{\mathscr{W}}_{T^*X}$ of unitary $\mathbf{k}$-algebras together with*

(i) *a natural isomorphism* $\operatorname{gr}\widehat{\mathscr{W}}_{T^*X} \simeq \mathscr{O}_{T^*X}[\hbar^{-1}, \hbar]$,

(ii) *monomorphisms of $\mathbb{C}$-algebras*

$$\pi^{-1}\mathcal{D}_X \hookrightarrow \widehat{\mathscr{E}}_{T^*X} \hookrightarrow \widehat{\mathscr{W}}_{T^*X}, \tag{3.3}$$

(iii) *and, when X is affine, a canonical isomorphism of $\mathbf{k}$-algebras*

$$\sigma_{\mathrm{tot}} \colon \widehat{\mathscr{W}}_{T^*X} \xrightarrow{\sim} (\mathcal{O}_{T^*X}((\hbar)), \star),$$

such that (i) *and* (ii) *are induced by* (iii) *and the product in $\widehat{\mathscr{W}}_{T^*X}$ induces the Leibniz product* (3.2).

One may construct intrinsically $\widehat{\mathscr{W}}_{T^*X}$ as follows (see [30]). Consider the complex line $\mathbb{C}$ endowed with the coordinate t and the subsheaf $\widehat{\mathscr{E}}_{T^*(\mathbb{C}\times X),\hat{t}}$ of $\widehat{\mathscr{E}}_{T^*(\mathbb{C}\times X)}$ consisting of sections which commute with ∂_t. Denote by $\rho\colon T^*_{\tau\neq 0}(\mathbb{C}\times X) \to T^*X$ the map $(t, x; \tau, \xi) \mapsto (x, \xi/\tau)$. Then $\widehat{\mathscr{W}}_{T^*X} = \rho_*\widehat{\mathscr{E}}_{T^*(\mathbb{C}\times X),\hat{t}}$.

3.2 Star-products and DQ-algebras

From now on, we still denote by X a complex manifold, but in fact, X will be endowed with a Poisson structure. When this Poisson structure is symplectic, X will locally play the role of the cotangent bundle encountered in the previous sections.

Let us recall a classical definition which generalizes the product in (3.2).

Definition 3.2.1. An associative multiplication law $\star$ on $\mathscr{O}_X[[\hbar]]$ is a star-product if it is $\mathbf{k}_0$-bilinear and satisfies

$$f \star g = \sum_{i \geq 0} P_i(f, g)\hbar^i \text{ for } f, g \in \mathscr{O}_X, \tag{3.4}$$

where the P_i's are bidifferential operators such that $P_0(f, g) = fg$ and $P_i(f, 1) = P_i(1, f) = 0$ for all $f \in \mathcal{O}_X$ and $i > 0$. We call $(\mathscr{O}_X[[\hbar]], \star)$ a star-algebra.

Note that $1 \in \mathcal{O}_X \subset \mathcal{O}_X[[\hbar]]$ is a unit with respect to $\star$. Note also that we have

$$(\sum_{i\geq 0} f_i\hbar^i) \star (\sum_{i\geq 0} g_i\hbar^i) = \sum_{n\geq 0}\big(\sum_{i+j=n} P_n(f_i, g_j)\big)\hbar^n.$$

A star-product defines a Poisson structure on $(X, \mathcal{O}_X)$, by setting for $f, g \in \mathcal{O}_X$:

$$\{f, g\} = P_1(f, g) - P_1(g, f) = \hbar^{-1}(f \star g - g \star f) \bmod \hbar\mathcal{O}_X[[\hbar]], \tag{3.5}$$

and that locally, (globally in the real case), any Poisson manifold $(X, \mathcal{O}_X)$ may be endowed with a star-product to which the Poisson structure is associated. This is a famous theorem of Kontsevich [26].

Example 3.2.2. (i) If the star product is commutative, then it is isomorphic to the usual product on $\mathcal{O}[[\hbar]]$.
(ii) If the Poisson structure is symplectic, then the star product is locally isomorphic to that given by the product (3.2).

DQ-algebras

Definition 3.2.3. A DQ-algebra $\mathscr{A}_X$ on X is a $\mathbf{k}_0$-algebra locally isomorphic to a star-algebra $(\mathscr{O}_X[[\hbar]], \star)$ as a $\mathbf{k}_0$-algebra.

Clearly a DQ-algebra $\mathscr{A}_X$ satisfies the conditions:

$$\begin{cases} \text{(i)} \ \hbar\colon \mathcal{A} \to \mathcal{A} \text{ is injective,} \\ \text{(ii)} \ \mathcal{A} \to \varprojlim_n \mathcal{A}/\hbar^n\mathcal{A} \text{ is an isomorphism,} \\ \text{(iii)} \ \mathcal{A}/\hbar\mathcal{A} \text{ is isomorphic to } \mathcal{O}_X \text{ as a } \mathbb{C}\text{-algebra.} \end{cases} \tag{3.6}$$

For a $\mathbf{k}_0$-algebra $\mathcal{A}$ satisfying (3.6), the $\mathbb{C}$-algebra isomorphism $\mathcal{A}/\hbar\mathcal{A} \xrightarrow{\sim} \mathcal{O}_X$ in (3.6) (iii) is unique. This follows from the fact that any $\mathbb{C}$-algebra endomorphism of $\mathcal{O}_X$ is equal to the identity.

We denote by

$$\sigma_0 \colon \mathcal{A} \to \mathcal{O}_X \tag{3.7}$$

the $\mathbf{k}_0$-algebra morphism $\mathcal{A} \to \mathcal{A}/\hbar\mathcal{A} \xrightarrow{\sim} \mathcal{O}_X$. If φ is a $\mathbb{C}$-linear section of $\sigma_0 \colon \mathcal{A} \to \mathcal{O}_X$, then φ extends to an isomorphism of $\mathbf{k}_0$-modules $\widetilde{\varphi} \colon \mathscr{O}_X[[\hbar]] \xrightarrow{\sim} \mathcal{A}$, given by $\widetilde{\varphi}(\sum_i f^i\hbar^i) = \sum_i \varphi(f_i)\hbar^i$.

One proves that a DQ-algebra is right and left Noetherian. As usual, one denotes by $\mathrm{D^b}(\mathscr{A}_X)$ the bounded derived category of left $\mathscr{A}_X$-modules and by $\mathrm{D^b_{coh}}(\mathscr{A}_X)$ the full triangulated subcategory consisting of objects with coherent cohomology.

DQ-algebroids

A DQ-algebra $\mathscr{A}_X$ on X endows X with a structure of a complex Poisson manifold. In the real setting, a famous theorem of Kontsevich [26] asserts that, conversely, any Poisson structure on a real manifold comes from a DQ-algebra. However, in the algebraic or complex analytic setting this is true locally but no more globally and one has to replace the the notion of a sheaf of algebra by that of an algebroid stack. This is done by Kashiwara [19] in the contact case and Kontsevich [21] in the algebraic Poisson case.

Since we do not want to enter into the theory of stacks, we simply give here a description "à la Cech" of an algebroid. For the notions of stacks, refer to the original book [16] and for an exposition, see [24].

Let X be a topological space and $\mathbb{K}$ a commutative unital ring. Consider an open covering $\mathcal{U} = \{U_i\}_{i\in I}$ of X and

- for each $i \in I$, a sheaf $\mathcal{A}_i$ of $\mathbb{K}$-algebra on U_i,
- for each $(i,j) \in I \times I$ an isomorphism of $\mathbb{K}$-algebras $f_{ij} \colon \mathcal{A}_j|_{U_{ij}} \xrightarrow{\sim} \mathcal{A}_i|_{U_{ij}}$,
- for each $(i,j,k) \in I \times I \times I$, invertible sections a_{ijk} of $\mathcal{A}_i(U_{ijk})$

these data satisfying

$$\begin{cases} f_{ij} \circ f_{jk} = \mathrm{Ad}(a_{ijk}) \circ f_{ik} \text{ on } U_{ijk}, \\ a_{ijk}a_{ikl} = f_{ij}(a_{jkl})a_{ijl} \text{ on } U_{ijkl}. \end{cases} \tag{3.8}$$

(Recall that $\mathrm{Ad}(a)(b) = aba^{-1}$.) One calls

$$\mathcal{A} := (\{\mathcal{A}_i\}_{i\in I}, \{f_{ij}\}_{i,j\in I}, \{a_{ijk}\}_{i,j,k\in I}) \tag{3.9}$$

a $\mathbb{K}$-algebroid (associated to gluing data on $\mathcal{U}$).

If X is a complex manifold, $\mathbb{K} = \mathbf{k}$ and each $\mathcal{A}_i$ is a DQ-algebra, then one calls $\mathcal{A}$ a DQ-algebroid.

In the sequel, we shall work for simplicity with DQ-algebras, but all constructions and results extend with suitable modification to DQ-algebroids.

Opposite DQ-algebra

Lemma 3.2.4. *Let $\mathcal{A}$ be a* DQ*-algebra. Then the opposite algebra $\mathcal{A}^{\mathrm{op}}$ is also a* DQ*-algebra.*

Proof. This follows from (3.4). q.e.d.

In the sequel, if X is endowed with a DQ-algebra $\mathscr{A}_X$, we denote by X^a the space X endowed with the DQ-algebra $\mathscr{A}_X^{\mathrm{op}}$.

Exterior product

Let X and Y be complex manifolds endowed with two star-products $\star_X$ and $\star_Y$. Denote by $\{P_i\}_i$ and $\{Q_j\}_j$ the bi-differential operators associated to these star-products as in (3.4). One defines $P_i \boxtimes Q_j$, the unique bi-differential operator on $X \times Y$ such that $(P_i \boxtimes Q_j)(f_1(x)g_1(y), f_2(x)g_2(y)) = P_i(f_1(x), f_2(x)) \cdot Q_j(g_1(y), g_2(y))$ for any $f_\nu(x) \in \mathcal{O}_X$ and $g_\nu(y) \in \mathcal{O}_Y$ ($\nu = 1, 2$).

One defines the external product of the star-products $\star_X$ and $\star_Y$ on $\mathscr{O}_{X\times Y}[[\hbar]]$ by setting

$$f \star g = \sum_{n\geq 0} \hbar^n \sum_{i+j=n} (P_i \boxtimes Q_j)(f, g).$$

Lemma 3.2.5. *Let X and Y be complex manifolds, and let $\mathscr{A}_X$ be a* DQ*-algebra on X and $\mathscr{A}_Y$ a* DQ*-algebra on Y. Then there exists a* DQ*-algebra $\mathcal{A}$ on $X \times Y$ which contains $\mathscr{A}_X \boxtimes_{\mathbf{k}_0} \mathscr{A}_Y$ as a $\mathbf{k}_0$-subalgebra. Moreover such an $\mathcal{A}$ is unique up to a unique isomorphism.*

One calls $\mathcal{A}$ the external product of the DQ-algebras $\mathscr{A}_X$ on X and the DQ-algebra $\mathscr{A}_Y$ on Y, and denotes it by $\mathscr{A}_X \boxtimes \mathscr{A}_Y$ or simply $\mathscr{A}_{X\times Y}$ if there is no risk of confusion.

Duality

Let $\mathscr{A}_X$ be a DQ-algebra on X and let $\mathcal{M} \in \mathrm{D}^{\mathrm{b}}(\mathscr{A}_X)$. Its *dual* $\mathbb{D}'_{\mathscr{A}}\mathcal{M} \in \mathrm{D}^{\mathrm{b}}(\mathscr{A}_{X^a})$ is given by

$$\mathbb{D}'_{\mathscr{A}}\mathcal{M} := R\mathcal{H}om_{\mathscr{A}_X}(\mathcal{M}, \mathscr{A}_X) \in \mathrm{D}^{\mathrm{b}}(\mathrm{gr}\,(\mathcal{A}_{X^a})). \tag{3.10}$$

Here, $\mathscr{A}_X$ is regarded as an $\mathscr{A}_{X^a} \otimes \mathscr{A}_X$-module.

Graded modules

Let $\mathscr{A}_X$ be a DQ-algebra on X. We denote by $\operatorname{gr}(\mathcal{A}_X)$ the sheaf of $\mathbb{C}$-algebras $\mathscr{A}_X/\hbar\mathscr{A}_X$. Then $\operatorname{Mod}(\operatorname{gr}(\mathcal{A}_X))$ is equivalent to the full subcategory of $\operatorname{Mod}(\mathscr{A}_X)$ consisting of objects $\mathcal{M}$ such that $\hbar\colon \mathcal{M} \to \mathcal{M}$ vanishes. The functor *for*: $\operatorname{Mod}(\operatorname{gr}(\mathcal{A}_X)) \to \operatorname{Mod}(\mathscr{A}_X)$ admits a left adjoint functor $\mathcal{M} \mapsto \mathcal{M}/\hbar\mathcal{M} \simeq \mathbb{C} \otimes \mathcal{M}$.

The left derived functor of the functor $\mathcal{M} \mapsto \mathcal{M}/\hbar\mathcal{M}$ is denoted by $\operatorname{gr} : \mathrm{D^b}(\mathscr{A}_X) \to \mathrm{D^b}(\operatorname{gr}(\mathcal{A}_X))$. For $\mathcal{M} \in \mathrm{D^b}(\mathscr{A}_X)$ we call $\operatorname{gr}(\mathcal{M})$ the graded module associated to $\mathcal{M}$.

The functor gr induces a functor:

$$\operatorname{gr} : \mathrm{D^b_{coh}}(\mathscr{A}_X) \to \mathrm{D^b_{coh}}(\operatorname{gr}(\mathcal{A}_X)). \tag{3.11}$$

Proposition 3.2.6. *The functor* gr *in* (3.11) *is conservative*

Simple modules

Let Λ be a smooth submanifold of X and let $\mathcal{L}$ be a coherent $\mathscr{A}_X$-module supported by Λ. One says that $\mathcal{L}$ is *simple* along Λ if $\operatorname{gr}(\mathcal{L})$ is concentrated in degree 0 and $H^0(\operatorname{gr}(\mathcal{L}))$ is an invertible $\mathcal{O}_\Lambda \otimes_{\mathcal{O}_X} \operatorname{gr}(\mathscr{A}_X)$-module. (In particular, $\mathcal{L}$ is without $\hbar$-torsion.)

Proposition 3.2.7. *Let Λ be a closed smooth submanifold of X of codimension l and let $\mathcal{L}$ be a coherent $\mathscr{A}_X$-module simple along Λ. Then $\mathcal{E}xt^j_{\mathscr{A}_X}(\mathcal{L}, \mathscr{A}_X)$ vanishes for $j \neq l$ and for $j = l$ this $\mathscr{A}_{X^a}$-module is simple along Λ.*

Homological dimension

Let d_X denote the complex dimension of X.

Proposition 3.2.8. *Let $\mathscr{A}_X$ be a* DQ*-algebra and let $\mathcal{M} \in \operatorname{Mod_{coh}}(\mathscr{A}_X)$. Then, locally, $\mathcal{M}$ admits a resolution by free modules of finite rank of length $\leq d_X + 1$.*

DQ-modules supported by the diagonal

We denote by Δ_X the diagonal of $X \times X^a$, by $\delta_X\colon X \hookrightarrow X \times X^a$ the diagonal embedding, and by $\operatorname{Mod}_{\Delta_X}(\mathscr{A}_X \boxtimes \mathscr{A}_{X^a})$ the category of $(\mathscr{A}_X \boxtimes \mathscr{A}_{X^a})$-modules supported by the diagonal. Then

$$\delta_{X*}\colon \operatorname{Mod}(\mathscr{A}_X \otimes \mathscr{A}_{X^a}) \to \operatorname{Mod}_{\Delta_X}(\mathscr{A}_X \boxtimes \mathscr{A}_{X^a})$$

gives an equivalence of categories and we shall often identify these two categories.

The algebra $\mathscr{A}_X$ is well defined as an object of $\mathrm{Mod}(\mathscr{A}_X \otimes \mathscr{A}_{X^a})$ and the $\mathscr{A}_X \boxtimes \mathscr{A}_{X^a}$-module $\delta_{X*}\mathscr{A}_X$ has a natural structure of an $\mathscr{A}_{X\times X^a}$-module, simple along the diagonal. We set

$$\mathscr{C}_X := \delta_{X*}\mathscr{A}_X, \text{ an object of } \mathrm{Mod}(\mathscr{A}_{X\times X^a}). \tag{3.12}$$

A coherent $\mathscr{A}_{X\times X^a}$-module simple along the diagonal is called a *bi-invertible* $\mathscr{A}_X \otimes \mathscr{A}_{X^a}$-module. Then, the category of bi-invertible $\mathscr{A}_X \otimes \mathscr{A}_{X^a}$-modules is a tensor category and $\mathscr{C}_X$ is a unit object. More generally, we say that $\mathcal{P} \in \mathrm{D}^{\mathrm{b}}(\mathscr{A}_X \otimes \mathscr{A}_{X^a})$ is bi-invertible if it is concentrated in a single degree, say n, and $H^n(\mathcal{P})$ is bi-invertible. If $\mathcal{P}$ is bi-invertible, we set

$$\mathcal{P}^{\otimes -1} := R\mathcal{H}om_{\mathscr{A}_X}(\mathcal{P}, \mathscr{A}_X).$$

Hence we have

$$\mathcal{P}^{\otimes -1} \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_X} \mathcal{P} \simeq \mathcal{P} \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_X} \mathcal{P}^{\otimes -1} \simeq \mathscr{C}_X.$$

$\hbar$-localization

To a DQ-algebra $\mathscr{A}_X$ we associate its *$\hbar$-localization*, the $\mathbf{k}$-algebra

$$\mathscr{A}_X^{\mathrm{loc}} = \mathbf{k} \otimes_{\mathbf{k}_0} \mathscr{A}_X. \tag{3.13}$$

There exists a pair of adjoint exact functors $(\bullet \otimes_{\mathbf{k}_0} \mathbf{k}, \mathit{for})$:

$$\mathrm{Mod}(\mathscr{A}_X^{\mathrm{loc}}) \underset{\bullet \otimes_{\mathbf{k}_0} \mathbf{k}}{\overset{\mathit{for}}{\rightleftarrows}} \mathrm{Mod}(\mathscr{A}_X). \tag{3.14}$$

The algebra $\mathscr{A}_X^{\mathrm{loc}}$ is Noetherian.

If $\mathcal{M}$ is an $\mathscr{A}_X^{\mathrm{loc}}$-module, $\mathcal{M}_0$ an $\mathscr{A}_X$-submodule and $\mathcal{M}_0 \otimes_{\mathbf{k}_0} \mathbf{k} \xrightarrow{\sim} \mathcal{M}$, we shall say that $\mathcal{M}_0$ generates $\mathcal{M}$.

A coherent $\mathscr{A}_X^{\mathrm{loc}}$-module $\mathcal{M}$ is *good* if, for any open relatively compact subset U of X, there exists a coherent $(\mathscr{A}_X|_U)$-module which generates $\mathcal{M}|_U$.

One denotes by $\mathrm{Mod}_{\mathrm{gd}}(\mathscr{A}_X^{\mathrm{loc}})$ the full subcategory of $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_X^{\mathrm{loc}})$ consisting of good modules. Similarly as in [20, Prop. 4.23], one proves that $\mathrm{Mod}_{\mathrm{gd}}(\mathscr{A}_X^{\mathrm{loc}})$ is a thick subcategory of $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{A}_X^{\mathrm{loc}})$.

We denote by $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X^{\mathrm{loc}})$ (resp. $\mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{A}_X^{\mathrm{loc}})$) the full triangulated subcategory of $\mathrm{D}^{\mathrm{b}}(\mathscr{A}_X^{\mathrm{loc}})$ consisting of objects $\mathcal{M}$ such that $H^j(\mathcal{M})$ is coherent (resp. good) for all $j \in \mathbb{Z}$. The notion of good $\mathscr{A}_X$-module is similar to that of good $\mathcal{D}$-module of loc. cit.

Dualizing complex

One sets

$$\omega_X^{\mathscr{A}} := (\mathbb{D}'_{\mathscr{A}}(\mathscr{C}_{X^a}))^{\otimes -1} \quad \text{an object of } \mathrm{D^b}(\mathscr{A}_{X\times X^a}) \tag{3.15}$$

and calls $\omega_X^{\mathscr{A}}$ the dualizing complex for $\mathscr{A}_X$-modules. In this formula, $\mathbb{D}'_{\mathscr{A}}$ is the dual over $\mathscr{A}_{X\times X^a}$ and $(\bullet)^{\otimes -1}$ is the dual over $\mathscr{A}_X$. Note that dualizing complexes for DQ-moodules have already been introduced (in more restricted situations) in [11].

3.3 Convolution of kernels

The operations of direct or inverse images are not well suited for DQ-modules and should be replaced by the more general notion of correspondence. For example, when the associated Poisson structure on X is symplectic, a local model for X is an open subset of a cotangent bundle T^*M, and we have already noticed that a morphism of manifolds $M \to N$ gives rise to a correspondance

$$T^*M \leftarrow M \times_N T^*N \to T^*N.$$

Note that in case of $\mathcal{O}$-modules, the transformations we shall study are often referred as Fourier-Mukai transforms (see [18]).

For two complex manifolds X_i $(i = 1, 2)$ endowed with DQ-algebroids $\mathscr{A}_{X_i}$ and for $\mathcal{M}_i \in \mathrm{D^b}(\mathscr{A}_{X_i}$, we defined there external product

$$\mathcal{M}_1 \overset{\mathrm{L}}{\underline{\boxtimes}} \mathcal{M}_2 := \mathscr{A}_{X_1\times X_2} \otimes_{(\mathscr{A}_{X_1}\boxtimes\mathscr{A}_{X_2})} (\mathcal{M}_1 \overset{\mathrm{L}}{\boxtimes}_{\mathbf{k}_0} \mathcal{M}_2).$$

Consider now three complex manifolds X_i $(i = 1, 2, 3)$ endowed with DQ-algebroids $\mathscr{A}_{X_i}$. We denote by p_i the i-th projection and by p_{ij} the (i, j)-th projection. For $\Lambda_i \subset X_i \times X_{i+1}$ $(i = 1, 2)$, we set

$$\Lambda_1 \circ \Lambda_2 = p_{13}(p_{12}^{-1}\Lambda_1 \cap p_{23}^{-1}\Lambda_2). \tag{3.16}$$

We shall write for short $\mathscr{A}_i$ instead of $\mathscr{A}_{X_i}$, $\mathscr{A}_{ij^a}$ instead of $\mathscr{A}_{X_{ij^a}}$, etc.

Definition 3.3.1. Let $\mathcal{K}_i \in \mathrm{D^b}(\mathscr{A}_{X_i\times X_{i+1}^a})$ $(i = 1, 2)$. We set

$$\mathcal{K}_1 \otimes_{\mathscr{A}_2} \mathcal{K}_2 = (\mathcal{K}_1 \overset{\mathrm{L}}{\underline{\boxtimes}} \mathcal{K}_2) \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_2\boxtimes\mathscr{A}2^a} \mathscr{C}_2 \in \mathrm{D^b}(p_{13}^{-1}\mathscr{A}_{13^a}), \tag{3.17}$$

$$\mathcal{K}_1 \overset{\mathrm{L}}{\underline{\otimes}}_{\mathscr{A}_2} \mathcal{K}_2 = (\mathcal{K}_1 \overset{\mathrm{L}}{\underline{\boxtimes}} \mathcal{K}_2) \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_{22^a}} \mathscr{C}_2 \in \mathrm{D^b}(p_{13}^{-1}\mathscr{A}_{13^a}), \tag{3.18}$$

$$\mathcal{K}_1 \underset{X_2}{\circ} \mathcal{K}_2 = Rp_{13!}(\mathcal{K}_1 \overset{\mathrm{L}}{\underline{\otimes}}_{\mathscr{A}_2} \mathcal{K}_2) \in \mathrm{D^b}(\mathscr{A}_{X_1\times X_3^a}), \tag{3.19}$$

$$\mathcal{K}_1 \underset{X_2}{*} \mathcal{K}_2 = Rp_{13*}(\mathcal{K}_1 \overset{\mathrm{L}}{\underline{\otimes}}_{\mathscr{A}_2} \mathcal{K}_2) \in \mathrm{D^b}(\mathscr{A}_{X_1\times X_3^a}). \tag{3.20}$$

If there is no risk of confusion we write $\mathcal{K}_1 \circ \mathcal{K}_2$ for $\mathcal{K}_1 \underset{X_2}{\circ} \mathcal{K}_2$ and similarly with $*$.

One defines similarly the composition of $\mathrm{gr}\,(\mathcal{A}_X)$-modules. Then one easily proves:

Proposition 3.3.2. *For $\mathcal{K}_i \in \mathrm{D}^{\mathrm{b}}(\mathscr{A}_{X_i \times X_{i+1}^a})$ $(i = 1, 2)$, we have*

$$\mathrm{gr}\,(\mathcal{K}_1 \circ \mathcal{K}_2) \simeq \mathrm{gr}\,(\mathcal{K}_1) \circ \mathrm{gr}\,(\mathcal{K}_2). \tag{3.21}$$

When $X_1 = \mathrm{pt}$ or $X_3 = \mathrm{pt}$ we get $\mathcal{K}_1 \otimes_{\mathscr{A}_2} \mathcal{K}_2 \xrightarrow{\sim} \mathcal{K}_1 \overset{\mathrm{L}}{\underline{\otimes}}_{\mathscr{A}_2} \mathcal{K}_2)$. There are canonical isomorphisms

$$\mathcal{K}_1 \underset{X_2}{\circ} \mathscr{C}_{X_2} \simeq \mathcal{K}_1 \quad \text{and} \quad \mathscr{C}_{X_1} \underset{X_1}{\circ} \mathcal{K}_1 \simeq \mathcal{K}_1. \tag{3.22}$$

Since $\circ$ and $*$ are not associative in general, we also define for $\mathcal{K}_i \in \mathrm{D}^{\mathrm{b}}(\mathscr{A}_{X_i \times X_{i+1}^a})$ $(i = 1, \dots, n)$:

$$\mathcal{K}_1 \underset{2}{\circ} \cdots \underset{n}{\circ} \mathcal{K}_n \quad = \quad Rp_{1,n+1!}(\mathcal{K}_1 \overset{\mathrm{L}}{\underline{\otimes}}_{\mathscr{A}_2} \cdots \overset{\mathrm{L}}{\underline{\otimes}}_{\mathscr{A}_n} \mathcal{K}_n) \in \mathrm{D}^{\mathrm{b}}(\mathscr{A}_{X_1 \times X_{n+1}^a})$$

and similarly with $*$.

Note that the functor gr in (3.11) commutes with the convolution of kernels.

Theorem 3.3.3. *Let $\mathcal{K}_i \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_{X_i \times X_{i+1}^a})$ $(i = 1, 2)$. Assume that the projection p_{13} defined on $X_1 \times X_2 \times X_3$ is proper on $p_{12}^{-1} \operatorname{supp}(\mathcal{K}_1) \cap p_{23}^{-1} \operatorname{supp}(\mathcal{K}_2)$. Then*

(a) *the object $\mathcal{K}_1 \circ \mathcal{K}_2$ belongs to $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_{X_1 \times X_3^a})$ and $\operatorname{supp}(\mathcal{K}_1 \circ \mathcal{K}_2) \subset \operatorname{supp}(\mathcal{K}_1) \circ \operatorname{supp}(\mathcal{K}_2)$,*

(b) *we have a natural isomorphism*

$$\mathbb{D}'_{\mathscr{A}}(\mathcal{K}_1) \underset{X_2^a}{\circ} \omega^{\mathscr{A}}_{X_2^a} \underset{X_2^a}{\circ} \mathbb{D}'_{\mathscr{A}}(\mathcal{K}_2) \xrightarrow{\sim} \mathbb{D}'_{\mathscr{A}}(\mathcal{K}_1 \underset{X_2}{\circ} \mathcal{K}_2) \tag{3.23}$$

in $\mathrm{D}^{\mathrm{b}}(\mathscr{A}_{X_1^a \times X_3})$.

Remark 3.3.4. A nice application of the coherency of the composition of kernels in the case of symplectic manifolds may be found [1].

One defines similarly the composition for $\mathscr{A}_X^{\mathrm{loc}}$-modules. Using the isomorphism

$$(\mathcal{K}_1 \circ \mathcal{K}_2)^{\mathrm{loc}} \simeq \mathcal{K}_1^{\mathrm{loc}} \circ \mathcal{K}_2^{\mathrm{loc}},$$

we get

Corollary 3.3.5. *Let $\mathcal{K}_i \in \mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{A}^{\mathrm{loc}}_{X_i \times X^a_{i+1}})$ $(i = 1, 2)$. Assume that the projection p_{13} defined on $X_1 \times X_2 \times X_3$ is proper on $p_{12}^{-1} \operatorname{supp}(\mathcal{K}_1) \cap p_{23}^{-1} \operatorname{supp}(\mathcal{K}_2)$. Then $\mathcal{K}_1 \circ \mathcal{K}_2$ belongs to $\mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathscr{A}^{\mathrm{loc}}_{X_1 \times X^a_3})$.*

Corollary 3.3.6. *Let $\mathcal{M}$ and $\mathcal{N}$ be two objects of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X)$ and assume that $\operatorname{supp}(\mathcal{M}) \cap \operatorname{supp}(\mathcal{N})$ is compact. Then there is a natural isomorphism in $\mathrm{D}^{\mathrm{b}}_f(\mathbf{k}_0)$:*

$$\mathrm{RHom}_{\mathscr{A}_X}(\mathcal{N}, \omega_X^{\mathscr{A}} \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_X} \mathcal{M}) \simeq (\mathrm{RHom}_{\mathscr{A}_X}(\mathcal{M}, \mathcal{N}))^\star,$$

where * is the duality functor in $\mathrm{D}^{\mathrm{b}}_f(\mathbf{k}_0)$.

In particular, if X is compact, then $\mathcal{M} \mapsto \omega_X^{\mathscr{A}} \otimes_{\mathscr{A}_X} \mathcal{M}$ is a Serre functor on the triangulated category $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X)$.

The proof of the coherency of the direct image relies on two results which may be of independent interest.

Theorem 3.3.7. *Let $\mathcal{A}$ be a sheaf of $\mathbf{k}_0$-algebras satifying (3.6). Let $\mathcal{M}^\bullet$ be a complex of $\mathcal{A}$-modules bounded from below and assume the following conditions:*

(i) $\mathcal{M}^j \xrightarrow{\sim} \varprojlim_n (\mathcal{M}^j / \hbar^{n+1} \mathcal{M}^j)$ *for any $j \in \mathbb{Z}$,*

(ii) $\hbar \colon \mathcal{M}^j \to \mathcal{M}^j$ *is a monomorphism for any j,*

(iii) $H^k(U; \mathcal{M}^j / \hbar \mathcal{M}^j) = 0$ *for any U open Stein, any $j \in \mathbb{Z}$ and any $k > 0$,*

(iv) $H^j(\mathcal{M}^\bullet / \hbar \mathcal{M}^\bullet)$ *is a coherent $\mathcal{O}_X$-module for any j.*

Then $H^j(\mathcal{M}^\bullet)$ is a coherent $\mathcal{A}$-module for any j.

Let us take a family $\mathcal{S}$ of open subsets of X stable by intersection and which is a basis of the topology.

The additive category $\mathrm{Mod}^{\mathrm{af}}(\mathcal{A})$ of $\mathcal{S}$-almost free $\mathcal{A}$-modules is defined as follows.

(a) An object of $\mathrm{Mod}^{\mathrm{af}}(\mathcal{A})$ is the data of $\{I, \{U_i, U'_i, L_i\}_{i\in I}\}$ where I is an index set, U_i and U'_i are open subsets of X, $U_i \in \mathcal{S}$, $\overline{U_i} \subset U'_i$, the family $\{U'_i\}_{i\in I}$ is locally finite and L_i is an invertible $\mathcal{A}|_{U'_i}$-module.

(b) Let $N = \{J, \{V_j, V'_j, K_j\}_{j\in J}\}$ and $M = \{I, \{U_i, U'_i, L_i\}_{i\in I}\}$ be two objects of $\mathrm{Mod}^{\mathrm{af}}(\mathcal{A})$. A morphism $u\colon N \to M$ is the data for all $(i, j) \in I \times J$ of an element u_{ij} of $\Gamma(\overline{V_j}; \mathcal{H}om_{\mathcal{A}}(K_j, L_i))$ such that $u_{i,j} = 0$ if $V_j \not\subset U_i$. The composition of morphisms is the natural one.

(c) We denote by $\Phi\colon \mathrm{Mod}^{\mathrm{af}}(\mathcal{A}) \to \mathrm{Mod}(\mathcal{A})$ the functor which sends the object $\{I, \{U_i, U'_i, L_i\}_{i\in I}\}$ to $\bigoplus_{i\in I}(L_i)_{U_i}$ and which sends an element u_{ij} of $\Gamma(\overline{V_j}; \mathcal{H}om_{\mathcal{A}}(K_j, L_i))$ to its image in $\mathrm{Hom}_{\mathcal{A}}((K_j)_{V_j}, (L_i)_{U_i})$ if $V_j \subset U_i$ and 0 otherwise.

Theorem 3.3.8. *Let $\mathcal{A}$ be a left coherent algebra and let $\mathcal{M} \in \mathrm{D}^-_{\mathrm{coh}}(\mathcal{A})$. Then there exist $L^\bullet \in \mathrm{C}^-(\mathrm{Mod}^{\mathrm{af}}(\mathcal{A}))$ and an isomorphism $\Phi(L^\bullet) \xrightarrow{\sim} \mathcal{M}$ in $\mathrm{D}^-(\mathcal{A})$.*

There also exists a dual version, replacing $(L_i)_{U_i}$ with $\Gamma_{U_i}L_i$ and obtaining an isomorphism $\mathcal{M} \xrightarrow{\sim} \Psi(L^\bullet)$ in $\mathrm{D}^+(\mathcal{A})$.

In other words, an object $\mathcal{M} \in \mathrm{D}^-_{\mathrm{coh}}(\mathcal{A})$ is quasi-isomorphic to a complex bounded above of $\mathscr{A}_X$-modules, each of which is a locally finite sum of the type $(\mathscr{A}_X)_U$, U open relatively compact in X.

Remark 3.3.9. All results of this section hold for algebroids instead of sheaves of algebras.

Applications to $\mathcal{O}$-modules

Integral transforms for $\mathcal{O}$-modules are of course extremely important. They often are called Fourier-Mukai transforms, by extension of the case of abelian varieties. Such transform are intensively studied in the book [18].

Note that the first example of a generalization of the Fourier transform, passing from sets and functions to categories and sheaves, is the Fourier-Sato transform (see [23, Ch 3]).

The algebra $\mathcal{O}[[\hbar]]$ endowed with the usual commutative product is an example of a DQ-algebra. In this case, Theorem 3.3.3 is a variant with formal parameters of the Grauert theorem. It may be reformulated as follows.

Define the dualizing complex and the duality functor for $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}(\mathcal{O}_X[[\hbar]])$

$$
\begin{aligned}
&\omega_X[[\hbar]] := \Omega_X[[\hbar]]\,[d_X],\\
&\mathbb{D}_{\mathscr{O}}\mathcal{F} := R\mathcal{H}om_{\mathcal{O}[[\hbar]]}(\mathcal{F}, \omega_X[[\hbar]]),\\
&\mathbb{D}'_{\mathscr{O}}\mathcal{F} := R\mathcal{H}om_{\mathcal{O}[[\hbar]]}(\mathcal{F}, \mathcal{O}_X[[\hbar]]).
\end{aligned}
$$

Let X and Y be two complex manifolds and denote by p_i the i-th projection defined on $X \times Y$ $(i = 1, 2)$. Let $\mathcal{M} \in \mathrm{D}^{\mathrm{b}}(\mathcal{O}_X[[\hbar]])$ and $\mathcal{L} \in \mathrm{D}^{\mathrm{b}}(\mathcal{O}_{X\times Y}[[\hbar]])$. Set

$$\mathcal{L} \underset{X}{\circ} \mathcal{M} := Rp_{2!}(\mathcal{L} \overset{\mathrm{L}}{\otimes}_{\mathcal{O}[[\hbar]]} p_1^{-1}\mathcal{M}).$$

Corollary 3.3.10. *Assume that $\mathcal{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{O}_X[[\hbar]])$, $\mathcal{L} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{O}_{X\times Y}[[\hbar]])$ and that p_2 is proper on $p_1^{-1}\operatorname{supp}(\mathcal{M}) \cap \operatorname{supp}(\mathcal{L})$. Then $\mathcal{L} \circ \mathcal{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{O}_Y[[\hbar]])$ and moreover*

$$\mathbb{D}'_{\mathscr{O}}(\mathcal{L}) \underset{X}{\circ} \omega_X[[\hbar]] \underset{X}{\circ} \mathbb{D}'_{\mathscr{O}}(\mathcal{M}) \xrightarrow{\sim} \mathbb{D}'_{\mathscr{O}}(\mathcal{L} \underset{X}{\circ} \mathcal{M}).$$

Consider now a morphism of complex manifolds $f\colon X \to Y$. Denote by Γ_f the graph of f in $X \times Y$ and set $\mathcal{L} := \mathcal{O}_{\Gamma_f}$. Then $\mathcal{L} \underset{X}{\circ} \mathcal{M} \simeq Rf_!\mathcal{M}$. On the other hand,

$$\mathbb{D}'_{\mathscr{O}}(\mathcal{L}) \otimes_{\mathcal{O}[[\hbar]]} \omega_X[[\hbar]]^{\otimes -1} \simeq \mathcal{L}.$$

We obtain:

Corollary 3.3.11. *Let $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{O}_X[[\hbar]])$ and assume that f is proper on $\operatorname{supp}(\mathcal{F})$. Then $Rf_!\mathcal{F} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{O}_Y[[\hbar]])$ and there is a natural isomorphism*

$$Rf_!\mathbb{D}_{\mathscr{O}}\mathcal{F} \simeq \mathbb{D}_{\mathscr{O}}Rf_!\mathcal{F}.$$

Applications to $\widehat{\mathscr{W}}$-modules

Denote by M a complex manifold and set $X = T^*M$. The algebra $\widehat{\mathscr{W}}_X(0)$ is thus a DQ-algebra and the associated Poisson structure on X is the symplectic structure of T^*M. Moreover, $\widehat{\mathscr{W}}_X \simeq \widehat{\mathscr{W}}_X(0)^{\mathrm{loc}}$.

Let M and N be two complex manifolds, set $X = T^*M$, $Y = T^*N$. Denote by p_i the i-th projection defined on $X \times Y$ $(i = 1, 2)$. Let $\mathcal{M} \in \mathrm{D}^{\mathrm{b}}(\widehat{\mathscr{W}}_X(0))$ and $\mathcal{L} \in \mathrm{D}^{\mathrm{b}}(\widehat{\mathscr{W}}_{X^a\times Y}(0))$. Set

$$\mathcal{L} \underset{X}{\circ} \mathcal{M} := Rp_{2!}(\mathcal{L} \overset{\mathrm{L}}{\otimes}_{\widehat{\mathscr{W}}(0)} p_1^{-1}\mathcal{M})$$

and define similarly the convolution of $\widehat{\mathscr{W}}$-modules.

3.4 Applications to $\mathcal{D}$-modules

Denote by M a complex manifold and set $X = T^*M$. Note that M (identified to the zero section of T^*M) is Lagrangian, and any smooth Lagrangian submanifold of X is locally isomorphic to M by a complex symplectic isomorphism. We set

$$\begin{aligned}\mathcal{D}_M[[\hbar]] &:= \varprojlim_n \mathcal{D}_M \otimes (\mathbf{k}_0/\hbar^n\mathbf{k}_0) \simeq \prod_{n\geq 0} \mathcal{D}_M \hbar^n,\\ \mathcal{D}_M((\hbar)) &:= \mathcal{D}_M[[\hbar]] \otimes_{\mathbf{k}_0} \mathbf{k}.\end{aligned}$$

Note that $\mathcal{O}_M[[\hbar]]$ is a coherent $\widehat{\mathscr{W}_X}(0)$-module simple along M. Indeed, it is isomorphic to $\widehat{\mathscr{W}_X}(0) \otimes_{\pi_M^{-1}\mathcal{D}_M} \pi_M^{-1}\mathcal{O}_M$. Similarly, $\mathcal{O}_M((\hbar))$ is a coherent $\widehat{\mathscr{W}_X}$-module.

Proposition 3.4.1. *There is a natural monomorphism $\nu_M \colon \widehat{\mathscr{W}_X}|_M \hookrightarrow \mathcal{D}_M((\hbar))$ whose composition with the morphism $\mathcal{D}_M \to \widehat{\mathscr{W}_X}|_M$ induced by* (3.3) *is the natural embedding $\mathcal{D}_M \hookrightarrow \mathcal{D}_M((\hbar))$. Moreover, this embedding induces the embedding $\nu_M \colon \widehat{\mathscr{W}_X}(0)|_M \hookrightarrow \mathcal{D}_M[[\hbar]]$ whose restriction to $\mathscr{O}_M$ is the natural embedding $\mathcal{O}_M \hookrightarrow \mathcal{D}_M[[\hbar]]$.*

Proof. (i) Since $\mathcal{O}_M[[\hbar]]$ is a $\widehat{\mathscr{W}_X}(0)|_M$-module, there is a natural morphism $\widehat{\mathscr{W}_X}(0)|_M \to \mathcal{E}nd_{\mathbf{k}}(\mathcal{O}_M[[\hbar]])$. Similarly, there is a natural monomorphism $\mathcal{D}_M[[\hbar]] \to \mathcal{E}nd_{\mathbf{k}}(\mathcal{O}_M[[\hbar]])$.

(ii) Let us show that the morphism $\widehat{\mathscr{W}_X}(0)|_M \to \mathcal{E}nd_{\mathbf{k}}(\mathcal{O}_M[[\hbar]])$ is a monomorphism and factorizes through $\mathcal{D}_M[[\hbar]]$. This is a local problem and we may assume to be given a symplectic local coordinate system (x, u) on X such that $M = \{u = 0\}$ ($u \in \mathbb{C}^n$). In this case, one may identify $\widehat{\mathscr{W}_X}(0)$ with the DQ-algebra $(\mathcal{O}_X[[\hbar]], \star)$ where $\star$ is the Leibniz product (3.2) and the action of u_i on $\mathcal{O}_M[[\hbar]]$ is that of $\hbar\partial_i$.

(iii) Let $\mathscr{O}_X\hat{|}_M[[\hbar]]$ be the formal completion of $\mathcal{O}_X[[\hbar]]$ along M. The Leibniz product (3.2) extends to this sheaf. We denote by $\widehat{\mathscr{W}_X}(0)\hat{|}_M$ this new $\mathbf{k}_0$-algebra. Clearly, $\widehat{\mathscr{W}_X}(0)|_M \to \widehat{\mathscr{W}_X}(0)\hat{|}_M$ is a monomorphism. Now remark that $\mathcal{O}_M[[\hbar]]$ is a $\widehat{\mathscr{W}_X}(0)\hat{|}_M$-module and we have a morphism of $\mathbf{k}_0$-algebras:

$$\iota \colon \widehat{\mathscr{W}_X}(0)\hat{|}_M \to \mathcal{D}_M[[\hbar]] \tag{3.24}$$

which associates $\hbar\partial_i$ to u_i. Let us show that this morphism ι is an injective. Let

$$f = \sum_{j=0}^{\infty} \sum_{\alpha_j \in \mathbb{N}^n} f_{j,\alpha_j}(x) u^{\alpha_j} \hbar^j \in \mathscr{O}_X\hat{|}_M[[\hbar]] = \widehat{\mathscr{W}_X}(0)\hat{|}_M.$$

To f, we associate

$$\begin{aligned}\iota(f) &= \sum_{j=0}^{\infty}\sum_{\alpha_j\in\mathbb{N}^n} f_{j,\alpha_j}(x)\partial_x^{\alpha_j}\hbar^{|\alpha_j|+j} \\ &= \sum_{m=0}^{\infty}\Big(\sum_{|\alpha_j|+j=m} f_{j,\alpha_j}(x)\partial_x^{\alpha_j}\Big)\hbar^m.\end{aligned}$$

Then, ι is clearly injective.

(iv) We thus have got the monomorphisms of $\mathbf{k}_0$-algebras

$$\widehat{\mathscr{W}}_X(0)|_M \hookrightarrow \widehat{\mathscr{W}}_X(0)\hat{|}_M \hookrightarrow \mathcal{D}_M[[\hbar]]. \tag{3.25}$$

These morphisms define the monomorphisms

$$\mathcal{D}_M \hookrightarrow \widehat{\mathscr{W}}_X|_M \hookrightarrow \widehat{\mathscr{W}}_X\hat{|}_M \hookrightarrow \mathcal{D}_M((\hbar)). \tag{3.26}$$

q.e.d.

We now introduce the exact functor

$$(\bullet)^{\mathrm{W}}\colon \mathrm{Mod}(\mathcal{D}_M) \to \mathrm{Mod}(\widehat{\mathscr{W}}_X) \tag{3.27}$$

$$\mathcal{M} \mapsto \widehat{\mathscr{W}}_X \otimes_{\pi_M^{-1}\mathcal{D}_M} \pi_M^{-1}\mathcal{M}. \tag{3.28}$$

Clearly, $(\bullet)^W$ sends $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_M)$ to $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\widehat{\mathscr{W}}_X)$ and $\mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\mathcal{D}_M)$ to $\mathrm{D}^{\mathrm{b}}_{\mathrm{gd}}(\widehat{\mathscr{W}}_X)$.

Example 3.4.2. One has $\mathcal{O}_M^{\mathrm{W}} \simeq \mathcal{O}_M((\hbar))$.

The next result shows that one can, in some sense, reduce the study of $\mathcal{D}$-modules to that of $\widehat{\mathscr{W}}$-modules.

Proposition 3.4.3. *The functor $\mathcal{M} \mapsto \mathcal{M}^{\mathrm{W}}|_M$ is exact and faithful.*

Proof. The exactness follows from the fact that $\widehat{\mathscr{W}}_X|_M$ is flat over $\mathcal{D}_M$ and $\widehat{\mathscr{W}}_X\hat{|}_M$ is flat over $\widehat{\mathscr{W}}_X|_M$. This functor is faithful thanks to Proposition 3.4.1 and the fact that $\mathcal{D}_M((\hbar))$ is faithfully flat over $\mathcal{D}_M$. q.e.d.

Let M and N be two complex manifolds, set $X = T^*M$, $Y = T^*N$. Denote by q_i the i-th projection defined on $M \times N$ ($i = 1, 2$). Let $\mathcal{M} \in \mathrm{D}^{\mathrm{b}}(\mathcal{D}_M)$ and $\mathcal{L} \in \mathrm{D}^{\mathrm{b}}(\mathcal{D}_{M^a\times N})$. Set

$$\mathcal{L} \underset{M}{\circ} \mathcal{M} := Rq_{2!}(\mathcal{L} \overset{\mathrm{L}}{\otimes}_{\mathcal{D}} q_1^{-1}\mathcal{M}).$$

Theorem 3.4.4. *Assume that* $\mathcal{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_M)$, $\mathcal{L} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_{M^a \times N})$ *and that* p_2 *is proper on* $p_1^{-1}\operatorname{char}(\mathcal{M}) \cap \operatorname{char}(\mathcal{L})$. *Then*

$$(\mathcal{L} \underset{M}{\circ} \mathcal{M})^{\mathrm{W}} \xrightarrow{\sim} \mathcal{L}^{\mathrm{W}} \underset{X}{\circ} \mathcal{M}^{\mathrm{W}} \tag{3.29}$$

Proof. The natural morphism in (3.29) is easily constructed. To check it is an isomorphism, we remark that convolution may be obtained by combining the three operations: external product, inverse image and direct image. Moreover, in our case, the inverse image is non characteristic and the direct image is proper on the support of the module.

(i) External product. For $\mathcal{M} \in \mathrm{Mod}_{\mathrm{coh}}(\mathcal{D}_M)$ and $\mathcal{N} \in \mathrm{Mod}_{\mathrm{coh}}(\mathcal{D}_N)$, we have the isomorphism

$$(\mathcal{M} \underline{\boxtimes} \mathcal{N})^{\mathrm{W}} \simeq \mathcal{M}^{\mathrm{W}} \underline{\boxtimes} \mathcal{N}^{\mathrm{W}}.$$

Indeed, we may reduce to the case where $\mathcal{M} = \mathcal{D}_M$ and $\mathcal{N} = \mathcal{D}_N$. In this case, the result is obvious.

(ii) Let $f\colon M \to N$ be a morphism of complex manifolds. We set $X = T^*M$ and $Y = T^*N$. We define $\widehat{\mathscr{W}}_{X \to Y}$ as $(\mathcal{D}_{M \to N})^{\mathrm{W}}$. We may assume that f is the composition of a closed embedding and a smooth map and treat separately each case.

(iii) Non characteristic inverse image. Assume that $f\colon M \to N$ is non characteristic for $\mathcal{N} \in \mathrm{Mod}_{\mathrm{coh}}(\mathcal{D}_N)$. We have to prove that

$$(f_{\mathcal{D}}^{-1}\mathcal{N})^{\mathrm{W}} \simeq \widehat{\mathscr{W}}_{X \to Y} \overset{\mathrm{L}}{\otimes}_{\widehat{\mathscr{W}}} \mathcal{N}^{\mathrm{W}}.$$

(a) In the smooth case, we may assume that $M = N \times L$ and f is the first projection. Then $f_{\mathcal{D}}^{-1}\mathcal{N} \simeq \mathcal{N} \underline{\boxtimes} \mathcal{O}_L$ and the result follows from (i).

(b) Assume now that f is a closed embedding. Moreover, arguing by induction on the codimension, we may assume that X is a hypersurface given by an equation $x_1 = 0$. Finally, using standard arguments, we may assume that $\mathcal{N} = \mathcal{D}_N / \mathcal{D}_N \cdot P$, where P is a differential operator, say of order m, non characteristic for X. In this case $f_{\mathcal{D}}^{-1}\mathcal{N} \simeq \mathcal{D}_M^m$ and $\widehat{\mathscr{W}}_{X \to Y} \overset{\mathrm{L}}{\otimes}_{\widehat{\mathscr{W}}} \mathcal{N}^{\mathrm{W}} \simeq \widehat{\mathscr{W}}_Y / (x_1 \cdot \widehat{\mathscr{W}}_Y + \widehat{\mathscr{W}}_Y \cdot P) \simeq \widehat{\mathscr{W}}_X^m$.

(iv) Proper direct image. Assume that $f\colon M \to N$ is proper on the support of $\mathcal{M} \in \mathrm{Mod}_{\mathrm{gd}}(\mathcal{D}_M^{\mathrm{op}})$. We have to prove that

$$(f_*^{\mathcal{D}}\mathcal{M})^{\mathrm{W}} \simeq \mathcal{M}^{\mathrm{W}} \overset{\mathrm{L}}{\otimes}_{\widehat{\mathscr{W}}} \widehat{\mathscr{W}}_{X \to Y}.$$

(a) In case f is a closed embedding, we may assume that $N = M \times L$ for another submanifold L and that the embedding $f\colon M \hookrightarrow N$ is given by $M \xrightarrow{\sim} M \times \{a\}$, for some $a \in L$. In this case,

$$f_*^{\mathcal{D}}\mathcal{M} \simeq \mathcal{M}\underline{\boxtimes}\mathcal{B}_{a|L}$$

and the result follows from (i).

(b) Assume now that f is the projection $M = L \times N \to N$. In the sequel, we do not write the symbols π^{-1} for short and still denote by f the projection $L \times Y \to Y$ $(Y = T^*N)$. We have

$$\begin{aligned}
(f_*^{\mathcal{D}}\mathcal{M})^{\mathrm{W}} &\simeq Rf_!(\mathcal{M}\overset{\mathrm{L}}{\otimes}_{\mathcal{D}_M}(\mathcal{O}_L\underline{\boxtimes}\mathcal{D}_N)) \otimes_{\mathcal{D}_N} \widehat{\mathscr{W}_Y} \\
&\simeq Rf_!(\mathcal{M}\overset{\mathrm{L}}{\otimes}_{\mathcal{D}_M}(\mathcal{O}_L\underline{\boxtimes}\widehat{\mathscr{W}_Y})) \\
&\simeq Rf_!(\mathcal{M}\overset{\mathrm{L}}{\otimes}_{\mathcal{D}_M}\widehat{\mathscr{W}}_{X\to Y}) \\
&\simeq Rf_!(\mathcal{M}^{\mathrm{W}}\overset{\mathrm{L}}{\otimes}_{\widehat{\mathscr{W}_X}}\widehat{\mathscr{W}}_{X\to Y}).
\end{aligned}$$

Here, $\mathcal{O}_L\underline{\boxtimes}\widehat{\mathscr{W}_Y} = \mathcal{D}_M \otimes_{\mathcal{D}_L\boxtimes\mathcal{D}_N} \mathcal{O}_L \boxtimes \widehat{\mathscr{W}_Y}$. This completes the proof. q.e.d.

Chapter 4

Characteristic classes for $\mathbb{DQ}$-modules

The results of these Chapter are extracted from a paper in preparation, an extended version of [**?**]. However, all non-senses and mistakes are under the only responsibility of the author of these Notes.

4.1 Hochschild class for $\mathbb{DQ}$-modules

Let X be a complex manifold and let $\mathscr{A}_X$ be a DQ-algebroid. We define the Hochschild homology of $\mathscr{A}_X$ by setting:

$$\mathcal{HH}(\mathcal{A}_X) := \mathscr{C}_{X^a} \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_{X\times X^a}} \mathscr{C}_X, \text{ an object of } \mathrm{D^b}(\mathbf{k}_{0X}). \tag{4.1}$$

Note that, using (3.15), we get the isomorphisms:

$$\begin{aligned}
\mathcal{HH}(\mathcal{A}_X) &\simeq R\mathcal{H}om_{\mathscr{A}_{X\times X^a}}(\mathbb{D}'_{\mathscr{A}}(\mathscr{C}_{X^a}), \mathscr{C}_X)\\
&\simeq R\mathcal{H}om_{\mathscr{A}_{X\times X^a}}((\omega_X^{\mathscr{A}})^{\otimes -1}, \mathscr{C}_X)\\
&\simeq R\mathcal{H}om_{\mathscr{A}_{X\times X^a}}(\mathscr{C}_X, \omega_X^{\mathscr{A}}).
\end{aligned}$$

Here, the last isomorphism is associated with

$$\begin{aligned}
R\mathcal{H}om_{\mathscr{A}_{X\times X^a}}(\omega_X^{\mathscr{A}\otimes -1}, \mathscr{C}_X) &\to R\mathcal{H}om_{\mathscr{A}_{X\times X^a}}(\omega_X^{\mathscr{A}} \underset{X}{\circ} \omega_X^{\mathscr{A}\otimes -1}, \omega_X^{\mathscr{A}} \underset{X}{\circ} \mathscr{C}_X)\\
&\simeq R\mathcal{H}om_{\mathscr{A}_{X\times X^a}}(\mathscr{C}_X, \omega_X^{\mathscr{A}}).
\end{aligned}$$

One shall be aware that this isomorphism does not coincide in general with

$$\begin{aligned}
R\mathcal{H}om_{\mathscr{A}_{X\times X^a}}(\omega_X^{\mathscr{A}\otimes -1}, \mathscr{C}_X) &\to R\mathcal{H}om_{\mathscr{A}_{X\times X^a}}(\omega_X^{\mathscr{A}\otimes -1} \underset{X}{\circ} \omega_X^{\mathscr{A}}, \mathscr{C}_X \underset{X}{\circ} \omega_X^{\mathscr{A}})\\
&\simeq R\mathcal{H}om_{\mathscr{A}_{X\times X^a}}(\mathscr{C}_X, \omega_X^{\mathscr{A}}).
\end{aligned}$$

Let $\mathcal{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X)$. We have the chain of morphisms

$$R\mathcal{H}om_{\mathscr{A}_X}(\mathcal{M},\mathcal{M}) \xleftarrow{\sim} R\mathcal{H}om_{\mathscr{A}_X}(\mathcal{M},\mathscr{A}_X)\overset{\mathrm{L}}{\otimes}_{\mathscr{A}_X}\mathcal{M} \tag{4.2}$$

$$\simeq \mathscr{C}_{X^a}\overset{\mathrm{L}}{\otimes}_{\mathscr{A}_{X\times X^a}}(\mathcal{M}\underline{\boxtimes}\mathbb{D}'_{\mathscr{A}}\mathcal{M}) \tag{4.3}$$

$$\to \mathscr{C}_{X^a}\overset{\mathrm{L}}{\otimes}_{\mathscr{A}_{X\times X^a}}\mathscr{C}_X = \mathcal{HH}(\mathcal{A}_X). \tag{4.4}$$

We get a map

$$\mathrm{Hom}_{\mathscr{A}_X}(\mathcal{M},\mathcal{M}) \to H^0_{\mathrm{supp}(\mathcal{M})}(X;\mathcal{HH}(\mathcal{A}_X)).$$

For $u \in \mathrm{End}(\mathcal{M})$, the image of u gives an element

$$\mathrm{hh}_X((\mathcal{M},u)) \in H^0_{\mathrm{supp}(\mathcal{M})}(X;\mathcal{HH}(\mathcal{A}_X)). \tag{4.5}$$

Definition 4.1.1. Let $\mathcal{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X)$. We set $\mathrm{hh}_X(\mathcal{M}) = \mathrm{hh}_X((\mathcal{M},\mathrm{id}_{\mathcal{M}}))$ and call it the Hochschild class of $\mathcal{M}$.

Denote by $s\colon X\times X^a \to X^a\times X$ the map $(x,y)\mapsto(y,x)$ and recall that δ is the diagonal embedding. Then $s\circ\delta=\delta$, $s^{-1}\mathscr{C}_X \simeq \mathscr{C}_{X^a}$, $s^{-1}\mathscr{A}_{X\times X^a} \simeq \mathscr{A}_{X^a\times X}$ and we obtain the isomorphisms

$$\begin{aligned}
\delta^{-1}\mathcal{HH}(\mathcal{A}_X) &= \delta^{-1}(\mathscr{C}_{X^a}\overset{\mathrm{L}}{\otimes}_{\mathscr{A}_{X\times X^a}}\mathscr{C}_X)\\
&\simeq \delta^{-1}s^{-1}(\mathscr{C}_{X^a}\overset{\mathrm{L}}{\otimes}_{\mathscr{A}_{X\times X^a}}\mathscr{C}_X)\\
&\simeq \delta^{-1}(s^{-1}\mathscr{C}_{X^a}\overset{\mathrm{L}}{\otimes}_{s^{-1}\mathscr{A}_{X\times X^a}}s^{-1}\mathscr{C}_X)\\
&\simeq \delta^{-1}(\mathscr{C}_X\overset{\mathrm{L}}{\otimes}_{\mathscr{A}_{X^a\times X}}\mathscr{C}_{X^a}) = \delta^{-1}\mathcal{HH}(\mathcal{A}_{X^a}).
\end{aligned}$$

After identifying $\mathcal{HH}(\mathcal{A}_X)$ and $\mathcal{HH}(\mathcal{A}_{X^a})$ by the isomorphism above, we have

$$\mathrm{hh}_{X^a}(\mathbb{D}'_{\mathscr{A}}\mathcal{M}) = \mathrm{hh}_X(\mathcal{M}). \tag{4.6}$$

If $X = \mathrm{pt}$, then $\mathcal{HH}(\mathcal{A}_X)$ is isomorphic to $\mathbf{k}_0$, and for $\mathcal{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X)$ and $u \in \mathrm{End}(\mathcal{M})$, we have

$$\mathrm{hh}_{\mathrm{pt}}((\mathcal{M},u)) = \sum_{i\in\mathbb{Z}}(-1)^i\,\mathrm{tr}(u;\mathbf{k}\otimes_{\mathbf{k}_0}H^i(\mathcal{M})).$$

We get:

$$\begin{aligned}
\mathrm{hh}_{\mathrm{pt}}(\mathcal{M}) &= \chi(\mathcal{M}) := \sum_{i\in\mathbb{Z}}(-1)^i\dim_{\mathbf{k}}(\mathbf{k}\otimes_{\mathbf{k}_0}H^i(\mathcal{M}))\\
&= \sum_{i\in\mathbb{Z}}(-1)^i\big(\dim_{\mathbb{C}}(\mathbb{C}\otimes_{\mathbf{k}_0}H^i(\mathcal{M})) - \dim_{\mathbb{C}}\mathrm{Tor}_1^{\mathbf{k}_0}(\mathbb{C},H^i(\mathcal{M}))\big).
\end{aligned}$$

Proposition 4.1.2. *Let* $\mathcal{M}' \to \mathcal{M} \to \mathcal{M}' \xrightarrow{+1}$ *be a distinguished triangle in* $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X)$*. Then* $\mathrm{hh}_X(\mathcal{M}) = \mathrm{hh}_X(\mathcal{M}') + \mathrm{hh}_X(\mathcal{M}'')$.

Proof. This follows from a theorem of May [29]. q.e.d.

Let X_i be complex manifolds endowed with DQ-algebroids $\mathscr{A}_{X_i}$ ($i = 1,2,3$) and denote as usual by p_{ij} the projection from $X_1 \times X_2 \times X_3$ to $X_i \times X_j$ ($1 \le i < j \le 3$).

Proposition 4.1.3. *There is a natural morphism*

$$\circ : Rp_{13!}(p_{12}^{-1}\mathcal{HH}(\mathcal{A}_{X_1\times X_2^a}) \overset{\mathrm{L}}{\otimes} p_{23}^{-1}\mathcal{HH}(\mathcal{A}_{X_2\times X_3^a})) \to \mathcal{HH}(\mathcal{A}_{X_1\times X_3^a}).$$

Proof. (i) Set $Z_i = X_i \times X_i^a$. We shall denote by the same letter p_{ij} the projection from $Z_1 \times Z_2 \times Z_3$ to $Z_i \times Z_j$.

We have

$$\begin{aligned}
&\mathcal{HH}(\mathcal{A}_{X_i\times X_j^a}) \\
&\qquad \simeq (\mathscr{C}_{X_i^a} \overset{\mathrm{L}}{\underline{\boxtimes}} \mathscr{C}_{X_j}) \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_{Z_i\times Z_j^a}} (\mathscr{C}_{X_i} \overset{\mathrm{L}}{\underline{\boxtimes}} \mathscr{C}_{X_j^a}) \\
&\qquad \simeq R\mathcal{H}om_{\mathscr{A}_{Z_i\times Z_j^a}}(\omega_{X_i}^{\mathscr{A}\otimes -1} \overset{\mathrm{L}}{\underline{\boxtimes}} \omega_{X_j^a}^{\mathscr{A}\otimes -1}, \mathscr{C}_{X_i} \overset{\mathrm{L}}{\underline{\boxtimes}} \mathscr{C}_{X_j{}^a}) \\
&\qquad \simeq R\mathcal{H}om_{\mathscr{A}_{Z_i\times Z_j^a}}\big((\omega_{X_i}^{\mathscr{A}\otimes -1} \overset{\mathrm{L}}{\underline{\boxtimes}} \omega_{X_j^a}^{\mathscr{A}\otimes -1}) \overset{\mathrm{D}}{\otimes}_{\mathscr{A}_{X_j^a}} \omega_{X_j^a}^{\mathscr{A}}, (\mathscr{C}_{X_i} \overset{\mathrm{L}}{\underline{\boxtimes}} \mathscr{C}_{X_j{}^a}) \overset{\mathrm{D}}{\otimes}_{\mathscr{A}_{X_j^a}} \omega_{X_j^a}^{\mathscr{A}}\big) \\
&\qquad \simeq R\mathcal{H}om_{\mathscr{A}_{Z_i\times Z_j^a}}(\omega_{X_i}^{\mathscr{A}\otimes -1} \overset{\mathrm{L}}{\underline{\boxtimes}} \mathscr{C}_{X_j^a}, \mathscr{C}_{X_i} \overset{\mathrm{L}}{\underline{\boxtimes}} \omega_{X_j{}^a}^{\mathscr{A}}).
\end{aligned}$$

Set $S_{ij} := \omega_{X_i}^{\mathscr{A}\otimes -1} \overset{\mathrm{L}}{\underline{\boxtimes}} \mathscr{C}_{X_j^a} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_{Z_i\times Z_j^a})$ and $K_{ij} := \mathscr{C}_{X_i} \overset{\mathrm{L}}{\underline{\boxtimes}} \omega_{X_j^a}^{\mathscr{A}} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_{Z_i\times Z_j^a})$. Then we get

$$\mathcal{HH}(\mathcal{A}_{X_i\times X_j^a}) \simeq R\mathcal{H}om_{\mathscr{A}_{Z_i\times Z_j^a}}(S_{ij}, K_{ij}).$$

Thus we obtain

$$\begin{aligned}
&p_{12}^{-1}\mathcal{HH}(\mathcal{A}_{X_1\times X_2^a}) \overset{\mathrm{L}}{\otimes} p_{23}^{-1}\mathcal{HH}(\mathcal{A}_{X_2\times X_3^a}) \\
&\qquad\qquad \simeq p_{12}^{-1} R\mathcal{H}om_{\mathscr{A}_{Z_1\times Z_2^a}}(S_{12}, K_{12}) \overset{\mathrm{L}}{\otimes} p_{23}^{-1} R\mathcal{H}om_{\mathscr{A}_{Z_2\times Z_3^a}}(S_{23}, K_{23}) \\
&\qquad\qquad \to R\mathcal{H}om_{\mathscr{A}_{Z_1\times Z_3^a}}\big(S_{12} \overset{\mathrm{D}}{\otimes}_{\mathscr{A}_{Z_2}} S_{23}, K_{12} \overset{\mathrm{D}}{\otimes}_{\mathscr{A}_{Z_2}} K_{23}\big).
\end{aligned}$$

We get a morphism

$$\begin{aligned}
&Rp_{13!}(p_{12}^{-1}\mathcal{HH}(\mathcal{A}_{X_1\times X_2^a}) \overset{\mathrm{L}}{\otimes} p_{23}^{-1}\mathcal{HH}(\mathcal{A}_{X_2\times X_3^a})) \\
&\qquad \to Rp_{13!} R\mathcal{H}om_{\mathscr{A}_{Z_1\times Z_3^a}}\big(S_{12} \overset{\mathrm{D}}{\otimes}_{\mathscr{A}_{Z_2}} S_{23}, K_{12} \overset{\mathrm{D}}{\otimes}_{\mathscr{A}_{Z_2}} K_{23}\big).
\end{aligned} \tag{4.7}$$

(ii) We have a morphism

$$\mathbf{k}_{0X_2} \to R\mathcal{H}om_{\mathscr{A}_{Z_2^a}}(\mathscr{C}_{X_2^a}, \mathscr{C}_{X_2^a}) \simeq \mathscr{C}_{X_2^a} \overset{\mathrm{D}}{\otimes}_{\mathscr{A}_{Z_2}} \omega_{X_2}^{\mathscr{A}\otimes -1},$$

which induces the morphism:

$$p_{13}^{-1}(\omega_{X_1}^{\mathscr{A}\otimes -1} \overset{\mathrm{L}}{\underline{\boxtimes}} \mathscr{C}_{X_3^a}) \to ((\omega_{X_1}^{\mathscr{A}})^{\otimes -1} \overset{\mathrm{L}}{\underline{\boxtimes}} \mathscr{C}_{X_2^a}) \overset{\mathrm{D}}{\otimes}_{\mathscr{A}_{Z_2}} (\omega_{X_2}^{\mathscr{A}\otimes -1} \overset{\mathrm{L}}{\underline{\boxtimes}} \mathscr{C}_{X_3^a}),$$

that is, the morphism in $\mathrm{D}^{\mathrm{b}}(\mathscr{A}_{Z_1 \times Z_3^a})$:

$$S_{13} \to Rp_{13*}(S_{12} \overset{\mathrm{D}}{\otimes}_{\mathscr{A}_{Z_2}} S_{23}). \tag{4.8}$$

(iii) We have a morphism:

$$(\mathscr{C}_{X_1} \overset{\mathrm{L}}{\underline{\boxtimes}} \omega_{X_2^a}^{\mathscr{A}}) \overset{\mathrm{D}}{\otimes}_{\mathscr{A}_{Z_2}} (\mathscr{C}_{X_2} \overset{\mathrm{L}}{\underline{\boxtimes}} \omega_{X_3^a}^{\mathscr{A}}) \to p_{13}^{-1}(\mathscr{C}_{X_1} \overset{\mathrm{L}}{\underline{\boxtimes}} \omega_{X_3^a}^{\mathscr{A}})[2d_Y],$$

which induces the morphism in $\mathrm{D}^{\mathrm{b}}(\mathscr{A}_{Z_1 \times Z_3^a})$:

$$Rp_{13!}(K_{12} \overset{\mathrm{D}}{\otimes}_{\mathscr{A}_{Z_2}} K_{23}) \to K_{13}. \tag{4.9}$$

(iv) Using (4.8) and (4.9) we obtain

$$\begin{aligned} &Rp_{13!} R\mathcal{H}om_{\mathscr{A}_{Z_1 \times Z_3^a}}\big(S_{12} \overset{\mathrm{D}}{\otimes}_{\mathscr{A}_{Z_2}} S_{23}, K_{12} \overset{\mathrm{D}}{\otimes}_{\mathscr{A}_{Z_2}} K_{23}\big) \\ &\qquad \to R\mathcal{H}om_{\mathscr{A}_{Z_1 \times Z_3^a}}\big(Rp_{13*}(S_{12} \overset{\mathrm{D}}{\otimes}_{\mathscr{A}_{Z_2}} S_{23}), Rp_{13!}(K_{12} \overset{\mathrm{D}}{\otimes}_{\mathscr{A}_{Z_2}} K_{23})\big) \\ &\qquad \to R\mathcal{H}om_{\mathscr{A}_{Z_1 \times Z_3^a}}(S_{13}, K_{13}) \simeq \mathcal{HH}(\mathcal{A}_{X_1 \times X_3^a}). \end{aligned} \tag{4.10}$$

Combining (4.7) and (4.10), we get the result. q.e.d.

Consider four manifolds X_i endowed with DQ-algebroids $\mathscr{A}_{X_i}$ $(i = 1, 2, 3, 4)$.

Notation 4.1.4. In the sequel and until the end of this section, when there is no risk of confusion, we use the following conventions.
(i) For $i, j \in \{1, 2, 3, 4\}$, we set $X_{ij^a} := X_i \times X_j^a$ and similarly with X_{ij}, X_{ijk}, etc.
(ii) We do not write the symbols $p_{ij}, p_{ij*}, p_{ij}^{-1}$, etc.
(iii) We write $\mathscr{A}_i$ instead of $\mathscr{A}_{X_i}$, $\mathscr{A}_{ij^a}$ instead of $\mathscr{A}_{X_{ij^a}}$ and similarly with $\mathscr{C}_X, \omega_X^{\mathscr{A}}$, etc. Moreover, we even sometimes write $\mathcal{H}om_i$ instead of $\mathcal{H}om_{\mathscr{A}_i}$ and $\otimes_i$ instead of $\otimes_{\mathscr{A}_i}$ and similarly with ij^a, ijk, etc.

(iv) We write $\mathbb{D}'_{\mathcal{D}}$ instead of $\mathbb{D}'_{\mathscr{A}}$ and ω_X instead of $\omega_X^{\mathscr{A}}$.
(v) We write $\omega_{ij/j}$ instead of $\omega_i \boxtimes \mathscr{C}_j$ and similarly with ij^a, ijk, etc. and with $\omega_X^{\otimes -1}$ instead of ω_X.
(vi) We often identify an invertible object of $\mathrm{D^b}(\mathscr{A}_X)$ with an object of $\mathrm{D^b}(\mathscr{A}_{X\times X^a})$ supported by the diagonal.

Let Z be a closed subset of X. We set

$$\mathrm{HH}_Z(\mathcal{A}_X) := H^0\mathrm{R}\Gamma_Z(X;\mathcal{HH}(\mathcal{A}_X)). \tag{4.11}$$

Let $\Lambda_i \subset X_{ij}$ $(i=1,2,\ j=i+1)$ and assume that $p_{12}^{-1}\Lambda_1 \cap p_{23}^{-1}\Lambda_2$ is proper over $X_1 \times X_3$. Using Proposition 4.1.3, we get a map

$$\circ\colon \mathrm{HH}_{\Lambda_1}(\mathcal{A}_{X_{12^a}}) \times \mathrm{HH}_{\Lambda_2}(\mathcal{A}_{X_{23^a}}) \longrightarrow \mathrm{HH}_{\Lambda_1\circ\Lambda_2}(\mathcal{A}_{X_{23^a}}). \tag{4.12}$$

Let $C_i \in \mathrm{HH}_{\Lambda_i}(\mathcal{A}_{X_{ij^a}})$ $(i=1,2,\ j=i+1)$. We obtain a class

$$C_1 \circ C_2 \in \mathrm{HH}_{\Lambda_1\circ\Lambda_2}(\mathcal{A}_{X_{13^a}}). \tag{4.13}$$

Now let $\Lambda_i \subset X_{ij}$ $(i=1,2,3,\ j=i+1)$ and assume that $p_{ij}^{-1}\Lambda_i \cap p_{jk}^{-1}\Lambda_j$ is proper over X_{ik} $(i=1,2,\ j=1+1,\ k=j+1)$.

Lemma 4.1.5. *Let $C_i \in \mathrm{HH}_{\Lambda_i}(\mathcal{A}_{X_{ij^a}})$ $(i=1,2,3,\ j=i+1)$.*

(a) *One has $(C_1\circ C_2)\circ C_3 = C_1\circ(C_2\circ C_3)$.*

(b) *Set $C_{\Delta_i} = \mathrm{hh}_{X_{ii^a}}(\mathscr{C}_{X_i})$. Then $(C_1\circ C_2)\circ C_{\Delta_3} = C_1\circ C_2$ and $C_{\Delta_1}\circ(C_2\circ C_3) = C_2\circ C_3$.*

The main result of [**?**] is:

Theorem 4.1.6. *Let $\mathcal{K}_i \in \mathrm{D^b_{coh}}(\mathscr{A}_{X_i\times X_{i+1}^a})$ $(i=1,2)$ set $\Lambda_i = \mathrm{supp}(\mathcal{K}_i)$ and assume that $\Lambda_1 \times_{X_2} \Lambda_2$ is proper over $X_1\times X_3$. Then*

$$\mathrm{hh}_{X_{13^a}}(\mathcal{K}_1\circ\mathcal{K}_2) = \mathrm{hh}_{X_{12^a}}(\mathcal{K}_1)\circ \mathrm{hh}_{X_{23^a}}(\mathcal{K}_2) \tag{4.14}$$

as elements of $\mathrm{HH}_{\Lambda_1\circ\Lambda_2}(\mathcal{A}_{X_1\times X_3^a})$.

Remark 4.1.7. (i) The fact that Hochschild homology of $\mathscr{O}$-modules is functorial is well-known, see *e.g.,* [8].
(ii) In [9], its authors interpret Hochschild homology as a morphism of functors and the action of kernels as a 2-morphism in a suitable 2-category. Their result applies in a general framework including in particular $\mathscr{O}$-modules in the algebraic case and presumably DQ-modules but the precise axiomatic is not specified in loc. cit. Note that, as far as we understand, even if their framework include DQ-modules, their result would not give Theorem 4.1.6.

As a particular case of Theorem 4.1.6, consider two objects $\mathcal{M}$ and $\mathcal{N}$ in $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X)$ and assume that $\operatorname{supp}(\mathcal{M}) \cap \operatorname{supp}(\mathcal{N})$ is compact. Then the cohomology groups of the complex $\mathrm{RHom}_{\mathscr{A}_X}(\mathcal{M},\mathcal{N})$ are finitely generated $\mathbf{k}_0$-modules and

$$\chi(\mathrm{RHom}_{\mathscr{A}_X}(\mathcal{M},\mathcal{N})) = \mathrm{hh}_{X^a}(\mathbb{D}'_{\mathscr{A}}\mathcal{M}) \circ \mathrm{hh}_X(\mathcal{N}).$$

4.2 Hochschild class for $\mathcal{O}$-modules

Note that the results of this section are well known from the specialists.

Let $(X, \mathscr{O}_X)$ be a complex manifold of complex dimension d_X. As usual, we denote by $\delta_X \colon X \hookrightarrow X \times X$ the diagonal embedding. We denote by Ω^i_X the sheaf of holomorphic i-forms and one sets $\Omega_X := \Omega^{d_X}_X$. We set

$$\omega_X := \Omega_X\,[d_X].$$

When there is no risk of confusion, we identify Ω_X and $\delta_{X*}\Omega_X$, an $\mathscr{O}_{X\times X}$-module, and similarly with ω_X. We denote by $\mathbb{D}'_{\mathscr{O}}$ and $\mathbb{D}_{\mathscr{O}}$ the duality functors

$$\mathbb{D}_{\mathscr{O}}(\mathcal{F}) = R\mathcal{H}om_{\mathscr{O}_X}(\mathcal{F}, \mathscr{O}_X), \quad \mathbb{D}'_{\mathscr{O}}(\mathcal{F}) = R\mathcal{H}om_{\mathscr{O}_X}(\mathcal{F}, \omega_X).$$

When there is no risk of confusion, we write $\mathbb{D}'_{\mathcal{D}}$ and D instead of $\mathbb{D}_{\mathscr{O}}$ and $\mathbb{D}'_{\mathscr{O}}$, respectively.

Let $f \colon X \to Y$ be a morphism of complex manifolds. For $\mathcal{G} \in \mathrm{D}^{\mathrm{b}}(\mathscr{O}_Y)$, we set

$$f^*\mathcal{G} := \mathscr{O}_X \overset{\mathrm{L}}{\otimes}_{f^{-1}\mathscr{O}_Y} f^{-1}\mathcal{G}.$$

The Hochschild homology of $\mathscr{O}_X$ is given by:

$$\mathcal{HH}(\mathcal{O}_X) := \delta_{X*}\mathscr{O}_X \overset{\mathrm{L}}{\otimes}_{\mathscr{O}_{X\times X}} \delta_{X*}\mathscr{O}_X = \delta^*_X \delta_{X*}\mathscr{O}_X, \text{ an object of } \mathrm{D}^{\mathrm{b}}(\mathscr{O}_X).$$

By reformulating the construction of the Hochschild class for modules over DQ-algebroids, we get

Definition 4.2.1. For $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_X)$, we define its Hochschild class $\mathrm{hh}_X(\mathcal{F}) \in H^0_{\operatorname{supp}\mathcal{F}}(X; \delta^*_X\delta_{X*}\mathscr{O}_X)$ as the composition

$$\mathscr{O}_X \to R\mathcal{H}om_{\mathscr{O}_X}(\mathcal{F},\mathcal{F}) \xrightarrow{\sim} \delta^*_X(\mathcal{F} \underline{\boxtimes} \mathbb{D}'_{\mathcal{D}}\mathcal{F}) \to \delta^*_X\delta_{X*}\mathscr{O}_X. \tag{4.15}$$

Here the morphism $\mathcal{F} \underline{\boxtimes} \mathbb{D}'_{\mathcal{D}}\mathcal{F} \to \delta_{X*}\mathscr{O}_X$ is deduced from the morphism $\delta^*_X(\mathcal{F} \underline{\boxtimes} \mathbb{D}'_{\mathcal{D}}\mathcal{F}) \xrightarrow{\sim} \mathcal{F} \overset{\mathrm{L}}{\otimes}_{\mathscr{O}_X} \mathbb{D}'_{\mathcal{D}}\mathcal{F} \to \mathscr{O}_X$.

Applying Theorem 4.1.6, we get that for two complex manifolds X and Y and for $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_X)$ and $\mathcal{G} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_Y)$, we have

$$\mathrm{hh}_{X\times Y}(\mathcal{F}\boxtimes\mathcal{G}) = \mathrm{hh}_X(\mathcal{F}) \boxtimes \mathrm{hh}_Y(\mathcal{G}).$$

Let $f\colon X \to Y$ be a morphism of complex manifolds and denote by $\Gamma_f \subset X \times Y$ its graph. We denote by $\mathrm{hh}_{X\times Y}(\mathscr{O}_{\Gamma_f})$ the Hochschild class of the coherent $\mathscr{O}_{X\times Y}$-module $\mathscr{O}_{\Gamma_f}$. Hence

$$\mathrm{hh}_{X\times Y}(\mathscr{O}_{\Gamma_f}) \in H^0(X \times Y; \mathcal{HH}(\mathcal{O}_{X\times Y})).$$

Applying Theorem 4.1.6, we get

Corollary 4.2.2. (i) *Let $\mathcal{G} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_Y)$. Then*

$$\mathrm{hh}_X(f^*\mathcal{G}) = \mathrm{hh}_{X\times Y}(\mathscr{O}_{\Gamma_f}) \circ \mathrm{hh}_Y(\mathcal{G}).$$

(ii) *Let $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_X)$ and assume that f is proper on* $\mathrm{supp}(\mathcal{F})$. *Then*

$$\mathrm{hh}_Y(f_!\mathcal{F}) = \mathrm{hh}_X(\mathcal{F}) \circ \mathrm{hh}_{X\times Y}(\mathscr{O}_{\Gamma_f}).$$

In Proposition 4.2.3 and 4.2.6 below, we give a more direct description of the maps $\mathrm{hh}_{X\times Y}(\mathscr{O}_{\Gamma_f})\circ$ and $\circ\,\mathrm{hh}_{X\times Y}(\mathscr{O}_{\Gamma_f})$.

Proposition 4.2.3. *Let $f\colon X \to Y$ be a morphism of complex manifolds.*

(i) *There is a canonical morphism*

$$f^*\delta_Y^*\delta_{Y*}\mathscr{O}_Y \to \delta_X^*\delta_{X*}\mathscr{O}_X. \tag{4.16}$$

(ii) *This morphism together with the isomorphism $\mathscr{O}_X \xleftarrow{\sim} f^*\mathscr{O}_Y$ induces a morphism*

$$f^*\colon \mathrm{R}\Gamma(Y; \delta_Y^*\delta_{Y*}\mathscr{O}_Y) \to \mathrm{R}\Gamma(X; \delta_X^*\delta_{X*}\mathscr{O}_X) \tag{4.17}$$

and for $\mathcal{G} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_Y)$, we have

$$\mathrm{hh}_X(f^*\mathcal{G}) = f^*\mathrm{hh}_Y(\mathcal{G}). \tag{4.18}$$

Hochschild-Euler class

Definition 4.2.4. For $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_X)$, we define its Hochschild-Euler class $\mathrm{hh}^{\mathrm{e}}_{\mathrm{X}}(\mathcal{F}) \in \mathrm{H}^0_{\mathrm{supp}\,\mathcal{F}}(\mathrm{X}; \delta^!_{\mathrm{X}}\delta_{\mathrm{X}!}\omega_{\mathrm{X}})$ as the composition

$$\mathscr{O}_X \to R\mathcal{H}om_{\mathscr{O}_X}(\mathcal{F},\mathcal{F}) \simeq \delta^!_X(\mathcal{F}\underline{\boxtimes}\mathrm{D}\mathcal{F}) \to \delta^!_X\delta_{X!}\omega_X. \tag{4.19}$$

Here, the morphism $(\mathcal{F}\underline{\boxtimes}\mathrm{D}\mathcal{F}) \to \delta_{X!}\omega_X$ is induced by $\delta^*_X(\mathcal{F}\underline{\boxtimes}\mathrm{D}\mathcal{F}) \simeq \mathcal{F}\overset{\mathrm{L}}{\otimes}_{\mathscr{O}_X}\mathrm{D}\mathcal{F} \to \omega_X$.

Consider the sequence of isomorphisms

$$\begin{aligned}
\delta^*_X\delta_{X*}\mathscr{O}_X &\xrightarrow{\sim} \mathscr{O}_X\overset{\mathrm{L}}{\otimes}_{\mathscr{O}_X}\delta^*_X\delta_{X*}\mathscr{O}_X \xrightarrow{\sim} \delta^!_X(\mathscr{O}_X\underline{\boxtimes}\omega_X)\overset{\mathrm{L}}{\otimes}_{\mathscr{O}_X}\delta^*_X\delta_{X*}\mathscr{O}_X\\
&\xrightarrow{\sim} \delta^!_X((\mathscr{O}_X\underline{\boxtimes}\omega_X)\overset{\mathrm{L}}{\otimes}_{\mathscr{O}_X}\delta_{X*}\mathscr{O}_X) \xrightarrow{\sim} \delta^!_X\delta_{X*}(\delta^*_X(\mathscr{O}_X\underline{\boxtimes}\omega_X\overset{\mathrm{L}}{\otimes}_{\mathscr{O}_X}\mathscr{O}_X)\\
&\xrightarrow{\sim} \delta^!_X\delta_{X!}\omega_X.
\end{aligned}$$

We denote by $\widetilde{\mathrm{td}}$ the isomorphism

$$\widetilde{\mathrm{td}}\colon \delta^*_X\delta_{X*}\mathscr{O}_X \xrightarrow{\sim} \delta^!_X\delta_{X!}\omega_X \tag{4.20}$$

constructed above.

Proposition 4.2.5. *For $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_X)$, we have*

$$\mathrm{hh}^{\mathrm{e}}_{\mathrm{X}}(\mathcal{F}) = \widetilde{\mathrm{td}} \circ \mathrm{hh}_{\mathrm{X}}(\mathcal{F}). \tag{4.21}$$

Proof. The proof follows from the commutativity of the diagram below in which we use the natural morphism $\mathscr{O}_X \to \delta^!_X(\mathscr{O}_X \boxtimes \omega_X)$

$$\begin{tikzcd}
\mathscr{O}_X \arrow[r] \arrow[ddddr] & \delta^*_X(\mathcal{F}\boxtimes\mathbb{D}'_{\mathcal{D}}\mathcal{F}) \arrow[r] \arrow[d] & \delta^*_X\delta_{X*}\mathscr{O}_X \arrow[d, "\sim"] \\
 & \delta^!_X(\mathscr{O}_X\boxtimes\omega_X)\otimes_{\mathcal{O}}\delta^*_X(\mathcal{F}\boxtimes\mathbb{D}'_{\mathcal{D}}\mathcal{F}) \arrow[r] \arrow[d] & \delta^!_X(\mathscr{O}_X\boxtimes\omega_X)\otimes_{\mathcal{O}}\delta^*_X\delta_{X*}\mathscr{O}_X \arrow[d, "\sim"] \\
 & \delta^!_X((\mathscr{O}_X\boxtimes\omega_X)\otimes_{\mathcal{O}}(\mathcal{F}\boxtimes\mathbb{D}'_{\mathcal{D}}\mathcal{F})) \arrow[r] \arrow[dd] & \delta^!_X((\mathscr{O}_X\boxtimes\omega_X)\otimes_{\mathcal{O}}\delta_{X*}\mathscr{O}_X) \arrow[d, "\sim"] \\
 & & \delta^!_X\delta_{X*}(\delta^!_X\mathscr{O}_X\boxtimes\omega_X)\otimes_{\mathcal{O}}\mathscr{O}_X) \arrow[d, "\sim"] \\
 & \delta^!_X(\mathcal{F}\boxtimes\mathrm{D}\mathcal{F}) \arrow[r] & \delta^!_X\delta_{X!}\omega_X
\end{tikzcd}$$

q.e.d.

For a morphism $f\colon X \to Y$ of complex manifolds, we denote by $\Gamma_{\mathrm{f-pr}}(X;\bullet)$ the functor of global sections with f-proper supports.

Proposition 4.2.6. *Let $f\colon X \to Y$ be a morphism of complex manifolds.*

(i) *There is a canonical morphism*

$$f_!\delta_X^!\delta_{X!}\omega_X \to \delta_Y^!\delta_{Y!}\omega_Y. \tag{4.22}$$

(ii) *This morphism together with the morphism $\mathscr{O}_Y \to f_*\mathscr{O}_X$ induces a morphism*

$$f_!\colon \mathrm{R}\Gamma_{\mathrm{f-pr}}(X;\delta_X^!\delta_{X!}\omega_X) \to \mathrm{R}\Gamma(Y;\delta_Y^!\delta_{Y!}\omega_Y) \tag{4.23}$$

and for $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_X)$ such that f is proper on $\mathrm{supp}(\mathcal{F})$*, we have*

$$\mathrm{hh}^{\mathrm{e}}_{\mathrm{Y}}(\mathrm{f}_!\mathcal{F}) = \mathrm{f}_!\,\mathrm{hh}^{\mathrm{e}}_{\mathrm{X}}(\mathcal{F}). \tag{4.24}$$

Chern and Euler classes of $\mathcal{O}$-modules

We shall make the link between the Hochschild class and the Chern and Euler classes of coherent $\mathscr{O}_X$-modules, following a unpublished letter from Masaki Kashiwara to P.S, dated 18/11/1991.

The Hodge cohomology of $\mathscr{O}_X$ is given by:

$$\mathcal{HD}_X := \bigoplus_{i=0}^{d_X} \Omega_X^i\,[i],\ \text{an object of } \mathrm{D}^{\mathrm{b}}(\mathscr{O}_X). \tag{4.25}$$

Lemma 4.2.7. *Let $f\colon X \to Y$ be a morphism of complex manifolds. There are canonical morphisms*

$$\underline{\boxtimes} \quad : \quad \mathcal{HD}_X \boxtimes \mathcal{HD}_Y \to \mathcal{HD}_{X\times Y}, \tag{4.26}$$

$$f^* \quad : \quad f^*\mathcal{HD}_Y \to \mathcal{HD}_X, \tag{4.27}$$

$$f_! \quad : \quad Rf_!\mathcal{HD}_X \to \mathcal{HD}_Y. \tag{4.28}$$

Proof. The morphism (4.26), (4.27) and (4.28) are respectively associated with the morphisms

$$\begin{aligned}
&\Omega_X^i\,[i] \boxtimes \Omega_Y^j\,[j] \to \Omega_{X\times Y}^{i+j}\,[i+j],\\
&f^*\Omega_Y^i\,[i] \to \Omega_X^i\,[i],\\
&Rf_!\Omega_X^{i+d_X}\,[i+d_X] \to \Omega_Y^{i+d_Y}\,[i+d_Y].
\end{aligned}$$

q.e.d.

Theorem 4.2.8. *There is a commutative diagram in* $\mathrm{D^b}(\mathscr{O}_X)$*:*

$$\begin{tikzcd} \delta_X^*\delta_{X*}\mathscr{O}_X \arrow[r, "\sim", "\widetilde{\mathrm{td}}"'] \arrow[d, "\alpha_X"', "\sim"] & \delta_X^!\delta_{X!}\omega_X \\ \mathcal{HD}_X \arrow[r, "\sim", "\tau"'] & \mathcal{HD}_X. \arrow[u, "\sim", "\beta_X"'] \end{tikzcd} \tag{4.29}$$

In this diagram, all arrows are isomorphisms, α_X *commutes to the functor* $\boxtimes$ *and the functor* f^* *and* β_X *commutes to the functor* $\boxtimes$ *and the functor* $f_!$.

Definition 4.2.9. For $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_X)$, we set

$$\mathrm{ch}(\mathcal{F}) = \alpha_X \circ \mathrm{hh}_X(\mathcal{F}) \in \bigoplus_{i=0}^{d_X} H^i_{\mathrm{supp}(\mathcal{F})}(X;\Omega^i_X), \tag{4.30}$$

$$\mathrm{eu}(\mathcal{F}) = \beta_X \circ \mathrm{hh}^{\mathrm{e}}_{\mathrm{X}}(\mathcal{F}) \in \bigoplus_{\mathrm{i}=0}^{\mathrm{d_X}} \mathrm{H}^{\mathrm{i}}_{\mathrm{supp}(\mathcal{F})}(\mathrm{X};\Omega^{\mathrm{i}}_{\mathrm{X}}). \tag{4.31}$$

We call $\mathrm{ch}(\mathcal{F})$ the Chern class of $\mathcal{F}$ and $\mathrm{eu}(\mathcal{F})$ the Euler class of $\mathcal{F}$.

Of course, $\mathrm{ch}(\mathcal{F})$ coincides with the classical Chern character and the morphism α_X is the so-called Hochschild-Kostant-Rosenberg map.

Conjecture 4.2.10. *One has* $\mathrm{eu}(\mathcal{F}) = \mathrm{ch}(\mathcal{F}) \cup \mathrm{td}(X)$*, where* $\mathrm{td}(X)$ *is the Todd class of the tangent bundle* TX*.*

Note that this conjecture (of Kashiwara in 1991) is considered as a well known result by many specialists, but it is not clear for us that it has been really proved in this formulation.

4.3 Hochschild class on symplectic manifolds

In all this section, X denotes a complex manifold endowed with a DQ-algebroid $\mathscr{A}_X$ such that the Poisson structure attached to $\mathscr{A}_X$ is symplectic. Hence, X is symplectic and we denote by α_X the symplectic form on X.

We set $2n = d_X$, $Z = X \times X^a$ and we denote by dv the volume form on X, $dv = \alpha_X^n/n!$.

Recall that the object $\omega_X^{\mathscr{A}}$ is defined in (3.15).

Theorem 4.3.1. *Assume that* X *is symplectic.*

(i) $L \simeq \hbar^{d_X/2}\mathbf{k}_{0X}$.

(ii) *There is a canonical $\mathscr{A}_Z$-linear isomorphism* $\Omega_X^{\mathscr{A}} \xrightarrow{\sim} \hbar^{d_X/2}\mathbf{k}_0 \otimes_{\mathbf{k}_0} \mathscr{C}_X$.

(iii) *This isomorphism induces canonical morphisms*

$$\hbar^{d_X/2}\mathbf{k}_{0X}[d_X] \xrightarrow{\iota_X} \mathcal{HH}(\mathcal{A}_X) \xrightarrow{\tau_X} \hbar^{-d_X/2}\mathbf{k}_{0X}[d_X] \tag{4.32}$$

and the composition $\tau_X \circ \iota_X$ is the canonical morphism $\hbar^{d_X/2}\mathbf{k}_{0X}[d_X] \to \hbar^{-d_X/2}\mathbf{k}_{0X}[d_X]$.

(iv) $H^j(\mathcal{HH}(\mathcal{A}_X)) \simeq 0$ *unless* $-d_X \le j \le 0$ *and the morphism ι_X induces an isomorphism*

$$\iota_X\colon \hbar^{d_X/2}\mathbf{k}_{0X} \xrightarrow{\sim} H^{-d_X}(\mathcal{HH}(\mathcal{A}_X)). \tag{4.33}$$

In particular, there is a canonical non-zero section in $H^{-d_X}(X;\mathcal{HH}(\mathcal{A}_X))$.

(v) *The complex $\mathcal{HH}(\mathcal{A}_X^{\mathrm{loc}})$ is concentrated in degree $-d_X$ and the morphisms ι_X and τ_X induce isomorphisms*

$$\mathbf{k}_X[d_X] \xrightarrow[\iota_X]{\sim} \mathcal{HH}(\mathcal{A}_X^{\mathrm{loc}}) \xrightarrow[\tau_X]{\sim} \mathbf{k}_X[d_X]. \tag{4.34}$$

Remark 4.3.2. The existence of a canonical section in $H^{-d_X}(X;\mathcal{HH}(\mathcal{A}_X^{\mathrm{loc}}))$ is well known when $X = T^*M$ is a cotangent bundle, see in particular [7, 12]. It is intensively used in [6] where these authors call it the “trace density map”.

Lemma 4.3.3. *Let X_i be complex symplectic manifolds endowed with* DQ-*algebroids $\mathscr{A}_{X_i}$ $(i = 1, 2, 3)$, and set $d_{ij} = d_{X_i} + d_{X_j}$. We have a commutative diagram*

$$\begin{tikzcd}
Rp_{13!}(p_{12}^{-1}\mathcal{HH}(\mathcal{A}^{\mathrm{loc}}_{X_1\times X_2^a})\overset{\mathrm{L}}{\otimes} p_{23}^{-1}\mathcal{HH}(\mathcal{A}^{\mathrm{loc}}_{X_2\times X_3^a})) \arrow[r, "\circ"] \arrow[d, "\sim"] & \mathcal{HH}(\mathcal{A}^{\mathrm{loc}}_{X_1\times X_3^a}) \arrow[d, "\sim"] \\
Rp_{13!}(p_{12}^{-1}\mathbf{k}_{X_{12}}[d_{12}] \otimes p_{23}^{-1}\mathbf{k}_{X_{23}}[d_{23}]) \arrow[r] & \mathbf{k}_{X_{13}}[d_{13}].
\end{tikzcd}$$

Here, the horizontal arrow on the top is the morphism given by Proposition 4.1.3, *the two vertical arrows are given by* Theorem 4.3.1 *and the horizontal arrow on the bottom is obtained by Poincaré duality, using the fact that the manifold X_2 has real dimension $2\,d_{X_2}$ and is oriented.*

We may summarize our results with the diagram below.

$$\begin{tikzcd}
 & & \mathcal{HH}(\mathcal{A}_X) \arrow[dl, "\mathrm{gr}"'] \arrow[dr, "loc"] & \\
 & \mathcal{HH}(\mathcal{O}_X) \arrow[dl, "\alpha", "\sim"'] & & \mathcal{HH}(\mathcal{A}_X^{\mathrm{loc}}) \arrow[d, "b", "\sim"'] \\
\bigoplus_i \Omega_X^i[i] & & \bigoplus_i \Omega_X^i[i] \arrow[ul, "\beta"', "\sim"] & \mathbf{k}_X[d_X]
\end{tikzcd}$$

The morphism gr, *loc* and b commute with the composition of kernels.

4.4 Links with $\mathcal{D}$-modules

Recall that the functoriality of the Chern class of the graded modules associated to coherent $\mathcal{D}$-modules (in the algebraic settings) is proved in [28] as a corollary of the Riemann-Roch-Grothendieck theorem.

On the other hand the Euler class (in De Rham cohomology) of coherent $\mathcal{D}$-modules is constructed in [35] and its functoriality is proved. Let us briefly explain how it is possible to recover these results from Theorem 4.1.6.

Denote by M a complex manifold and by $\pi_M \colon T^*M \to M$ its cotangent bundle. We set $X := T^*M$. Recall that M is endowed with the $\mathbb{C}_M$-algebra $\mathcal{D}_M$ of differential operators and X is endowed with the $\mathbb{C}_X$-algebra $\widehat{\mathscr{E}}_X$ and the DQ-algebra $\widehat{\mathscr{W}}_X$. There are natural morphisms of algebras

$$\pi_M^{-1}\mathcal{D}_M \hookrightarrow \widehat{\mathscr{E}}_X \hookrightarrow \widehat{\mathscr{W}}_X.$$

We set $\mathcal{D}_{M^a} := (\mathcal{D}_M)^{\mathrm{op}}$ and we define similarly $\widehat{\mathscr{E}}_{X^a}$ and $\widehat{\mathscr{W}}_{X^a}$. We define the Hochschild homology of $\widehat{\mathscr{E}}_X$ by setting:

$$\mathcal{HH}(\widehat{\mathcal{E}}_X) := \widehat{\mathscr{E}}_{X^a} \overset{\mathrm{L}}{\otimes}_{\widehat{\mathscr{E}}_{X \times X^a}} \widehat{\mathscr{E}}_X.$$

One defines the Hochschild class of a coherent $\mathcal{E}$-module as usual, and we define Hochschild class $\mathrm{hh}_{\mathcal{D}}(\mathcal{M})$ of a coherent $\mathcal{D}$-module $\mathcal{M}$ as the Hochschild class of $\widehat{\mathscr{E}}_X \otimes_{pi_M^{-1}\mathcal{D}_M} pi_M^{-1}\mathcal{M}$. Hence

$$\mathrm{hh}_{\mathcal{D}}(\mathcal{M}) \in H^{d_X}_{\mathrm{char}(\mathcal{M})}(X; \mathbb{C}_X). \tag{4.35}$$

One has an isomorphism $\mathcal{HH}(\widehat{\mathcal{E}}_X) \simeq \mathbb{C}_X[d_X]$ and a commutative diagram

$$\begin{tikzcd}
\mathcal{HH}(\widehat{\mathcal{E}}_X) \arrow[r] \arrow[d, "\sim"] & \mathcal{HH}(\widehat{\mathcal{W}}_X) \arrow[d, "\sim"] \\
\mathbb{C}_X[d_X] \arrow[r] & \mathbb{C}_X((\hbar))[d_X].
\end{tikzcd}$$

Recall that in (3.27) we introduced the exact functor

$$\begin{array}{rcl} (\bullet)^W \colon \mathrm{Mod}(\mathcal{D}_M) & \to & \mathrm{Mod}(\widehat{\mathscr{W}}_X) \\ \mathcal{M} & \mapsto & \widehat{\mathscr{W}}_X \otimes_{\pi_M^{-1}\mathcal{D}_M} \pi_M^{-1}\mathcal{M}. \end{array}$$

and this functor sends $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_M)$ to $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\widehat{\mathscr{W}_X})$. One checks easily that for $\mathcal{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_M)$:

$$\mathrm{hh}(\mathcal{M}^W) = \iota(\mathrm{hh}_{\mathcal{D}}(\mathcal{M})) \in H^{d_X}_{\mathrm{char}(\mathcal{M})}(X; \mathbf{k}_X)$$

where ι is the morphism induced by $\mathbb{C} \to \mathbf{k}$. This formula, together with the functoriality of the Hochschild classes of $\mathcal{W}$-modules and Theorem 3.4.4 allow us to recover the results of [35] on the functoriality of the Euler class of $\mathcal{D}$-modules. Similarly, by taking the associated graded modules, one could recover the results in [28].

Note that the results of [35] also deal with constructible sheaves, which are not relevant to these methods.

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