Abstract

On a finite-dimensional real vector space, we give a microlocal characterization of (derived) piecewise linear sheaves (PL sheaves) and prove that the category of such sheaves is generated by sheaves associated with convex polytopes. We then give a similar theorem for PL γ-sheaves, that is PL sheaves associated with the γ-topology, for a closed convex polyhedral proper cone γ. Recall that convex polytopes may be considered as building blocks for higher dimensional barcodes.

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Introduction

In a previous paper [KS18] we have interpreted persistent homology in higher dimension in the language of (derived) γ-sheaves, where γ is a closed convex proper cone in a finite-dimensional real vector space $\mathbb{V}$. We have also proved that such γ-sheaves may be approximated (for a kind of derived bottleneck distance) by PL γ-sheaves (PL for piecewise linear).

Here, we start with a systematic study of PL-sheaves. We show in particular that a sheaf is PL if and only if its microsupport is a PL Lagrangian variety or, equivalently, is contained in such a Lagrangian variety. We also show that the triangulated category of PL-sheaves is generated by the family of constant sheaves on locally closed polytopes.

Then, after recalling the main results of [KS18], we prove that the triangulated category of PL-γ-sheaves is generated by the family of constant sheaves on γ-locally closed polytopes. Recall that convex polytopes may be considered as building blocks for higher dimensional barcodes [KS18].

1 PL geometry

1.1 PL sets and PL stratifications

Let $\mathbb{V}$ be a real finite-dimensional vector space.

Definition 1.1. (a) A convex polytope $P$ in $\mathbb{V}$ is the intersection of a finite family of open or closed affine half-spaces.

(b) A PL set is a finite\(^3\) union of convex polytopes.

Note that a PL set is subanalytic.

The next result is obvious.

Lemma 1.2. (a) The family of PL-sets in $\mathbb{V}$ is stable by finite unions and finite intersections.

(b) If $Z$ is PL, then its closure $\overline{Z}$, its interior $\text{Int}(Z)$ and its complementary set $\mathbb{V} \setminus Z$ are PL.

(c) Let $u: \mathbb{V} \rightarrow \mathbb{W}$ be a linear map.

(i) If $S \subset \mathbb{V}$ is PL, then $u(S) \subset \mathbb{W}$ is PL.

(ii) If $Z \subset \mathbb{W}$ is PL, then $u^{-1}(Z) \subset \mathbb{V}$ is PL.

For a locally closed submanifold $Z \subset \mathbb{V}$, one sets for short

$$T^*_Z \mathbb{V} := T^*_Z U$$ where $U$ is an open subset of $\mathbb{V}$ containing $Z$ as a closed subset.

\(^3\)In all this paper, we consider finite union of polytopes, contrarily to [KS18] in which we consider locally finite union. The same remark applies in particular to Definitions 1.3, 2.3 and 3.7.
Definition 1.3. A PL-stratification of a set $S$ of $V$ is a finite family $Z = \{Z_a\}_{a \in A}$ of non-empty convex polytopes such that

(i) each $Z_a$ is a locally closed submanifold,

(ii) $Z_a \cap Z_b = \emptyset$ for $a \neq b$,

(iii) $Z_a \cap Z_b \neq \emptyset$ implies $Z_a \subset Z_b$.

Recall the operation $\hat{\oplus}$ and the notion of a $\mu$-stratification of [KS90, Def. 6.2.4, 8.3.19].

Proposition 1.4. Let $Z = \{Z_a\}_{a \in A}$ be a PL stratification. Then

(i) $\{Z_a\}_{a \in A}$ is a $\mu$-stratification, that is, $Z_a \subset Z_b$ implies $(T_{Z_a}^* V \hat{\oplus} T_{Z_b}^* V) \cap \pi^{-1}(Z_a) \subset T_{Z_a}^* V$.

(ii) Set $\Lambda = \bigsqcup_{a \in A} T_{Z_a}^* V$. Then $\Lambda \hat{\oplus} \Lambda = \Lambda$.

Proof. We shall prove both statements together.

Assume that $P_a \subset P_b \cap P_c$. We may assume (in a neighborhood of a point of $P_a$) that $V = W \oplus W'$ for two linear spaces $W$ and $W'$ and $P_a = W$. Then $P_b = W \times S$ and $P_c = W \times Z$ where $S$ is open in some linear subspace $W''$ of $W'$ and similarly for $P_c$.

Then one immediately checks that $(T_{P_b}^* V \hat{\oplus} T_{P_c}^* V) \cap \pi^{-1}(P_a) \subset T_{P_a}^* V$. \hfill \Box

Proposition 1.5. Consider a finite family $\{P_b\}_{b \in B}$ of convex polytopes. Then there exists a PL-stratification $V = \bigsqcup_{a \in A} Z_a$ such that for any $b \in B$, $P_b$ is a union of strata.

In the sequel, an interval of $\mathbb{R}$ means a convex subset of $\mathbb{R}$.

Proof. There exists a finite family $\{f_1, \ldots, f_l\}$ of linear forms and a finite family $\{I_c\}_{c \in C}$ such that each $I_c$ is either an open interval or a point and that $\mathbb{R} = \bigcup_{c \in C} I_c$ and for all $b \in B$,

$$P_b = \bigcap_{1 \leq j \leq l} f_j^{-1}(J_j),$$

where $J_j$ is a union of some $I_c$, $c \in C$.

For any family $d = \{c_1, \ldots, c_l\} \in C^l$, set

$$Z_d = \bigcap_{j=1}^l f_j^{-1}(I_{c_j}).$$

Then the family $\{Z_d\}_{d \in C^l}$ is a PL-stratification of $V$ finer that the family $\{P_b\}_{b \in B}$. \hfill \Box
1.2 PL Lagrangian subvarieties

Recall that the notions of co-isotropic, isotropic and Lagrangian subanalytic subvarieties are given in [KS90, Def. 6.5.1, 8.3.9].

Proposition 1.6. Let $\Lambda$ be a locally closed conic PL isotropic subset of $T^*V$. Then for any $p \in \Lambda_{\text{reg}}$ there exists a linear affine subspace $L \subset V$ with $\Lambda \subset T^*_LV$ in a neighborhood of $p$.

Proof. If $\lambda$ is a linear affine isotropic subspace of $T^*V$, then there exists a linear affine subspace $L$ of $V$ such that $\lambda \subset T^*_LV$. \qed

Lemma 1.7. Let $\{L_a\}_{a \in A}$ be a finite family of affine linear subspaces of $V$. Set $X = \bigcup_{a \in A} L_a$ and let $S \subset X$. Assume that $X$ is a closed subset of $V$, $S \cap X_{\text{reg}}$ is open in $X_{\text{reg}}$ and $S$ is the closure of $S \cap X_{\text{reg}}$. Then $S$ is PL.

Proof. Indeed $S \cap \Lambda_{\text{reg}}$ is a union of connected components of $\Lambda_{\text{reg}}$. \qed

Theorem 1.8. (a) Let $\Lambda$ be a locally closed conic PL isotropic subset of $T^*V$. Then there exists a PL stratification $\{P_a\}_{a \in A}$ of $V$ such that $\Lambda \subset \bigsqcup_{a \in A} T^*P_aV$.

(b) Let $\Lambda$ be a locally closed conic subanalytic Lagrangian subset of $T^*V$ and assume that $\Lambda$ is contained in a closed conic PL isotropic subset. Then $\Lambda$ is PL.

Proof. (a) Let $\{\Omega_i\}_{i \in I}$ be the family of connected components of $\Lambda_{\text{reg}}$. Note that the $\Omega_i$’s are PL. Then there exists an affine linear subspace $L_i$ such that $\Omega_i = \subset T^*_L\mathbb{V}$ by Proposition 1.6. Set $\omega_i = \pi(\Omega_i)$ and choose a PL stratification $\{P_a\}_{a \in A}$ finer than the family $\{\omega_i\}_{i \in I}$. Then $\Lambda_{\text{reg}} \subset \bigsqcup P_aV$ and Proposition 1.4 (a) implies that this last set is closed, hence contains $\Lambda$.

(b) follows from Lemma 1.7. \qed

2 PL sheaves

2.1 Review on sheaves

Let us recall some definitions extracted from [KS90] and a few notations.

- Throughout this paper, $k$ denotes a field. We denote by $\text{Mod}(k)$ the abelian category of $k$-vector spaces.

- For an abelian category $\mathcal{C}$, we denote by $\text{D}^b(\mathcal{C})$ its bounded derived category. However, we write $\text{D}^b(k)$ instead of $\text{D}^b(\text{Mod}(k))$.

- If $\pi: E \to M$ is a vector bundle over $M$, we identify $M$ with the zero-section of $E$ and we set $\hat{E} := E \setminus M$. We denote by $\hat{\pi}: \hat{E} \to M$ the restriction of $\pi$ to $\hat{E}$.
For a vector bundle $E \rightarrow M$, we denote by $a: E \rightarrow E$ the antipodal map, $a(x, y) = (x, -y)$. For a subset $Z \subset E$, we simply denote by $Z^a$ its image by the antipodal map. In particular, for a cone $\gamma$ in $E$, we denote by $\gamma^a = -\gamma$ the opposite cone. For such a cone, we denote by $\gamma^o$ the polar cone (or dual cone) in the dual vector bundle $E^*$:

$$\gamma^o = \{(x; \xi) \in E^*; \langle \xi, v \rangle \geq 0 \text{ for all } v \in \gamma_x\}. \quad (2.1)$$

Let $M$ be a real manifold of dimension $\dim M$. We shall use freely the classical notions of microlocal sheaf theory, referring to [KS90]. We denote by $\text{Mod}(k_M)$ the abelian category of sheaves of $k$-modules on $M$ and by $D^b(k_M)$ its bounded derived category. For short, an object of $D^b(k_M)$ is called a “sheaf” on $M$.

For a locally closed subset $Z \subset M$, one denotes by $k_Z$ the constant sheaf with stalk $k$ on $Z$ extended by 0 on $M \setminus Z$. One defines similarly the sheaf $L_Z$ for $L \in D^b(k)$.

We denote by $or_M$ the orientation sheaf on $M$ and by $\omega_M$ the dualizing complex on $M$. Recall that $\omega_M \cong or_M [\dim M]$. One shall use the duality functors

$$D'_M(\bullet) = R\mathcal{H}om(\bullet, k_M), \quad D_M(\bullet) = R\mathcal{H}om(\bullet, \omega_M). \quad (2.2)$$

For $F \in D^b(k_M)$ we denote by $\mu_{\text{supp}}(F)^4$ its microsupport, a closed conic co-isotropic subset of $T^*M$.

Constructible sheaves
We refer the reader to [KS90] for terminologies not explained here.

**Definition 2.1.** Let $M$ be a real analytic manifold and let $F \in \text{Mod}(k_M)$. One says that $F$ is weakly $\mathbb{R}$-constructible if there exists a subanalytic stratification $M = \bigsqcup_{a \in A} M_a$ such that for each stratum $M_a$, the restriction $F|_{M_a}$ is locally constant. If moreover, the stalk $F_x$ is of finite rank for all $x \in M$, then one says that $F$ is $\mathbb{R}$-constructible.

**Notation 2.2.**
(i) One denotes by $\text{Mod}_{\mathbb{R}c}(k_M)$ the abelian category of $\mathbb{R}$-constructible sheaves, a thick abelian subcategory of $\text{Mod}(k_M)$.

(ii) One denotes by $D^b_{\mathbb{R}c}(k_M)$ the full triangulated subcategory of $D^b(k_M)$ consisting of sheaves with $\mathbb{R}$-constructible cohomology and by $D^b_{\mathbb{R}c,c}(k_M)$ the full triangulated subcategory of $D^b_{\mathbb{R}c}(k_M)$ consisting of sheaves with compact support.

Recall that the natural functor $D^b(\text{Mod}_{\mathbb{R}c}(k_M)) \rightarrow D^b_{\mathbb{R}c,c}(k_M)$ is an equivalence of categories.

$^4 \mu_{\text{supp}}(F)$ was denoted by $\text{SS}(F)$ in [KS90].
2.2 Microlocal characterization of PL sheaves

Definition 2.3. One says that \( F \in \text{D}^b(k_V) \) is PL if there exists a finite family \( \{ P_a \}_{a \in A} \) of convex polytopes such that \( V = \bigcup_{a \in A} P_a \) and \( F|_{P_a} \) is constant of finite rank for any \( a \in A \).

By this definition

\( F \) is PL if and only if \( H^j(F) \) is PL for all \( j \in \mathbb{Z} \).

One sets

\[
\begin{align*}
\text{D}^b_{\text{PL}}(k_V) &:= \{ F \in \text{D}^b(k_V); \; F \text{ is PL} \}, \\
\text{Mod}_{\text{PL}}(k_V) &:= \text{Mod}(k_V) \cap \text{D}^b_{\text{PL}}(k_V).
\end{align*}
\]

Of course, \( \text{D}^b_{\text{PL}}(k_V) \) is a subcategory of \( \text{D}^b_{\text{Rc}}(k_V) \) and \( \text{Mod}_{\text{PL}}(k_V) \) is a subcategory of \( \text{Mod}_{\text{Rc}}(k_V) \).

Proposition 2.4. The natural functor \( \text{D}^b(\text{Mod}_{\text{PL}}(k_V)) \to \text{D}^b_{\text{PL}}(k_V) \) is an equivalence.

Proof. There exists a triangulation \( S = (S, \Delta) \) and a homeomorphism \( f: |S| \to V \) such that its restriction to \( |\sigma| \) is linear for any \( \sigma \in \Delta \). Then the result follow from [KS90, Th. 8.1.10]. \( \square \)

Theorem 2.5. Let \( F \in \text{D}^b_{\text{Rc}}(k_V) \). Then the conditions below are equivalent.

(a) \( F \in \text{D}^b_{\text{PL}}(k_V) \),

(b) \( \mu\text{supp}(F) \) is a closed conic PL Lagrangian subset of \( T^*V \),

(c) \( \mu\text{supp}(F) \) is contained in a closed conic PL Lagrangian subset of \( T^*V \).

Proof. (a)\( \Rightarrow \)(c). Consider a covering \( \{ P_b \}_{b \in B} \) by convex polytopes such that \( F|_{P_b} \) is constant and choose a finer PL stratification \( V = \bigcup_{a \in A} Z_a \). This is a \( \mu \)-stratification and this implies \( \mu\text{supp}(F) \subset \bigcup_{a \in A} T_{Z_a}^*V \) by [KS90, Prop. 8.4.1].

(b)\( \Rightarrow \)(a) By Theorem 1.8 (a), there exists a PL stratification \( V = \bigcup_{a \in A} Z_a \) such that \( \mu\text{supp}(F) \subset \bigcup_{a \in A} T_{Z_a}^*V \). Then \( F|_{Z_a} \) is locally constant for each \( a \in A \) by [KS90, Prop. 8.4.1].

(b)\( \Leftrightarrow \)(c) in view of Theorem 1.8 (b). \( \square \)

The next result immediately follows from Definition 2.3. It can also easily be deduced from [KS90], Theorem 1.8 and Theorem 2.5.

Corollary 2.6. (i) The category \( \text{D}^b_{\text{PL}}(k_V) \) is a full triangulated subcategory of the category \( \text{D}^b(k_V) \) and the category \( \text{Mod}_{\text{PL}}(k_V) \) is a full thick abelian subcategory of the category \( \text{Mod}(k_V) \).

(ii) If \( F_1 \) and \( F_2 \) are PL, then so are \( F_1 \otimes F_2 \) and \( \text{RHom}(F_1, F_2) \).

(iii) Let \( f: V \to W \) be a linear map.
(a) If $G$ is a PL sheaf on $\mathcal{W}$, then $f^{-1}G$ is a PL sheaf on $V$.
(b) If $F$ is a PL sheaf on $V$ then $Rf_\ast F$ is a PL sheaf on $\mathcal{W}$.

**Proposition 2.7.** Let $Z$ be a locally closed subset of $V$. Then $Z$ is PL if and only if $\text{supp}(k_Z)$ is PL.

**Proof.** (i) Assume that $Z$ is PL. Then the sheaf $k_Z$ is PL and its microsupport is PL by Theorem 2.5.
(ii) Conversely, assume that $\text{supp}(k_Z)$ is PL. Set $\partial Z = \overline{Z} \setminus Z$. Since $Z$ is locally closed, $\partial Z$ is closed.
(ii)-(a) First, notice that $\overline{Z} = \pi(\text{supp}(k_Z))$ is PL.
(ii)-(b) Now consider the exact sequence of sheaves $0 \to k_Z \to k_{\overline{Z}} \to k_{\partial Z} \to 0$. Since $k_Z$ and $k_{\overline{Z}}$ are PL sheaves, the sheaf $k_{\partial Z}$ is PL. Therefore, $\partial Z$ is PL and it follows that $Z$ is PL.

### 2.3 Generators for PL sheaves

Consider a triangulated category $\mathcal{D}$ and a family of objects $\mathcal{G}$. Consider the full subcategory $\mathcal{T}$ of $\mathcal{D}$ defined as follows. An object $F \in \mathcal{D}$ belongs to $\mathcal{T}$ if there exists a finite sequence $F_0, \ldots, F_N$ in $\mathcal{D}$ with $F_0 = 0$, $F_N = F$ and distinguished triangles $F_k \to F_{k+1} \to G_k \xrightarrow{+1} 0$, $0 \leq k < N$ with $G_k \in \mathcal{G}$. Clearly, $\mathcal{T}$ is a triangulated subcategory of $\mathcal{D}$. In this paper, we shall say that $\mathcal{G}$ generates $\mathcal{D}$ if $\mathcal{D} = \mathcal{T}$.

**Theorem 2.8.** The triangulated category $\text{D}_{\text{PL}}^b(k_V)$ is generated by the family $\{k_P\}$ where $P$ ranges over the family of locally closed convex polytopes.

**Proof.** (i) We denote by $\mathcal{G}$ the family of sheaves isomorphic to some $k_P$, $P$ a locally closed convex polytope, and denote by $\mathcal{T}$ the triangulated subcategory of $\text{D}_{\text{PL}}^b(k_V)$ generated by $\mathcal{G}$, that is, the smallest triangulated subcategory of $\text{D}_{\text{PL}}^b(k_V)$ which contains $\mathcal{G}$.
(ii) We argue by induction on $\dim V$. The case where $\dim V$ is 0 or 1 is clear.
(iii) Let $F \in \text{D}_{\text{PL}}^b(k_V)$. By truncation, we may reduce to the case where $F$ is concentrated in degree 0.
(iv) There exists a finite family $\{H_a\}_{a \in A}$ such that, setting $U = V \setminus \bigcup_a H_a$, the restriction of $F$ to $U$ is locally constant. Let $U = \bigcup_i U_i$ be the decomposition of $U$ into connected component. Each $U_i$ is an open convex polytope. Set $Z = \bigcup_a H_a$ and consider the exact sequence $0 \to F_U \to F \to F_Z \to 0$. The sheaf $F_U$ is finite direct sum of sheaves of the type $k_{U_i}$. Hence $F_U \in \mathcal{T}$ and it remains to show that $F_Z$ belongs to $\mathcal{T}$.
(v) We argue by induction on $\# A$. If $\# A = 1$, then the result follows from the induction hypothesis on the dimension of $V$ since we may identify $F_{H_a}$ with a sheaf on the affine space $H_a$. Let $a \in A$ and define $G$ by the exact sequence $0 \to G \to F_Z \to F_{H_a} \to 0$. By the induction hypothesis $G$ and $F_{H_a}$ belong to $\mathcal{T}$ and the result follows. \qed
3 PL $\gamma$-sheaves

3.1 Review on $\gamma$-sheaves

In this subsection we shall review some definitions and results extracted from [KS90, KS18]. The so-called $\gamma$-topology has been studied with some details in [KS90, §3.4].

Let $V$ be a finite-dimensional real vector space. We denote by $a: V \to V$ the antipodal map $x \mapsto -x$.

Hence, for two subsets $A, B$ of $V$, one has $A + B = s(A \times B)$. A subset $A$ of $V$ is called a cone if $0 \in A$ and $\mathbb{R}_{>0}A \subseteq A$. A convex cone $A$ is proper if $A \cap A^a = \{0\}$.

Throughout the paper, we consider a cone $\gamma \subset V$ and we assume:

(3.1) $\gamma$ is closed proper convex with non-empty interior.

In §3.2 we shall make the extra assumption that $\gamma$ is polyhedral, meaning that it is a finite intersection of closed half-spaces.

We say that a subset $A$ of $V$ is $\gamma$-invariant if $A + \gamma = A$. Note that a subset $A$ is $\gamma$-invariant if and only if $V \setminus A$ is $\gamma^a$-invariant.

The family of $\gamma$-invariant open subsets of $V$ defines a topology, which is called the $\gamma$-topology on $V$. One denotes by $V_\gamma$ the space $V$ endowed with the $\gamma$-topology and one denotes by

$$\varphi_\gamma: V \to V_\gamma$$

the continuous map associated with the identity. Note that the closed sets for this topology are the $\gamma^a$-invariant closed subsets of $V$.

Definition 3.1. Let $A$ be a subset of $V$.

(a) One says that $A$ is $\gamma$-open (resp. $\gamma$-closed) if $A$ is open (resp. closed) for the $\gamma$-topology.

(b) One says that $A$ is $\gamma$-locally closed if $A$ is the intersection of a $\gamma$-open subset and a $\gamma$-closed subset.

(c) One says that $A$ is $\gamma$-flat if $A = (A + \gamma) \cap (A + \gamma^a)$.

(d) One says that a closed set $A$ is $\gamma$-proper if the map $s$ is proper on $A \times \gamma^a$.

Remark that a closed subset $A$ is $\gamma$-proper if and only if $A \cap (x + \gamma)$ is compact for any $x \in V$.

Proposition 3.2 ([KS18, Prop. 3.4]). The set of $\gamma$-flat open subsets $\Omega$ of $V$ and the set of $\gamma$-locally closed subsets $Z$ of $V$ are isomorphic by the correspondence

$$\begin{align*}
\Omega &\longrightarrow (\Omega + \gamma) \cap \Omega + \gamma^a \\
\text{Int}(Z) &\longleftarrow Z.
\end{align*}$$

In particular, $\gamma$-locally closed subsets are $\gamma$-flat.
We shall use the notations:

\[
\begin{align*}
D^b_{\gamma^a}(k_V) := \{ F \in D^b(k_V); \mu \text{supp}(F) \subset V \times \gamma^a \}, \\
D^b_{\text{Re},\gamma^a}(k_V) := D^b_{\text{Re}}(k_V) \cap D^b_{\gamma^a}(k_V), \\
\text{Mod}^a_{\gamma^a}(k_V) := \text{Mod}(k_V) \cap D^b_{\gamma^a}(k_V), \\
\text{Mod}^\text{Re,}\gamma^a_{\gamma^a}(k_V) := \text{Mod}^\text{Re}(k_V) \cap \text{Mod}^a_{\gamma^a}(k_V).
\end{align*}
\]

(3.3)

We call an object of $D^b_{\gamma^a}(k_V)$ a $\gamma$-sheaf.

It follows from [KS90, Prop. 5.4.14] that for $F, G \in D^b_{\gamma^a}(k_V)$ and $H \in D^b_{\gamma^a}(k_V)$, the sheaves $F \otimes G$ and $R\text{Hom}(H, F)$ belong to $D^b_{\gamma^a}(k_V)$.

The next result is implicitly proved in [KS90] and explicitly in [KS18]. (In this statement, the hypothesis that $\text{Int}(\gamma)$ is non empty is not necessary.)

**Theorem 3.3.** Let $\gamma$ be a closed convex proper cone in $V$. The functor $R\varphi_{\gamma^a}: D^b_{\gamma^a}(k_V) \to D^b(k_{V,\gamma})$ is an equivalence of triangulated categories with quasi-inverse $\varphi_{\gamma}^{-1}$. Moreover, this equivalence preserves the natural t-structures of both categories. In particular, for $F \in D^b(k_V)$, the condition $F \in D^b_{\gamma^a}(k_V)$ is equivalent to the condition: $\mu \text{supp}(H^j(F)) \subset V \times \gamma^a$ for any $j \in \mathbb{Z}$.

Thanks to this theorem, the reader may ignore microlocal sheaf theory, at least in a first reading.

**Corollary 3.4 ([KS18, Cor. 1.8]).** Let $A$ be a $\gamma$-locally closed subset of $V$. Then $\mu \text{supp}(k_A) \subset V \times \gamma^a$.

**Proposition 3.5.** Assume (3.1). Let $U = U + \gamma$ be a $\gamma$-open set and let $x_0 \in \partial U$. Then there exist a linear coordinate system $(x_1, \ldots, x_n)$ on $U$, an open neighborhood $V$ of $x_0$, an open subset $W$ of $V$ and a bi-Lipschitz isomorphism $\varphi: V \cong W$ such that $\varphi(V \cap U) = W \cap \{ x \in V; x_n > 0 \}$.

**Proof.** The proofs of [GS16, Lem. 2.36, 2.37] (which was formulated for subanalytic open subsets) extend immediately to our situation. \qed

**Corollary 3.6.** Let $Z$ be a $\gamma$-locally closed subset of $V$. Then, $D^b_M(k_Z)$ is concentrated in degree 0. Moreover, $D^b_M(k_Z) \simeq k_S$ with $\Omega = \text{Int}(Z)$ and $S = \Omega + \gamma \cap (\Omega + \gamma^a)$.

**Proof.** It follows from Proposition 3.5 that $D^b_M(k_{\Omega + \gamma}) \simeq k_{\Omega + \gamma^a}$ and $D^b_M(k_{\Omega + \gamma^a}) \simeq k_{\Omega + \gamma^a}$.

Set $A = \Omega + \gamma$ and $B = \Omega + \gamma^a$. Then $k_A$ and $k_B$ are cohomologically constructible. By Corollary 3.4, $\mu \text{supp}(k_A) \cap \mu \text{supp}(k_B) \subset T^*_\gamma V$. Then $D^b_M(k_A \otimes k_B) \simeq D^b_M(k_A) \otimes D^b_M(k_B)$ by [KS90, Cor. 6.4.3]. \qed

### 3.2 Review on PL $\gamma$-sheaves

From now on, we shall assume that the cone $\gamma$ satisfies:

(3.4) $\gamma$ is a closed proper convex polyhedral cone with non-empty interior.

**Definition 3.7.** Assume (3.4).
(a) A $\gamma$-barcode $(A, Z)$ in $V$ is the data of a finite set of indices $A$ and a family $Z = \{Z_a\}_{a \in A}$ of non-empty, $\gamma$-locally closed, convex polytopes of $V$.

(b) A $\gamma$-partition $(A, Z)$ is a $\gamma$-barcode $(A, Z)$ such that $Z_a \cap Z_b = \emptyset$ for $a \neq b$.

(c) The support of a $\gamma$-barcode or a $\gamma$-partition $(A, Z)$, denoted by $\text{supp}(A, Z)$, is the set $\bigcup_{a \in A} Z_a$.

**Remark 3.8.** In [KS18], we defined a PL $\gamma$-stratification of a closed set $S$ as a barcode $(A, Z)$ such that $\text{supp}(A, Z) = S$ and $Z_a \cap Z_b = \emptyset$ for $a \neq b$. However, since a PL $\gamma$-stratification is not a PL stratification (see Definition 1.3), we prefer here to avoid this terminology and use the notion of a $\gamma$-partition.

We shall use the notations:

\[
\begin{aligned}
D_{PL, \gamma a}(k_V) &:= D_{PL}(k_V) \cap D_{\gamma a}(k_V), \\
\text{Mod}_{PL, \gamma a}(k_V) &:= \text{Mod}(k_V) \cap D_{PL, \gamma a}(k_V).
\end{aligned}
\]

Note that, in view of (2.3) and Theorem 3.3:

\[
F \in D_{PL, \gamma a}(k_V) \iff H^j(F) \in \text{Mod}_{PL, \gamma a}(k_V) \text{ for all } j \in \mathbb{Z}.
\]

**Lemma 3.9.** Assume (3.4). If $F \in D_{PL, \gamma a}(k_V)$ then $\varphi_1^{-1} R \varphi_{\gamma a} F \in D_{PL, \gamma a}(k_V)$.

**Proof.** By Theorem 3.3, it remains to prove that $\varphi_1^{-1} R \varphi_{\gamma a} F$ is PL. Denote by $Z(\gamma)$ the set $\{(x, y) \in V \times V; y - x \in \gamma\}$ and denote by $q_1$ and $q_2$ the first and second projections defined on $V \times V$. Then (see [KS90, Prop. 3.5.4]):

\[
\varphi_1^{-1} R \varphi_{\gamma a} F \simeq R q_1, (k_{Z(\gamma)}) \otimes q_2^{-1} F.
\]

Then the result follows from Corollary 2.6.

**Theorem 3.10** ([KS18, Th. 3.10]). Assume (3.4) and let $F \in D_{PL, \gamma a}(k_V)$. Then for each $x \in V$, there exists an open neighborhood $U$ of $x$ such that $F|_{(x + \gamma a) \cap U}$ is constant.

**Theorem 3.11** ([KS18, Th. 3.14]). Assume (3.4) and let $F \in D_{PL, \gamma a}(k_V)$. Let $\Omega$ be a $\gamma$-flat open set and let $Z = (\Omega + \gamma) \cap \Omega + \gamma a$, a $\gamma$-locally closed subset. Assume that $F|_{\Omega}$ is locally constant. Then $F|_{Z}$ is locally constant.

**Theorem 3.12** ([KS18, Lem. 3.16, Th. 3.17]). Assume (3.4) and let $F \in D_{PL, \gamma a}(k_V)$. Then there exists a $\gamma$-partition $(A, Z)$ with $\text{supp}(A, Z) = \text{supp}(F)$ and such that $F|_{Z_a}$ is constant for each $a \in A$. Moreover, $F|_{Z} \simeq 0$ for $x \notin \bigcup_{a \in A} Z_a$.

### 3.3 Generators for PL $\gamma$-sheaves

In [KS18] we have constructed a category $\text{Bar}_\gamma$ whose objects are the $\gamma$-barcodes and a fully faithful functor

\[
\Psi : \text{Bar}_\gamma \to \text{Mod}_{PL, \gamma a}(k_V), \quad Z = \{Z_a\}_{a \in A} \mapsto \bigoplus_{a \in A} k_{Z_a}.
\]

However, as shown in [KS18, Ex. 2.14, 2.15], the functor $\Psi$ is not essentially surjective as soon as $\dim V > 1$. 

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Definition 3.13. An object of \( \text{Mod}_{PL, \gamma \circ a}(k_V) \) is a barcode \( \gamma \)-sheaf if it is in the essential image of \( \Psi \).

In [KS18] we made the following conjecture.

Conjecture 3.14. Let \( F \in D^b_{PL, \gamma \circ a}(k_V) \) and assume that \( F \) has compact support. Then there exists a bounded complex \( F^* \in C^b(\text{Mod}_{PL, \gamma \circ a}(k_V)) \) whose image in \( D^b_{PL, \gamma \circ a}(k_V) \) is isomorphic to \( F \) and such that each component \( F^j \) of \( F^* \) is a barcode \( \gamma \)-sheaf with compact support.

As usual, for an additive category \( \mathcal{C} \), \( C^b(\mathcal{C}) \) denotes the category of bounded complexes of objects of \( \mathcal{C} \).

In this subsection, we shall prove a weaker form of this conjecture, namely:

Theorem 3.15. The triangulated category \( D^b_{PL, \gamma \circ a}(k_V) \) is generated by the family \( \{k_P\}_P \) where \( P \) ranges over the family of \( \gamma \)-locally closed convex polytopes.

In particular, the category \( D^b_{PL, \gamma \circ a}(k_V) \) is generated by the barcodes \( \gamma \)-sheaves.

Proof. Let \( F \in \text{Mod}_{PL, \gamma \circ a}(k_V) \). There exists \( \{\xi_k\}_{1 \leq k \leq l} \in \gamma^{0,a} \) and \( \{c_j\}_{0 \leq j \leq N} \in \mathbb{R} \) with \(-\infty = c_0 < c_1 < \cdots < c_{N-1} < c_N = +\infty \) such that, setting

\[
H_{k,j} = \{x \in V; \langle x, \xi_k \rangle = c_j\}, \quad U := V \setminus \bigcup_{k,j} H_{k,j},
\]

the sheaf \( F|_U \) is locally constant.

Let \( n = (n_1, \ldots, n_l) \) with \( 0 \leq n_k < N \) and define

\[
Z_n = \bigcap_{k=1}^l \{x; c_{n_k} \leq \langle x, \xi_k \rangle < c_{n_k+1}\}, \quad \Omega_n = \bigcap_{k=1}^l \{x; c_{n_k} < \langle x, \xi_k \rangle < c_{n_k+1}\}.
\]

Then \( Z_n = \Omega_n + \gamma^{0,a} \cap (\Omega_n + \gamma) \) and \( Z_n \) is \( \gamma \)-locally closed.

Since \( F|_{\Omega_n} \) is constant, \( F|_{Z_n} \) is constant by Theorem 3.11.

Now we have \( V = \bigcup_{n \in [0,N-1]} Z_n \). Let \( W \) be a \( \gamma \)-closed subset of \( V \) which is a union of \( Z_n \)'s.

Lemma 3.16. There exists an \( n \) such that \( Z_n \) is open in \( W \).

Proof of the lemma. We order the set of \( n \)'s by \( n \leq n' \) if \( n_j \leq n'_j \) for all \( j \in [1, \ldots, l] \). Let \( n \) be minimal among the \( n \)'s such that \( Z_n \subset W \). Then \( Z_n \) is open in \( W \). Indeed,

\[
W \cap Z_n = W \cap \bigcap_k \{x; c_{n_k} \leq \langle x, \xi_k \rangle < c_{n_k+1}\}
= W \cap \bigcap_k \{x; \langle x, \xi_k \rangle < c_{n_k+1}\}.
\]

Indeed, the second equality is true, otherwise there exists \( y \in W \) with \( \langle y, \xi_k \rangle < c_{n_k} \) and \( n \) would not be minimal. \( \square \)
Now we can complete the proof of Theorem 3.15. Let $Z_n \subset W$ open in $W$ and assume that $\text{supp} \, F \subset W$. We have an exact sequence

$$0 \to k_{Z_n} \otimes F(Z_n) \to F \to F'' \to 0$$

and $\text{supp} \, F'' \subset W \setminus Z_n$.

Then the proof goes by induction on $\# \{n; Z_n \subset W\}$. \qed

References

