A short review on microlocal sheaf theory

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Abstract

This is a brief survey of the microlocal theory of sheaves of [KS90], with some complements and variants.

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1 Introduction

Between 1960 and 1970, Mikio Sato (see [Sat59, Sat70]) introduced what is now called algebraic analysis and microlocal analysis. The idea of algebraic analysis is to use the tools of algebraic geometry (categories, sheaves) to interpret and to treat problems of analysis and the idea of microlocal analysis is, given a manifold $M$, to look at its cotangent bundle $T^*M$ to better understand the phenomena on $M$ and to treat some objects living on $M$ (e.g., distributions, hyperfunctions, differential operators) as the projection on $M$ of objects living on $T^*M$.

The microlocal theory of sheaves, due to Masaki Kashiwara and the author, has appeared in [KS82] and was developed in [KS85, KS90]. It is an illustration of Sato’s philosophy since it shows that it is possible to associate to an abelian sheaf $F$ (in the derived sense) on a real manifold $M$, a closed conic co-isotropic subset $SS(F)$ of the cotangent bundle $T^*M$, its singular support, which describes the set of non-propagation of $F$. This theory has applications in various domains, such as singularity theory, D-module theory and symplectic topology.

In these notes we make a very brief survey of this theory.

Denote by $D^b(k_M)$ the bounded derived category of sheaves of $k$-modules, for a unital commutative ring $k$. First we recall the equivalent definitions of $SS(F)$ and discuss with some proofs the behavior of the singular support with respect to the six operations, with a glance at Morse theory. We recall the construction of the specialization functor and its Fourier–Sato transform,
the microlocalization functor, as well as a variant of this last functor, the functor $\mu_{hom}$ which plays a central role in the whole theory. Then we introduce the localization $D^b(k_M; A)$ of the category $D^b(k_M)$ with respect to some subset $A \subset T^*M$ and make quantized contact transforms operate on it for $A = \{p\}$, the functor $\mu_{hom}$ playing then the role of the internal $\mathcal{H}hom$. Following [GKS12], we briefly show how Hamiltonian isotopies operate and deduce a very short proof of Arnold’s non displaceability theorem (after the pioneering work of Tamarkin [Tam08]). We also introduce simple and pure sheaves along a smooth Lagrangian submanifold. Finally, we treat applications of this theory to the study of holomorphic solutions of D-modules, in particular elliptic pairs and hyperbolic systems.

We assume the reader familiar with classical sheaf theory (in the derived setting).

2 Microsupport of sheaves

In this section, we recall some definitions and results from [KS90], following its notations with the exception of slight modifications. We consider a real manifold $M$ of class $C^\infty$.

2.1 Some geometrical notions

For a locally closed subset $A$ of $M$, one denotes by $\text{Int}(A)$ its interior and by $\overline{A}$ its closure.

One denotes by $\Delta_M$ or simply $\Delta$ the diagonal of $M \times M$ and by $\delta_M$, or simply $\delta$, the diagonal embedding $M \hookrightarrow M \times M$.

For two manifolds $M$ and $N$, one denotes by $q_1$ and $q_2$ the projections from $M \times N$ to $M$ and $N$, respectively.

One denotes by $\text{pt}$ a set with one element. When necessary, we look at $\text{pt}$ as a manifold.

Vector bundles

Let $\tau: E \to M$ be a real (finite dimensional) vector bundle over $M$.

The antipodal map $a_M$ on $E$ is defined by:

$$a_M: E \to E, \quad (x; \xi) \mapsto (x; -\xi).$$
If $A$ is a subset of $E$, we write $A^a$ instead of $a_M(A)$.

One denotes by $\mathbb{R}^+$ the multiplicative group $\mathbb{R}_{>0}$. Then $\mathbb{R}^+$ acts on $E$. We say that a subset $A$ of $E$ is $\mathbb{R}^+$-conic, or simply conic, if $\mathbb{R}^+ \cdot A = A$.

One denotes by $R^+$ the multiplicative group $\mathbb{R}^+$. Then $R^+$ acts on $E$.

We say that a subset $A$ of $E$ is $R^+$-conic, or simply conic, if $R^+ \cdot A = A$.

One denotes by $\tau_M : T^*M \to M$ and $\pi_M : T^*M \to M$ the tangent and cotangent bundles to $M$. If $L \subset M$ is a submanifold, we denote by $T_L^* M$ its normal bundle and by $T^*_L M$ its conormal bundle. They are defined by the exact sequences

$$0 \to TL \to L \times_M TM \to T_L^* M \to 0,$$

$$0 \to T_L^* M \to L \times_M T^* M \to T^* L \to 0.$$

One identifies $M$ with $T^*_M M$, the zero-section of $T^* M$. One sets

$$(2.2) \quad \hat{T}^* M := T^* M \setminus M, \quad \hat{\pi}_M := \pi_M | \hat{T}^* M.$$

If there is no risk of confusion, one simply writes $\tau$ and $\pi$ instead of $\tau_M$ and $\pi_M$.

Let $f : M \to N$ be a morphism of real manifolds. To $f$ are associated the tangent morphisms

$$(2.3) \quad T^* M \xrightarrow{\tau} M \times_N T^* N \xrightarrow{f^*} T^* N \xleftarrow{\pi} T^* M$$

By duality, we deduce the diagram:

$$(2.4) \quad T^* M \xleftarrow{\pi} M \times_N T^* N \xrightarrow{f^*} T^* N \xrightarrow{\pi} M \xrightarrow{f} N.$$

One sets

$$T^*_M N := \text{Ker} f_d = f_d^{-1}(T^*_M M).$$

Note that, denoting by $\Gamma_f$ the graph of $f$ in $M \times N$, the projection $T^*(M \times N) \to M \times T^* N$ identifies $T^*_{\Gamma_f}(M \times N)$ and $M \times N T^* N$.

\footnote{In these notes, a submanifold is always smooth and locally closed}
Whitney’s normal cones

The intrinsic construction of the Whitney’s normal cones will be recalled in Definition 4.4.

Let $S \subset M$ be a locally closed subset and let $L$ be a submanifold of $M$. The Whitney normal cone $C_L(S)$ is a closed conic subset of $T_L M$ given in a local coordinate system $(x) = (x', x'')$ on $M$ with $N = \{x' = 0\}$ by

$$\{ (x''; v_0) \in C_N(S) \subset T_N M \text{ if and only if there exists a sequence } x_n \text{ such that } x_n \nrightarrow x', x_n \nrightarrow x'' \text{ and } c_n(x'_n) \nrightarrow v_0. \}$$

For two subsets $S_1, S_2 \subset M$, their Whitney’s normal cone is given in a local coordinate system $(x)$ on $M$ by:

$$\{ (x_0; v_0) \in C(S_1, S_2) \subset TM \text{ if and only if there exists a sequence } x_n \text{ such that } x_n \nrightarrow x_0, x_n \nrightarrow x_0 \text{ and } c_n(x_n - y_n) \nrightarrow v_0. \}$$

Example 2.1. Let $\mathbb{V}$ be a real finite dimensional vector space and let $\gamma$ be a closed cone (unless otherwise specified, a cone is always centred at the origin). Then $C_0(\gamma) = \gamma$ and $C_0(\gamma, \gamma)$ is the vector space generated by $\gamma$.

Liouville form and Hamiltonian isomorphism

The map $\pi_M : T^* M \to M$ induces the maps

$$T^* T^* M \leftarrow T^* M \times_M T^* M \to T^* M.$$  

By sending $T^* M$ to $T^* M \times_M T^* M$ by the diagonal map, we get a map $\alpha_M : T^* M \to T^* T^* M$, that is a section of $T^*(T^* M)$. This is the Liouville 1-form, given in a local homogeneous symplectic coordinate system $(x; \xi)$ on $T^* M$, by

$$\alpha_M = \sum_{j=1}^n \xi_j \, dx_j.$$  

The differential $d\alpha_M$ of the Liouville form is the symplectic form $\omega_M$ on $T^* M$ given in a local symplectic coordinate system $(x; \xi)$ on $T^* M$ by

$$\omega_M = \sum_{j=1}^n d\xi_j \wedge dx_j.$$
Hence $T^*M$ is not only a symplectic manifold, it is a homogeneous (or exact) symplectic manifold.

We shall use the Hamiltonian isomorphism $H: T^*(T^*M) \rightarrow T(T^*M)$ given in a local symplectic coordinate system $(x; \xi)$ by

$$H((\lambda, dx) + (\mu, d\xi)) = -\langle \lambda, \partial_\xi \rangle + \langle \mu, \partial_x \rangle.$$ 

Co-isotropic subsets

**Definition 2.2** (See [KS90, Def. 6.5.1]). A subset $S$ of $T^*M$ is co-isotropic (one also says involutive) at $p \in T^*M$ if $C_p(S, S)^\perp \subset C_p(S)$. Here we identify the orthogonal $C_p(S, S)^\perp$ to a subset of $T_pT^*M$ via the Hamiltonian isomorphism.

When $S$ is smooth, one recovers the usual notion.

### 2.2 Microsupport of sheaves

References are made to [KS90, §5.1-5.3].

**Sheaves**

We consider a commutative unital ring $k$ of finite global dimension (e.g. $k = \mathbb{Z}$ or $k$ a field). We denote by $k_M$ the sheaf of $k$-valued locally constant functions on $M$ and by $D(k_M)$ (resp. $D^b(k_M)$) the derived category (resp. bounded derived category) of sheaves of $k$-modules on $M$. We shall identify $D(k_{pt})$ with $D(k)$ and we denote by $a_M$ the unique map $M \rightarrow pt$.

We assume that the reader is familiar with the Grothendieck six operations on sheaves.

For $V \in D(k)$, we set $V_M := a_M^{-1}V$. For a locally closed subset $A$ of $M$, we still denote by $k_A$, the sheaf which is $k$ on $A$ and 0 on $M \setminus A$. For $F \in D(k_M)$, one sets $F_A := F \otimes k_A$ and $R\Gamma_A F := R\mathcal{H}om(k_A,F)$.

The dualizing complex $\omega_M$ is defined as $\omega_M = a_M^!(k)$. One has the isomorphism

$$\omega_M \simeq or_M [\dim M]$$

(2.7) where $or_M$ is the orientation sheaf on $M$. 
The duality functors $D'_M$ and $D_M$ are given by

\[ D'_M F = R\mathcal{H}om(F, k_M), \]
\[ D_M F = R\mathcal{H}om(F, \omega_M). \]

One also sets $\omega_M^{-1} = D'_M \omega_M$.

If $f : M \to N$ is a morphism of manifolds, one sets

\[ \omega_{M/N} := f^! k_N \simeq \omega_M \otimes f^{-1} \omega_N^{-1}. \]

We shall have to consider cohomologically constructible sheaves and, in the case $M$ is real analytic, $\mathbb{R}$-constructible sheaves. We don’t recall here their definitions and refer to [KS90, § 3.4, § 8.4]. In the case $M = \text{pt}$, $F \in D^b(k)$ is cohomologically constructible if and only if it is represented by a bounded complex of finitely generated projective $k$-modules.

**Micro-support**

To $F \in D^b(k_M)$ one associates $SS(F)$, its singular support or micro-support as follows.

**Definition 2.3.** Let $F \in D^b(k_M)$ and let $p \in T^*M$. One says that $p \notin SS(F)$ if there exists an open neighborhood $U$ of $p$ such that for any $x_0 \in M$ and any real $C^1$-function $\varphi$ on $M$ defined in a neighborhood of $x_0$ satisfying $d\varphi(x_0) \in U$ and $\varphi(x_0) = 0$, one has $(R\Gamma\{x : \varphi(x) \geq 0\})(F)_{x_0} \simeq 0$.

In other words, $p \notin SS(F)$ if the sheaf $F$ has no cohomology supported by “half-spaces” whose conormals are contained in a neighborhood of $p$.

- By its construction, the microsupport is closed and is conic, that is, invariant by the action of $\mathbb{R}^+$ on $T^*M$.
- $SS(F) \cap T^*_M = \pi_M(SS(F)) = \text{Supp}(F)$.
- $SS(F) = SS(F [j])$ ($j \in \mathbb{Z}$).
- The microsupport satisfies the triangular inequality: if $F_1 \to F_2 \to F_3 \xrightarrow{+1} \rightarrow$ is a distinguished triangle in $D^b(k_M)$, then $SS(F_i) \subset SS(F_j) \cup SS(F_k)$ for all $i, j, k \in \{1, 2, 3\}$ with $j \neq k$.  


An essential properties of the micro-support is given by the next theorem. The proof is beyond the scope of these notes and will not be even sketched here.

**Theorem 2.4** (See [KS90, Th. 6.5.4]). Let \( F \in D^b(k_M) \). Then its micro-support \( \text{SS}(F) \) is co-isotropic.

**Example 2.5.**

(i) \( \text{SS}(F) \subset T^*M \) if and only if \( F \) is a local system, that is, \( H^j(F) \) is locally constant on \( M \) for all \( j \in \mathbb{Z} \).

(ii) If \( N \) is a smooth closed submanifold of \( M \) and \( F = k_N \), then \( \text{SS}(F) = T^*_N M \), the conormal bundle to \( N \) in \( M \).

(iii) Let \( \varphi \) be \( C^1 \)-function with \( d\varphi(x) \neq 0 \) when \( \varphi(x) = 0 \). Let \( U = \{ x \in M; \varphi(x) > 0 \} \) and let \( Z = \{ x \in M; \varphi(x) \geq 0 \} \). Then

\[
\begin{align*}
\text{SS}(k_U) &= U \times_M T^*_M M \cup \{(x; \lambda d\varphi(x)); \varphi(x) = 0, \lambda \leq 0\}, \\
\text{SS}(k_Z) &= Z \times_M T^*_M M \cup \{(x; \lambda d\varphi(x)); \varphi(x) = 0, \lambda \geq 0\}.
\end{align*}
\]

(iv) Let \( (X, \mathcal{O}_X) \) be a complex manifold and let \( \mathcal{M} \) be a coherent \( \mathcal{D}_X \)-module (see § 6.2). Set \( F = R\mathcal{H}om_{\mathcal{D}_X} (\mathcal{M}, \mathcal{O}_X) \). Then \( \text{SS}(F) = \text{char}(\mathcal{M}) \), the characteristic variety of \( \mathcal{M} \). See § 6.2 for details. Note that this result together with Theorem 2.4 gives a totally new proof of the involutivity of the characteristic variety of coherent D-modules.

There are other equivalent definitions of the microsupport.

Let \( V \) be a real finite dimensional vector space and let \( \gamma \) be a closed convex cone centred at the origin. The \( \gamma \)-topology on \( V \) is the topology for which the open sets \( U \) are the open subsets \( U \) such that \( U = U + \gamma \). One denotes by \( V_\gamma \) the space \( V \) endowed with the \( \gamma \)-topology. For an open subset \( X \subset V \), one denotes by \( X_\gamma \) the space \( X \) endowed with the topology induced by \( V_\gamma \) and one denotes by

\[ \Phi_\gamma : X \to X_\gamma. \]
the continuous map associated with the identity on $X$.

**Theorem 2.6.** Assume that $M$ is an open subset of a vector space $V$ and let $F \in D^b(k_M)$. Let $p = (x_0; \xi_0) \in T^*M$. Then the following conditions are equivalent

(a) $p \notin \text{SS}(F)$,

(b) there exist a neighborhood $U$ of $x_0$, an $\varepsilon > 0$ and a proper closed convex cone $\gamma$ with $0 \in \gamma$ such that

$$\gamma \setminus \{0\} \subset \{v; \langle v, \xi_0 \rangle < 0\} \quad \text{(equivalently, } \xi_0 \in \text{Int}(\gamma^\circ))$$

(2.8) and setting

$$H = \{x \in V; \langle x - x_0, \xi_0 \rangle \geq -\varepsilon\},$$

$$L = \{x \in V; \langle x - x_0, \xi_0 \rangle = -\varepsilon\},$$

then $H \cap (U + \gamma) \subset M$ and we have the natural isomorphism for any $x \in U$:

$$\text{R} \Gamma(H \cap (x + \gamma); F) \cong \text{R} \Gamma(L \cap (x + \gamma); F),$$

(c) there exist a proper closed convex cone $\gamma$ with $0 \in \gamma$ satisfying (2.8) and $F' \in D^b(k_\gamma)$ such that $F'|_U \simeq F|_U$ for a neighborhood $U$ of $x_0$ and $R\Phi_{\gamma_*}F' \simeq 0$.

### 2.3 Functorial operations

References are made to [KS90, §5.4].

Let $M$ and $N$ be two manifolds. We denote by $q_i$ ($i = 1, 2$) the $i$-th projection defined on $M \times N$ and by $p_i$ ($i = 1, 2$) the $i$-th projection defined on $T^*(M \times N) \simeq T^*M \times T^*N$. 

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Definition 2.7. Let \( f : M \to N \) be a morphism of manifolds and let \( \Lambda \subset T^*N \) be a closed \( \mathbb{R}^+ \)-conic subset. One says that \( f \) is non-characteristic for \( \Lambda \) (or else, \( \Lambda \) is non-characteristic for \( f \), or \( f \) and \( \Lambda \) are transverse) if
\[
   f^{-1}_\pi(\Lambda) \cap T^*_M N \subset M \times_N T^*_N N.
\]
- Clearly, if \( f \) is submersive then it is non-characteristic for any \( \Lambda \).
- A morphism \( f : M \to N \) is non-characteristic for a closed \( \mathbb{R}^+ \)-conic subset \( \Lambda \) of \( T^*N \) if and only if \( f_d : M \times_N T^*N \to T^*M \) is proper on \( f^{-1}_\pi(\Lambda) \) and in this case \( f_df^{-1}_\pi(\Lambda) \) is closed and \( \mathbb{R}^+ \)-conic in \( T^*M \).

Recall that \( \bigotimes\bigotimes \cdot := q_1^{-1}(\cdot) \otimes q_2^{-1}(\cdot) \).

Theorem 2.8. Let \( F \in D^b(k_M) \) and let \( G \in D^b(k_N) \). One has
\[
   SS\left( F \bigotimes\bigotimes G \right) \subset SS\left( F \right) \times SS\left( G \right),
   SS\left( R\mathcal{H}om\left(q_1^{-1}F, q_2^{-1}G\right)\right) \subset SS\left( F \right)^a \times SS\left( G \right).
\]

Theorem 2.9. Let \( f : M \to N \) be a morphism of manifolds, let \( F \in D^b(k_M) \) and assume that \( f \) is proper on \( \text{Supp}(F) \). Then \( Rf_! F \isom Rf_* F \) and
\[
   SS\left( Rf_* F \right) \subset f^*_\pi f_d^{-1} SS\left( F \right).
\]

Moreover, if \( f \) is a closed embedding, this inclusion is an equality.

Proof. (i) The isomorphism \( Rf_! F \isom Rf_* F \) is obvious.
(ii) Let \( y \in N \) and let \( \varphi : N \to \mathbb{R} \) be a \( C^1 \)-function such that \( \varphi(y) = 0 \) and \( d(\varphi \circ f)(x) \notin SS\left( F \right) \) for all \( x \in f^{-1}(y) \). By the definition of \( SS\left( F \right) \) we get
\[
   R\Gamma_{\{\varphi \geq 0\}}\left( F \right)|_{f^{-1}(y)} = 0.
\]
On the other hand, we have
\[
   R\Gamma_{\{\varphi \geq 0\}}\left( Rf_* F \right)_y \quad \simeq \quad \left( Rf_* R\Gamma_{\{\varphi \geq 0\}}\left( F \right) \right)_y
   \quad \simeq \quad R\Gamma\left( f^{-1}(y); R\Gamma_{\{\varphi \geq 0\}}\left( F \right) \right) \simeq 0.
\]
Here, the second isomorphism follows from the fact that one may replace \( f_* \) with \( f_! \).
(iii) Assume $f$ is a closed embedding and let $p \notin \text{SS}(Rf_*F)$. We may assume that $N$ is a vector space, $M$ is a vector subspace and $f$ is the inclusion. Let $H, L, \gamma, U$ be as in Theorem 2.6 (b). Hence, we have for $x \in U \cap M$:

$$R\Gamma(H \cap (x + \gamma); Rf_*F) \cong R\Gamma(L \cap (x + \gamma); Rf_*F).$$

On the other hand, we have

$$R\Gamma(H \cap (x + \gamma); Rf_*F) \cong R\Gamma((M \cap H) \cap (x + (M \cap \gamma)); F),$$

and a similar formula with $H$ replaced with $L$. Therefore, $p \notin \text{SS}(F)$ again by Theorem 2.6 (b). Q.E.D.

On Figure 3, one sees that the inclusion in Theorem 2.9 may be strict.

**Corollary 2.10.** Let $I$ be an open interval of $\mathbb{R}$, let $q: M \times I \to I$ be the projection and let $\iota_s$ is the embedding $M \times \{s\} \hookrightarrow M \times I$. Let $F \in D^b(k_{M \times I})$ such that $\text{SS}(F) \cap (T^*_M M \times T^*I) \subset T^*_M (M \times I)$ and $q$ is proper on $\text{Supp}(F)$. Set $F_s := \iota_s^{-1}F$. Then we have isomorphisms $R\Gamma(M; F_s) \simeq R\Gamma(M; F_t)$ for any $s, t \in I$.

**Proof.** Consider the Cartesian square in which $\iota_s$ also denotes the embedding $\{s\} \hookrightarrow I$:

$$
\begin{array}{ccc}
M \times \{s\} & \xrightarrow{\alpha} & \{s\} \\
\downarrow_{\iota_s} & & \downarrow_{\iota_s} \\
M \times I & \xrightarrow{q} & I.
\end{array}
$$
By the “base change formula” for sheaves, $R\alpha_{M!}t^{-1}F \simeq \iota^{-1}_s Rq_{s!}F$. Since $q$ is proper on $\text{Supp}(F)$, we get $\Gamma(M; F_s) \simeq (Rq_{s!}F)_s$.

On the other hand, it follows from Theorem 2.9 that $\text{SS}(Rq_{s!}F) \subset T^*_I$. Hence, there exists $V \in \mathbb{D}^b(k)$ and an isomorphism $Rq_{s!}F \simeq V_I$. Therefore, $(Rq_{s!}F)_s \simeq (Rq_{s!}F)_t$ for any $s, t \in I$.

Q.E.D.

**Theorem 2.11.** Let $f: M \to N$ be a morphism of manifolds, let $G \in \mathbb{D}^b(k_N)$ and assume that $f$ is non-characteristic with respect to $\text{SS}(G)$. Then the natural morphism $f^{-1}G \otimes \omega_{M/N} \to f^!G$ is an isomorphism and

\[(2.10) \quad \text{SS}(f^{-1}G) \subset f_d f^{-1}_\pi(\text{SS}(G)).\]

Moreover, if $f$ is submersive, this inclusion is an equality.

Note that $f^{-1}G \otimes \omega_{M/N}$ being locally isomorphic to $f^{-1}G$ up to a shift, we get that $\text{SS}(f^!G) = \text{SS}(f^{-1}G)$ in this case.

**Sketch of proof.**

(i) By decomposing $f$ by its graph, it is enough to check separately the case of a closed embedding and the case of a submersion.

(ii) First, we assume that $f$ is submersive.

(ii)-(a) In this case, the isomorphism $f^{-1}G \otimes \omega_{M/N} \sim \to f^!G$ is well-known. Locally on $M$, $f$ is isomorphic to the projection $M = N \times T \to N$ and $f^{-1}G \simeq G \boxtimes k_T$. Hence the inclusion $\text{SS}(f^{-1}G) \subset f_d f^{-1}_\pi(\text{SS}(G))$ is a particular case of Theorem 2.8.

(ii)-(b) The converse inclusion follows from the isomorphism

\[(\Gamma_{\{\varphi \geq 0\}}(G))_y \simeq (\Gamma_{\{\varphi \circ f \geq 0\}}f^{-1}G)_x\]

for any $x \in M$ with $f(x) = y$. Indeed, if $(y_0; \eta_0) \in \text{SS}(G)$, then there exists a sequence $y_n \to y_0$ and functions $\varphi_n$ such that $d\varphi_n(y_n) \to \eta_0$ and $(\Gamma_{\{\varphi_n \geq 0\}}G)_{y_n} \neq 0$ and the result follows by the definition of the microsupport.

(iii) We may reduce the proof to the case where $M$ is a closed hypersurface of $N$.

(iii)-(a) Let us prove the isomorphism $f^{-1}G \otimes \omega_{M/N} \sim \to f^!G$. This is a local problem on $M$ and we may assume that $N = M \sqcup M^+ \sqcup M^-$ for $M^+$ and $M^-$ two closed half-spaces with boundary $M$. The exact sequence $0 \to k_M \to k_{M^+} \oplus k_{M^-} \to k_M \to 0$ gives rise to the distinguished triangle

\[\Lambda M^+ G \xrightarrow{\alpha} (\Gamma M^+ G)_M \oplus (\Gamma M^- G)_M \xrightarrow{\beta} G_M \xrightarrow{\gamma}.\]
The map $\beta$ is given by $(u,v) \mapsto \varepsilon u|_M - \varepsilon v|_M$ with $\varepsilon = \pm$, and this sign is given by the relative orientation sheaf or $\mathcal{O}_{M/N}$. The hypothesis that $M$ is non-characteristic implies the vanishing of $(R\Gamma^+_M G) \oplus (R\Gamma^-_M G)_M$ and we get the isomorphism $f^{-1}G \otimes \mathcal{O}_{M/N} [-1] \xrightarrow{\sim} (R\Gamma_M G)_M$.

(iii)-(b) The proof of the inclusion (2.10) is more technical and we refer to [KS90, Prop. 5.4.13]. Q.E.D.

**Corollary 2.12.** Let $F_1, F_2 \in D^b(k_M)$.

(i) Assume that $\text{SS}(F_1) \cap \text{SS}(F_2)^a \subset T^*_M M$. Then

$$\text{SS}(F_1 \otimes F_2) \subset \text{SS}(F_1) + \text{SS}(F_2).$$

(ii) Assume that $\text{SS}(F_1) \cap \text{SS}(F_2) \subset T^*_M M$. Then

$$\text{SS}(R\mathcal{H}om (F_2, F_1)) \subset \text{SS}(F_2)^a + \text{SS}(F_1).$$

**Proof.** One has the isomorphisms $R\mathcal{H}om (F_2, F_1) \simeq \delta^! R\mathcal{H}om (q^{-1}_2 F_2, q_1^! F_1)$ and $F_1 \otimes F_2 \simeq \delta^{-1}(F_1 \boxtimes F_2)$. Hence, the result follows from Theorems 2.8 and 2.11. Q.E.D.

### 2.4 Kernels

References for this subsection are made to [KS90, §3.6].

Let $M_i (i = 1, 2, 3)$ be manifolds. For short, we write $M_{ij} := M_i \times M_j$ $(1 \leq i, j \leq 3)$ and $M_{123} = M_1 \times M_2 \times M_3$. We denote by $q_i$ the projection $M_{ij} \to M_i$ or the projection $M_{123} \to M_i$ and by $q_{ij}$ the projection $M_{123} \to M_{ij}$. Similarly, we denote by $p_i$ the projection $T^* M_{ij} \to T^* M_i$ or the projection $T^* M_{123} \to T^* M_i$ and by $p_{ij}$ the projection $T^* M_{123} \to T^* M_{ij}$. We also need to introduce the map $p_{12}^*$, the composition of $p_{12}$ and the antipodal map on $T^* M_2$.

Let $\Lambda_1 \subset T^* M_{12}$ and $\Lambda_2 \subset T^* M_{23}$. We set

$$\Lambda_1 \circ \Lambda_2 := p_{13}(p_{12}^{-1} \Lambda_1 \cap p_{23}^{-1} \Lambda_2).$$

(2.11) We consider the operation of convolution of kernels:

$$\circ : D^b(k_{M_{12}}) \times D^b(k_{M_{23}}) \to D^b(k_{M_{13}})$$

$$(K_1, K_2) \mapsto K_1 \circ K_2 := Rq_{13!}(q_{12}^{-1} K_1 \otimes q_{23}^{-1} K_2).$$
Let $\Lambda_i = \text{SS}(K_i) \subset T^*M_{i,i+1}$ and assume that

\[
\begin{align*}
(i) & \quad q_{13} \text{ is proper on } q_{12}^{-1} \text{Supp}(K_1) \cap q_{23}^{-1} \text{Supp}(K_2), \\
(ii) & \quad p_{12}^{-1}\Lambda_1 \cap p_{23}^{-1}\Lambda_2 \cap (T^*_{M_1}M_1 \times T^*_{M_2}M_2 \times T^*_{M_3}M_3) \\
& \quad \subset T^*_M(M_1 \times M_2 \times M_3) \\
& \quad (M_1 \times M_2 \times M_3).
\end{align*}
\]

(2.12)

It follows from the functorial properties of the microsupport, namely Theorems 2.8, 2.9 and 2.11, that under the assumption (2.12) we have:

\[
\text{SS}(K_1 \circ_i K_2) \subset \Lambda_1 \circ \Lambda_2.
\]

(2.13)

If there is no risk of confusion, we write $\circ$ instead of $\circ_i$.

### 3 Morse theory for sheaves

References are made to [KS90, § 1.12, § 5.4].

#### 3.1 A basic lemma

The next lemma, although elementary, is extremely useful. It is due to M. Kashiwara.

Let $\{X_s, \rho_{s,t}\}_{s \in \mathbb{R}}$ be a projective system of sets indexed by $\mathbb{R}$. Hence, the $X_s$ are sets and $\rho_{s,t}: X_t \to X_s$ are maps defined for $s \leq t$, satisfying the natural compatibility conditions. Set

\[
\lambda_s: X_s \to \lim_{t < s} X_t, \quad \mu_s: \lim_{t > s} X_t \to X_s.
\]

**Lemma 3.1** (See [KS90, Pro. 1.12.6]). Assume that for each $s \in \mathbb{R}$, both maps $\lambda_s$ and $\mu_s$ are injective (resp. surjective). Then all maps $\rho_{s_0,s_1}$ ($s_0 \leq s_1$) are injective (resp. surjective).

#### 3.2 Mittag-Leffler theorem

References are made to [Gro61] (see [KS90, § 1.12]). Consider a projective system of abelian groups indexed by $\mathbb{N}$, $\{M_n, \rho_{n,p}\}_{n \in \mathbb{N}}$, with $\rho_{n,p}: M_p \to M_n$ ($p \geq n$). (In the sequel we shall simply denote such a system by $\{M_n\}_n$.) Recall that one says that this system satisfies the Mittag-Leffler condition (ML
for short) if for any \( n \in \mathbb{N} \) the decreasing sequence \( \{ \rho_{n,p}M_p \} \) of subgroups of \( M_n \) is stationary.

Of course, this condition is in particular satisfied if all maps \( \rho_{n,p} \) are surjective.

**Notation 3.2.** For a projective system of abelian groups \( \{ M_n \} \), we set \( M_\infty = \lim_{\leftarrow n} M_n \).

Consider a projective system of exact sequences of abelian groups indexed by \( \mathbb{N} \). For each \( n \in \mathbb{N} \) we have an exact sequence

\[
E_n: 0 \rightarrow M'_n \rightarrow M_n \rightarrow M''_n \rightarrow 0,
\]

and we have morphisms \( \rho_{n,p}: E_p \rightarrow E_n \) satisfying the compatibility conditions. Recall:

**Lemma 3.3.** If the projective system \( \{ M'_n \} \) satisfies the ML condition, then the sequence

\[
E_\infty: 0 \rightarrow M'_\infty \rightarrow M_\infty \rightarrow M''_\infty \rightarrow 0
\]

is exact.

Now consider a projective system of complexes

\[
E_n^\bullet: \cdots \rightarrow M_{n-1}^j \rightarrow M_n^j \rightarrow M_{n+1}^j \rightarrow \cdots
\]

and its projective limit

\[
E_\infty^\bullet: \cdots \rightarrow M_{\infty-1}^j \rightarrow M_\infty^j \rightarrow M_{\infty+1}^j \rightarrow \cdots
\]

Denote by

\[
\Phi_k: H^k(E_\infty^\bullet) \rightarrow \lim_{\leftarrow n} H^k(E_n^\bullet)
\]

the natural morphism.

**Lemma 3.4.** Assume that for all \( j \in \mathbb{Z} \), the system \( \{ M_j^j \} \) satisfies the ML condition. Then

(a) for each \( k \in \mathbb{Z} \), the map \( \Phi_k \) in (3.5) is surjective,

(b) if moreover, for a given \( i \) the system \( \{ H^{i-1}(E_n^\bullet) \} \) satisfies the ML condition, then \( \Phi_i \) is bijective.
3.3 Morse lemma in dimension one

Let $F \in D^b(k_M)$, let $U \subset M$ be an open subset, $\partial U$ its boundary, $\overline{U}$ its closure. The exact sequence $0 \to k_U \to k_M \to k_{\overline{U}} \to 0$ gives rise to the distinguished triangle $\mathcal{R}\Gamma_{M\setminus U} F \to F \to \mathcal{R}\Gamma_U F \xrightarrow{+1}$. Applying the functor $(\cdot)_{\overline{U}}$ we get the distinguished triangle

\[(\mathcal{R}\Gamma_{M\setminus U} F)|_{\partial U} \to F_{\overline{U}} \to \mathcal{R}\Gamma_U F \xrightarrow{+1} . \tag{3.6}\]

Until the end of this subsection, $M = \mathbb{R}$. For $t \in \mathbb{R}$, we set $Z_t = ]-\infty, t], \quad I_t = ]-\infty, t[.$

**Lemma 3.5.** Let $-\infty < a < b < +\infty$. Let $G \in D^b(k_{\mathbb{R}})$ and assume that $(\mathcal{R}\Gamma_{[t, +\infty]}(G))_t \simeq 0$ for all $t \in [a, b]$. Then one has the natural isomorphism

\[\mathcal{R}\Gamma(I_b; G) \xrightarrow{\sim} \mathcal{R}\Gamma(I_a; G). \]

Note that if $(t; dt) \notin SS(G)$ for all $t \in [a, b]$, then $(\mathcal{R}\Gamma_{[t, +\infty]}(G))_t \simeq 0$ for all $t \in [a, b]$.

**Proof.** As a particular case of (3.6) with $U = I_t$, we have the distinguished triangle

\[(\mathcal{R}\Gamma_{[t, +\infty]}(G))_t \to G_{Z_t} \to \mathcal{R}\Gamma_{t, G} \xrightarrow{+1} . \tag{3.7}\]

Applying the functor $\mathcal{R}\Gamma(\mathbb{R}; \cdot)$, we deduce the distinguished triangle

\[(\mathcal{R}\Gamma_{[t, +\infty]}(G))_t \to \mathcal{R}\Gamma(Z_t; G) \to \mathcal{R}\Gamma(I_t; G) \xrightarrow{+1} . \tag{3.8}\]

Set

\[E^k_s = H^k(I_s; G). \]

Consider the assertions

\[\lim_{t > s} E^k_t \xrightarrow{\sim} E^k_s \text{ for all } s \in [a, b], \tag{3.9}\]

\[\lim_{t < s} E^k_s \xleftarrow{\sim} E^k_t \text{ for all } t \in ]a, b[. \tag{3.10}\]
By the hypothesis, \( R\Gamma(Z_t; G) \cong \rightarrow R\Gamma(I_t; G) \) for \( t \in [a, b] \). Therefore, \((3.9)\) holds for any \( k \in \mathbb{Z} \). Moreover, \((3.10)\) holds for \( k \ll 0 \). Let us argue by induction on \( k \) and assume \((3.10)\) holds for all \( t \in [a, b] \) and all \( k \leq k_0 \). Applying Lemma 3.1, we find the isomorphisms

\[
(3.11) H^k(I_s; G) \cong \rightarrow H^k(I_t; G) \quad \text{for all} \quad k \leq k_0 \quad \text{and all} \quad a < s \leq t < b.
\]

On the other hand, we may represent \( G \) by a complex of flabby sheaves \( G^\bullet \). Let \( t \in [a, b] \) be given. Consider the complex \( E^\bullet_n = \Gamma(I_{t-1/n}; G^\bullet) \).

Since \( G^\bullet \) is a complex of flabby sheaves, the projective systems \( \{ \Gamma(I_{t-1/n}; G^j) \}_n \) satisfies the ML condition for all \( j \in \mathbb{Z} \).

By \((3.11)\), the projective system \( \{ H^{k_0}(E^\bullet_n) \}_n \) satisfies the ML condition. Applying Lemma 3.4 to \( \{ E^\bullet_n \}_n \), we get that \((3.10)\) is satisfied for \( k = k_0 + 1 \) and the induction proceeds. Again by Lemma 3.1, we get the isomorphisms \( H^k(I_s; G) \cong \rightarrow H^k(I_t; G) \) for all \( k \) all \( a < s \leq t \leq b \).

Finally, using \((3.8)\) and the hypothesis, we have the isomorphims

\[
H^k(I_a; G) \leftarrow \rightarrow H^k(Z_a; G) \cong \rightarrow H^k(I_s; G) \quad \text{for all} \quad k \in \mathbb{Z}, \ a < s \leq b.
\]

Q.E.D.

### 3.4 Morse theorem

We consider a function \( \psi : M \to \mathbb{R} \) of class \( C^1 \).

**Theorem 3.6.** Let \( F \in D^b(k_M) \), let \( \psi : M \to \mathbb{R} \) be a function of class \( C^1 \) and assume that \( \psi \) is proper on \( \text{Supp}(F) \). Let \( a < b \) in \( \mathbb{R} \) and assume that \( d\psi(x) \notin SS(F) \) for \( a \leq \psi(x) < b \). For \( t \in \mathbb{R}, \) set \( M_t = \psi^{-1}([-\infty, t]) \). Then the restriction morphism \( R\Gamma(M_t; F) \to R\Gamma(M_a; F) \) is an isomorphism.

The classical Morse theorem corresponds to the constant sheaf \( F = k_M \).

**Proof.** Set \( G = R\psi_* F \). Then \( R\Gamma(M_t; F) \cong \rightarrow R\Gamma([-\infty, t]; G) \). By Theorem 2.9, \( SS(G) \subset \psi_d\psi^{-1}(SS(F)) \). In other words,

\[
SS(G) \subset \{(t; \tau) ; \text{there exists} \ x \in M, \psi(x) = t, d\psi(x) \in SS(F)\}.
\]

Therefore, \( (t; dt) \notin SS(G) \) for \( a \leq t < b \) and it remains to apply Lemma 3.5.

Q.E.D.
Set

\[(3.12) \quad \Lambda_\psi = \{(x; d\psi(x))\} \subset T^*M.\]

The next corollary will be an essential tool when proving non-displaceability theorems (see Theorem 5.20) below.

**Corollary 3.7.** Let \(F \in \mathcal{D}^b(k_M)\) and let \(\psi: M \to \mathbb{R}\) be a function of class \(C^1\). Let \(\Lambda_\psi\) be given by (3.12). Assume that

(i) \(\text{Supp}(F)\) is compact,

(ii) \(R\Gamma(M; F) \neq 0\).

Then \(\Lambda_\psi \cap \text{SS}(F) \neq \emptyset\).

**Proof.** Assume that \(\Lambda_\psi \cap \text{SS}(F) = \emptyset\). It follows from Theorem 3.6 that \(R\Gamma(M_t; F)\) does not depend on \(t \in \mathbb{R}\). Since \(F\) has compact support, we get \(R\Gamma(M_t; F) \simeq 0\) for \(t \ll 0\) and \(R\Gamma(M; F) \simeq R\Gamma(M_t; F)\) for \(t \gg 0\). This is a contradiction. Q.E.D.

### 3.5 Propagation

References are made to [KS90, §5.2].

The microsupport is a tool to obtain global propagation results.

**Theorem 3.8.** Let \(V\) be a real finite dimensional vector space and let \(\gamma\) be a proper closed convex cone centred at 0 \(\in V\). Let \(U \subset V\) be an open subset and let \(\Omega_0 \subset \Omega_1\) be two \(\gamma\)-open subsets of \(V\). Let \(F \in \mathcal{D}^b(k_U)\). Assume

\[
\begin{cases}
\text{SS}(F) \cap (U \times \text{Int}(\gamma^0)) = \emptyset, \\
\Omega_1 \setminus \Omega_0 \subset U, \\
\text{for any } x \in \Omega_1, (x + \gamma) \setminus \Omega_0 \text{ is compact.}
\end{cases}
\]

Then

\[(3.13) \quad (R\Phi_{\gamma^0} R\Gamma_{U \setminus \Omega_0} F)|_{\Omega_1} \simeq 0\]

and the natural morphism

\[(3.14) \quad R\Gamma(\Omega_1 \cap U; F) \to R\Gamma(\Omega_0 \cap U; F)\]

is an isomorphism.
Sketch of proof. (i) Let us prove (3.13) and set \( \tilde{F} := R\Gamma_{U \setminus \Omega_0} F \). In order to check that \( (R\Phi_{\gamma} \tilde{F})|_{\Omega_1} \simeq 0 \), one reduces to the case where \( \Omega_0 = \{ x \in \mathbb{V}; \langle x, \xi_0 \rangle < 0 \} \), for some \( \xi_0 \in V^* \), \( \xi_0 \neq 0 \). Then one is reduced to prove

\[
R\Gamma((x + \gamma) \cap \{ \langle x, \xi_0 \rangle \geq 0 \}; F) \sim R\Gamma((x + \gamma) \cap \{ \langle x, \xi_0 \rangle = 0 \}; F).
\]

This last isomorphism is obtained by constructing a family of open sets \( \{ U_t \}_{t \geq 0} \) such that \( U_0 \) is a neighborhood of \( (x + \gamma) \cap \langle x, \xi_0 \rangle = 0 \), \( U_1 \) is a neighborhood of \( (x + \gamma) \cap \langle x, \xi_0 \rangle \geq 0 \) and the conormals to the boundary of \( U_t \) do not belong to \( SS(F) \) on \( \Omega_1 \setminus \Omega_0 \).

(ii) We have to prove that \( R\Gamma_{\Omega_1 \setminus \Omega_0}(U; F) \simeq 0 \). This follows from

\[
R\Gamma_{\Omega_1 \setminus \Omega_0}(U; F) \simeq R\Gamma(U; R\Gamma_{\Omega_1 \setminus \Omega_0} F) \simeq R\Gamma(\Omega_1; \tilde{F}) \simeq R\Gamma(\Omega_1; R\Phi_{\gamma} \tilde{F}|_{\Omega_1}).
\]

Q.E.D.

4 The functor \( \mu_\text{hom} \)

References for this section are made to [KS90, § 3.7, Ch. 4, §6.2 §7.2].

4.1 Fourier-Sato transform

The classical Fourier transform interchanges (generalized) functions on a vector space \( \mathbb{V} \) and (generalized) functions on the dual vector space \( \mathbb{V}^* \). The idea of extending this formalism to sheaves, hence of replacing an isomorphism of spaces with an equivalence of categories, seems to have appeared first in Mikio Sato’s construction of microfunctions in [Sat70].

Let \( \tau : E \to M \) be a finite dimensional real vector bundle over a real manifold \( M \) with fiber dimension \( n \) and let \( \pi : E^* \to M \) be the dual vector bundle. Denote by \( p_1 \) and \( p_2 \) the first and second projection defined on \( E \times_M E^* \), and define:

\[
P = \{(x, y) \in E \times_M E^*; \langle x, y \rangle \geq 0 \},
P' = \{(x, y) \in E \times_M E^*; \langle x, y \rangle \leq 0 \}.
\]
Consider the diagram:

\[
\begin{array}{ccc}
E & \times & M \\
p_1 & \downarrow & \downarrow p_2 \\
\downarrow \tau & \downarrow & \downarrow \pi \\
E & \rightarrow & E^* \\
\end{array}
\]

Denote by \( D_{\mathbb{R}^+}^b(k_E) \) the full triangulated subcategory of \( D^b(k_E) \) consisting of conic sheaves, that is, objects with locally constant cohomology on the orbits of the action of \( \mathbb{R}^+ \).

**Definition 4.1.** Let \( F \in D_{\mathbb{R}^+}^b(k_E) \), \( G \in D_{\mathbb{R}^+}^b(k_E^*) \). One sets:

\[
F^\wedge := R\pi_2!(p_1^{-1}F)_{\mathcal{P}'} \simeq R\pi_2^*(R\Gamma_{\mathcal{P}'}p_1^{-1}F),
\]

\[
G^\vee := R\pi_1^*(R\Gamma_{\mathcal{P}'}p_2^1G) \simeq R\pi_1!(p_2^1G)_{\mathcal{P}'}.
\]

The main result of the theory is the following.

**Theorem 4.2.** The two functors \( (\cdot)^\wedge \) and \( (\cdot)^\vee \) are inverse to each other, hence define an equivalence of categories \( D_{\mathbb{R}^+}^b(k_E) \simeq D_{\mathbb{R}^+}^b(k_E^*) \) and for \( F_1, F_2 \in D_{\mathbb{R}^+}^b(k_E) \), one has the isomorphism

\[
(4.1) \quad R\text{Hom}(F_1^\wedge, F_2^\wedge) \simeq R\text{Hom}(F_1, F_2).
\]

**Example 4.3.** (i) Let \( \gamma \) be a closed proper convex cone in \( E \) with \( M \subset \gamma \). Then:

\[
(k_\gamma)^\wedge \simeq k_{\text{Int}(\gamma^\circ)}.
\]

Here \( \gamma^\circ \) is the polar cone to \( \gamma \), a closed convex cone in \( E^* \) and \( \text{Int}\gamma^\circ \) denotes its interior.

(ii) Let \( \gamma \) be an open convex cone in \( E \). Then:

\[
(k_\gamma)^\wedge \simeq k_{\gamma^\circ \circ} \otimes_{E^*/M} [-n].
\]

Here \( \lambda^\circ = -\lambda \), the image of \( \lambda \) by the antipodal map.

(iii) Let \( (x) = (x', x'') \) be coordinates on \( \mathbb{R}^n \) with \( (x') = (x_1, \ldots, x_p) \) and \( (x'') = (x_{p+1}, \ldots, x_n) \). Denote by \( (y) = (y', y'') \) the dual coordinates on \( (\mathbb{R}^n)^* \). Set

\[
\gamma = \{ x; x'^2 - x''^2 \geq 0 \}, \quad \lambda = \{ y; y'^2 - y''^2 \leq 0 \}.
\]

Then \( (k_\gamma)^\wedge \simeq k_\lambda[-p] \). (See [KS97].)
4.2 Specialization

Let $\iota: N \hookrightarrow M$ be the embedding of a closed submanifold $N$ of $M$. Denote by $\tau_M: T_NM \to M$ the normal bundle to $N$.

If $F$ is a sheaf on $M$, its restriction to $N$, denoted $F|_N$, may be viewed as a global object, namely the direct image by $\tau_M$ of a sheaf $\nu_N F$ on $T_NM$, called the specialization of $F$ along $N$. Intuitively, $T_NM$ is the set of light rays issued from $N$ in $M$ and the germ of $\nu_N F$ at a normal vector $(x; v) \in T_NM$ is the germ at $x$ of the restriction of $F$ along the light ray $v$.

One constructs a new manifold $\tilde{M}_N$, called the normal deformation of $M$ along $N$, together with the maps

$$
\begin{array}{ccc}
T_NM & \xrightarrow{s} & \tilde{M}_N \\
\downarrow M & & \downarrow \Omega \\
N & \xrightarrow{\iota} & M \\
\end{array}
$$

with the following properties. Locally, after choosing a local coordinate system $(x', x'')$ on $M$ such that $N = \{x' = 0\}$, we have $\tilde{M}_N = M \times \mathbb{R}$, $t: \tilde{M}_N \to \mathbb{R}$ is the projection, $\Omega = \{(x; t) \in M \times \mathbb{R}; t > 0\}$, $p(x', x'', t) = (tx', x'')$, $T_NM = \{t = 0\}$.

**Definition 4.4.** (a) Let $S \subset M$ be a locally closed subset. The Whitney normal cone $C_N(S)$ is a closed conic subset of $T_NM$ given by

$$
C_N(S) = \overline{p^{-1}(S)} \cap T_NM.
$$

(b) For two subsets $S_1, S_2 \subset M$, their Whitney’s normal cone is given by

$$
C(S_1, S_2) = C_\Delta(S_1 \times S_2)
$$

where $\Delta$ is the diagonal of $M \times M$ and $TM$ is identified to $T_\Delta(M \times M)$ by the first projection $T(M \times M) \to TM$.

One defines the specialization functor

$$
\nu_N: D^b(k_M) \to D^b(k_{T_NM})
$$

by a formula mimicking Definition 4.4, namely:

$$
\nu_N F := s^{-1}Rj_*p^{-1}F.
$$
Clearly, $\nu_N F \in D^b_{\mathbb{R}^+}(k_{T_N^* M})$, that is, $\nu_N F$ is a conic sheaf for the $\mathbb{R}^+$-action on $T_N^* M$. Moreover,

$$R\tau_{M*} \nu_N F \simeq \nu_N F|_N \simeq F|_N.$$ 

For an open cone $V \subset T_N^* M$, one has

$$H^j(V; \nu_N F) \simeq \lim_{\to \downarrow} H^j(U; F)$$

where $U$ ranges through the family of open subsets of $M$ such that

$$C_N(M \setminus U) \cap V = \emptyset.$$

### 4.3 Microlocalization

Denote by $\pi_M: T_N^* M \to M$ the conormal bundle to $N$ in $M$, that is, the dual bundle to $\tau_M: T_N M \to M$.

If $F$ is a sheaf on $M$, the sheaf of sections of $F$ supported by $N$, denoted $R\Gamma_N F$, may be viewed as a global object, namely the direct image by $\pi_M$ of a sheaf $\mu_M F$ on $T_N^* M$. Intuitively, $T_N^* M$ is the set of “walls” (half-spaces) in $M$ containing $N$ in their boundary and the germ of $\mu_N F$ at a conormal vector $(x; \xi) \in T_N^* M$ is the germ at $x$ of the sheaf of sections of $F$ supported by closed tubes with edge $N$ and which are almost the half-space associated with $\xi$.

More precisely, the microlocalization of $F$ along $N$, denoted $\mu_N F$, is the Fourier-Sato transform of $\nu_N F$, hence is an object of $D^b_{\mathbb{R}^+}(k_{T_N^* M})$. It satisfies:

$$R\pi_{M*} \mu_N F \simeq \mu_N F|_N \simeq R\Gamma_N F.$$ 

For a convex open cone $V \subset T_N^* M$, one has

$$H^j(V; \mu_N F) \simeq \lim_{\to \downarrow} H^j(U \cap Z; F),$$

where $U$ ranges over the family of open subsets of $M$ such that $U \cap N = \pi_M(V)$ and $Z$ ranges over the family of closed subsets of $M$ such that $C_M(Z) \subset V^o$ where $V^o$ is the polar cone to $V$.

If $H \in D^b_{\mathbb{R}^+}(k_{T^* M})$ is a conic sheaf on $T^* M$, then $R\pi_{M!} H \simeq R\Gamma_M H$ and one gets Sato’s distinguished triangle

$$R\pi_{M!} H \to R\pi_{M*} H \to R\tilde{\pi}_{M*} H \xrightarrow{+1}. \tag{4.4}$$
Applying this result to the conic sheaf $\mu_N F$, one gets the distinguished triangle
\[
F|_N \otimes \omega_{N/M} \to R\Gamma_N F|_N \to R\hat{\pi}_{*\mathcal{N}}\mu_N F \xrightarrow{+1}.
\]  
(4.5)

4.4 The functor $\mu hom$

Let us briefly recall the main properties of the functor $\mu hom$, a variant of Sato’s microlocalization functor.

Recall that $\Delta$ denotes the diagonal of $M \times M$. We shall denote by $\tilde{\delta}$ the isomorphism
\[
\tilde{\delta}: T^* M \sim \to T^* M\left(M \times M\right), \quad (x; \xi) \mapsto (x, x; \xi, -\xi).
\]

**Definition 4.5.** One defines the functor $\mu hom: D^b(k_M)^{\text{op}} \times D^b(k_M) \to D^b(k_{T^* M})$ by
\[
\mu hom(F_2, F_1) = \tilde{\delta}^{-1} \mu_\Delta \mathcal{H}om(q^{-1}_2 F_2, q^{-1}_1 F_1)
\]
where $q_i (i = 1, 2)$ denotes the $i$-th projection on $M \times M$.

Note that

- $R\hat{\pi}_{*\mathcal{N}} \mu hom(F_2, F_1) \simeq R\mathcal{H}om(F_2, F_1)$,
- $\mu hom(k_N, F) \simeq \mu_N (F)$ for $N$ a closed submanifold of $M$,
- assuming that $F_2$ is cohomologically constructible, there is a distinguished triangle $D' F_2 \otimes F_1 \to R\mathcal{H}om(F_2, F_1) \to R\hat{\pi}_{*\mathcal{N}} \mu hom(F_2, F_1) \xrightarrow{+1}$.

Moreover
\[
(4.6) \quad \text{Supp} \mu hom(F_2, F_1) \subset SS(F_2) \cap SS(F_1).
\]

We shall see in the next section that, in some sense, $\mu hom$ is the sheaf of microlocal morphisms.

**Corollary 4.6.** (The Petrowsky theorem for sheaves.) Assume that $F_2$ is cohomologically constructible and $SS(F_2) \cap SS(F_1) \subset T^*_M M$. Then the natural morphism
\[
R\mathcal{H}om(F_2, k_M) \otimes F_1 \to R\mathcal{H}om(F_2, F_1)
\]
is an isomorphism.
For two subsets $A$ and $B$ of $T^*M$, we still denote by $C(A,B)$ the inverse image in $T^*T^*M$ of their Whitney normal cone by the Hamiltonian isomorphism $H: T^*T^*M \cong TT^*M$.

**Theorem 4.7** (See [KS90, Cor. 6.4.3]). Let $F_1, F_2 \in \mathcal{D}^b(k_M)$. Then

$$SS(\muhom(F_2, F_1)) \subset C(SS(F_2), SS(F_1)). \tag{4.7}$$

Consider a vector bundle $\tau: E \to N$ over a manifold $N$. It gives rise to a morphism of vector bundles over $N$, $\tau': TE \to E \times_N TN$ which by duality gives the map $\tau_d: E \times_N T^*N \to T^*E$. By restricting to the zero-section of $E$, we get the map:

$$T^*N \hookrightarrow T^*E.$$

Applying this construction to the bundle $T^*_N M$ above $N$, and using the Hamiltonian isomorphism we get the maps

$$T^*N \hookrightarrow T^*_N T^*M \cong T^*_N T^*M \tag{4.8}.$$

**Corollary 4.8.** (See [KS90, Cor. 6.4.4].) One has

$$SS(R\Gamma_N F) \subset T^*N \cap C_{T^*_N M}(SS(F)), \quad SS(F|_N) \subset T^*N \cap C_{T^*_N M}(SS(F)).$$

**Microlocal Serre functor**

There is an interesting phenomenon which holds with $\muhom$ and not with $R\mathcal{H}om$. Indeed, assume $M$ is real analytic. Then, although the category $\mathcal{D}^b_{\mathbb{R}-c}(k_M)$ of $\mathbb{R}$-constructible sheaves does not admit a Serre functor, it admits a kind of microlocal Serre functor, as shown by the isomorphism, functorial in $F_1$ and $F_2$ (see [KS90, Prop. 8.4.14]):

$$(D_{T^*_M \muhom}(F_2, F_1) \simeq \muhom(F_1, F_2) \otimes \pi^{-1}_M \omega_M).$$

This confirms the fact that to fully understand $\mathbb{R}$-constructible sheaves, it is natural to look at them microlocally, that is, in $T^*M$. This is also in accordance with the “philosophy” of Mirror Symmetry which interchanges the category of coherent $\mathcal{O}_X$-modules on a complex manifold $X$ with the Fukaya category on a symplectic manifold $Y$. In case of $Y = T^*M$, the Fukaya category is equivalent to the category of $\mathbb{R}$-constructible sheaves on $M$, according to Nadler-Zaslow [Nad09,NZ09].

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**Microlocal Fourier-Sato transform**

The Fourier-Sato transform is by no means local: it interchanges sheaves on a vector bundle $E$ and sheaves on $E^*$. However, this transformation is *microlocal* in the following sense.

Let $E \to Z$ be vector bundle over a manifold $Z$. There is a natural isomorphism $T^*E \cong T^*E^*$ given in local coordinates

\[(4.9) \quad T^*E \ni (z, x; \zeta, \xi) \mapsto (z, \xi; \zeta, -x) \in T^*E^*.\]

**Theorem 4.9** ([KS90, Exe. VII.2]). Let $F_1, F_2 \in D^b_{\mathbb{R}^+}(k_E)$. There is a natural isomorphism

\[(4.10) \quad \mu_{\text{hom}}(F_2, F_1) \cong \mu_{\text{hom}}(F_2^\wedge, F_1^\wedge).\]

### 5 Microlocal theory

#### 5.1 Localization

Let $A$ be a subset of $T^*M$ and let $Z = T^*M \setminus A$. The full subcategory $D^b_Z(k_M)$ of $D^b(k_M)$ consisting of sheaves $F$ such that $\text{SS}(F) \subset Z$ is a triangulated subcategory. One sets

\[D^b(k_M; A) := D^b(k_M)/D^b_Z(k_M),\]

the localization of $D^b(k_M)$ by $D^b_Z(k_M)$. Hence, the objects of $D^b(k_M; A)$ are those of $D^b(k_M)$ but a morphism $u: F_1 \to F_2$ in $D^b(k_M)$ becomes an isomorphism in $D^b(k_M; A)$ if, after embedding this morphism in a distinguished triangle $F_1 \to F_2 \to F_3 \to \to$, one has $\text{SS}(F_3) \cap A = \emptyset$. When $A = \{ p \}$ for some $p \in T^*M$, one simply writes $D^b(k_M; p)$ instead of $D^b(k_M; \{ p \})$.

The functor $\mu_{\text{hom}}$ describes in some sense the microlocal morphisms of the category $D^b(k_M)$. More precisely, for $U$ open in $T^*M$, it follows from (4.6) that $\mu_{\text{hom}}$ induces a bifunctor:

\[\mu_{\text{hom}}: D^b(k_M; U)^{\text{op}} \times D^b(k_M; U) \to D^b(k_U).\]

Moreover, the sequence of morphisms

\[
\begin{align*}
\text{RHom}_U(G,F) & \cong \text{R} \Gamma(M; \text{R Hom}(G,F)) \\
& \cong \text{R} \Gamma(T^*M; \mu_{\text{hom}}(G,F)) \\
& \to \text{R} \Gamma(U; \mu_{\text{hom}}(G,F))
\end{align*}
\]
define the morphism
\[ \text{Hom}_{\mathcal{D}^b(k_M;U)}(G, F) \to H^0 \Gamma(U; \mu_{\text{hom}}(G, F)). \]

The morphism (5.1) is not an isomorphism, but it induces an isomorphism at each \( p \in T^*M \):

**Theorem 5.1** (See [KS90, Th. 6.1.2]). Let \( p \in T^*M \). Then
\[ \text{Hom}_{\mathcal{D}^b(k_M;p)}(G, F) \simeq H^0(\mu_{\text{hom}}(G, F)_p). \]

5.2 Pure and simple sheaves

Let \( S \) be a smooth submanifold of \( M \) and let \( \Lambda = T^*_S M \). Let \( p \in \Lambda, p \notin T^*_M M \) and let \( F \in \mathcal{D}^b(k_M;p) \). Let us say that \( F \) is pure at \( p \) if \( F \simeq V[d] \) for some \( k \)-module \( V \) and some shift \( d \) and let us say that \( F \) is simple if moreover \( V \) is free of rank one. A natural question is to generalize this definition to the case where \( \Lambda \) is a smooth Lagrangian submanifold of \( T^*M \) but is no more necessarily a conormal bundle. Another natural question would be to calculate the shift \( d \). This last point makes use of the Maslov index and we refer to [KS90, § 7.5].

**Notation 5.2.** Let \( \Lambda \) be a smooth \( \mathbb{R}^+ \)-conic Lagrangian locally closed submanifold of \( T^*M \), closed in an open conic neighborhood \( W \) of \( \Lambda \).
(i) We denote by \( \mathcal{D}^b(\Lambda)(k_M) \) the full triangulated subcategory of \( \mathcal{D}^b(k_M) \) consisting of objects \( F \) such that there exists an open neighborhood \( W \) of \( \Lambda \) (containing \( \Lambda \) as a closed subset) in \( T^*M \) such that \( \text{SS}(F) \cap W \subset \Lambda \).
(ii) One denotes by \( \text{DLoc}(k_{\Lambda}) \) the full triangulated subcategory of \( \mathcal{D}^b(k_{\Lambda}) \) consisting of objects \( F \) such that for each \( j \in \mathbb{Z}, H^j(F) \) is a local system on \( \Lambda \). Equivalently, \( \text{DLoc}(k_{\Lambda}) \) is the subcategory of \( \mathcal{D}^b(k_{\Lambda}) \) consisting of sheaves with microsupport contained in the zero-section \( T^*_\Lambda \Lambda \).

Applying Theorem 4.7, we get

**Corollary 5.3.** The functor \( \mu_{\text{hom}} \) induces a functor
\[ \mu_{\text{hom}}: \mathcal{D}^b(\Lambda)(k_M)^{\text{op}} \times \mathcal{D}^b(\Lambda)(k_M) \to \text{DLoc}(k_{\Lambda}). \]

**Lemma 5.4.** Let \( L \in \mathcal{D}^b(\Lambda)(k_M;W) \). There is a natural morphism \( k_{\Lambda} \to \mu_{\text{hom}}(L, L) \).
Proof. Represent \( L \in D^b_{(\Lambda)}(k_M) \) by \( F \in D^b(k_M) \). The morphism \( k_M \to R\mathcal{H}om(F, F) \simeq R\pi_*\mu hom(F, F) \) defines the morphism \( k_{T^*M} \to \mu hom(F, F) \). Since \( \mu hom(L, L) \) is supported by \( \Lambda \) in a neighborhood of \( \Lambda \), this last morphism factorizes through \( k_\Lambda \). Q.E.D.

The notions of pure and simple sheaves are introduced and intensively studied in [KS90, § 7.5].

For a \( C^\infty \)-function \( \varphi \) on \( M \) we denote by \( \Lambda_{\varphi} \) the (non conic) Lagrangian submanifold of \( T^*M \) given by

\[
\Lambda_{\varphi} := \{(x; d\varphi(x)); x \in M\}.
\]

Let \( p \in \Lambda \). One says that \( \varphi \) is transverse to \( \Lambda \) at \( p \) if \( \varphi(\pi_M(p)) = 0 \) and the manifolds \( \Lambda \) and \( \Lambda_{\varphi} \) intersect transversally at \( p \). We define the Lagrangian planes in \( T_pT^*M \):

\[
\begin{align*}
\lambda_0(p) &= T_p(\pi_M^{-1}\pi_M(p)), \\
\lambda_\Lambda(p) &= T_p\Lambda, \\
\lambda_\varphi(p) &= T_p\Lambda_{\varphi}.
\end{align*}
\]

(5.2)

Lemma 5.5. Let \( p \in \Lambda \) and let \( \varphi \) be transverse to \( \Lambda \) at \( p \). The property that \( R\Gamma_{\{\varphi \geq 0\}}(F)_{\pi_M(p)} \) is concentrated in a single degree (resp. and is free of rank one over \( k \)) does not depend on the choice of \( \varphi \).

Proof. See [KS90, Prop. 7.5.3, 7.5.6]. Q.E.D.

In loc. cit., the shift of \( R\Gamma_{\{\varphi \geq 0\}}(F)_{\pi_M(p)} \) is related to the Maslov index of the Lagrangian planes of (5.2)

By this lemma, one can state:

Definition 5.6. Let \( F \in D^b_{(\Lambda)}(k_M) \) and let \( \varphi \) be transverse to \( \Lambda \) at \( p \).

(a) One says that \( F \) is pure on \( \Lambda \) if \( R\Gamma_{\{\varphi \geq 0\}}(F)_{\pi(p)} \) is concentrated in a single degree. One denotes by Pure(\( \Lambda, k \)) the subcategory of \( D^b_{(\Lambda)}(k_M) \) consisting of pure sheaves.

(b) One says that \( F \) simple on \( \Lambda \) if \( R\Gamma_{\{\varphi \geq 0\}}(F)_{\pi(p)} \) is concentrated in a single degree and is free of rank one. One denotes by Simple(\( \Lambda, k \)) the subcategory of \( D^b_{(\Lambda)}(k_M) \) consisting of simple sheaves.

When \( k \) is a field, there is an easy criterion of purity and simplicity.

Proposition 5.7. Assume that \( k \) is a field and let \( F \in D^b_{(\Lambda)}(k_M) \). Then
(a) \(F\) is pure on \(\Lambda\) if and only if \(\mu_{\text{hom}}(F,F)|_\Lambda\) is concentrated in degree 0,

(b) \(F\) simple on \(\Lambda\) if and only if \(k_\Lambda \xrightarrow{\sim} \mu_{\text{hom}}(L,L)|_\Lambda\).

Sketch of proof. By using a quantized contact transformation (see § 5.3 below) one reduces the problem to the case where \(\Lambda = T^*_NM\) for a closed submanifold \(N\) of \(M\). Then locally, \(F \simeq A_N\) for some \(A \in D^b(k)\) and the result is obvious in this case. Q.E.D.

Remark 5.8. Let \(L \in \text{Simple}(\Lambda, k)\). Then the functor

\[
\mu_{\text{hom}}(L, \cdot): \text{Pure}(\Lambda, k) \to \text{DLoc}(k_\Lambda)
\]  

is well-defined. One shall be aware that:
(i) the category \(\text{Simple}(\Lambda, k)\) may be empty,
(ii) the functor in (5.3) is not fully faithful in general,
(iii) the categories \(\text{Pure}(\Lambda, k)\) and \(\text{Simple}(\Lambda, k)\) are not additive.

Proposition 5.9 (see [KS90, Cor.7.5.4]). Let \(F \in D^b_{(\Lambda)}(k_M)\). Then the set of \(p \in \Lambda\) in a neighborhood of which \(F\) is pure (resp. simple) is open and closed in \(\Lambda\).

Proof. We shall only give a proof when assuming that \(k\) is a field.
One knows by Corollary 5.3 that \(L := \mu_{\text{hom}}(F,F)\) is a local system on \(\Lambda\).
Then the set of \(p \in \Lambda\) in a neighborhood of which \(L\) is concentrated in degree 0 (resp. is of rank one) is open and closed in \(\Lambda\). Q.E.D.

Remark 5.10. Pure sheaves are intensively (and implicitely) used in [STZ14] in their study of Legendrain knots.

5.3 Quantized contact transformations

References for this subsection are made to [KS90, §7.2].
Consider two manifolds \(M\) and \(N\), two conic open subsets \(U \subset T^*M\) and \(V \subset T^*N\) and a homogeneous contact transformation \(\chi:\)

\[
T^*N \supset V \xrightarrow{\chi} U \subset T^*M.
\]

Denote by \(V^a\) the image of \(V\) by the antipodal map \(a_N\) on \(T^*N\) and by \(\Lambda\) the image of the graph of \(\chi\) by \(id_U \times a_N\). Hence \(\Lambda\) is a conic Lagrangian
submanifold of $U \times V^a$. Consider $K \in D^b(k_{M \times N})$ and the hypotheses

\begin{equation}
(5.5) \begin{cases}
K \text{ is cohomologically constructible}, \\
K \text{ is simple along } \Lambda, \\
(p_1^{-1}U \cup p_2^{-1}V^a) \cap SS(K) \subset \Lambda.
\end{cases}
\end{equation}

**Theorem 5.11** (See [KS90, Th. 7.2.1]). If $K$ satisfies the hypotheses (5.5), then the functor $K \circ$ induces an equivalence

\begin{equation}
(5.6) \quad K \circ : D^b(k_N; V) \isom D^b(k_M; U).
\end{equation}

Moreover, for $G_1, G_2 \in D^b(k_N; V)$

\begin{equation}
(5.7) \quad \chi_\ast (\mu_{hom}(G_1, G_2)|_V) \isom \mu_{hom}(K \circ G_1, K \circ G_2)|_U.
\end{equation}

One calls $(\chi, K)$ a quantized contact transformation (a QCT, for short).

**Corollary 5.12.** Let $\Lambda_i$ be a conic smooth Lagrangian submanifold of $\dot{T}^*M_i$ ($i = 1, 2$) with $\chi(\Lambda_2) = \Lambda_1$. Then $K \circ$ induces an equivalence $\text{Pure}(\Lambda_2; k) \isom \text{Pure}(\Lambda_1; k)$ and similarly when Pure is replaced with Simple.

**Corollary 5.13.** Consider a homogeneous contact transformation $\chi : T^*M \supset U \isom V \subset T^*N$. Then for any $p \in U$, there exists a conic open neighborhood $W$ of $p$ in $U$ and a quantized contact transform $(\chi|_W, K)$ where $\chi|_W : W \isom \chi(W)$ is the restriction of $\chi$.

**Proof.** Locally any contact transform $\chi$ is the composition $\chi_1 \circ \chi_2$ where the graph of each $\chi_i$ ($i = 1, 2$) is the Lagrangian manifold associated with the conormal to a hypersurface $S$. In this case, one can choose $K = k_S$. Q.E.D.

## 5.4 Quantization of Hamiltonian isotopies

References for this subsection are made to [GKS12].

**Hamiltonian symplectic isotopies**

- A symplectic manifold $(X, \omega_X)$, or simply $X$, is a real $C^\infty$-manifold $X$ endowed with a closed non-degenerate 2-form $\omega_X$. 

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• For two symplectic manifolds \((X, \omega_X)\) and \((Y, \omega_Y)\), one endows \(X \times Y\) with the symplectic form \(\omega_X + \omega_Y\).

• One denotes by \(X^a\) the symplectic manifold for which \(\omega_{X^a} = -\omega_X\).

• The symplectic form \(\omega_X\) defines the Hamiltonian isomorphism \(H^{-1}: TX \cong T^*X\) by the formula (up to a sign) \(H^{-1}(v) = \iota_v(\omega_X)\) where \(\iota_v\) is the interior product.

• If \(f: X \to \mathbb{R}\) is a \(C^\infty\)-map, the image by \(H\) of its differential \(df\) is a vector field on \(X\), called the Hamiltonian vector field and denoted \(H_f\). Hence, \(H_f = H(df)\).

• A symplectic isomorphism \(\varphi\) is a \(C^\infty\) isomorphism \(\varphi: X \to X\) such that \(\varphi^*\omega_X = \omega_X\). Its graph \(\Lambda_{\varphi}\) is a Lagrangian submanifold of \(X \times X^a\).

• Consider an open interval \(I\) and a map \(f: X \times I \to X\). We shall write for short \(f_s = f(\cdot, s)\).

**Definition 5.14.** A Hamiltonian isotopy \(\Phi\) on \(X\) is the data of an open interval \(I\) containing 0 and a \(C^\infty\)-map \(\Phi: X \times I \to X\) such that

\[
\begin{cases}
(a)~ \Phi = \{\varphi_s\}_{s \in I}, \varphi_s \text{ is a symplectic isomorphism for each } s \in I, \\
(b)~ \varphi_0 = \text{id}_X, \\
(c) \text{there exists a } C^\infty\text{-function } f: X \times I \to \mathbb{R} \text{ such that } \frac{\partial \Phi}{\partial s} = H_{f_s}.
\end{cases}
\]

Let \(\Phi\) be as in (5.8) satisfying conditions (a) and (b). Denote by \(\Lambda'\) its graph in \(X \times X^a \times I\). Then \(\Phi\) is an Hamiltonian isotopy if and only if there exists a Lagrangian manifold

\[
(5.9) \quad \Lambda' \subset X \times X^a \times T^*I
\]

such that \(\Lambda'\) is the image of \(\Lambda\) by the projection \(\pi: X \times X^a \times T^*I \to X \times X^a \times I\).

**Homogeneous Hamiltonian isotopies**

• An exact symplectic manifold \((X, \alpha_X)\), or simply \(X\), is a real \(C^\infty\)-manifold \(X\) endowed with a non-degenerate 1-form \(\alpha_X\) such that \(\omega_X := d\alpha_X\) is symplectic.
• The Hamiltonian isomorphism on \((X, \omega_X)\) sends \(\alpha_X\) to a vector field that we call the Euler vector field and denote by \(e_{\!\!u_X}\). A submanifold \(Y\) of \(X\) is homogeneous (or conic) if the Euler vector field is tangent to it.

• A homogeneous symplectic isomorphism \(\varphi\) is a \(C^\infty\) isomorphism \(\varphi: X \to X\) such that \(\varphi^* \alpha_X = \alpha_X\). Its graph \(\Lambda_{\varphi}\) is a homogeneous Lagrangian submanifold of \(X \times X^a\).

Of course a homogeneous symplectic isomorphism induces a symplectic isomorphism.

**Example 5.15.** Let \(M\) be a real manifold of class \(C^\infty\). Set \(X := \dot{T}^*M\) the space \(T^*M \setminus T_M M\) and by \(\pi_M: \dot{T}^*M \to M\) the projection. Then \(X\) is an exact symplectic manifold when endowed with the Liouville form \(\alpha_X\) on \(\dot{T}^*M\). If \((x) = (x_1, \ldots, x_n)\) is a local coordinate system on \(M\), \((x; \xi)\) the associated coordinate system on \(T^*M\), then

\[
\alpha_X = \sum_j \xi_j dx_j, \quad e_{\!\!u_X} = - \sum_j \xi_j \frac{\partial}{\partial \xi_j}.
\]

We consider a \(C^\infty\)-map \(\Phi: X \times I \to X\).

**Definition 5.16.** A homogeneous Hamiltonian isotopy \(\Phi\) on \(X\) is the data of an open interval \(I\) containing 0 and a \(C^\infty\)-map \(\Phi: X \times I \to X\) such that

\[
\begin{cases}
(a) \Phi = \{\varphi_s\}_{s \in I}, \varphi_s \text{ is a homogeneous symplectic isomorphism for each } s \in I, \\
(b) \varphi_0 = \text{id}_X.
\end{cases}
\]

Let \(\Phi\) be a homogeneous Hamiltonian isotopy. Set

\[
v_\Phi := \frac{\partial \Phi}{\partial t}: X \times I \to TX,
\]

\[
f = \langle \alpha_M, v_\Phi \rangle: X \times I \to \mathbb{R}, f_s = f(. , s).
\]

Then

\[
\frac{\partial \Phi}{\partial s} = H_{f_s}.
\]

In other words, if \(\Phi\) satisfies conditions (a) and (b) of Definition 5.16 then it satisfies condition (a), (b) and (c) of Definition 5.14.
Now we assume \( X = \tilde{T}^*M \). If \( \varphi: X \rightarrow X \) is a homogeneous symplectic isomorphism, its graph \( \Gamma_{\varphi} \) is Lagrangean in \( X \times X^o \) and we denote by \( \Lambda_{\varphi} \) the image of \( \Gamma_{\varphi} \) by the antipodal map on the second group of variables, \( (x, \xi, y, \eta) \mapsto (x, \xi, y, -\eta) \). Then \( \Lambda_{\varphi} \) is Lagrangian in \( X \times X \). For short, we call \( \Lambda_{\varphi} \) the Lagrangian graph of \( \varphi \).

Let \( \Phi \) be a homogeneous Hamiltonian isotopy on \( X = \tilde{T}^*M \). Then there exists a unique conic Lagrangian submanifold \( \Lambda \) of \( \tilde{T}^*M \times \tilde{T}^*M \times T^*I \) such that

- \( \Lambda \) is closed in \( \tilde{T}^*(M \times M \times I) \)
- for any \( s \in I \), the inclusion \( i_s: M \times M \rightarrow M \times M \times I \) is non-characteristic for \( \Lambda \)
- the Lagrangian graph of \( \varphi_s \) is \( \Lambda_s = \Lambda \circ T^*_sI \).

**Quantization of homogeneous isotopies**

When applying kernels associated with homogeneous isotopies we may encounter objects of the derived category of sheaves which are locally bounded but not globally. Hence we denote by \( \text{D}^b(k_M) \) the full subcategory of \( \text{D}(k_M) \) consisting of objects \( F \) such that for any open relatively compact subset \( U \subset M, F|_U \in \text{D}^b(k_U) \).

For \( K \in \text{D}^b(k_{M \times M \times I}) \) and \( s_0 \in I \), we set

\[
K_{s_0} := K|_{s=s_0}.
\]

**Theorem 5.17** ([GKS12]). Let \( \Phi: \tilde{T}^*M \times I \rightarrow \tilde{T}^*M \) be a homogeneous Hamiltonian isotopy. Then there exists \( K \in \text{D}^b(k_{M \times M \times I}) \) satisfying

(a) \( SS(K) \subset \Lambda \cup T^*_{M \times M \times I}(M \times M \times I) \),
(b) \( K_0 \simeq k_{\Delta} \).

Moreover:

(i) both projections \( \text{Supp}(K) \Rightarrow M \times I \) are proper,
(ii) setting \( K^{-1}_s := v^{-1}R\text{Hom}(K_s, \omega_M \otimes k_M) \), we have \( K_s \circ K^{-1}_s \simeq K^{-1}_s \circ K_s \simeq k_{\Delta} \) for all \( s \in I \),
such a $K$ satisfying the conditions (a) and (b) above is unique up to a unique isomorphism.

Example 5.18. Let $M = \mathbb{R}^n$ and denote by $(x; \xi)$ the homogeneous symplectic coordinates on $T^*\mathbb{R}^n$. Consider the isotopy $\varphi_s(x; \xi) = (x - s\frac{\xi}{|\xi|}; \xi)$, $s \in I = \mathbb{R}$. Then

$$\Lambda_s = \{(x, y, \xi, \eta); |x - y| = |s|, \xi = -\eta = \lambda(x - y), s\lambda < 0\} \text{ for } s \neq 0,$$

$$\Lambda_0 = \mathcal{T}_\Delta^*(M \times M).$$

For $s \in \mathbb{R}$, the morphism $k_{\{ |x - y| \leq s \}} \to k_{\{ |x - y| = s \}}$ gives by duality (replacing $s$ with $-s$) $k_{\{ |x - y| = s \}} \to k_{\{ |x - y| < -s \}}[n + 1]$. We get a morphism $k_{\{ |x - y| \leq s \}} \to k_{\{ |x - y| < -s \}}[n + 1]$ and we define $K$ by the distinguished triangle in $\mathbb{D}^b(k_{M \times M \times I})$:

$$k_{\{ |x - y| < -s \}}[n] \to K \to k_{\{ |x - y| \leq s \}} \xrightarrow{+1}.$$

One can show that $K$ is a quantization of the Hamiltonian isotopy $\{\varphi_s\}_s$. We have the isomorphisms in $\mathbb{D}^b(k_{M \times M})$: $K_s \simeq k_{\{ |x - y| \leq s \}}$ for $s \geq 0$ and $K_s \simeq k_{\{ |x - y| < -s \}}[n]$ for $s < 0$.

Corollary 5.19. Let $\Phi$ be a homogeneous Hamiltonian isotopy as in Theorem 5.17. Let $\Lambda_0$ be a smooth closed conic Lagrangian submanifold of $\mathcal{T}^*M$ and let $\Lambda_1 = \varphi_1(\Lambda_0)$. The the categories Pure($\Lambda_0; k$) and Pure($\Lambda_1; k$) are equivalent. The same result holds with Simple instead of Pure.

5.5 Application to non displaceability

In [Tam08] (see also [GS14] for an exposition and some developments), Dmitry Tamarkin shows that microlocal sheaf theory may be applied to solve some problems of symplectic topology. He gives in particular a new proof of Arnold’s non displaceability conjecture/theorem as well as other results of non displaceability. One difficulty is that the objects appearing in microlocal sheaf theory are conic for the $\mathbb{R}^+$-action, or, equivalently, this theory uses the homogeneous symplectic structure of the cotangent bundle, contrarily to the problems encountered in classical symplectic topology. This difficulty is overcome by Tamarkin who add a variable $t$ and, denoting by $(t; \tau)$ the coordinates on $T^*\mathbb{R}$, works in $\mathbb{D}^b(k_{M \times \mathbb{R}}; \tau > 0)$. However, it is sometimes possible to “translate” non conic problem to conic ones, and then to use the tools of sheaves. This is the approach of [GKS12], that we shall recall now.
Theorem 5.20 ([GKS12]). Consider a homogeneous Hamiltonian isotopy \( \Phi = \{ \varphi_s \}_{s \in I} : \dot{T}^*M \times I \to T^*M \) and a \( C^1 \)-map \( \psi : M \to \mathbb{R} \) such that the differential \( d\psi(x) \) never vanishes. Set
\[
\Lambda_\psi := \{ (x; d\psi(x)); \ x \in M \} \subset \dot{T}^*M.
\]
Let \( F \in D^b(\mathbb{k}_M) \) with compact support and such that \( R\Gamma(M; F) \neq 0 \). Then for any \( s \in I \), \( \varphi_s(\text{SS}(F) \cap \dot{T}^*M) \cap \Lambda_\psi \neq \emptyset \).

Proof. Let \( K \in D^b(\mathbb{k}_{M \times M \times I}) \) be the quantization of \( \Phi \) given by Theorem 5.17.

Set:
\[
F_s := K_s \circ F \in D^b(\mathbb{k}_M) \quad \text{for } s \in I.
\]
We have \( F_0 = F \), \( F_t \) has compact support and \( R\Gamma(M; F_s) \simeq R\Gamma(M; F) \neq 0 \) by Corollary 2.10. Applying Corollary 3.7, we get \( \Lambda_\psi \cap \text{SS}(F_s) \neq \emptyset \). Finally, \( \text{SS}(F_s) \cap \dot{T}^*M = \varphi_s(\text{SS}(F) \cap \dot{T}^*M) \).

Q.E.D.

Corollary 5.21. Let \( \Phi = \{ \varphi_t \}_{t \in I} \) and \( \psi : M \to \mathbb{R} \) be as in Theorem 5.22. Let \( N \) be a non-empty compact submanifold of \( M \). Then for any \( t \in I \), \( \varphi_t(T^*_NM) \cap \Lambda_\psi \neq \emptyset \).

Consider a compact manifold \( N \) and a (no more homogeneous) Hamiltonian isotopy \( \Phi = \{ \varphi_s \}_{s \in I} \)

Theorem 5.22 (Arnold’s non displaceability conjecture/theorem). In the above situation, \( \varphi_s(T^*_NN) \cap T^*_NN \neq \emptyset \) for all \( s \in I \).

This theorem can be deduced from Theorem 5.22 by choosing \( M = N \times \mathbb{R} \) and \( \psi : N \times \mathbb{R} \), but this is not totally straightforward.

Remark 5.23. There is now a vast literature in the field of symplectic topology in which microlocal sheaf theory plays an essential role. Let us quote among others [Chi14, Gui12, Gui13, Nad16, Tam15].

6 Applications to Analysis

In this section, \( \mathbb{k} = \mathbb{C} \).
6.1 Generalized functions

In the sixties, people were used to work with various spaces of generalized functions constructed with the tools of functional analysis. Sato’s construction of hyperfunctions in 59-60 (see [Sat59]) is at the opposite of this practice: he uses purely algebraic tools and complex analysis. The importance of Sato’s definition is twofold: first, it is purely algebraic (starting with the analytic object $\mathcal{O}_X$), and second it highlights the link between real and complex geometry. (See [Sat59] and see [Sch07] for an exposition of Sato’s work.)

Consider first the case where $M$ is an open subset of the real line $\mathbb{R}$ and let $X$ an open neighborhood of $M$ in the complex line $\mathbb{C}$ satisfying $X \cap \mathbb{R} = M$. The space $\mathcal{B}(M)$ of hyperfunctions on $M$ is given by

$$\mathcal{B}(M) = \mathcal{O}(X \setminus M)/\mathcal{O}(X).$$

It is easily proved, using the solution of the Cousin problem, that this space depends only on $M$, not on the choice of $X$, and that the correspondence $U \mapsto \mathcal{B}(U)$ ($U$ open in $M$) defines a flabby sheaf $\mathcal{B}_M$ on $M$.

With Sato’s definition, the boundary values always exist and are no more a limit in any classical sense.

**Example 6.1.** (i) The Dirac function at 0 is

$$\delta(0) = \frac{1}{2i\pi} \left( \frac{1}{x-i0} - \frac{1}{x+i0} \right).$$

Indeed, if $\varphi$ is a $C^0$-function on $\mathbb{R}$ with compact support, one has

$$\varphi(0) = \lim_{\varepsilon \to 0} \frac{1}{2i\pi} \int_{\mathbb{R}} \left( \frac{\varphi(x)}{x-i\varepsilon} - \frac{\varphi(x)}{x+i\varepsilon} \right) dx.$$  

(ii) The holomorphic function $\exp(1/z)$ defined on $\mathbb{C} \setminus \{0\}$ has a boundary value as a hyperfunction (supported by $\{0\}$) not as a distribution.

On a real analytic manifold $M$ of dimension $n$ with complexification $X$, Sato first proved that the complex $R\Gamma_M \mathcal{O}_X [n]$ is concentrated in degree 0 and he defined the sheaf $\mathcal{B}_M$ as

$$\mathcal{B}_M = H^n_M(\mathcal{O}_X) \otimes_{\mathcal{O}_M}$$
where \(\mathfrak{o}_M\) is the orientation sheaf on \(M\). Since \(X\) is oriented, Poincaré’s duality gives the isomorphism \(D'_X(\mathbb{C}_M) \simeq \mathfrak{o}_M[-n]\). An equivalent definition of hyperfunctions is thus given by

\[
(6.1) \quad \mathcal{B}_M = R\mathcal{H}om(D'_X(\mathbb{C}_M), \mathcal{O}_X).
\]

Let us define the notion of “boundary value” in this setting. Consider a subanalytic open subset \(\Omega\) of \(X\) and denote by \(\overline{\Omega}\) its closure. Assume that:

\[
\begin{cases}
  D'_X(\mathbb{C}_\Omega) \simeq \mathbb{C}_\Omega, \\
  M \subset \overline{\Omega}.
\end{cases}
\]

The morphism \(\mathbb{C}_{\overline{\Omega}} \to \mathbb{C}_M\) defines by duality the morphism \(D'_X(\mathbb{C}_M) \to D'_X(\mathbb{C}_{\overline{\Omega}}) \simeq \mathbb{C}_\Omega\). Applying the functor \(R\mathcal{H}om(\cdot, \mathcal{O}_X)\), we get the boundary value morphism

\[
(6.2) \quad b: \mathcal{O}(\Omega) \to \mathcal{B}(M) \text{ where } \mathcal{B}(M) := \Gamma(M; \mathcal{B}_M).
\]

When considering operations on hyperfunctions such as integral transforms, one is naturally lead to consider more general sheaves of generalized functions such as \(R\mathcal{H}om(G, \mathcal{O}_X)\) where \(G\) is an \(\mathbb{R}\)-constructible sheaf.

Similarly as in dimension one, one can represent the sheaf \(\mathcal{B}_M\) by using Čech cohomology of coverings of \(X \setminus M\). For example, let \(X\) be a Stein open subset of \(\mathbb{C}^n\) and set \(M = \mathbb{R}^n \cap X\). Denote by \(x\) the coordinates on \(\mathbb{R}^n\) and by \(x+iy\) the coordinates on \(\mathbb{C}^n\). One can recover \(\mathbb{C}^n \setminus \mathbb{R}^n\) by \(n+1\) open half-spaces \(V_i = \langle y, \xi_i \rangle > 0 \ (i = 1, \ldots, n+1)\). For \(J \subset \{1, \ldots, n+1\}\) set \(V_J = \bigcap_{j \in J} V_j\).

Assuming \(n > 1\), we have the isomorphism \(H^M_M(X; \mathcal{O}_X) \simeq H^{n-1}(X \setminus M; \mathcal{O}_X)\).

Therefore, setting \(U_J = V_J \cap X\), one has

\[
\mathcal{B}(M) \simeq \sum_{|J|=n} \mathcal{O}_X(U_J)/ \sum_{|K|=n-1} \mathcal{O}_X(U_K).
\]

On a real analytic manifold \(M\), any hyperfunction \(u \in \Gamma(M; \mathcal{B}_M)\) is a (non unique) sum of boundary values of holomorphic functions defined in tubes with edge \(M\). Such a decomposition leads to the so-called Edge of the Wedge theorem and was intensively studied in the seventies.

Then comes naturally the following problem: how to recognize the directions associated with these tubes? This is at the origin of the construction of Sato’s microlocalization functor. Sato introduced in [Sat70] the sheaf \(\mathcal{C}_M\) of microfunctions on \(T^*_M X\) as

\[
(6.3) \quad \mathcal{C}_M = \mu hom(D'_X(\mathbb{C}_M), \mathcal{O}_X).
\]
It is proved that again, this complex is concentrated in degree 0. Thus \( C_M \) is a conic sheaf on \( T^*_M X \) and one has by its construction

\[
\mathcal{B}_M \cong \pi_{M*} \mathcal{C}_M.
\]

Denote by \( \text{spec} \) the natural map.

\[
\text{spec}: \Gamma(M; \mathcal{B}_M) \cong \Gamma(T^*_M X; \mathcal{C}_M).
\]

**Definition 6.2.** The (analytic) wave front set \( \text{WF}(u) \) of a hyperfunction \( u \in \mathcal{B}(M) \) is the support of \( \text{spec}(u) \).

Soon after Mikio Sato has defined the analytic wave front set of hyperfunction, Lars Hörmander defined the \( C^\infty \)-wave front set of distributions, by using classical Fourier transform (see [Hör83]).

### 6.2 Holomorphic solutions of D-modules

References for D-modules are made to [Kas03].

**Characteristic variety**

Let \( X \) be a complex manifold. One denotes by \( \mathcal{D}_X \) the sheaf of rings of holomorphic (finite order) differential operators. It is a right and left coherent ring. A system of linear partial differential equations on \( X \) is a left coherent \( \mathcal{D}_X \)-module \( \mathcal{M} \). The link with the intuitive notion of a system of linear partial differential equations is as follows. Locally on \( X \), \( \mathcal{M} \) may be represented as the cokernel of a matrix \( \cdot P_0 \) of differential operators acting on the right:

\[
\mathcal{M} \cong \mathcal{D}^{N_0}_X / \mathcal{D}^{N_1}_X \cdot P_0.
\]

By classical arguments of analytic geometry (Hilbert’s syzygies theorem), one shows that \( \mathcal{M} \) is locally isomorphic to the cohomology of a bounded complex

\[
(6.4) \quad \mathcal{M}^* := 0 \rightarrow \mathcal{D}^{N_r}_X \rightarrow \cdots \rightarrow \mathcal{D}^{N_1}_X \xrightarrow{\cdot P_0} \mathcal{D}^{N_0}_X \rightarrow 0.
\]

For a coherent \( \mathcal{D}_X \)-module \( \mathcal{M} \), one sets for short

\[
\text{Sol}(\mathcal{M}) := \text{RHom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X).
\]
Representing (locally) $\mathcal{M}$ by a bounded complex $\mathcal{M}^\bullet$, we get

\[(6.5) \quad \text{Sol}(\mathcal{M}) \cong 0 \to \mathcal{O}_X^{N_0} \xrightarrow{P_0} \mathcal{O}_X^{N_1} \to \cdots \to \mathcal{O}_X^{N_r} \to 0,\]

where now $P_0$ operates on the left.

One defines naturally the characteristic variety of $\mathcal{M}$, denoted $\text{char}(\mathcal{M})$, a closed complex analytic subset of $T^*_X$, conic with respect to the action of $\mathbb{C}^\times$ on $T^*X$. For example, if $\mathcal{M}$ has a single generator $u$ with relation $\mathcal{I} u = 0$, where $\mathcal{I}$ is a locally finitely generated left ideal of $\mathcal{D}_X$, then

\[\text{char}(\mathcal{M}) = \{ (z; \zeta) \in T^*X; \sigma(P)(z; \zeta) = 0 \text{ for all } P \in \mathcal{I} \},\]

where $\sigma(P)$ denotes the principal symbol of $P$.

The fundamental result below was first obtained in [SKK73].

**Theorem 6.3.** Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. Then $\text{char}(\mathcal{M})$ is a closed conic complex analytic involutive (i.e., co-isotropic) subset of $T^*X$.

The proof of the involutivity is really difficult: it uses microdifferential operators of infinite order and quantized contact transformations. Later, Gabber [Gab81] gave a purely algebraic (and much simpler) proof of this result and we shall give in Theorem 6.4 below another totally different proof.

After identifying $X$ with its real underlying manifold, the link between the microsupport of sheaves and the characteristic variety of coherent $\mathcal{D}_X$-modules is given by:

**Theorem 6.4.** (See [KS90, Th. 11.3.3].) Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. Then

\[(6.6) \quad \text{SS}(F) = \text{char}(\mathcal{M}).\]

As a corollary of Theorems 2.4 and 6.4, one recovers the fact that the characteristic variety of a coherent $\mathcal{D}_X$-module is co-isotropic.

We shall only prove the inclusion $\subset$ in (6.6), the most useful for applications.

**Sketch of proof.** Let $p \notin \text{char}(\mathcal{M})$.

(i) Assume first that $\mathcal{M} = \mathcal{D}_X/\mathcal{D}_X \cdot P$ for a section $P$ of $\mathcal{D}_X$, say of order $m$. Hence, $\sigma(P)(p) \neq 0$, where $\sigma(P)$ is the principal symbol of $P$. If $p = (x_0; 0) \in T^*_X X$, then $P$ is an invertible function at $x_0$ and the result is clear.
Assume \( p \notin T_X^*X \). We choose a local holomorphic coordinate system \((x) = (x_1, \ldots, x_n)\) so that \( p = (x_0; \xi_0) \) with \( \xi_0 = (1, 0, \ldots, 0) \) and we set \( x = (x_1, x') \).

Set
\[
\begin{align*}
\gamma_\delta &= \{ x; 3x_1 = 0, \Re x_1 \geq \delta |x'| \}, \\
H_\varepsilon &= \{ x; \Re \langle x, \xi_0 \rangle \geq -\varepsilon \}, \\
L_\varepsilon &= \{ x; \Re (x - x_0, \xi_0) = -\varepsilon \}.
\end{align*}
\]

We choose \( 0 < R \ll 1 \) and \( \delta \gg 0 \) such that
\[
\sigma(P)(x; \xi) \neq 0 \text{ for } |x - x_0| \leq R, \xi \in \gamma_\delta \setminus \{0\}.
\]

Let \( K \) be a compact convex subset of \( X = \mathbb{C}^n \). Since \( R\Gamma(K; \mathcal{O}_X) \) is concentrated in degree 0, the object \( R\Gamma(K; \text{Sol}(\mathcal{M})) \) is represented by the complex
\[
0 \to \mathcal{O}_X(K) \xrightarrow{p} \mathcal{O}_X(K) \to 0.
\]

Applying Theorem 2.6, we are reduced to prove that the two complexes
\[
0 \to \mathcal{O}_X((x + \gamma_\delta) \cap H_\varepsilon) \xrightarrow{p} \mathcal{O}_X((x + \gamma_\delta) \cap H_\varepsilon) \to 0
\]
and
\[
0 \to \mathcal{O}_X((x + \gamma_\delta) \cap L_\varepsilon) \xrightarrow{p} \mathcal{O}_X((x + \gamma_\delta) \cap L_\varepsilon) \to 0
\]
are quasi-isomorphic for \( |x - x_0| \ll 1 \). This follows from the precise version of the Cauchy-Kowalevski\(^2\) theorem of Petrowsky, Leray, Zerner (see [Hör83, Vol 1, Th. 11.4.7]).

(ii) In order to reduce to the case (i), one mimics the proof of the Cauchy-Kowalevski theorem for systems of [Kas95]. In a neighborhood of \( x_0 \) the \( \mathcal{D} \)-module \( \mathcal{M} \) admits a system of generators \((u_1, \ldots, u_N)\) and \( p \notin \text{char}(\mathcal{D}_X \cdot u_j) \) \((j = 1, \ldots, N)\). For each \( j \) there exists a section \( P_j \) of \( \mathcal{D}_X \) such that \( p \) is non characteristic for \( P_j \) and \( P_ju_j = 0 \). Hence there is a natural \( \mathcal{D}_X \)-linear morphism \( \mathcal{D}_X / \mathcal{D}_X \cdot P_j \to \mathcal{D}_X / u_j \). Define the coherent \( \mathcal{D}_X \)-module \( \mathcal{K} \) by the exact sequence
\[
0 \to \mathcal{K} \to \bigoplus_{j=1}^N (\mathcal{D}_X / \mathcal{D}_X \cdot P_j) \to \mathcal{M} \to 0.
\]

\(^2\)we use the name “Kowalevski”, according to Sofia Kovalevskaya’s practice.
Then \( p \notin \text{char}(\mathcal{K}) \). Set for short \( \mathcal{L} = \bigoplus_{j=1}^{N}(\mathcal{D}_{X}/\mathcal{D}_{X} \cdot P_{j}) \). Let \( \varphi: X \to \mathbb{R} \) be a \( C^{1} \)-function as in Definition 2.3 and denote for short by \( \text{Sol}_{\varphi} \) the functor \((\Gamma\{x;\varphi(x) \geq 0\})_{x_{0}} \simeq 0 \). We have a distinguished triangle

\[
\text{Sol}_{\varphi}(\mathcal{M}) \to \text{Sol}_{\varphi}(\mathcal{L}) \to \text{Sol}_{\varphi}(\mathcal{K}) \xrightarrow{+1} \]

from which one deduces the long exact sequence

\[
0 \to H^{0}\text{Sol}_{\varphi}(\mathcal{M}) \to H^{0}\text{Sol}_{\varphi}(\mathcal{L}) \to H^{0}\text{Sol}_{\varphi}(\mathcal{K}) \to H^{1}\text{Sol}_{\varphi}(\mathcal{M}) \to \cdots.
\]

It follows from (i) that \( \text{Sol}_{\varphi}(\mathcal{L}) \simeq 0 \). Therefore, \( H^{0}\text{Sol}_{\varphi}(\mathcal{M}) \simeq 0 \) and \( H^{j}\text{Sol}_{\varphi}(\mathcal{K}) \simeq H^{j+1}\text{Sol}_{\varphi}(\mathcal{M}) \). Since \( \mathcal{K} \) satisfies the same hypotheses as \( \mathcal{M} \), we get by induction that \( H^{j}\text{Sol}_{\varphi}(\mathcal{M}) \simeq 0 \) for all \( j \in \mathbb{Z} \).

Q.E.D.

**Cauchy problem**

Let \( Y \) be a complex submanifold of the complex manifold \( X \) and let \( \mathcal{M} \) be a coherent \( \mathcal{D}_{X} \)-module. One can define the induced \( \mathcal{D}_{Y} \)-module \( \mathcal{M}_{Y} \), but in general it is an object of the derived category \( \mathcal{D}^{b}(\mathcal{D}_{Y}) \) which is neither concentrated in degree zero nor coherent. Nevertheless, there is a natural morphism

\[
(6.7) \quad R\mathcal{H}\text{om}_{\mathcal{D}_{X}}(\mathcal{M}, \mathcal{O}_{X})|_{Y} \to R\mathcal{H}\text{om}_{\mathcal{D}_{Y}}(\mathcal{M}_{Y}, \mathcal{O}_{Y}).
\]

Recall that one says that \( Y \) is non-characteristic for \( \mathcal{M} \) if

\[
\text{char}(\mathcal{M}) \cap T^{*}_{Y}X \subset T^{*}_{X}X.
\]

With this hypothesis, the induced system \( \mathcal{M}_{Y} \) by \( \mathcal{M} \) on \( Y \) is a coherent \( \mathcal{D}_{Y} \)-module and one has the Cauchy-Kowalevski-Kashiwara theorem [Kas95]:

**Theorem 6.5.** Assume \( Y \) is non-characteristic for \( \mathcal{M} \). Then \( \mathcal{M}_{Y} \) is a coherent \( \mathcal{D}_{Y} \)-module and the morphism (6.7) is an isomorphism.

**Sketch of proof.** (i) Similarly as in the proof of Theorem 6.4, one reduces to the case where \( \mathcal{M} = \mathcal{D}_{X}/\mathcal{D}_{X} \cdot P \) for a differential operator \( P \) of order \( m \) and \( Y \) is a hypersurface.

(ii) Choose a local coordinate system \( z = (z_{0}, z_{1}, \ldots, z_{n}) = (z_{0}, z') \) on \( X \) such that \( Y = \{z_{0} = 0\} \). Then \( Y \) is non-characteristic with respect to \( P \) (i.e., for the \( \mathcal{D}_{X} \)-module \( \mathcal{D}_{X}/\mathcal{D}_{X} \cdot P \)) if and only if \( P \) is written as

\[
(6.8) \quad P(z_{0}, z'; \partial_{z_{0}}, \partial_{z'}) = \sum_{0 \leq j \leq m} a_{j}(z_{0}, z') \partial^{j}_{z_{0}}
\]
where $a_j(z_0, z', \partial z')$ is a differential operator not depending on $\partial z_0$ of order $\leq m - j$ and $a_m(z_0, z')$ (which is a holomorphic function on $X$) satisfies: $a_m(0, z') \neq 0$. By the definition of the induced system $\mathcal{M}_Y$ we obtain

$$\mathcal{M}_Y \simeq \mathcal{D}_X/(z_0 \cdot \mathcal{D}_X + \mathcal{D}_X \cdot P).$$

By the Späth-Weierstrass division theorem for differential operators, any $Q \in \mathcal{D}_X$ may be written uniquely in a neighborhood of $Y$ as

$$Q = R \cdot P + \sum_{j=0}^{m-1} S_j(z, \partial z') \partial^j z_0,$$

hence, as

$$Q = z_0 \cdot Q_0 + R \cdot P + \sum_{j=0}^{m-1} R_j(z', \partial z') \partial^j z_0.$$

Therefore $\mathcal{M}_Y$ is isomorphic to $\mathcal{D}_Y^m$. Theorem 6.5 gives:

$$\text{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)|_Y \simeq \mathcal{O}^m_Y, \quad \text{Ext}^1_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)|_Y \simeq 0.$$

In other words, the morphism which to a holomorphic solution $f$ of the homogeneous equation $Pf = 0$ associates its $m$-first traces on $Y$ is an isomorphism and one can solve the equation $Pf = g$ is a neighborhood of each point of $Y$.

This is exactly the classical Cauchy-Kowalevski theorem. Q.E.D.

### 6.3 Elliptic pairs

Let us apply Corollary 4.6 when $X$ is a complex manifold. For $G \in \mathcal{D}^b_{\mathcal{R-c}}(\mathbb{C}_X)$, set

$$\mathcal{A}_G = \mathcal{O}_X \otimes G, \quad \mathcal{B}_G := R\text{Hom}(\mathcal{D}'_X G, \mathcal{O}_X).$$

Note that if $X$ is the complexification of a real analytic manifold $M$ and we choose $G = \mathcal{C}_M$, we recover the sheaf of real analytic functions and the sheaf of hyperfunctions:

$$\mathcal{A}_{\mathcal{C}_M} = \mathcal{A}_M, \quad \mathcal{B}_{\mathcal{C}_M} = \mathcal{B}_M.$$

Now let $\mathcal{M} \in \mathcal{D}^b_{\text{coh}}(\mathcal{D}_X)$. According to [SS94], one says that the pair $(G, \mathcal{M})$ is elliptic if $\text{char}(\mathcal{M}) \cap \text{SS}(G) \subset T^*_X X$. 41
Corollary 6.6. [SS94] Let $(\mathcal{M}, G)$ be an elliptic pair.

(a) We have the canonical isomorphism:

\[
\mathcal{R} \mathcal{H}om_{\mathfrak{D}}(\mathcal{M}, A_G) \xrightarrow{\sim} \mathcal{R} \mathcal{H}om_{\mathfrak{D}}(\mathcal{M}, B_G).
\]

(b) Assume moreover that $\text{Supp}(\mathcal{M}) \cap \text{Supp}(G)$ is compact and $\mathcal{M}$ admits a global presentation as in (6.4). Then the cohomology of the complex $\mathcal{R} \mathcal{H}om_{\mathfrak{D}}(\mathcal{M}, A_G)$ is finite dimensional.

Proof. (a) This is a particular case of Corollary 4.6.

(b) One represents the left hand side of the global sections of (6.9) by a complex of topological vector spaces of type DFN and the right hand side by a complex of topological vector spaces of type FN. Q.E.D.

Let us particularize Corollary 4.6 to the usual case of an elliptic system. Let $M$ be a real analytic manifold, $X$ a complexification of $M$ and let us choose $G = D'_X \mathbb{C}_M$. Then $(G, \mathcal{M})$ is an elliptic pair if and only if

\[
T^*_M X \cap \text{char}(\mathcal{M}) \subset T^*_X X.
\]

(6.10)

In this case, one simply says that $\mathcal{M}$ is an elliptic system. Then one recovers a classical result:

Corollary 6.7. Let $\mathcal{M}$ be an elliptic system.

(a) We have the canonical isomorphism:

\[
\mathcal{R} \mathcal{H}om_{\mathfrak{D}}(\mathcal{M}, A_M) \xrightarrow{\sim} \mathcal{R} \mathcal{H}om_{\mathfrak{D}}(\mathcal{M}, B_M).
\]

(b) Assume moreover that $M$ is compact and $\mathcal{M}$ admits a global presentation as in (6.4). Then the cohomology of the complex $\mathcal{R} \mathcal{H}om_{\mathfrak{D}}(\mathcal{M}, A_M)$ is finite dimensional.

There is a more precise result, due to Sato [Sat70].

Proposition 6.8. Let $\mathcal{M}$ be a coherent $\mathfrak{D}_X$-module, let $j \in \mathbb{Z}$ and let $u \in \mathfrak{E}xt^j_{\mathfrak{D}}(\mathcal{M}, \mathcal{O}_X)$. Then $WF(u) \subset T^*_M X \cap \text{char}(\mathcal{M})$.

Proof. One has

\[
\mathcal{R} \mathcal{H}om_{\pi^{-1}_M \mathfrak{D}_X}(\pi^{-1}_M \mathcal{M}, \mathcal{O}_M) \cong \mu \text{hom}(D'_M \mathbb{C}_M, \mathcal{R} \mathcal{H}om(\mathcal{M}, \mathcal{O}_X))
\]

and the support of the right-hand side is contained in $\text{SS}(\mathcal{R} \mathcal{H}om(\mathcal{M}, \mathcal{O}_X)) \cap \text{SS}(\mathbb{C}_M)$, that is, in $T^*_M X \cap \text{char}(\mathcal{M})$. Q.E.D.
6.4 Hyperbolic systems

Let again $M$ be a real analytic manifold and $X$ a complexification of $M$.

Recall that we have constructed in (4.8) (with other notations) the maps

$$T^*M \hookrightarrow T^*T_M^*X \simeq T_{T_M^*X}T^*X.$$  

(6.12)

**Definition 6.9.** Let $M$ be a coherent left $\mathcal{D}_X$-module. We set

$$\text{hypchar}_M(M) = T^*M \cap C_{T_M^*X}(\text{char}(M))$$

and call $\text{hypchar}_M(M)$ the hyperbolic characteristic variety of $M$ along $M$. A vector $\theta \in T^*M$ such that $\theta \notin \text{hypchar}_M(M)$ is called hyperbolic with respect to $M$. In case $M = \mathcal{D}_X/\mathcal{D}_X \cdot P$ for a differential operator $P$, one says that $\theta$ is hyperbolic for $P$.

**Example 6.10.** Assume we have a local coordinate system $z = x + \sqrt{-1}y$ and $M = \{y = 0\}$. Denote by $(z; \xi)$ the symplectic coordinates on $T^*X$ with $\xi = \xi + \sqrt{-1}\eta$. Let $(x_0; \theta_0) \in T^*M$ with $\theta_0 \neq 0$. Let $P$ be a differential operator with principal symbol $\sigma(P)$. Applying the definition of the normal cone, we find that $(x_0; \theta_0)$ is hyperbolic for $P$ if and only if

$$\left\{ \begin{array}{l}
\text{there exist an open neighborhood } U \text{ of } x_0 \text{ in } M \text{ and an open conic neighborhood } \gamma \text{ of } \theta_0 \in \mathbb{R}^n \text{ such that } 
\sigma(P)(x; \theta_0 + \sqrt{-1}\eta) \neq 0 \text{ for all } \eta \in \mathbb{R}^n, x \in U \text{ and } \theta \in \gamma. 
\end{array} \right.$$  

(6.13)

As noticed in the 70’s by M. Kashiwara, it follows from the local Bochner’s tube theorem that condition (6.13) will be satisfied as soon as $\sigma(P)(x; \theta_0 + \sqrt{-1}\eta) \neq 0$ for all $\eta \in \mathbb{R}^n$ and $x \in U$. Hence, one recovers the classical notion of a (weakly) hyperbolic operator (see [Ler53]).

**Theorem 6.11.** Let $M$ be a coherent $\mathcal{D}_X$-module. Then

$$\text{SS}(\mathcal{H}om_{\mathcal{D}_X}(M, \mathcal{B}_M)) \subset \text{hypchar}_M(M).$$

The same result holds with $\mathcal{A}_M$ instead of $\mathcal{B}_M$.

**Proof.** This follows from Corollary 4.8 and the isomorphisms

$$\Gamma_M \mathcal{H}om_{\mathcal{D}_X}(M, \mathcal{O}_X) \simeq \mathcal{H}om_{\mathcal{D}_X}(M, \Gamma_M \mathcal{O}_X),$$

$$\mathcal{H}om_{\mathcal{D}_X}(M, \mathcal{O}_X)|_M \simeq \mathcal{H}om_{\mathcal{D}_X}(M, \mathcal{O}_X|_M).$$

Q.E.D.
We consider the following situation: $M$ is a real analytic manifold of dimension $n$, $X$ is a complexification of $M$, $N \hookrightarrow M$ is a real analytic smooth closed submanifold of $M$ of codimension $d$ and $Y \hookrightarrow X$ is a complexification of $N$ in $X$.

**Theorem 6.12.** Let $M, X, N, Y$ be as above and let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. We assume

\begin{equation}
T^*_N M \cap \text{hypchar}_M(\mathcal{M}) \subset T^*_M M.
\end{equation}

In other words, any non zero vector $\theta \in T^*_N M$ is hyperbolic for $M$. Then $Y$ is non characteristic for $M$ in a neighborhood of $N$ and the isomorphism (6.7) induces the isomorphism

\begin{equation}
R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M)|_N \cong R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{B}_N).
\end{equation}

**Proof.** (i) We shall not prove here that the hypothesis implies that $Y$ is non characteristic for $M$.

(ii) We have the chain of isomorphisms

\[
R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M)|_N \cong R\Gamma_N R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M) \otimes \mathcal{O}_M[d] \\
\cong R\Gamma_N R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X) \otimes \mathcal{O}_N[n + d] \\
\cong R\Gamma_N R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X) \otimes \mathcal{O}_N[n + d] \\
\cong R\Gamma_N R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{O}_Y) \otimes \mathcal{O}_N[n - d] \\
\cong R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{B}_N).
\]

Here, the first isomorphism follows from Theorem 6.11 and Corollary 4.6, since the micro-support of $R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M)$ does not intersect $T^*_N M$ outside the zero-section. The second uses the definition of the sheaf $\mathcal{B}_M$, the third is obvious since $N$ is both contained in $M$ and in $Y$, the fourth follows from Theorem 6.4 and Corollary 4.6, the fifth is Theorem 6.5 and the last one uses the definition of the sheaf $\mathcal{B}_N$. Q.E.D.

Consider for simplicity the case where $\mathcal{M} = \mathcal{D}_X/I$ where $I$ is a coherent left ideal of $\mathcal{D}_X$. A section $u$ of $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M)$ is a hyperfunction $u$ such that $Qu = 0$ for all $Q \in I$. It follows that the analytic wave front set of $u$ does not intersect $T^*_Y X \cap T^*_N X$ and this implies that the restriction of $u$ (and its derivative) to $N$ is well-defined as a hyperfunction on $N$. One can show that the morphism (6.15) is then obtained using this restriction morphism, similarly as in Theorem 6.5.

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References


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